# Applied Algebra

Dr. Suzanne Dorée
Professor of Mathematics
Augsburg College
Minneapolis, MN 55454

Fall 2012 edition (early July DRAFT for JODY)

# Contents

# Chapter 1

# Variables

Believe it or not, algebra is useful. Really useful. It's useful in later courses you might take in mathematics, statistics, science, social science, or business. But it's also useful in real life. A lot of what happens in the world around us is easier to understand using algebra. That's what this course is all about: using algebra to answer questions.

In this first chapter, we will introduce the key concepts of variable and function that help us translate between problems stated in words and the mathematics explaining the situation. We explain the important tools of units, tables, and graphs. We also describe how functions change and use the rate of change to approximate answers to questions. Throughout this chapter we keep a careful eye on evaluating the reasonableness of answers by connecting what we learn from algebra with our own life experience. After all, an answer to a real problem should make sense, right?

Some of our approach may feel very different to you. It is possibly quite different from what you've seen in mathematics classes before. It might take you a little time to get used to, but it will be worth it.

## 1.1 Variables and functions

Things change, like the price of gasoline, and just about every day it seems. What does it mean when the price of a gallon of gas drops from \$3.999/gal to \$3.299/gal? The symbol / is short for "per" or "for each," so that means each gallon costs

$$\$3.999 - \$3.299 = \$.70 = 70¢$$

less. Does this 70¢ truly matter?

Before we answer that question, are you wondering why there's that extra 9 at the end of the price? We might think a gallon costs \$3.99 but there's really a small 9 following it. Sometimes that 9 is raised up slightly on the gas station sign. You have to read the fine print. What it means is an extra $\frac{9}{10}$¢ for each gallon. So the true price of a gallon gas would be \$3.999. Gas costs a tiny bit more than you thought. Good grief.

Back to our question. Does 70¢ truly matter to us? Probably not. Can't even buy a bag of potato chips for 70¢. But, how often do you buy just one gallon of gas? Typically you might put five, or ten, or even twenty gallons of gas into the tank. We want to understand how the price of gasoline influences what it really costs us at the pump. To do that let's compare our costs when we buy ten gallons of gas. There's no good reason for picking ten; it's just a nice number to work with.

If gas costs \$3.999/gal and we buy 10 gallons, it costs

$$10 \text{ gallons} * \frac{\$3.999}{\text{gallon}} = 10 \times 3.999 = \$39.99$$

See how we described the computation twice? First, with units, fractions, and $*$ for multiplication in what's sometimes called "algebraic notation." Then, with just numbers and $\times$ for multiplication – that's what you can type into a calculator.

If gas drops to \$3.299/gal and we buy 10 gallons, it costs

$$10 \text{ gallons} * \frac{\$3.299}{\text{gallon}} = 10 \times 3.299 = \$32.99$$

That's \$7 less. For \$7 savings on gas you could buy that bag of potato chips, and an iced tea to go with it, and still have change. That amount matters. I mean, especially since it's \$7 savings every time you put 10 gallons in the tank.

Gas prices have been changing wildly, and along with them, the price of 10 gallons of gas. In mathematics, things that change are called **variables**. The two variables we're focusing on in this story are

$$\begin{aligned} P &= \text{price of gasoline (\$/gal)} \\ C &= \text{total cost (\$)} \end{aligned}$$

Notice that we gave each variable a letter name. It is helpful to just use a single letter chosen from the word it stands for. In our example, $P$ stands for "price" and $C$ stands for "cost". In this course we rarely use the letter $X$ simply because so few words begin with $X$. Whenever we name a variable ($P$) we also describe in words what it represents (the price of gasoline), and we state what units it's measured in (\$/gal).

In talking about the relationship between these variables we might say "the cost depends on the price of gas," so $C$ depends on $P$. That tells us that $C$ is the **dependent variable** and $P$ is the **independent variable**. In general, the variable we really care about is the dependent variable, in this case $C$ the total amount of money it costs us. The concept of dependence is so important that there's yet another word for it. We say that $C$ is a **function** of $P$, as in "cost is a function of price."

Knowing which variable is independent or dependent is helpful to us. To emphasize the dependence, we often make a notation next to the variable name.

$$\begin{aligned} &P = \text{price of gasoline (\$/gal)} \sim \text{indep} \\ &C = \text{total cost (\$)} \sim \text{dep} \end{aligned}$$

This labeling is rarely used outside this textbook, so add it in for yourself if you need it. In some situations dependency can be viewed either way; there might not be one correct way to do it. Labeling the dependence is extra important then, so anyone reading your work knows which way you are thinking of it.

Given a choice, we usually assign dependence such that given a value of the independent variable, it is easy to calculate the corresponding value for the dependent variable. In our example it's easy to use the price per gallon, $P$, to figure out the total cost, $C$. We can work backwards – from $C$ to $P$ – but it's not as easy.

For example, suppose we buy 10 gallons of gas and it costs \$28.99. We can figure out that the price per gallon must be

$$P = \frac{\$28.99}{10 \text{ gallons}} = 28.99 \div 10 = \$2.899/\text{gal}$$

Notice that we use the fraction as part of the algebraic notation, but we use $\div$ to indicate division on the calculator. Your calculator key for division may be / instead, which we reserve as a shorthand for "per."

From our experience we have a sense of what gas might cost. In my lifetime, I've seen gas prices as low as 35.9¢ /gallon in the 1960s to a high of \$4.099/gallon recently. This range of values sounds too specific, so it would sound better to say something general like

"Gas prices are (definitely) between \$0/gal and \$5/gal."

The mathematical shorthand for this sentence is

$$0 \leq P \leq 5$$

The inequality symbol $\leq$ is pronounced "less than or equal to". Formally, the range of realistic values of the independent variable is called the **domain** of the function $C$. In this text, we rarely write the domain because it's usually clear from the story what realistic values would be. The exercises in this section ask you to do so for practice.

Be aware that there are often many different numbers in a story. Some numbers are examples of values the variables take on, such as \$3.999/gal or \$39.99 in our example. Other numbers are **constants**; they do not change (at least not during the story). The one constant in our story is that we are always buying 10 gallons of gas. Occasionally there are other numbers in a story that turn out not to be relevant at all, so be on the lookout.

Back to our story. A report says that the average price of gasoline in Minnesota was \$2.900/gal in 2010 and increased approximately 20% per year for the next several years. We would like to check what that says about the average price of gasoline in 2011 and 2012, say. (It is unlikely that the price increase continued much longer at that rate.)

To understand what that report is saying, we need to remember how percents work. Luckily, the word "percent" is very descriptive. The "cent" part means "hundred," like 100 cents in a dollar or 100 years in a century. And, as usual, "per" means "for each." Together, **percent** means "per hundred." The number 20% means 20 for each hundred. Written as a fraction it is $\frac{20}{100}$. Divide to get the decimal $20 \div 100 = 0.20$.

Think money: 20% is like 20¢, and 0.20 is like \$0.20

Bottom line: 20%, $\frac{20}{100}$, and 0.20 mean exactly the same number.

$$20\% = \frac{20}{100} = 20 \div 100 = 0.20$$

To calculate the percent of a number we multiply by the decimal version. For example,

$$20\% \text{ of } \$2.900 = 0.20 \times 2.900 = \$.58$$

The report says the price increased by 20% each year, so by 2011 the price had increased an average of \$.58. That's not what gas cost in 2011. It's how much *more* gas cost in 2011 compared to 2010. To see what the report projected for the 2011 cost we need to add that increase on to the original 2010 price.

$$\$2.099 + \$0.58 = \$3.48 \text{ per gallon}$$

Sounds about right. Expensive, to be sure, but fairly accurate.

For 2012, the price increased by 20% again. That means 20% of what it was in 2011. We can't just add \$.58 again. That was 20% of the 2010 value, and we want 20% *of the 2011 value*. Going to have to calculate that.

$$20\% \text{ of } \$3.48 = 0.20 \times 3.48 = \$.696$$

so the projected 2012 value was

$$\$3.48 + \$.696 = \$4.176 \text{ per gallon}$$

Yikes.

One last note. The number 20% in the report sounds like a rough approximation. The report probably means the increase was around 20%, maybe a little less, maybe a little more. So our answers of \$3.48/gal and \$4.176/gal could be a little less or a little more too. But they sound so perfectly correct. To be safe, we really ought to round off these answers, to something more general like around \$3.50/gal in 2011 or approximately \$4.20/gal in 2012.

When we want someone reading our calculation to know that we mean approximately, not exactly, we use the **approximately equal to** symbol $\approx$. We save the equal sign, $=$, for when we have not rounded off the number at all. So, according to the report $P \approx \$3.50$/gal in 2011 and $P \approx \$4.20$/gal in 2012.

A lot of realistic problems involve percentages and so we use them often in this text.

## 1.2 Tables and graphs

Lung cancer, chronic bronchitis, bad breath, stains on your clothes, and the expense. These are just a few of the consequences of smoking cigarettes. With what we know now about the dangers of smoking, are people smoking more or less than they were ten years ago, fifty years ago, or even one hundred years ago?

Reality is, we don't have information on each individual person's smoking rate, so we can't answer this question exactly. We do have information on the total number of cigarettes sold each year. So maybe we should look at that total. Uh oh, that isn't going to work. There are way more people now than there were fifty or a hundred years ago. So, even if the same percentage of people smoke, and even if they each smoke the same amount as their predecessors did, we would have a much bigger number of cigarettes smoked now just because there are more people now.

Turns out a reasonable measure is to compare the number of cigarettes smoked per year *per person*. By taking into account the number of people we will be able to see whether people are smoking more or less, on average. That's what we want.

Here are some representative years from the Center for Disease Control for the United States. The smoking rate is the average cigarettes per year per person. (Here "person" only includes adults.)

| Year | 1900 | 1915 | 1930 | 1940 | 1950 | 1965 | 1975 | 1990 | 2000 | 2006 |
|---|---|---|---|---|---|---|---|---|---|---|
| Smoking rate | 54 | 285 | 1,485 | 1,976 | 3,552 | 4,258 | 4,122 | 2,834 | 2,049 | 1,619 |

To make sense of these numbers, suppose there are five friends. Three don't smoke at all, so that is 0 cigarettes in a year. Another smokes only occasionally, maybe 100 cigarettes a year. The fifth smokes "a pack a day," which adds up to 7,300 cigarettes in a year because

$$\frac{1 \text{ pack}}{\text{day}} * \frac{20 \text{ cigarettes}}{\text{pack}} * \frac{365 \text{ days}}{\text{year}} = 20 \times 365 = \frac{7{,}300 \text{ cigarettes}}{\text{year}}$$

(Not sure about this calculation? Not to worry. More about unit conversions in §1.4.) These five people smoke a total of

$$0 + 0 + 0 + 100 + 7{,}300 = 7{,}400 \text{ cigarettes per year}$$

so when we divide by the number of people we get

$$\frac{7{,}400 \text{ cigarettes per year}}{5 \text{ people}} = 7{,}400 \div 5 = 1{,}480 \text{ cigarettes per year per person}$$

This group is fairly typical for the United States in 2012. They smoke less than the average of 1,619 cigarettes per year per person for 2006 (the last year the CDC published the data).

We can tell a lot of information from this table. For example, what was the smoking rate in 1964, and how does that compare to 2006? The answers appears in the table, a whopping 3,552 cigarettes per person in 1964 and 1,619 cigarettes per person in 2006.

When did the consumption first pass 3,000? That answer does not appear in the table, but we can use the information in the table to make a good guess. In 1940, there were an average of 1,976 cigarettes per person per year and by 1950, there were 3,552. Somewhere between 1940 and 1950 the number first climbed above 3,000. More specifically, the number we're looking for (3,000) is a lot closer to the 1950 figure (3,552) than to the 1940 figure (1,976). So, it would be reasonable to guess close to 1950. I'd say 1947. Of course, you might guess 1946 or 1948, or even 1949 and those would be good guesses too.

When did the consumption drop below 3,000 again? This answer also does not appear in the table, but falls somewhere between 1975 when consumption was 4,122 and 1990 when consumption was 2,834 . Here I'd guess just before 1990, say in 1989.

What's changing are the number of cigarettes smoked per person per year and the year. Those are our variables. The smoking rate is a function of year, and it's what we care about, so it's the dependent variable. Time, as measured in years, is the independent variable.

$$S = \text{smoking rate (cigarettes per year per person)} \sim \text{dep}$$
$$Y = \text{year (years since 1900)} \sim \text{indep}$$

Quick note on how we deal with actual years. Since the year 0 doesn't make sense in this problem, it is convenient to measure time in years since 1900. Officially we should rewrite our table as:

| $Y$ | 0 | 15 | 30 | 40 | 50 | 65 | 75 | 90 | 100 | 106 |
|---|---|---|---|---|---|---|---|---|---|---|
| $S$ | 54 | 285 | 1,485 | 1,976 | 3,552 | 4,258 | 4,122 | 2,834 | 2,049 | 1,619 |

Notice where the variable names are listed in the table. In a horizontal format like this table, the independent variable ($Y$) is in the top row, with the dependent variable ($S$) is in the bottom row. If you want to write your table in a vertical format, that's okay too. Just put the independent variable in the left column, with the dependent variable in the right column. It might help to remember that the independent variable goes first (either top or left) and the dependent variable follows (either bottom or right).

