PS#7 – Chain rule - Answer key
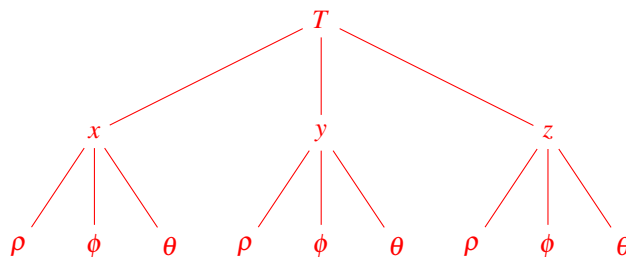
1. (AC Multi 10.5 Exercise 13) Suppose that $T = x^2 + y^2 - 2z$ where

$$x = \rho \sin(\phi) \cos(\theta)$$
$$y = \rho \sin(\phi) \sin(\theta)$$
$$z = \rho \cos(\phi)$$

a. Construct a tree diagram representing the dependencies among the variables.

(In case you're curious, I generated this tree diagram using the very good TikZ library "trees". I can share the LaTeX source if you're interested.)

b. Apply the chain rule to find the partial derivatives $\dfrac{\partial T}{\partial \rho}$, $\dfrac{\partial T}{\partial \phi}$, and $\dfrac{\partial T}{\partial \theta}$.

We'll just read the chain rule off the tree:

$$\frac{\partial T}{\partial \rho} = \frac{\partial T}{\partial x} \cdot \frac{\partial x}{\partial \rho} + \frac{\partial T}{\partial y} \cdot \frac{\partial y}{\partial \rho} + \frac{\partial T}{\partial z} \cdot \frac{\partial z}{\partial \rho}$$
$$= 2x \cdot (\sin\phi \cos\theta) + 2y \cdot (\sin\phi \sin\theta) - 2 \cdot (\cos\phi)$$
$$= 2(\rho \sin\phi \cos\theta) \cdot (\sin\phi \cos\theta) + 2(\rho \sin\phi \sin\theta) \cdot (\sin\phi \sin\theta) - 2 \cdot (\cos\phi)$$
$$= 2\rho \sin^2\phi \cos^2\theta + 2\rho \sin^2\phi \sin^2\theta - 2\cos\phi$$
$$= 2\rho \sin^2\phi (\cos^2\theta + \sin^2\theta) - 2\cos\phi$$
$$= 2\rho \sin^2\phi - 2\cos\phi$$
$$\frac{\partial T}{\partial \phi} = \frac{\partial T}{\partial x} \cdot \frac{\partial x}{\partial \phi} + \frac{\partial T}{\partial y} \cdot \frac{\partial y}{\partial \phi} + \frac{\partial T}{\partial z} \cdot \frac{\partial z}{\partial \phi}$$
$$= 2x \cdot (\rho \cos\phi \cos\theta) + 2y \cdot (\rho \cos\phi \sin\theta) - 2 \cdot (-\rho \sin\phi)$$
$$= 2(\rho \sin\phi \cos\theta) \cdot (\rho \cos\phi \cos\theta) + 2(\rho \sin\phi \sin\theta) \cdot (\rho \cos\phi \sin\theta) + 2\rho \sin\phi$$
$$= 2\rho^2 \sin\phi \cos\phi \cdot (\cos^2\theta + \sin^2\theta) + 2\rho \sin\phi$$
$$= 2\rho^2 \sin\phi \cos\phi + 2\rho \sin\phi$$
$$\frac{\partial T}{\partial \theta} = \frac{\partial T}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial T}{\partial y} \cdot \frac{\partial y}{\partial \theta} + \frac{\partial T}{\partial z} \cdot \frac{\partial z}{\partial \theta}$$
$$= 2x \cdot (\rho \sin\phi (-\sin\theta)) + 2y \cdot (\rho \sin\phi \cos\theta) - 2 \cdot (0)$$
$$= -2(\rho \sin\phi \cos\theta) \cdot (\rho \sin\phi \sin\theta) + 2(\rho \sin\phi \sin\theta) \cdot (\rho \sin\phi \cos\theta)$$
$$= -2\rho^2 \sin^2\phi \cos\theta \sin\theta + 2\rho^2 \sin^2\phi \sin\theta \cos\theta = 0$$

Learning Targets Reflection: Maybe S1, maybe S2, D5, D6.