

PS#4: Calculus with space curves - Answer key

1. (AC Multi 9.7 Exercise 13) Compute the derivative of each of the following functions in two different ways: (1) use the rules provided in the theorem stated just after Activity 9.7.3, and (2) rewrite each given function so that it is stated as a single function (either a scalar function or a vector-valued function with three components), and differentiate component-wise. Compare your answers to ensure that they are the same.

(a) $\mathbf{r}(t) = \sin(t)\langle 2t, t^2, \arctan(t) \rangle$

Using the scalar product rule:

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= \left(\frac{d}{dt} \sin(t) \right) \langle 2t, t^2, \arctan(t) \rangle + \sin(t) \left(\frac{d}{dt} \langle 2t, t^2, \arctan(t) \rangle \right) \\ &= \cos(t) \langle 2t, t^2, \arctan(t) \rangle + \sin(t) \left\langle 2, 2t, \frac{1}{1+t^2} \right\rangle \\ &= \left\langle 2t \cos(t) + 2 \sin(t), t^2 \cos(t) + 2t \sin(t), \arctan(t) \cos(t) + \frac{\sin(t)}{1+t^2} \right\rangle\end{aligned}$$

And rewriting first:

$$\mathbf{r}(t) = \langle 2t \sin(t), t^2 \sin(t), \arctan(t) \sin(t) \rangle$$

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= \left\langle \left(\frac{d}{dt} 2t \right) \sin(t) + 2t \left(\frac{d}{dt} \sin(t) \right), \left(\frac{d}{dt} t^2 \right) \sin(t) + t^2 \left(\frac{d}{dt} \sin(t) \right), \left(\frac{d}{dt} \arctan(t) \right) \sin(t) + \arctan(t) \left(\frac{d}{dt} \sin(t) \right) \right\rangle \\ &= \left\langle 2t \cos(t) + 2 \sin(t), t^2 \cos(t) + 2t \sin(t), \frac{\sin(t)}{1+t^2} + \arctan(t) \cos(t) \right\rangle\end{aligned}$$

(b) $\mathbf{s}(t) = \mathbf{r}(2^t)$, where $\mathbf{r}(t) = \langle t + 2, \ln(t), 1 \rangle$ – Note that $\mathbf{r}'(t) = \langle 1, \frac{1}{t}, 0 \rangle$

Using the chain rule first:

$$\begin{aligned}\mathbf{s}'(t) &= \mathbf{r}'(2^t) \frac{d}{dt} 2^t \\ &= \left\langle 1, \frac{1}{2^t}, 0 \right\rangle 2^t \ln(2) \\ &= \langle 2^t \ln(2), \ln(2), 0 \rangle\end{aligned}$$

And simplifying first:

$$\begin{aligned}\mathbf{s}(t) &= \mathbf{r}(2^t) = \langle 2^t + 2, \ln(2^t), 1 \rangle = \langle 2^t + 2, t \ln(2), 1 \rangle \\ \mathbf{s}'(t) &= \langle 2^t \ln(2), \ln(2), 0 \rangle\end{aligned}$$

(c) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle \cdot \langle -\sin(t), \cos(t), 1 \rangle$

Using the dot product rule:

$$\begin{aligned}r'(t) &= \left(\frac{d}{dt} \langle \cos(t), \sin(t), t \rangle \right) \cdot \langle -\sin(t), \cos(t), 1 \rangle + \langle \cos(t), \sin(t), t \rangle \cdot \left(\frac{d}{dt} \langle -\sin(t), \cos(t), 1 \rangle \right) \\ &= \langle -\sin(t), \cos(t), 1 \rangle \cdot \langle -\sin(t), \cos(t), 1 \rangle + \langle \cos(t), \sin(t), t \rangle \cdot \langle -\cos(t), -\sin(t), 0 \rangle \\ &= [\sin^2(t) + \cos^2(t) + 1] + [-\cos^2(t) - \sin^2(t) + 0] = 1 + 1 - 1 + 0 = 1 (!!)\end{aligned}$$

