$J_{-n}(x) = (-1)^n J_n(x)$ ① [wikipedia]

SE(2) HARMONIC TRANSFORMS

$$u^{(1)}_{ab} = i^{b-a} e^{i[a\theta_1 + (b-a)\phi_1]} J_{a-b}(pr_1)$$

$$u^{(2)}_{bc} = i^{c-b} e^{i[b\theta_2 + (c-b)\phi_2]} J_{b-c}(pr_2)$$

$$\sum_b u^{(1)}_{ab} u^{(2)}_{bc} = i^{c-a} e^{ia\theta_1 + \cancel{\ldots} b(\phi_1-\phi_2) - a\phi_1 + c\phi_2} \sum_{b=\infty}^{\infty} e^{ib[\theta_2+\phi_2-\phi_1]} J_{a-b}(pr_1) J_{b-c}(pr_2)$$

Neumann's Addition theorem ← Abramowitz Stegun.

$$J_\nu(u \pm v) = \sum_{k=-\infty}^{\infty} J_{\nu \mp k}(u) J_k(v)$$

if

~~$\sum_b u_{ab} u_{bc} = e^{\ldots} e^{i\theta + (c-a)\phi}$~~

$$\sum J_{a-b}(pr) J_{b-c}(pr)$$

$$= \sum J_{a-c-k}(pr_1) J_k(pr_2) \qquad \begin{matrix} k=b-c \\ b=k+c \end{matrix}$$

$$= J_{a-c}[p(r_1+r_2)]$$

So if $\theta_2, \phi_1, \phi_2 = 0$ ...

$$u_{ab} u_{bc} = i^{c-a} e^{ia\theta_1} J_{a-c}[p(\underbrace{r_1+r_2}_{r})]$$

Looks O.K. ... Rotation by θ_1, then translation by r_{ϕ_1} then r_2 ... $= r_1 + r_2$.

Need expression for:

$$? = \sum_{k=-\infty}^{\infty} \left[J_{\nu-k}(u) J_k(v) e^{ik\theta} \right]$$

$$u_{mn}(g(r,\phi,\theta),p) = i^{n-m} e^{-i[n\theta+(m-n)\phi]} J_{n-m}(pr)$$

$$u_{mn}(g^{-1}(r,\phi,\theta),p) = i^{n-m} e^{i[m\theta+(n-m)\phi]} J_{m-n}(pr),$$

INVERSE... $-r$, $-\theta$, $\phi-\theta$

$$u_{mn}(I,p) = u_{ma}(g(r,\phi,\theta,p))\,u_{an}(g^{-1}(r,\phi,\theta))$$

$$= \sum_a i^{a-m} e^{-i[a\theta+(m-a)\phi]} J_{a-m}(pr)$$

$$i^{n-a} e^{+i[a\theta+(n-a)\phi]} J_{a-n}(pr)$$

$$= \sum_a i^{n-m} e^{-i(m-n)\phi} J_{a-m}(pr) J_{a-n}(pr)$$

$$\overset{?}{=} i^{n-m} e^{i(m-n)\phi} J$$

uh... still having issues with expansion.

WHAT IS DELTA FUNCTION?

RxT

ROTATE $-\theta$

TRANSLATE

$-r$
$\phi-\theta$

$$e^{-i[n(-\theta)+(m-n)(\phi-\theta)]} J(pr)$$

$$= e^{-i[m\theta+(m-n)\phi]} J(pr)$$

$$= e^{[+im\theta+(n-m)\phi]} J_{m-n}(pr)$$

O.K.

?

Are odd Bessel functions also odd functions? YES

$$J_k(-pr) = (-1)^k J_k(-pr) = J_{-k}(pr)$$

For a δ Function,

$$u_{mn} \overset{?}{=} i^{n-m} e^{i[m\theta+(n-m)\phi]} J_{m-n}(pr)$$

Suppose:

$$f(g(r,\phi,\theta)) = \delta(\theta-\theta_0)\,\delta(\phi-\phi_0)\,\delta(r-r_0)/r.$$

$$\hat{f}_{mn}(p) = \int\left[\delta(\theta-\theta_0)\delta(\phi-\phi_0)\delta(r-r_0)/r\right] r\,dr\,d\phi\,d\theta$$

$$\times\, i^{n-m} e^{-i[n\theta+(m-n)\phi]} J_{n-m}(pr)$$

$$= i^{n-m} J_{m-n}(pr_0)\, e^{-in\theta_0+(m-n)\phi_0}$$

Deriving key identity for SE(2) transform

First, note "Bessel's First integral":

$$J_n(z) = \frac{1}{2\pi i^n}\int_0^{2\pi} e^{iz\cos\phi}\, e^{in\phi}\, d\phi .$$

