PS#4: Calculus with space curves - Answer key

1. (AC Multi 9.7 Exercise 13) Compute the derivative of each of the following functions in two different ways: (1) use the rules provided in the theorem stated just after Activity 9.7.3, and (2) rewrite each given function so that it is stated as a single function (either a scalar function or a vector-valued function with three components), and differentiate component-wise. Compare your answers to ensure that they are the same.

   (a) $\mathbf{r}(t) = \sin(t)\langle 2t, t^2, \arctan(t)\rangle$

   Using the scalar product rule:

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= \left(\frac{d}{dt}\sin(t)\right)\langle 2t, t^2, \arctan(t)\rangle + \sin(t)\left(\frac{d}{dt}\langle 2t, t^2, \arctan(t)\rangle\right)\\ &= \cos(t)\langle 2t, t^2, \arctan(t)\rangle + \sin(t)\left\langle 2, 2t, \frac{1}{1+t^2}\right\rangle\\ &= \left\langle 2t\cos(t) + 2\sin(t), t^2\cos(t) + 2t\sin(t), \arctan(t)\cos(t) + \frac{\sin(t)}{1+t^2}\right\rangle\end{aligned}$$

   And rewriting first:

$$\begin{aligned}\mathbf{r}(t) &= \langle 2t\sin(t), t^2\sin(t), \arctan(t)\sin(t)\rangle\\ \frac{d\mathbf{r}}{dt} &= \left\langle \left(\frac{d}{dt}2t\right)\sin(t) + 2t\left(\frac{d}{dt}\sin(t)\right), \left(\frac{d}{dt}t^2\right)\sin(t) + t^2\left(\frac{d}{dt}\sin(t)\right), \left(\frac{d}{dt}\arctan(t)\right)\sin(t) + \arctan(t)\left(\frac{d}{dt}\sin(t)\right)\right\rangle\\ &= \left\langle 2t\cos(t) + 2\sin(t), t^2\cos(t) + 2t\sin(t), \frac{\sin(t)}{1+t^2} + \arctan(t)\cos(t)\right\rangle\end{aligned}$$

   (b) $\mathbf{s}(t) = \mathbf{r}(2^t)$, where $\mathbf{r}(t) = \langle t+2, \ln(t), 1\rangle$ – Note that $\mathbf{r}'(t) = \langle 1, \frac{1}{t}, 0\rangle$

   Using the chain rule first:

$$\begin{aligned}\mathbf{s}'(t) &= \mathbf{r}'(2^t)\frac{d}{dt}2^t\\ &= \left\langle 1, \frac{1}{2^t}, 0\right\rangle 2^t\ln(2)\\ &= \langle 2^t\ln(2), \ln(2), 0\rangle\end{aligned}$$

   And simplifying first:

$$\begin{aligned}\mathbf{s}(t) &= \mathbf{r}(2^t) = \langle 2^t+2, \ln(2^t), 1\rangle = \langle 2^t+2, t\ln(2), 1\rangle\\ \mathbf{s}'(t) &= \langle 2^t\ln(2), \ln(2), 0\rangle\end{aligned}$$

   (c) $\mathbf{r}(t) = \langle\cos(t), \sin(t), t\rangle \cdot \langle -\sin(t), \cos(t), 1\rangle$

   Using the dot product rule:

$$\begin{aligned}r'(t) &= \left(\frac{d}{dt}\langle\cos(t), \sin(t), t\rangle\right)\cdot\langle -\sin(t), \cos(t), 1\rangle + \langle\cos(t), \sin(t), t\rangle\cdot\left(\frac{d}{dt}\langle -\sin(t), \cos(t), 1\rangle\right)\\ &= \langle -\sin(t), \cos(t), 1\rangle\cdot\langle -\sin(t), \cos(t), 1\rangle + \langle\cos(t), \sin(t), t\rangle\cdot\langle -\cos(t), -\sin(t), 0\rangle\\ &= \left[\sin^2(t) + \cos^2(t) + 1\right] + \left[-\cos^2(t) - \sin^2(t) + 0\right] = 1 + 1 - 1 + 0 = 1(!!)\end{aligned}$$

