Aalto University

Reflection and Refraction in Fluid Boundaries

S-89.3310 Acoustics and the Physics of Sound, Lecture 7

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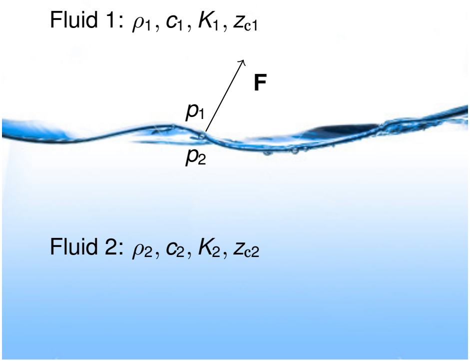
Fluid Boundary

1 Continuity at the Boundary

Let us consider the boundary between two fluids.

- pressures p_1 and p_2 create a force **F**
- $p = F/A \Rightarrow |\mathbf{F}| = p_1 - p_2$
- boundary is infinitely thin $\Rightarrow$ **F** must be zero (why?)



Conclusion: **pressure must be continuous at the boundary**!

1 Continuity at the Boundary II

Also, the **normal component of particle velocity** (or displacement) **must be continuous at the boundary**.

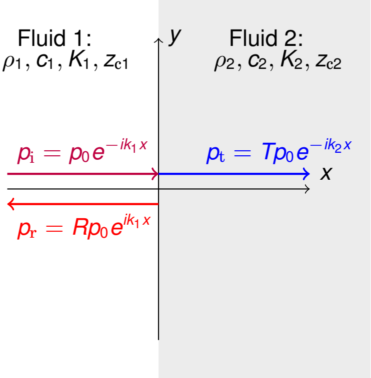
- if it were not, there would either become a vacuum at the boundary, or the fluids would overlap each other
- $\Rightarrow$ the normal component of particle velocity must be zero at the boundary of a fluid and an infinitely rigid wall

2 Perpendicular Reflection

2 Perpendicular Reflection: Pressure

If the incident angle is 90° w.r.t boundary, a *perpendicular reflection* occurs

- consider an incoming plane wave p_i (temporal term omitted)
- sound pressure p_r reflects
  - R is the reflection coefficient for pressure
  - sign of the exponential changes (why?)
- pressure p_t propagates into fluid 2
  - T is the transmission coefficient for pressure



Note: $k_1 = \frac{\omega}{c_1} \neq \frac{\omega}{c_2} = k_2$

2 Perpendicular Reflection: Particle Velocity

Particle velocities behave in a similar manner

- plane wave: $u = \frac{p}{z_\mathrm{c}}$
- fluid 1:
  - incident particle velocity $u_\mathrm{i} = \frac{p_0}{z_{\mathrm{c}1}} e^{-ik_1 x}$
  - reflecting particle velocity $u_\mathrm{r} = -\frac{Rp_0}{z_{\mathrm{c}1}} e^{ik_1 x}$
  - note: the signs of the exponential and the term itself change (why?)
- fluid 2:
  - transmitted particle velocity $u_\mathrm{t} = \frac{Tp_0}{z_{\mathrm{c}2}} e^{-ik_2 x}$

2 Perpendicular Reflection: Relation between Coefficients

Pressure continuity:

$$p_0 + Rp_0 = Tp_0 \Rightarrow T = 1 + R$$

Continuity for the normal component of particle velocity:

$$\frac{p_0}{z_{c1}} - \frac{Rp_0}{z_{c1}} = \frac{Tp_0}{z_{c2}} \Rightarrow \frac{1-R}{z_{c1}} = \frac{T}{z_{c2}}$$

Thus,

$$R = \frac{z_{c2} - z_{c1}}{z_{c2} + z_{c1}} \text{ (always } \leq 1) \qquad (1)$$

$$T = \frac{2z_{c2}}{z_{c2} + z_{c1}} \text{ (can be } \geq 1) \qquad (2)$$

2 Perpendicular Reflection: Relation between Coefficients II

- Since T can be over unity, sound pressure can increase when the wave passes through the boundary

"How can this be? Energy can't increase!"

