Just Enough Algebra

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Contents

Chapter 1

Variables

Believe it or not, algebra is useful. Really useful. It's useful in later courses you might take in mathematics, statistics, science, social science, or business. But it's also useful in real life. A lot of what happens in the world around us is easier to understand using algebra. That's what this course is all about: using algebra to answer questions.

The philosophy of this course is captured in this quote by Ethan Bolker from his text *Using algebra*.

> With common sense and some algebra you can understand the world better than you can with common sense alone.

In this first chapter we introduce the key concepts of variable and function that help us translate between problems stated in words and the mathematics explaining a situation. We explain the important tools of units, tables, and graphs. We also describe how functions change using the fundamental concept of rate of change. Throughout this chapter we keep a careful eye on evaluating the reasonableness of answers by connecting what we learn from algebra with our own life experience. After all, an answer to a real problem should make sense, right?

Some of our approach may feel strange to you. It is possibly very different from what you've seen in mathematics classes before. It might take you a little time to get used to, but it will be worth it.

1.1 Variables and functions

Things change, like the price of gasoline, and just about every day it seems. What does it mean when the price of a gallon of gas drops from \$3.999/gal to \$3.299/gal? The symbol / is short for "per" or "for each," so that means each gallon costs

$$\$3.999 - \$3.299 = \$.70 = 70¢$$

less. Does this 70¢ truly matter?

Before we answer that question, are you wondering why there's that extra 9 at the end of the price? We might think a gallon costs \$3.99 but there's really a small 9 following it. Sometimes that 9 is raised up slightly on the gas station sign. You have to read the fine print. What it means is an extra $\frac{9}{10}$¢ for each gallon. So the true price of a gallon gas would be \$3.999. Gas costs a tiny bit more than you thought. Good grief.

Back to our question. Does 70¢ truly matter to us? Probably not. Can't even buy a bag of potato chips for 70¢. But, how often do you buy just one gallon of gas? Typically you might put five, or ten, or even twenty gallons of gas into the tank. We want to understand how the price of gasoline influences what it really costs us at the pump. To do that let's compare our costs when we buy ten gallons of gas. There's no good reason for picking ten; it's just a nice number to work with.

If gas costs \$3.999/gal and we buy 10 gallons, it costs

$$10 \text{ gallons} * \frac{\$3.999}{\text{gallon}} = 10 \times 3.999 = \$39.99$$

See how we described the computation twice? First, with units, fractions, and $*$ for multiplication in what's sometimes called "algebraic notation." Then, with just numbers and $\times$ for multiplication – that's what you can type into a calculator.

If gas drops to \$3.299/gal and we buy 10 gallons, it costs

$$10 \text{ gallons} * \frac{\$3.299}{\text{gallon}} = 10 \times 3.299 = \$32.99$$

That's \$7 less. For \$7 savings on gas you could buy that bag of potato chips, and an iced tea to go with it, and still have change. That amount matters. I mean, especially since it's \$7 savings every time you put 10 gallons in the tank.

Gas prices have been changing wildly, and along with them, the price of 10 gallons of gas. In mathematics, things that change are called **variables**. The two variables we're focusing on in this story are

$$P = \text{price of gasoline (\$/gal)}$$
$$C = \text{total cost (\$)}$$

Notice that we gave each variable a letter name. It is helpful to just use a single letter chosen from the word it stands for. In our example, P stands for "price" and C stands for "cost". In this course we rarely use the letter X simply because so few words begin with X. Whenever we name a variable (P) we also describe in words what it represents (the price of gasoline), and we state what units it's measured in ($/gal).

In talking about the relationship between these variables we might say "the cost depends on the price of gas," so C depends on P. That tells us that C is the **dependent variable** and P is the **independent variable**. In general, the variable we really care about is the dependent variable, in this case C the total amount of money it costs us. The concept of dependence is so important that there's yet another word for it. We say that C is a **function** of P, as in "cost is a function of price."

Knowing which variable is independent or dependent is helpful to us. To emphasize the dependence, we often make a notation next to the variable name.

$$P = \text{price of gasoline (\$/gal)} \sim \text{indep}$$
$$C = \text{total cost (\$)} \sim \text{dep}$$

This labeling is rarely used outside this textbook, so add it in for yourself if you need it. In some situations dependency can be viewed either way; there might not be one correct way to do it. Labeling the dependence is extra important then, so anyone reading your work knows which way you are thinking of it.

Given a choice, we usually assign dependence such that given a value of the independent variable, it is easy to calculate the corresponding value for the dependent variable. In our example it's easy to use the price per gallon, P, to figure out the total cost, C. We can work backwards – from C to P – but it's not as easy.

For example, suppose we buy 10 gallons of gas and it costs $28.99. We can figure out that the price per gallon must be

$$P = \frac{\$28.99}{10 \text{ gallons}} = 28.99 \div 10 = \$2.899/\text{gal}$$

Notice that we use the fraction as part of the algebraic notation, but we use ÷ to indicate division on the calculator. Your calculator key for division may be / instead, which we reserve as a shorthand for "per."

From our experience we have a sense of what gas might cost. In my lifetime, I've seen gas prices as low as 35.9¢ /gallon in the 1960s to a high of $4.099/gallon recently. This range of values sounds too specific, so it would sound better to say something general like

"Gas prices are (definitely) between $0/gal and $5/gal."

The mathematical shorthand for this sentence is

$$0 \leq P \leq 5$$

The inequality symbol $\leq$ is pronounced "less than or equal to". Formally, the range of realistic values of the independent variable is called the **domain** of the function C. In this text, we rarely write the domain because it's usually clear from the story what realistic values would be. The exercises in this section ask you to do so for practice.

Be aware that there are often many different numbers in a story. Some numbers are examples of values the variables take on, such as \$3.999/gal or \$39.99 in our example. Other numbers are **constants**; they do not change (at least not during the story). The one constant in our story is that we are always buying 10 gallons of gas. Occasionally there are other numbers in a story that turn out not to be relevant at all, so be on the lookout.

Back to our story. A report says that the average price of gasoline in Minnesota was \$2.900/gal in 2010 and increased approximately 20% per year for the next several years. We would like to check what that says about the average price of gasoline in 2011 and 2012, say. (It is unlikely that the price increase continued much longer at that rate.)

To understand what that report is saying, we need to remember how percents work. Luckily, the word "percent" is very descriptive. The "cent" part means "hundred," like 100 cents in a dollar or 100 years in a century. And, as usual, "per" means "for each." Together, **percent** means "per hundred." The number 20% means 20 for each hundred. Written as a fraction it is $\frac{20}{100}$. Divide to get the decimal $20 \div 100 = .20$.

Think money: 20% is like 20¢, and .20 is like \$.20

Bottom line: 20%, $\frac{20}{100}$, and .20 mean exactly the same number.

$$20\% = \frac{20}{100} = 20 \div 100 = .20$$

To calculate the percent of a number we multiply by the decimal version. For example,

$$20\% \text{ of } \$2.900 = .20 \times 2.900 = \$.58$$

The report says the price increased by 20% each year, so by 2011 the price had increased an average of \$.58. That 58 cents is not what gas cost in 2011. It's how much *more* gas cost in 2011 compared to 2010. To see what the report projected for the 2011 cost we need to add that increase on to the original 2010 price.

$$\$2.099 + \$.58 = \$3.48 \text{ per gallon}$$

Sounds about right. Expensive, to be sure, but fairly accurate.

For 2012, the price increased by 20% again. That means 20% of what it was in 2011. We can't just add \$.58 again. That was 20% of the 2010 value, and we want 20% *of the 2011 value*. Going to have to calculate that.

$$20\% \text{ of } \$3.48 = .20 \times 3.48 = \$.696$$

so the projected 2012 value was

$$\$3.48 + \$.696 = \$4.176 \text{ per gallon}$$

One last note. The number 20% in the report sounds like a rough approximation. The report probably means the increase was around 20%, maybe a little less, maybe a little more. So our answers of \$3.48/gal and \$4.176/gal could be a little less or a little more too. But they sound so perfectly correct. To be safe, we really ought to round off these answers, to something more general like around \$3.50/gal in 2011 or approximately \$4.20/gal in 2012. Using our " approximately equal to" symbol we write $P \approx \$3.50$/gal in 2011 and $P \approx \$4.20$/gal in 2012.

Homework

Start by doing Practice exercises #1-4 in the workbook.

Do you know ...

- The difference between a variable and a constant?
- The information needed to "name" a variable?
- Which variable is dependent and which variable is independent?
- What "domain" means?
- How to calculate percent increase?

⋆ The symbol for "approximately equal to"?

⋆ Why an approximate answer is often as good as we can get?

⋆ When to round your answer up or down instead of off?

⋆ What the term "precisely" refers to?

⋆ How to decide how precisely to round your answer?

⋆ indicates question based on *Prelude: approximation*

If you're not sure, work the rest of exercises and then return to these questions. Or, ask your instructor or a classmate for help.

Exercises

5. It's about time! For each story, try to figure out the answer to the question(s).

   (a) The Nussbaums planted a walnut tree years ago when they first bought their house. The tree was 5 feet tall then and has grown around 2 feet a year. The tree is now 40 feet tall. How long ago did the Nussbaums plant their walnut tree?