   And rewriting first:

$$\begin{aligned}r(t) &= cos(t)\cdot(-\sin(t)) + \sin(t)\cdot\cos(t) + t\cdot 1\\ r(t) &= t\,(!!)\\ r'(t) &= 1\end{aligned}$$

(d) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t\rangle \times \langle -\sin(t), \cos(t), 1\rangle$

Using the cross product rule:

$$\begin{aligned}\mathbf{r}'(t) &= \left(\frac{d}{dt}\langle \cos(t), \sin(t), t\rangle\right) \times \langle -\sin(t), \cos(t), 1\rangle + \langle \cos(t), \sin(t), t\rangle \times \left(\frac{d}{dt}\langle -\sin(t), \cos(t), 1\rangle\right) \\ &= \langle -\sin(t), \cos(t), 1\rangle \times \langle -\sin(t), \cos(t), 1\rangle + \langle \cos(t), \sin(t), t\rangle \times \langle -\cos(t), -\sin(t), 0\rangle\end{aligned}$$

The first two vectors are parallel, so their cross product is $\mathbf{0}$.

$$= \mathbf{0} + \langle \cos(t), \sin(t), t\rangle \times \langle -\cos(t), -\sin(t), 0\rangle = \langle t\sin(t), -t\cos(t), 0\rangle \quad \text{(Thanks, WA!)}$$

And finding the cross product first:

$$\begin{aligned}\mathbf{r}(t) &= \langle \sin(t) - t\cos(t), -t\sin(t) - \cos(t), \sin^2(t) + \cos^2(t)\rangle = \langle \sin(t) - t\cos(t), -t\sin(t) - \cos(t), 1\rangle \\ \mathbf{r}'(t) &= \frac{d}{dt}\langle \sin(t) - t\cos(t), -t\sin(t) - \cos(t), 1\rangle \\ &= \langle \cos(t) - (1\cos(t) + t(-\sin(t))), -(1\sin(t) + t\cos(t)) - (-\sin(t)), 0\rangle \\ &= \langle t\sin(t), -t\cos(t), 0\rangle\end{aligned}$$

2. (AC Multi 9.7 Exercise 18) A central force is one that acts on an object so that the force $\mathbf{F}$ is parallel to the object's position $\mathbf{r}$. Since Newton's Second Law says that an object's acceleration is proportional to the force exerted on it, the acceleration $\mathbf{a}$ of an object moving under a central force will be parallel to its position $\mathbf{r}$. For instance, the Earth's acceleration due to the gravitational force that the sun exerts on the Earth is parallel to the Earth's position vector (see figure in the textbook).

(a) If an object of mass m is moving under a central force, the angular momentum vector is defined to be $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$. Assuming the mass is constant, show that the angular momentum is constant by showing that $\frac{d\mathbf{L}}{dt} = \mathbf{0}$.

Some stuff to keep track of:

- m is a constant scalar;
- $\mathbf{r}$ is a variable vector (depends on t);
- $\mathbf{v}$ is a variable vector (depends on t).

Seems like the natural thing to do is to use the product rule to compute $\frac{d\mathbf{L}}{dt}$:

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}[m\mathbf{r} \times \mathbf{v}] = \frac{d}{dt}[m\mathbf{r}] \times \mathbf{v} + m\mathbf{r} \times \frac{d}{dt}[\mathbf{v}]$$

Well, the derivative of position is velocity, and the derivative of velocity is acceleration:

$$= [m\mathbf{v}] \times \mathbf{v} + m\mathbf{r} \times \mathbf{a}$$

Now we're getting somewhere. Remember that the cross product of two parallel vectors is $\mathbf{0}$. $m\mathbf{v}$ is certainly parallel to $\mathbf{v}$, and we're assuming in the context of the problem that $\mathbf{a}$ is parallel to $\mathbf{r}$, so it's also parallel to $m\mathbf{r}$.

$$= \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Cool, so that tells us that $\mathbf{L}$ is a constant vector.