Horizontal table format:

| indep | | | | |
|---|---|---|---|---|
| dep | | | | |

Vertical table format:

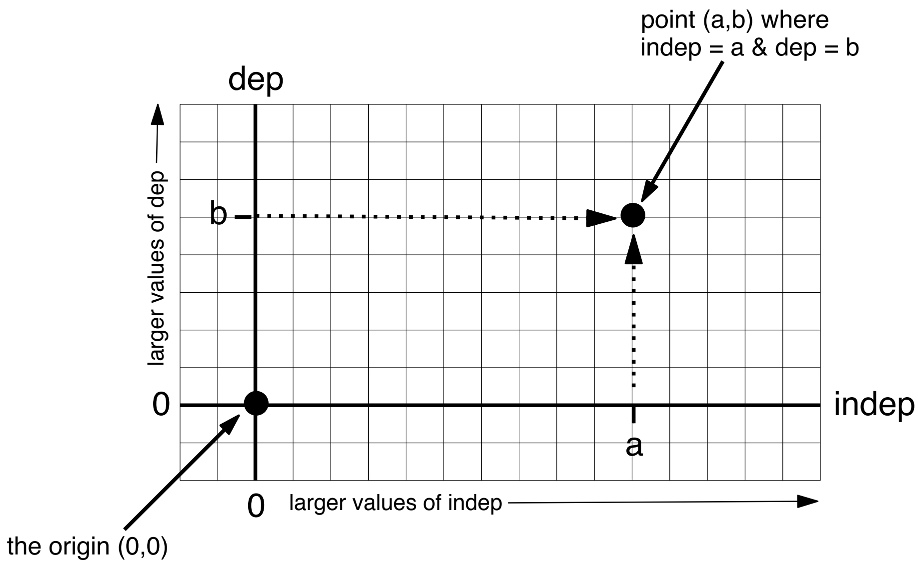
| indep | dep |
|---|---|
| | |
| | |
| | |
| | |

Where the variables go in a table is not something you can figure out. It's a **convention** – a custom, practice, or standard used within the mathematical community. Though based on reason, it often involves some arbitrary choice, which is why we can't figure it out. So, whenever some practice is introduced to you as a "convention", you need to memorize it.

Tables are useful because they contain specific numbers, but it can be difficult to guess or see general trends. For that, a picture is worth a thousand words – or numbers, in this case. By picture we mean the graph of the function.

Throughout this text, we draw graphs by hand. On graph paper. Seriously. You might wonder why we do that when graphing calculators, spreadsheet programs, graphing "apps," or computer algebra systems all can draw graphs for us. The answer is we want to understand graphs better. I promise – drawing them by hand will help you do that. Different folks have different opinions on the importance of graphing by hand, so be sure to ask your instructor what you are expected to do. Even if you're allowed to use some type of graphing technology, I strongly encourage you to practice drawing graphs by hand as well.

There is a standard set up for a graph.



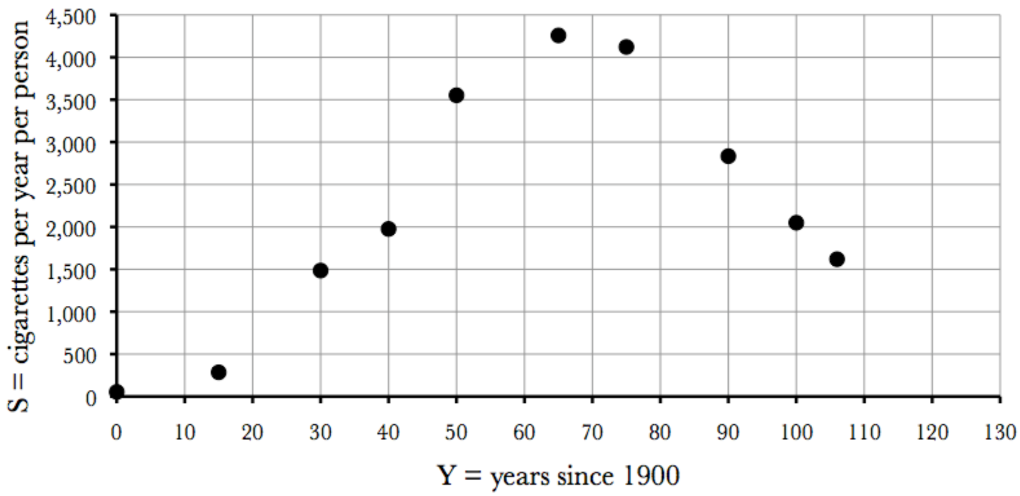

The graph is based on a horizontal line and vertical line, called the **axes**. Where they cross is a point called the **origin**. It represents where each variable is 0. By convention, the independent variable is measured along the horizontal axis, with larger values progressing to the right of the origin, and negatives to the left. Similarly, the dependent variable is measured along the vertical axis, with larger values progressing up from the origin, and negatives down. Each gridline counts the same number, called the **scale**, but the scale for the vertical may be different from the scale for the horizontal. Each pair of values of the independent and dependent variable from our table correspond to a point on our graph.

In the graph of smoking rates, the independent variable is $Y$, the year, so that goes on the horizontal axis for our graph. Our dependent variable is $S$, the smoking rate, so that goes on the vertical axis. For the scale, it works nicely to count by 10 years and count by 500s for the smoking rate.

There's a certain amount of guess and check involved in figuring out a good scale for each axis. As a general rule of thumb we would like the graph to be as large as possible so we can see all of its features clearly. But, not so big that it runs off the graph paper. What matters is that the gridlines are evenly scaled and that they can handle large enough numbers. Speaking of which, it's a good idea to leave a little room to extend the graph a little further than the information we have in the table, in case we get curious about values beyond what we have already.

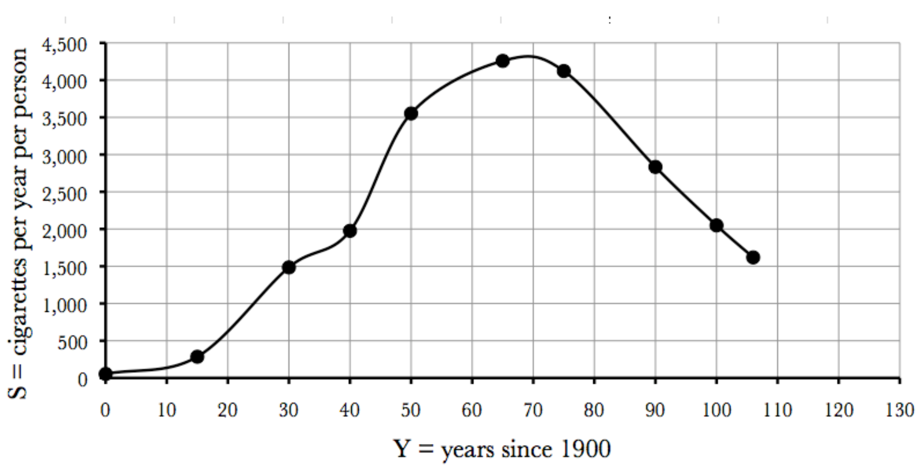
With realistic numbers it's normal to have numbers in the table that are not exactly where the gridlines are. It is very helpful to count by round numbers (2s, 5s, 10s, etc.)

because it makes guessing in between easier. Easier for you drawing the graph. Easier for someone reading your graph.




To plot each point, we start at the origin and move right to that $Y$-value, and then up to that $S$-value. When a value doesn't land exactly on a grid mark, we have to guess in between. For example, in 1900, when $Y = 0$ so we don't move right at all, just up to $S = 54$. The first labeled gridline on our graph is 500. Where's 54? It's between 0 and 500, very close to 0. Our point is just a tiny bit above the origin. In 1915 we have $Y = 15$. Our labeled gridlines are for 10 and 20, so 15 must land halfway in between. The smoking rate to 285, which is around halfway between 0 and 500. Etc.

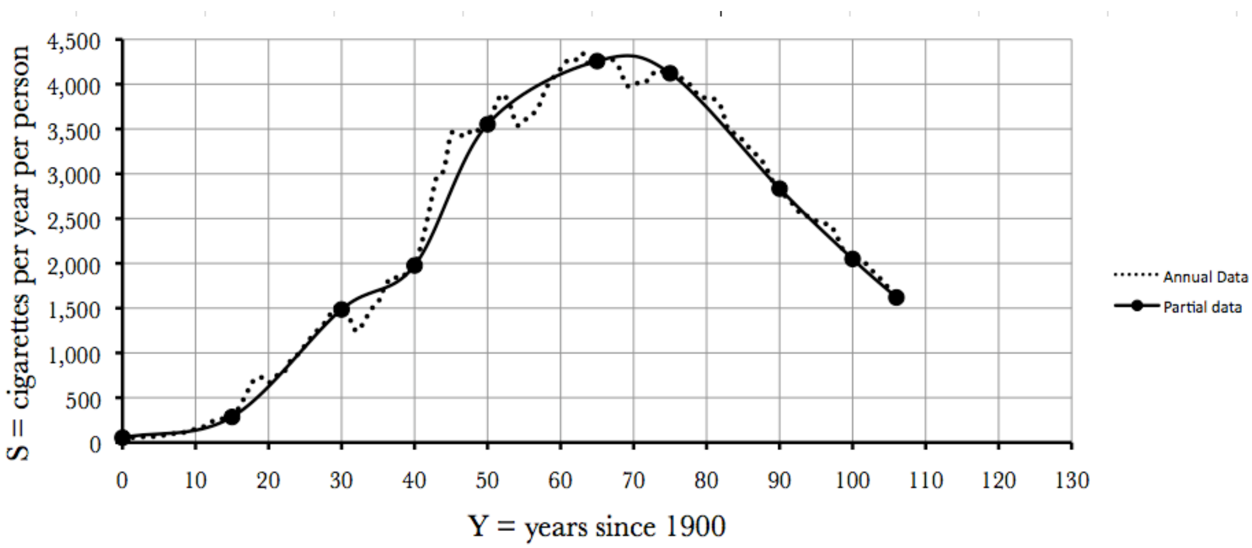
What we have so far is a **scatter plot** of points. Can you see why it's called that? Anyway, our whole goal here was to be able to understand smoking rates better by having a graph. You may already begin to see a curve suggested by the points. Time to draw it in. I don't mean "connecting the dots" – drawing a line between each pair of points. That isn't quite right. . It was probably more of a continuous trend and so the graph should be smoother.



When we draw in this smooth curve for the graph, what we are really doing is making a whole lot of guesses all at once. For example, from the table we guessed that the smoking rate passed 3,000 in around 1947, and dropped back to that level in around 1989. What does the graph show? If we look where the horizontal gridline for 3,000 crosses our graph, it crosses in two places. First, between the vertical gridlines for 40 and 50, and perhaps slightly closer to 50. I'd say $Y \approx 47$, in the year 1947. Sure. The second time is between the gridlines for 80 and 90, much closer to 90. Looks like $Y \approx 88$, in the year 1988. We guessed 1989. Close enough.

Don't forget that when we drew in that curve it was really just a guess. We're sure about the points we plotted, but we're only guessing about where to draw the curve in. That means we're not sure about the other points. If we knew a lot more points we could have a more accurate graph.

Turns out more data is available from the CDC. The full table of data from the CDC shows that consumption first topped 3,000 as early as 1944. Here's an example where the history tells you more than the mathematics as cigarette consumption rose sharply during World War II. Our guess about 1988 or 1989 was spot on. Look at how the graph from the full data compares to our guess.




July 6, 2012

## 1.3 Rate of change (and interpolation)

A diver bounces on a 3-meter springboard. Up she goes. A summersault, a twist, then whoosh, into the water. The table shows the diver's height, measured as $H$ meters above the water, as a function of time, $T$ seconds.

| $T$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2 | 1.4 |
|---|---|---|---|---|---|---|---|---|
| $H$ | 3.00 | 3.88 | 4.38 | 4.48 | 4.20 | 3.52 | 2.45 | 1.00 |

In case you're wondering, 3 meters is nearly 10 feet up and the highest height listed, 4.48 meters, is close to 15 feet above the water. More on how we figured those numbers out in the next section, but thought you might like to know.

How fast is she moving? During the first 0.2 seconds, her height changes from 3.00 meters to 3.88 meters. She rose $3.88 - 3.00 = 0.88$ meters in that 0.2 seconds. Measured in meters per second, her speed is

$$\frac{0.88 \text{ meters}}{0.2 \text{ seconds}} = 4.4 \text{ meters/sec.}$$

Her speed is called the *rate of change*, and is calculated as

$$\text{Rate of change} = \frac{\text{change in height}}{\text{change in time}}$$

$$= \frac{3.88 - 3.00 \text{ meters}}{0.2 \text{ seconds}} = \frac{0.88 \text{ meters}}{0.2 \text{ seconds}} = 4.4 \text{ meters/sec.}$$

What about during the next 0.2 seconds? Does she move faster, slower, or the same? This time her height changed from 3.88 meters to 4.38 meters. In these 0.2 seconds she rose $4.38 - 3.88 = .50$ meters. That's less than before (since $0.50 < 0.88$), which means so she is going slower. Officially, we can calculate that her speed is

$$\text{Rate of change} = \frac{\text{change in height}}{\text{change in time}}$$

$$= \frac{4.38 - 3.88 \text{ meters}}{0.4 - 0.2 \text{ seconds}} = \frac{0.50 \text{ meters}}{0.2 \text{ seconds}} = 2.5 \text{ meters/sec.}$$

To calculate the rate of change we figured out the top of the fraction (0.50) and the bottom of the fraction (0.2), and then divided $0.50 \div 0.2 = 2.5$. There is a way to do the entire calculation at once on the calculator, but you need to use parentheses:

$$(4.38 - 3.88) \div (0.4 - 0.2) =$$

We need those parentheses to force the calculator to do the subtractions first and division second. The usual order of operations would do it the other way around: multiplication and division before addition and subtraction.