~~Sum~~ Addition theorem:

$$\sum_{m=-\infty}^{\infty} J_m(y)\, J_{n-m}(z)$$

$$= \sum_{m=-\infty}^{\infty} \int_0^{2\pi}\!\!\int_0^{2\pi} d\phi\, d\phi'\; e^{iy\cos\phi}\, e^{im\phi}\; e^{iz\cos\phi'}\, e^{i(n-m)\phi'} \times \frac{1}{2\pi i^m}\,\frac{1}{2\pi i^{n-m}}$$

$$= \frac{1}{2\pi i^n}\int_0^{2\pi}\!\!\int_0^{2\pi} d\phi\, d\phi'\; e^{iy\cos\phi}\, e^{iz\cos\phi'}\, e^{in\phi'}\; \underbrace{\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}}_{\delta(\phi-\phi')}$$

$$= \frac{1}{2\pi i^n}\int_0^{2\pi} d\phi\; e^{i(y+z)\cos\phi}\, e^{in\phi}$$

$$= J_n(y+z)$$

sweet.

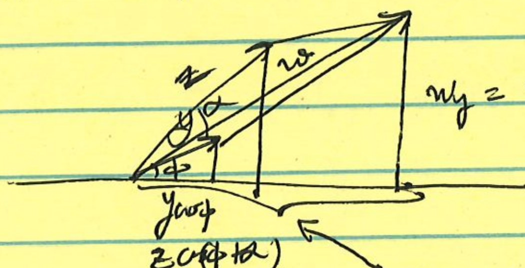
Now, what if I introduce additional $e^{im\alpha}$ term?

④

$$\sum_{m=-\infty}^{\infty} J_m(y)\, J_{n-m}(z)\, e^{im\alpha}$$

$$= \frac{1}{2\pi i^n} \int_0^{2\pi}\int_0^{2\pi} d\phi\, d\phi'\, e^{iy\cos\phi}\, e^{iz\cos\phi'}\, e^{in\phi'}\, \underbrace{\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi'+\alpha)}}_{\delta(\phi'-(\phi+\alpha))}$$

$$= \frac{1}{2\pi i^n} \int_0^{2\pi} d\phi\, e^{iy\cos\phi}\, e^{iz\cos(\phi+\alpha)}\, e^{in\phi}\, e^{in\alpha}$$



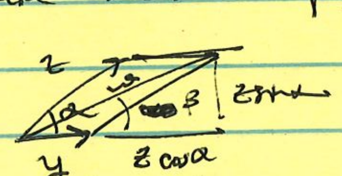
$w_y = y \sin\phi + z \sin(\phi+\alpha)$

$w_x = y\cos\phi + z\cos(\phi+\alpha)$

Check: $w^2 = w_x^2 + w_y^2 = y^2 + z^2 + 2yz\underbrace{[\sin\phi\sin(\phi+\alpha) + \cos\phi\cos(\phi+\alpha)]}_{= \cos\alpha}$

$$= y^2 + 2yz\cos\alpha + z^2. \qquad \boxed{\text{YES}}$$

But there is also a phase shift...