(b) Explain why $\mathbf{L} \cdot \mathbf{r} = 0$.

$\mathbf{L}$ was defined as the cross product of $m\mathbf{r}$ and $\mathbf{v}$, so it's perpendicular to both of those. Since m is a scalar, $m\mathbf{r}$ points in the same direction as $\mathbf{r}$, so if $\mathbf{L}$ is perpendicular to $m\mathbf{r}$, it's also perpendicular to $\mathbf{r}$. Therefore, their dot product is zero.

(c) Explain why we may conclude that the object is constrained to lie in the plane passing through the origin and perpendicular to $\mathbf{L}$.

The equation $\mathbf{L} \cdot \mathbf{r} = 0$ reminds me of the vector equation of a plane, with $\mathbf{L}$ as the (constant) normal vector and $\mathbf{r}$ playing the role of $\overrightarrow{PP_0}$. Since $\mathbf{r}$'s initial point is the origin, we thus have the plane that's perpendicular to $\mathbf{L}$ and passing through the origin.

(Another way to see this: Certainly all the position vectors are perpendicular to $\mathbf{L}$. Also, certainly all the position vectors emanate from the origin. Also, we've just found that $\mathbf{L}$ is constant. So the only way this is going to happen is if all the position vectors lie in the **same** plane – specifically, the plane containing the origin and perpendicular to the constant vector $\mathbf{L}$.)

3. (AC Multi 9.8 Exercise 14) Consider the standard helix parameterized by $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$.

(a) Recall that the unit tangent vector $\mathbf{T}(t)$ is the vector tangent to the curve at time t that points in the direction of motion and has length 1. Find $\mathbf{T}(t)$.

$$\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 1\rangle$$
$$|\mathbf{r}'(t)| = \left[\sin^2(t) + \cos^2(t) + 1^2\right] = \sqrt{2}$$
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{2}}\langle -\sin(t), \cos(t), 1\rangle$$

(b) Explain why the fact that $|\mathbf{T}(t)| = 1$ implies that $\mathbf{T}$ and $\mathbf{T}'$ are orthogonal vectors for every value of t.

The hint is to look at the dot product of $\mathbf{T}$ and $\mathbf{T}'$. Where would that have come from? Well, certainly if we did the derivative of $\mathbf{T} \cdot \mathbf{T}$, then the product rule would make that pop out: $\frac{d}{dt}[\mathbf{T} \cdot \mathbf{T}] = \mathbf{T}' \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{T}'$. But since $\mathbf{T}' \cdot \mathbf{T} = \mathbf{T} \cdot \mathbf{T}'$, this is $2(\mathbf{T} \cdot \mathbf{T}')$.

The other thing we know about a dot product of a vector with itself is that it's the magnitude of that vector squared: $\mathbf{T} \cdot \mathbf{T} = |\mathbf{T}|^2 = 1^2 = 1$. Therefore, its derivative must be zero.

So now let's combine the two things we know about $\frac{d}{dt}[\mathbf{T} \cdot \mathbf{T}]$:

$$\begin{aligned} & \frac{d}{dt}[\mathbf{T} \cdot \mathbf{T}] &=& \quad 2(\mathbf{T} \cdot \mathbf{T}') \\ \frac{d}{dt}[1] &= \frac{d}{dt}[\mathbf{T} \cdot \mathbf{T}] && \\ 0 &= && \quad 2(\mathbf{T} \cdot \mathbf{T}') \end{aligned}$$

Therefore $\mathbf{T}$ is orthogonal to $\mathbf{T}'$.