Notice that the top of our fraction is

height at 0.4 seconds - height at 0.2 seconds and the bottom of our fraction is

$$0.4 \text{ seconds - } 0.2 \text{ seconds}$$

It's important that they match up – the 0.4s first on top and bottom and the 0.2s second on top and bottom.

During the next time interval she's moving even slower.

$$\text{Rate of change} = \frac{4.48 - 4.38 \text{ meters}}{0.6 - 0.4 \text{ seconds}} = 0.5 \text{ meters/sec.}$$

And look what happens when we calculate her speed during the next time interval.

$$\text{Rate of change} = \frac{4.20 - 4.48 \text{ meters}}{0.8 - 0.6 \text{ seconds}} = \frac{-.28 \text{ meters}}{0.2 \text{ seconds}} = -1.4 \text{ meters/sec.}$$

What does a negative speed mean? During this time interval her height drops. She's headed down towards the water. Her speed is 1.4 meters/sec downward. The negative tells us her height is falling. We can add these speeds in our table.

| $T$ | 0.0 | | 0.2 | | 0.4 | | 0.6 | | 0.8 | | 1.0 | | 1.2 | | 1.4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $H$ | 3.00 | | 3.88 | | 4.38 | | 4.48 | | 4.20 | | 3.52 | | 2.45 | | 1.00 |
| speed | | 4.4 | | 2.5 | | 0.5 | | -1.4 | | -3.4 | | -5.35 | | -7.25 | |

To be perfectly correct, these are her "average" speeds over the interval. Instead of saying "rate of change" people will often say "average rate of change," but the formula is the same.

Over any interval where increasing the independent variable corresponds to an increase in the dependent variable, we say the function is *increasing*. The diver's height is increasing for $0 \leq T \leq 0.6$ seconds. It is possible that she continues to rise a little longer, but we can't tell from just the numbers in our table.

On the other hand, over any interval where increasing the dependent variable corresponds to a decrease in the dependent variable, we say the function is *decreasing*. The diver's height is decreasing for $0.8 \leq T \leq 1.4$ seconds. It is possible that her height starts decreasing sooner, and it certainly continues decreasing until she hits the water, but we don't know exactly when.

When does the diver's height stop increasing and start decreasing? When she's at the highest height, some time between 0.6 and 0.8 seconds into her dive. Before then her rate of change is positive. After that time her rate of change is negative. So, at the highest height her rate of change is probably equal to zero. Does that make sense? Think about watching a diver on film in very slow motion. Up, up she goes, then almost a pause at the top, and then down, down, into the water. At the top of her dive it's as if she stands still for an instant. That would correspond to zero speed.

We can use the rate of change to estimate values missing from the table. For example, let's guess her height at 0.3 seconds. During the time interval between 0.2 and 0.4 seconds, we figured out that her average speed was 2.5 meters/sec. From 0.2 to 0.3 is 0.1 extra seconds. During that 0.1 extra second, she goes about

$$0.1 \text{ extra seconds } * \frac{2.5 \text{ meters}}{\text{second}} = 0.25 \text{ extra meters,}$$

so her height would be approximately

$$3.88 \text{ meters} + 0.25 \text{ extra meters} = 4.13 \text{ meters.}$$

We expected something in between 3.88 and 4.38 meters. In fact, since 0.3 was right in the middle of 0.2 and 0.4, we actually get the average $\frac{3.88+4.38}{2} = 4.13$. If it wasn't right in the middle we wouldn't get the average, but a more weighted average. By the way, we could do this estimate all at once as

$$\begin{aligned} \text{Estimated value} &= \text{original value} + \text{extra independent variable } * \text{ rate of change} \\ &= 3.88 \text{ meters} + 0.1 \text{ extra seconds } * \frac{2.5 \text{ meters}}{\text{second}} \\ &= 3.88 \text{ meters} + 0.25 \text{ extra meters} \\ &\approx 4.13 \text{ meters.} \end{aligned}$$

A photographer snapped a picture at exactly 1.03 seconds. How high was the diver then? We expect the answer to be just a little bit less than her height at 1.0 seconds, which was 3.52 meters, but not nearly as low as after 1.2 seconds, which was 2.45 meters. Let's see what the rate of change estimate is. First,

$$0.03 \text{ extra seconds } * \frac{-5.35 \text{ meters}}{\text{second}} = -0.1605 \text{ extra meters,}$$

which means about 0.1605 lower. Her height would be approximately

$$3.52 \text{ meters} - 0.1605 \text{ extra meters} = 3.3595 \text{ meters} \approx 3.36 \text{ meters.}$$

As before we can calculate this estimate in one fell swoop as

$$\begin{aligned} \text{Estimated value} &= 3.52 \text{ meters} + 0.03 \text{ extra seconds } * \frac{-5.35 \text{ meters}}{\text{second}} \\ &= 3.3595 \text{ meters} \\ &\approx 3.36 \text{ meters.} \end{aligned}$$

Notice that we rounded off to two decimal places for our approximation because all the numbers in the table were rounded off. That's a reasonable answer, much closer to her height at 1.0 seconds (3.52 meters) than her height at 1.2 seconds (2.45 seconds).

How long is the diver in the air? At 1.4 seconds she's 1.00 meter up, so she must enter the water soon after that. We can use the rate of change to estimate her height after 1.5

and 1.6 seconds to see. We don't know the average speed past 1.4 seconds, so we'll just have to use the closest value we know, her speed was -7.25 meters/sec during the preceding interval. Bear in mind that we're really guessing about that, and so our estimate is even less accurate than usual. For both estimates we start with 1.00 meter at 1.4 seconds.

$$\begin{aligned} \text{Estimated height at 1.5 seconds} &= 1.00 \text{ meters} + 0.1 \text{ extra seconds } * \frac{-7.25 \text{ meters}}{\text{second}} \\ &= 0.275 \text{ meters} \\ &\approx 0.3 \text{ meters.} \end{aligned}$$

$$\begin{aligned} \text{Estimated height at 1.6 seconds} &= 1.00 \text{ meters} + 0.2 \text{ extra seconds } * \frac{-7.25 \text{ meters}}{\text{second}} \\ &= -0.45 \text{ meters} \\ &\Longrightarrow \text{already hit the water.} \end{aligned}$$

Here is a graph showing the diver's height. The variables are

$$T = \text{time (seconds), indep, } 0 \leq T \leq 1.6$$
$$H = \text{diver's height (meters), dep, } 0 \leq H \leq 5$$

As usual we drew in a smooth curve connecting the points, which illustrates our best guesses for the points we don't know. We also drew in the straight lines connecting each pair of points. As you can see, the first line segment is steepest – that's where the rate of change was 4.4 meters/sec. The next line segment was less steep – that's where the rate of change was less, down to 2.5 meters/sec. The third line segment is almost flat – that's where the rate of change was only 0.5 meters/sec.

We notice the same connection between the rate of change and steepness of the curve for the decreasing portion, only this time all the rate of changes are negative. The first downhill line segment is fairly flat – that's where the rate of change was -1.4 meters/sec. The next downhill line segment was much steeper – that's where the rate of change was -3.4 meters/sec. The next two downhill line segments were each steeper yet – this time with rates of change -5.35 and -7.25 meters/sec.

In each case we can visualize the rate of change as the steepness of the graph.

SHOULD WE INCLUDE THIS: When the rate of change is constant, the graph is a line and the function is called *linear*. OR MAYBE WITHIN THE PRACTICE PROBLEM ABOUT WEDDING?

DO WE DO ENOUGH INTERPOLATION IN THIS EXAMPLE – check old solutions from first version of 1.3Epsilon

Whenever we use the rate of change to estimate values it's as if we're assuming the rate of change is constant, at least for that interval of values. If the function really is linear and the graph is a line, then this estimate is excellent. If the function is pretty far off from linear and the graph curves a lot, then this estimate might not be as accurate. Something to keep in mind.

There's a formal name for what we're doing here. When we use the rate of change to estimate a number in between two numbers that we know it's called *linear interpolation*. When we estimate a number beyond what we know (smaller than the smallest number or larger than the largest number), it's called *linear extrapolation*. For both terms, the word "linear" reminds us that it only works perfectly for a line. In general interpolation is often a reasonable guess. Extrapolation can be pretty far off, but sometimes it's the only guess we have.

Sometimes we just want to know when a function is increasing, decreasing, at a maximum or minimum, how steep it is, or how much it seems to change. Even if there aren't specific numbers on the graph, we can sometimes learn a lot about a story. The next example looks at this sort of numberless graphs, sometimes called *qualitative* graphs.

4 in

## 1.4 Units

We know 5 city blocks and 5 miles are very different lengths to walk; \$5 and 5¢ are very different values of money; 5 minutes and 5 years are very different amounts of time to wait – even though all of these quantities are represented by the number 5. Every variable is measured in terms of some unit. Since there are often several different units available to use it is important when naming a variable to state which units we are choosing to measure it in.

In the last section we examined the height of a diver and her speed in the air. But, how high is 3 meters? How fast is 4.4 meters per second?

The metric unit of length called the *meter* is just over 3 feet (a yard). Let's use the conversion

$$1 \text{ meter} \approx 3.281 \text{ feet}.$$

We can convert

$$3 \text{ meters} * \frac{3.281 \text{ feet}}{1 \text{ meter}} = 9.843 \text{ feet} \approx 9.8 \text{ feet}.$$

Since our conversion is just approximate, we rounded off our calculation too.

See how the meters on the top and bottom cancel, leaving the units as feet? It might help to view it as

$$\frac{3 \cancel{\text{meters}}}{1} * \frac{3.281 \text{ feet}}{1 \cancel{\text{meter}}}.$$

Notice that we multiplied the quantity we were interested in (3 meters) by the fraction $\frac{3.281 \text{ feet}}{1 \text{ meter}}$. Since 3.281 feet and 1 meter are just two different ways of saying the same distance, the top and bottom of that fraction are equal. In other words,

$$\frac{3.281 \text{ feet}}{1 \text{ meter}} = 1.$$

A fraction where the top and bottom are equal quantities expressed in different units is sometimes called a *unit conversion fraction*. Because it's equal to 1, multiplying by the unit conversion fraction doesn't change the value, just the units.

One more thing to keep in mind when converting units: a few large things equals a lot of small things. Think calories here – instead of eating a few large cookies, you can eat a larger number of small cookies for the same caloric intake. So a small number of meters (3 meters) equalled a larger number of feet (9.8 feet). That might seem backwards, but that's how it works.

Of course, 9.843 feet might sound like a funny answer. We're much more used to a whole number of feet and then the fraction in inches. It's 9 feet and some number of inches. To figure out the inches we calculate

$$0.843 \text{ feet} * \frac{12 \text{ inches}}{1 \text{ foot}} = 10.116 \text{ inches} \approx 10 \text{ inches}.$$

So, the board is about 9 feet and 10 inches high, or 9'10" for short.

The highest height we had recorded for the diver was 4.48 meters. Now we know that's

$$4.48 \text{ meters} * \frac{3.281 \text{ feet}}{1 \text{ meter}} = 14.69888 \text{ feet} \approx 14.7 \text{ feet}.$$

In feet and inches, that's about 14 feet, 8 inches because

$$0.69888 \text{ feet} * \frac{12 \text{ inches}}{1 \text{ foot}} = 8.38656 \text{ inches} \approx 8 \text{ inches}.$$

What about the diver's speed? During the first 0.2 seconds we calculated her speed as 4.4 meters per second. How fast is that? We can certainly convert to feet per second.

$$\frac{4.4 \text{ meters}}{\text{second}} * \frac{3.281 \text{ feet}}{1 \text{ meter}} = \frac{14.4364 \text{ feet}}{\text{second}}.$$

Does that help us understand how fast she's going? Maybe a little. But, we're probably most familiar with speeds measured in miles per hour.

Let's convert to miles per hour. First,

$$\frac{14.4364 \text{ feet}}{\text{second}} * \frac{60 \text{ seconds}}{1 \text{ minute}} = \frac{866.184 \text{ feet}}{\text{minute}}.$$

The larger number makes sense here because she can go more feet in a minute than in just 1 second. Next,

$$\frac{866.184 \text{ feet}}{\text{minute}} * \frac{60 \text{ feet}}{1 \text{ hour}} = \frac{51,971.04 \text{ feet}}{\text{hour}}.$$

Again, the larger number makes sense because she can go more feet in an hour than in just 1 minute. Last,

$$\frac{51,971.04 \text{ feet}}{\text{hour}} * \frac{1 \text{ mile}}{5,280 \text{ feet}} = \frac{9.843 \text{ miles}}{\text{hour}} \approx 10 \text{ mph}.$$

This time we got a smaller number because she can go a lot fewer miles in an hour than feet.

