PS#7 - Partial derivatives - Answer key

1. (Activity 10.2.4) The speed of sound C traveling through ocean water is a function of temperature, salinity and depth. It may be modeled by the function

$$C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34 - 0.01T)(S - 35) + 0.016D.$$

Here C is the speed of sound in meters/second, T is the temperature in degrees Celsius, S is the salinity in grams/liter of water, and D is the depth below the ocean surface in meters.

   (a) State the units in which each of the partial derivatives, C_T, C_S, and C_D, are expressed and explain the physical meaning of each.

   C_T is measured in $\frac{\text{m/s}}{°C}$, and it tells us how much the speed of sound in water changes as the temperature changes.

   C_S is measured in $\frac{\text{m/s}}{\text{gm/L}}$, and it tells us how much the speed of sound in water changes as the salinity changes.

   C_D is measured in $\frac{\text{m/s}}{\text{m}}$, and it tells us how much the speed of sound in water changes as the depth changes.

   (b) Find the partial derivatives C_T, C_S, and C_D.

$$C_T = 4.6 - 0.11T + 0.00087T^2 - 0.01(S - 35)$$
$$C_S = 1.34 - 0.01T$$
$$C_D = 0.016$$

   (c) Evaluate each of the three partial derivatives at the point where $T = 10$, $S = 35$ and $D = 100$. What does the sign of each partial derivative tell us about the behavior of the function C at the point $(10, 35, 100)$?

   $C_T(10, 35, 100) = 3.587$ – so the speed of sound **increases** as the temperature goes up from this point.

   $C_S(10, 35, 100) = 1.24$ – so the speed of sound **increases** as the salinity goes up from this point.

   $C_D(10, 35, 100) = 0.016$ – so the speed of sound goes **up** as the depth increases from this point.

2. (Activity 10.2.5) The wind chill, as frequently reported, is a measure of how cold it feels outside when the wind is blowing. In the table below, the wind chill w, measured in degrees Fahrenheit, is a function of the wind speed v, measured in miles per hour, and the ambient air temperature T, also measured in degrees Fahrenheit. We thus view w as being of the form $w = w(v, T)$.

$v \backslash T$	−30	−25	−20	−15	−10	−5	0	5	10	15	20
5	−46	−40	−34	−28	−22	−16	−11	−5	1	7	13
10	−53	−47	−41	−35	−28	−22	−16	−10	−4	3	9
15	−58	−51	−45	−39	−32	−26	−19	−13	−7	0	6
20	−61	−55	−48	−42	−35	−29	−22	−15	−9	−2	4
25	−64	−58	−51	−44	−37	−31	−24	−17	−11	−4	3
30	−67	−60	−53	−46	−39	−33	−26	−19	−12	−5	1
35	−69	−62	−55	−48	−41	−34	−27	−21	−14	−7	0
40	−71	−64	−57	−50	−43	−36	−29	−22	−15	−8	−1

   (a) Estimate the partial derivative $w_v(20, -10)$. What are the units on this quantity and what does it mean?

$$w_v(20, -10) \approx \frac{w(25, -10) - w(15, -10)}{25 - 15} = \frac{-37 - (-32)}{10} = \frac{-5}{10} = -0.5 \frac{°\text{F}}{\text{mph}}.$$

   That is, if the windspeed goes up by 1 mph, the perceived temperature will drop by 0.5° F.

(b) Estimate the partial derivative $w_T(20,-10)$. What are the units on this quantity and what does it mean?

$$w_T(20,-10) \approx \frac{w(20,-5)-w(20,-15)}{-5-(-15)} = \frac{-29-(-42)}{10} = \frac{13}{10} = 1.3\frac{^\circ\text{F}}{^\circ\text{F}}.$$

That is, if the ambient temperature goes up by 1° F, the perceived temperature will go up by 1.3° F.

(c) Use your results to estimate the wind chill $w(18,-10)$.

The windspeed has gone down by 2 mph. Each 1 mph decrease in windspeed causes an 0.5° F **increase** in perceived temperature. Therefore, $w(18,-10)$ should be about 1° F warmer than $w(20,-10)$, which is -35, so it should be -34° F.