And rewriting first:

$$\begin{aligned}r(t) &= \cos(t) \cdot (-\sin(t)) + \sin(t) \cdot \cos(t) + t \cdot 1 \\ r(t) &= t (!!)\end{aligned}$$

$$r'(t) = 1$$

(d) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle \times \langle -\sin(t), \cos(t), 1 \rangle$

Using the cross product rule:

$$\begin{aligned}\mathbf{r}'(t) &= \left(\frac{d}{dt} \langle \cos(t), \sin(t), t \rangle \right) \times \langle -\sin(t), \cos(t), 1 \rangle + \langle \cos(t), \sin(t), t \rangle \times \left(\frac{d}{dt} \langle -\sin(t), \cos(t), 1 \rangle \right) \\ &= \langle -\sin(t), \cos(t), 1 \rangle \times \langle -\sin(t), \cos(t), 1 \rangle + \langle \cos(t), \sin(t), t \rangle \times \langle -\cos(t), -\sin(t), 0 \rangle\end{aligned}$$

The first two vectors are parallel, so their cross product is $\mathbf{0}$.

$$= \mathbf{0} + \langle \cos(t), \sin(t), t \rangle \times \langle -\cos(t), -\sin(t), 0 \rangle = \langle t \sin(t), -t \cos(t), 0 \rangle \quad (\text{Thanks, WA!})$$

And finding the cross product first:

$$\begin{aligned}\mathbf{r}(t) &= \langle \sin(t) - t \cos(t), -t \sin(t) - \cos(t), \sin^2(t) + \cos^2(t) \rangle = \langle \sin(t) - t \cos(t), -t \sin(t) - \cos(t), 1 \rangle \\ \mathbf{r}'(t) &= \frac{d}{dt} \langle \sin(t) - t \cos(t), -t \sin(t) - \cos(t), 1 \rangle \\ &= \langle \cos(t) - (1 \cos(t) + t(-\sin(t))), -(1 \sin(t) + t \cos(t)) - (-\sin(t)), 0 \rangle \\ &= \langle t \sin(t), -t \cos(t), 0 \rangle\end{aligned}$$

2. (AC Multi 9.7 Exercise 18) A central force is one that acts on an object so that the force \mathbf{F} is parallel to the object's position \mathbf{r} . Since Newton's Second Law says that an object's acceleration is proportional to the force exerted on it, the acceleration \mathbf{a} of an object moving under a central force will be parallel to its position \mathbf{r} . For instance, the Earth's acceleration due to the gravitational force that the sun exerts on the Earth is parallel to the Earth's position vector (see figure in the textbook).

- (a) If an object of mass m is moving under a central force, the angular momentum vector is defined to be $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$. Assuming the mass is constant, show that the angular momentum is constant by showing that $\frac{d\mathbf{L}}{dt} = \mathbf{0}$.

Some stuff to keep track of:

- m is a constant scalar;
- \mathbf{r} is a variable vector (depends on t);
- \mathbf{v} is a variable vector (depends on t).

Seems like the natural thing to do is to use the product rule to compute $\frac{d\mathbf{L}}{dt}$:

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt} [m\mathbf{r} \times \mathbf{v}] = \frac{d}{dt} [m\mathbf{r}] \times \mathbf{v} + m\mathbf{r} \times \frac{d}{dt} [\mathbf{v}]$$

Well, the derivative of position is velocity, and the derivative of velocity is acceleration:

$$= [m\mathbf{v}] \times \mathbf{v} + m\mathbf{r} \times \mathbf{a}$$

Now we're getting somewhere. Remember that the cross product of two parallel vectors is $\mathbf{0}$. $m\mathbf{v}$ is certainly parallel to \mathbf{v} , and we're assuming in the context of the problem that \mathbf{a} is parallel to \mathbf{r} , so it's also parallel to $m\mathbf{r}$.

$$= \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Cool, so that tells us that \mathbf{L} is a constant vector.

- (b) Explain why $\mathbf{L} \cdot \mathbf{r} = 0$.

\mathbf{L} was defined as the cross product of $m\mathbf{r}$ and \mathbf{v} , so it's perpendicular to both of those. Since m is a scalar, $m\mathbf{r}$ points in the same direction as \mathbf{r} , so if \mathbf{L} is perpendicular to $m\mathbf{r}$, it's also perpendicular to \mathbf{r} . Therefore, their dot product is zero.

