

PS#7 - Partial derivatives - Answer key

1. (Activity 10.2.4) The speed of sound C traveling through ocean water is a function of temperature, salinity and depth. It may be modeled by the function

$$C(T, S, D) = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34 - 0.01T)(S - 35) + 0.016D.$$

Here C is the speed of sound in meters/second, T is the temperature in degrees Celsius, S is the salinity in grams/liter of water, and D is the depth below the ocean surface in meters.

- (a) State the units in which each of the partial derivatives, C_T , C_S , and C_D , are expressed and explain the physical meaning of each.

C_T is measured in $\frac{\text{m/s}}{^\circ\text{C}}$, and it tells us how much the speed of sound in water changes as the temperature changes.

C_S is measured in $\frac{\text{m/s}}{\text{gm/L}}$, and it tells us how much the speed of sound in water changes as the salinity changes.

C_D is measured in $\frac{\text{m/s}}{\text{m}}$, and it tells us how much the speed of sound in water changes as the depth changes.

- (b) Find the partial derivatives C_T , C_S , and C_D .

$$C_T(T, S, D) = 4.6 - 0.11T + 0.00087T^2 - 0.01(S - 35)$$

$$C_S(T, S, D) = 1.34 - 0.01T$$

$$C_D(T, S, D) = 0.016$$

- (c) Evaluate each of the three partial derivatives at the point where $T = 10$, $S = 35$ and $D = 100$. What does the sign of each partial derivative tell us about the behavior of the function C at the point $(10, 35, 100)$?

$C_T(10, 35, 100) = 3.587$ – so the speed of sound **increases** as the temperature goes up from this point.

$C_S(10, 35, 100) = 1.24$ – so the speed of sound **increases** as the salinity goes up from this point.

$C_D(10, 35, 100) = 0.016$ – so the speed of sound **increases** as the depth increases from this point.

2. (Activity 10.2.5) The wind chill, as frequently reported, is a measure of how cold it feels outside when the wind is blowing. In the table below, the wind chill w , measured in degrees Fahrenheit, is a function of the wind speed v , measured in miles per hour, and the ambient air temperature T , also measured in degrees Fahrenheit. We thus view w as being of the form $w = w(v, T)$.

$v \backslash T$	-30	-25	-20	-15	-10	-5	0	5	10	15	20
5	-46	-40	-34	-28	-22	-16	-11	-5	1	7	13
10	-53	-47	-41	-35	-28	-22	-16	-10	-4	3	9
15	-58	-51	-45	-39	-32	-26	-19	-13	-7	0	6
20	-61	-55	-48	-42	-35	-29	-22	-15	-9	-2	4
25	-64	-58	-51	-44	-37	-31	-24	-17	-11	-4	3
30	-67	-60	-53	-46	-39	-33	-26	-19	-12	-5	1
35	-69	-62	-55	-48	-41	-34	-27	-21	-14	-7	0
40	-71	-64	-57	-50	-43	-36	-29	-22	-15	-8	-1

- (a) Estimate the partial derivative $w_v(20, -10)$. What are the units on this quantity and what does it mean?

$$w_v(20, -10) \approx \frac{w(25, -10) - w(15, -10)}{25 - 15} = \frac{-37 - (-32)}{10} = \frac{-5}{10} = -0.5 \frac{^\circ\text{F}}{\text{mph}}.$$

That is, if the windspeed goes up by 1 mph, the perceived temperature will drop by 0.5°F .

- (b) Estimate the partial derivative $w_T(20, -10)$. What are the units on this quantity and what does it mean?

$$w_T(20, -10) \approx \frac{w(20, -5) - w(20, -15)}{-5 - (-15)} = \frac{-29 - (-42)}{10} = \frac{13}{10} = 1.3 \frac{^\circ\text{F}}{^\circ\text{F}}.$$

That is, if the ambient temperature goes up by 1°F , the perceived temperature will go up by 1.3°F .

- (c) Use your results to estimate the wind chill $w(18, -10)$.

The windspeed has gone down by 2 mph. Each 1 mph decrease in windspeed causes an 0.5°F **increase** in perceived temperature. Therefore, $w(18, -10)$ should be about 1°F warmer than $w(20, -10)$, which is -35 , so it should be -34°F .

In symbols:

$$\begin{aligned} w(18, -10) &\approx w(20, -10) + (18 - 20) \cdot w_v(20, -10) \\ &= -35^\circ\text{F} + (-2 \text{ mph}) \cdot \left(-0.5 \frac{^\circ\text{F}}{\text{mph}}\right) \\ &= -35^\circ\text{F} + 1^\circ\text{F} = -34^\circ\text{F}. \end{aligned}$$

- (d) Use your results to estimate the wind chill $w(20, -12)$.