- Remember: the characteristic impedances are different: if the particle velocity decreases, sound pressure increases
- of course, intensity can't increase

In fact, also the characteristic impedance is continuous at the boundary (see this by comparing the pressures and particle velocities on both sides)

2 Perpendicular Reflection: Intensity

- Incoming intensity: $I_\mathrm{i} = \frac{|p_\mathrm{i}|^2}{z_\mathrm{c1}}$
- Reflecting intensity: $I_\mathrm{r} = \frac{|p_\mathrm{r}|^2}{z_\mathrm{c1}} = \frac{|Rp_\mathrm{i}|^2}{z_\mathrm{c1}} = |R|^2 I_\mathrm{i}$
- Transmitted intensity: $I_\mathrm{t} = \frac{|p_\mathrm{t}|^2}{z_\mathrm{c2}} = \frac{|Tp_\mathrm{i}|^2}{z_\mathrm{c2}} = \frac{z_\mathrm{c1}}{z_\mathrm{c2}}|T|^2 I_\mathrm{i}$
- $I_\mathrm{i} - I_\mathrm{r} = I_\mathrm{t}$ (check the math to verify)
- Transmitted intensity can also be given as

$$I_\mathrm{t} = I_\mathrm{i} - I_\mathrm{r} = (1 - |R|^2) I_\mathrm{i} \tag{3}$$

  - in other words, obtained using the incoming intensity and characteristic impedances!

2 Perpendicular Reflection: Intensity II

The result in Eq. (3) holds for the active intensity also when z_{c2} is complex

- denote $z_{c2} = r_2 + ix_2$, $z_{c1} = r_1$

Remember Eq. (1): $R = \frac{z_{c2}-z_{c1}}{z_{c2}+z_{c1}} \Rightarrow |R|^2 = \frac{|r_2+ix_2-r_1|^2}{|r_2+ix_2+r_1|^2}$

- the energy transfer coefficient becomes

$$1-|R|^2 = \frac{|r_2 + ix_2 + r_1|^2 - |r_2 + ix_2 - r_1|^2}{|r_2 + ix_2 + r_1|^2} = \frac{4r_1 r_2}{(r_1 + r_2)^2 + x_2^2}$$

- if also z_{c2} is real, $x_2 = 0$ and we get

$$1 - |R|^2 = \frac{4r_1 r_2}{(r_1 + r_2)^2}$$

2 Example: Sound from Air to Water

Let's consider a case where a plane wave arrives perpendicularly from air to water. What is the ratio between the transmitted and incoming

- (a) sound pressures?
- (b) intensities?

$\rho_{\text{air}} = 1.2\ \text{kg/m}^3$, $\rho_{\text{water}} = 1000\ \text{kg/m}^3$,
$c_{\text{air}} = 343\ \text{m/s}$, $c_{\text{water}} = 1500\ \text{m/s}$.

$z_{\text{c,air}} = \rho_{\text{air}} c_{\text{air}} = 412\ \text{Pas/m}$,

$z_{\text{c,water}} = \rho_{\text{water}} c_{\text{water}} = 1.5\ \text{MPas/m}$,

Eq. (2): $T = \frac{2z_{\text{c,water}}}{z_{\text{c,water}} + z_{\text{c,air}}} \approx 2$

Eq. (3): $\frac{I_{\text{t}}}{I_{\text{i}}} = 1 - |R|^2$, Eq. (1): $R = \frac{z_{\text{c,water}} - z_{\text{c,air}}}{z_{\text{c,water}} + z_{\text{c,air}}} \Rightarrow$

$\frac{I_{\text{t}}}{I_{\text{i}}} = 1 - |R|^2 \approx 0.0011$

$\Rightarrow$ large impedance mismatch, poor power transfer.

2 Table: Sound from Air to Different Materials

Table: Perpendicular reflection from air to different materials

Material	z_c [Pas/m]	I_t/I_i	TL [dB]
steel	$3.9 \cdot 10^7$	$4 \cdot 10^{-5}$	44
concrete	$9 \cdot 10^6$	$18 \cdot 10^{-5}$	38
glass	$1.4 \cdot 10^7$	$11 \cdot 10^{-5}$	40
oak	$2.8 \cdot 10^6$	$6 \cdot 10^{-4}$	32
mineral wool	1000	0.83	1

Note: TL gives here only the effect of a single boundary, not dissipative losses, etc.

