PS#4: Calculus with space curves - Answer key

1. (AC Multi 9.7 Exercise 13) Compute the derivative of each of the following functions in two different ways: (1) use the rules provided in the theorem stated just after Activity 9.7.3, and (2) rewrite each given function so that it is stated as a single function (either a scalar function or a vector-valued function with three components), and differentiate component-wise. Compare your answers to ensure that they are the same.

   (a) $\mathbf{r}(t) = \sin(t)\langle 2t, t^2, \arctan(t)\rangle$

   Using the scalar product rule:

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= \left(\frac{d}{dt}\sin(t)\right)\langle 2t, t^2, \arctan(t)\rangle + \sin(t)\left(\frac{d}{dt}\langle 2t, t^2, \arctan(t)\rangle\right)\\ &= \cos(t)\langle 2t, t^2, \arctan(t)\rangle + \sin(t)\left\langle 2, 2t, \frac{1}{1+t^2}\right\rangle\\ &= \left\langle 2t\cos(t) + 2\sin(t), t^2\cos(t) + 2t\sin(t), \arctan(t)\cos(t) + \frac{\sin(t)}{1+t^2}\right\rangle\end{aligned}$$

   And rewriting first:

$$\begin{aligned}\mathbf{r}(t) &= \langle 2t\sin(t), t^2\sin(t), \arctan(t)\sin(t)\rangle\\ \frac{d\mathbf{r}}{dt} &= \left\langle \left(\frac{d}{dt}2t\right)\sin(t) + 2t\left(\frac{d}{dt}\sin(t)\right), \left(\frac{d}{dt}t^2\right)\sin(t) + t^2\left(\frac{d}{dt}\sin(t)\right), \left(\frac{d}{dt}\arctan(t)\right)\sin(t) + \arctan(t)\left(\frac{d}{dt}\sin(t)\right)\right\rangle\\ &= \left\langle 2t\cos(t) + 2\sin(t), t^2\cos(t) + 2t\sin(t), \frac{\sin(t)}{1+t^2} + \arctan(t)\cos(t)\right\rangle\end{aligned}$$

   (b) $\mathbf{s}(t) = \mathbf{r}(2^t)$, where $\mathbf{r}(t) = \langle t+2, \ln(t), 1\rangle$ – Note that $\mathbf{r}'(t) = \langle 1, \frac{1}{t}, 0\rangle$

   Using the chain rule first:

$$\begin{aligned}\mathbf{s}'(t) &= \mathbf{r}'(2^t)\frac{d}{dt}2^t\\ &= \left\langle 1, \frac{1}{2^t}, 0\right\rangle 2^t\ln(2)\\ &= \langle 2^t\ln(2), \ln(2), 0\rangle\end{aligned}$$

   And simplifying first:

$$\begin{aligned}\mathbf{s}(t) &= \mathbf{r}(2^t) = \langle 2^t+2, \ln(2^t), 1\rangle = \langle 2^t+2, t\ln(2), 1\rangle\\ \mathbf{s}'(t) &= \langle 2^t\ln(2), \ln(2), 0\rangle\end{aligned}$$

   (c) $\mathbf{r}(t) = \langle\cos(t), \sin(t), t\rangle \cdot \langle -\sin(t), \cos(t), 1\rangle$

   Using the dot product rule:

$$\begin{aligned}r'(t) &= \left(\frac{d}{dt}\langle\cos(t), \sin(t), t\rangle\right)\cdot\langle -\sin(t), \cos(t), 1\rangle + \langle\cos(t), \sin(t), t\rangle\cdot\left(\frac{d}{dt}\langle -\sin(t), \cos(t), 1\rangle\right)\\ &= \langle -\sin(t), \cos(t), 1\rangle\cdot\langle -\sin(t), \cos(t), 1\rangle + \langle\cos(t), \sin(t), t\rangle\cdot\langle -\cos(t), -\sin(t), 0\rangle\\ &= \left[\sin^2(t) + \cos^2(t) + 1\right] + \left[-\cos^2(t) - \sin^2(t) + 0\right] = 1 + 1 - 1 + 0 = 1(!!)\end{aligned}$$

   And rewriting first:

$$\begin{aligned}r(t) &= cos(t)\cdot(-\sin(t)) + \sin(t)\cdot\cos(t) + t\cdot 1\\ r(t) &= t\ (!!)\\ r'(t) &= 1\end{aligned}$$

(d) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t\rangle \times \langle -\sin(t), \cos(t), 1\rangle$

Using the cross product rule:

$$\begin{aligned}\mathbf{r}'(t) &= \left(\frac{d}{dt}\langle \cos(t), \sin(t), t\rangle\right) \times \langle -\sin(t), \cos(t), 1\rangle + \langle \cos(t), \sin(t), t\rangle \times \left(\frac{d}{dt}\langle -\sin(t), \cos(t), 1\rangle\right)\\ &= \langle -\sin(t), \cos(t), 1\rangle \times \langle -\sin(t), \cos(t), 1\rangle + \langle \cos(t), \sin(t), t\rangle \times \langle -\cos(t), -\sin(t), 0\rangle\end{aligned}$$

The first two vectors are parallel, so their cross product is $\mathbf{0}$.

$$= \mathbf{0} + \langle \cos(t), \sin(t), t\rangle \times \langle -\cos(t), -\sin(t), 0\rangle = \langle t\sin(t), -t\cos(t), 0\rangle \quad \text{(Thanks, WA!)}$$