The ambient temperature has dropped by 2 degrees. Each 1 degree decrease in ambient temperature causes a 1.3 degree decrease in perceived temperature. Therefore, $w(20, -12)$ should be about 2.6 degrees colder than $w(20, -10)$, which is -35°F , so it should be -37.6°F .

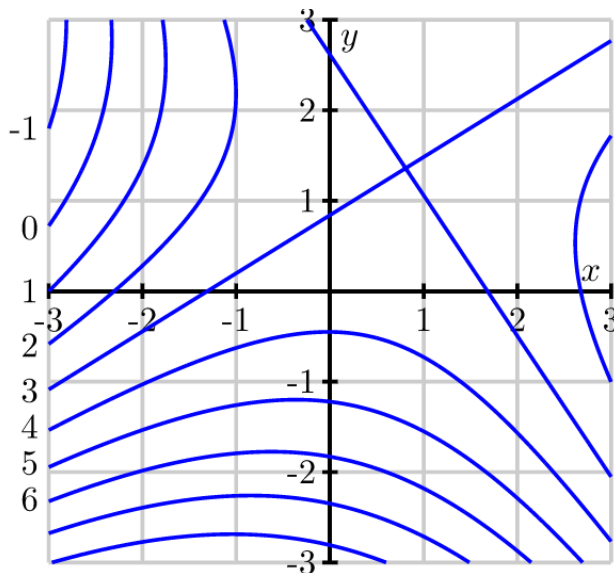
In symbols:

$$\begin{aligned} w(20, -12) &\approx w(20, -10) + (-12 - (-10)) \cdot w_T(20, -10) \\ &= -35^\circ\text{F} + (-2^\circ\text{F}) \cdot \left(1.3 \frac{^\circ\text{F}}{^\circ\text{F}}\right) \\ &= -35^\circ\text{F} - 2.6^\circ\text{F} = -37.6^\circ\text{F}. \end{aligned}$$

- (e) Consider how you might combine your previous results to estimate the wind chill $w(18, -12)$. Explain your process.

From the original -35°F , the perceived temperature should go up by 1 (from the decrease in windspeed) and down by 2.6 (from the decrease in temperature). So I bet $w(18, -12) \approx -35 + 1 - 2.6 = -36.6^\circ\text{F}$.

3. (Activity 10.2.6) Shown in the figure below is a contour plot of a function f . The values of the function on a few of the contours are indicated to the left of the figure.



- (a) Estimate the partial derivative $f_x(-2, -1)$.

We can use the contours to read off some approximate values and use a symmetric difference.

NOTE: I'm using a step size of 1 – your answer may be a little different from this if you chose a different step size. Same deal in parts (b) and (c).

$$\begin{aligned} f(-3, -1) &\approx 3 \\ f(-1, -1) &\approx 4.5 \\ f_x(-2, -1) &\approx \frac{f(-1, -1) - f(-3, -1)}{-1 - (-3)} \\ &= \frac{4.5 - 3}{2} = 0.75 \end{aligned}$$

- (b) Estimate the partial derivative $f_y(-2, -1)$.

Same game – we can use the contours to read off some approximate values and use a symmetric difference:

$$\begin{aligned} f(-2, -2) &\approx 6 \\ f(-2, 0) &\approx 2.5 \\ f_y(-2, -1) &\approx \frac{f(-2, 0) - f(-2, -2)}{0 - (-2)} \\ &= \frac{2.5 - 6}{2} = -1.75 \end{aligned}$$

- (c) Estimate the partial derivatives $f_x(-1, 2)$ and $f_y(-1, 2)$.

$$\begin{aligned} f(0, 2) &\approx 2.5 & f(-1, 1) &\approx 2.5 \\ f(-2, 2) &\approx 0.5 & f(-1, 3) &\approx 2.5 \\ f_x(-1, 2) &\approx \frac{f(0, 2) - f(-2, 2)}{0 - (-2)} & f_y(-1, 2) &\approx \frac{f(-1, 3) - f(-1, 1)}{3 - 1} \\ &= \frac{2.5 - 0.5}{2} = 1 & &= \frac{2.5 - 2.5}{2} = 0 \end{aligned}$$

- (d) Locate, if possible, one point (x, y) where $f_x(x, y) = 0$.

Looks like at about $(0, -0.5)$, if I move in the x direction, the heights on the contour map don't change much.

- (e) Locate, if possible, one point (x, y) where $f_x(x, y) < 0$.

Looks like at about $(1, -2)$, if I move in the x direction, I'm going downhill.

- (f) Locate, if possible, one point (x, y) where $f_y(x, y) > 0$.

Looks like at about $(1, 2)$, if I move in the y direction, I'm going (very slightly) uphill.

NOTE: Of course there are other points that will work for all three of these parts.

4. (AC Multi 10.2 Exercise 14) Let $f(x, y) = \frac{1}{2}xy^2$ represent the kinetic energy in Joules of an object of mass x in kilograms with velocity y in meters per second. Let (a, b) be the point $(4, 5)$ in the domain of f .

