PS#6 - Partial derivatives - Answer key

1. (Activity 10.2.3) Throughout this solution, constants are in blue.

(a) If $f(x,y) = 3x^3 - 2x^2y^5$, find the partial derivatives f_x and f_y.

$$\begin{aligned} f_x(x,y) &= \frac{\partial}{\partial x}(3x^3 - 2x^2y^5) \\ &= 9x^2 - 2(2x)y^5 \\ &= 9x^2 - 4xy^5 \end{aligned}$$

$$\begin{aligned} f_y(x,y) &= \frac{\partial}{\partial y}(3x^3 - 2x^2y^5) \\ &= 0 - 2x^2(5y^4) \\ &= -10x^2y^4 \end{aligned}$$

(b) If $f(x,y) = \dfrac{xy^2}{x+1}$, find the partial derivatives f_x and f_y.

$$\begin{aligned} f_x(x,y) &= \frac{\partial}{\partial x}\left(\frac{xy^2}{x+1}\right) \\ &= y^2\frac{(x+1)(1)-(x)(1)}{(x+1)^2} \\ &= \frac{y^2}{(x+1)^2} \end{aligned}$$

$$\begin{aligned} f_y(x,y) &= \frac{\partial}{\partial y}\left(\frac{x}{x+1}y^2\right) \\ &= \frac{x}{x+1}\cdot 2y \\ &= \frac{2xy}{x+1} \end{aligned}$$

(c) If $g(r,s) = rs\cos(r)$, find the partial derivatives g_r and g_s.

$$\begin{aligned} g_r(r,s) &= \frac{\partial}{\partial r}(sr\cos(r)) \\ &= s(r(-\sin(r)) + (1)\cos(r)) \\ &= s\cos(r) - sr\sin(r) \end{aligned}$$

$$\begin{aligned} g_s(r,s) &= \frac{\partial}{\partial s}(r\cos(r)s) \\ &= r\cos(r)(1) \\ &= r\cos(r) \end{aligned}$$

(d) Assuming $f(w,x,y) = (6w+1)\cos(3x^2+4xy^3+y)$, find the partial derivatives f_w, f_x, and f_y.

$$\begin{aligned} f_w(w,x,y) &= \frac{\partial}{\partial w}[(6w+1)\cos(3x^2+4xy^3+y)] \\ &= (6)\cos(3x^2+4xy^3+y) \\ &= 6\cos(3x^2+4xy^3+y) \end{aligned}$$

$$\begin{aligned} f_x(w,x,y) &= \frac{\partial}{\partial x}[(6w+1)\cos(3x^2+4xy^3+y)] \\ &= (6w+1)(-\sin(3x^2+4xy^3+y))\cdot(6x+4y^3) \\ &= -(6w+1)(6x+4y^3)\sin(3x^2+4xy^3+y) \end{aligned}$$

$$\begin{aligned} f_y(w,x,y) &= \frac{\partial}{\partial y}[(6w+1)\cos(3x^2+4xy^3+y)] \\ &= (6w+1)(-\sin(3x^2+4xy^3+y))\cdot(4x\cdot 3y^2+1) \\ &= -(6w+1)(12xy^2+1)\sin(3x^2+4xy^3+y) \end{aligned}$$

(e) Find all possible first-order partial derivatives of $q(x,t,z) = \dfrac{x2^tz^3}{1+x^2}$.

$$\begin{aligned} q_x(x,t,z) &= \frac{\partial}{\partial x}\left(2^tz^3\frac{x}{1+x^2}\right) \\ &= 2^tz^3\frac{(1+x^2)(1)-(x)(2x)}{(1+x^2)^2} \\ &= \frac{2^tz^3(1-x^2)}{(1+x^2)^2} \end{aligned}$$

$$\begin{aligned} q_t(x,t,z) &= \frac{\partial}{\partial t}\left(\frac{xz^3}{1+x^2}2^t\right) \\ &= \frac{xz^3}{1+x^2}(2^t\ln(2)) \\ &= \frac{xz^32^t\ln(2)}{1+x^2} \end{aligned}$$

$$\begin{aligned} q_z(x,t,z) &= \frac{\partial}{\partial z}\left(\frac{x2^t}{1+x^2}z^3\right) \\ &= \frac{x2^t}{1+x^2}(3z^2) \\ &= \frac{3x2^tz^2}{1+x^2} \end{aligned}$$

2. (Activity 10.2.4) The speed of sound C traveling through ocean water is a function of temperature, salinity and depth. It may be modeled by the function

$$C(T,S,D) = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34 - 0.01T)(S-35) + 0.016D.$$

Here C is the speed of sound in meters/second, T is the temperature in degrees Celsius, S is the salinity in grams/liter of water, and D is the depth below the ocean surface in meters.

