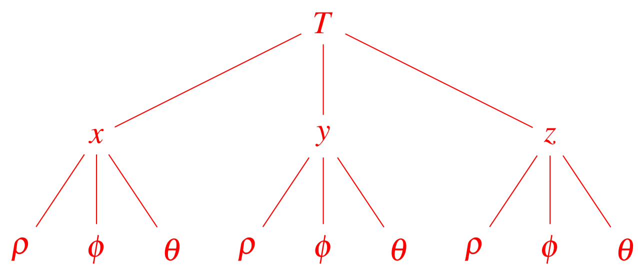
PS#7 – Chain rule - Answer key

1. (AC Multi 10.5 Exercise 13) Suppose that $T = x^2 + y^2 - 2z$ where

$$\begin{aligned} x &= \rho \sin(\phi)\cos(\theta) \\ y &= \rho \sin(\phi)\sin(\theta) \\ z &= \rho \cos(\phi) \end{aligned}$$

a. Construct a tree diagram representing the dependencies among the variables.



(In case you're curious, I generated this tree diagram using the very good TikZ library "trees". I can share the LaTeX source if you're interested.)

b. Apply the chain rule to find the partial derivatives $\frac{\partial T}{\partial \rho}$, $\frac{\partial T}{\partial \phi}$, and $\frac{\partial T}{\partial \theta}$.

We'll just read the chain rule off the tree:

$$\begin{aligned}
\frac{\partial T}{\partial \rho} &= \frac{\partial T}{\partial x}\cdot\frac{\partial x}{\partial \rho} + \frac{\partial T}{\partial y}\cdot\frac{\partial y}{\partial \rho} + \frac{\partial T}{\partial z}\cdot\frac{\partial z}{\partial \rho} \\
&= 2x\cdot(\sin\phi\cos\theta) + 2y\cdot(\sin\phi\sin\theta) - 2\cdot(\cos\phi) \\
&= 2(\rho\sin\phi\cos\theta)\cdot(\sin\phi\cos\theta) + 2(\rho\sin\phi\sin\theta)\cdot(\sin\phi\sin\theta) - 2\cdot(\cos\phi) \\
&= 2\rho\sin^2\phi\cos^2\theta + 2\rho\sin^2\phi\sin^2\theta - 2\cos\phi \\
&= 2\rho\sin^2\phi(\cos^2\theta + \sin^2\theta) - 2\cos\phi \\
&= 2\rho\sin^2\phi - 2\cos\phi \\
\frac{\partial T}{\partial \phi} &= \frac{\partial T}{\partial x}\cdot\frac{\partial x}{\partial \phi} + \frac{\partial T}{\partial y}\cdot\frac{\partial y}{\partial \phi} + \frac{\partial T}{\partial z}\cdot\frac{\partial z}{\partial \phi} \\
&= 2x\cdot(\rho\cos\phi\cos\theta) + 2y\cdot(\rho\cos\phi\sin\theta) - 2\cdot(-\rho\sin\phi) \\
&= 2(\rho\sin\phi\cos\theta)\cdot(\rho\cos\phi\cos\theta) + 2(\rho\sin\phi\sin\theta)\cdot(\rho\cos\phi\sin\theta) + 2\rho\sin\phi \\
&= 2\rho^2\sin\phi\cos\phi\cdot(\cos^2\theta + \sin^2\theta) + 2\rho\sin\phi \\
&= 2\rho^2\sin\phi\cos\phi + 2\rho\sin\phi \\
\frac{\partial T}{\partial \theta} &= \frac{\partial T}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial T}{\partial y}\cdot\frac{\partial y}{\partial \theta} + \frac{\partial T}{\partial z}\cdot\frac{\partial z}{\partial \theta} \\
&= 2x\cdot(\rho\sin\phi(-\sin\theta)) + 2y\cdot(\rho\sin\phi\cos\theta) - 2\cdot(0) \\
&= -2(\rho\sin\phi\cos\theta)\cdot(\rho\sin\phi\sin\theta) + 2(\rho\sin\phi\sin\theta)\cdot(\rho\sin\phi\cos\theta) \\
&= -2\rho^2\sin^2\phi\cos\theta\sin\theta + 2\rho^2\sin^2\phi\sin\theta\cos\theta = 0
\end{aligned}$$