Just Enough Algebra

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Chapter 3

Solving equations

One of the great powers of symbolic algebra is being able to solve equations exactly. Quick, accurate, and easily generalized, symbolic techniques have long been the backbone of algebra. The good news is that linear equations, power equations, exponential equations, and quadratic equations can all be solved exactly. That means we can find the answer to all of our problems. What more could you want? Okay, slight exaggeration there, but our ability to solve equations is worthy of some enthusiasm.

Each type of equation has its own method, and so this chapter takes on each type of equation in turn. In each case, we need special operations that "undo" what's in our equation. For example, you probably know that subtraction $(-)$ undoes addition $(+)$ and division $(\div)$ undoes multiplication $(\times)$. And both vice versa. That's all we need to know to solve linear equations and inequalities. Do you know what undoes powers $(\wedge n)$ or exponentials $(g\wedge)$? Turns out it's roots $(\sqrt[n]{})$ or logarithms (log). We discuss those in more detail along the way.

Everything we learned about tables, graphs, and approximating solutions is still important. Even if we solve an equation exactly, we still want to know that the answer makes sense. That's the one thing an equation cannot do. We continue to rely on words, numbers, and graphs for that. You will likely find it a good habit to use those tools to estimate the answer before solving an equation as well doing the "reality check" after.

Okay, I have to mention something here. That list of types of equations we know how to solve. That's the good news. The bad news? Many other types of equations, including polynomials in general, do not have a sure way to solve them. So what do we do in practice? Well, for starters, successive approximation works for any type of equation. Not always quickly or easily, but if there's a solution (and the function is nice enough), we can find it. Happily there are much fancier numeric method to approximate solutions of equations, especially using a computer.

But for now, let's solve what we can.

3.1 Solving linear equations

Your kitchen sink keeps getting clogged. Very annoying. Last time the plumber was able to fix it pretty quickly. But now the sink is clogged again. This time when the plumber comes and unclogs the sink, he suggests redoing the trap and a few other things that were causing the problem. You are pretty tired of it clogging up and tell him to "go ahead." While you're glad that the sink works when he's done, you're a bit shocked when his bill arrives a few days later for parts plus \$278.75 in labor. Does that seem right?

Remember our plumber charged \$100 for just showing up and then \$75 per hour for the service. Using the variables

$$T = \text{time plumber takes (hours)} \sim \text{indep}$$
$$P = \text{total plumber's charge (\$)} \sim \text{dep}$$

we found that the equation was

$$P = 100 + 75T$$

Let's figure out how many hours of work would add up to a bill of \$278.75. Our first approach might be to look at a table. From earlier we had

T	0	1/2	1	2	3
P	100.00	137.50	175.00	250.00	325.00

Since \$278.75 is between \$250.00 and \$325.00, we see that the time must be between 2 and 3 hours. You remember the plumber being there over 2 hours, so this is certainly a reasonable answer. Well, a lot of money, but mathematically it makes sense.

Still curious, you'd like to know exactly how many hours and minutes he worked. We could use successive approximations. For example, for 2.5 hours the bill would have been

$$P = 100 + 75 * 2.5 = 100 + 75 \times \underline{2.5} = \$287.50$$

which is more than the charge. Continuing to guess and check we get

T	2	3	2.5	2.3	2.4	2.35	2.37	2.38
P	250.00	325.00	287.50	272.50	280.00	276.25	277.75	278.50
vs. 278.75	low	high	high	low	high	low	low	close enough

The plumber worked approximately 2.38 hours. Converting units we calculate

$$.38\ \cancel{\text{hours}} * \frac{60 \text{ minutes}}{1\ \cancel{\text{hour}}} = .38 \times 60 = 22.8 \approx 23 \text{ minutes}$$

The plumber took about 2 hours, 23 minutes. Thinking back, the plumber had arrived around 10:30 in the morning and stayed past lunch, probably until around 1:00 p.m. That's about right.

Wait a minute! We could have figured this out much more quickly. If the labor cost was \$278.75, we know the first \$100 was the trip charge. That leaves

$$\$278.75 - \$100.00 = \$178.75$$

in hourly charges. At \$75 per hour that comes to

$$\$178.75 * \frac{1 \text{ hour}}{\$75} = 178.75 \div 75 = 2.3883333\ldots \approx 2.388 \text{ hours}$$

which comes to around 2 hours, 23 minutes as before. See how we used the \$75/hour as a unit conversion here? Very clever.

That worked well. But, can we figure out this sort of calculation in other problems? What is the general method we're using? Can we write down our method in an organized fashion so that someone else could follow our thinking here? Turns out there is a formal way to show this calculation, called **(symbolically) solving** the equation. Officially *any* method of getting a solution to an equation is considered solving the equation, but in the rest of this book, and in most places that use algebra, when we refer to "solving the equation" or give the instruction to **solve**, we mean *symbolically*.

Here's how it works. We want to figure out when $P = 278.75$. We know from our equation that $P = 100 + 75T$. So we want to find the time T where

$$100 + 75T = 278.75$$

Remember that the equal sign indicates that the two sides are the same number. On the left-hand side we have $100 + 75T$. On the right-hand side we have 278.75. Looks different, but same thing. We are looking for the value of T that makes these two sides equal.

The first thing we did was subtract the \$100 trip charge. In this formal method, we subtract 100 from each side of our equation. If the left-hand side and the right-hand side are the same number, then they will still be equal when we take away 100 from each side. We get

$$\begin{array}{rcl} \cancel{100} + 75T & = & 278.75 \\ -\cancel{100} & & -100 \end{array}$$

which simplifies to

$$75T = 278.75 - 100 = 178.75$$

because the $+100$ and -100 cancelled.

The next thing we did to figure out the answer was divide by the \$75/hour charge. In this formal method, we can divide each side of our equation by 75. Again, if the left-hand side and right-hand side are the same number, then they will still be equal when we divide by 75. Here goes.

$$\frac{\cancel{75}T}{\cancel{75}} = \frac{178.75}{75}$$

Notice that we wrote the division in fraction form (instead of using $\div$). To understand why the 75's cancelled, remember that $75T$ is short for $75 * T$ and so

$$\frac{75T}{75} = \frac{75 * T}{75} = 75 \times T \div 75 = T$$

because the $\times 75$ and $\div 75$ cancelled. So we have

$$T = \frac{178.75}{75} = 178.75 \div 75 = 2.3883333\ldots$$

as before. Yet again, our answer is around 2 hours, 23 minutes.

Let's practice working with this symbolic way of solving equations. Suppose instead the plumber went to my neighbor's house and billed her for \$160 in labor costs. How long did the plumber work at my neighbor's? As before, we begin with our equation

$$P = 100 + 75T$$

and we are looking for $P = 160$. Put these together to get

$$100 + 75T = 160$$

Subtract 100 from each side to get

$$\begin{array}{rcl} \cancel{100} + 75T & = & 160 \\ -\cancel{100} & & -100 \end{array}$$

which simplifes to

$$75T = 160 - 100 = 60$$

Last, divide each side by 75 to get

$$\frac{\cancel{75}T}{\cancel{75}} = \frac{60}{75}$$

which simplifies to

$$T = \frac{60}{75} = 60 \div 75 = .8 \text{ hours}$$

We have solved the equation, but it would make more sense to report our answer in minutes so we convert

$$.8\ \cancel{\text{hours}} * \frac{60 \text{ minutes}}{1\ \cancel{\text{hour}}} = .8 \times 60 = 48 \text{ minutes}$$

The plumber worked for 48 minutes at my neighbor's house.

Let's quick check. Since T is measured in hours we need to go back and use $T = .8$ hours, not 48 which is in minutes. Evaluating in our original equation we get

$$P = 100 + 75 * .8 = 100 + 75 \times \underline{.8} = 160 \quad \checkmark$$

You might be wondering how we knew to subtract the 100 first and then later divide by 75. In this particular situation we had figured it out already and knew it made sense to take the \$100 right off the top. But, in general, how would we know?

It turns out that when solving an equation we do the *opposite* operations in the *reverse* order from the usual order of operations for evaluating. To evaluate a linear equation we would first multiply and then add. To solve a linear equation we first subtract (that is the opposite of adding) and then we divide (that is the opposite of multiplying).

Homework

Start by doing Practice exercises #1-4 in the workbook.

Do you know . . .

- When you solve an equation, as opposed to just evaluating?
- Why we "do the same thing to each side" of an equation when solving?
- How to solve a linear equation?
- The advantages and disadvantages of solving versus successive approximation?
- How to check that a solution is correct using the equation?

