

## PS#6 - Partial derivatives - Answer key

1. (Activity 10.2.3) Throughout this solution, constants are in blue.

(a) If  $f(x, y) = 3x^3 - 2x^2y^5$ , find the partial derivatives  $f_x$  and  $f_y$ .

$$\begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x}(3x^3 - 2x^2y^5) \\ &= 9x^2 - 2(2x)y^5 \\ &= 9x^2 - 4xy^5 \end{aligned} \qquad \begin{aligned} f_y(x, y) &= \frac{\partial}{\partial y}(3x^3 - 2x^2y^5) \\ &= 0 - 2x^2(5y^4) \\ &= -10x^2y^4 \end{aligned}$$

(b) If  $f(x, y) = \frac{xy^2}{x+1}$ , find the partial derivatives  $f_x$  and  $f_y$ .

$$\begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x} \left( \frac{xy^2}{x+1} \right) \\ &= y^2 \frac{(x+1)(1) - (x)(1)}{(x+1)^2} \\ &= \frac{y^2}{(x+1)^2} \end{aligned} \qquad \begin{aligned} f_y(x, y) &= \frac{\partial}{\partial y} \left( \frac{x}{x+1} y^2 \right) \\ &= \frac{x}{x+1} \cdot 2y \\ &= \frac{2xy}{x+1} \end{aligned}$$

(c) If  $g(r, s) = rs \cos(r)$ , find the partial derivatives  $g_r$  and  $g_s$ .

$$\begin{aligned} g_r(r, s) &= \frac{\partial}{\partial r}(sr \cos(r)) \\ &= s(r(-\sin(r)) + (1)\cos(r)) \\ &= s \cos(r) - sr \sin(r) \end{aligned} \qquad \begin{aligned} g_s(r, s) &= \frac{\partial}{\partial s}(r \cos(r)s) \\ &= r \cos(r)(1) \\ &= r \cos(r) \end{aligned}$$

(d) Assuming  $f(w, x, y) = (6w+1)\cos(3x^2+4xy^3+y)$ , find the partial derivatives  $f_w$ ,  $f_x$ , and  $f_y$ .

$$\begin{aligned} f_w(w, x, y) &= \frac{\partial}{\partial w}[(6w+1)\cos(3x^2+4xy^3+y)] & f_x(w, x, y) &= \frac{\partial}{\partial x}[(6w+1)\cos(3x^2+4xy^3+y)] \\ &= (6)\cos(3x^2+4xy^3+y) & &= (6w+1)(-\sin(3x^2+4xy^3+y)) \cdot (6x+4y^3) \\ &= 6\cos(3x^2+4xy^3+y) & &= -(6w+1)(6x+4y^3)\sin(3x^2+4xy^3+y) \end{aligned}$$

$$\begin{aligned} f_y(w, x, y) &= \frac{\partial}{\partial y}[(6w+1)\cos(3x^2+4xy^3+y)] \\ &= (6w+1)(-\sin(3x^2+4xy^3+y)) \cdot (4x \cdot 3y^2 + 1) \\ &= -(6w+1)(12xy^2+1)\sin(3x^2+4xy^3+y) \end{aligned}$$

(e) Find all possible first-order partial derivatives of  $q(x, t, z) = \frac{x2^t z^3}{1+x^2}$ .

$$\begin{aligned} q_x(x, t, z) &= \frac{\partial}{\partial x} \left( 2^t z^3 \frac{x}{1+x^2} \right) & q_t(x, t, z) &= \frac{\partial}{\partial t} \left( \frac{xz^3}{1+x^2} 2^t \right) & q_z(x, t, z) &= \frac{\partial}{\partial z} \left( \frac{x2^t}{1+x^2} z^3 \right) \\ &= 2^t z^3 \frac{(1+x^2)(1) - (x)(2x)}{(1+x^2)^2} & &= \frac{xz^3}{1+x^2} (2^t \ln(2)) & &= \frac{x2^t}{1+x^2} (3z^2) \\ &= \frac{2^t z^3 (1-x^2)}{(1+x^2)^2} & &= \frac{xz^3 2^t \ln(2)}{1+x^2} & &= \frac{3x2^t z^2}{1+x^2} \end{aligned}$$

