PS#5 - Arc length, curvature, limits - Answer key

1. Choose your favorite 3D space curve from the Examples menu in CalcPlot3D. Calculate its arc length. Use Wolfram|Alpha or similar to calculate the integral numerically because it's probably impossible to find an antiderivative. Look at your space curve and say why the number you got makes sense.

   I chose Viviani's curve (which, interestingly, is the intersection between a sphere and a cylinder that's tangent to the sphere and goes through the center of the sphere).

   The integrand is $|\mathbf{r}'(t)|$:

$$\begin{aligned}
\mathbf{r}(t) &= \left\langle 1+\cos t, \sin t, 2\sin\tfrac{t}{2}\right\rangle \\
\mathbf{r}'(t) &= \left\langle -\sin t, \cos t, \cos\tfrac{t}{2}\right\rangle \\
|\mathbf{r}'(t)| &= \sqrt{(-\sin t)^2+(\cos t)^2+\left(\cos\tfrac{t}{2}\right)^2} \\
&= \sqrt{\sin^2 t+\cos^2 t+\cos^2\left(\tfrac{t}{2}\right)} \\
&= \sqrt{1+\cos^2\left(\tfrac{t}{2}\right)}
\end{aligned}$$

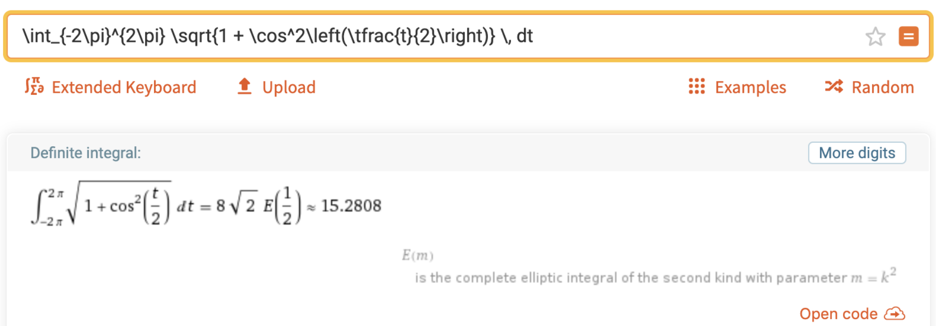
   (I imagine I could use some trig identities to simplify this further, but I don't ever remember any trig identities besides the Pythagorean identity, lol.)

   My bounds of integration should be from -2π to 2π, based on the bounds that are programmed into CalcPlot3D. As I look at the "trace" arrows moving around the curve, this does indeed appear to produce one full trip around the curve.

   So, my integral should be:

$$\int_{-2\pi}^{2\pi}\sqrt{1+\cos^2\left(\tfrac{t}{2}\right)}\,dt$$

   This integrand probably does not have an antiderivative. That means I can't use the fundamental theorem of calculus to compute the definite integral. Darn! I'll just use Wolfram|Alpha instead. Fun fact: you can type LaTeX directly into Wolfram|Alpha and it will interpret it correctly. Here's my result:



   This whole business about "the complete elliptic integral of the second kind" is W|A telling you that there's no elementary antiderivative of this particular integrand. Fortunately, it's smart enough to compute a pretty good numerical approximation, 15.2808.

   This value does seem plausible. I think you could reasonably approximate this curve by gluing a couple of circles with radius $\sqrt{2}$ together, and each one of those is going to have circumference $2\cdot\pi\cdot\sqrt{2}\approx 2\cdot 3\cdot 1.5 = 9$. So getting something close to 18 is pretty good, I guess.

2. (AC Multi 10.1 Exercise 15) Use the properties of continuity to determine the set of points at which each of the following functions is continuous. Justify your answers.

   (a) The function f defined by $f(x,y) = \dfrac{x+2y}{x-y}$

   This function is continuous everywhere the denominator is nonzero – that is, whenever $x \neq y$.

(b) The function g defined by $g(x,y) = \dfrac{\sin(x)}{1+e^y}$

The only issue here would be if the denominator was ever zero – that is, if $1+e^y = 0$. However, since e^y is always positive, $1+e^y$ is never 0, so this function is continuous for all $(x,y) \in \mathbb{R}^2$.

(c) The function h defined by

$$h(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

This function is clearly ok everywhere but $(0,0)$, which is going to require a little more investigation.