   (b) After his first beer, Stephen's blood alcohol content (BAC) was already .04 and as he continued to drink, his BAC level rose 45% per hour. Note that

   $$45\% = \frac{45}{100} = 45 \div 100 = .45$$

   What was Stephen's BAC after 1 hour? After 2 hours?

   Story also appears in 2.4 Exercises and 3.4 #1

   (c) When McKenna drives 60 mph (miles per hour) it takes her 20 minutes on the highway to get between exits, but when traffic is bad it can take her an hour. How slow is McKenna driving when traffic is bad? *Hint: can you figure out the distance between exits?*

   (d) The sun set at 6:00 p.m. today and I heard on the radio that it sets about 2 minutes earlier each day this time of year. In how many days will the sun set at 4:30 p.m.? *Bonus question: in what month is the story set?*

   Stories also appear in 1.1 #4

6. The temperature was 40°F at noon yesterday downtown Minneapolis but it dropped 3°F an hour in the afternoon. *Story also appears in 1.2 and 4.1 Exercises*

   (a) Which number is a constant in this story: the temperature (40) or the rate at which the temperature dropped (3)?

   (b) Name the variables, including units and dependence.

   (c) When did the temperature drop below freezing (32°F)?

7. Mrs. Nystrom's Social Security benefit was \$746.17/month when she retired from teaching in 2009. She had taught in elementary school since I was a girl. Benefits have increased by 4% per year. *Story also appears in 1.2 and 5.1 Exercises*

(a) Name the variables, including units and dependence.

(b) What was her benefit in 2012?

(c) When will her benefit pass \$900/month? A reasonable guess is fine.

8. Between e-mail, automatic bill pay, and online banking, it seems like I hardly ever actually mail something. But for those times, I need postage stamps. The corner store sells as many (or few) stamps as I want for 44¢ each but they charge a 75¢ convenience fee for the whole purchase. *Story also appears in 3.1 Exercises*

   (a) Identify and name the variables, including the units.

   (b) Which variable is dependent and which is independent?

   (c) How many stamps could I buy for \$10? Try to figure it out from the story.

9. Sofía bought her car new for \$22,500. Now the car is fairly old and just passed 109,000 miles. Sofía looked online and estimates the car is still worth \$5,700.

   Story also appears in Section 5.4

   (a) Identify and name the variables, including the units

   (b) Explain the dependence using a sentence of the form "___ is a function of ___"

   (c) What is a realistic number of miles for a car to drive? Express the domain as an inequality.

   (d) Sofía wonders when the car would be practically worthless, meaning under \$500. Make a reasonable guess.

10. For each story, name the variables including units and dependence.

   (a) The closer you sit to a lamp, the brighter the light is.

   Story also appears in 2.3 and 3.3 Exercises.

   (b) The thicker the piece of fish, the longer it takes to grill it.

   Story also appears in 2.3 and 3.5 Exercises.

   (c) Wind turbines are used to generate electricity. The faster the wind, the more power they generate. *Story also appears in 1.3, 2.4, and 3.3 Exercises.*

1.2 Tables and graphs

Lung cancer, chronic bronchitis, bad breath, stains on your clothes, and the expense. These are just a few of the consequences of smoking cigarettes. With what we know now about the dangers of smoking, are people smoking more or less than they were ten years ago, fifty years ago, or even one hundred years ago?

Reality is, we don't have information on each individual person's smoking rate, so we can't answer this question exactly. We do have information on the total number of cigarettes sold each year. So maybe we should look at that total. Uh oh, that isn't going to work. There are way more people now than there were fifty or a hundred years ago. So, even if the same percentage of people smoke, and even if they each smoke the same amount as their predecessors did, we would have a much bigger number of cigarettes smoked now just because there are more people now.

Turns out a reasonable measure is to compare the number of cigarettes smoked per year *per person*. By taking into account the number of people we will be able to see whether people are smoking more or less, on average. That's what we want.

The table showing the smoking rate for select years. The smoking rate is the average cigarettes per year per person (adults).

Year	1900	1915	1930	1940	1950	1965	1975	1990	2000	2006
Smoking rate	54	285	1,485	1,976	3,552	4,258	4,122	2,834	2,049	1,619

Source: Center for Disease Control and Prevention

To make sense of these numbers, suppose there are five friends. Three don't smoke at all, so that is 0 cigarettes in a year. Another smokes only occasionally, maybe 100 cigarettes a year. The fifth smokes "a pack a day," which adds up to 7,300 cigarettes in a year because

$$\frac{1 \text{ pack}}{\text{day}} * \frac{20 \text{ cigarettes}}{\text{pack}} * \frac{365 \text{ days}}{\text{year}} = 20 \times 365 = \frac{7{,}300 \text{ cigarettes}}{\text{year}}$$

(Not sure about this calculation? Not to worry. More about unit conversions in §1.4.) These five people smoke a total of

$$0 + 0 + 0 + 100 + 7{,}300 = 7{,}400 \text{ cigarettes per year}$$

so when we divide by the number of people we get

$$\frac{7{,}400 \text{ cigarettes per year}}{5 \text{ people}} = 7{,}400 \div 5 = 1{,}480 \text{ cigarettes per year per person}$$

which is less than the average of 1,619 cigarettes per year per person for 2006 (the last year the CDC published the data), which is the way things seem to be heading.

We can tell a lot of information from this table. For example, what was the smoking rate in 1965, and how does that compare to 2006? The answers appears in the table, a whopping 4,258 cigarettes per person in 1965 and 1,619 cigarettes per person in 2006.

When did the consumption first pass 3,000? That answer does not appear in the table, but we can use the information in the table to make a good guess. In 1940, there were an average of 1,976 cigarettes per person per year and by 1950, there were 3,552. Somewhere between 1940 and 1950 the number first climbed above 3,000. More specifically, the number we're looking for (3,000) is a lot closer to the 1950 figure (3,552) than to the 1940 figure (1,976). So, it would be reasonable to guess close to 1950. I'd say 1947. Of course, you might guess 1946 or 1948, or even 1949 and those would be good guesses too.

When did the consumption drop below 3,000 again? This answer also does not appear in the table, but falls somewhere between 1975 when consumption was 4,122 and 1990 when consumption was 2,834 . Here I'd guess just before 1990, say in 1989.

What's changing are the number of cigarettes smoked per person per year and the year. Those are our variables. The smoking rate is a function of year, and it's what we care about, so it's the dependent variable. Time, as measured in years, is the independent variable.

$$\begin{aligned} S &= \text{smoking rate (cigarettes per year per person)} \sim \text{dep} \\ Y &= \text{year (years since 1900)} \sim \text{indep} \end{aligned}$$

Quick note on how we deal with actual years. Since the year 0 doesn't make sense in this problem, it is convenient to measure time in years since 1900, since that's the earliest year mentioned. Officially we should rewrite our table as:

Y	0	15	30	40	50	65	75	90	100	106
S	54	285	1,485	1,976	3,552	4,258	4,122	2,834	2,049	1,619

Notice where the variable names are listed in the table. In a horizontal format like this table, the independent variable (Y) is in the top row, with the dependent variable (S) is in the bottom row. If you want to write your table in a vertical format, that's okay too. Just put the independent variable in the left column, with the dependent variable in the right column. It might help to remember that the independent variable goes first (either top or left) and the dependent variable follows (either bottom or right).

Where the variables go in a table is not something you can figure out. It's a **convention** – a custom, practice, or standard used within the mathematical community. Though based on reason, it often involves some arbitrary choice, which is why we can't figure it out. So, whenever some practice is introduced to you as a "convention," you need to memorize it.

Horizontal table format:

indep				
dep				

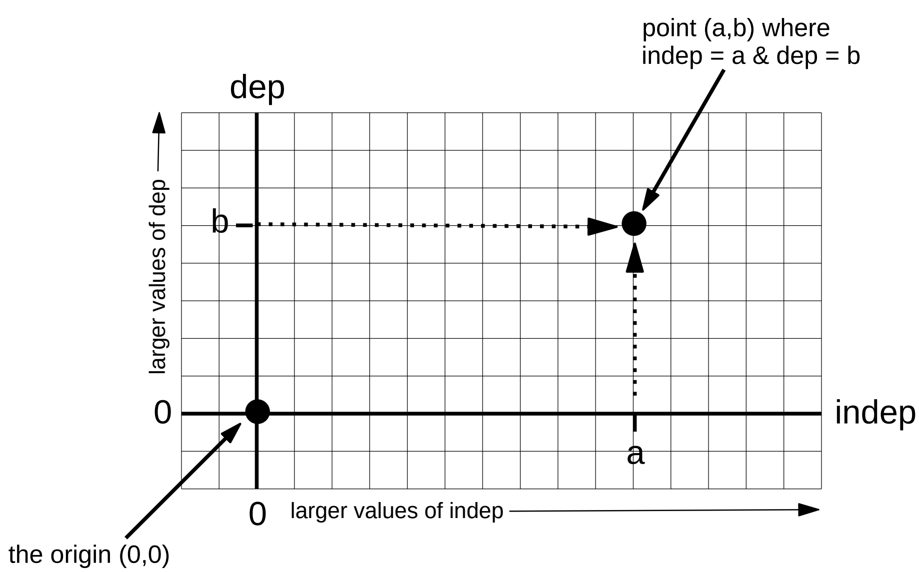
Vertical table format:

indep	dep

Tables are useful because they contain specific numbers, but it can be difficult to guess or see general trends. For that, a picture is worth a thousand words – or numbers, in this case. By "picture" we mean a graph of the function.