2. (Activity 10.2.4) The speed of sound  $C$  traveling through ocean water is a function of temperature, salinity and depth. It may be modeled by the function

$$C(T, S, D) = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34 - 0.01T)(S - 35) + 0.016D.$$

Here  $C$  is the speed of sound in meters/second,  $T$  is the temperature in degrees Celsius,  $S$  is the salinity in grams/liter of water, and  $D$  is the depth below the ocean surface in meters.

- (a) State the units in which each of the partial derivatives,  $C_T$ ,  $C_S$ , and  $C_D$ , are expressed and explain the physical meaning of each.

$C_T$  is measured in  $\frac{\text{m/s}}{^\circ\text{C}}$ , and it tells us how much the speed of sound in water changes as the temperature changes.

$C_S$  is measured in  $\frac{\text{m/s}}{\text{gm/L}}$ , and it tells us how much the speed of sound in water changes as the salinity changes.

$C_D$  is measured in  $\frac{\text{m/s}}{\text{m}}$ , and it tells us how much the speed of sound in water changes as the depth changes.

- (b) Find the partial derivatives  $C_T$ ,  $C_S$ , and  $C_D$ .

$$C_T(T, S, D) = 4.6 - 0.11T + 0.00087T^2 - 0.01(S - 35)$$

$$C_S(T, S, D) = 1.34 - 0.01T$$

$$C_D(T, S, D) = 0.016$$

- (c) Evaluate each of the three partial derivatives at the point where  $T = 10$ ,  $S = 35$  and  $D = 100$ . What does the sign of each partial derivative tell us about the behavior of the function  $C$  at the point  $(10, 35, 100)$ ?

$C_T(10, 35, 100) = 3.587$  – so the speed of sound **increases** as the temperature goes up from this point.

$C_S(10, 35, 100) = 1.24$  – so the speed of sound **increases** as the salinity goes up from this point.

$C_D(10, 35, 100) = 0.016$  – so the speed of sound **increases** as the depth increases from this point.

3. (Activity 10.2.5) The wind chill, as frequently reported, is a measure of how cold it feels outside when the wind is blowing. In the table below, the wind chill  $w$ , measured in degrees Fahrenheit, is a function of the wind speed  $v$ , measured in miles per hour, and the ambient air temperature  $T$ , also measured in degrees Fahrenheit. We thus view  $w$  as being of the form  $w = w(v, T)$ .

$v \backslash T$	-30	-25	-20	-15	-10	-5	0	5	10	15	20
5	-46	-40	-34	-28	-22	-16	-11	-5	1	7	13
10	-53	-47	-41	-35	-28	-22	-16	-10	-4	3	9
15	-58	-51	-45	-39	-32	-26	-19	-13	-7	0	6
20	-61	-55	-48	-42	-35	-29	-22	-15	-9	-2	4
25	-64	-58	-51	-44	-37	-31	-24	-17	-11	-4	3
30	-67	-60	-53	-46	-39	-33	-26	-19	-12	-5	1
35	-69	-62	-55	-48	-41	-34	-27	-21	-14	-7	0
40	-71	-64	-57	-50	-43	-36	-29	-22	-15	-8	-1

- (a) Estimate the partial derivative  $w_v(20, -10)$ . What are the units on this quantity and what does it mean?

$$w_v(20, -10) \approx \frac{w(25, -10) - w(15, -10)}{25 - 15} = \frac{-37 - (-32)}{10} = \frac{-5}{10} = -0.5 \frac{^\circ\text{F}}{\text{mph}}.$$

That is, if the windspeed goes up by 1 mph, the perceived temperature will drop by  $0.5^\circ\text{F}$ .