2 Plane-Wave Generated Sound Field in Front of a Boundary

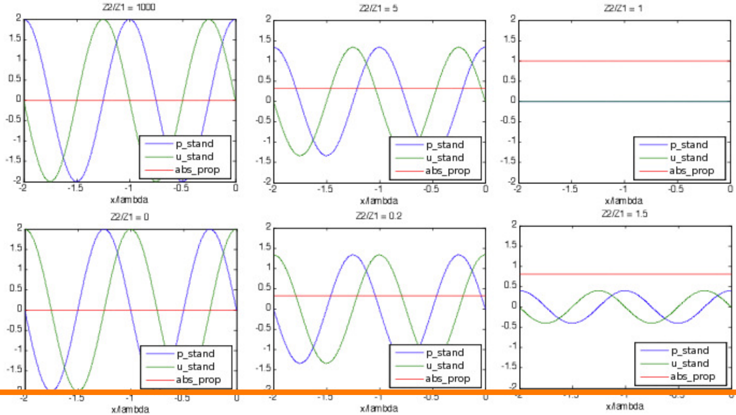
The sound field in front of the boundary consists of **the incoming wave** and **the reflecting wave**. Another interpretation is **a standing wave** plus **a propagating wave**

$$p = p_{\mathrm{i}} + p_{\mathrm{r}} = p_0(e^{-ik_1x} + Re^{ik_1x})$$

$$= \begin{cases} p_0(2R\cos(k_1x) + (1-R)e^{-ik_1x}), \text{ when } R \geq 0 \\ p_0(2iR\sin(k_1x) + (1+R)e^{-ik_1x}), \text{ when } R \leq 0 \end{cases}$$

Remember: a standing wave transmits no energy (active intensity is zero). Similar equations can be derived for the particle velocity.

2 Example: Standing and Propagating Waves with Changing Impedance Ratio



2 Impedance in front of a Boundary

Specific acoustic impedance at the distance d from the boundary can be given as

$$
\begin{aligned}
z_c = \frac{p_x}{u_x} &= z_{c1} \frac{e^{-ik_1 x} + Re^{ik_1 x}}{e^{-ik_1 x} - Re^{ik_1 x}} \\
&= z_{c1} \frac{z_{c2} \cos(k_1 d) + i z_{c1} \sin(k_1 d)}{z_{c1} \cos(k_1 d) + i z_{c2} \sin(k_1 d)}, \ d = -x \\
&= z_{c1} \frac{z_{c2} + i z_{c1} \tan(k_1 d)}{z_{c1} + i z_{c2} \tan(k_1 d)} \qquad (4)
\end{aligned}
$$

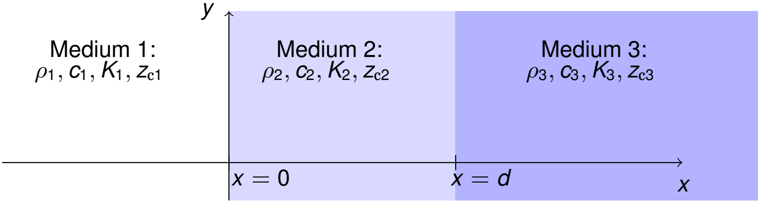
2 Reflection from Multilayer Medium

Consider a structure consisting of a multilayer medium. The impedance to the right at $x = 0$ depends on

- properties of media 2 and 3
- reflection at $x = d$

$$z_0 = z_{c2} \frac{z_{c3} + i z_{c2} \tan(k_2 d)}{z_{c2} + i z_{c3} \tan(k_2 d)}$$

With multiple layers, one starts evaluating the boundary impedances from right to left using the above equation

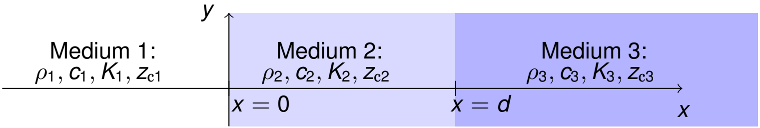


2 Reflection from Multilayer Medium II

Is it possible to select the material for medium 2 and thickness d so that there is no reflection back to medium 1? I. e. this would mean $z_0 = z_{c1}$ for $z_0 = z_{c2}\frac{z_{c3}+iz_{c2}\tan(k_2 d)}{z_{c2}+iz_{c3}\tan(k_2 d)}$.

Answer: Yes, there are three such cases (check the math):

- Trivial case: $z_{c1} = z_{c2} = z_{c3}$
- Half-wave layer: $z_{c1} = z_{c3}$, $d = \frac{n\lambda_2}{2}$, $n = 1, 2, 3...$
- Quarter-wave transformer:
  $z_{c2} = \sqrt{z_{c1} z_{c3}}$, $d = \frac{(2n+1)\lambda_2}{4}$, $n = 1, 2, 3...$



