Reassessment Carnival Problems – MATH 210 – Spring 2019

1. (**P1**) Here's a sketch of a proof of a theorem: if n^2 is a multiple of 5, then n is a multiple of 5. The proof is by contrapositive: assume n is not a multiple of 5. Then n is of the form $5k + j$, where k and j are integers and $j = 1, 2, 3, 4$. Now we'd like to show n^2 is not a multiple of 5. $n^2 = (5k + j)^2 = 25k^2 + 10kj + j^2$. So then we can just look at what happens when we square j. The squares of 1, 2, 3, and 4 are all not multiples of 5 (we should check this). So you can't write n^2 as a multiple of 5 (we should check this too). Yay, we win.

Use the idea of this proof to show that if n^2 is divisible by 7, then n is divisible by 7.

2. **(P2, L1)** Let $P(n)$ be the proposition "n is a prime number". Consider the following statement:

$$\forall n \in \mathbb{N}, P(n) \to (\exists k \in \mathbb{N}, n = 2k + 1).$$

This statement is false. Explain why, and give a counterexample.

3. **(P3)** Give a combinatorial proof of the fact that each row of Pascal's triangle adds up to a power of 2:

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n.$$

4. **(P4)** The following is true for all integers $n \geq 1$. Prove it using mathematical induction.

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

5. **(P5)** Here is a true theorem: If n^2 is a multiple of 3, then n is a multiple of 3.

Here is an **incorrect** proof of this theorem:

> This proof is by contrapositive. Suppose that n is not a multiple of 3. Then there is some integer k such that $n = 3k + 1$. Therefore, $n^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$. Let $\ell = 3k^2 + 2k$; then ℓ is also an integer. Thus $n^2 = 3\ell + 1$, and so n^2 is also not a multiple of 3. □

Explain why this proof is incorrect. For optional **P1** credit, fix this proof.

6. (**L1, L3**) Consider these two very similar-looking statements:

$\exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z}, x + y = 0$

$\forall y \in \mathbb{Z}\ \exists x \in \mathbb{Z}, x + y = 0$

(a) Translate both statements into words.

(b) Write in symbols the formal negation of both statements.

(c) Precisely one of these two statements is true. Which is it, and why?
(Moral: the order of quantifiers is super important.)

7. (**L2**) We have a number of logical connectives but we really could have gotten away with fewer. For example, any time we wanted to say $P \iff Q$ we could have just said it with **AND**, **OR**, and **NOT** with $(P \wedge Q) \vee (\neg P \wedge \neg Q)$. Write a truth table to show that these two statements are logically equivalent.

8. **(L3, S4)** Consider the following statement about a relationship R on some set A:

$$\exists x \in A\ \exists y \in A,\ xRy \wedge \neg(yRx)$$

(a) Write the formal simplified negation of this statement.

(b) Explain the meaning of the original statement and the meaning of its negation using familiar properties of relations.

9. **(S1, S2)** Consider the sets $A = \{a \in \mathbb{N} : \exists k \in \mathbb{N}, a = 5k\}$ and $B = \{b \in \mathbb{N} : \exists j \in \mathbb{N}, b = 6j + 1\}$.

(a) Give an example of some number in $A \cap B$. Explain why you're right.

(b) Give an example of some number in $A \cap \overline{B}$. Explain why you're right.

10. **(S3)** Another way functions (and more general relations) are sometimes represented is as a set of ordered pairs. For instance, if $f : \{1, 2, 3, 4\} \to \mathbb{N}$ was given by $f(x) = x^2$, then we could write f as the set $\{(1, 1), (2, 4), (3, 9), (4, 16)\}$.

(a) Let $g : \{a, b, c, d\} \to \{v, w, x, y, z\}$ be given by the set of ordered pairs

$$\{(d, v), (a, x), (c, y), (b, w), (a, x), (c, z), (d, v)\}.$$

Is g a function? Why or why not?

(b) Let $h : \{a, b, c, d\} \to \{v, w, x, y, z\}$ be given by the set of ordered pairs

$$\{(c, x), (d, y), (a, y), (c, x), (b, z)\}.$$

I'll tell you for free that h is a function. Is it injective? Is it surjective? Explain why or why not.

11. **(S4, S5)** Recall that the power set of A, written $\mathcal{P}(A)$, is the set of all subsets of A.

(a) Write all of the elements of $\mathcal{P}(\{1, 2, 3\})$.

(b) We define a relation on $\mathcal{P}(\{1, 2, 3\})$ as follows: For all sets X and Y in the power set, X is related to Y if and only if X and Y have the same cardinality. Explain why this relation is reflexive, symmetric, and transitive (thus, is an equivalence relation).

(c) Write the equivalence classes for this relation.

12. **(C1, C3)** Suppose you have a huge box of animal crackers containing plenty of each of 10 different animals. Write a counting problem about giving animal crackers to people whose answer is $\binom{10}{6}$, and carefully explain why the answer to your problem is $\binom{10}{6}$.

13. **(C2)** Consider five-digit numbers $a = a_1a_2a_3a_4a_5$, where each digit a_i comes from the set $\{1, 2, 3, 4\}$. How many such numbers contain more even digits than odd digits? Carefully explain your answer.

14. (**Q1**) Find $3 + 7 + 11 + 15 + \cdots + 427$. Show all your work.

15. **(Q1, Q2)** Recall that K_n is the complete graph on k vertices. Consider the sequence defined below on all natural numbers $n \geq 1$:

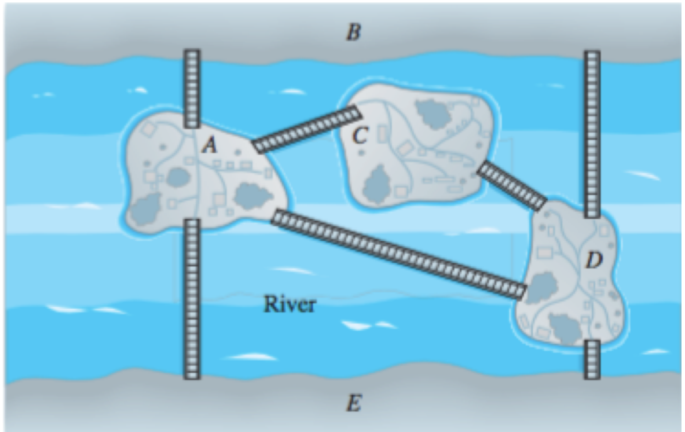
$$s_n \text{ is the number of edges in } K_n$$

(a) Find s_1, s_2, s_3, s_4, and s_5.

(b) Find a recurrence relation for s_n.

(c) Find a closed formula for s_n.

16. **(G1, G3)** The city of Queenigsberg spans both sides of a river and includes three islands with bridges as shown below. (Picture from Epp 2011)



(a) Draw a graph accurately representing the city and its bridges.

(b) Is it possible to take a walk around the city starting and ending at the same point and crossing each bridge exactly once? Explain.

17. **(G2)** Draw two non-isomorphic graphs with 4 vertices. Carefully explain how you know they are not isomorphic.

18. **(G3)** Is it possible for a citizen of Königsberg to go on a walking tour of the city crossing each bridge exactly twice? Explain.

19. **(G4)** Draw a graph with 7 vertices that has a chromatic number of 5.