Throughout this text, we draw graphs by hand. On graph paper. Seriously. You might wonder why we do that when graphing calculators, spreadsheet programs, graphing "apps," or computer algebra systems all can draw graphs for us. The answer is we want to understand graphs better, and I promise that drawing them by hand will help you do that. (Different instructors have different opinions on the importance of graphing by hand, so be sure to ask your instructor what you are expected to do. Even if you're allowed to use some type of graphing technology, I strongly encourage you to practice drawing graphs by hand as well.)

There is a standard set-up for a graph.

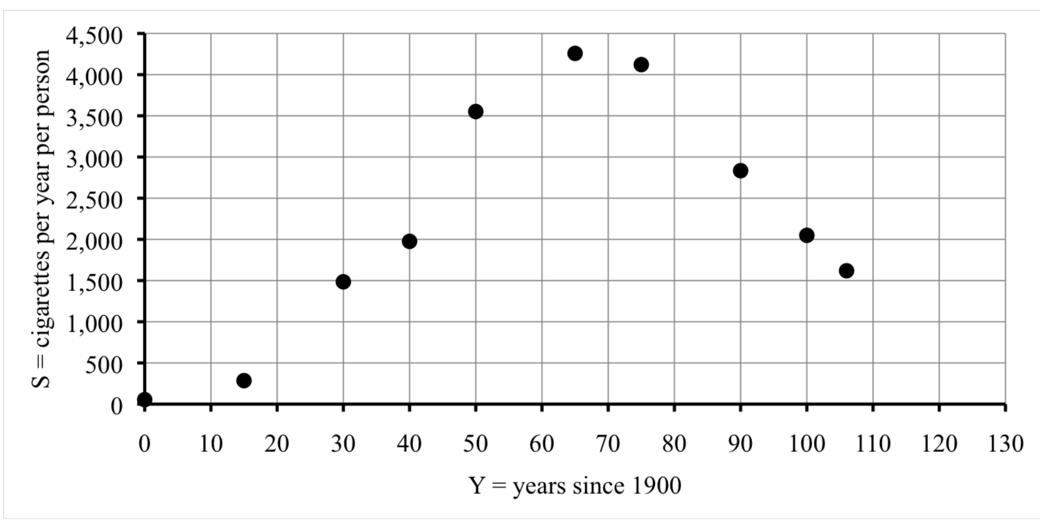


The graph is based on a horizontal line and vertical line, called the **axes**. Where they cross is a point called the **origin**. It represents where each variable is 0. By convention, the independent variable is measured along the horizontal axis, with larger values progressing to the right of the origin, and negatives to the left. Similarly, the dependent variable is measured along the vertical axis, with larger values progressing up from the origin, and negatives down. Each gridline counts the same number, called the **scale**, but the scale for the vertical may be different from the scale for the horizontal. Each pair of values of the independent and dependent variable from our table correspond to a point on our graph.

In the graph of smoking rates, the independent variable is Y, the year, so that goes on the horizontal axis for our graph. Our dependent variable is S, the smoking rate, so that goes on the vertical axis. For the scale, it works nicely to count by 10s for years and count by 500s for the smoking rate.

There's a certain amount of guess and check involved in figuring out a good scale for each axis. As a general rule of thumb we would like the graph to be as large as possible so we can see all of its features clearly. But, not so big that it runs off the graph paper. What matters is that the gridlines are evenly scaled and that they can handle large enough numbers. Speaking of which, it's a good idea to leave a little room to extend the graph a little further than the information we have in the table, in case we get curious about values beyond what we have already.

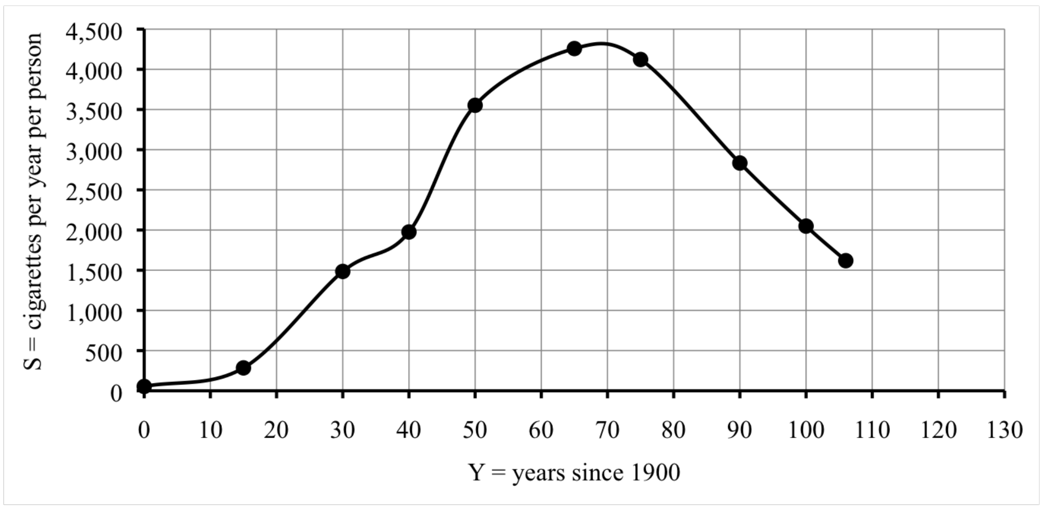
With realistic numbers it's normal to have numbers in the table that are not exactly where the gridlines are. It is very helpful to count by round numbers (2s, 5s, 10s, etc.) because it makes guessing in between easier. Easier for you drawing the graph. Easier for someone reading your graph.



To plot each point, we start at the origin and move right to that Y-value, and then up to that S-value. When a value doesn't land exactly on a grid mark, we have to guess in between. For example, in 1900, when $Y = 0$ so we don't move right at all, just up to $S = 54$. The first labeled gridline on our graph is 500. Where's 54? It's between 0 and 500, very close to 0. Our point is just a tiny bit above the origin. In 1915 we have $Y = 15$.

Our labeled gridlines are for 10 and 20, so 15 must land halfway in between. The smoking rate to 285, which is around halfway between 0 and 500. Etc.

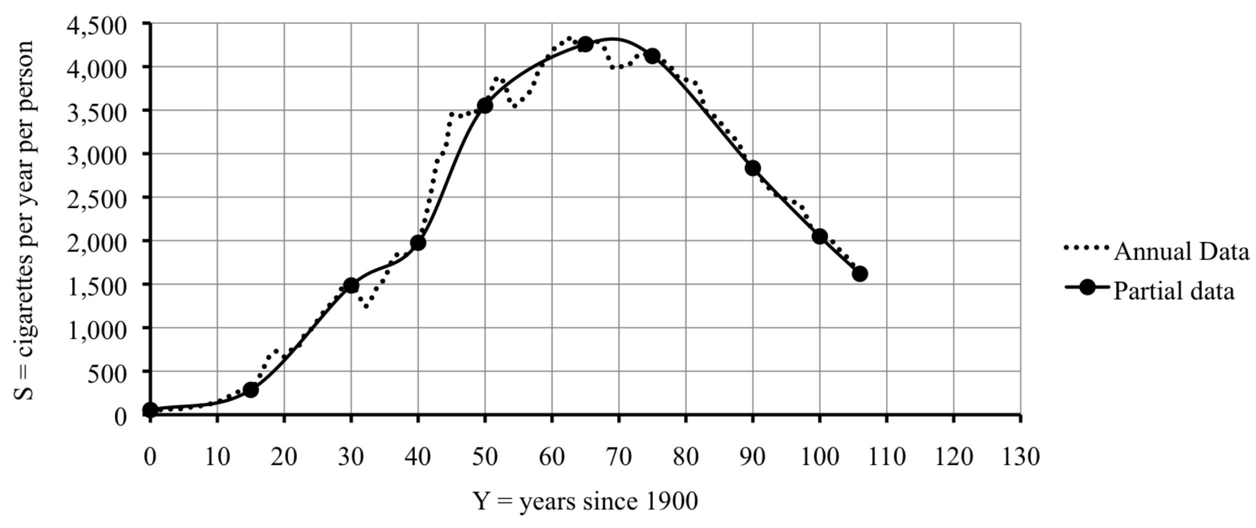
What we have so far is a **scatter plot** of points. Can you see why it's called that? Anyway, our whole goal here was to be able to understand smoking rates better by having a graph. You may already begin to see a curve suggested by the points. Time to draw it in. I don't mean drawing a line between each pair of points, like you do in the children's game "connect the dots." That isn't quite right. It was probably more of a continuous trend and so the graph should be smoother.



When we draw in this smooth curve for the graph, what we are really doing is making a whole lot of guesses all at once. For example, from the table we guessed that the smoking rate passed 3,000 in around 1947, and dropped back to that level in around 1989. What does the graph show? If we look where the horizontal gridline for 3,000 crosses our graph, it crosses in two places. First, between the vertical gridlines for 40 and 50, and perhaps slightly closer to 50. I'd say $Y \approx 47$, in the year 1947. Sure. The second time is between the gridlines for 80 and 90, much closer to 90. Looks like $Y \approx 88$, in the year 1988. We guessed 1989. Close enough.