  If you're not sure, work the rest of exercises and then return to these questions. Or, ask your instructor or a classmate for help.

Exercises

5. A charter boat tour costs \$$C$ for P passengers, where

$$C = 135.00 + 11.95P$$

   (a) Make a table of values showing the charges for no passengers, 4 passengers, 10 passengers, and 20 passengers.

(b) What does the 135.00 represent and what are its units?

(c) What does the 11.95 represent and what are its units?

(d) If Freja was charged \$326.20 for use of the boat, how many passengers were there? Set up and solve an equation to answer the question.

(e) Graph and check.

6. Abduwali has just opened a restaurant. He spent \$82,500 to get started but hopes to earn back \$6,300 each month. Earlier we determined that

$$A = 6{,}300M - 82{,}500$$

describes how Abduwali's profit $\$A$ is a function of how long he works (M months).

Story also appears in 2.1 Exercises

(a) Set up and solve an equation to determine how long it will take Abduwali to **break even**, meaning make a profit of \$0?

(b) Aduwali will consider the restaurant a success once he's earned \$100,000. According to our equation, when will that be?

7. Between e-mail, automatic bill pay, and online banking, it seems like I hardly ever actually mail something. But for those times, I need postage stamps. The corner store sells as many (or few) stamps as I want for 44¢ each but they charge a 75¢ convenience fee for the whole purchase. *Story also appears in 1.1 Exercises*

(a) Make a table showing the cost to buy 5 stamps, 10 stamps, or 20 stamps from the corner store.

(b) Name the variables and write a linear equation showing how the total price depends on the number of stamps I buy.

(c) My partner bought postage stamps at the corner store and it cost him \$7.35. Solve your equation to determine how many stamps she bought.

(d) How many stamps could I buy for \$10? Solve your equation and check your answer.

8. When Gretchen walks on her treadmill, she burns 125 calories per mile. Recall

$$C = 125M$$

where C is the number of calories Gretchen burns by walking M miles.

Story also appears in 2.1 and 3.2 Exercises

(a) Set up and solve an equation to calculate how far Gretchen has to walk to burn 300 calories.

(b) If Gretchen walks 3.4 miles per hour on her treadmill, how long will it take her to burn those 300 calories? Report your answer to the nearest minute.

(c) Pecan pie? Yum. Not fitting into your favorite jeans? No fun. How far does Gretchen have to walk to burn off the calories from those two slices of pecan pie she ate last night? Each slice has approximately 456 calories.

9. The more expensive something is, the less likely we are to buy it. Well, if we have a choice. For example, when strawberries are in the peak of season, they cost about \$2.50 per pint at my neighborhood farmer's market and demand is approximately 180 pints. (That means, people want to buy about 180 pints at that price.) We approximate that the demand, D pints, depends on the price, $\$P$, as described by the equation

$$D = 305 - 50P$$

(a) How many pints of strawberries are in demand when the price is \$3.19 per pint?

(b) Make a table of values showing the demand for strawberries priced at \$2.00/pint, \$2.25/pint, \$2.50/pint, \$2.75/pint, \$3.00/pint, \$3.25/pint, \$3.50/pint.

(c) Draw a graph illustrating the function. Start at \$0/pint even though that's not realistic.

(d) It's been a great week for strawberries and there are 240 pints to be sold at my neighborhood farmer's. What price should the farmer charge for her strawberries in order to sell them all? Estimate your answer from the graph. Then set up and solve an equation to answer the question.

10. The stretch of interstate highway through downtown averages 1,450 cars per hour during the morning rush hour. The economy is improving (finally), but with that the county manager predicts traffic levels with increase around 130 cars per hour more each week for the next couple of years. Earlier we found the equation

$$C = 1{,}450 + 130W$$

where C is the number of cars per hour during the morning rush W weeks since the country manage made her projection. *Story also appears in 2.1 Exercises*

(a) Significant slowdown are expected if traffic levels exceed 2,000 cars per hour. When do they expect that will happen? Set up and solve an equation. Don't forget to check your answer by evaluating.

(b) If traffic levels exceed 2,500 the county plans to install control lights at the on ramps. When is that expected to happen? Set up and solve an equation. Don't forget to check your answer by evaluating.

3.2 Solving linear inequalities

In the United States temperatures for everyday things like the weather or cooking are given in Fahrenheit, denoted °F. In this system, water freezes into ice at 32°F and boils into steam at 212°F. A common setting for room temperature is 68°F whereas average human body temperature is around 98.6°F. And, most importantly, chocolate brownies bake at 350°F.

In the sciences, medicine, and most other countries, temperatures are measured in Celsius, denoted °C. (For those of us who grew up in the 1960s or earlier, "Celsius" is the temperature scale formerly known as "centigrade.") For comparison's sake, it's useful to know that water freezes at 0°C and boils at 100°C. Not coincidentally – it was set up that way. Room temperature is 20°C whereas now average human body temperature is 37°C. And those brownies?

A common conversion is given by the equation

$$F = 1.8C + 32$$

where

$$\begin{aligned} F &= \text{Fahrenheit temperature (°F)} \sim \text{dep} \\ C &= \text{Celsius temperature (°C)} \sim \text{indep} \end{aligned}$$

You may have seen this equation before with fractions in it: $F = \frac{9}{5}C + 32$. Just another way to write the equation, since $\frac{9}{5} = 9 \div 5 = 1.8$. For example, when $C = 100$ we have

$$F = 1.8 * 100 + 32 = 1.8 \times \underline{100} + 32 = 212 \quad \checkmark$$

You can (and should check) the other examples in our story.

What about those chocolate brownies? We are looking for $F = 350$. That's the dependent variable, so you can practice your linear equation solving skills to find the independent variable, C. It turns out that chocolate brownies bake at around $177°C$.

But, actually, chocolate brownies just need to bake in a **moderate oven**, which means between $325°F$ and $375°F$. Let's first figure out when the oven temperature is under $375°F$. We want to know when

$$F \leq 375$$

so we have an inequality instead of an equation. Remember $\leq$ stands for **less than or equal to**. Using $F = 1.8C + 32$ we get

$$1.8C + 32 \leq 375$$

We're looking for values of C that make the left-hand side a number that's smaller than, or maybe as large as, 375, but no larger. Quick vocabulary: equations have equal signs (=). When we have inequality signs ($\leq$, $\geq$, $>$, or $<$), it's called an **inequality** instead.

To solve this inequality we begin the same way as we would if we were solving the equation, by subtracting 32 from each side to get

$$\begin{array}{rcl} 1.8C + \cancel{32} & \leq & 375 \\ -\cancel{32} & & -32 \end{array}$$

which simplifes to

$$1.8C \leq 375 - 32 = 343$$

To understand why the inequality stays the same when we subtract, think of the inequality as "little" $\leq$ "big." If one number is smaller than the other, the same will be true if we subtract the same amount from each number. For example, $18 \leq 21$. To make it real, suppose I have \$18 and you have \$21. Then imagine we each buy a movie ticket for \$12. I would have $\$18 - \$12 = \$6$ and you would have $\$21 - \$12 = \$9$. And still $6 \leq 9$.

Back to our example. We had $1.8C \leq 343$. Divide each side by 1.8 to get

$$\frac{\cancel{1.8}C}{\cancel{1.8}} \leq \frac{293}{1.8}$$

which simplifies to

$$C \leq \frac{293}{1.8} = 293 \div 1.8 = 190.555555\ldots \approx 190°C$$

The oven should be set at most $190°C$. We rounded down because we do not want the brownies to burn.

To understand why the inequality stays the same when we divide, again think of the inequality as "little" $\leq$ "big." If one number is littler than the other, the same will be true when we divide each number by the same divisor. For example, $6 \leq 9$, which we imagined as my having \$6 and your having \$9 after we each bought a movie ticket. While we're making up stories, suppose we each have three children who want some money from us for treats. We each divide our remaining cash among our three children, respectively. My kids each get $\$6 \div 3 = \$2$ and your kids each get $\$9 \div 3 = \$3$. And $2 \leq 3$ still.

There is a bit of caution when solving inequalities. When symbolically solving an equation, any operation you do to each side preserves the equality: start with equal amounts, do the same thing to each, end with equal amounts. But, when symbolically solving an inequality, only some operations you do to each side preserves the inequality: add or subtract from each side or multiply or divide each side by the same (positive) number. But other operations can reverse the inequality.

For example, we can swap sides of an equation, but if we swap sides of an inequality then the direction of the sign reverses. In this brownie example, we want

$$F \geq 325$$

Remember $\geq$ stands for **greater than or equal to**. That's like "big" $\geq$ "little." We can rewrite that inequality as "little" $\leq$ ‘big," or equivalently

$$325 \leq F$$

In each case, 325 is "little" and F is "big". Make sense?