- (c) Explain why we may conclude that the object is constrained to lie in the plane passing through the origin and perpendicular to \mathbf{L} .

The equation $\mathbf{L} \cdot \mathbf{r} = 0$ reminds me of the vector equation of a plane, with \mathbf{L} as the (constant) normal vector and \mathbf{r} playing the role of $\overrightarrow{PP_0}$. Since \mathbf{r} 's initial point is the origin, we thus have the plane that's perpendicular to \mathbf{L} and passing through the origin.

(Another way to see this: Certainly all the position vectors are perpendicular to \mathbf{L} . Also, certainly all the position vectors emanate from the origin. Also, we've just found that \mathbf{L} is constant. So the only way this is going to happen is if all the position vectors lie in the **same** plane – specifically, the plane containing the origin and perpendicular to the constant vector \mathbf{L} .)

3. (AC Multi 9.8 Exercise 14) Consider the standard helix parameterized by $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$.

- (a) Recall that the unit tangent vector $\mathbf{T}(t)$ is the vector tangent to the curve at time t that points in the direction of motion and has length 1. Find $\mathbf{T}(t)$.

$$\begin{aligned}\mathbf{r}'(t) &= \langle -\sin(t), \cos(t), 1 \rangle \\ |\mathbf{r}'(t)| &= [\sin^2(t) + \cos^2(t) + 1] = \sqrt{2} \\ \mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle\end{aligned}$$

- (b) Explain why the fact that $|\mathbf{T}(t)| = 1$ implies that \mathbf{T} and \mathbf{T}' are orthogonal vectors for every value of t .

The hint is to look at the dot product of \mathbf{T} and \mathbf{T}' . Where would that have come from? Well, certainly if we did the derivative of $\mathbf{T} \cdot \mathbf{T}$, then the product rule would make that pop out: $\frac{d}{dt}[\mathbf{T} \cdot \mathbf{T}] = \mathbf{T}' \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{T}'$. But since $\mathbf{T}' \cdot \mathbf{T} = \mathbf{T} \cdot \mathbf{T}'$, this is $2(\mathbf{T} \cdot \mathbf{T}')$.

The other thing we know about a dot product of a vector with itself is that it's the magnitude of that vector squared: $\mathbf{T} \cdot \mathbf{T} = |\mathbf{T}|^2 = 1^2 = 1$. Therefore, its derivative must be zero.

So now let's combine the two things we know about $\frac{d}{dt}[\mathbf{T} \cdot \mathbf{T}]$:

$$\begin{aligned}\frac{d}{dt}[\mathbf{T} \cdot \mathbf{T}] &= 2(\mathbf{T} \cdot \mathbf{T}') \\ \frac{d}{dt}[1] &= \frac{d}{dt}[\mathbf{T} \cdot \mathbf{T}] \\ 0 &= 2(\mathbf{T} \cdot \mathbf{T}')\end{aligned}$$

Therefore \mathbf{T} is orthogonal to \mathbf{T}' .

- (c) For the given function \mathbf{r} with unit tangent vector $\mathbf{T}(t)$ (from part (a)), determine $\mathbf{N}(t) = \frac{1}{|\mathbf{T}'(t)|} \mathbf{T}'(t)$.

$$\begin{aligned}\mathbf{T}(t) &= \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle \\ \mathbf{T}'(t) &= \frac{1}{\sqrt{2}} \langle -\cos(t), -\sin(t), 0 \rangle \\ |\mathbf{T}'(t)| &= \frac{1}{\sqrt{2}} \sqrt{\cos^2(t) + \sin^2(t) + 0} = \frac{1}{\sqrt{2}} \\ \mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \langle -\cos(t), -\sin(t), 0 \rangle\end{aligned}$$

- (d) What geometric properties does $\mathbf{N}(t)$ have? How long is this vector, and how is it situated in comparison to $\mathbf{T}(t)$?

Since $\mathbf{N}(t)$ is the unitized version of $\mathbf{T}'(t)$, it has length 1, and it's orthogonal to $\mathbf{T}(t)$ (because we proved that $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$ in part b!).