We can do this entire calculation all at once. Notice how the units cancel to leave us with miles per hour (mph).

$$\frac{4.4 \cancel{\text{meters}}}{\cancel{\text{second}}} * \frac{3.281 \cancel{\text{feet}}}{1 \cancel{\text{meter}}} * \frac{60 \cancel{\text{seconds}}}{1 \cancel{\text{minute}}} * \frac{60 \cancel{\text{minutes}}}{1 \text{ hour}} * \frac{1 \text{ mile}}{5,280 \cancel{\text{feet}}} = 9.843 \text{ mph} \approx 10 \text{ mph}.$$

On the calculator we enter

$$4.4 \times 3.281 \times 60 \times 60 \div 5,280 =$$

Right before the diver hit the water she was going around 7.25 meters per second. How fast is that in mph?

$$\frac{7.25 \text{ meters}}{\text{second}} * \frac{3.281 \text{ feet}}{1 \text{ meter}} * \frac{60 \text{ seconds}}{1 \text{ minute}} * \frac{60 \text{ minutes}}{1 \text{ hour}} * \frac{1 \text{ mile}}{5,280 \text{ feet}} = 16.2185\ldots \approx 16 \text{ mph}.$$

On the calculator we enter

$$7.25 \times 3.281 \times 60 \times 60 \div 5,280 =$$

If you're having trouble setting up unit conversions, remember to write down the units so you can see how they cancel. If you can't remember a number to unit conversion, like 5280 feet for one mile, try a search engine on the Internet (like Google). A dictionary also has some conversions.

## 1.5 The metric system and scientific notation*

INTRODUCTORY EXAMPLE

SU – look at old 3.1 and old 3.4 and also exercises that combined these with unit conversions. Maybe something about acid rain and pH?

SU – try distance to sun here. Pitch scientific notation and metric prefixes and words like millions and billons as shorthand for really large and really small numbers.

OLD NOTES:

- The standard form of scientific notation. Converting between expanded decimal notation and scientific notation and comparing the relative size of numbers from their scientific notation.
- Converting units involving scientific notation such as millions, billions, and trillions and units involving standard metric prefixes.
- Imbedded topics: Exponents, order of operations, and the scientific calculator
- The definition of exponential notation (for integral powers). Recognizing the effect of multiplying by positive and negative powers of 10.
- Evaluating powers on a calculator, interpreting calculator display of numbers in scientific notation, and entering numbers in scientific notation into a calculator. [Formal properties of exponents and simplifying expressions are addressed in the exercises.]

# Chapter 2

# Equations

For most of us the word "algebra" brings to mind equations, formulas, and all those symbols. One chapter into a book on algebra and we haven't seen any equations. What gives?

Remember, this course is all about using algebra to answer questions. Equations are going to be a very important part of that work. It turns out that equations are helpful algebraic tool for at least three reasons. First, equations provide a nice shorthand for describing a function – much quicker to write down than making an extensive table, careful graph, or describing the dependence in words. Second, equations help us categorize problems which, in turn, helps us know what to expect in that type of situation. Lastly, there are lots of powerful "symbolic" techniques we can use to solve problems when we have an equation.

So why haven't we used equations yet? Why did the first chapter focus on describing functions using words (verbal), tables (numeric), and graphs (graphical)? It turns out that there's one thing equations can't do – it's hard to tell from an equation whether an answer makes sense in the real world. If we just worked with equations we might find an answer calling for us to produce a negative number of tables or wait 300 years for an investment to reach our payoff level, or similar nonsense.

Even as we add equations to our list of tools for describing and working with functions, we will rely on words, numbers, and graphs to help evaluate the reasonableness of our answer. You will likely find it a good habit to use those tools to estimate the answer before using the equation as well doing the "reality check" after. Thus most problems will ask you to work with all of these modes.

In this chapter we introduce equations but taking a first look at the two most important types of equations – linear and exponential. Our emphasis will be on understanding where these equations arise and how to interpret them in context. Next, we work with a variety of equations learning how to use equations and general methods for approximating solutions to equations. In later chapters we will solve equations exactly (Chapter 3) and return to study linear and exponential equations each in greater depth (Chapters 4 and 5), so don't worry if we leave a few questions unresolved for now.

## 2.1 A first look at linear equations

Your sink, I'm sorry to say, is clogged. The bottle of drain opener didn't clear it out and you're expecting dinner guests in a few hours. Your brother-in-law has offered to help, but last time he tried he only made it worse. The plumber will charge you \$100 just to come to your house. In addition, there will be a charge of \$75 per hour for the service. If you decide to call the plumber, what will it cost?

What will it cost if the plumber takes 1 hour, 2 hours, or 3 hours? If the plumber takes one hour, then he'll charge you \$100 for showing up and \$75 for the one hour of work. So, the total plumber's bill will be

$$100 + 75 = \$175$$

For two hours, there's still the \$100 charge, but also \$75 for each of the two hours. That's an additional charge of

$$2 \text{ hours} * \frac{\$75}{\text{hour}} = 2 \times 75 = \$150$$

So, the total plumber's bill will be

$$\$100 + \$150 = \$250$$

Try this calculation all at once.

$$\$100 + 2 \text{ hours} * \frac{\$75}{\text{hour}} = 100 + 2 \times 75 = \$250$$

Let's hope it wouldn't take the plumber as long as three hours, but if it did, we can do a similar calculation. Add the fixed charge of \$100 to the additional charge of \$75 for each of the three hours. The plumber's bill would be

$$\$100 + 3 \text{ hours} * \frac{\$75}{\text{hour}} = 100 + 3 \times 75 = \$325$$

What would it cost if the plumber takes only $\frac{1}{2}$ hour? The plumber's bill would be

$$\$100 + \frac{1}{2} \text{ hours} * \frac{\$75}{\text{hour}} = 100 + .5 \times 75 = \$100 + \$225 = \$325$$

Notice we used $\frac{1}{2} = 1 \div 2 = .5$, bet you knew that.

What would happen if the plumber was taking so long that before he got there you dumped another bottle of drain opener in the sink and that did the trick. But before you could call and cancel the plumber, wouldn't you know it, but there he was. What do you owe him for that 0 hours of work? Probably \$100. Unless your plumber is super sympathetic and tells you to "forget it."

The plumber's charge will depend on the amount of time it takes to unclog the sink. We can name these variables.

$$T = \text{time plumber takes (hours)} \sim \text{indep}$$
$$P = \text{total plumber's charge (\$)} \sim \text{dep}$$

Look at the the relationship between $T$ and $P$ by making a table to describe how the plumber's bill is a function of the time.

| $T$ | 0 | $\frac{1}{2}$ | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $P$ | 100.00 | 137.50 | 175.00 | 250.00 | 325.00 |

Each time we knew how long the plumber spent and calculated the plumber's bill $P$ by starting with the trip charge of \$100 and adding in \$75 times the number of hours. For example, for 3 hours we calculated

$$\$100 + 3\text{ hours} * \frac{\$75}{\text{hour}} = \$325$$

We have a name for the number of hours in general; it is $T$. So for $T$ hours, we would calculate

$$\$100 + T\text{ hours} * \frac{\$75}{\text{hour}} = P$$

See how we just put the $P$ in for \$325 and $T$ where the 3 hours was? We're just generalizing from our example. Drop the units and we have our equation. If the plumber works for $T$ hours, then the cost is $\$P$ where

$$P = 100 + T * 75$$

We started the equation "$P =$" because it is a convention to begin equations with the dependent variable, when possible.

An **equation** is a formula that shows how the value of the dependent variable (like $P$) depends on the value of the independent variable (like $T$). Usually an equation is in the form dependent variable equals a formula involving the independent variable. We remember this template as

**Equation:** dep= formula involving indep

An equation is another way to describe a function. It carries a lot of information in only a few symbols.

There is a mathematical convention that we write numbers before letters in an equation. So, instead of $T * 75$ we should write $75 * T$. There's a conventional shorthand for this product: when a number and letter are next to each other, it means that they are multiplied. So, instead of $75 * T$ we should write $75T$. Thus our equation is normally written as

$$P = 100 + 75T.$$

You'll have to remember the hidden multiplication when you're calculating.

If you wanted to write the equation as

$$P = 75T + 100,$$

that would be okay too. We can add the \$100 trip charge first, like we did in our examples, or at the end. Same answer.

Suppose the plumber shows up at your house and fixed the sink in 25 minutes. Whew! No sooner do you pay your bill than your first dinner guest arrives. How much do you owe the plumber? Notice that

$$25 \text{ minutes} * \frac{1 \text{ hour}}{60 \text{ minutes}} = 25 \div 60 = .4166\ldots \text{ hours}$$

Therefore for 25 minutes we have $T \approx .4166$ Using our equation we get

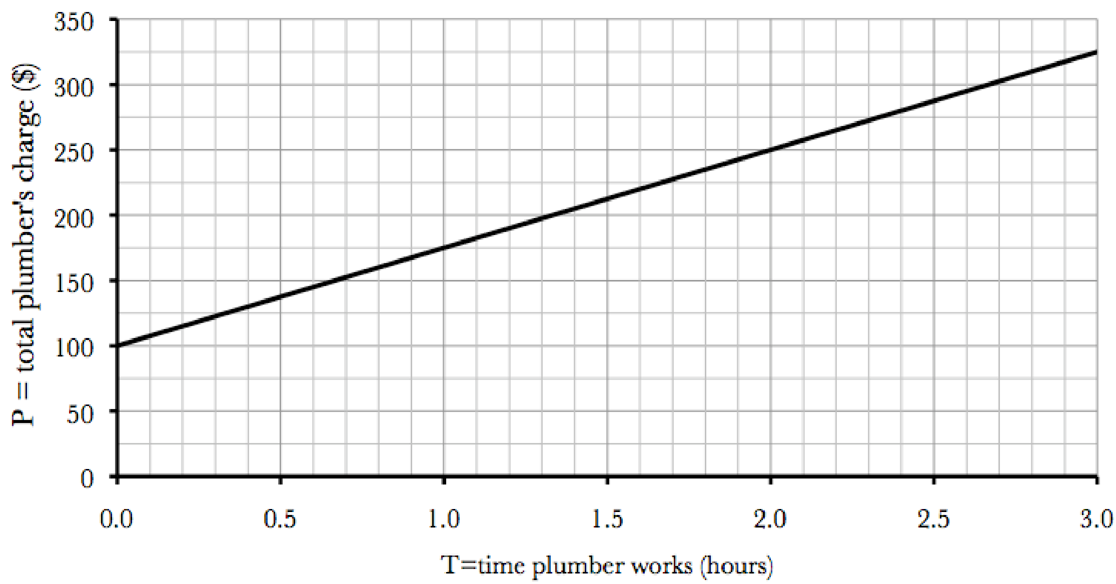
$$P = 100 + 75 * .4166 = 100 + 75 \times \underline{.4166} = 131.245 \approx \$131.25.$$

It was important that we rounded off our final answer because we had rounded off to get .4166 along the way. We could have done the entire calculation at once (avoiding the round off error) as

$$100 + 75 \times 25 \div 60 = 131.25$$

Either way, we owe the plumber \$131.25.

If we plot the points from the table of values in a graph, we see that the points lie on a line.




Why is the graph a line? Remember that the rate of change tells us how steep the graph is. For example, let's find the rate of change between 1 hour and 2 hours.

$$\text{rate of change} = \frac{\text{change dep}}{\text{change indep}} = \frac{\$250 - \$175}{2 \text{ hours} - 1 \text{ hour}} = \frac{\$75}{1 \text{ hour}} = \$75 \text{ per hour}$$

Sure! We knew that. The plumber charges an extra \$75 for each extra hour he works. The rate of change is precisely \$75/hour, no matter where we calculate it. Since the rate of change is constant, the graph is the same steepness everywhere. So, the graph is a line.

Recall that a function with constant rate of change is **linear**. The plumber's total charge is a linear function of time.

Look back at our equation.

$$P = 100 + 75T$$

Any linear equation fits this template.

| |
|---|
| LINEAR EQUATION TEMPLATE: dep = start + slope ∗ indep |

Notice our two variables are in our equation and there are two constants. Each constant has its own meaning. The first constant is 100 and it is measured in dollars. It is the trip charge, the fixed amount we would owe the plumber even if he does 0 hours work. In our standard form we refer to this quantity as the **starting value** (or **start** for short), but it's official name is **intercept**. On the graph it's where the line crosses the vertical access. Think of a football player (running along the vertical axis) intercepting a pass (coming in the line). We can find the intercept from our equation by plugging in $T = 0$:

$$P = 100 + 75 \times 0 = 100$$

The second constant is 75 and though its tempting to say it is measured in dollars, it is really measured in \$ per hour. This number is the rate of change and in the context of linear equations it gets its own name too. Its called the **slope**. Since the rate of change measures the steepness of any curve or line, the word "slope", like mountain slope, makes sense. In our plumber example the intercept was \$100 and the slope was \$75/hour.

## 2.2 A first look at exponential equations

Jocelyn got a job right out of college, as an administrative assistant earning \$28,000 a year. The position turned out to be a great fit for her, and after one year she was promoted to data analyst with a 15% raise. The next year Joceyln was promoted again, to senior data analyst along with a 21% raise. "Not bad," her friend Russell said, "a 36% raise in two years." But Jocelyn quickly corrected him. "Russ, it's even better than that! Over 39% raise."