Don't forget that when we drew in that curve it was really just a guess. We're sure about the points we plotted, but we're only guessing about where to draw the curve in. That means we're not sure about the other points. If we knew a lot more points we could have a more accurate graph.

Turns out more data is available from the CDC. The full table of data from the CDC shows that consumption first topped 3,000 as early as 1944. Here's an example where the history tells you more than the mathematics as cigarette consumption rose sharply during World War II. Our guess about 1988 or 1989 was spot on. Look at how the graph from the full data (the dotted line) compares to our guess.



Homework

Start by doing Practice exercises #1-4 in the workbook.

Do you know ...

- Where the independent and dependent variables appear in a table and in a graph?
- How to guess values from a table or from a graph?
- How to make a graph from a table?
- Why we start each axis at 0?
- What we mean by scaling an axis evenly?
- How to make a table and then a graph from a story?
- Why we draw in a smooth line or curve connecting the points?
- What type of graphing technology, if any, you're allowed to use? *Ask your instructor.*

  If you're not sure, work the rest of exercises and then return to these questions. Or, ask your instructor or a classmate for help.

Exercises

5. The table lists estimates of Earth's population, in billions, for select years since 1800.

Year	1800	1850	1900	1950	1970	1990	2000
Population	.98	1.26	1.65	2.52	3.70	5.27	6.06

Source: "The World at Six Billion" United Nations report

Story also appears in 1.3 Exercises

Use the table to find or reasonably guess the answers to the following questions.

(a) What was the population of Earth in 1850?

(b) What do you think the population of Earth was in 1860?

(c) What do you think the population of Earth was in 1960?

(d) In what year do you think the population of Earth first exceeded 2 billion?

(e) In what year do you think the population of the world will exceed 7 billion?

(f) Identify the variables, including units and dependence.

6. Your local truck rental agency lists what it costs to rent a truck (for one day) based on the number of miles you drive the truck.

Distance driven (miles)	50	100	150	200
Rental cost ($)	37.50	55.00	72.50	90.00

Story also appears in 1.3 and 4.4 Exercises

Use the table to find or reasonably guess the answers to the following questions.

(a) How much does it cost to rent a truck if you drive it 100 miles?

(b) How many miles did you drive a truck costing $90.00 to rent?

(c) If you rent a truck and drive it 75 miles, how much do you think it will cost?

(d) If you rent a truck and drive it 10 miles, how much do you think it will cost?

(e) If you rent a truck and it costs $60.00, about how many miles was it driven?

(f) Identify the variables, including units, realistic domain, and dependence.

(g) Draw a detailed graph illustrating the dependence based on the points given in the table. Be sure your axes are labeled and evenly scaled. Sketch in a smooth curve connecting the points.

(h) Use your graph to check your answers to the questions. Modify, if necessary.

7. The temperature was 40°F at noon yesterday downtown Minneapolis but it dropped 3°F an hour in the afternoon. *Story also appears in 1.1 and 4.1 Exercises*

   (a) Make a table of reasonable values.

   (b) Draw a graph illustrating the dependence. Count time in hours after noon.

   (c) According to your table and graph, when did the temperature drop below freezing (32°F)?

   (d) According to your graph, when did the temperature drop below 0°F. Does that seem realistic? *Here in the midwest there are no oceans or mountains to moderate large temperature changes.*

8. Mrs. Nystrom's Social Security benefit was $746.17/month when she retired from teaching in 2009. She had taught in elementary school since I was a girl. Benefits have increased by 4% per year. *Story also appears in 1.1 and 5.1 Exercises*

   (a) Make a table of reasonable values using N for Mrs. Nystrom's benefits (in dollars) and Y for time (in years since 2009).

   (b) Draw a graph illustrating the dependence. Scale your graph to show up through the year 2020 and $1,200.

   (c) According to your table and graph, when did her benefit pass $900/month?

   (d) If you extend your graph to 2020, what would you estimate Mrs. Nystrom's benefit will be then, assuming these increases continue?

9. The table adapted from shows the "heat index" as a function of humidity at an air temperature of 88°F. With up to about 40% humidity, 88°F feels like it's 88°F. But if the humidity rises to 60%, then it feels like it is 95°F; that is, the heat index is 95°F.

Humidity (%)	50	60	70	85	90	95
Heat index (°F)	91	95	100	110	113	117

Source: National Oceanic and Atmospheric Administration

All of the following questions refer to situations when the air temperature is 88°F.

   (a) What is the heat index when the humidity level is 70%?

   (b) At what humidity level does it feel more like 98°F?

   (c) Heat exhaustion is likely to occur when the heat index reaches 105°F. At what humidity level will heat exhaustion likely occur?

   (d) The heat index is considered danger in the range from 105°F to 129°F. What range of humidity levels are considered dangerous?

(e) What do you think the heat index would be at 99% humidity?

(f) Identify the variables, including units, realistic domain, and dependence.

(g) Draw a detailed graph illustrating the dependence based on the points given in the table. Be sure your axes are labeled and evenly scaled. Sketch in a smooth curve connecting the points.

(h) Use your graph to check your answers to the questions. Modify, if necessary.

1.3 Rate of change

A diver bounces on a 3-meter springboard. Up she goes. A somersault, a twist, then whoosh, into the water. The table shows the diver's height as a function of time,

T	0	.2	.4	.6	.8	1.0	1.2	1.4
H	3.00	3.88	4.38	4.48	4.20	3.52	2.45	1.00

where

$$H = \text{diver's height (meters)} \sim \text{dep}$$
$$T = \text{time (seconds)} \sim \text{indep}$$

In case you're wondering, 3 meters is nearly 10 feet up and the highest height listed, 4.48 meters, is close to 15 feet above the water. More on how we figured those numbers out in the next section, but thought you might like to know.

How fast is she moving? The diver starts at 3 meters, which is the height of the springboard, and .2 seconds later she's up to 3.88 meters. That means during the first .2 seconds, the diver went up

$$3.88 - 3 = .88 \text{ meters}$$

Her speed is

$$\frac{.88 \text{ meters}}{.2 \text{ seconds}} = 0.88 \div .2 = 4.4 \text{ meters/sec}$$

What about during the next .2 seconds? Does she move faster, slower, or the same? During this time, her height changed from 3.88 meters to 4.38 meters. In these .2 seconds she rose

$$4.38 - 3.88 = .50 \text{ meters}$$

That's less than before (since $.50 < .88$), which means so she is going slower. Let's double check by calculating her speed.

$$\frac{.50 \text{ meters}}{.2 \text{ seconds}} = .50 \div .2 = 2.5 \text{ meters/sec}$$

Yup, slower.

Let's take a look at this calculation again. Here's what we did.

$$\text{speed} = \frac{4.38 - 3.88 \text{ meters}}{.4 - .2 \text{ seconds}} = \frac{.50 \text{ meters}}{.2 \text{ seconds}} = .50 \div .2 = 2.5 \text{ meters/sec}$$

There is a way to do the entire calculation at once on your calculator.

$$(4.38 - 3.88) \div (.4 - .2) = 2.5 \text{ meters/sec}$$

See how we put parentheses around both the top and bottom of the fraction? We needed them to force the calculator to do the subtractions first and division second. The usual order of operations would do it the other way around: multiplication and division before addition and subtraction. (If you need a reminder, the full list of the order of operations appears later, in Section 1.5.) Because the top and bottom of the fraction each have meaning in the story, we continue to calculate them separately, but feel free to do the whole calculation at once if you prefer.

Notice that we are subtracting **like terms**: meters from meters and then seconds from seconds. It would not make sense to mix. Think:

$$\text{children} - \text{cookies} = \text{crying}$$

so we don't want to mix units because that would be like taking cookies away from children.

In our story, we calculated the speed of the diver. In general, that number is the **rate of change** of the function over that interval of values.

RATE OF CHANGE FORMULA: $\quad \text{rate of change} = \dfrac{\text{change dep}}{\text{change indep}}$

Notice how the change in dependent variable (height, in meters) is on top of the fraction and the change in independent variable (time, in seconds) is on the bottom. That makes sense in our example because speed is measured in meters/second. The units can help you keep that straight.

$$\text{units for rate of change} = \frac{\text{units for dep}}{\text{units for indep}}$$

Back to our diver. During the next time interval she's moving even slower.

$$\text{speed} = \frac{4.48 - 4.38 \text{ meters}}{.6 - .4 \text{ seconds}} = \frac{.1 \text{ meters}}{.2 \text{ seconds}} = .1 \div .2 = .5 \text{ meters/sec.}$$

And look what happens when we calculate her speed during the next time interval.

$$\text{speed} = \frac{4.20 - 4.48 \text{ meters}}{.8 - .6 \text{ seconds}} = \frac{-.28 \text{ meters}}{.2 \text{ seconds}} = (\text{-}).28 \div .2 = -1.4 \text{ meters/sec}$$

What does a negative speed mean? During this time interval her height drops. She's headed down towards the water. Her speed is 1.4 meters/sec downward. The negative tells us her height is falling. What goes up, must come down. Sure enough.