- (b) Estimate the partial derivative  $w_T(20, -10)$ . What are the units on this quantity and what does it mean?

$$w_T(20, -10) \approx \frac{w(20, -5) - w(20, -15)}{-5 - (-15)} = \frac{-29 - (-42)}{10} = \frac{13}{10} = 1.3 \frac{^\circ\text{F}}{^\circ\text{F}}.$$

That is, if the ambient temperature goes up by  $1^\circ\text{F}$ , the perceived temperature will go up by  $1.3^\circ\text{F}$ .

- (c) Use your results to estimate the wind chill  $w(18, -10)$ .

The windspeed has gone down by 2 mph. Each 1 mph decrease in windspeed causes an  $0.5^\circ \text{ F}$  **increase** in perceived temperature. Therefore,  $w(18, -10)$  should be about  $1^\circ \text{ F}$  warmer than  $w(20, -10)$ , which is  $-35$ , so it should be  $-34^\circ \text{ F}$ .

In symbols:

$$\begin{aligned} w(18, -10) &\approx w(20, -10) + (18 - 20) \cdot w_v(20, -10) \\ &= -35^\circ \text{F} + (-2 \text{ mph}) \cdot \left( -0.5 \frac{^\circ \text{F}}{\text{mph}} \right) \\ &= -35^\circ \text{F} + 1^\circ \text{F} = -34^\circ \text{F}. \end{aligned}$$

- (d) Use your results to estimate the wind chill  $w(20, -12)$ .

The ambient temperature has dropped by 2 degrees. Each 1 degree decrease in ambient temperature causes a 1.3 degree decrease in perceived temperature. Therefore,  $w(20, -12)$  should be about 2.6 degrees colder than  $w(20, -10)$ , which is  $-35^\circ \text{ F}$ , so it should be  $-37.6^\circ \text{ F}$ .

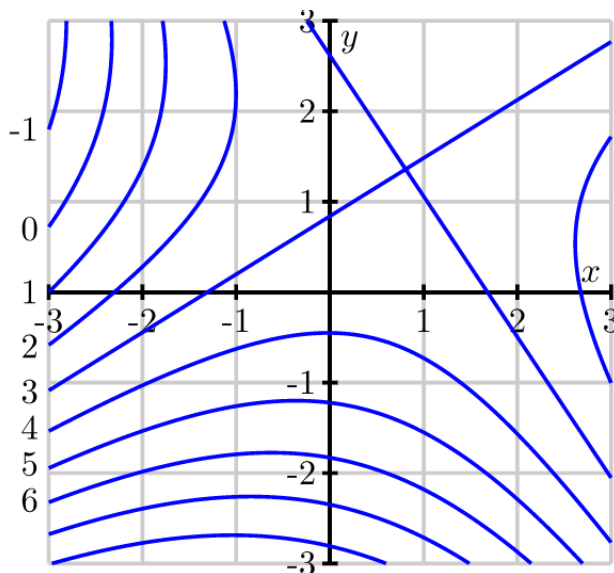
In symbols:

$$\begin{aligned} w(20, -12) &\approx w(20, -10) + (-12 - (-10)) \cdot w_T(20, -10) \\ &= -35^\circ \text{F} + (-2^\circ \text{F}) \cdot \left( 1.3 \frac{^\circ \text{F}}{^\circ \text{F}} \right) \\ &= -35^\circ \text{F} - 2.6^\circ \text{F} = -37.6^\circ \text{F}. \end{aligned}$$

- (e) Consider how you might combine your previous results to estimate the wind chill  $w(18, -12)$ . Explain your process.

From the original  $-35^\circ \text{F}$ , the perceived temperature should go up by 1 (from the decrease in windspeed) and down by 2.6 (from the decrease in temperature). So I bet  $w(18, -12) \approx -35 + 1 - 2.6 = -36.6^\circ \text{F}$ .

4. (Activity 10.2.6) Shown in the figure below is a contour plot of a function  $f$ . The values of the function on a few of the contours are indicated to the left of the figure.