- (e) Let $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$, and compute $\mathbf{B}(t)$ in terms of your results in (a) and (c).

$$\begin{aligned}\mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle \times \langle -\cos(t), -\sin(t), 0 \rangle \\ &= \frac{1}{\sqrt{2}} \langle \sin(t), -\cos(t), 1 \rangle \quad (\text{Thanks, WolframAlpha!})\end{aligned}$$

- (f) What geometric properties does $\mathbf{B}(t)$ have? How long is this vector, and how is it situated in comparison to $\mathbf{T}(t)$ and $\mathbf{N}(t)$?

The length of \mathbf{B} is 1, which you can tell either by computing directly or by noting that $|\mathbf{T} \times \mathbf{B}| = |\mathbf{T}| \cdot |\mathbf{B}| \cdot \sin \theta$, and the vectors \mathbf{T} and \mathbf{B} are at right angles, so $\sin \theta = 1$.

\mathbf{B} is perpendicular to both \mathbf{T} and \mathbf{N} . (\mathbf{B} here stands for “binormal” – it’s “the other” normal vector.)

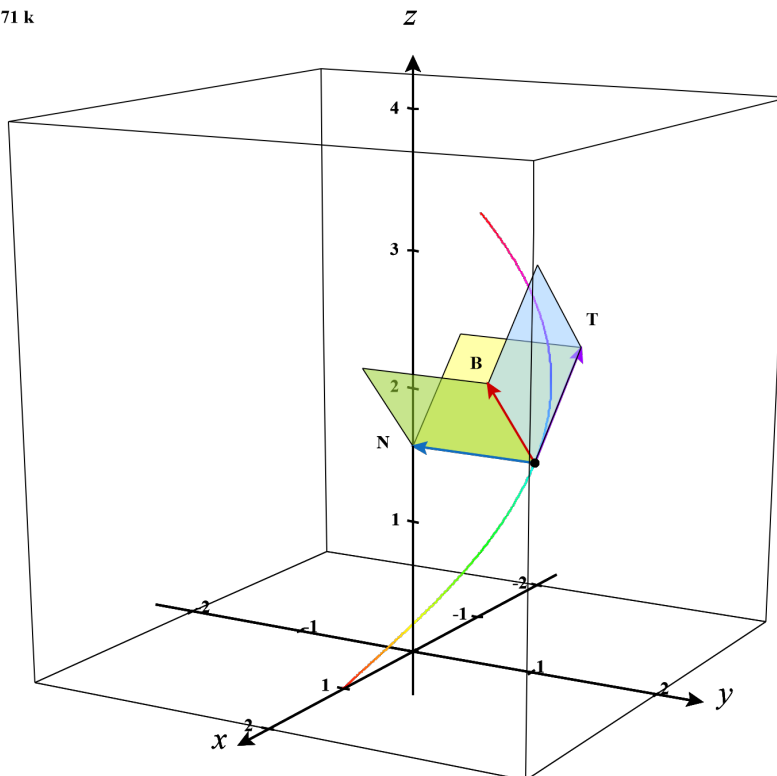
- (g) Sketch a plot of the given helix, and compute and sketch $\mathbf{T}\left(\frac{\pi}{2}\right)$, $\mathbf{N}\left(\frac{\pi}{2}\right)$, and $\mathbf{B}\left(\frac{\pi}{2}\right)$.

Plugging and chugging:

$$\begin{aligned}\mathbf{T}\left(\frac{\pi}{2}\right) &= \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle \\ \mathbf{N}\left(\frac{\pi}{2}\right) &= \langle 0, -1, 0 \rangle \\ \mathbf{B}\left(\frac{\pi}{2}\right) &= \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle\end{aligned}$$

And here’s a plot from CalcPlot3D:

$$\begin{aligned}\mathbf{T} &= -0.71 \mathbf{i} + 0.71 \mathbf{k} \\ \mathbf{N} &= -1 \mathbf{j} \\ \mathbf{B} &= 0.71 \mathbf{i} + 0.71 \mathbf{k}\end{aligned}$$



4. (AC Multi 9.7 Exercise 16) For each given function \mathbf{r} , determine $\int \mathbf{r}(t) dt$. In addition, recalling the Fundamental Theorem of Calculus for functions of a single variable, also evaluate $\int_0^1 \mathbf{r}(t) dt$ for each given function r . Is the resulting quantity a scalar or a vector? What does it measure?