Multiplying or dividing each side of an inequality by a negative number switches the inequality sign as well. Watch out for that with decreasing functions because that's where the slope is negative. And the number we're dividing by is actually the slope.

Remember that the recipe for chocolate brownies says to bake in a moderate oven, between $325°F$ and $375°F$. We just figured out that $F \leq 375$ corresponds to $C \leq 190$. But that's only half of the story. We also wanted $F \geq 325$. While we could solve that inequality separately, it turns out there's an easier way.

Inequalities are a very useful notation for indicating "between". We want between $325°F$ and $375°F$ to bake the brownies. We can write

$$325 \leq F \leq 375$$

which is read

"F is between 325 and 375 (inclusive)"

The word **inclusive** indicates that we're allowing $F = 325$ or $F = 375$.

The good news is that we can solve this chain of inequalities all at once using the same steps as before but now being sure to do the same thing to all *three* sides. "*Three* sides?" you ask. Yes, "three," I confirm. Watch how this works. Start with

$$325 \leq F \leq 375$$

Using $F = 1.8C + 32$ we get

$$325 \leq 1.8C + 32 \leq 375$$

Subtract 32 from each of the three sides to get

$$\begin{array}{ccccc} 325 & \leq & 1.8\text{C} + \cancel{32} & \leq & 375 \\ -32 & & -\cancel{32} & & -32 \end{array}$$

which simplifies to

$$293 \leq 1.8C \leq 343$$

Next, divide all three sides by 1.8 to get

$$\frac{293}{1.8} \leq \frac{\cancel{1.8}C}{\cancel{1.8}} \leq \frac{343}{1.8}$$

which simplifies to

$$293 \div 1.8 \leq C \leq 343 \div 1.8$$

so,

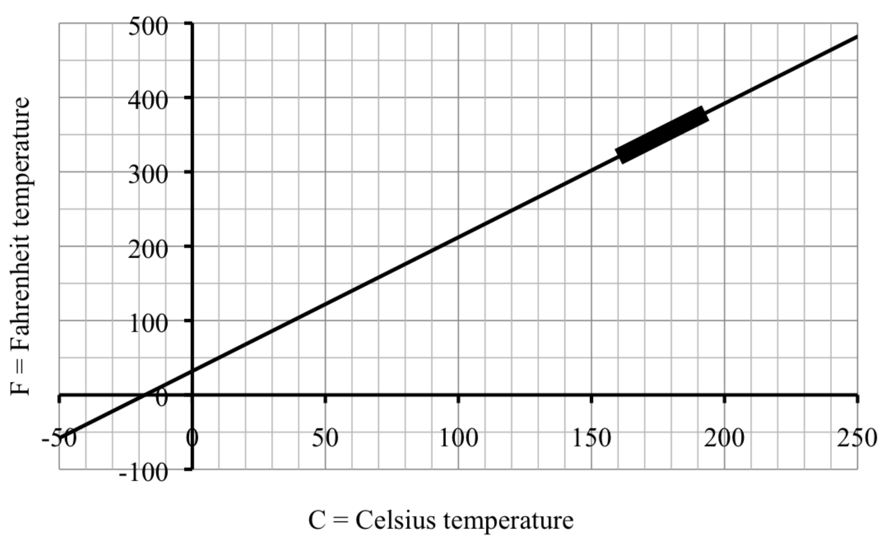
$$162.777777... \leq C \leq 190.555555...$$

Probably best to say

$$163 \leq C \leq 190$$

Chocolate brownies bake between $163°C$ and $190°C$. Oven actually aren't that precise, so somewhere between $170°C$ and $190°C$ should do the job.

If we graph our linear function $F = 1.8C + 32$, we can check our answer for the right temperature range for our brownies. Since we want F between 325 and 375 we start on the vertical axis and then use the graph to find the right range on the horizontal axis. You can see from the highlighted region that our answer is reasonable. Now, who wants brownies?



Homework

Start by doing Practice exercises #1-4 in the workbook.

Do you know . . .

- Common phrases that indicate an inequality?

- How to represent the idea of “between” using a double-sided inequality?
- Why we “do the same thing to each side” of an inequality when solving?
- How to solve a linear inequality? A chain of inequalities?
- Why the inequality sign is reversed if we switch sides of the equation?
- When to solve an inequality, as opposed to solving an equation?

If you’re not sure, work the rest of exercises and then return to these questions. Or, ask your instructor or a classmate for help.

Exercises

5. Recall that the conversion between Fahrenheit (F) and Celsius (C) temperatures is given by the equation
$$F = 1.8C + 32$$
   (a) Evaluate the equation at the appropriate values to check that 0°C= 32°F, 20°C= 68°F, and 37°C= 98.6°F.
   (b) Set up and solve an equation to find the Celsius equivalent of brownie-baking temperature (350°F).
   (c) You’re planning a trip to Norway over Christmas and have heard it’s will be around 10°C. What sort of jacket will you need? Convert to Fahrenheit to decide.
   (d) You want to explain to your Norwegian hosts that back in Minnesota this time of year temperatures can range between -20°F and 40°F. Express this range in Celsius instead. Set up and solve a chain of inequalities.
   (e) Your Norwegian hosts ask about the temperature in Minnesota during the summer. You explain that summer temperatures typically range from 55°F and 105°F. Express this range in Celsius instead. Set up and solve a chain of inequalities.

6. After borrowing some money through a line of credit on my bank account, I started paying off the interest plus \$250 a month. Even once the loan is paid off I plan to continue to deposit \$250 each month to start savins. That means my balance is given by the equation
$$A = 250M - 2{,}189.57$$
where M is the number of months since the loan and A is the account balance, in dollars. *Story also appears in 2.1 #3*

(a) Set up and solve an inequality to determine when I will have paid off my line of credit. That means the account balance will be \$0 or more.

(b) Set up and solve an inequality to determine when I will have saved at least \$2,000.

7. When Gretchen walks on her treadmill, the number of calories per mile that she burns is given by the equation

$$C = 125M$$

where C is the number of calories Gretchen burns by walking M miles.

Story also appears in 2.1 and 3.1 Exercises

(a) Draw a graph showing the number of calories Gretchen burns if she walks 0, 1, 2, or 3 miles.

(b) How far does she need to walk to burn at least 200 calories? Set up and solve an inequality.

(c) Highlight the part of your graph where she burns at least 200 calories.

8. The water in the local reservoir has been dropping steadily. In fact,

$$D = 47 - 1.5W$$

where D is the depth of the water (in feet) after W weeks. Any depth below 20 feet is considered dangerously low. When will that happen, assuming no change in the weather? Set up and solve an inequality. And, check your answer.

Story also appears in 2.1 #2 and 4.1 #3

9. A manufacturer makes family-sized bags of potato chips, advertised as containing 200 grams each. In fact it's difficult to control the exact weight of a bag of potato chips, so it varies. The standard deviation is rather high, about 3.8 grams per bag. The company would rather have bags too heavy than too light, lest they be accused of false advertising, so their average bag actually weighs 207 grams. It turns out that approximately 97% of all bags of chips weigh 200 grams or more. We can compute the standard Z-score of a given bag of chips weighing B grams using the equation

$$Z = \frac{B - 207}{3.8}$$

(a) What is the Z-score for a bag of potato chips weighing the advertised 200 grams? Remember above average Z-scores are positive and below average Z-scores are negative, so your answer should be negative.

(b) About 3/4 of all bags of chips will have $Z \geq -.67$. What weight bag has Z-score of $-.67$? Set up and solve an inequality.

(c) A standard Z-score between -1 and +1 is considered ordinary. What weight bags are considered ordinary?

(d) Oh, and if a serving size is 28 grams (approximately 1 ounce), how many servings are in a bag that weights 207 grams?

10. The cost of vacation to Cork, Ireland from the Minneapolis/St. Paul airport for two people is given by the equation formula

$$C = 2{,}828 + 310N$$

where C is the total cost in U.S. dollars and N is the number of days. Ciara wants to take her boyfriend Seamus to Cork to meet Ciara's grandmother.

(a) What would it cost Ciara to go to Ireland with Seamus for six days?

(b) What might the number 2,828 mean in terms of the story, and what are its units?

(c) What might the number 310 mean in terms of the story, and what are its units?

(d) If Ciara has budgeted up to \$10,000, how many days can they afford to spend in Ireland? Set up and solve an inequality to find the answer.