- (a) Estimate the partial derivative  $f_x(-2, -1)$ .

We can use the contours to read off some approximate values and use a symmetric difference.

**NOTE:** I'm using a step size of 1 – your answer may be a little different from this if you chose a different step size. Same deal in parts (b) and (c).

$$\begin{aligned}f(-3, -1) &\approx 3 \\f(-1, -1) &\approx 4.5 \\f_x(-2, -1) &\approx \frac{f(-1, -1) - f(-3, -1)}{-1 - (-3)} \\&= \frac{4.5 - 3}{2} = 0.75\end{aligned}$$

- (b) Estimate the partial derivative  $f_y(-2, -1)$ .

Same game – we can use the contours to read off some approximate values and use a symmetric difference:

$$\begin{aligned}f(-2, -2) &\approx 6 \\f(-2, 0) &\approx 2.5 \\f_y(-2, -1) &\approx \frac{f(-2, 0) - f(-2, -2)}{0 - (-2)} \\&= \frac{2.5 - 6}{2} = -1.75\end{aligned}$$

- (c) Estimate the partial derivatives  $f_x(-1, 2)$  and  $f_y(-1, 2)$ .

$$\begin{aligned}f(0, 2) &\approx 2.5 & f(-1, 1) &\approx 2.5 \\f(-2, 2) &\approx 0.5 & f(-1, 3) &\approx 2.5 \\f_x(-1, 2) &\approx \frac{f(0, 2) - f(-2, 2)}{0 - (-2)} & f_y(-1, 2) &\approx \frac{f(-1, 3) - f(-1, 1)}{3 - 1} \\&= \frac{2.5 - 0.5}{2} = 1 & &= \frac{2.5 - 2.5}{2} = 0\end{aligned}$$

- (d) Locate, if possible, one point  $(x, y)$  where  $f_x(x, y) = 0$ .

Looks like at about  $(0, -0.5)$ , if I move in the  $x$  direction, the heights on the contour map don't change much.

- (e) Locate, if possible, one point  $(x, y)$  where  $f_x(x, y) < 0$ .

Looks like at about  $(1, -2)$ , if I move in the  $x$  direction, I'm going downhill.

- (f) Locate, if possible, one point  $(x, y)$  where  $f_y(x, y) > 0$ .

Looks like at about  $(1, 2)$ , if I move in the  $y$  direction, I'm going (very slightly) uphill.

**NOTE:** Of course there are other points that will work for all three of these parts.

5. (AC Multi 10.2 Exercise 14) Let  $f(x, y) = \frac{1}{2}xy^2$  represent the kinetic energy in Joules of an object of mass  $x$  in kilograms with velocity  $y$  in meters per second. Let  $(a, b)$  be the point  $(4, 5)$  in the domain of  $f$ .

- (a) Calculate  $f_x(a, b)$ .

First of all, let's calculate  $f_x(x, y)$ . Holding  $y$  constant, we'll take the derivative with respect to  $x$ :

$$f_x(x, y) = \frac{1}{2}y^2.$$

Now we'll evaluate this at our point:  $f_x(a, b) = f_x(4, 5) = \frac{1}{2} \cdot 5^2 = \frac{25}{2}$ . (By the way, the units on this quantity are Joules per kilogram.)

- (b) Explain as best you can in the context of kinetic energy what the partial derivative

$$f_x(a, b) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h}$$

tells us about kinetic energy.

This essentially tells us how much the kinetic energy changes per unit change in **mass** of the object.

Dimensional analysis helps: Note here since  $h$  is being added to the  $x$  term, it's a small change in **mass**. Thus, since the numerator has the same output as  $f$ , it's Joules; since the denominator is an  $h$ , it's kilograms. So, the units on this quantity are Joules per kilogram.

- (c) Calculate  $f_y(a, b)$ .