After the first year, Jocelyn's salary of \$28,000 was increased by 15%. That doesn't mean it was 0.15 more, but rather that it was 15% of \$28,000 more. To calculate 15% of \$28,000 we multiply using the decimal form to get

$$15\% \text{ of } \$28{,}000 = 0.15 \times 28000 = \$4{,}200$$

That's how much Jocelyn's raise was that first year. By adding that amount to the original salary we get

$$28000 + 4200 = 32200$$

After one year Jocelyn's salary was \$32,200.

After the second year, Jocelyn got a 21% raise. This means her rose by 21% from what it was just before the raise, that is, from the \$32,200. (The 21% does not refer back to the original \$28,000 value.) So, to calculate the increase, we take 21% of \$32,200, which is

$$21\% \text{ of } \$32{,}200 = 0.21 \times 32200 = \$6{,}762$$

By adding on this raise we get

$$32200 + 6762 = 38962$$

After the second year Jocelyn was earning \$38,962.

Since Jocelyn's original salary was \$28,000, the net increase in her salary is the difference

$$38962 - 28000 = \$10{,}962$$

The corresponding percentage increase was

$$\text{percentage increase} = \frac{\text{net increase}}{\text{original value}} = \frac{10962}{28000} = 0.3915 = 39.15\%$$

As Jocelyn said, that's over 39% increase.

What's going on here? Russell thought that 15% and 21% would be 36% because

$$15 + 21 = 36$$

The reason it doesn't work that was is that while the 15% is of the original \$28,000, the 21% was actually calculated on the \$32,200. So, we can't just combine percentages by adding.

There's a quicker way to calculate the percentage increase and to combine percentages. Notice that each time we figured out the value of Jocelyn's house, we did a two-step process. First, we calculated the amount of the increase, and second we found the new value by adding on. Notice that when we increase a number by 15%, then what we'll have at the end is the 100% we started with plus the 15% more. That is, we'll have 115% of what we started with. So we can just multiply by 1.15, which is 115% written in decimal. (Looks weird, works great.)

So, in our example, we can just do

$$28000 \times 1.15 = 32200$$

We can do the same thing for the next calculation

$$32200 \times 1.21 = 38962$$

Here we multiplied by 1.21 because after a 21% increase you'll have 121% of what you started with. And 121% in decimal form is just 1.21. A quicker way to calculate the growth factor is

$$1 + .21 = 1.21$$

or from the percentage

$$1 + 21 \div 100 = 1.21$$

.

There are a lot of important applications in which we'll consider repeated percentage increase. So, hang on to your hat because we can combine both of these parts together. In our example, we started with $28,000. Then we multiplied by 1.15, which gave us $32,200. And then we multiplied that answer by 1.21, to get our final answer of $38,962. So really we just did

$$28000 \times 1.15 \times 1.21 = 38962$$

Same answer. A lot less effort.

And check it out,

$$1.15 \times 1.21 = 1.3915.$$

That's where the 39.15% is hidden. Cool.

There are names for the different numbers we are using. The percentage increase is called the ***growth rate*** and the number we multiply in the one-step method is called the ***growth factor***. For example, in calculating 15% increase, the growth rate is 15% and the growth factor is 1.15.

Jocelyn's most recent assignment has been analyzing information on rising health care costs. In 2007 the United States spent $2.26 trillion on health care. For all you zero-philes that's

$$\$2.26 \times 10^{12} = 2.26 \times 10 \wedge 12 = \$2{,}260{,}000{,}000{,}000$$

Did your calculator leave the answer in scientific notation? We saw how to convert in Section SU FILL IN.

Health care costs were projected to increase at an average of 6.7% annually for the subsequent decade. Let's write an equation showing how health care costs are projected to grow. Our variables are

Constant: 6.7% increase

Variables: $H$ = health care costs ($ trillions), dep, $0 \leq H \leq 10$

$Y$ = time (years from 2007), indep, $0 \leq Y \leq 20$

We chose up to 20 years and up to $10 trillion. Even though those numbers might not be realistic, they are definitely large enough.

Since

$$6.7\% = 6.7 \div 100 = .067$$

the growth factor is 1.067. That tells us that to find the effect of a 6.7% increase, we can just multiply by 1.067. In 2007 the United States spent $2.26 trillion on health care. The projection for one year later was

$$2.26 \times 1.067 = 2.41142 \approx \$2.41 \text{ trillion}$$

Another year later, projected health care costs were

$$2.41 \times 1.067 = 2.57147 \approx \$2.57 \text{ trillion}$$

And so on. For each year we multiply by another 1.067.

For example,by 2017 (ten years later), health care cost is projected to be

$$2.26 \times 1.067 \times 1.067 \times 1.067 \times 1.067 \times 1.067 \times 1.067 \times 1.067 \times 1.067 \times 1.067 \times 1.067$$

Don't know about you, but I would rather not type that all into a calculator. A more economical way to write and calculate this product is with an exponent. Health care costs in 2017 are projected to be

$$2.26 * 1.067^{10}$$

On a calculator we can do this calculation in one step as

$$2.26 \times 1.067 \wedge 10 = 4.322675488 \approx \$4.32 \text{ trillion}$$

The order of operations on the calculator does the power before the multiplication, which is exactly what we want.

Most calculators use the $\wedge$ symbol for exponents, as do most computer software packages. Two other notations calculators sometimes use are $y^x$ or $x^y$. Sometimes that operation is accessible through the 2nd or shift key; something like SHIFT $\times$. If you're not sure, ask a classmate or your instructor.

We're so close to the equation now, we can smell it. We just found cost after 10 years. It was

$$\$2.26 * 1.067^{10} \approx \$4.32 \text{ trillion}$$

We can generalize to get the equation by putting in $Y$ years (instead of 10) and $H$ for the health care cost (instead of $4.32). When we do we get

$$\$2.26 * 1.067^{Y} = H$$

Rewriting the equation to begin with the dependent variable we get

$$H = 2.26 * 1.067^Y$$

By the way, there are two other standard ways of writing this equation

$$H = 2.26(1.067)^Y \text{ or also } H = 2.26\left(1.067^Y\right)$$

For example, in 2027 we have $Y = 2027 - 2007 = 20$ years and our equation projects health care cost at

$$2.26 \times 1.067 \wedge 20 = 8.821882053 \approx \$8.82 \text{ trillion}$$

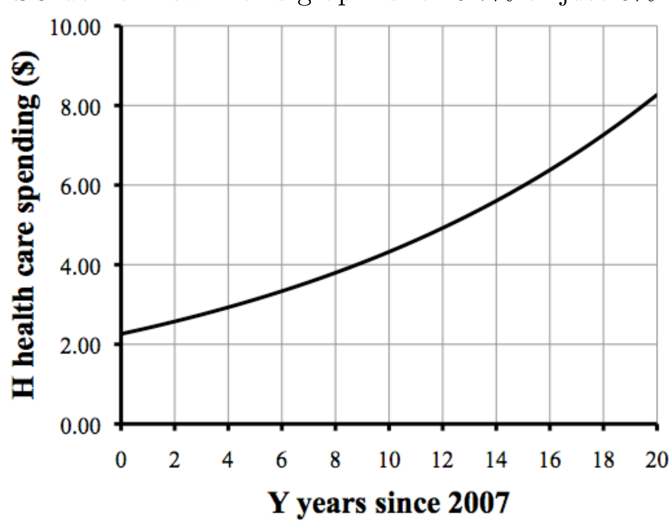
By the way, this type of equation is called an ***exponential equation*** because the independent variable is in the exponent. More specifically, for our purposes an exponential equation has the form

$$\text{dep. var.} = \text{starting amount (growth factor)}^{\text{indep. var.}}$$

Exponential equations are not linear and so their graphs are not lines. Sometimes the graph of an exponential equation looks like a line, especially if you only plot a few points. So, be sure to plot five or more points to see the curve in the graph of an exponential equation. For example, here are five points and the graph. Admittedly expecting a the constant increase of 6.7% over a 20 year time period is not realistic but we need that much of the graph to see the curve.

| year | 2007 | 2008 | 2009 | 2017 | 2027 |
|---|---|---|---|---|---|
| $Y$ | 0 | 1 | 2 | 10 | 20 |
| $H$ | 2.26 | 2.41 | 2.57 | 4.32 | 8.82 |

SU don't know if this graph is for 6.7% or just 6%



## 2.3 Using equations

The Cadillac Escalade is a cross between a sports utility vehicle (SUV) and luxury car. Either way, it's a big car. And it takes awhile to stop. One study showed that the 2010 Escalade traveling at 60 miles per hour takes about 144 feet to come to a complete stop from when you first hit the brakes.

In fact, the braking distance of any car depends on how fast it is going. If you were driving 30 mph on a residential street, then it wouldnt take nearly as far to stop, for example. We would like to be able to calculate the braking distances at other speeds.

Let $S$ be the speed, in miles per hour, and let $F$ be the braking distance that it takes to stop, in feet. Using the data and equations from physics, automobile analysts were able to determine that the equation relating these two variables is

$$F = 0.04S^2$$

Remember that the 0.04 written next to the $S^2$ means they are multiplied. We might equally well have written

$$F = 0.04 * S^2$$

You may be a little surprised to see the variable $S$ squared or wonder what the number 0.04 means. This equation is definitely not easily figured out because it relies both on the data and knowledge of the physics involved. But, we can still work with this equation to find the braking distances at any speed. (If you must know, this equation is only approximate since things like tire and road conditions are a factor, but for what we want itll be good enough.)

By the way, although in the last couple of sections we were able to find the equation for linear or exponential functions by generalizing from examples, in reality there are many different mathematical and statistical techniques for finding equations. A scientist might use lab experiments and some theory to figure it out. An economist might recognize that the equation is from a particular family because of the underlying economics. A store manager might know from years of experience that a certain equation works well to predict sales. It can be comforting to know where an equation comes from but whether we find an equation for ourselves or get it from an expert, we can use it to answer our questions and make predictions.

Now that we have an equation we can calculate the braking distance for a Cadillac Escalade traveling 30, 50, 70 or 90 miles per hour. For 30 miles per hour, we have $S = 30$. So, we substitute 30 in place of the $S$ in the equation to get

$$F = 0.04(30)^2 = 0.04 \times \underline{30} \wedge 2 = 36 \text{ feet}$$

At 30 mph, it will take the Cadillac Escalade 36 feet to stop. As we expected, it doesnt take nearly as far to stop as it did at 60 mph.

For the other speeds we do the same thing: substitute the appropriate value of $S$. At 50 mph, $S = 50$ and so

$$F = 0.04(50)^2 = 0.04 \times \underline{50} \wedge 2 = 100 \text{ feet}$$

At 70 mph, $S = 70$ and so

$$F = 0.04(70)^2 = 0.04 \times \underline{70} \wedge 2 = 196 \text{ feet}$$

At 90 mph, $S = 90$ and so

$$F = 0.04(90)^2 = 0.04 \times \underline{90} \wedge 2 = 324 \text{ feet}$$

What does our equation tell us when the speed is 0 mph? We evaluate at $S = 0$ and so

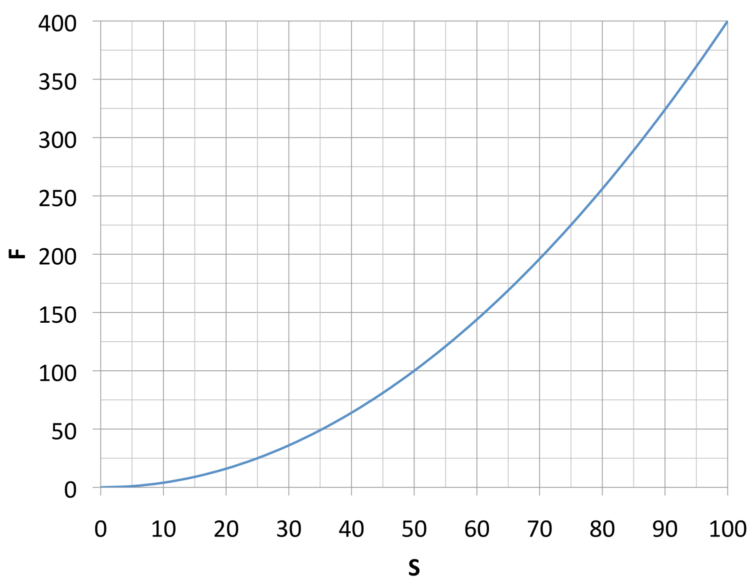
$$F = 0.04(0)^2 = 0.04 \times \underline{0} \wedge 2 = 0 \text{ feet}$$

Well, sure! If the car isn't moving, then it won't need any distance to stop.

A bit of terminology is useful here. When we substituted 30 in for $S$ in the equation, we say we ***evaluated*** at $S = 30$. The verb evaluate is used when we know the value of the independent variable and we substitute it into the equation to find the value of the dependent variable.

We can summarize our data in a table and graph.

| $S$ | 0 | 30 | 50 | 60 | 70 | 90 |
|---|---|---|---|---|---|---|
| $F$ | 0.0 | 36 | 100 | 144 | 196 | 324 |




My neighbor Jeff happens to drive a 2010 Cadillac Escalade. The other day he almost was in an accident on the highway. Luckily no one was hurt, but he had to slam on the brakes to stop. The police report mentioned that the skid marks from Jeff's tires indicate that it took him 183 feet to stop. Jeff says he was not driving over the posted speed limit of 65 mph. Should we believe him?

