PS#7 – Linearization and differentials - Answer key

(Assignment: Pick your two favorite parts of Activity 10.4.4, Exercise 10.4.13, and Exercise 10.4.14.)

10.4.4a Suppose that the elevation of a landscape is given by the function h, where we additionally know that $h(3,1) = 4.35$, $h_x(3,1) = 0.27$, and $h_y(3,1) = -0.19$. Assume that x and y are measured in miles in the easterly and northerly directions, respectively, from some base point $(0,0)$. Your GPS device says that you are currently at the point $(3,1)$. However, you know that the coordinates are only accurate to within 0.2 units; that is, $dx = \Delta x = 0.2$ and $dy = \Delta y = 0.2$. Estimate the uncertainty in your elevation using differentials.

The total differential in h at the point $(3,1)$ is given by

$$dh = h_x(3,1)\,dx + h_y(3,1)\,dy.$$

All we have to do is throw in our values:

$$dh = 0.27 \cdot 0.2 + (-0.19) \cdot 0.2 = 0.016.$$

Notice, however, that dy could also be negative, which would provide a slightly larger value for dh:

$$dh = 0.27 \cdot 0.2 + (-0.19) \cdot (-0.2) = 0.092.$$

So, our actual height could be anywhere between $4.35 - 0.092 = 4.258$ and $4.35 + 0.092 = 4.442$.

10.4.4b The pressure, volume, and temperature of an ideal gas are related by the equation

$$P = P(T,V) = \frac{8.31T}{V},$$

where P is measured in kilopascals, V in liters, and T in kelvin. Find the pressure when the volume is 12 liters and the temperature is 310 K. Use differentials to estimate the change in the pressure when the volume increases to 12.3 liters and the temperature decreases to 305 K.

We'll start by computing some stuff we'll need:

$$\begin{aligned}
P(310,12) &= \frac{8.31 \cdot 310}{12} = 214.675\,\text{kPa} \\
P_T(T,V) &= \frac{8.31}{V} & P_T(310,12) &= \frac{8.31}{12} = 0.6925\,\text{kPa/K} \\
P_V(T,V) &= \frac{-8.31 \cdot T}{V^2} & P_V(310,12) &= \frac{-8.31 \cdot 310}{12^2} \approx -17.89\,\text{kPa/L}
\end{aligned}$$

If the volume increases to $12.3\,\text{L}$, then $dV = 0.3\,\text{L}$, and if the temperature decreases to $305\,\text{K}$, then $dT = -5\,\text{K}$. The total differential in P is given by

$$\begin{aligned}
dP &= P_V(310,12) \cdot dV + P_T(310,12) \cdot dT \\
&= -17.89\,\text{kPa/L} \cdot 0.3\,\text{L} + 0.6925\,\text{kPa/K} \cdot -5\,\text{K} \\
&= -8.8295\,\text{kPa}
\end{aligned}$$

So, the pressure should drop by $8.8295\,\text{kPa}$, down to $205.8455\,\text{kPa}$ or so. (Indeed, $P(305, 12.3) \approx 206.061\,\text{kPa}$.)

10.4.4c Refer to the table below, the table of values of the wind chill $w(v,T)$, in degrees Fahrenheit, as a function of temperature, also in degrees Fahrenheit, and wind speed, in miles per hour. Suppose your anemometer says the wind is blowing at 25 miles per hour and your thermometer shows a reading of -15°F. However, you know your thermometer is only accurate to within 2°F and your anemometer is only accurate to within 3 miles per hour. What is the wind chill based on your measurements? Estimate the uncertainty in your measurement of the wind chill.

$v\backslash T$	−30	−25	−20	−15	−10	−5	0	5	10	15	20
5	−46	−40	−34	−28	−22	−16	−11	−5	1	7	13
10	−53	−47	−41	−35	−28	−22	−16	−10	−4	3	9
15	−58	−51	−45	−39	−32	−26	−19	−13	−7	0	6
20	−61	−55	−48	−42	−35	−29	−22	−15	−9	−2	4
25	−64	−58	−51	−44	−37	−31	−24	−17	−11	−4	3
30	−67	−60	−53	−46	−39	−33	−26	−19	−12	−5	1
35	−69	−62	−55	−48	−41	−34	−27	−21	−14	−7	0
40	−71	−64	−57	−50	−43	−36	−29	−22	−15	−8	−1

$w(25,-15) = -44$°F, and we know that $dT = \pm 2$°F and $dv = \pm 3$ mph. We need to estimate the partial derivatives:

$$w_v(25,-15) \approx \frac{w(30,-15)-w(20,-15)}{30-20} = \frac{-46-(-42)}{10} = -0.4\ {}^\circ\text{F/mph}$$
$$w_T(25,-15) \approx \frac{w(25,-10)-w(25,-20)}{-10-(-20)} = \frac{-37-(-51)}{10} = 1.4\ {}^\circ\text{F}/{}^\circ\text{F}$$

The total differential is

$$\begin{aligned} dw &= w_v(25,-15)\cdot dv + w_T(25,-15)\cdot dT \\ &= -0.4\ {}^\circ\text{F/mph}\cdot dv + 1.4\ {}^\circ\text{F}/{}^\circ\text{F}\cdot dT \end{aligned}$$

Note that we get the largest errors if dv is negative and dT is positive:

$$\begin{aligned} dw &= -0.4\ {}^\circ\text{F/mph}\cdot(-3\text{ mph}) + 1.4\ {}^\circ\text{F}/{}^\circ\text{F}\cdot 2^\circ\text{F} \\ &= 4\ {}^\circ\text{F} \end{aligned}$$

So, the windchill might be anywhere between $-44-4=-48$ °F and $-44+4=-40$ °F.