3.3 Solving power equations (and roots)

There's an old saying – "when life gives you lemons, make lemonade." But how many lemons do you need? It turns out a reasonable equation describing the juice from a single lemon is given by

$$J = .0056C^3$$

where

$$J = \text{juice (tablespoons)} \sim \text{dep}$$
$$C = \text{circumference (inches)} \sim \text{indep}$$

In case you've forgotten, the **circumference** is the distance *around* the lemon. Think of taking wrapping a piece of string around the middle part of the lemon. Then stretch it out on a ruler to see how long it is.

Recipes for lemonade vary widely, but for my recipe calls for 4 tablespoons of lemon juice for each 12 ounce glass. The rest is a syrup made of hot water and sugar, mulled with a sweet herb like basil or mint, then finished with ice and cold water. Yum.

So, how large a lemon would yield 4 tablespoons of juice? Let's try to guess the answer. For example, a small lemon 7 inches in circumference would yield

$$J = .0056 * 7^3 = .0056 \times \underline{7} \wedge 3 = 1.9208 \approx 2 \text{ tablespoons}$$

A medium size lemon 8 inches in circumference would yield

$$J = .0056 * 8^3 = .0056 \times \underline{8} \wedge 3 = 2.8672 \approx 3 \text{ tablespoons}$$

Bet 9 inches is a good next guess. We get

$$J = .0056 * 9^3 = .0056 \times \underline{9} \wedge 3 = 4.0824 \approx 4 \text{ tablespoons}$$

That was quick! A lemon 9 inches around should produce just over 4 tablespoons of juice.

Much as we have learned to love successive approximation, this chapter is all about solving equations. Remember,

$$J = .0056C^3$$

is a power equation because it fits the template

$$\text{dep} = k * \text{indep}^n$$

with power $n = 3$ and proportionality constant $k = .0056$. Turns out we can solve any power equation symbolically.

Here's how. We're looking for $J = 4$. Use our equation $J = .0056C^3$ to get

$$.0056C^3 = 4$$

We want to find the value of C, so we can divide both sides by .0056 to get

$$\frac{\cancel{.0056}C^3}{\cancel{.0056}} = \frac{4}{.0056}$$

which simplifies to

$$C^3 = \frac{4}{.0056} = 4 \div .0056 = 714.2857\ldots$$

We have found C^3. How can we "undo" the $\wedge 3$ to find C? The answer: take the cube root of each side. (More about roots at the end of this section.) That means

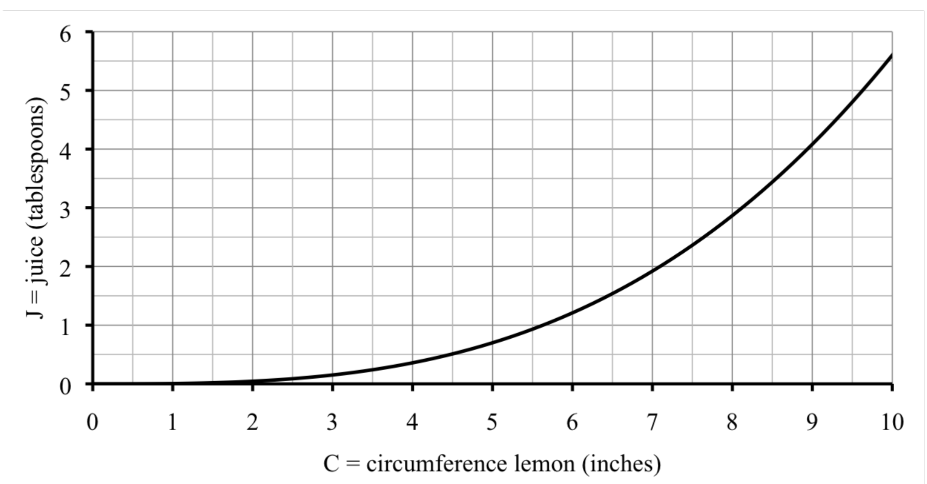
$$\sqrt[3]{C^3} = \sqrt[3]{714.2857\ldots}$$

which simplifies to

$$C = \sqrt[3]{714.2857\ldots} = 3 \sqrt[x]{\ \ }\ 714.2857 = 8.9390\ldots \approx 8.9 \text{ inches}$$

as expected, just under 9 inches. (More about the $\sqrt[x]{\ \ }$ key later too.)

A look at the graph confirms our result.



Now, what goes better with lemonade than lemon cheesecake? For that we need lemon zest. **Zest** is what you get when you grate the lemon peel in long skinny strips. As with juice, the amount of lemon zest depends on the size of the lemon. Our variables are

Z = amount of lemon zest (tablespoons) $\sim$ dep
C = circumference (inches) $\sim$ indep *as before*

and an equation is

$$Z = .018C^2$$

We have another power equation, this time with power $n = 2$ and proportionality constant $k = .018$.

My lemon cheesecake recipe calls for 1½ tablespoons of zest. There are various sized lemons at the store. How large a lemon should I buy? A small lemon of circumference 7 inches produces less than 1 tablespoon of zest because

$$Z = .018 * 7^2 = .018 \times \underline{7} \wedge 2 = .882 < 1$$

so that's not large enough.

Let's use successive approximations, summarizing our guesses in a table. Of course, we don't really need this precise an answer, but it's good practice. Notice $1\frac{1}{2} = 1 + 1 \div 2 = 1.5$

C	7	8	9	10	9.5	9.3	9.2	9.1
Z	.882	1.152	1.458	1.8	1.6245	1.55682	1.52352	1.49058
vs. 1.5	low	high	low	high	high	high	high	low

We need a large lemon, somewhere between 9.1 and 9.2 inches around. Truth is, I'll just buy the biggest lemon I can find because extra lemon zest looks wonderful on top of the cheesecake.

We are supposed to be practicing solving the equation. Here goes. We want $Z = 1.5$. Use our equation $Z = .018C^2$ to get

$$.018C^2 = 1.5$$

We want to find the value of C, so we can divide both sides by .018 to get

$$\frac{\cancel{.018}C^2}{\cancel{.018}} = \frac{1.5}{.018}$$

which simplifies to

$$C^2 = \frac{1.5}{.018} = 1.5 \div .018 = 83.333333\ldots$$

Take the square root of each side to get

$$\sqrt{C^2} = \sqrt{83.333333\ldots}$$

which simplifies to

$$C = \sqrt{83.333333\ldots} = \sqrt{}\,83.333333 = 9.128709292\ldots \approx 9.13 \text{ inches}$$

as expected, between 9.1 and 9.2 inches.

As when solving linear equations, notice that we do the opposite operation in reverse order from the usual order of operations. To evaluate a power equation we would first raise to the power and then multiply. To solve a power equation we first divide (that is the opposite of multiplying) and then we take a root (that is the opposite of raising to a power).

As promised, a brief discussion of roots is in order. Here's the deal. Roots essential "undo" powers. What this means is, for example, we know

$$10^2 = 10 \times 10 = 100$$

but it's quicker to calculate it using powers as

$$10^2 = 10 \wedge 2 = 100$$

We say 10 **squared** is 100. The **square root** of a number is just whatever number you would square to get that number. So, for example, $\sqrt{100} = 10$ because you would square 10 to get 100. Many calculators have a special square root key that looks like $\sqrt{}$ so we get

$$\sqrt{100} = \sqrt{}\ 100 = 10$$

Your calculator might insert a parenthesis with the square root, in which case you should (but don't need to) close it before hitting =, like this

$$\sqrt{100} = \sqrt{}\,(100) = 10$$

Your calculator might not have this key, or might need the square root after the number. Ask a classmate or your instructor or search online if you can't figure it out.

The same idea works for higher powers. Like

$$10^3 = 10 \times 10 \times 10 = 1{,}000$$

That's really

$$10^3 = 10 \wedge 3 = 1{,}000$$

and we say 10 **cubed** is 1,000. The **cube root** of a number is whatever number you would cube to get that number. So, for example, $\sqrt[3]{1{,}000} = 10$. Many calculators have a special root key that looks like $\sqrt[x]{}$. That x looks similar to multiplication ($\times$), but it isn't. The x is like a placeholder for the real root you want – for a cube root x is just 3.

Here's how to use that root key. First you type in the root you want (3), second you use that key ($\sqrt[x]{}$), and last you type in the number you're taking the root of (1,000) like this

$$\sqrt[3]{1{,}000} = 3 \sqrt[x]{}\ 1{,}000 = 10$$

Like with squareroots, your calculator might introduce a parenthesis, or you might do a slightly different order. You might have to use a shift or second key to get to the root key. On many graphing calculators the $\sqrt[x]{\ }$ key is one of the MATH functions, so you have to type something like MATH 5 to get it. Again, ask if you need help figuring it out.