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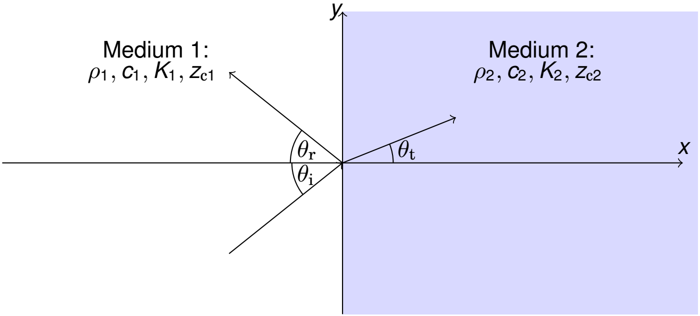
Reflection with Arbitrary Angle

3 Snell's Law



Generally,

- $\theta_i = \theta_r$
- $\theta_i \neq \theta_t \Rightarrow$ refraction happens!
- furthermore, $\frac{c_1}{c_2} = \frac{\sin\theta_i}{\sin\theta_t}$

3 Impedance in front of a Boundary

For arbitrary angles, the impedance at distance *d* from the boundary can be given as (compare to Eq. (4))

$$z = \left(\frac{z_{c1}}{\cos\theta_\mathrm{i}}\right) \frac{\left(\frac{z_{c2}}{\cos\theta_\mathrm{t}}\right) + i\left(\frac{z_{c1}}{\cos\theta_\mathrm{i}}\right)\tan(k_1 d\cos\theta_\mathrm{i})}{\left(\frac{z_{c1}}{\cos\theta_\mathrm{i}}\right) + i\left(\frac{z_{c2}}{\cos\theta_\mathrm{t}}\right)\tan(k_1 d\cos\theta_\mathrm{i})} \tag{5}$$

Reflection coefficient in this case is

$$R = \frac{\frac{z_{c2}}{\cos\theta_\mathrm{t}} - \frac{z_{c1}}{\cos\theta_\mathrm{i}}}{\frac{z_{c2}}{\cos\theta_\mathrm{t}} + \frac{z_{c1}}{\cos\theta_\mathrm{i}}} \tag{6}$$

3 Total Reflection

If $c_1 < c_2$, there is a **critical angle**

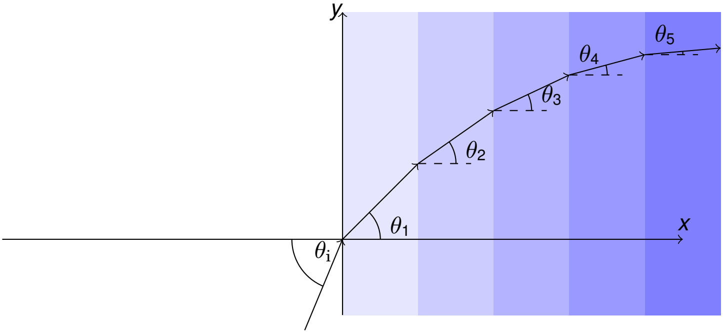
$$\theta_{\mathrm{cr}} = \sin^{-1}\left(\frac{c_1}{c_2}\right), \tag{7}$$

for which the wave in medium 2 propagates *parallel to the surface*. If $\theta_{\mathrm{i}} > \theta_{\mathrm{cr}}$, there is an inhomogeneous plane wave propagating along the surface, and it decays exponentially when moving to medium 2.

- sound energy does not propagate into medium 2 $\Rightarrow$ total reflection of sound energy!

`http://www.phy.ntnu.edu.tw/ntnujava/index.php?topic=16`

3 Snell's Law in Multilayer or Gradually Changing Media



Bending of sound due to temperature gradients!

See also demo at `http://www.falstad.com/ripple/`

3 Locally Reacting Surface

Generally, when a wave propagates from one medium to another, it creates a surface wave that propagates along the boundary. I. e., the particle velocity on any point on the surface depends both on the

- local sound pressure
- vibration caused by the surface wave

Thus, evaluation of the surface impedance becomes complicated. However, the surface wave may be neglected if

- the surface is anisotropic so that it is hard for the surface wave to propagate
- $c_1 >> c_2$
- medium 2 is highly dissipative, so that the surface wave attenuates quickly

3 Locally Reacting Surface II

If some of these conditions hold, the surface is called **locally reactive**. Properties of locally reactive surfaces:

- the particle velocity at any point on the surface depends only on the local sound pressure
- the direction of an incoming plane wave becomes irrelevant

Examples of locally reacting media (when air is medium 1):

- most acoustic absorption materials, such as mineral wools
- perforated materials
- soft soil

3 Locally Reacting Surface III

Example of a locally reacting surface: perforated material.