(c) For the given function $\mathbf{r}$ with unit tangent vector $\mathbf{T}(t)$ (from part (a)), determine $\mathbf{N}(t) = \frac{1}{|\mathbf{T}'(t)|}\mathbf{T}'(t)$.

$$\mathbf{T}(t) = \frac{1}{\sqrt{2}}\langle -\sin(t), \cos(t), 1\rangle$$
$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}}\langle -\cos(t), -\sin(t), 0\rangle$$
$$|\mathbf{T}'(t)| = \frac{1}{\sqrt{2}}\sqrt{\cos^2(t) + \sin^2(t) + 0} = \frac{1}{\sqrt{2}}$$
$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)} = \langle -\cos(t), -\sin(t), 0\rangle$$

(d) What geometric properties does $\mathbf{N}(t)$ have? How long is this vector, and how is it situated in comparison ton $\mathbf{T}(t)$?

Since $\mathbf{N}(t)$ is the unitized version of $\mathbf{T}'(t)$, it has length 1, and it's orthogonal to $\mathbf{T}(t)$ (because we proved that $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$ in part b!).

(e) Let $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$, and compute $\mathbf{B}(t)$ in terms of your results in (a) and (c).

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{2}}\langle -\sin(t), \cos(t), 1\rangle \times \langle -\cos(t), -\sin(t), 0\rangle$$
$$= \frac{1}{\sqrt{2}}\langle \sin(t), -\cos(t), 1\rangle \quad \text{(Thanks, WolframAlpha!)}$$

(f) What geometric properties does $\mathbf{B}(t)$ have? How long is this vector, and how is it situated in comparison to $\mathbf{T}(t)$ and $\mathbf{N}(t)$?

The length of $\mathbf{B}$ is 1, which you can tell either by computing directly or by noting that $|\mathbf{T} \times \mathbf{B}| = |\mathbf{T}| \cdot |\mathbf{B}| \cdot \sin\theta$, and tthe vectors $\mathbf{T}$ and $\mathbf{B}$ are at right angles, so $\sin\theta = 1$.

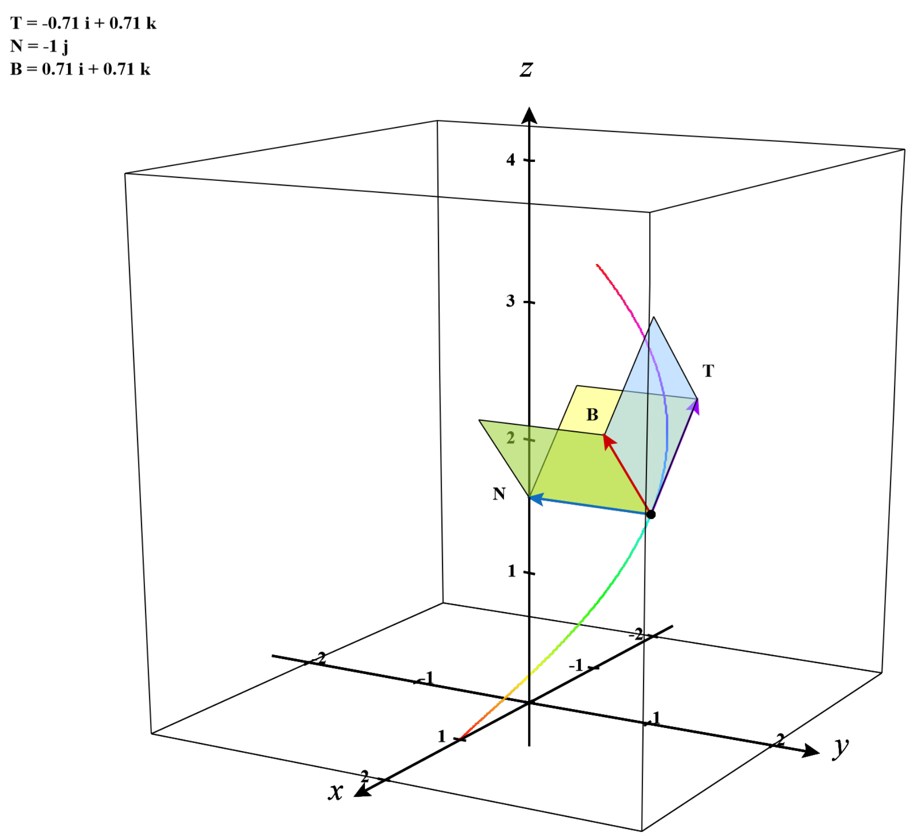
$\mathbf{B}$ is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$. ($\mathbf{B}$ here stands for "binormal" – it's "the other" normal vector.)