(a) State the units in which each of the partial derivatives, C_T, C_S, and C_D, are expressed and explain the physical meaning of each.

C_T is measured in $\frac{\text{m/s}}{^\circ C}$, and it tells us how much the speed of sound in water changes as the temperature changes.

C_S is measured in $\frac{\text{m/s}}{\text{gm/L}}$, and it tells us how much the speed of sound in water changes as the salinity changes.

C_D is measured in $\frac{\text{m/s}}{\text{m}}$, and it tells us how much the speed of sound in water changes as the depth changes.

(b) Find the partial derivatives C_T, C_S, and C_D.

$$C_T(T,S,D) = 4.6 - 0.11T + 0.00087T^2 - 0.01(S-35)$$
$$C_S(T,S,D) = 1.34 - 0.01T$$
$$C_D(T,S,D) = 0.016$$

(c) Evaluate each of the three partial derivatives at the point where $T = 10$, $S = 35$ and $D = 100$. What does the sign of each partial derivative tell us about the behavior of the function C at the point $(10,35,100)$?

$C_T(10,35,100) = 3.587$ – so the speed of sound **increases** as the temperature goes up from this point.

$C_S(10,35,100) = 1.24$ – so the speed of sound **increases** as the salinity goes up from this point.

$C_D(10,35,100) = 0.016$ – so the speed of sound **increases** as the depth increases from this point.

3. (Activity 10.2.5) The wind chill, as frequently reported, is a measure of how cold it feels outside when the wind is blowing. In the table below, the wind chill w, measured in degrees Fahrenheit, is a function of the wind speed v, measured in miles per hour, and the ambient air temperature T, also measured in degrees Fahrenheit. We thus view w as being of the form $w = w(v,T)$.

$v\backslash T$	-30	-25	-20	-15	-10	-5	0	5	10	15	20
5	-46	-40	-34	-28	-22	-16	-11	-5	1	7	13
10	-53	-47	-41	-35	-28	-22	-16	-10	-4	3	9
15	-58	-51	-45	-39	-32	-26	-19	-13	-7	0	6
20	-61	-55	-48	-42	-35	-29	-22	-15	-9	-2	4
25	-64	-58	-51	-44	-37	-31	-24	-17	-11	-4	3
30	-67	-60	-53	-46	-39	-33	-26	-19	-12	-5	1
35	-69	-62	-55	-48	-41	-34	-27	-21	-14	-7	0
40	-71	-64	-57	-50	-43	-36	-29	-22	-15	-8	-1

(a) Estimate the partial derivative $w_v(20,-10)$. What are the units on this quantity and what does it mean?

$$w_v(20,-10) \approx \frac{w(25,-10) - w(15,-10)}{25-15} = \frac{-37-(-32)}{10} = \frac{-5}{10} = -0.5\frac{^\circ\text{F}}{\text{mph}}.$$

That is, if the windspeed goes up by 1 mph, the perceived temperature will drop by 0.5° F.

(b) Estimate the partial derivative $w_T(20,-10)$. What are the units on this quantity and what does it mean?

$$w_T(20,-10) \approx \frac{w(20,-5) - w(20,-15)}{-5-(-15)} = \frac{-29-(-42)}{10} = \frac{13}{10} = 1.3\frac{^\circ\text{F}}{^\circ\text{F}}.$$

That is, if the ambient temperature goes up by 1° F, the perceived temperature will go up by 1.3° F.

(c) Use your results to estimate the wind chill $w(18,-10)$.

The windspeed has gone down by 2 mph. Each 1 mph decrease in windspeed causes an 0.5° F **increase** in perceived temperature. Therefore, $w(18,-10)$ should be about 1° F warmer than $w(20,-10)$, which is -35, so it should be -34° F.

In symbols:

$$\begin{aligned} w(18,-10) &\approx w(20,-10) + (18-20)\cdot w_v(20,-10) \\ &= -35^\circ\text{F} + (-2\text{ mph})\cdot\left(-0.5\frac{{}^\circ\text{F}}{\text{mph}}\right) \\ &= -35^\circ\text{F} + 1^\circ\text{F} = -34^\circ\text{F}. \end{aligned}$$

(d) Use your results to estimate the wind chill $w(20,-12)$.

The ambient temperature has dropped by 2 degrees. Each 1 degree decrease in ambient temperature causes a 1.3 degree decrease in perceived temperature. Therefore, $w(20,-12)$ should be about 2.6 degrees colder than $w(20,-10)$, which is -35° F, so it should be -37.6° F.