There is a small chance that your calculator doesn't have roots. In that case there is a strange-looking alternative

$$\sqrt[3]{1{,}000} = 1{,}000 \wedge (1 \div 3) = 10$$

Note the necessary parentheses. This process works for square roots too.

$$\sqrt{100} = 100 \wedge (1 \div 2) = 10$$

Bet you see how this idea of roots generalizes. The **nth root** of a number is whatever number you would raise to the nth power to get the number. Stated in terms of equations we have

ROOT FORMULA: The equation $C^n = v$ has solution $C = \sqrt[n]{v}$

Homework

Start by doing Practice exercises #1-4 in the workbook.

Do you know . . .

- What a "power" equation is?
- What we mean by square root, cube root, and nth root?
- How to calculate square roots, cube roots, and nth roots on your calculator?
- How to evaluate the ROOT FORMULA on your calculator?
- When to use the ROOT FORMULA? *Ask your instructor if you need to remember the* ROOT FORMULA *or it will be provided during the exam.*
- How to solve a power equation?
- What the graph of a power function looks like?

  If you're not sure, work the rest of exercises and then return to these questions. Or, ask your instructor or a classmate for help.

Exercises

5. Recall our lemon zest formula $Z = .018C^2$ where C is the circumference of the lemon, in inches, and Z is the amount of lemon zest, in tablespoons.

   (a) Use the information we found earlier to draw a graph of the function. Include values $0 \le C \le 10$.

   (b) Set up and solve an equation to find the size lemon needed for 1 tablespoon of zest.

   (c) Suppose the formula holds for grapefruit too. I don't know of any recipe that calls for grapefruit zest; it is very bitter! But grapefruit is notorious for interacting with certain medications, and so we're collecting some zest for an experiment. Let's say we need $1/4$ cup of zest. How large a grapefruit will we need? Set up and solve an equation to answer. Use that 1 cup = 16 tablespoons.

6. Wind turbines are used to generate electricity. For a particular wind turbine, the equation

$$W = 2.4S^3$$

   can be used to calculate the amount of electricity generated (W watts) for a given wind speed (S mph), over a fixed period of time.

   Story also appears in 1.1, 1.3, and 2.4 Exercises

   (a) Set up and solve an equation to determine the wind speed that will generate 12,500 watts of electricity.

   (b) Repeat for 45,000 watts.

7. Mom always said to sit close to the lamp when I was reading. The intensity of light L, measured in percentage (%) that you see from a lamp depends on your distance from the lamp, F feet as described by the formula

$$L = \frac{100}{F^2}$$

   Story also appears in 1.1 and 2.3 Exercises

   (a) I am most comfortable reading in good light, say 70% intensity. According to the equation, how far away can I sit from the lamp? Use successive approximation to guess the answer to the nearest $1/10$ foot. Then set up and solve an equation. Answer to the nearest inch.

(b) For reading a magazine 35% intensity is enough light. According to the equation, how far away can I sit from the lamp? Use successive approximation to guess the answer to the nearest 1/10 foot. Then set up and solve an equation. Answer to the nearest inch.

8. The lake by Rodney's condo was stocked with bass (fish) 10 years ago. There were initially 400 bass introduced. Rodney wonders what the annual percent increase of the bass has been and realizes he can calculate it from the number of fish now. He will use the equation

$$B = 400g^{10}$$

where B is the number of bass in the lake now and g is the annual growth factor. For each number of bass, first solve for g using the ROOT FORMULA, then calculate $r = g - 1$. The percent increase is r written as a percent.

Story also appears in 5.5 Exercises

(a) Find the annual percent increase if there are $B = 3{,}000$ bass now.

(b) Find the annual percent increase if there are $B = 4{,}000$ bass now.

9. If you drop a rock from a high place, it falls R feet in T seconds where

$$R = 16T^2$$

(a) How far does the rock fall in 2 seconds? In 4 seconds?

(b) Is the rock falling faster during the first two seconds ($T = 0$ to $T = 2$) or during the second two seconds ($T = 2$ to $T = 4$)? Calculate the rate of change to decide.

(c) Tia dropped a rock from her apartment window that's 300 feet above ground. Will the rock have hit the ground by 4 seconds after it was dropped?

(d) If you evaluate at $T = 5$, what value of R do you get and what does it mean in the story, again assuming the rock is dropped from 300 feet up.

(e) When does the rock hit the ground? Set up and solve an equation. *Hint: what value of R do you solve for?*

(f) Now suppose we have a new variable, H, which represents the height of the rock Tia dropped after T seconds, write a new equation for H as a function of T.

(g) Show how to set up and solve an equation using this new equation to find when the rock hits the ground. *Hint: what value of H do you solve for now?*

10. Wynter has a pretty decent job. He is paid a salary of $780 per week but his hours vary week-to-week. Even though Wynter is not paid by the hour, he can figure out

what his hourly wage would be depending on the number of hours he works using the equation

$$E = \frac{780}{H}$$

where if he works H hours, then he's earning the equivalent of \$$E$/hour.

Story also appears in 2.3 Exercises

(a) Make a table showing Wynter's equivalent hourly wage if he works 40, 50, or 60 hours a week.

(b) Wynter was complaining that things have been so busy lately at work that he's earning the equivalent of only \$9.25/hr. How many hours a week does that correspond to?

(c) Wynter was hoping to earn the equivalent of \$14/hour. How many hours a week does that correspond to?

(d) Draw a graph illustrating how Wynter's equivalent hourly wage decreases as a function of the number of hours he works. Include a few extreme values such as 10 hours and 100 hours to better see the shape of the graph.

3.4 Solving exponential equations (and logs)

Remember Jocelyn? She was asked to analyze information on rising health care costs. In 2007 the United States spent \$2.26 trillion on health care and costs were projected to increase at an average of 6.7% annually for the subsequent decade. For the variables

$$H = \text{health care costs (\$ trillions)} \sim \text{dep}$$
$$Y = \text{time (years since 2007)} \sim \text{indep}$$

she found the exponential equation

$$H = 2.26 * 1.067^Y$$

In what year did health care costs first pass \$3 trillion? We can use successive approximation to find the answer, starting with the values she found earlier.

year	2007	2008	2009	2017	2027
Y	0	1	2	10	20
H	2.26	2.41	2.57	4.32	8.82
vs. 3	low	low	low	high	high

The answer must be between 2009 and 2017. Let's split the difference and guess 2013. For that year, $Y = 2013 - 2007 = 6$ and so

$$H = 2.26 * 1.067^6 = 2.26 \times 1.067 \wedge \underline{6} = 3.334993223 \approx \$3.33 \text{ trillion}$$

which is already over \$3 trillion. What about 2011? Use $Y = 4$ to check that $H \approx$ \$2.93 trillion, which is almost, but not quite there. Must be 2012 was the year. Sure enough when $Y = 5$ we get $H \approx$ \$3.12 trillion. That's it. Health care costs first passed \$3 trillion in 2012. Well, at least according to our equation. As usual, we summarize the numbers in a table.

year	2009	2017	2013	2011	2012
Y	2	10	6	4	5
H	2.57	4.32	3.33	2.93	3.12
vs. 3	low	high	high	low	high

Successive approximation gives us the answer fairly quickly. But there is an even quicker way – solving the exponential equation. Start with what we're looking for, which is $H = 3$. Use the equation $H = 2.26 * 1.067^Y$ to get

$$2.26 * 1.067^Y = 3$$

We want to find the value of Y, so divide each side by 2.26 to get

$$\frac{\cancel{2.26} * 1.067^Y}{\cancel{2.26}} = \frac{3}{2.26}$$

which simplifies to

$$1.067^Y = \frac{3}{2.26} = 3 \div 2.26 = 1.327433628\ldots$$

When the dust settles we're left with

$$1.067^Y = 1.327433628\ldots$$

Hmm. How do we find Y here? We saw how to use roots to solve power equations. In our lemonade example we had $C^3 = 714.2857\ldots$. We knew the exponent (3) and wanted to find the number being raised to that power (C). That's when we took the cube root to get

$$C = \sqrt[3]{714.2857\ldots} = 3\sqrt[x]{\ }714.2857 = 8.9390\ldots \approx 8.9 \text{ inches}$$

That approach is not going to work here because it's backwards now – we know the number being raised to a power (1.067) and are on the hunt for the exponent (Y) instead.