(g) Sketch a plot of the given helix, and compute and sketch $\mathbf{T}\left(\frac{\pi}{2}\right)$, $\mathbf{N}\left(\frac{\pi}{2}\right)$, and $\mathbf{B}\left(\frac{\pi}{2}\right)$.

Plugging and chugging:

$$\mathbf{T}\left(\frac{\pi}{2}\right) = \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$$
$$\mathbf{N}\left(\frac{\pi}{2}\right) = \langle 0, -1, 0\rangle$$
$$\mathbf{B}\left(\frac{\pi}{2}\right) = \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$$

And here's a plot from CalcPlot3D:



4. (AC Multi 9.7 Exercise 16) For each given function $\mathbf{r}$, determine $\int \mathbf{r}(t)\,dt$. In addition, recalling the Fundamental Theorem of Calculus for functions of a single variable, also evaluate $\int_0^1 \mathbf{r}(t)\,dt$ for each given function r. Is the resulting quantity a scalar or a vector? What does it measure?

If we label the antiderivative $\int \mathbf{r}(t)\,dt$ as $\mathbf{R}(t)$, then $\int_0^1 \mathbf{r}(t)\,dt$ will be a vector that points from $\mathbf{R}(0)$ to $\mathbf{R}(1)$.

(a) $\mathbf{r}(t) = \left\langle \cos(t), \dfrac{1}{t+1}, te^t \right\rangle$

$$\begin{aligned}
\int \mathbf{r}(t)\,dt &= \left\langle \int \cos(t)\,dt, \int \frac{1}{t+1}\,dt, \int te^t\,dt \right\rangle \\
&= \left\langle \sin(t), \ln|t+1|, te^t - e^t \right\rangle + \vec{C} \\
\int_0^1 \mathbf{r}(t)\,dt &= \left\langle \sin(t), \ln|t+1|, te^t - e^t \right\rangle\Big|_0^1 \\
&= \left\langle \sin(1) - \sin(0), \ln(2) - \ln(1), (1e^1 - e^1) - (0e^0 - e^0) \right\rangle = \langle \sin(1), \ln(2), 1 \rangle
\end{aligned}$$

(b) $\mathbf{r}(t) = \langle \cos(3t), \sin(2t), t \rangle$

$$\begin{aligned}
\int \mathbf{r}(t)\,dt &= \left\langle \frac{1}{3}\sin(3t), -\frac{1}{2}\cos(2t), \frac{1}{2}t^2 \right\rangle + \vec{C} \\
\int_0^1 \mathbf{r}(t)\,dt &= \left\langle \frac{1}{3}\sin(3t), -\frac{1}{2}\cos(2t), \frac{1}{2}t^2 \right\rangle\Big|_0^1 \\
&= \left\langle \frac{1}{3}\sin(3), -\frac{1}{2}(\cos(2) - 1), \frac{1}{2} \right\rangle
\end{aligned}$$

(c) $\mathbf{r}(t) = \left\langle \dfrac{t}{1+t^2}, te^{t^2}, \dfrac{1}{1+t^2} \right\rangle$

(The first two components require integration by substitution!)