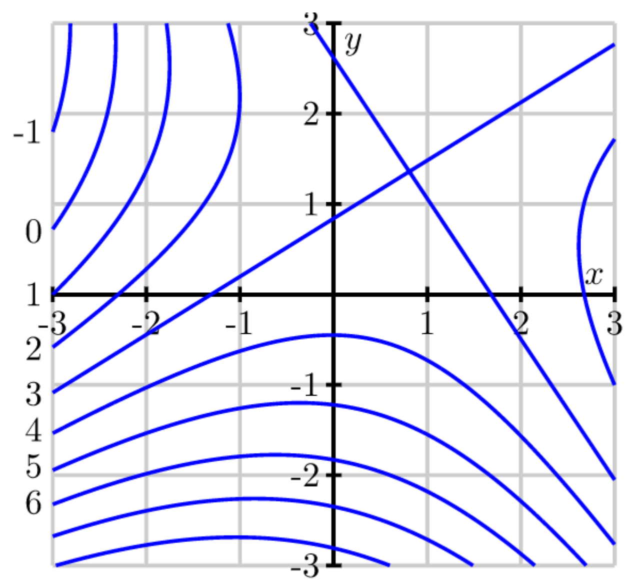
In symbols:

$$\begin{aligned} w(20,-12) &\approx w(20,-10) + (-12-(-10))\cdot w_T(20,-10) \\ &= -35^\circ\text{F} + (-2^\circ\text{F})\cdot\left(1.3\frac{{}^\circ\text{F}}{{}^\circ\text{F}}\right) \\ &= -35^\circ\text{F} - 2.6^\circ\text{F} = -37.6^\circ\text{F}. \end{aligned}$$

(e) Consider how you might combine your previous results to estimate the wind chill $w(18,-12)$. Explain your process.

From the original -35°F, the perceived temperature should go up by 1 (from the decrease in windspeed) and down by 2.6 (from the decrease in temperature). So I bet $w(18,-12) \approx -35+1-2.6-36.6$°F.

4. (Activity 10.2.6) Shown in the figure below is a contour plot of a function f. The values of the function on a few of the contours are indicated to the left of the figure.



(a) Estimate the partial derivative $f_x(-2,-1)$.

We can use the contours to read off some approximate values and use a symmetric difference.

NOTE: I'm using a step size of 1 – your answer may be a little different from this if you chose a different step size. Same deal in parts (b) and (c).

$$\begin{aligned}
f(-3,-1) &\approx 3 \\
f(-1,-1) &\approx 4.5 \\
f_x(-2,-1) &\approx \frac{f(-1,-1)-f(-3,-1)}{-1-(-3)} \\
&= \frac{4.5-3}{2} = 0.75
\end{aligned}$$

(b) Estimate the partial derivative $f_y(-2,-1)$.

Same game – we can use the contours to read off some approximate values and use a symmetric difference:

$$\begin{aligned}
f(-2,-2) &\approx 6 \\
f(-2,0) &\approx 2.5 \\
f_y(-2,-1) &\approx \frac{f(-2,0)-f(-2,-2)}{0-(-2)} \\
&= \frac{2.5-6}{2} = -1.75
\end{aligned}$$

(c) Estimate the partial derivatives $f_x(-1,2)$ and $f_y(-1,2)$.

$$\begin{aligned}
f(0,2) &\approx 2.5 \\
f(-2,2) &\approx 0.5 \\
f_x(-1,2) &\approx \frac{f(0,2)-f(-2,2)}{0-(-2)} \\
&= \frac{2.5-0.5}{2} = 1
\end{aligned}$$

$$\begin{aligned}
f(-1,1) &\approx 2.5 \\
f(-1,3) &\approx 2.5 \\
f_y(-1,2) &\approx \frac{f(-1,3)-f(-1,1)}{3-1} \\
&= \frac{2.5-2.5}{2} = 0
\end{aligned}$$

(d) Locate, if possible, one point (x,y) where $f_x(x,y)=0$.

Looks like at about $(0,-0.5)$, if I move in the x direction, the heights on the contour map don't change much.

(e) Locate, if possible, one point (x,y) where $f_x(x,y)<0$.

Looks like at about $(1,-2)$, if I move in the x direction, I'm going downhill.

(f) Locate, if possible, one point (x,y) where $f_y(x,y)>0$.

Looks like at about $(1,2)$, if I move in the y direction, I'm going (very slightly) uphill.

NOTE: Of course there are other points that will work for all three of these parts.

5. (AC Multi 10.2 Exercise 14) Let $f(x,y)=\frac{1}{2}xy^2$ represent the kinetic energy in Joules of an object of mass x in kilograms with velocity y in meters per second. Let (a,b) be the point $(4,5)$ in the domain of f.