Turns out there's a different formula for solving for the exponent that uses **logarithms** (nickname: **logs**). More about logs in a minute, but first let's write down the formula and practice working with it. The formula is

Log-Divides Formula: The equation $g^Y = v$ has solution $Y = \dfrac{\log(v)}{\log(g)}$

Quick aside about the name. Some formulas have well-known names. Not this one. We call it the "Log-Divides Formula" because it has logs and divides in it. Perhaps you already guessed that. Other math books do not have an name for this formula, although it is related to something called the "change of base formula".

Okay. Back to solving our equation. We got stuck trying to solve

$$1.067^Y = 1.327433628\ldots$$

We have growth factor $g = 1.067$ and value $v = 1.327433628\ldots$. So the formula says

$$\begin{aligned} Y &= \frac{\log(v)}{\log(g)} \\ &= \frac{\log(1.327433628\ldots)}{\log(1.067)} \\ &= \log(1.327433628\ldots) \div \log(1.067) = \\ &= 4.367667365 \approx 4.37 \end{aligned}$$

Your calculator should have a key that says "log" or maybe "LOG". Try typing

$$\log(1.327433628) \div \log(1.067) = 4.367667365 \approx 4.37$$

A small note here about parentheses. Some calculators give the first parenthesis for free when you type log but you have to type the closing parenthesis in yourself.

This answer of 4.37 means that costs are projected to exceed \$3 trillion just over 4 years after 2007. That's some time during 2011, or by 2012 for sure. Same answer as before.

Let's practice. Suppose instead we want to know when health care costs would exceed \$10 trillion instead. (By the way – wow!) That means $H = 10$. Using our equation $H = 2.26 * 1.067^Y$ we get

$$2.26 * 1.067^Y = 10$$

Before we can use the Log-Divides Formula, we need to get rid of that 2.26. To do so, we can divide both sides by 2.26

$$\frac{\cancel{2.26} * 1.067^Y}{\cancel{2.26}} = \frac{10}{2.26} = 10 \div 2.26 = 4.424778761\ldots$$

That means

$$1.067^Y = 4.424778761\ldots$$

Now our equation fits the format $g^Y = v$ for the LOG DIVIDES FORMULA with new value $v = 4.424778761\ldots$ (and the growth factor is $g = 1.067$ still). So the answer is

$$\begin{aligned} Y &= \frac{\log(v)}{\log(g)} \\ &= \frac{\log(4.424778761\ldots)}{\log(1.067)} \\ &= \log(4.424778761\ldots) \div \log(1.067) = \\ &= 22.932891 \approx 23 \end{aligned}$$

Want to avoid typing in the number 4.424778761...? Depending on your calculator, you might try this instead:

$$10 \div 2.26 = \log(\text{ANS}) \div \log(1.067) = 22.932891$$

where **ANS** stands for "answer".

Again that means 23 years after 2007, or 2007 + 23 = 2030. Health care costs are projected to exceed \$10 million in the year 2030. Well, unless we do something about that. (Helps explain why government folks are often discussing how to contain health care costs.)

Time to fill you in a bit more about logs. Look at these examples. Don't take my word for it; calculate them yourself.

$$\begin{aligned} \log(10) &= 1 \\ \log(100) &= 2 \\ \log(1{,}000) &= 3 \\ \log(10{,}000) &= 4 \end{aligned}$$

What do you see? In each case the logarithm is the number of zeros. For example, 10,000 has 4 zeros and $\log 10{,}000 = 4$. Another way to think of this connection is

$$10{,}000 = 10^4 \text{ and } \log 10{,}000 = 4$$

In other words, the logarithm is picking off the power of 10.

Wait a minute. The Log-Divides formula helped us find the value of Y which was an exponent. And now we see that the log of a power of 10 is that exponent. So a logarithm is just an exponent. And logarithms help us find the exponent. Makes sense.

What about logs of numbers that aren't just powers of 10? Here are some examples.

$$\begin{aligned} \log(25) &= 1.3979\ldots \\ \log(250) &= 2.3979\ldots \\ \log(2{,}500) &= 3.3979\ldots \\ \log(25{,}000) &= 4.3979\ldots \end{aligned}$$

To see what's happening we want to involve powers of 10. Scientific notation will do that for us. Let's write these numbers in scientific notation and see what we learn. For example.

$$25{,}000 = 2.5 \times 10^4 \text{ and } \log(25{,}000) = 4.3979\ldots \approx 4$$

We are back to the power of 10. Well, approximately. Let's check another number.

$$250 = 2.5 \times 10^2 \quad \text{and} \quad \log(250) = 2.3979\ldots \approx 2$$

Before we write down a general rule, let's check more numbers.

$$\begin{array}{lll} 7{,}420{,}000 = 7.42 \times 10^6 & \text{and} & \log(7{,}420{,}000) = 6.870403905\ldots \approx 6 \\ 4 = 4 \times 10^0 & \text{and} & \log(4) = 0.602059991\ldots \approx 0 \\ .00917 = 9.17 \times 10^{-3} & \text{and} & \log(.00917) = -2.037630664\ldots \approx -3 \end{array}$$

In every case we are rounding down, but it's always the same.

$$\log(\text{number}) \approx \text{power of 10 in the scientific notation for that number.}$$

Homework

Start by doing Practice exercises #1-4 in the workbook.

Do you know . . .

- What "log" means?
- The connection is between logs and scientific notation?
- How to evaluate logs on your calculator?
- How to evaluate the LOG DIVIDES FORMULA using your calcuator?
- When to use the LOG DIVIDES FORMULA? *Ask your instructor if you need to remember the* LOG DIVIDES FORMULA *or if it will be provided during the exam.*
- How to solve an exponential equation?

  If you're not sure, work the rest of exercises and then return to these questions. Or, ask your instructor or a classmate for help.

Exercises

5. The employee-paid cost of health insurance has risen dramatically, increasing by 7% each year since 2003 when it cost $420/month.

   (a) Name the variables and write an exponential equation relating them.

   (b) If this rate of increase continues, when will or did the employee-paid cost pass $550/month? Solve your equation.

   (c) Repeat for $600/month.

   (d) Graph the function.

6. The number of school children in the district from a single parent household has been on the rise. In one district there were 1,290 children from single parent households in 2010 and that number was expected to increase about 3% per year. Earlier, we found the equation was

   $$C = 1{,}290 * 1.03^Y$$

   where C is the number of children and Y is the years since 2010.

   Story also appears in 2.2 and 5.3 Exercises

(a) Use successive approximation to determine when there will be over 3,000 school children in the district from a single parent household. Display your work in a table. Round your answer to the nearest year.

(b) Show how to solve the equation to calculate when there will be over 3,000 school children in the district from a single parent household. Show how you solve the equation.

(c) Solve again to determine when there will be over 3,500 children. Check your answer.

7. Suppose a special kind of window glass is 1 inch thick and lets through only 75% of the light. If we use W inches of window glass, it lets $L\%$ of the light through where

$$L = 100 * .75^W$$

Story also appears in 2.4 and 5.3 Exercises

(a) What thickness glass should be used to let through less than 10% of the light? *Set up and solve an equation.*

(b) What about 50%? *Set up and solve an equation.*

(c) Check the graph (drawn before) to see if your answers make sense.

8. We saw that poultry population was estimated to grow according to the equation

$$P = 78 * 1.016^Y$$

where P is the poultry population in million tons and Y is the years starting in 2005.

Source: Worldwatch Institute *Story also appears in 2.2 Exercises*

(a) When will production rise above 95 million tons? Set up and solve an equation. Then use some other method to check.

(b) Repeat for 120 million tons.

9. Darcy likes to use temporary hair color in wild colors. Good thing it washes out. Her best guess is that 8% of the color washes out each time she washes her hair. That means the percentage of color remaining, C, is a function of the number of times she washes her hair, W, according to the equation

$$C = 100 * .92^W$$

(a) When will half the color be gone? That means find $C = 50\%$. Set up and solve an equation. Then check some other way.

(b) By the time only 10% of the color remains you really can't tell anymore if it was pink or orange or blue. So, she might as well switch to a new color then. How many washes before only 10% remains? Again, first solve. Then check.

(c) Draw a graph showing how the color washes out of Darcy's hair.

