PS#8 – Directional derivatives and the gradient - Answer key

1. Here's an absolutely classic problem that I think is kind of a rite of passage for students in a multivariate calculus course: the lighthouse problem.

   In the lighthouse at Point Gradient, the lamp has been knocked slightly out of vertical, so that the axis is tilted just a little. When the light points east, the beam of light is inclined upward at 5 degrees. When the light points north, the beam of light is inclined upward at 2 degrees.

   (a) The beam of light sweeps out a plane; let's call that plane $f(x,y)$ and say that the lighthouse is at the point $(0,0)$ What's $f_x(0,0)$, and what's $f_y(0,0)$? (Hint: the answers aren't 5° and 2°. Draw a picture and use some trigonometry to figure out the slopes.)

   Slope is rise over run, yeah? So $f_x(0,0) = \tan 5^\circ$, and $f_y(0,0) = \tan 2^\circ$.

   (b) What's $\nabla f(0,0)$?

   $\nabla f(0,0) = \langle f_x(0,0), f_y(0,0)\rangle = \langle \tan 5^\circ, \tan 2^\circ\rangle \approx \langle 0.087, 0.035\rangle$.

   (c) Looking down from above on a map, in which direction is the light beam pointing when it's most significantly inclined from the horizontal? Explain.

   $\nabla f = \langle \tan 5^\circ, \tan 2^\circ\rangle$. This is about 21.76° north of east.

   (d) What is the maximum angle of elevation of the plane of the light beam from horizontal? (Hint: you'll now have to do some inverse trigonometry.)

   $|\nabla f| = \sqrt{(\tan 5^\circ)^2 + (\tan 2^\circ)^2} \approx \sqrt{0.0089} \approx 0.0942$.

   The angle is $\arctan(0.0942) \approx 5.381^\circ$ or 0.0939 radians.