First of all, let's calculate  $f_y(x, y)$ . Holding  $x$  constant, we'll take the derivative with respect to  $y$ :

$$f_y(x, y) = \frac{1}{2}x \cdot 2y = xy.$$

Now we'll evaluate this at our point:  $f_y(a, b) = f_y(4, 5) = 4 \cdot 5 = 20$ . (By the way, the units on this quantity are Joules per (meters per second).)

- (d) Explain as best you can in the context of kinetic energy what the partial derivative

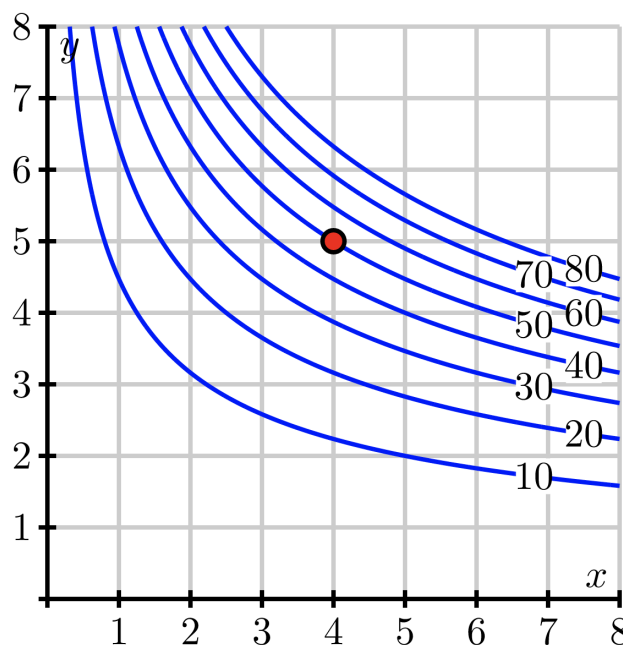
$$f_y(a, b) = \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h}$$

tells us about kinetic energy.

This essentially tells us how much the kinetic energy changes per unit change in **velocity** of the object.

Dimensional analysis helps: Note here since  $h$  is being added to the  $y$  term, it's a small change in **velocity**. Thus, since the numerator has the same output as  $f$ , it's Joules; since the denominator is an  $h$ , it's meters per second. So, the units on this quantity are Joules per (meters per second).

- (e) Often we are given certain graphical information about a function instead of a rule. We can use that information to approximate partial derivatives. For example, suppose that we are given a contour plot of the kinetic energy function (as in the figure below) instead of a formula. Use this contour plot to approximate  $f_x(4, 5)$  and  $f_y(4, 5)$  as best you can. Compare to your calculations from earlier parts of this exercise.



To find  $f_x(4, 5)$ : I'm going to stand at the point  $(4, 5)$  and, holding  $y$  constant, take a small step in the  $x$  direction, and see what happens to my output values.

When I'm on the point  $(4, 5)$ , I'm on the  $z = 50$  contour, so I know that  $f(4, 5) = 50$ . If I take a little step in the  $x$  direction, I land just a little above the  $z = 60$  contour, so I know that  $f(4 + 1, 5) \approx 60$  or 62ish. Therefore, my step of 1 kg has produced a change of 10 or 12 Joules, so  $f_x(4, 5) \approx 10/1$  or  $12/1$  Joules per kg. This is pretty close to the value we calculated earlier,  $\frac{25}{2} = 12.5$ .

To find  $f_y(4, 5)$ : I'm going to stand at the point  $(4, 5)$  and, holding  $x$  constant, take a small step in the  $y$  direction, and see what happens to my output values.

When I'm on the point  $(4, 5)$ , I'm on the  $z = 50$  contour, so I know that  $f(4, 5) = 50$ . If I take a little step in the  $y$  direction, I land just a little above the  $z = 70$  contour, so I know that  $f(4, 5 + 1) \approx 70$  or 72ish. Therefore, my step of 1 m/s has produced a change of 20 or 22 Joules, so  $f_y(4, 5) \approx 20/1$  or  $22/1$  Joules per m/s. This is pretty close to the value we calculated earlier, 20.