We can see from the table that braking distance of 183 feet falls in between the 144 and 196 on our table which leads us to believe that Jeff was traveling faster than 60 mph and slower than 70 mph, but we can't answer the question yet. We can figure out if Jeff were driving at 65 mph, then his braking distance would have been

$$F = 0.04(65)^2 = 0.04 \times \underline{65}\wedge^2 = 169 \text{ feet}$$

That's less than the 183 feet Jeff took to stop. So, it appears that Jeff was driving faster than 65 mph.

But wait a minute. The braking distance is just the time it takes from when your foot hits the brake until the car is stopped. That distance doesn't take into account the reaction time – how long it takes between when see the situation and when your foot actually hits the brake. Suppose it takes 1.5 seconds to react. If we write $D$ for the total stopping distance, measured in feet, then it turns out that a reasonable equation for $D$ is

$$D = 1.5S + 0.04S^2$$

Something interesting to note about this equation is that the independent variable, $S$, appears twice. That means when we evaluate the equation we will need to plug in the value of $S$ in two places. Let's do a few examples.

At 30 mph, $S = 30$ we have

$$D = 1.5(30) + 0.04(30)^2 = 1.5 \times \underline{30} + 0.04 \times \underline{30} \wedge 2 = 81 \text{ feet}$$

At 50 mph, $S = 30$ we have

$$D = 1.5(50) + 0.04(50)^2 = 1.5 \times \underline{50} + 0.04 \times \underline{50} \wedge 2 = 175 \text{ feet}$$

You can (and should) check the rest of the values in the table.

| $S$ | 0 | 30 | 50 | 60 | 70 | 90 |
|---|---|---|---|---|---|---|
| $D$ | 0 | 81 | 175 | 234 | 301 | 459 |

These numbers make us rethink Jeff's assertion. Given that he stopped in 183 feet, which is much less than the 234 feet it takes to stop at 60 mph, it looks like Jeff was driving less than 60 mph. If you like, calculate that at 65 mph (the speed limit), it would have taken Jeff

$$D = 1.5(65) + 0.04(65)^2 = 1.5 \times \underline{65} + 0.04 \times \underline{65} \wedge 2 = 266.5 \text{ feet}$$

Again, we should believe Jeff. And, be glad nobody was hurt.

Here's a graph of our new function

SU need graph here too.

## 2.4 Approximating solutions of equations

PLAN TO COMPLETELY REVAMP THIS SECTION USING SOME OF EXISTING 3.5 THIS COMPARISON WILL BECOME A HOMEWORK PROBLEM?

Which country on Earth has the most people? If you guess China and India, in that order, you'd be right. And by a lot compared to other countries. Number three on the list in 2010 was the United States. Here are the numbers from the CIA Factbook:

| Country | Population (2010) | Growth Rate |
|---|---|---|
| China | 1,330,141,295 | 0.494% |
| India | 1,173,108,018 | 1.376% |
| United States | 310,232,863 | 0.97% |

Notice that in comparison to China, the United States has only a fraction of the population. In fact, if we compare

$$\frac{\text{pop China}}{\text{pop U.S.}} = \frac{1,330,141,295}{310,232,863} = 4.287\ldots \approx 4.3$$

so the population of China is around 4.3 times that of the United States.

With the projected growth rates listed, when will the United States' population pass 400 million? When will China's population pass the 1.5 billion mark? Will India's pass it first? Let's tackle each question in turn.

We can measure the population in millions or billions. Either way we will have to convert some of the numbers. Let's use millions.

$$P = \text{population (measured in millions)}$$
$$Y = \text{year (years since 2010)}$$

The United States population began in 2010 at 310,232,863 which is approximately 310 million because

$$310,232,863 \text{ people} * \frac{\text{millions}}{1,000,000} = 310232863 \div 1000000 = 310.232863 \approx 310$$

Or, if you're in the mood to use scientific notation instead

$$310,232,863 \text{ people} * \frac{\text{millions}}{1 \times 10^6} = 310232863 \div (1 \times 10 \wedge 6) = 310.232863 \approx 310$$

Notice how we needed parentheses

As we saw in Section 2.2 SU CITE, since the United States population growth rate is 0.97%, each year we would have 100.97% of our previous population. That is, we multiply by the growth factor 1.0097. Then, our equation for the United States population is

$$P = 310(1.0097)^Y$$

Since we rounded off the population to the nearest million we will need to be sure to round off each of our calculations as well. Notice that we did not round off the growth factor of 1.0097 because that would lead to too much round off error.

Our first question asks when the U.S. population will pass 400 million. Let's try to figure out the answer by guessing. Since we're not sure where to start, let's see what the equation projects for 2011:

$$310(1.0097)^1 = 310 \times 1.0097 \wedge \underline{1} = 313.007 \approx 313 \text{ million}$$

Hardly budged. We'll have to make a much larger guess to get up near 400 million.

How about in 10 years, in the year 2020? The equation gives us

$$310(1.0097)^{10} = 310 \times 1.0097 \wedge \underline{10} \approx 342 \text{ million}$$

Still much less than 400 million.

By the way, if we had used the precise population of 310.232863 million people, we would estimate 341.67355263010 million here instead of 341.41709 million. As you can see these two numbers differ after the decimal point. That's why it's important to round off our answer. Even then, we probably should have 342 instead of 341 – but let's not be that picky about it.

Let's try 20 years, in the year 2030.

$$310(1.0097)^{20} = 310 \times 1.0097 \wedge \underline{20} \approx 376 \text{ million}$$

Still less.

This is going slowly. We would really like to find a point at which the equation gives us more than 400 million. Then we can work backwards from there to narrow things down. How about 50 years?

$$310(1.0097)^{50} = 310 \times 1.0097 \wedge \underline{50} \approx 502 \text{ million}$$

That's too much, but the good news is now we know the answer is between 20 years and 50 years.

Let's summarize what we have so far in a table. Notice how we've added a third row to keep track of our progress for our goal.

| $Y$ | 0 | 1 | 10 | 20 | 50 | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P$ | 310 | 313 | 341 | 376 | 502 | | | | | | |
| vs. 400 | low | low | low | low | high | | | | | | |

We know the answer is between 20 and 50 years, and it seems closer to 20, so let's guess 30 years which gives $\approx$ 414 million as you can check. That means the answer is between 20 years (a bit low) and 30 years (a bit high), so let's split the difference and guess 25 years which gives $\approx$ 395 million. Ooooh, we're getting close. It's between 25 and 30 years, and likely closer to 25 so let's guess 27 years which gives $\approx$ 402 million. Would 26 years have been enough? That gives $\approx$ 398 million, not quite enough. Let's add these numbers to our table.

| $Y$ | 0 | 1 | 10 | 20 | 50 | 30 | 25 | 27 | 26 |
|---|---|---|---|---|---|---|---|---|---|
| $P$ | 310 | 313 | 341 | 376 | 502 | 414 | 395 | 402 | 398 |
| vs. 400 | low | low | low | low | high | high | low | a bit high | a bit low |

According to our equation, the population of the United States should pass 400 million in about 27 years, which would be in the year 2037. By the way, if you're not sure how we got the year 2037, it works to add the numbers

$$2010 + 27 \text{ years } = 2037$$

SU graph with points labelled here, maybe horizontal line at value 400 and the words high and low written into their regions?

The overall strategy we used here is called ***successive approximation***. That's just a fancy way of saying "guess-and-check". It's called "successive" because we're trying to get a closer guess each time. Typically once we have a value that's too big and one that's too small, we guess a value in between (for example, their average). This sort of splitting the difference method of guessing is a rough version of the ***bisection method***. Now you know.

You might be surprised that you're supposed to guess the answer at this point in the course. I mean, in the beginning of the course we didn't have equations, just tables and graphs, and so guessing was all we had to work with. But now we have actual equations, right? In previous courses your instructor or textbook might have emphasized getting the "exact" answer.

Here's why it's different in this course. First, in almost every story in this book the numbers in the problem are approximations, or at least rounded off. If you start with approximations, no matter how exact your mathematics is, the answers will still be approximate.

Second, even if our numbers started out precisely exact, chances are that the equation is only approximating reality. Do we really know what the population growth rate will be in the U.S. over the next fifty years? And, if the equation is just approximate, then no matter how exact the numbers or the mathematics, the answer will again still be approximate.

Last, and this is good news – we really just want approximations. Do you really need to know that working out will burn 427.2889 calories? Isn't 430 calories close enough?

In previous mathematics courses you may have seen ways to solve equations "exactly," and we will talk about those methods in the next chapter of this text. It is true that successive approximations can taking a long time and, because of that, is a bit annoying. Solving techniques we'll learn later are much, much quicker.

But there are two important reasons for using successive approximations. First, the method of successive approximations works in most situations for any type of equation. Solving methods that we will see later on just work for one type of an equation or another

– one technique for linear equations, a different technique for exponential equations, etc. That's a lot of different methods to know.

Second, even if you're going to use a formal equation-solving technique to solve a problem it's a good habit to guess-and-check a bit first to make sure your answer is reasonable. It is easy to make mistakes when using those formal techniques. There's a rather famous quote here that I like, often falsely attributed to the famous economist John Maynard Keynes, but reportedly from logician Carveth Read in 1898.

> It is better to be vaguely right than exactly wrong.

Something to think about.

Okay, enough digression. Let's finish up our example. In 2010 the population of China was reported to be 1,330,141,295 and growing at 0.494% each year. Converting to millions we get the original population of approximately 1,330 million people. The growth rate of 0.494% means each year there will be 100.494% of the previous year and, so, the growth factor should be 1.00494. So the equation for China's population is

$$P = 1,330(1.00494)^Y$$

We are curious when China's population will exceed 1.5 billion. Oh shoot. That's billions and we need millions to use our equation. We'll have to convert. Maybe we should have used billions all along, but at this point we're kinda committed to millions, so here goes.

$$1.5 \text{ billion } * \frac{1 \times 10^9}{1 \text{ billion}} * \frac{1 \text{ million}}{1 \times 10^6} = 1.5 \times 10 \wedge 9 \div 10 \wedge 6 = 1,500$$

Does that make sense?

$$1.5 \text{ billion } = 1,500,000,000 \text{ and } 1,500 \text{ million} = 1,500,000,000$$

Okay.

Let's start with a guess of 30 years because that worked well last time

$$P = 1,330(1.00494)^{30} = 1330 \times 1.00494 \wedge 30 \approx 1,527$$

Since that turns out to be higher than we're looking for (though not too much higher), we can guess in between values as before. Here's our work displayed in a table.

| $Y$ | 0 | 30 | 20 | 25 | 27 | 26 |
|---|---|---|---|---|---|---|
| $P$ | 1,330 | 1,527 | 1,468 | 1,490 | 1,504 | 1,497 |
| vs. 1,500 | low | high | low | low | a bit high | a bit low |

Since

$$2010 + 26 \text{ years } = 2036$$

we estimate that the population of China should pass 1.5 billion in the year 2036.

Will India get there first? Remember India's population in 2010 was 1,173,108,018 and growing at 1.376% per year. The equation for India's population is approximately

$$P = 1173(1.01376)^Y$$

Notice where the decimal point is in our growth factor. Since 1.376% increase means there will be 101.376% of the previous year, our growth factor is 1.01376 as in our equation.

Here's a table of our guesses for India's population.

| $Y$ | 0 | 30 | 20 | 15 | 19 | 18 |
|---|---|---|---|---|---|---|
| $P$ | 1,173 | 1,720 | 1,542 | 1,440 | 1,521 | 1,500 |
| vs. 1500 | low | high | high | low | a bit high | yes! |

It looks like India's population will reach 1.5 billion by the year 2028 – eight years before China.

Which raises a new question. When will India's population pass China's? Let's put our guess together in a table. We need to keep our equations straight:

$$\textbf{China: } P = 1,330(1.00494)^Y \quad \text{and} \quad \textbf{India: } P = 1,173(1.01376)^Y$$

| $Y$ | 0 | 20 | 10 | 15 | 13 | 14 |
|---|---|---|---|---|---|---|
| China: $P$ | **1,330** | 1,468 | **1,397** | 1,432 | **1,418** | **1,425** |
| India: $P$ | 1,173 | **1,542** | 1,345 | **1,440** | 1,401 | 1,420 |
| larger | China | India | China | India | China | China |

It appears that the population of India will pass the population of China around the year 2025.

## 2.5 Finance formulas*

Addison is trying to figure out her finances – finding a good investment for her tax refund, saving for a down payment on a house, and dealing with her student loans. While there are various online tools that will "do the math" for her, Addison would really like to work out the formulas for herself.

First that tax refund. What a relief, \$1,040 back this year. Much as Addison is tempted to catch the next flight to Cancun play on the beach, she knows she should save that money. Her local bank offers her two choices: a savings account paying 1.2% interest compounded monthly or a 3-year certificate of deposit paying 3.0% interest compounded monthly.

Compounded monthly means that the bank will pay her 1/12th of the annual interest each month, and then use that new balance in computing her interest in the month that follows.

For example, the savings account will pay Addison $\frac{1.2\%}{12} = 0.1\%$ each month. As we know from our study of exponential equations, to increase by 0.1% we multiply the balance each month by the monthly growth factor $1 + \frac{0.1}{100} = 1.001$.