3.5 Solving quadratic equations

Claude likes to juggle. As he throws a beanbag up in the air, the height changes over time as described by the equation

$$H = 3 + 15T - 16T^2$$

where

$$H = \text{height of beanbag (feet)} \sim \text{dep}$$
$$T = \text{time (seconds)} \sim \text{indep}$$

Let's make a table and graph this function. For example, when $T = 0$ seconds we have

$$H = 3 + 15 * 0 - 16 * 0^2 = 3 + 15 \times \underline{0} - 16 \times \underline{0} \wedge 2 = 3 \text{ feet}$$

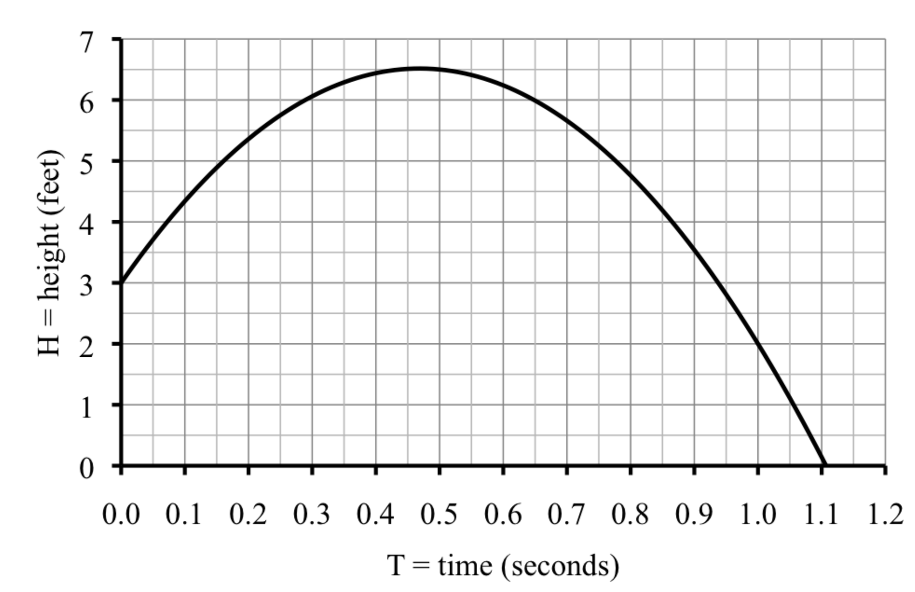
and when $T = 1$ second we have

$$H = 3 + 15 * 1 - 16 * 1^2 = 3 + 15 \times \underline{1} - 16 \times \underline{1} \wedge 2 = 2 \text{ feet}$$

Huh? I though the beanbag went up in the air. What's happening here? Oh, I know. The beanbag must be falling down by then. As we fill in the table with intermediate values we see how Claude's beanbag went up in the air and then back down.

T	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1	1.1	1.2
H	3	4.34	5.36	6.06	6.44	6.50	6.34	5.66	4.76	3.54	2	.14	~~−2.04~~

Notice by $T = 1.2$ seconds we got -2.04 feet. We can't have negative feet. The beanbag must hit the ground before 1.2 seconds. From the graph I'd say in just over 1.1 seconds.



Of course, we can refine our answer by successive approximations. The beanbag hits the ground when it's height is 0 feet. Looks a little strange but we want $H = 0$. We expect the answer is just a little bigger than 1.1, so we start our guess optimistically with 1.11.

T	1.1	1.11	1.105	1.107	1.106
H	.14	$\cancel{-.06}$	.0368	$\cancel{-.002}$	.018
vs. 0	high	low	high	low	good

The beanbag was in the air for approximately 1.106 seconds.

In this chapter we've seen how to solve linear, power, and exponential equations. Let's solve this equation too. By the way, our function is quadratic because

$$H = 3 + 15T - 16T^2$$

fits the template for a quadratic equation.

> QUADRATIC EQUATION TEMPLATE: $\quad \text{dep} = a * \text{indep}^2 + b * \text{indep} + c$

with constants

$$a = -16 \quad b = 15 \quad c = 3$$

(More on how we found those numbers in a moment.)

Back to our juggler. We are trying to figure out when $H = 0$. Using our equation $H = 3 + 15T - 16T^2$, we get

$$3 + 15T - 16T^2 = 0$$

We want to solve for T. Notice that because T occurs twice in the equation, nothing we have seen to do to each side of the equation can knock it down to just one T. That means none of our methods so far work. Luckily there's a way to solve any quadratic equation using the aptly-named QUADRATIC FORMULA.

> QUADRATIC FORMULA: The equation $aT^2 + bT + c = 0$ has solutions
>
> $$T = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Oh my! First thing to understand in this complicated formula is that we actually get two possible answers

$$T = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad T = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

Sometimes one answer makes sense in the story, other times they both might. Stay tuned.

For Claude's situation we had

$$3 + 15T - 16T^2 = 0$$

To fit the formula, we need the T^2 first, the T second, and then the constant. No sweat, just reorder to get

$$-16T^2 + 15T + 3 = 0$$

Notice how subtracting $16T^2$ became adding $-16T^2$ when we rearranged? That lines up perfectly with

$$aT^2 + bT + c = 0$$

That's how we knew the constants were

$$a = -16 \quad b = 15 \quad c = 3$$

The first fraction in the formula is

$$\frac{-b}{2a} = \frac{-15}{2 * -16} = \text{(-)}15 \div (2 \times \text{(-)}16) = .46875$$

As usual, we needed parentheses around the denominator (bottom) of our fraction to override the normal order of operations. As a reminder, (-) refers to negation. Remember, this does not mean you type in parentheses, just hit the one key that is labeled (-) already.

The second fraction is

$$\begin{aligned} \frac{\sqrt{b^2 - 4ac}}{2a} &= \frac{\sqrt{(15)^2 - 4 * -16 * 3}}{2 * -16} \\ &= \sqrt{\ }((15) \wedge 2 - 4 \times \text{(-)}16 \times 3) \div (2 \times \text{(-)}16) \\ &= -.6381431\ldots \approx -.63814 \end{aligned}$$

Check out the parentheses now. Three sets here. First, around the quantity we're taking the square root of. Maybe your calculator included the open parentheses along with the square root, but either way we need them. Second, around the number (15) that we are squaring. That didn't matter here but if b were negative it would have. Last, we added parentheses around the bottom of the fraction, as always.

Oh, and we're not done yet. Remember there are two possible answers. One is the sum of these two numbers

$$.46875 + \text{(-)}.63814 = -.16939 \text{ seconds}$$

which doesn't make any sense because time is never negative. The other is the difference

$$.46875 - (\text{-}).63814 = 1.10689 \text{ seconds}$$

We had guessed around 1.106 seconds, so that is definitely the right answer: Claude's beanbag will hit the ground after 1.10689 seconds. Yeah, too precise. But you get the idea.

Wait a minute! Any good juggler isn't about to let the beanbag fall on the ground. He's going to catch it again, perhaps at about 3½ feet above ground. That means we're looking for $H = 3.5$. Using our equation $H = 3 + 15T - 16T^2$, we get

$$3 + 15T - 16T^2 = 3.5$$

The QUADRATIC FORMULA only works if the equation has $= 0$, but we have $= 3.5$. It might seem that we're out of luck, but it's an easy fix. Just subtract 3.5 from each side.

$$\begin{aligned} 3 + 15T - 16T^2 &= \cancel{3.5} \\ -3.5 \qquad\qquad &= -\cancel{3.5} \end{aligned}$$

which simplifies to

$$-.5 + 15T - 16T^2 \quad = \quad 0$$

So now we have $= 0$. Yes!

We can write the new equation as

$$-16T^2 + 15T - .5 = 0$$

from which we see that

$$a = -16 \quad b = 15$$

as before, but now we have a new value

$$c = -.5$$

We're all set to use the QUADRATIC FORMULA.

The first fraction is

$$\frac{-b}{2a} = \frac{-(15)}{2 * -16} = (\text{-})15 \div (2 \times (\text{-})16) = .46875$$

No surprise here. We used the same values of a and b as before, so we should have the same number here. The second fraction is

$$\begin{aligned} \frac{\sqrt{b^2 - 4ac}}{2a} &= \frac{\sqrt{(15)^2 - 4 * -16 * -.5}}{2 * -16} \\ &= \sqrt{\;}((15) \wedge 2 - 4 \times (\text{-})16 \times (\text{-}).5) \div (2 \times (\text{-})16) \\ &= -.43413887\ldots \approx -.43414 \end{aligned}$$

Don't forget we need to put together these parts to find the possible answers. The sum gives us

$$.46875 + (\text{-}).43414 = .03461 \approx .03 \text{ seconds}$$

and the difference gives us

$$.46875 - (\text{-}).43414 = .90289 \approx .9 \text{ seconds}$$

Both answers seem to make sense. Let's look at the graph to confirm that they're reasonable. We first find $H = 3.5$ while the beanbag is going up in the air, just before the unlabeled gridline for .05 (midway between 0 and .1) so an answer of $T \approx .03$ makes sense. Then, on the way back down, the beanbag is 3.5 feet up at what looks like .9 seconds, so our answer of $T \approx .9$ makes sense too. Since Claude catches the beanbag on the way down, we want that second answer, after .90289 seconds (which is before 1.106 seconds when it hits the ground, by the way).