$$\begin{aligned}
\int \mathbf{r}(t)\,dt &= \left\langle \frac{1}{2}\ln(1+t^2), \frac{1}{2}e^{t^2}, \arctan(t) \right\rangle + \vec{C} \\
\int \mathbf{r}(t)\,dt &= \left\langle \frac{1}{2}\ln(1+t^2), \frac{1}{2}e^{t^2}, \arctan(t) \right\rangle\Big|_0^1 \\
&= \left\langle \frac{1}{2}\ln(2), \frac{1}{2}(e-1), \frac{\pi}{4} \right\rangle
\end{aligned}$$

5. (AC Multi 9.7 Exercise 17) In this exercise, we develop the formula for the position function of a projectile that has been launched at an initial speed of $|\mathbf{v}_0|$ and a launch angle of θ. Recall that $\mathbf{a}(t) = \langle 0, -g \rangle$ is the constant acceleration of the projectile at any time t.

(a) Find all velocity vectors for the given acceleration vector $\mathbf{a}(t)$. When you anti-differentiate, remember that there is an arbitrary constant that arises in each component.

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \left\langle \int 0\,dt, \int -g\,dt \right\rangle = \langle c_1, -gt + c_2 \rangle$$

(b) Use the given information about initial speed and launch angle to find $\mathbf{v}_0$, the initial velocity of the projectile. You will want to write the vector in terms of its components, which will involve $\sin\theta$ and $\cos\theta$.

$$\mathbf{v}_0 = \langle |\mathbf{v}_0|\cos\theta, |\mathbf{v}_0|\sin\theta \rangle$$

(c) Next, find the specific velocity vector function $\mathbf{v}(t)$ for the projectile. That is, combine your work in (a) and (b) in order to determine expressions in terms of $|\mathbf{v}_0|$ and θ for the constants that arose when integrating.

On the one hand, $\mathbf{v}_0 = \langle |\mathbf{v}_0|\cos\theta, |\mathbf{v}_0|\sin\theta\rangle$. But on the other hand, $\mathbf{v}_0 = \mathbf{v}(0) = \langle c_1, c2\rangle$. Therefore, $c_1 = |\mathbf{v}_0|\cos\theta$ and $c_2 = |\mathbf{v}_0|\sin\theta$. So,

$$\mathbf{v}(t) = \langle |\mathbf{v}_0|\cos\theta, -gt + |\mathbf{v}_0|\sin\theta\rangle.$$

(d) Find all possible position vectors for the velocity vector $\mathbf{v}(t)$ you determined in (c).

$$\begin{aligned}\mathbf{r}(t) = \int \mathbf{v}(t)\,dt &= \left\langle \int |\mathbf{v}_0|\cos\theta\,dt, \int -gt + |\mathbf{v}_0|\sin\theta\,dt \right\rangle \\ &= \left\langle (|\mathbf{v}_0|\cos\theta)t + c_3, -\frac{g}{2}t^2 + (|\mathbf{v}_0|\sin\theta)t + c_4 \right\rangle\end{aligned}$$

(e) Let $\mathbf{r}(t)$ denote the position vector function for the given projectile. Use the fact that the object is fired from the position (x_0, y_0) to show it follows that

$$\mathbf{r}(t) = \left\langle |\mathbf{v}_0|\cos(\theta)t + x_0, -\frac{g}{2}t^2 + |\mathbf{v}_0|\sin(\theta)t + y_0 \right\rangle.$$

On the one hand,

$$\mathbf{r}(0) = \left\langle (|\mathbf{v}_0|\cos\theta)\cdot 0 + c_3, -\frac{g}{2}\cdot 0^2 + (|\mathbf{v}_0|\sin\theta)\cdot 0 + c_4 \right\rangle = \langle c_3, c_4\rangle.$$

On the other hand, $\mathbf{r}(0) = \mathbf{r}_0 = \langle x_0, y_0\rangle$. Therefore, $c_3 = x_0$ and $c_4 = y_0$; plugging those in gives us the requested thing, yay!