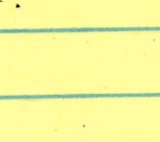
$$\tan\beta = \frac{z\sin\alpha}{y + z\cos\alpha}$$

$$\beta = \tan^{-1}\frac{z\sin\alpha}{y+z\cos\alpha}$$ not quite symmetric But o.k.

$$\sum_{m=-\infty}^{\infty} J_m(y)\, J_{n-m}(z)\, e^{im\alpha} = \frac{1}{2\pi i^n}\int_0^{2\pi} d\phi\, e^{iw\cos(\phi+\beta)}\, e^{in\phi}\, e^{in\alpha}$$

$$= e^{in\alpha}\, e^{-in\beta}\, J_n(w)$$

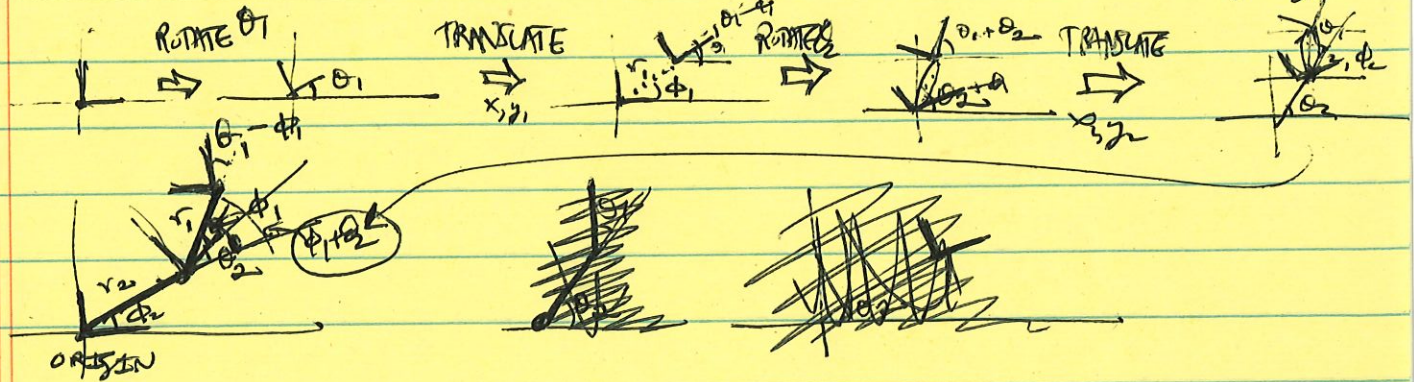
with $w = [y^2 + 2yz\cos\alpha + z^2]^{1/2}$



$$\beta = \tan^{-1}\frac{z\sin\alpha}{y+z\cos\alpha}$$

i.e. given by vector sum

Now let's re-orient composition of two translation/rotations but in SE(2), HARMONIC TRANSFORMS (see p. 1)



And ... actually need to apply $\hat{u}^{(2)}$ to $\hat{u}^{(1)}$ [HAD REVERSE ORDER on p. 1 ?]

RE-WRITE CLEANLY ON NEXT PAGE

$$u^{(2)}_{ab} = i^{b-a} e^{i[a\theta_2 + (b-a)\phi_2]} J_{a-b}(p r_2)$$

$$u^{(1)}_{bc} = i^{c-b} e^{i[b\theta_1 + (c-b)\phi_1]} J_{b-c}(p r_1)$$

$$u_{ac} = \sum_b u_{ab} u_{bc} = i^{c-a} e^{ia\theta_2} e^{ic\phi_1 - a\phi_2} \sum_b e^{ib[\theta_1 - \phi_1 + \phi_2]} \underbrace{J_{a-b}}_{(a-c)-m \text{ ``}m\text{''}}(pr_2) \underbrace{J_{b-c}}_{b = m+c}(pr_1)$$

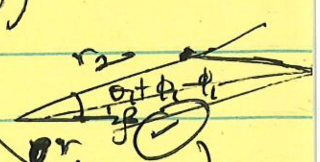
$$= i^{c-a} e^{ia\theta_2} e^{\cancel{ic\phi_1} - a\phi_2} \times e^{ic[\theta_1 - \cancel{\phi_1} + \phi_2]} \sum e^{im[\theta_1 + \phi_2 - \phi_1]} J_m(pr_1) J_{(a-c)-m}(pr_2)$$

$$= i^{c-a} e^{i[a\theta_2 + c\theta_1 + (c-a)\phi_2]} e^{-i(a-c)\beta} J_{a-c}(p\omega)$$

$$e^{i[a(\theta_1+\theta_2) + (c-a)(\phi_2+\theta_1)]} \quad e^{i(a-c)(\theta_1+\phi_2-\phi_1)}$$