Since we know the monthly growth factor, it would make sense to measure time in months. It is customary, however, to measure time in years instead. No big deal. To get the number of months we just multiply by 12. So, after $Y$ years, we have $12Y$ months.

With those subtleties in mind, Addison's account balance of $\$A$ is given by the equation

$$A = 1040(1.001)^{12Y}$$

After one year, Addison would have

$$1040(1.001)^{12} = 1040 \times 1.001 \wedge 12 = 1052.54887\ldots \approx \$1052.55$$

After three years, she would have a not terribly impressive

$$1040(1.001)^{12*3} = 1040 \times 1.001 \wedge (12 \times 3) = 1078.10269\ldots \approx \$1078.10$$

This is complicated. It is probably easier to work with a general formula rather than having to figure it out the equation each time. The general formula that gives an account balance when interest is compounded monthly is

MONTHLY COMPOUND INTEREST FORMULA

$$A = P\left(1 + \frac{r}{12}\right)^{12Y}$$

where the variables are

$$A = \text{account balance (\$)}$$
$$Y = \text{time invested (years)}$$

and the constants we have to plug in are

$$P = \text{initial deposit or "principal" (\$)}$$
$$r = \text{interest rate compounded monthly (as a decimal)}$$

For Addison's savings account we have $p = \$1,040$ and $r = 1.2\% = 0.012$, so we get

$$A = 1040\left(1 + \frac{.012}{12}\right)^{12Y}$$

Since $1 + \frac{.012}{12} = 1.001$, we have the same equation as before. If we use this general form of the equation to find her account balance after one year we have

$$1040\left(1 + \frac{.012}{12}\right)^{12*1} = 1040 \times (1 + .012 \div 12) \wedge (12 \times \underline{1}) = 1052.54887\ldots \approx \$1052.55$$

and after three years we have

$$1040\left(1 + \frac{.012}{12}\right)^{12*3} = 1040 \times (1 + .012 \div 12) \wedge (12 \times \underline{3}) = 1078.10269\ldots \approx \$1078.10$$

as before. And for something new, after six years Addison's balance will be

$$1040\left(1 + \frac{.012}{12}\right)^{12*6} = 1040 \times (1 + .012 \div 12) \wedge (12 \times \underline{6}) = \$1117.60135 \approx \$1117.60$$

But wait. Addison might want to choose that certificate of deposit instead. That pays 3.0% interest compounded monthly, so now $r = 3.0\% = .03$. Equipped with our fancy new formula, we calculate that this account balance would be given by

$$A = 1040\left(1 + \frac{.03}{12}\right)^{12Y}$$

After one year Addison would have

$$1040\left(1 + \frac{.03}{12}\right)^{12*1} = 1040 \times (1 + .03 \div 12) \wedge (12 \times \underline{1}) = 1071.6326\ldots \approx \$1071.63$$

after three years she would have

$$1040\left(1 + \frac{.03}{12}\right)^{12*3} = 1040 \times (1 + .03 \div 12) \wedge (12 \times \underline{3}) = 1137.81346\ldots \approx \$1137.81$$

and after six years

$$1040\left(1 + \frac{.03}{12}\right)^{12*6} = 1040 \times (1 + .03 \div 12) \wedge (12 \times \underline{6}) = \$1244.82641\ldots \approx \$1244.83$$

It looks like the certificate of deposit is a clear winner, but there is a catch. If Addison wants her money before the three year term is up, she'll lose all (or most) of the interest earned. Ouch. So Addison should decide not only based on what the accounts pay – \$1,078.10 after 3 years in the savings account versus \$1,137.81 after three years in the certificate of deposit (a whopping \$59.71 more) – but also on whether she is comfortable leaving the money alone for three years or not.

Unimpressed by the \$59.71 difference after three years and uncomfortable locking her money in for that long, Addison decides on the savings account. When she reads the account information carefully she is surprised to see the account pays "1.207% APR." What does that mean?

APR stands for "annual percentage rate". It means that 1.2% interest compounded monthly has the same net effect as paying 1.207% at the end of each year. Where does that number come from? Imagine \$1 in the account. Silly, yes, but watch what we learn. After one year the balance would be

$$1\left(1+\frac{.012}{12}\right)^{12*1} = (1+.012 \div 12) \wedge 12 = 1.01206622\ldots$$

That tells us the annual growth factor is $1.01206622\ldots \approx 1.01207$ which corresponds to an annual growth rate of $.01207 = 1.207\%$ APR. There's a formula for this too.

EQUIVALENT ANNUAL PERCENTAGE RATE (APR) FORMULA

$$\text{APR} = \left(1+\frac{r}{12}\right)^{12} - 1$$

To check our work, use $r = 1.2\% = .012$ to get

$$\text{APR} = (1+\underline{.012} \div 12) \wedge 12 - 1 = .01206622 \approx .01207 = 1.207\%$$

All this thinking about savings reminds Addison that she wants to own her own place someday. She promised herself that she would start putting away some money each month to save for a down payment on a house, or maybe condo. Living back in her parent's house rent-free, postponing buying her first car, and bringing lunch from home most days leaves Addison nearly \$1,000 per month to save. Her bank offers a special savings account paying 4.2% compounded monthly if she commits to depositing \$1,000 every month.

Suppose Addison deposits \$1,000 to open the account at the end of the month. At the end of the next month, the account adds $\frac{4.2\%}{12} = .35\%$ so we multiply by $1 + \frac{.35}{100} = 1.0035$ to add in the interest and add another \$1,000 deposit, bringing the balance to

$$1000(1.0035) + 1000 = \$2003.50$$

At the end of the third month, we add another .35% interest (this time on the \$2003.50) and then add another \$1,000 deposit to get

$$2003.50(1.0035) + 1000 = \$3010.51$$

Any sequence of regular deposits is called an **annuity**. The general formula that gives the account balance for an annuity is

VALUE MONTHLY DEPOSITS FORMULA

$$A = P\,\frac{\left(1+\frac{r}{12}\right)^{12Y} - 1}{\frac{r}{12}}$$

where the variables are

$$A = \text{account balance } (\$)$$
$$Y = \text{time invested (years)}$$

and the constants we have to plug in are

$$P = \text{regular deposit } (\$)$$
$$r = \text{interest rate compounded monthly (as a decimal)}$$

In Addison's situation, we have $P = \$1000$ and $r = 4.2\% = .042$ so the equation becomes

$$A = 1000 \, \frac{\left(1 + \frac{.042}{12}\right)^{12Y} - 1}{\frac{.042}{12}}$$

Let's check our calculations for the end of three months. Remembering that time is measured in years, we convert

$$3 \text{ months } = 3 \cancel{\text{months}} \, \frac{1 \text{ year}}{12 \cancel{\text{months}}} = \frac{3}{12} \text{ years} = .25 \text{ years}$$

That means $Y = .25$ and so

$$A = 1000 \, \frac{\left(1 + \frac{.042}{12}\right)^{12*.25} - 1}{\frac{.042}{12}}$$

$$= 1000 \times ((1 + .042 \div 12) \wedge (12 \times \underline{.25}) - 1) \div (.042 \div 12)$$

$$= 3010.5123 \ldots \approx \$3010.51$$

as before. Notice how we need parentheses not only where they appear in the formula, but also around the entire numerator (top) of the fraction, around the entire denominator (bottom) of the fraction, and around the exponent. That's going to take some practice to get used to.

Let's practice by finding her balance after two years. Now $Y = 2$ so

$$A = 1000 \, \frac{\left(1 + \frac{.042}{12}\right)^{12*2} - 1}{\frac{.042}{12}}$$

$$= 1000 \times ((1 + .042 \div 12) \wedge (12 \times \underline{2}) - 1) \div (.042 \div 12)$$

$$= 24991.2560 \ldots \approx \$24{,}991.26$$

Wow. She'll be buying her own house in no time.

Oh, but wait, there's those looming student loans. Addison currently owes $16,700 at 5.75% interest compounded monthly. She's ready to start paying it back every month, which means this loan repayment is another example of an annuity. The general formula that gives the monthly payment due for an annuity is

MONTHLY LOAN PAYMENT FORMULA

$$P = \frac{A * \frac{r}{12}}{1 - \left(1 + \frac{r}{12}\right)^{-12Y}}$$

where the variables are

$$P = \text{regular payment (\$)}$$
$$Y = \text{time invested (years)}$$

and the constants we have to plug in are

$$A = \text{loan amount (\$)}$$
$$r = \text{interest rate compounded monthly (as a decimal)}$$

In Addison's situation, $A = 16700$ and $r = 5.75\% = .0575$ so the equation is

$$P = \frac{16700 * \frac{.0575}{12}}{1 - \left(1 + \frac{.0575}{12}\right)^{-12Y}}$$

If she wants to pay back the loan in two years ($Y = 2$), she will need to pay

$$P = \frac{16700 * \frac{.0575}{12}}{1 - \left(1 + \frac{.0575}{12}\right)^{-12*\underline{2}}}$$

$$= (16700 \times .0575 \div 12) \div (1 - (1 + .0575 \div 12) \wedge (-12 \times \underline{2}))$$

$$= 738.2744\ldots \approx \$738.27$$

That \$738.27 per month payment on her student loan won't leave much money for her to save for that down payment.

If she takes the full five years to pay back the loan (as allowed in her loan agreement), then we plug in $Y = 5$ to get her new monthly payment of

$$P = \frac{16700 * \frac{.0575}{12}}{1 - \left(1 + \frac{.0575}{12}\right)^{-12*\underline{5}}}$$

$$= (16700 \times .0575 \div 12) \div (1 - (1 + .0575 \div 12) \wedge (-12 \times \underline{5}))$$

$$= 320.9200\ldots \approx \$320.92$$

That's more like it, \$320.92 per month.

# Chapter 3

# Solving equations

Yada yada

## 3.1 Solving linear equations

Your kitchen sink keeps getting clogged. Very annoying. Last time the plumber was able to fix it pretty quickly, well ahead of when your dinner guests were due. But now the sink is clogged again. This time when the plumber comes and unclogs the sink, he suggests replacing the trap and a few other things that were causing the problem. You are pretty tired of it clogging up and tell him to "go ahead." While you're glad that the sink works when he's done, you're a bit shocked when his bill arrives a few days later for \$278.75. Does that seem right?

Remember our plumber charged \$100 for just showing up and then \$75 per hour for the service. Using the variables $T$ for the time the plumber takes, measured in hours, and $P$ for the total plumber's charge, measured in dollars, we found that the equation was

$$P = 100 + 75T$$

Let's figure out how many hours of work would add up to a bill of \$278.75. Our first approach might be to look at a table. From earlier we had

| $T$ | 0 | $\frac{1}{2}$ | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $P$ | 100.00 | 137.50 | 175.00 | 250.00 | 325.00 |

Since \$278.75 is between \$250.00 and \$325.00, we see that the time must be between 2 and 3 hours. You remember the plumber being there over 2 hours, so this is certainly a reasonable answer. Well, a lot of money, but mathematically it makes sense.

Still curious, you'd like to know exactly how many hours and minutes he worked. We could use successive approximations.For example, for 2.5 hours the bill would have been

$$100 + 75 \times \underline{2.5} = \$287.50$$

which is more than our bill. Continuing to guess and check, and displaying our work in a table, we get

| $T$ | 2 | 3 | 2.5 | 2.3 | 2.4 | 2.35 | 2.37 | 2.38 |
|---|---|---|---|---|---|---|---|---|
| $P$ | 250.00 | 325.00 | 287.50 | 272.50 | 280.00 | 276.25 | 277.75 | 278.50 |
| vs. 278.75 | low | high | high | low | high | low | low | close enough |

So now we know that the plumber took approximately 2.38 hours.

Converting units we calculate

$$.38 \text{ hours} * \frac{60 \text{ minutes}}{1 \text{ hour}} = .38 \times 60 = 22.8 \approx 23 \text{ minutes}$$

The plumber took about 2 hours, 23 minutes. Thinking back, the plumber had arrived around 10:30 in the morning and stayed past lunch, probably until around 1:00 p.m. That's about right.

Wait a minute! We could have figured this out much more quickly. If the bill was \$278.75, we know the first \$100 was the trip charge. That leaves

$$\$278.75 - \$100.00 = \$178.75$$

in hourly charges. At \$75 per hour that comes to

$$\$178.75 * \frac{1 \text{ hour}}{\$75} = 178.75 \div 75 = 2.388\ldots \approx 2.388 \text{ hours}$$

which comes to around 2 hours, 23 minutes as before. See how we used the \$75/hour as a unit conversion here? Very clever.

That worked well. But, can we figure out this sort of calculation in other problems? What is the general method we're using? Can we write down our method in an organized fashion so that someone else could follow our thinking here? Turns out there is a formal way to show this calculation, called ***symbolically solving the equation***. Officially *any* method of getting a solution to an equation is considered solving the equation, but in the rest of this book, and in most places that use algebra, when we refer to "solving the equation" we mean *symbolically*.

Here's how it works. We want to figure out when $P = 278.75$. We know from our equation that $P = 100 + 75T$. Replace $P$ by $100 + 75T$ in the equation to get

$$100 + 75T = 278.75$$

Remember that the equal sign indicates that the two sides are the same number. On the left-hand side we have $100 + 75T$. On the right-hand side we have 278.75. Looks different, but same thing. That is, we're looking for the value of $T$ that makes these two sides the same number.