10.4.13a. Find the linearization $L(x,y)$ for the function f defined by $f(x,y)=\cos(x)(2e^{2y}+e^{-2y})$ at the point $(x_0,y_0)=(0,0)$. Hence use the linearization to estimate the value of $f(0.1,0.2)$. Compare your estimate to the actual value of $f(0.1,0.2)$.

First let's compute all the values we'll need:

$$\begin{aligned} f(0,0) &= \cos(0)(2e^{2\cdot 0}+e^{-2\cdot 0}) = 3 \\ f_x(x,y) &= -\sin(x)(2e^{2y}+e^{-2y}) \\ f_x(0,0) &= -\sin(0)(2e^{2\cdot 0}+e^{-2\cdot 0}) = 0 \\ f_y(x,y) &= \cos(x)(4e^{2y}-2e^{-2y}) \\ f_y(0,0) &= \cos(0)(4e^{2\cdot 0}-2e^{-2\cdot 0}) = 2 \end{aligned}$$

Cool, so the linearization $L(x,y) = 3+0\cdot(x-0)+2\cdot(y-0) = 3+2y$.

So, $f(0.1,0.2)\approx L(0.1,0.2) = 3+2\cdot 0.2 = 3.4$.

The actual value is $f(0.1,0.2) = 3.63571$, per W|A. 3.4 is pretty dang close!

10.4.13b. The Heat Index, I (measured in apparent degrees F), is a function of the actual temperature T outside (in degrees F) and the relative humidity H (measured as a percentage). A portion of the table which gives values for this function,

$I = I(T,H)$, is provided below:

$T \downarrow \backslash H \rightarrow$	70	75	80	85
90	106	109	112	115
92	112	115	119	123
94	118	122	127	132
96	125	130	135	141

Suppose you are given that $I_T(94,75) = 3.75$ and $I_H(94,75) = 0.9$. Use this given information and one other value from the table to estimate the value of $I(93.1,77)$ using the linearization at $(94,75)$. Using proper terminology and notation, explain your work and thinking.

Okay, so we'll also need a starting value. From the table, $I(94,75) = 122$. (Yikes!)

Now we have enough information to write down the linearization:

$$\begin{aligned} L(T,H) &= I(94,75) + I_T(94,75)(T-94) + I_H(94,75)(H-75) \\ &= 122 + 3.75(T-94) + 0.9(H-75) \end{aligned}$$

I could simplify this, but I think this will actually be easier to work with as is:

$$\begin{aligned} I(93.1,77) \approx L(93.1,77) &= 122 + 3.75(93.1-94) + 0.9(77-75) \\ &= 122 + 3.75(-0.9) + 0.9(2) \\ &= 122 + (-3.375) + 1.8 = 122 - 1.575 \\ &= 120.425 \end{aligned}$$

What I really like about this is it shows the contribution of each change from our starting point to the overall heat index. Dropping the temperature to 93.1°, a decrease of 0.9°, caused the heat index to drop by 3.375°, and increasing the humidity to 77%, an increase of 2%, caused the heat index to go up by 1.8°, for a total change of -1.575°. So we end up dropping 1.575° from 122° to land at 120.425°.

10.4.13c. Just as we can find a local linearization for a differentiable function of two variables, we can do so for functions of three or more variables. By extending the concept of the local linearization from two to three variables, find the linearization of the function $h(x,y,z) = e^{2x}(y+z^2)$ at the point $(x_0,y_0,z_0) = (0,1,-2)$. Then, use linearization to estimate the value of $h(-0.1,0.9,-1.8)$.

First let's compute all the values we'll need:

$$\begin{aligned} h(0,1,-2) &= e^{2\cdot 0}(1+(-2)^2) = 5 \\ h_x(x,y,z) &= 2e^{2x}(y+z^2) \\ h_x(0,1,-2) &= 2e^{2\cdot 0}(1+(-2)^2) = 10 \\ h_y(x,y,z) &= e^{2x}(1) \\ h_y(0,1,-2) &= e^{2\cdot 0} = 1 \\ h_z(x,y,z) &= e^{2x}(2z) \\ h_z(0,1,-2) &= e^{2\cdot 0}(2\cdot(-2)) = -4 \end{aligned}$$

Putting this all together,

$$\begin{aligned} L(x,y,z) &= 5 + 10(x-0) + 1(y-1) + (-4)(z+2) \\ h(-0.1,0.9,-1.8) \approx L(-0.1,0.9,-1.8) &= 5 + 10(-0.1-0) + 1(0.9-1) + (-4)(-1.8+2) \\ &= 5 - 1 - 0.1 - 0.8 = 3.1 \end{aligned}$$

Just for fun, the actual value of $h(-0.1,0.9,-1.8)$ is 3.38955. Pretty decent approximation!