Approaching $(0,0)$ along $y=0$, we see that $h(x,y) = h(x,0) = \dfrac{0}{x^2} = 0$, but approaching $(0,0)$ along $y=x$, we see that $h(x,y) = h(x,x) = \dfrac{x^2}{x^2+x^2} = \dfrac{1}{2}$. Therefore, $\lim\limits_{(x,y)\to(0,0)} h(x,y)$ does not exist, so h is not continuous at $(0,0)$ – no matter what value we define for h there.

So, overall, h is continuous everywhere except $(0,0)$.

(d) The function k defined by

$$k(x,y) = \begin{cases} \dfrac{x^2y^4}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Again, this function is clearly ok everywhere but $(0,0)$, which is going to require a little more investigation.

By Example 10.1.14, we know that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2+y^2} = 0$. Note that $k(x,y) = y^2 \cdot \dfrac{x^2y^2}{x^2+y^2}$, and that y^2 is continuous everywhere. By the properties of limits,

$$\lim_{(x,y)\to(0,0)} \left(y^2 \cdot \frac{x^2y^2}{x^2+y^2} \right) = \left(\lim_{(x,y)\to(0,0)} y^2 \right) \cdot \left(\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} \right) = 0 \cdot 0 = 0$$

Therefore, since the limit and the value of $k(0,0)$ match, k is indeed continuous at $(0,0)$.

3. OPTIONAL FOR A BONUS TOKEN: Let's think about **the** unit normal vector $\mathbf{N}(t)$.

For any space curve $\mathbf{r}(t)$, you can always find the unit tangent vector $\mathbf{T}(t)$ by simply "unitizing" $\mathbf{r}'(t)$:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Explain why $|\mathbf{T}(t)| = 1$ means that $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$ at every time t.

(Hint: Consider $\mathbf{T} \cdot \mathbf{T}$. It might be nice to take the derivative of this, so that $\mathbf{T}'$ shows up. Use the product rule. But also, note that $\mathbf{T} \cdot \mathbf{T} = |\mathbf{T}|^2 = 1$, which is a constant. What's the derivative of a constant?)

So, if you were going to define **the** unit normal vector $\mathbf{N}(t)$, how might you define it? Why does your definition make sense?

Following the hint, I'm going to compute $\frac{d}{dt}[\mathbf{T} \cdot \mathbf{T}]$:

$$\begin{aligned} \frac{d}{dt}[\mathbf{T} \cdot \mathbf{T}] &= \frac{d}{dt}[\mathbf{T}] \cdot \mathbf{T} + \mathbf{T} \cdot \frac{d}{dt}[\mathbf{T}] \\ &= \mathbf{T}' \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{T}' \\ &= 2(\mathbf{T}' \cdot \mathbf{T}) \end{aligned}$$

But on the other hand, since $\mathbf{T} \cdot \mathbf{T} = 1$, $\frac{d}{dt}[\mathbf{T} \cdot \mathbf{T}]$ has to equal 0. Therefore,

$$0 = 2(\mathbf{T}' \cdot \mathbf{T}),$$

so $\mathbf{T}'$ must be orthogonal to $\mathbf{T}$. Yay!

So, if I was going to define **the** unit normal vector $\mathbf{N}(t)$, I might say $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$. It's always orthogonal to $\mathbf{T}(t)$, so it's always normal to the space curve $\mathbf{r}(t)$.

4. OPTIONAL FOR A BONUS TOKEN (AC Multi 9.8 Exercise 15) In this exercise we verify the curvature formula

$$\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

(a) Explain why $|\mathbf{r}'(t)| = \frac{ds}{dt}$.

Let's start by remembering what $s(t)$ is: it's the function that tells us how far we've gone by time t, so it's the accumulation function of the speed of the particle:

$$s(t) = \int_0^t |\mathbf{r}'(u)|\, du$$

Therefore,

$$\frac{ds}{dt} = \frac{d}{dt}\left[\int_0^t |\mathbf{r}'(u)|\, du\right] = |\mathbf{r}'(t)|,$$

by the fundamental theorem of calculus.

(b) Use the fact that $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ and $|\mathbf{r}'(t)| = \frac{ds}{dt}$ to explain why $\mathbf{r}'(t) = \frac{ds}{dt}\mathbf{T}(t)$.