- (a) Calculate $f_x(a, b)$.

First of all, let's calculate $f_x(x, y)$. Holding y constant, we'll take the derivative with respect to x :

$$f_x(x, y) = \frac{1}{2}y^2.$$

Now we'll evaluate this at our point: $f_x(a, b) = f_x(4, 5) = \frac{1}{2} \cdot 5^2 = \frac{25}{2}$. (By the way, the units on this quantity are Joules per kilogram.)

- (b) Explain as best you can in the context of kinetic energy what the partial derivative

$$f_x(a, b) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h}$$

tells us about kinetic energy.

This essentially tells us how much the kinetic energy changes per unit change in **mass** of the object.

Dimensional analysis helps: Note here since h is being added to the x term, it's a small change in **mass**. Thus, since the numerator has the same output as f , it's Joules; since the denominator is an h , it's kilograms. So, the units on this quantity are Joules per kilogram.

- (c) Calculate $f_y(a, b)$.

First of all, let's calculate $f_y(x, y)$. Holding x constant, we'll take the derivative with respect to y :

$$f_y(x, y) = \frac{1}{2}x \cdot 2y = xy.$$

Now we'll evaluate this at our point: $f_y(a, b) = f_y(4, 5) = 4 \cdot 5 = 20$. (By the way, the units on this quantity are Joules per (meters per second).)

- (d) Explain as best you can in the context of kinetic energy what the partial derivative

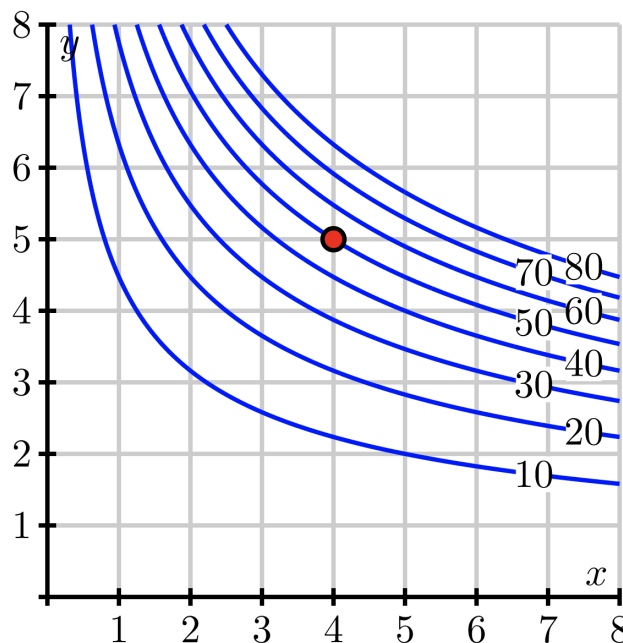
$$f_y(a, b) = \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h}$$

tells us about kinetic energy.

This essentially tells us how much the kinetic energy changes per unit change in **velocity** of the object.

Dimensional analysis helps: Note here since h is being added to the y term, it's a small change in **velocity**. Thus, since the numerator has the same output as f , it's Joules; since the denominator is an h , it's meters per second. So, the units on this quantity are Joules per (meters per second).

- (e) Often we are given certain graphical information about a function instead of a rule. We can use that information to approximate partial derivatives. For example, suppose that we are given a contour plot of the kinetic energy function (as in the figure below) instead of a formula. Use this contour plot to approximate $f_x(4, 5)$ and $f_y(4, 5)$ as best you can. Compare to your calculations from earlier parts of this exercise.



To find $f_x(4, 5)$: I'm going to stand at the point $(4, 5)$ and, holding y constant, take a small step in the x direction, and see what happens to my output values.

When I'm on the point $(4, 5)$, I'm on the $z = 50$ contour, so I know that $f(4, 5) = 50$. If I take a little step in the x direction, I land just a little above the $z = 60$ contour, so I know that $f(4 + 1, 5) \approx 60$ or 62ish. Therefore, my step of 1 kg has produced a change of 10 or 12 Joules, so $f_x(4, 5) \approx 10/1$ or $12/1$ Joules per kg. This is pretty close to the value we calculated earlier, $\frac{25}{2} = 12.5$.

To find $f_y(4, 5)$: I'm going to stand at the point $(4, 5)$ and, holding x constant, take a small step in the y direction, and see what happens to my output values.

When I'm on the point $(4, 5)$, I'm on the $z = 50$ contour, so I know that $f(4, 5) = 50$. If I take a little step in the y direction, I land just a little above the $z = 70$ contour, so I know that $f(4, 5 + 1) \approx 70$ or 72ish. Therefore, my step of 1 m/s has produced a change of 20 or 22 Joules, so $f_y(4, 5) \approx 20/1$ or $22/1$ Joules per m/s. This is pretty close to the value we calculated earlier, 20.