10.4.14a. Let f represent the vertical displacement in centimeters from the rest position of a string (like a guitar string) as a function of the distance x in centimeters from the fixed left end of the string and y the time in seconds after the string has been plucked. A simple model for f could be

$$f(x,y) = \cos(x)\sin(2y).$$

Use the differential to approximate how much more this vibrating string is vertically displaced from its position at $(a,b) = \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$ if we decrease a by 0.01 cm and increase the time by 0.1 seconds. Compare to the value of f at the point $\left(\frac{\pi}{4} - 0.01, \frac{\pi}{3} + 0.1\right)$.

First let's compute the partial derivatives:

$$\begin{aligned}
f_x(x,y) &= -\sin(x)\sin(2y) \\
f_x\left(\frac{\pi}{4}, \frac{\pi}{3}\right) &= -\sin\left(\frac{\pi}{4}\right)\sin\left(2\cdot\frac{\pi}{3}\right) = -\frac{\sqrt{6}}{4} \approx -0.6124 \\
f_y(x,y) &= 2\cos(x)\cos(2y) \\
f_y\left(\frac{\pi}{4}, \frac{\pi}{3}\right) &= 2\cos\left(\frac{\pi}{4}\right)\cos\left(2\cdot\frac{\pi}{3}\right) = -\frac{\sqrt{2}}{2} \approx -0.7071
\end{aligned}$$

Now we can write down the differential:

$$\begin{aligned}
df &= f_x\left(\frac{\pi}{4}, \frac{\pi}{3}\right)dx + f_y\left(\frac{\pi}{4}, \frac{\pi}{3}\right)dy \\
&= -0.6124\,dx - 0.7071\,dy
\end{aligned}$$

We're given that $dx = -0.01$ and $dy = 0.1$, so

$$df = -0.6124\cdot(-0.01) - 0.7071\cdot 0.1 = -0.064586.$$

The value of f at the point $\left(\frac{\pi}{4} - 0.01, \frac{\pi}{3} + 0.1\right)$ is about 0.5351982. The value of f at the point $\left(\frac{\pi}{4}, \frac{\pi}{3}\right)$ is about 0.6123724. That makes for an actual change Δf of $0.5351982 - 0.6123724 = -0.0771742$, which is pretty close to the approximate change df we calculated.

10.4.14b. Resistors used in electrical circuits have colored bands painted on them to indicate the amount of resistance and the possible error in the resistance. When three resistors, whose resistances are R_1, R_2, and R_3, are connected in parallel, the total resistance R is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

Suppose that the resistances are $R_1 = 25\Omega$, $R_2 = 40\Omega$, and $R_3 = 50\Omega$. Find the total resistance R. If you know each of R_1, R_2, and R_3 with a possible error of 0.5%, estimate the maximum error in your calculation of R.

So, first of all, the total resistance:

$$\begin{aligned}
\frac{1}{R} &= \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \\
&= \frac{1}{25} + \frac{1}{40} + \frac{1}{50} = \frac{17}{200} \\
R &= \frac{200}{17} \approx 11.7647.
\end{aligned}$$

We now need to figure out each of the differentials, which should be 0.5% of each resistance value. So, $dR_1 = 0.005\cdot 25 = 0.125$, $dR_2 = 0.005\cdot 40 = 0.2$, and $dR_3 = 0.005\cdot 50 = 0.25$.

Writing down the total differential is going to be a little tricky because we don't have R by itself. We could certainly write

$$R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}},$$

but I would really rather not find partial derivatives of that beast. Instead, I'm going to use implicit differentiation:

$$\begin{aligned}
-\frac{1}{R^2}\,dR &= -\frac{1}{R_1^2}\,dR_1 - \frac{1}{R_2^2}\,dR_2 - \frac{1}{R_3^2}\,dR_3 \\
dR &= R^2 \cdot \left(\frac{1}{R_1^2}\,dR_1 + \frac{1}{R_2^2}\,dR_2 + \frac{1}{R_3^2}\,dR_3\right) \\
&= \left(\frac{200}{17}\right)^2 \cdot \left(\frac{1}{25^2}\cdot 0.125 + \frac{1}{40^2}\cdot 0.2 + \frac{1}{50^2}\cdot 0.25\right) \\
&\approx 0.058824.
\end{aligned}$$

(This is actually just about 0.5% of the computed value of R, which is interesting.)

If you're slightly freaked out about this implicit differentiation thing, I don't blame you, because I've hidden some details. Here's what's really going on. Consider a new function $W(R, R_1, R_2, R_3) = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} - \frac{1}{R}$. Our total resistance function is just the 0-contour of this function. So what we're doing is, we're finding dW in terms of dR, dR_1, dR_2, and dR_3. Then we just note that $dW = 0$ since we're moving around on the same contour, and then we can solve for dR in terms of dR_1, dR_2, and dR_3.