And finding the cross product first:

$$\begin{aligned}\mathbf{r}(t) &= \langle \sin(t) - t\cos(t), -t\sin(t) - \cos(t), \sin^2(t) + \cos^2(t)\rangle = \langle \sin(t) - t\cos(t), -t\sin(t) - \cos(t), 1\rangle\\ \mathbf{r}'(t) &= \frac{d}{dt}\langle \sin(t) - t\cos(t), -t\sin(t) - \cos(t), 1\rangle\\ &= \langle \cos(t) - (1\cos(t) + t(-\sin(t))), -(1\sin(t) + t\cos(t)) - (-\sin(t)), 0\rangle\\ &= \langle t\sin(t), -t\cos(t), 0\rangle\end{aligned}$$

2. (AC Multi 9.7 Exercise 18) A central force is one that acts on an object so that the force $\mathbf{F}$ is parallel to the object's position $\mathbf{r}$. Since Newton's Second Law says that an object's acceleration is proportional to the force exerted on it, the acceleration $\mathbf{a}$ of an object moving under a central force will be parallel to its position $\mathbf{r}$. For instance, the Earth's acceleration due to the gravitational force that the sun exerts on the Earth is parallel to the Earth's position vector (see figure in the textbook).

(a) If an object of mass m is moving under a central force, the angular momentum vector is defined to be $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$. Assuming the mass is constant, show that the angular momentum is constant by showing that $\frac{d\mathbf{L}}{dt} = \mathbf{0}$.

Some stuff to keep track of:

- m is a constant scalar;
- $\mathbf{r}$ is a variable vector (depends on t);
- $\mathbf{v}$ is a variable vector (depends on t).

Seems like the natural thing to do is to use the product rule to compute $\frac{d\mathbf{L}}{dt}$:

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}[m\mathbf{r} \times \mathbf{v}] = \frac{d}{dt}[m\mathbf{r}] \times \mathbf{v} + m\mathbf{r} \times \frac{d}{dt}[\mathbf{v}]$$

Well, the derivative of position is velocity, and the derivative of velocity is acceleration:

$$= [m\mathbf{v}] \times \mathbf{v} + m\mathbf{r} \times \mathbf{a}$$

Now we're getting somewhere. Remember that the cross product of two parallel vectors is $\mathbf{0}$. $m\mathbf{v}$ is certainly parallel to $\mathbf{v}$, and we're assuming in the context of the problem that $\mathbf{a}$ is parallel to $\mathbf{r}$, so it's also parallel to $m\mathbf{r}$.

$$= \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Cool, so that tells us that $\mathbf{L}$ is a constant vector.

(b) Explain why $\mathbf{L} \cdot \mathbf{r} = 0$.

$\mathbf{L}$ was defined as the cross product of $m\mathbf{r}$ and $\mathbf{v}$, so it's perpendicular to both of those. Since m is a scalar, $m\mathbf{r}$ points in the same direction as $\mathbf{r}$, so if $\mathbf{L}$ is perpendicular to $m\mathbf{r}$, it's also perpendicular to $\mathbf{r}$. Therefore, their dot product is zero.

(c) Explain why we may conclude that the object is constrained to lie in the plane passing through the origin and perpendicular to $\mathbf{L}$.

The equation $\mathbf{L} \cdot \mathbf{r} = 0$ reminds me of the vector equation of a plane, with $\mathbf{L}$ as the (constant) normal vector and $\mathbf{r}$ playing the role of $\overrightarrow{PP_0}$. Since $\mathbf{r}$'s initial point is the origin, we thus have the plane that's perpendicular to $\mathbf{L}$ and passing through the origin.

(Another way to see this: Certainly all the position vectors are perpendicular to $\mathbf{L}$. Also, certainly all the position vectors emanate from the origin. Also, we've just found that $\mathbf{L}$ is constant. So the only way this is going to happen is if all the position vectors lie in the **same** plane – specifically, the plane containing the origin and perpendicular to the constant vector $\mathbf{L}$.)

3. (AC Multi 9.8 Exercise 14) Consider the standard helix parameterized by $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$.

(a) Recall that the unit tangent vector $\mathbf{T}(t)$ is the vector tangent to the curve at time t that points in the direction of motion and has length 1. Find $\mathbf{T}(t)$.

(b) Explain why the fact that $|\mathbf{T}(t)| = 1$ implies that $\mathbf{T}$ and $\mathbf{T}'$ are orthogonal vectors for every value of t.

(c) For the given function $\mathbf{r}$ with unit tangent vector $\mathbf{T}(t)$ (from part (a)), determine $\mathbf{N}(t) = \dfrac{1}{|\mathbf{T}'(t)|}\mathbf{T}'(t)$.

(d) What geometric properties does $\mathbf{N}(t)$ have? How long is this vector, and how is it situated in comparison ton $\mathbf{T}(t)$?

(e) Let $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$, and compute $\mathbf{B}(t)$ in terms of your results in (a) and (c).

(f) What geometric properties does $\mathbf{B}(t)$ have? How long is this vector, and how is it situated in comparison ton $\mathbf{T}(t)$ and $\mathbf{N}(t)$?

(g) Sketch a plot of the given helix, and compute and sketch $\mathbf{T}\left(\frac{\pi}{2}\right)$, $\mathbf{N}\left(\frac{\pi}{2}\right)$, and $\mathbf{B}\left(\frac{\pi}{2}\right)$.