$-\theta_1 + \phi_1 - \phi_2$

$$= i^{c-a} e^{i[a(\theta_1+\theta_2) + (c-a)(\phi_2 + \theta_1 + \beta)]} J_{a-c}(p\omega)$$



$\boxed{\beta = \theta_1 + \phi_2 - \phi_1}$

Should be $(\theta_1 - \phi_1) + (\phi_1 + \theta_2) = \theta_1 + \theta_2$ OK

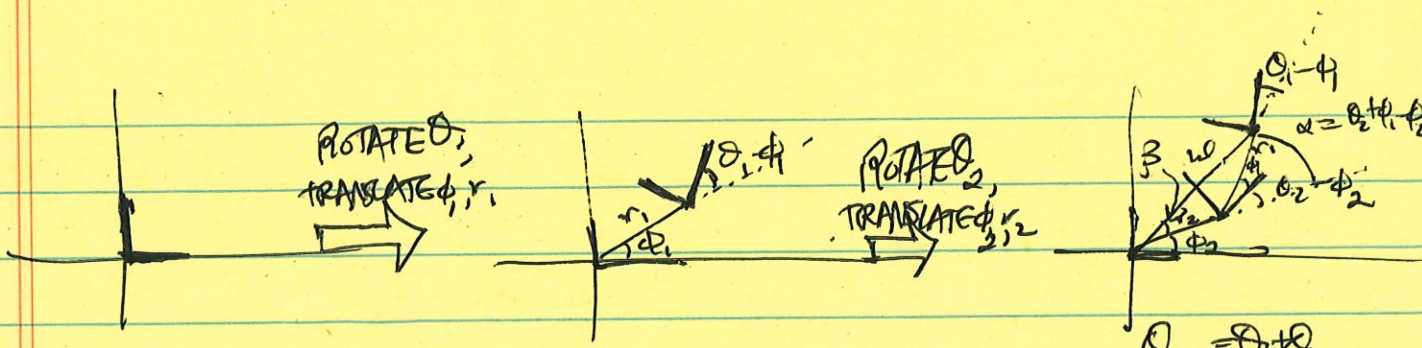
Almost

Should be

DIFFERENCE ??

but ϕ_1

Hmmm almost — actually should NOT HAVE REVERSED ORDER.



$\theta_{tot} = \theta_1 + \theta_2$

$\phi_{tot} = \beta + \phi_2$

$r_{tot} = w$

$$u^{tot}_{ac} = \sum_b u^{(1)}_{ab} u^{(2)}_{bc} \quad \text{with}$$

$$u^{(1)}_{ab} = i^{b-a} e^{i[a\theta_1 + (b-a)\phi_1]} J_{a-b}(pr_1)$$

$$u^{(2)}_{bc} = i^{c-b} e^{i[b\theta_2 + (c-b)\phi_2]} J_{b-c}(pr_2)$$

$$u^{tot}_{ac} = \sum_b i^{c-a} e^{i[a\theta_1 - a\phi_1 + c\phi_2]} e^{ib\underbrace{[\phi_1 - \phi_2 + \theta_2]}_{\alpha ?}} J_{a-b}(pr_1) J_{b-c}(pr_2)$$

(substitute $m = b-c$)

$$= i^{c-a} e^{i[a\theta_1 - a\phi_1 + c\phi_2]} e^{ic[\phi_1 - \phi_2 + \theta_2]} \sum_m e^{im[\phi_1 - \phi_2 + \theta_2]} J_{(a-c)-m}(pr_1) J_{b-c}(pr_2)$$

$$= i^{c-a} e^{i[a\theta_1 - \cancel{a\phi_1} + c\phi_2]} e^{i[\cancel{c\phi_1} - \cancel{c\phi_2} + \cancel{c\theta_2}]} e^{i(a-c)[\phi_1 - \phi_2 + \theta_2]} e^{-i(a-c)\beta} J_{a-c}(pw)$$

$$e^{i[\cancel{a\phi_1} - a\phi_2 + a\theta_2 - \cancel{c\phi_1} + c\phi_2 - \cancel{c\theta_2}]}$$

$$= i^{c-a} e^{i[a(\theta_1 + \theta_2) + (c-a)(\beta + \phi_2)]} J_{a-c}(pw)$$