You may notice that the signs for subtraction and negation each look like $-$. On the calculator these are two different keys. The subtraction key reads just $-$. The negation key often reads (-) and is done before the number. This does not mean you type in parentheses, just hit the one key that is labeled (-) already. (If your calculator does not have a key labeled (-), look for a key labeled +/- instead. That is not three keys, just one labeled +/- Often that key needs to follow the number, so enter

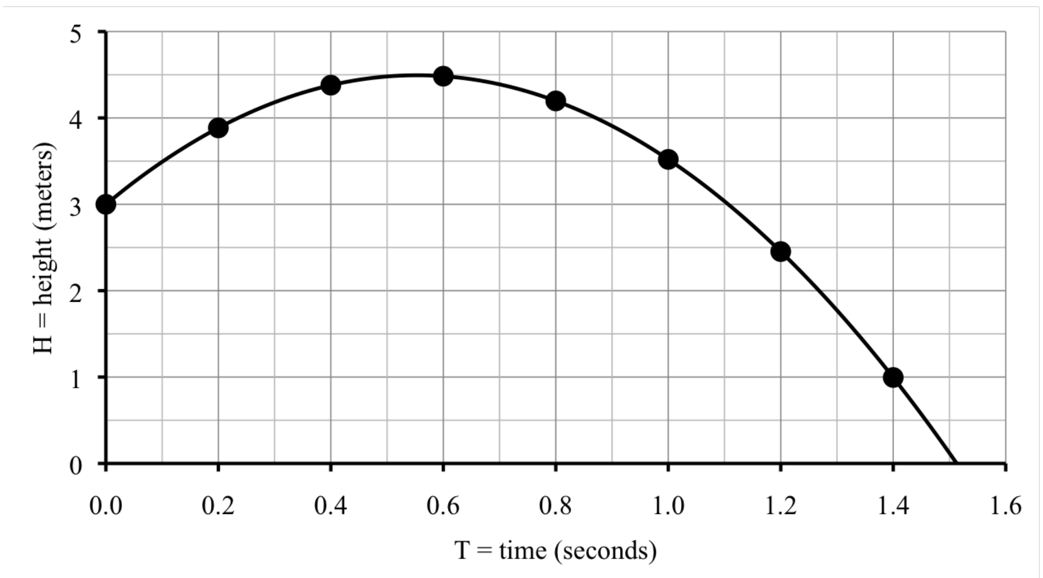
$$.28 \; +/- \; \div .2 =$$

You should get -1.4 meters/sec again.)

Here are the speeds included in our table.

T	0	.2	.4	.6	.8	1.0	1.2	1.4
H	3	3.88	4.38	4.48	4.20	3.52	2.45	1
Speed		$\rightarrow$ 4.4	$\rightarrow$ 2.5	$\rightarrow$ .5	$\rightarrow$ -1.4	$\rightarrow$ -3.4	$\rightarrow$ -5.35	$\rightarrow$ -7.25

Let's graph our function. Notice that time is on the horizontal axis because it's the independent variable and height is on the vertical axis because it's our dependent variable: height depends on time.



As usual we drew in a smooth curve connecting the points, which illustrates our best guesses for the points we don't know and we continued the graph until the height was zero (when the diver hits the water). The values from our table are indicated with big points to help explain what's going on.

There is a way to see the rate of change from the graph. In the case of our diver, the graph looks like a hill. The curve goes uphill at first. Between the first two points it is rather steep and the rate of change is 4.4 meters/sec there. The next segment is less steep and that's where the rate of change is less, down to 2.5 meters/sec. The third line segment is almost flat and that's where the rate of change is only 0.5 meters/sec. Aha. The rate of change corresponds to how steep the curve is.

We notice the same connection between the rate of change and steepness of the curve for the downhill portion, only this time all the rates of change are negative. The first downhill segment is not very steep and the rate of change is -1.4 meters/sec there. The next downhill segment has rate of change -3.4 meters/sec and the graph is steeper. The next two downhill segments are steeper and steeper yet and this time with rates of change -5.35 and -7.25 meters/sec.

A little more vocabulary here. For the uphill portion of the graph, from 0 to just before .6 seconds, the rate of change is positive. The function is **increasing** there: as the independent variable gets larger, so does the dependent variable. After about .6 seconds, the graph is downhill and the rate of change is negative. The function is **decreasing** there: as the independent variable gets larger, the dependent variable gets smaller.

When does the diver's height stop increasing and start decreasing? When she's at the highest height, some time just before .6 seconds into her dive. Before then her rate of change is positive. After that time her rate of change is negative. So, at the highest height her rate of change is probably equal to zero. Does that make sense? Think about watching a diver on film in very slow motion. Up, up she goes, then almost a pause at the top, and then down, down, into the water. At the top of her dive it's as if she stands still for an instant. That would correspond to zero speed.

Homework

Start by doing Practice exercises #1-4 in the workbook.

Do you know . . .

- How to calculate rate of change between two points? *Ask your instructor if you need to remember the formula or if it will be provided during the exam.*
- What the rate of change means in the story?
- How we can use the rate of change to estimate values?
- When a function is increasing or decreasing, and the connection to the rate of change?
- Why the rate of change is zero at the maximum (or minimum) value of a function?
- What the connection is between rate of change and the steepness of the graph?

  If you're not sure, work the rest of exercises and then return to these questions. Or, ask your instructor or a classmate for help.

Exercises

5. Look back at the springboard diver example in this section.

   (a) Check the other rates of change given in the table.

   (b) Approximately how fast is the diver moving as she enters the water? Use that her height at 1.4 seconds is 1 foot above water (given earlier), but also her height at 1.5 seconds is just .12 feet above water.

6. Your local truck rental agency lists what it costs to rent a truck (for one day) based on the number of miles you drive the truck.

Distance driven (miles)	50	100	150	200
Rental cost ($)	37.50	55.00	72.50	90.00

Story also appears in 1.2 and 4.4 Exercises

   (a) Calculate the rate of change for each time period.

   (b) Can you use figure out what it probably costs to rent a truck to drive 75 miles?

   (c) There must be some sort of fixed price plus a per mile price. Can you figure out what that fixed price must be?

7. Wind turbines are used to generate electricity. A few values are recorded in the table

wind speed (mph)	0	10	20	30
electricity (watts)	0	2,400	19,200	64,800

Story also appears in 1.1, 2.4, and 3.3 Exercises

   (a) Name the variables, including units and dependence.

   (b) Plot the points from the table on a graph.

   (c) Calculate the rate of change in electricity as a function of wind speeds from 0 to 10 mph. Sketch in the line segment connecting those two points on the graph.

   (d) Repeat for wind speeds from 10 to 20 mph. Is the electricity produced increasing faster or increasing slower than for lower wind speeds.

   (e) Repeat for wind speeds from 20 to 30 mph. Comment again on how the rate of change compares to earlier rates of change.

8. The table lists estimates of Earth's population, in billions, for select years since 1800.

year	1800	1850	1900	1950	1970	1990	2000
population	.98	1.26	1.65	2.52	3.70	5.27	6.06

Source: "The World at Six Billion" United Nations report, 1999

Story also appears in 1.2 Exercises

During which period of time was the Earth's population increasing the fastest? Calculate the rates of change for each time period to decide. (Or, explain some other way of deciding.)

9. A company produces backpacks. The more they make, the less it costs for each one. For example, if they produce 10 backpack it would cost \$39 each. For 40 backpacks, they would cost \$18 each. By 70 backpacks, the unit cost is down to \$15 each. At 100 backpacks, the unit cost is \$30 each. *Story also appears in 3.5 Exercises*

   (a) Name the variables and summarize the information in a table.

   (b) Calculate the rates of change between 10 and 40 backpacks, between 40 and 70 backpacks, and between 70 and 100 backpacks.

   (c) For which range of values does the cost per backpack decrease?

   (d) Any ideas why the cost might increase?

   (e) Draw a graph illustrating the dependence. Try for a nice, smooth curve.

   (f) Approximately how many backpacks does the company have to make to keep the cost per backpack as small as possible?

10. The public beach near Paloma's house has lost depth (measured from the dunes to the high water mark) due to erosion since they started keeping records 60 years ago. The table shows a few values. There D is the depth of the beach in feet, and Y is the year, measured since 60 years ago.

year	60 years ago	30 years ago	10 years ago	now
Y	0	30	50	60
D	435	322.5	247.5	210

Story also appears in 4.3 Exercises

   (a) Calculate the rates of change for each time period.

   (b) Explain why the rates of change should be negative.

   (c) Approximately how many feet a year is the beach eroding?

   (d) Draw a graph showing how the beach depth has changed over the past 60 years.

1.4 Units

We know 5 city blocks and 5 miles are very different lengths to walk; \$5 and 5¢ are very different values of money; 5 minutes and 5 years are very different amounts of time to wait – even though all of these quantities are represented by the number 5. Every variable is measured in terms of some unit. Since there are often several different units available to use it is important when naming a variable to state which units we are choosing to measure it in.

In the last section we examined the height of a springboard diver and her speed in the air. But, how high is 3 meters? How fast is 4.4 meters per second?

The metric unit of length called a meter is just over 3 feet (a yard). Let's use

$$1 \text{ meter} \approx 3.281 \text{ feet}$$

We can use this conversion to change 3 meters to feet.

$$3 \text{ meters} * \frac{3.281 \text{ feet}}{1 \text{ meter}} = 3 \times 3.281 = 9.843 \text{ feet} \approx 9.8 \text{ feet}$$

Since our conversion is just approximate, we rounded off our answer too.

