SCRATCH (iv) — 1

Feb 3, 2019

$\boxed{\text{REVISIT FOURIER TRANSFORMS in } SE(1)}$

$\delta(\theta)\ \delta(x-x_0)$ FORWARD, or

$\delta(\theta-\pi)\ \delta(x-x_0)$ REVERSE

$\rightarrow \delta(\theta-\theta_0)\,\delta(x-x_0)$

$\boxed{\theta_0 = 0, \pi}$

$$u(\theta,x) \stackrel{?}{=} \frac{e^{-im\theta}}{\sqrt{2}} \frac{e^{-ipx}}{\sqrt{2\pi}}$$

TRANSFORM $\theta_0$, $x_0$

$$f(\theta,x) = \delta(\theta-\theta_0)\,\delta(x-x_0)$$

$$\hat{f}(m,p) = \frac{e^{+im\theta_0}}{\sqrt{2}} \frac{e^{+ipx_0}}{\sqrt{2\pi}} \quad \text{OK.}$$

$m \stackrel{?}{=} 0, 1.$

$0$: $s$ wave (symmetric); $1$: $p$ wave [antisymmetric]

Check convolution:

$$\hat{f}_2(m,p) * \hat{f}_1(m,p) = e^{+[im(\theta_1+\theta_2)]}\, e^{+ip[x_0+x_1]}$$

~~OK!~~ NO, ACTUALLY NOT CORRECT ... see p. 7 – 8

BACK CONVERT?

$$\int dp \sum_{m=0}^{1} \left[\frac{e^{+im\theta_0}}{\sqrt{2}} \frac{e^{+ipx_0}}{\sqrt{2\pi}}\right] \frac{e^{-im\theta}}{\sqrt{2}} \frac{e^{-ipx}}{\sqrt{2\pi}}$$

$$\uparrow \hat{f}_{\theta_0,x_0}(m,p)$$

$$= \frac{1}{2}\left[1 + e^{i[\theta_0-\theta]}\right] \int dp\, \frac{e^{ip(x_0-x)}}{2\pi}$$

check $\theta_0 = 0 \Rightarrow \delta_{\theta,0}$, $\theta_0 = \pi \Rightarrow \delta_{\theta,\pi}$

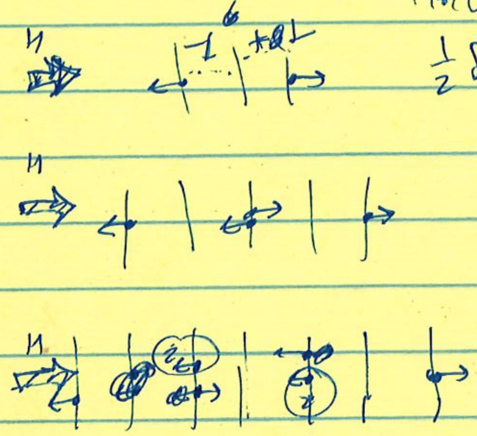
$$= \delta_{\theta,\theta_0}\ \delta(x-x_0). \qquad \text{o.K.}$$

Might want to choose a different convention for $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2\pi}}$ normalization factors.

WORK OUT AN EXAMPLE

TRANSFORM

$f(\theta, x) = \frac{1}{2}\delta_{\theta,0}\delta(x-L) + \frac{1}{2}\delta_{\theta,\pi}\delta(x+L)$



$$\hat{f}(m,p) = \frac{1}{\sqrt{4\pi}}\cdot\frac{1}{2}\left[e^{ipL} + e^{i\pi m}e^{-ipL}\right]$$

$$\hat{f}\cdot\hat{f}(m,p) = \frac{1}{4\pi}\cdot\frac{1}{4}\left[e^{2ipL} + e^{-2ipL} + 2e^{i\pi m} + \cancel{\ldots}\right) \quad ?$$

NO CROSS DIRECTION...

Need a different multiplication rule...

WANT:

$$e^{2ipL}, \quad e^{i\pi m}e^{-2ipL}, \quad e^{i\pi m}, \quad 1$$

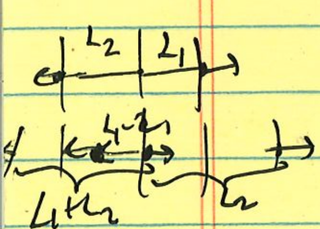
- Try MATRIX multiplication... $\left[e^{i\pi m}\ e^{-ipL}\right] \otimes \left[e^{-ipL}\right]$

Guess:

$$\begin{bmatrix} e^{ipL} & e^{i\pi m}e^{-ipL} \\ e^{ipL} & e^{-i\pi m}e^{-ipL} \end{bmatrix}\begin{bmatrix} e^{ipL} & e^{i\pi m}e^{-ipL} \\ e^{-ipL} & e^{-i\pi m}e^{ipL} \end{bmatrix} = \begin{bmatrix} e^{2ipL} + e^{i\pi m}e^{-2ipL} & e^{i\pi m} + 1 \\ 1 + e^{i\pi m} & e^{2ipL} + e^{i\pi m}e^{-2ipL} \end{bmatrix}$$

Complex conjugate?

- NOPE. Need $e^{i\pi m}$ to force complex conjugation.



$$\begin{pmatrix} e^{ipL_1} & e^{ipL_2} \\ e^{-ipL_2} & e^{-ipL_1} \end{pmatrix}\begin{pmatrix} e^{ipL_1} & e^{ipL_2} \\ e^{-ipL_2} & e^{-ipL_1} \end{pmatrix} = \begin{pmatrix} e^{ip(2L_1)} + 1 & e^{ip(L_1+L_2)} + e^{ip(L_2-L_1)} \\ e^{-ip(L_2-L_1)} + e^{-ip(L_1+L_2)} & e^{-2ipL_1} + 1 \end{pmatrix}$$

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BASIS FUNCTIONS

$e^{-ipx}\begin{pmatrix}1&0\\0&1\end{pmatrix}$ → $u_{mn}$ ?

$e^{-ipx}\begin{pmatrix}0&1\\1&0\end{pmatrix}$

$\begin{pmatrix}e^{ipx}&0\\0&e^{-ipx}\end{pmatrix}$ "FORWARD" $\delta_{\theta 0}$

$\begin{pmatrix}0&e^{ipx}\\e^{-ipx}&0\end{pmatrix}$ "REVERSE" $\delta_{\theta,\pi}$

What if we further transform $\theta \leftrightarrow m$?

$\begin{pmatrix}e^{ipx} & e^{ipx+im\theta}\\ e^{-ipx+im\theta} & e^{-ipx}\end{pmatrix}$, $\begin{pmatrix}e^{ipx} & -e^{ipx+im\theta}\\ -e^{-ipx+im\theta} & e^{-ipx}\end{pmatrix}$

what if we used same functions as SE(2)? Something like in

$e^{in\theta}$ $e^{im(\theta-\phi)}$ $J_{m-n}(pr)$