Let's start with $\frac{ds}{dt} \cdot \mathbf{T}(t)$ and just substitute some stuff in.

$$\frac{ds}{dt} \cdot \mathbf{T}(t) = |\mathbf{r}'(t)| \cdot \mathbf{T}(t) = |\mathbf{r}'(t)| \cdot \left(\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right) = \mathbf{r}'(t).$$

(c) The Product Rule shows that $\mathbf{r}''(t) = \frac{d^2s}{dt^2}\mathbf{T}(t) + \frac{ds}{dt}\mathbf{T}'(t)$. Explain why $\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(\frac{ds}{dt}\right)^2 (\mathbf{T}(t) \times \mathbf{T}'(t))$.

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left[\frac{ds}{dt} \cdot \mathbf{T}(t)\right] \times \left[\frac{d^2s}{dt^2}\mathbf{T}(t) + \frac{ds}{dt}\mathbf{T}'(t)\right]$$

$\frac{ds}{dt}$ is a scalar, so we can factor it out of the first term:

$$= \frac{ds}{dt}\left([\mathbf{T}(t)] \times \left[\frac{d^2s}{dt^2}\mathbf{T}(t) + \frac{ds}{dt}\mathbf{T}'(t)\right]\right)$$

We know the cross product distributes:

$$= \frac{ds}{dt}\left(\left[\mathbf{T}(t) \times \frac{d^2s}{dt^2}\mathbf{T}(t)\right] + \left[\mathbf{T}(t) \times \frac{ds}{dt}\mathbf{T}'(t)\right]\right)$$

The first cross product is $\mathbf{0}$, since $\mathbf{T}(t)$ is certainly parallel to $\frac{d^2s}{dt^2}\mathbf{T}(t)$.

$$= \frac{ds}{dt}\left(\mathbf{T}(t) \times \frac{ds}{dt}\mathbf{T}'(t)\right)$$

We can again factor out a $\frac{ds}{dt}$, this time from the second term:

$$= \left(\frac{ds}{dt}\right)^2 (\mathbf{T}(t) \times \mathbf{T}'(t))$$

Yay, we have arrived at what we wanted.

(d) In the previous exercise (#3 on this Problem Set), we explained why $|\mathbf{T}(t)| = 1$ means that $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$ at every time t. Explain what this tells us about $|\mathbf{T}(t) \times \mathbf{T}'(t)|$ and conclude that

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \left(\frac{ds}{dt}\right)^2 |\mathbf{T}'(t)|.$$

I'm going to use the fact that $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin(\theta)$. In this case, $|\mathbf{T}(t) \times \mathbf{T}'(t)| = |\mathbf{T}(t)| \cdot |\mathbf{T}'(t)| \cdot \sin(\theta)$, and $\sin(\theta) = 1$ since the two vectors are always orthogonal. Therefore, $|\mathbf{T}(t) \times \mathbf{T}'(t)| = |\mathbf{T}(t)| \cdot |\mathbf{T}'(t)| = 1 \cdot |\mathbf{T}'(t)| = |\mathbf{T}'(t)|$, since $|\mathbf{T}(t)| = 1$. Therefore,

$$\begin{aligned}
|\mathbf{r}'(t) \times \mathbf{r}''(t)| &= \left|\left(\frac{ds}{dt}\right)^2 (\mathbf{T}(t) \times \mathbf{T}'(t))\right| \\
&= \left(\frac{ds}{dt}\right)^2 |\mathbf{T}(t) \times \mathbf{T}'(t)| \\
&= \left(\frac{ds}{dt}\right)^2 |\mathbf{T}'(t)|.
\end{aligned}$$

(e) Finally, use the fact that $\kappa = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$ to verify that $\kappa = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$.

Since $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \left(\frac{ds}{dt}\right)^2 |\mathbf{T}'(t)|$, it's certainly true that $|\mathbf{T}'(t)| = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{\left(\frac{ds}{dt}\right)^2}$, and since $\frac{ds}{dt} = |\mathbf{r}'(t)|$, it's certainly true that $\dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{\left(\frac{ds}{dt}\right)^2} = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^2}$. Therefore,

$$\begin{aligned}
\kappa &= \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \\
&= \frac{1}{|\mathbf{r}'(t)|} \cdot |\mathbf{T}'(t)| \\
&= \frac{1}{|\mathbf{r}'(t)|} \cdot \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^2} \\
&= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.
\end{aligned}$$

Phew, that was some serious algebra!!