See that fraction? The 3.281 feet on the top and the 1 meter on the bottom are just two different ways of saying approximately the same distance. In other words,

$$\frac{3.281 \text{ feet}}{1 \text{ meter}} \approx 1$$

A fraction where the top and bottom are equal quantities expressed in different units is sometimes called a **unit conversion fraction**. Because it's equal to 1 (or at least very close to 1), multiplying by the unit conversion fraction doesn't change the value, just the units.

You might wonder how we knew to put the feet on the top and the meters on the bottom. One reminder for how this works is to think fractions. It's like the meters on the top and bottom cancel, leaving the units as feet.

$$\frac{3 \cancel{\text{meters}}}{1} * \frac{3.281 \text{ feet}}{1 \cancel{\text{meter}}} = 3 \times 3.281 \approx 9.8 \text{ feet}$$

One more idea to keep in mind when converting units: a few large things equals a lot of small things. Instead of buying a lot of small bags of ice to fill your cooler, you can buy a few larger bags of ice instead. In our example, a meter is much bigger than a foot. So it makes sense that a small number of meters (3 meters) equalled a larger number of feet (9.8 feet). That might seem backwards, but that's how it works.

Of course, 9.8 feet might sound like a funny answer. We're much more used to a whole number of feet and then the fraction in inches. It's 9 feet and some number of inches. To figure out the inches we look at the decimal part $9.8 - 9 = .8$. That's the part we need to convert to inches. Since there are 12 inches in a foot, we use the (exact) conversion

$$1 \text{ foot} = 12 \text{ inches}$$

to get

$$.8 \cancel{\text{feet}} * \frac{12 \text{ inches}}{1 \cancel{\text{foot}}} = .8 \times 12 = 9.6 \text{ inches} \approx 10 \text{ inches}$$

Quick caution here. We rounded off 9.843 to get 9.8 and then just used the .8 to find the extra inches. Maybe we should have used the .843 instead. Here's what happens.

$$.843 \cancel{\text{feet}} * \frac{12 \text{ inches}}{1 \cancel{\text{foot}}} = .843 \times 12 = 10.116 \text{ inches} \approx 10 \text{ inches}$$

Phew! Either way, the board is about 9 feet and 10 inches high. The common shorthand for this answer is 9'10". (That's pronounced *9 foot 10*, as in our team's new center is *6 foot 7*.) The ' symbol indicates feet and " indicates inches.

The highest height we had recorded for the diver was 4.48 meters. Now we know that's

$$4.48 \cancel{\text{meters}} * \frac{3.281 \text{ feet}}{1 \cancel{\text{meter}}} = 4.48 \times 3.281 = 14.69888\ldots \text{ feet} \approx 14.7 \text{ feet}$$

In feet and inches, that's about 14 feet, 8 inches because

$$.69888 \cancel{\text{feet}} * \frac{12 \text{ inches}}{1 \cancel{\text{foot}}} = .69888 \times 12 = 8.38656 \text{ inches} \approx 8 \text{ inches}$$

The diver's highest height was around 14'8".

You might have guessed that 14.7 feet would be 14'7". I mean, that sort of looks obvious. The reason it's not is because decimal numbers are based on 10. The .7 really means $\frac{7}{10}$. But inches are based on 12. Seven inches means

$$7'' = \frac{7}{12} = 7 \div 12 = .58333\ldots \approx .6$$

We wanted .7 so that's not it. Our answer of 8" worked just fine since

$$8\text{"} = \frac{8}{12} = 8 \div 12 = .66666\ldots \approx .7$$

What about the diver's speed? During the first .2 seconds we calculated her speed as 4.4 meters per second. How fast is that? We can certainly convert to feet per second.

$$\frac{4.4 \cancel{\text{meters}}}{\text{second}} * \frac{3.281 \text{ feet}}{1 \cancel{\text{meter}}} = 4.4 \times 3.281 = \frac{14.4364 \text{ feet}}{\text{second}}$$

Does that help us understand how fast she's going? Maybe a little. But, we're probably most familiar with speeds measured in miles per hour, that's what **mph** stands for.

Let's convert to miles per hour. First, use that

$$1 \text{ minute} = 60 \text{ seconds}$$

to get

$$\frac{14.4364 \text{ feet}}{\cancel{\text{second}}} * \frac{60 \cancel{\text{seconds}}}{1 \text{ minute}} = 14.4364 \times 60 = \frac{866.184 \text{ feet}}{\text{minute}}$$

The larger number makes sense here because she can go more feet in a minute than in just one second.

Next, use that

$$1 \text{ hour} = 60 \text{ minutes}$$

to get

$$\frac{866.184 \text{ feet}}{\cancel{\text{minute}}} * \frac{60 \cancel{\text{minutes}}}{1 \text{ hour}} = 866.184 \times 60 = \frac{51{,}971.04 \text{ feet}}{\text{hour}}$$

Again, the larger number makes sense because she can go more feet in an hour than in just one minute.

Last, we need to convert to miles. Turns out that

$$1 \text{ mile} = 5{,}280 \text{ feet}$$

and so

$$\frac{51{,}971.04 \cancel{\text{feet}}}{\text{hour}} * \frac{1 \text{ mile}}{5{,}280 \cancel{\text{feet}}} = 51,971.04 \div 5{,}280 = \frac{9.843 \text{ miles}}{\text{hour}} \approx 10 \text{ mph}.$$

This time we got a smaller number because she can go a lot fewer miles in an hour compared to feet in an hour. Notice how we needed to divide by 5,280. Numbers on top of the fraction multiply. Those on the bottom divide.

We can do this entire calculation all at once. Notice how all of the units cancel to leave us with miles per hour.

$$\frac{4.4 \cancel{\text{meters}}}{\cancel{\text{second}}} * \frac{3.281 \cancel{\text{feet}}}{1 \cancel{\text{meter}}} * \frac{60 \cancel{\text{seconds}}}{1 \cancel{\text{minute}}} * \frac{60 \cancel{\text{minutes}}}{1 \text{ hour}} * \frac{1 \text{ mile}}{5,280 \cancel{\text{feet}}}$$
$$= 4.4 \times 3.281 \times 60 \times 60 \div 5{,}280 = 9.843 \text{ mph} \approx 10 \text{ mph}.$$

Right before the diver hit the water she was going around 7.25 meters per second. How fast is that in mph? Ready for it all in one line? Here it is.

$$\frac{7.25 \cancel{\text{meters}}}{\cancel{\text{second}}} * \frac{3.281 \cancel{\text{feet}}}{1 \cancel{\text{meter}}} * \frac{60 \cancel{\text{seconds}}}{1 \cancel{\text{minute}}} * \frac{60 \cancel{\text{minutes}}}{1 \text{ hour}} * \frac{1 \text{ mile}}{5,280 \cancel{\text{feet}}}$$
$$= 7.25 \times 3.281 \times 60 \times 60 \div 5,280 = 16.2185\ldots \approx 16 \text{ mph}$$

If you're having trouble setting up unit conversions, remember to write down the units so you can see how they cancel. If you can't remember a number for a unit conversion, like 5,280 feet for one mile, try searching online.

Homework

Start by doing Practice exercises #1-4 in the workbook.

Do you know ...

- How to convert from one unit of measurement to another?
- What a unit conversion fraction is?
- Why multiplying by a unit conversion fraction doesn't change the amount, just the units?
- How to connect repeated conversions into one calculation?
- Why if we convert an amount to a larger unit, we use a smaller number?
- How many seconds in a minute, minutes in an hour, hours in a day, days in a year, inches in a foot, feet in a mile, and other common conversions?

  Ask your instructor which common conversions you need to remember, and whether any conversion formulas will be provided during the exam.
- How to convert between English and metric measurements?

  Again, ask your instructor which metric conversions you need to remember, and whether any conversion formulas will be provided during the exam.

  If you're not sure, work the rest of exercises and then return to these questions. Or, ask your instructor or a classmate for help.

Exercises

5. In August 2008, United States swimmer Michael Phelps set the world record for the 200 meter individual medley swimming it in 1 minute, 54.80 seconds.

   Source: Wikipedia (World record progression 200 metres IM)

   (a) Convert Phelps' time into minutes.

(b) How fast did Phelps' swim, as measured in meters/min?

(c) Convert Phelps' speed to mph. Use 1 mile $\approx 1,609$ meters.

In August 2012, Phelps improved his time and won Olympic gold, but failed to break the world record his teammate Ryan Lochte has set a year earlier of 1 minute, 54 seconds.

(d) Convert Lochte' time into minutes.

(e) How fast did Lochte' swim, as measured in meters/min?

(f) Convert Lochte' speed to mph.

6. (a) The typical weight limit for a suitcase on flights within Africa is 20 kg. How many pounds is that? Use 1 kilogram ≈ 2.2 pounds.

   (b) How many servings are in a 20 ounce package of cookies where a serving size is 3 cookies and each cookie weighs 11 grams? Use 1 ounce = 28.3 grams.

   (c) My corner convenience store sells a "thirst quencher" size of soft drink; it holds 64 (fluid) ounces. If a can of soft drink is 12 (fluid) ounces, how many cans are in the "thirst quencher"?

