PS#8 – Directional derivatives and the gradient - Answer key

1. Here's an absolutely classic problem that I think is kind of a rite of passage for students in a multivariate calculus course: the lighthouse problem.

   In the lighthouse at Point Gradient, the lamp has been knocked slightly out of vertical, so that the axis is tilted just a little. When the light points east, the beam of light is inclined upward at 5 degrees. When the light points north, the beam of light is inclined upward at 2 degrees.

   (a) The beam of light sweeps out a plane; let's call that plane $f(x,y)$ and say that the lighthouse is at the point $(0,0)$ What's $f_x(0,0)$, and what's $f_y(0,0)$? (Hint: the answers aren't $5°$ and $2°$. Draw a picture and use some trigonometry to figure out the slopes.)

   Slope is rise over run, yeah? So $f_x(0,0) = \tan 5°$, and $f_y(0,0) = \tan 2°$.

   (b) What's $\nabla f(0,0)$?

   $\nabla f(0,0) = \langle f_x(0,0), f_y(0,0)\rangle = \langle \tan 5°, \tan 2°\rangle \approx \langle 0.087, 0.035\rangle$.

   (c) Looking down from above on a map, in which direction is the light beam pointing when it's most significantly inclined from the horizontal? Explain.

   $\nabla f = \langle \tan 5°, \tan 2°\rangle$. This is about $21.76°$ north of east.

   (d) What is the maximum angle of elevation of the plane of the light beam from horizontal? (Hint: you'll now have to do some inverse trigonometry.)

   $|\nabla f| = \sqrt{(\tan 5°)^2 + (\tan 2°)^2} \approx \sqrt{0.0089} \approx 0.0942$.

   The angle is $\arctan(0.0942) \approx 5.381°$ or 0.0939 radians.

2. (Activity 10.7.6 quick reference)

   (a) Find all critical points in the region. $(2,1)$ is a saddle point.

   (b) Parameterize the horizontal leg and find critical points. $x = x$ and $y = 0$, so $f(x,y) = f(x,0) = x^2 - 4x$. Critical point at $(2,0)$; also need to check endpoints $(0,0)$ and $(4,0)$.

   (c) Parameterize the vertical leg and find critical points. $x = 0$ and $y = y$, so $f(0,y) = -3y^2 + 6y$. Critical point at $(0,1)$; also need to check endpoints $(0,0)$ and $(0,4)$.

   (d) Parameterize the hypotenuse and find critical points. $x = x$ and $y = 4 - x$, so $f(x, 4-x) = x^2 - 3(4-x)^2 - 4x + 6(4-x)$. Critical point at $x = 7/2$, so $y = 1/2$.

   (e) Find absolute max and absolute min.

   - $f(2,1) = -1$ – interior critical point
   - $f(2,0) = -4$ – horizontal leg critical point
   - $\mathbf{f(0,1) = 3}$ – vertical leg critical point
   - $f(7/2,1/2) = 1/2$ – hypotenuse critical point
   - $f(0,0) = 0$ – corner
   - $f(4,0) = 0$ – corner
   - $\mathbf{f(0,4) = -24}$ – corner