If we label the antiderivative $\int \mathbf{r}(t) dt$ as $\mathbf{R}(t)$, then $\int_0^1 \mathbf{r}(t) dt$ will be a vector that points from $\mathbf{R}(0)$ to $\mathbf{R}(1)$.

(a) $\mathbf{r}(t) = \left\langle \cos(t), \frac{1}{t+1}, te^t \right\rangle$

$$\begin{aligned} \int \mathbf{r}(t) dt &= \left\langle \int \cos(t) dt, \int \frac{1}{t+1} dt, \int te^t dt \right\rangle \\ &= \langle \sin(t), \ln|t+1|, te^t - e^t \rangle + \vec{C} \\ \int_0^1 \mathbf{r}(t) dt &= \langle \sin(t), \ln|t+1|, te^t - e^t \rangle \Big|_0^1 \\ &= \langle \sin(1) - \sin(0), \ln(2) - \ln(1), (1e^1 - e^1) - (0e^0 - e^0) \rangle = \langle \sin(1), \ln(2), 1 \rangle \end{aligned}$$

(b) $\mathbf{r}(t) = \langle \cos(3t), \sin(2t), t \rangle$

$$\begin{aligned} \int \mathbf{r}(t) dt &= \left\langle \frac{1}{3} \sin(3t), -\frac{1}{2} \cos(2t), \frac{1}{2} t^2 \right\rangle + \vec{C} \\ \int_0^1 \mathbf{r}(t) dt &= \left\langle \frac{1}{3} \sin(3t), -\frac{1}{2} \cos(2t), \frac{1}{2} t^2 \right\rangle \Big|_0^1 \\ &= \left\langle \frac{1}{3} \sin(3), -\frac{1}{2} (\cos(2) - 1), \frac{1}{2} \right\rangle \end{aligned}$$

(c) $\mathbf{r}(t) = \left\langle \frac{t}{1+t^2}, te^{t^2}, \frac{1}{1+t^2} \right\rangle$

(The first two components require integration by substitution!)

$$\begin{aligned} \int \mathbf{r}(t) dt &= \left\langle \frac{1}{2} \ln(1+t^2), \frac{1}{2} e^{t^2}, \arctan(t) \right\rangle + \vec{C} \\ \int \mathbf{r}(t) dt &= \left\langle \frac{1}{2} \ln(1+t^2), \frac{1}{2} e^{t^2}, \arctan(t) \right\rangle \Big|_0^1 \\ &= \left\langle \frac{1}{2} \ln(2), \frac{1}{2} (e - 1), \frac{\pi}{4} \right\rangle \end{aligned}$$

5. (AC Multi 9.7 Exercise 17) In this exercise, we develop the formula for the position function of a projectile that has been launched at an initial speed of $|\mathbf{v}_0|$ and a launch angle of θ . Recall that $\mathbf{a}(t) = \langle 0, -g \rangle$ is the constant acceleration of the projectile at any time t .

- Find all velocity vectors for the given acceleration vector $\mathbf{a}(t)$. When you anti-differentiate, remember that there is an arbitrary constant that arises in each component.
- Use the given information about initial speed and launch angle to find \mathbf{v}_0 , the initial velocity of the projectile. You will want to write the vector in terms of its components, which will involve $\sin \theta$ and $\cos \theta$.
- Next, find the specific velocity vector function $\mathbf{v}(t)$ for the projectile. That is, combine your work in (a) and (b) in order to determine expressions in terms of $|\mathbf{v}_0|$ and θ for the constants that arose when integrating.
- Find all possible position vectors for the velocity vector $\mathbf{v}(t)$ you determined in (c).
- Let $\mathbf{r}(t)$ denote the position vector function for the given projectile. Use the fact that the object is fired from the position (x_0, y_0) to show it follows that

$$\mathbf{r}(t) = \left\langle |\mathbf{v}_0| \cos(\theta)t + x_0, -\frac{g}{2}t^2 + |\mathbf{v}_0| \sin(\theta)t + y_0 \right\rangle.$$