(a) Calculate $f_x(a,b)$.

First of all, let's calculate $f_x(x,y)$. Holding y constant, we'll take the derivative with respect to x:

$$f_x(x,y)=\tfrac{1}{2}y^2.$$

Now we'll evaluate this at our point: $f_x(a,b)=f_x(4,5)=\frac{1}{2}\cdot 5^2=\frac{25}{2}$. (By the way, the units on this quantity are Joules per kilogram.)

(b) Explain as best you can in the context of kinetic energy what the partial derivative

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

tells us about kinetic energy.

This essentially tells us how much the kinetic energy changes per unit change in **mass** of the object.

Dimensional analysis helps: Note here since h is being added to the x term, it's a small change in **mass**. Thus, since the numerator has the same output as f, it's Joules; since the denominator is an h, it's kilograms. So, the units on this quantity are Joules per kilogram.

(c) Calculate $f_y(a,b)$.

First of all, let's calculate $f_y(x,y)$. Holding x constant, we'll take the derivative with respect to y:

$$f_y(x,y) = \tfrac{1}{2}x \cdot 2y = xy.$$

Now we'll evaluate this at our point: $f_y(a,b) = f_y(4,5) = 4 \cdot 5 = 20$. (By the way, the units on this quantity are Joules per (meters per second).)

(d) Explain as best you can in the context of kinetic energy what the partial derivative

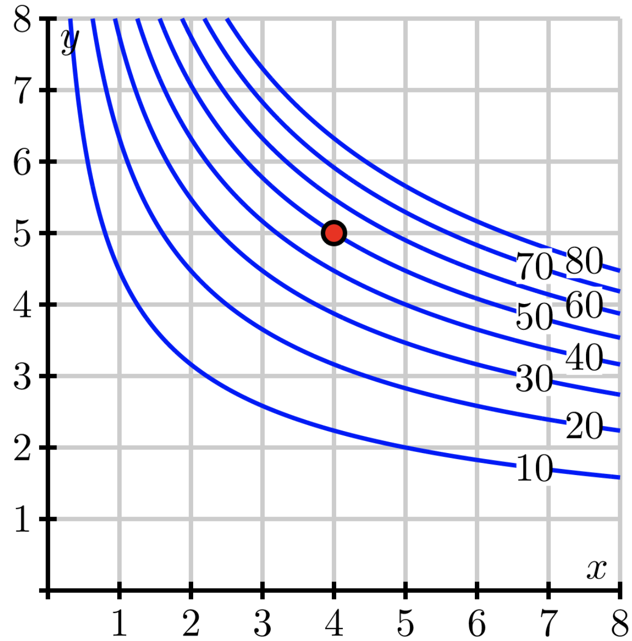
$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

tells us about kinetic energy.

This essentially tells us how much the kinetic energy changes per unit change in **velocity** of the object.

Dimensional analysis helps: Note here since h is being added to the y term, it's a small change in **velocity**. Thus, since the numerator has the same output as f, it's Joules; since the denominator is an h, it's meters per second. So, the units on this quantity are Joules per (meters per second).

(e) Often we are given certain graphical information about a function instead of a rule. We can use that information to approximate partial derivatives. For example, suppose that we are given a contour plot of the kinetic energy function (as in the figure below) instead of a formula. Use this contour plot to approximate $f_x(4,5)$ and $f_y(4,5)$ as best you can. Compare to your calculations from earlier parts of this exercise.



To find $f_x(4,5)$: I'm going to stand at the point $(4,5)$ and, holding y constant, take a small step in the x direction, and see what happens to my output values.

When I'm on the point $(4,5)$, I'm on the $z = 50$ contour, so I know that $f(4,5) = 50$. If I take a little step in the x direction, I land just a little above the $z = 60$ contour, so I know that $f(4+1,5) \approx 60$ or 62ish. Therefore, my step of 1 kg has produced a change of 10 or 12 Joules, so $f_x(4,5) \approx 10/1$ or $12/1$ Joules per kg. This is pretty close to the value we calculated earlier, $\frac{25}{2} = 12.5$.

To find $f_y(4,5)$: I'm going to stand at the point $(4,5)$ and, holding x constant, take a small step in the y direction, and see what happens to my output values.

When I'm on the point $(4,5)$, I'm on the $z = 50$ contour, so I know that $f(4,5) = 50$. If I take a little step in the y direction, I land just a little above the $z = 70$ contour, so I know that $f(4,5+1) \approx 70$ or 72ish. Therefore, my step of 1 m/s has produced a change of 20 or 22 Joules, so $f_y(4,5) \approx 20/1$ or $22/1$ Joules per m/s. This is pretty close to the value we calculated earlier, 20.