7. (a) The football coach wants everyone to sprint three-quarters of a mile, up and back on the field which is labeled in yards. How many yards are in three-quarters of a mile?

   (b) The quilt pattern calls for .375 yards of calico fabric. How many feet is .375 yards?

   (c) The website said that basil plants should be .35 feet tall a month after germinating. How many inches is .35 feet?

8. Authorities are tracking down the source of a pollution spill on a nearby river. They suspect that the local plant is inadvertently leaking waste water. Last week they found 35 minutes of waste water flow on Monday, 1 hour and 11 minutes on Tuesday, 1/4 hour on Wednesday (that's .25 hours in decimal), none on Thursday, and then 98 minutes Friday.

   (a) Convert units as needed to complete the following table showing each time in minutes, each time in hours, and each time in hours and minutes (H:MM format). *Hint: 15 minutes in H:MM format would be 0:15*

Day	Mon	Tue	Wed	Thu	Fri
Minutes	35			0	98
Hours			.25		
H:MM		1:11			

(b) Calculate the total waste water flow measured last week.

9. If your heart beats around 70 times a minute, how many times does it beat in a week? A year?

10. (a) Harold's Physics textbook says an object is thrown into the air at 36 feet per second. To understand how fast that is, convert to mph.

    (b) Harold's History textbook mentions that in 1800 the city encompassed about 6,000 acres. How many square miles is that? Use 1 square mile = 640 acres.

    (c) Harold's Economics textbooks lists the recent high price of crude oil at $100 per barrel. He'd like to know what that means in $/gallon of gasoline. It turns out that 1 barrel of crude oil produces about 19.4 gallons of gasoline.

1.5 Metric prefixes and scientific notation

Tara is working on a big project at work. She wants to back up her files to her online drop box. The site says she has 72 GB of memory remaining. Tara has about 200 files at an average of 42.3 MB each that she would like to upload. Will she have room?

To answer Tara's question we need to know that GB is short for "gigabyte" and MB is short for "megabyte." A **byte** is a very small unit of computer memory storage space just enough for about one letter. You may have heard the word "mega" used to mean "really big." There's a reason for that. **Mega** is short for 1 million. That's pretty big. But **giga** stands for 1 billion, so that's even bigger. (Maybe it's time for a gigamall?)

$$\begin{array}{lcll} \textbf{mega} & = & 1\ \textbf{million} & = \quad 1{,}000{,}000 \\ \textbf{giga} & = & 1\ \textbf{billion} & = \quad 1{,}000{,}000{,}000 \end{array}$$

What all this means is Tara has

$$72 \text{ GB} = 72 \text{ billion bytes} = 72{,}000{,}000{,}000 \text{ bytes}$$

of memory remaining. She would like to save 200 files at 42.3 MB each which comes to

$$200 \times 42.3 = 8{,}460 \text{ MB}$$

which is really

$$8{,}460 \text{ MB} = 8{,}460 \text{ million bytes} = 8{,}460{,}000{,}000 \text{ bytes}$$

Finding it hard to compare all those zeros? Try this.

$$8{,}460 \text{ MB} = 8{,}460{,}000{,}000 \text{ bytes} = 8.46 \text{ GB} \ < 9 \text{ GB}$$

So Tara wants to store less than 9 GB of information and she has 72 GB remaining. She has plenty of room. Save away.

Tara also needs to download about 700 MB of rather high quality photos. Her computer downloads photos at 187 kbps. How long will it take? (And does she have time to run for a coffee?) The mysterious **kbps** stands for kilobits (Kb) per second. Like mega and giga, the word "kilo" stands for a large number, in this case 1,000.

$$\textbf{kilo} \quad = \quad 1\ \textbf{thousand} \quad = \quad 1{,}000$$

That's the same word "kilo" as in kilometer (about $1/2$ mile) or kilogram (about $1/2$ pound) and there's good reason for that as

$$\begin{aligned}\textbf{kilometer} &= \text{1,000 meters}\\ \textbf{kilogram} &= \text{1,000 gram}\end{aligned}$$

Perhaps you've seen the letter **K** as short for a thousand? That's where it comes from.

(Okay, I have to mention something here. Kilo by itself is pronounced "KEE-loh," but kilogram is pronounced "KIL-uh gram," and kilometer is pronounced "ki-LOM-i-ter." Well, around these parts at least.)

Back to Tara. Her download speed is 187 kilobits per second. Perhaps this is the right moment to mention that a **bit** is even smaller than a **byte**.

$$1 \text{ byte} = 8 \text{ bits}$$

How long will it take Tara to download 700 MB? We can think of this calculation as a unit conversion by imagining.

$$187 \text{ kilobits} = 1 \text{ second}$$

Watch.

$$700\ \cancel{\text{MB}} * \frac{1{,}000{,}000\ \cancel{\text{bytes}}}{1\cancel{\text{MB}}} * \frac{8\ \cancel{\text{bits}}}{1\ \cancel{\text{byte}}} * \frac{1\ \cancel{\text{kilobit}}}{1{,}000\ \cancel{\text{bits}}} * \frac{\text{second}}{187\ \cancel{\text{kilobits}}}$$
$$= 700 \times 1{,}000{,}000 \times 8 \div 1{,}000 \div 187 = 29,946.524\ldots \text{ seconds}$$

Let's convert to a more reasonable unit.

$$29{,}946.524\ldots\cancel{\text{seconds}} * \frac{1\ \cancel{\text{minute}}}{60\ \cancel{\text{seconds}}} * \frac{1 \text{ hour}}{60\ \cancel{\text{minutes}}}$$
$$= 29,946.524\ldots \div 60 \div 60 = 8.318\ldots \approx 8.32 \text{ hours}$$

It will take Tara over 8 hours to download those photos. Perhaps Tara should compress the photos into a zip file or use lower resolution or find a way to download faster. Or, she can just set it up to download overnight.

Quick note. The **metric system of measurement**, or **international system of units (ISU)**, is the official system of nearly all countries, the United States being a notable exception. Science, international trade, and most international sporting events like the Olympics are based in the metric system. In the United States system (known officially at the **British system** or, since the British stopped using it, the **imperial system of measurement**), we have all sorts of difficult to remember conversions. One notable feature of the metric system is that most units come in sizes ranging from small to large: the **(metric) prefixes** like kilo, mega, or giga tell us which size.

Really large numbers, like 8,460,000,000, are awkward to read and awkward to work with. We have seen how metric prefixes allow us to rewrite these large numbers in a way that's much easier both to read and to work with. There's another option that's used often in the sciences (and by your calculator). To explain it we need to understand powers of 10.

Perhaps you know what happens when we multiply a number by 10, like $5 \times 10 = 50$ or, more appropriate to our example,

$$8.46 \times 10 = 84.6$$

The effect of multiplying by 10 is to move the decimal point one place to the right. When we multiply by 1,000 we get $5 \times 1{,}000 = 5{,}000$ or, for our example,

$$8.46 \times 1{,}000 = 8{,}460$$

The effect of multiplying by 1,000 is to move the decimal point three places to the right. The connection is that

$$10 \times 10 \times 10 = 1{,}000$$

Each $\times 10$ has the effect of moving the decimal point one place to the right so $\times 1{,}000$ has the same effect as multiplying by 10 three times, so the decimal point moves three places to the right. That means

$$\begin{aligned} 8{,}460{,}000{,}000 &= 8.46 \underbrace{\times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10}_{9 \text{ times}} \\ &= 8.46 * 10^9 \end{aligned}$$

Since we're multiplying by the same number (10) over and over again, it's easier to use **exponential notation**. Here 10 is the **base** and 9 is the **exponent** (or **power**). In this context, the exponent is also called the **order of magnitude**.

The point of this calculation was that

$$8{,}460{,}000{,}000 = 8.46 * 10^9$$

This shorthand is called **scientific notation**. The base is always 10. The exponent is always a whole number. The number out front, like 8.46 in our example, is always between 1 and 10, which means there's exactly one digit before the decimal point (and any others must come afterwards). It is customary to use $\times$ instead of $*$ in scientific notation, so we should write

$$8{,}460{,}000{,}000 = 8.46 \times 10^9$$

As another example, we saw earlier that

$$5{,}000 = 5 \times 1{,}000 = 5 \times 10^3$$

Most calculators use the $\wedge$ symbol for exponents, as do most computer software packages. Two other notations calculators sometimes use are y^x or x^y. Sometimes that operation is accessible through the 2nd or shift key; something like SHIFT ×. If you're not sure, ask a classmate or your instructor. For practice, check that

$$5 \times 10^3 = 5 \times 10 \wedge 3 = 5{,}000 \quad \checkmark$$

Notice that the order of operations is exactly what we wanted here: $5 \times 10 \wedge 3$ first raises 10 to the 3rd power and then multiplies by 5. So we can enter it all at once without needing parentheses.

Here's the full list of the **order of operations**, the priority ranking for arithmetic operations.

ORDER OF OPERATIONS:

First, calculate anything inside **P**arentheses.
Next, calculate **E**xponents ∧, in order from left to right.
Then, **M**ultiply × and **D**ivide ÷, in order from left to right.
Last, **A**dd + and **S**ubtract −, in order from left to right.