One interesting note. What happens in the story at the point where the beanbag stops going up in the air and starts falling down? That must be when the beanbag is at its highest point. What is the speed at that highest point? Well, I guess 0. For a split second it's almost frozen in midair, neither rising nor falling. (If we were able to compute the rate of change for an interval really, small time then we would find the rate of change ≈ 0.)

Turns out it's easy to find that point for a quadratic equation, just plug in the first fraction from the Quadratic Formula! Check it out. When

$$T = \frac{-b}{2a} = \frac{-15}{2 * -16} = (\text{-})15 \div (2 \times (\text{-})16) = .46875 \text{ seconds}$$

we get

$$\begin{aligned} H &= 3 + 15 * .46875 - 16 * .46875^2 \\ &= 3 + 15 \times \underline{.46875} - 16 \times \underline{.46875} \wedge 2 \\ &= 6.515625\ldots \approx 6.516 \text{ feet} \end{aligned}$$

Claude throws the beanbag about 6.516 feet up. Converting to more normal units we get

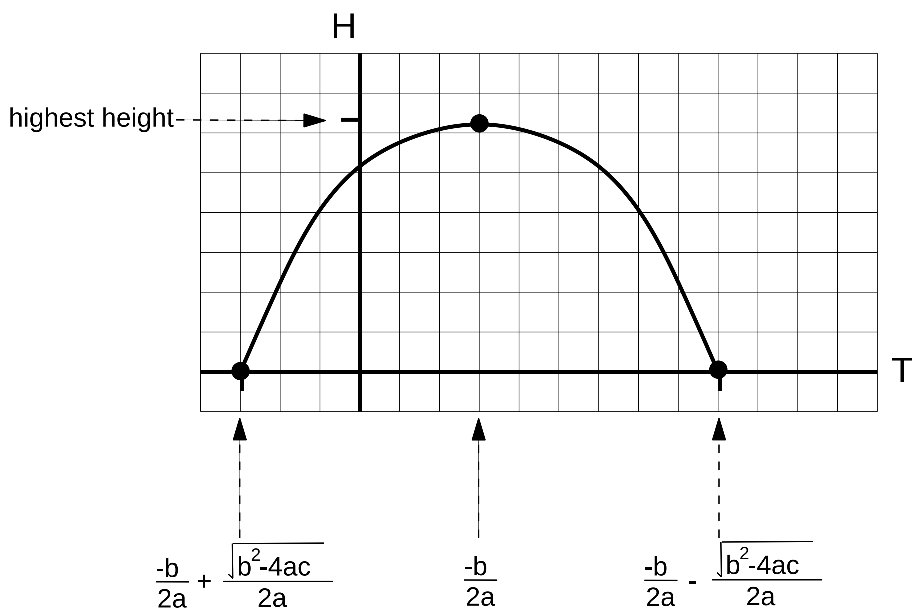
$$.516 \cancel{\text{feet}} * \frac{12 \text{ inches}}{\cancel{\text{feet}}} = .516 \times 12 = 6.192 \approx 6 \text{ inches}$$

The beanbag goes up to about 6'6". You can check that the graph shows just over 6.5 feet.

In general, the graph of $H = aT^2 + bT + c$ is a **parabola**. The two solutions from the QUADRATIC FORMULA are both places where $H = 0$ and, so, the graph crosses the T-axis there. Might not make sense in the real problem, but the equation and formula don't know that. (Okay, equations and formulas don't actually "know" anything. But you get my point.) Turns out the graph is symmetric about the highest point, so that must be midway in between the roots which is exactly where

$$T = \frac{-b}{2a}$$

Because a is negative, the answer we got by adding is to the left and the answer we got by subtracting is to the right.



Our graph was ∩ shaped parabola and so we found a maximum value. The graph of a quadratic function might be ∪ shaped instead. In that case evaluating at $T = \frac{-b}{2a}$ would give the minimum value.

Homework

Start by doing Practice exercises #1-4 in the workbook.

Do you know . . .

- What a "quadratic" function is?
- How to solve a quadratic equation?
- When we use the QUADRATIC FORMULA? *Ask your instructor if you need to remember the* QUADRATIC FORMULA *or if it will be provided during the exam.*

- How to solve a quadratic equation when the function is not set equal to zero?
- How to identify the values of a, b, c in the formula?
- How to evaluate the formula (using your calculator)?
- Why there are (usually) two solutions to a quadratic equation?
- How to decide which solution(s) from the QUADRATIC FORMULA are correct?
- What the graph of a quadratic function looks like?
- The value for the independent variable to find the highest (or lowest) value of a quadratic function?

If you're not sure, work the rest of exercises and then return to these questions. Or, ask your instructor or a classmate for help.

Exercises

5. Claude is an excellent juggler. Remember that the height H feet of Claude's beanbag T seconds after he throws it in the air is described by the equation $H = 3+15T-16T^2$. Answer each of the following question by the suggested method and then look back at the graph from earlier to make sure your answers make sense.

   (a) Use the QUADRATIC FORMULA to find when is the bean bag is 5 feet above ground? Why do both answers make sense in the story?

   (b) When is the beanbag 8 feet above ground? Try to use the QUADRATIC FORMULA to find the answer. What happens? Explain why it makes sense in the story that you can't solve this quadratic equation.

   (c) Claude decided that the beanbag was too high in the air, so he modified his throw slightly. Now the height is given by $H = 3 + 14T - 16T^2$. What is the maximum height the beanbag will reach now? *Hint: what number can you evaluate at?*

6. The stopping distance for Jeff 's Cadillac Escalade is given by

$$D = .04S^2 + 1.47S$$

   where S is the speed of the car (in miles per hour) and D is the stopping distance (in feet). Jeff took 183 feet to stop. How fast was he going?

   Story also appears in Section 2.3

(a) Use successive approximation to estimate the answer to the nearest miles per hour. Display your work in a table.

(b) Show how to use the QUADRATIC FORMULA to solve the equation.

7. A company produces backpacks. The more they make, the less it costs for each one. The cost per backpack is given by the equation

$$C = .01B^2 - 1.2B + 50$$

where C = cost per backpack ($ per backpack) and B = number of backpacks.

Story appears in 1.3 Exercises

(a) How many backpacks do they need to produce in order to hold costs to $20/backpack? Set up and solve a quadratic equation to find the answer.

(b) Make a table of values and draw a graph of the function. Does your answer agree with your table and graph?

(c) What is the minimum price per backpack? *Hint: evaluate at* $T = \frac{-b}{2a}$.

8. Mrs. Weber's cooking class came up with the equation

$$M = 1.2F^2 + 4F + 7$$

to approximate the grilling time of a piece of fish depending on its thickness. Here M is the number of minutes to grill the fish and F is the thickness of the fish in inches.

Story appears in 1.1 and 2.3 Exercises

(a) If we want to make sure the fish will cook in under 20 minutes, what thickness steak can we have? Set up and solve a quadratic equation to find the answer.

(b) Repeat for 10 minutes.

9. A company who makes electronics was doing great business in 1996, but sales quickly slid after 2000. Their sales M in millions of $ Y years from 1996 is given by the following equation

$$M = 104.4 + 11.5Y - 1.4Y^2$$

Story appears in 2.4 Exercises

(a) The company decided to declare bankruptcy when sales fell below $20 billion. In what year was that? Show how to solve using the QUADRATIC FORMULA.

(b) An analyst had suggested that they close down shop earlier, once sales were below $50 billion. In what year did sales fall that low? Show how to solve using the QUADRATIC FORMULA.

(c) What year did sales **peak** (reach their highest value)?

10. A kangaroo hops up in the air (and out) from a 10 foot cliff. Her height above the ground, K feet, after T seconds is given by the equation

$$K = 10 + 5.2T - 4.88T^2$$

(a) Calculate the missing values in the table.

T	0	.3	.6	.9	1.2	1.5
K	10	11.1208	11.3632			6.82

(b) According to this equation, how high up in the air does the kangaroo get? Choose the appropriate value to plug into the equation.

(c) When does the kangaroo land on the ground? Set up and solve an equation.

(d) If she jumps up, but not out, when will she land on the cliff itself again, assuming the same equation holds?