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But on the other hand, since $\mathbf{T} \cdot \mathbf{T} = 1$, $\frac{d}{dt}[\mathbf{T} \cdot \mathbf{T}]$ has to equal 0. Therefore,

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So, if I was going to define **the** unit normal vector $\mathbf{N}(t)$, I might say $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$. It's always orthogonal to $\mathbf{T}(t)$, so it's always normal to the space curve $\mathbf{r}(t)$.

6. For each of the following prompts, provide an example of a function of two variables with the desired properties (with justification), or explain why such a function does not exist.

(a) A function p that is defined at $(0,0)$, but $\lim_{(x,y)\to(0,0)} p(x,y)$ does not exist.

How about we just take the example in Preview Activity 2.1.1, and define it a value at $(0,0)$?

$$p(x,y) = \begin{cases} \frac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 17, & (x,y) = (0,0) \end{cases}$$

(Nothing special about the number 17. I just kinda picked it out of thin air.)

Preview Activity 2.1.1 explains why the limit doesn't exist.

(b) A function q that does not have a limit at $(0,0)$, but that has the same limiting value along any line $y = mx$ as $x \to 0$.

Webwork 2.1 #2 will be helpful here:

$$q(x,y) = \frac{x^5 y}{x^{10} + y^5}$$

If we approach along $y = mx$, we're looking at $q(x,mx)$:

$$\begin{aligned} q(x,mx) &= \frac{x^5 \cdot (mx)}{x^{10} + (mx)^5} = \frac{mx^6}{x^{10} + m^5 x^5} \\ &= \frac{mx^6}{x^5 \cdot (x^5 + m^5)} = \frac{mx}{x^5 + m^5} \end{aligned}$$

So now we can let x go to 0 without anything bad happening:

$$\lim_{x\to 0} q(x,mx) = \lim_{x\to 0} \frac{mx}{x^5 + m^5} = \frac{m \cdot 0}{0^5 + m^5} = 0,$$

no matter what the value of m.

However, something different will happen if we approach $(0,0)$ along $y = x^5$:

$$\begin{aligned} q(x,x^5) &= \frac{x^5 \cdot (x^5)}{x^{10} + (x^5)^5} = \frac{x^{10}}{x^{10} + x^{25}} \\ &= \frac{x^{10}}{x^{10} \cdot (1 + x^{15})} = \frac{1}{1 + x^{15}}. \end{aligned}$$

Again, we can now let x go to 0 without anything bad happening, but we get something different:

$$\lim_{x\to 0} q(x,x^5) = \lim_{x\to 0} \frac{1}{1 + x^{15}} = \frac{1}{1+0} = 1.$$

Therefore, the limit doesn't exist, because we've found two paths that give us different limiting values.

(c) A function r that is continuous at $(0,0)$, but $\lim_{(x,y)\to(0,0)} r(x,y)$ does not exist.

This one's not gonna work. If r is continuous at $(0,0)$, the limit **must** exist – and must in fact be the same value as the function value $r(0,0)$.

(d) A function s such that

$$\lim_{(x,x)\to(0,0)} s(x,x) = 3 \quad \text{and} \quad \lim_{(x,2x)\to(0,0)} s(x,2x) = 6,$$

for which $\lim_{(x,x)\to(0,0)} s(x,y)$ exists.

This one's not gonna work either. We've shown that there's two directions along which we can approach $(0,0)$ that give us two different limiting values: along $y = x$, the limiting value is 3, but along $y = 2x$, the limiting value is 6. Since there's two directions with two different limiting values, the overall limit can't exist.

(e) A function t that is not defined at $(1,1)$, but $\lim_{(x,x)\to(1,1)} t(x,y)$ does exist.

One such function:

$$t(x,y) = \frac{(x^2-1)(y^2-1)}{(x-1)(y-1)}$$

Note that the numerator factors as $(x-1)(x+1)(y-1)(y+1)$. As long as $x \neq 1$, the $(x-1)$s that appear in the numerator and denominator can divide to 1. (This won't work when $x = 1$, because $0/0$ is indeterminate.) When I'm taking the limit as x **approaches** 1, x doesn't **equal** 1 – so we can divide that problematic term out. The same logic applies to the y terms – the overall value of the limit is 4.

(Note that this is the same logic by which the limit definition of the derivative works.)

Learning Targets Reflection: V1, V2, maybe S1, D2, D5. Maybe some others.