We highlighted the letters PEMDAS which often helps people remember this order. ("Please Excuse My Dear Aunt Sally" is how I learned it.) The good news is that your calculator does the operations in exactly this order. And if you want something in a different order, all you need to do is use parentheses around quantities you want calculated first.

Back to our large number. Enter

$$8.46 \times 10 \wedge 9 =$$

What do you see? Some calculators correctly list out 8,460,000,000 while others report the number back in scientific notation, which is not particularly useful. (Sigh.)

Let's try a number so big that (nearly) every calculator will switch to scientific notation. Enter

$$2.7 \times 10 \wedge 30 =$$

Look carefully at the screen. Your calculator might display something like

$$\boxed{2.70000000 \quad \text{E} \; 30} \quad \text{or} \quad \boxed{2.70000000 \quad {\scriptstyle \times 10} \; 30}$$

Whatever shorthand your calculator uses, you should write

$$2.7 \times 10^{30}$$

Interested in what that number is in our usual decimal notation? It's

$$2,\underbrace{700{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000}_{\text{decimal point moves 30 places}}$$

Enough of that. Poor Tara is pulling her hair out over this project. Well, not literally, but she is quite frustrated over how slowly the project is going. She wonders: how thick is a human hair? And, how many hairs would you need to lay out to span an inch?

Turns out that a typical human hair is about .00012 meters across. Very small numbers are also awkward to read and awkward to work with. In this section, we write .000 12 where the strange-looking space is to help you read the number. Of course, a better solution is to use metric prefixes to get more appropriate units, just as we did for large numbers.

For example, **centi** is short for 1 in a hundred, or .01. Not surprising since one cent is \$.01, or one percent is 1%=.01. That's the same word "centi" as in centimeter (about 1/2 inch) and there's good reason for that as

$$1 \text{ meter} \quad = \quad 100 \textbf{ centimeter}$$

Similarly, **milli** is short for 1 in a thousand and **micro** is short for 1 in a million.

$$\begin{array}{lcl cr} \textbf{centi} & = & \text{1 in a hundred} & = & .01 \\ \textbf{milli} & = & \text{1 in a thousand} & = & .001 \\ \textbf{micro} & = & \text{1 in a million} & = & .000\ 001 \end{array}$$

What about that human hair? It is convenient to measure in micrometers using that

$$1 \text{ meter} = 1{,}000{,}000 \text{ micrometers}$$

The width of a human hair in micrometers (abbreviated μm in the sciences) is

$$.000\ 12 \cancel{\text{meters}} * \frac{1{,}000{,}000\ \mu\text{m}}{1 \cancel{\text{meter}}} = .000\ 12 \times 1{,}000{,}000 = 120\mu\text{m}$$

The μ symbol is the Greek letter *mu*, but we'll just read μm as micrometers.

To answer Tara's question about how many hairs in an inch, we recall that

$$1 \text{ inch} \approx 2.54 \text{ cm}$$

where cm is short for centimeter. Ready to convert?

$$1 \cancel{\text{inch}} * \frac{2.54 \cancel{\text{cm}}}{1 \cancel{\text{inch}}} * \frac{1 \cancel{\text{meter}}}{100 \cancel{\text{cm}}} * \frac{1{,}000{,}000 \cancel{\mu\text{m}}}{1 \cancel{\text{meter}}} * \frac{1 \text{ hair}}{120 \cancel{\mu\text{m}}}$$

$$= 2.54 \div 100 \times 1{,}000{,}000 \div 120 = 211.66666\ldots \approx 200 \text{ hairs}$$

We can also describe really small numbers using scientific notation. Perhaps you know what happens when we divide a number by 10, like $50 \div 10 = 5$ or, more appropriate to our example,

$$1.2 \div 10 = .12$$

The effect of dividing by 10 is to move the decimal point one place to the left. If we divide by 1,000,000 instead, we get

$$1.2 \div 1{,}000{,}000 = .000\ 001\ 2$$

The connection is that

$$1{,}000{,}000 = 10 \wedge 6$$

and so dividing by 1,000,000 moves the decimal point six places to the left. Notice that we have to introduce zeros as placeholders.

The width of a hair was .00012 meters. To get that number from 1.2, we need to move the decimal point 4 places to the left.

$$1.2 \div 10^4 = 1.2 \div 10{,}000 = .000\ 12$$

The shorthand for dividing by a power is to use negative exponents. For example

$$\div 10^4 = \times 10^{-4}$$

It has nothing to do with negative numbers. It's just a shorthand. The point of this calculation was that

$$.00012 = 1.2 * 10^{-4}$$

Once again we have scientific notation. The base is still 10. The exponent is still a whole number, although now it's negative. The number out front, like 1.2 in our example, is still between 1 and 10, which means there's exactly one digit before the decimal point (and any others must come afterwards). As before it customary to use $\times$ instead of $*$ in scientific notation, so we should write

$$.000\ 12 = 1.2 \times 10^{-4}$$

When you see a number written in scientific notation, the power of 10 tells you a lot. For example, $6.7 \times 10^4 = 67,000$ and $6.7 \times 10^{-3} = .006\ 7$. A positive power of 10 says you have a big number, and a negative power of 10 says you're dealing with a very small number.

Homework

Start by doing Practice exercises #1-4 in the workbook.

Do you know . . .

- How to calculate powers on your calculator?
- What million, billion, and trillion mean?
- Why metric prefixes are used?
- What common metric prefixes (mega, giga, kilo, centi, milli, micro, nano) mean?

  Ask your instructor which prefixes you need to remember, and whether any prefixes will be provided during the exam.
- Why scientific notation is used?
- The standard format for scientific notation?
- What kinds of numbers have a positive order of magnitude, and which have a negative order of magnitude?
- How to convert between decimal notation and scientific notation?
- How your calculator reports numbers in scientific notation, and what (might be) different when you're reporting that number?
- The usual order of operations (PEMDAS) and how to use parentheses when you want a different order?

  If you're not sure, work the rest of exercises and then return to these questions. Or, ask your instructor or a classmate for help.

Exercises

5. (a) How many files at an average of 42.3 MB each can each gig (1 GB) of computer memory hold?

   (b) Tara's coworker Brandon has a much faster Internet connection on his computer at 1,500 kbps. How long would it take Brandon to download 700 MB?

   (c) At that rate, how much information could Brandon upload in 8 hours? Express your answer in kilobytes (KB).

6. (a) Convert each of these amounts of time into an understandable unit of time: 1 million seconds, 1 billion seconds, 1 trillion seconds.
   (b) Billy Bob wants to throw a party when he turns 1 billion seconds old. About how many years old will he be?
   (c) *Bonus question:* On what date were you or will you be 1 billion seconds old? Don't forget leap years! Source: Mathew Foss, North Hennepin Community College

7. A proton has mass of about 1.67262×10^{-27} kg, while an electron has mass of about 9.10938×10^{-31} kg.
   (a) Write out the mass of a proton and that of an electron in normal decimal notation.
   (b) Which is heavier (has greater mass)?
   (c) How many times heavier is it? To calculate the answer take the mass of the heavier particle and divide it by the mass of the lighter particle.
   (d) How many protons would it take to weigh an ounce? Use 1 ounce $\approx$ 28.3 grams and, as always, 1 kg = 1,000 grams. *Because $\times$ and $\div$ are at the same level in the order of operations, you should put parentheses around each number in scientific notation before dividing.*

8. How many servings are in
   (a) A 2-liter bottle of a soft drink where the serving size is 250 mL?
   (b) A 750 mL bottle of wine where a serving size is 5 (fluid) ounces? Use 1 quart = 32 (fluid) ounces and 1 liter $\approx$ 1.056 quarts.

9. Rayka weighs 140 pounds. She would like to approximate how many cells are in her body. Use the following information: 1 cell $\approx 1 \times 10^{-15}$g, 1 kg $\approx$ 2.2 pounds, and, as always, 1 kg = 1,000 g.
   (a) How many cells are in Rayka's body? Write your answer in scientific notation.
   (b) Rewrite your answer in the most appropriate unit: millions (10^6), billions (10^9), trillions (10^{12}), quadrillions (10^{15}), or quintillions (10^{18}).

10. **Body Mass Index** (BMI) is one indicator of whether a person is a healthy weight. BMI are between 18.5 and 24.9 are considered "normal". Jared is 6'4" and weighs 200 pounds. He would like to calculate his BMI from this guide:

$$\text{BMI} = \text{weight in kilograms} \div \text{height in meters} \wedge 2$$

Source: Center for Disease Control and Prevention

(a) Check that Jared is around 1.93 meters tall and weighs around 90.91 kilograms. Use 1 inch $\approx$ 2.54 cm and 1 kilogram $\approx$ 2.2 pounds

(b) Jared entered the following keystrokes on his calculator:

$$90.91 \div 1.93 \wedge 2 =$$

and got the answer

$$\text{Jared's BMI } = 24.4060243\ldots$$

Is his BMI considered "normal"?

(c) Suppose Jared had rounded off his height to 1.9 meters and his weight to 91 kilograms. Calculate his BMI by entering the following keystrokes your calculator:

$$91 \div 1.9 \wedge 2 =$$

What do you get? Round your answer to one decimal place. Is Jared's BMI considered "normal"?

(d) What would you tell Jared?