In symbols:

$$\begin{aligned} w(18,-10) &\approx w(20,-10) + (18-20)\cdot w_v(20,-10) \\ &= -35^\circ\text{F} + (-2 \text{ mph})\cdot\left(-0.5\frac{^\circ\text{F}}{\text{mph}}\right) \\ &= -35^\circ\text{F} + 1^\circ\text{F} = -34^\circ\text{F}. \end{aligned}$$

(d) Use your results to estimate the wind chill $w(20,-12)$.

The ambient temperature has dropped by 2 degrees. Each 1 degree decrease in ambient temperature causes a 1.3 degree decrease in perceived temperature. Therefore, $w(20,-12)$ should be about 2.6 degrees colder than $w(20,-10)$, which is -35° F, so it should be -37.6° F.

In symbols:

$$\begin{aligned} w(20,-12) &\approx w(20,-10) + (-12-(-10))\cdot w_T(20,-10) \\ &= -35^\circ\text{F} + (-2^\circ\text{F})\cdot\left(1.3\frac{^\circ\text{F}}{^\circ\text{F}}\right) \\ &= -35^\circ\text{F} - 2.6^\circ\text{F} = -37.6^\circ\text{F}. \end{aligned}$$

(e) Consider how you might combine your previous results to estimate the wind chill $w(18,-12)$. Explain your process.

From the original -35°F, the perceived temperature should go up by 1 (from the decrease in windspeed) and down by 2.6 (from the decrease in temperature). So I bet $w(18,-12) \approx -35+1-2.6-36.6^\circ$F.

3. (Activity 10.2.6) Shown in the figure below is a contour plot of a function f. The values of the function on a few of the contours are indicated to the left of the figure.

203-keys/ps7-answer-key/10-2-6.png

(a) Estimate the partial derivative $f_x(-2,-1)$.

We can use the contours to read off some approximate values and use a symmetric difference.

NOTE: I'm using a step size of 1 – your answer may be a little different from this if you chose a different step size. Same deal in parts (b) and (c).

$$\begin{aligned} f(-3,-1) &\approx 3 \\ f(-1,-1) &\approx 4.5 \\ f_x(-2,-1) &\approx \frac{f(-1,-1)-f(-3,-1)}{-1-(-3)} \\ &= \frac{4.5-3}{2} = 0.75 \end{aligned}$$

(b) Estimate the partial derivative $f_y(-2,-1)$.

Same game – we can use the contours to read off some approximate values and use a symmetric difference:

$$\begin{aligned} f(-2,-2) &\approx 6 \\ f(-2,0) &\approx 2.5 \\ f_y(-2,-1) &\approx \frac{f(-2,0)-f(-2,-2)}{0-(-2)} \\ &= \frac{2.5-6}{2} = -1.75 \end{aligned}$$

(c) Estimate the partial derivatives $f_x(-1,2)$ and $f_y(-1,2)$.

$$= \frac{2.5-0.5}{2} = 1$$

$$\begin{aligned} f(0,2) &\approx 2.5 \\ f(-2,2) &\approx 0.5 \\ f_x(-1,2) &\approx \frac{f(0,2)-f(-2,2)}{0-(-2)} \end{aligned}$$

$$\begin{aligned} f(-1,1) &\approx 2.5 \\ f(-1,3) &\approx 2.5 \end{aligned}$$

$$f_y(-1,2) \approx \frac{f(-1,3) - f(-1,1)}{3-1} = \frac{2.5 - 2.5}{2} = 0$$

(d) Locate, if possible, one point (x,y) where $f_x(x,y) = 0$.
Looks like at about $(0,-0.5)$, if I move in the x direction, the heights on the contour map don't change much.

(e) Locate, if possible, one point (x,y) where $f_x(x,y) < 0$.
Looks like at about $(1,-2)$, if I move in the x direction, I'm going downhill.

(f) Locate, if possible, one point (x,y) where $f_y(x,y) > 0$.
Looks like at about $(1,2)$, if I move in the y direction, I'm going (very slightly) uphill.
NOTE: Of course there are other points that will work for all three of these parts.