The first thing we did to figure out the answer was subtract the \$100 trip charge. In this formal method, we can subtract 100 from each side of our equation. I mean, if the left-hand side and the right-hand side are the same number, then we sure better get the same answer when we take away 100 from each side, right? When we subtract 100 from each side we get

$$\begin{aligned} \cancel{100} + 75T &= 278.75 \\ -\cancel{100} \quad &\quad\; -100 \end{aligned}$$

which simplifies to

$$75T = 178.75$$

because the $+100$ and $-100$ cancelled.

The next thing we did to figure out the answer was divide by the \$75/hour charge. In this formal method, we can divide both side of our equation by 75. Again, if the left-hand side and right-hand side are the same number, then we will definitely get the same answer when we divide by 75. Here goes:

$$\frac{\cancel{75}T}{\cancel{75}} = \frac{178.75}{75}$$

Notice that we wrote the division in fraction form (instead of using $\div$). To understand why the 75's cancelled, remember that $75T$ is short for $75 * T$ and so

$$\frac{75T}{75} = \frac{75 * T}{75} = 75 \times T \div 75 = T$$

because the $\times 75$ and $\div 75$ cancelled.

So we have

$$T = \frac{178.75}{75} = 178.75 \div 75 = 2.388\ldots$$

as before. Yet again our answer is around 2 hours, 23 minutes.

Let's practice working with this symbolic way of solving equations. Suppose instead the plumber went to my neighbor's house and billed her for \$160. How long did the plumber work at my neighbor's? As before, we begin with our equation

$$P = 100 + 75T$$

and we are looking for $P = 160$. Putting these together we get

$$100 + 75T = 160$$

Then, we subtract 100 from both sides

$$\begin{aligned} \cancel{100} + 75T &= 160 \\ -\cancel{100} \quad &\quad -100 \end{aligned}$$

which simplifes to

$$75T = 60$$

Last, we divide both sides by 75 to get

$$\frac{\cancel{75}T}{\cancel{75}} = \frac{60}{75}$$

which simplifies to

$$T = 60 \div 75 = 0.8 \text{ hours}$$

We have solved the equation, but it would make more sense to report our answer in minutes so we convert

$$0.8 \text{ hours} * \frac{60 \text{ minutes}}{1\text{hour}} = 0.8 \times 60 = 48 \text{ minutes}$$

The plumber worked for 48 minutes at my neighbor's house.

Let's quick check this answer. Since $T$ is measured in hours we need to go back and use $T = 0.8$, not 48 which is in minutes. Evaluating in our original equation we get

$$P = 100 + 75 \times \underline{0.8} = 160$$

Yes!

You might be wondering how we knew to subtract the 100 first and then later divide by 75. In this particular situation we had figured it out already and knew it made sense to take the \$100 right off the top. But, in general, how would we know? It turns out that when solving an equation we do the operations in the *opposite* order from evaluating:

When evaluating, first do $\times, \div$, and then do $+, -$

When solving, first do $+, -$, and then do $\times, \div$

When we solve more complicated equations we will see that this reverse order of operations for solving works more generally.

## 3.2 Solving linear inequalities

In the United States temperatures for everyday things like the weather or cooking are given in Fahrenheit, denoted °F. In this system, water freezes into ice at 32°F and boils into steam at 212°F. A common setting for room temperature is 68°F whereas average human body temperature is around 98.6°F. And, most importantly, chocolate brownies bake at 350°F.

In the sciences, medicine, and most other countries, temperatures are measured in Celsius, denoted °C, instead. ( For those of us who grew up in the 1960s or earlier, "Celsius" is the temperature scale formerly known as "centigrade.") For comparison's sake, it's useful to know that water freezes at 0°C and boils at 100°C. (Not coincidentally. It was set up that way.) Room temperature is 20°C whereas now average human body temperature is 37°C. And those brownies?

A common conversion is given by the equation

$$F = 1.8C + 32$$

where $F$ is the temperature measured in Fahrenheit and $C$ is the temperature measured in Celsius. (You may have seen this equation before with fractions in it: $F = \frac{9}{5}C + 32$. Just another way to write the equation, since $\frac{9}{5} = 9 \div 5 = 1.8$.)

Let's check the information given to us in the story. When $C = 0$ we have

$$F = 1.8 \times \underline{0} + 32 = 32 \quad \checkmark$$

When $C = 100$ we have

$$F = 1.8 \times \underline{100} + 32 = 212 \quad \checkmark$$

When $C = 20$ we have

$$F = 1.8 \times \underline{20} + 32 = 68 \quad \checkmark$$

When $C = 37$ we have

$$F = 1.8 \times \underline{37} + 32 = 98.6 \quad \checkmark$$

What about those chocolate brownies? We are looking for $F = 350$. That's the dependent variable, so we can practice our newly found linear equation solving skills to find the independent variable, $C$. Using $F = 1.8C + 32$ we get

$$1.8C + 32 = 350$$

Subtract 32 from each side to get

$$\begin{array}{rcl} 1.8C + \cancel{32} & = & 350 \\ -\cancel{32} & & -32 \end{array}$$

which simplifes to

$$1.8C = 318$$

Then, divide each side by 1.8 to get

$$\frac{\cancel{1.8}C}{\cancel{1.8}} = \frac{318}{1.8} = 176.666666... \approx 177°C$$

Chocolate brownies bake at around $177°C$.

Actually, the recipe for chocolate brownies says to bake in a "moderate oven". That means between $325°F$ and $375°F$. Let's first figure out when the oven is under $375°F$. In this situation we want to know when

$$F \leq 375$$

so we have an inequality instead of an equation. (Remember $\leq$ stands for "less than or equal to.") Using $F = 1.8C + 32$ we get

$$1.8C + 32 \leq 375$$

We're looking for values of $C$ that make the left-hand side a number that's smaller than, or maybe as large as, 375, but no larger.

To solve this inequality we begin the same way as when we were solving the equation, by subtracting 32 from each sides to get

$$\begin{array}{rcl} 1.8C + \cancel{32} & \leq & 375 \\ -\cancel{32} & & -32 \end{array}$$

which simplifes to

$$1.8C \leq 343$$

To understand why the inequality stays the same when we subtract the same number from both sides, think of the inequality as

$$\text{“little”} \leq \text{“big”}$$

If one number is littler than the other, the same will be true when we take away equal amounts. So, for example, say that you have more cash than I do and we each buy a \$12 movie ticket – maybe you have \$21 to start with but I only have \$18. Afterwards, it will still be true that you have more cash than I do. You will have \$21 - \$12 = \$9 left but I will only have \$18 - \$12 = \$6. I mean, we each will have less cash than before we bought the ticket, but you still have more than I do. Which is why you should buy the popcorn. But I digress.

Back to our example. We had $1.8C \leq 343$. Divide each side by 1.8 to get

$$\frac{\cancel{1.8}C}{\cancel{1.8}} \leq \frac{293}{1.8} = 190.555555... \approx 190°C$$

The oven should be set at most $190°C$. We rounded down because we do not want the brownies to burn.

To understand why the inequality stays the same when we divided each side by the same number, think again of the inequality as

$$\text{“little”} \leq \text{“big”}$$

If one number is littler than the other, the same will be true when we divide both amounts by the same number. So, for example, suppose we decided against buying popcorn so that that you have \$9 left from the movie and I only had \$6. While we're making up stories, suppose we each have three children, who bought their own tickets but want some money from us fortreats. We each divide our remaining cash among our three children, respectively. Yours each get $\$9 \div 3 = \$3$ but mine only get $\$6 \div 3 = \$2$. So mine get less than yours.

There is a little bit of caution when solving inequalities. When symbolically solving an equation, any operation you do to both sides preserves the equality – start with equal amounts, do same thing to each, end with equal amounts. But, when symbolically solving an inequality, only some operations you do to both sides preserve the inequality – add or subtract from both sides, multiply or divide both sides by the same (positive) number. But other operations can reverse the inequality.

For example, we can swap sides of an equation, but if we swap sides of an inequality then the direction of the sign reverses. In this brownie example, we want

$$F \geq 325$$

(Remember $\leq$ stands for "less than or equal to.") We can rewrite that inequality as

$$325 \leq F$$

In each case, 325 is "little" and $F$ is "big". Make sense? Multiplying or dividing by a negative numbers switches the inequality sign as well, as and fancier operations that we won't see in this course.

Remember that the recipe for chocolate brownies says to bake in a moderate oven, between $325°F$ and $375°F$. We just figured out that $F \leq 375$ corresponds to $C \leq 190$. But that's only half of the story. We also wanted $F \geq 325$. While we could solve that inequality separately, it turns out there's an easier way.

Inequalities are a very useful notation for indicating "between". We want between $325°F$ and $325°F$ to bake the brownies. We can write

$$325 \leq F \leq 375$$

which is read

"$F$ is between 325 and 375 (inclusive)"

The word "inclusive" indicates that we're allowing $F = 325$ or $F = 375$.

Anyway, we can solve this chain of inequalities all at once using the same steps as before but now being sure to do the same thing to all *three* sides. "*Three* sides?" you ask. Yes, "three," I confirm. Watch how this works.

Start with

$$325 \leq F \leq 375$$

Using $F = 1.8C + 32$ we get

$$325 \leq 1.8C + 32 \leq 375$$

Subtract 32 from each of the three sides to get

$$\begin{array}{ccccc} 325 & \leq & 1.8\text{C} + \cancel{32} & \leq & 375 \\ \text{-32} & & \text{-}\cancel{32} & & \text{-32} \end{array}$$

which simplifies to

$$293 \leq 1.8C \leq 343$$

Next, divide all three sides by 1.8 to get

$$\frac{293}{1.8} \quad \leq \quad \frac{\cancel{1.8}C}{\cancel{1.8}} \quad \leq \quad \frac{343}{1.8}$$

which simplifies to

$$293 \div 1.8 \leq C \leq 343 \div 1.8$$

so,

$$162.777777... \leq C \leq 190.555555...$$

Best yet, say

$$163 \leq C \leq 190$$

Chocolate brownies bake between $163°C$ and $190°C$. Oven actually aren't that precise, so somewhere between $170 - 190°C$ should do the job.

## 3.3 Solving power equations (and roots)

There's an old saying – "when life gives you lemons, make lemonade." But how many lemons do you need? It turns out a reasonable equation describing the yield of lemonade from a single lemon is given by

$$J = 0.0185C^3$$

where $J$ is the juice, measured in tablespoons, and $C$ is the circumference of the lemon, measured in inches. (In case you've forgotten, the circumference is the distance *around* the lemon. Think of taking a piece of string and wrapping it around the middle part of the lemon. Then lay the string on a rule to see how long it is.)

A small lemon might measure 6 inches in circumference. According to our equation, it would yield

$$J = 0.0185(6)^3 = 0.0185 \times \underline{6} \wedge 3 = 3.996 \approx 4 \text{ tablespoons}$$

A regular-sized lemon, say 8 inches in circumference, would yield

$$J = 0.0185(8)^3 = 0.0185 \times \underline{8} \wedge 3 = 9.472 \approx 9.5 \text{ tablespoons}$$

Let's make a table of values and look at a graph of this function. We've added some values (including some unrealistic ones) to see the shape better.

| $C$ | 0 | 2 | 4 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| $J$ | 0 | 0.148 | 1.184 | 3.996 | 6.3455 | 9.472 | 13.4865 | 18.5 |

SU GRAPH

How large a lemon would yield half a cup of juice? Remember 1 cup is 16 tablespoons so

$$\frac{1}{2} \text{ cup} * \frac{16 \text{ tablespoons}}{1 \text{ cup}} = 16 \div 2 = 8 \text{ tablespoons}$$

From our graph it look like 7.5 inches in circumference should be pretty close. Let's use successive approximation to get the answer to two decimal places. We'll round our value of $J$ to two decimal places to fit easily into the table.

| $C$ | 7 | 8 | 7.5 | 7.6 | 7.55 | 7.56 | 7.57 |
|---|---|---|---|---|---|---|---|
| $J$ | 6.35 | 9.47 | 7.80 | 8.12 | 7.96 | 7.99 | 8.03 |
| vs. 8 | low | high | low | high | low | not quite | just over |

So a lemon with circumference of approximately 7.57 inches should yield about half a cup of juice.

That didn't take too much work. But this chapter is all about solving equations. Much as we have learned to love successive approximation (and for good reason as it works on any type of equation), you might be happy to know that there is a way to solve power equations exactly. Here's how.

Start with what we're looking for

$$J = 8$$

Next, use our equation $J = 0.0185C^3$ to get

$$0.0185C^3 = 8$$

We want to find the value of $C$, so we can divide both sides by 0.0185 to get

$$\frac{\cancel{0.0185}C^3}{\cancel{0.0185}} = \frac{8}{0.0185} = 432.432\ldots$$

Thus we have

$$C^3 = 432.432\ldots$$

How can we find $C$ just knowing $C^3$? We take cube roots of both sides to get

$$\sqrt[3]{C^3} = \sqrt[3]{432.432\ldots} = 432.432\ldots^{1/3} = 432.432\ldots \wedge (1 \div 3) = 7.56204\ldots$$

As before we see a lemon of about 7.57 inches should yield half a cup of juice.

Perhaps a brief digression on roots is in order. (And, that mysterious $\wedge(1 \div 3)$ business needs some explanation, I expect.) For example, we know

$$2^3 = 2 \wedge 3 = 8$$

which tells us

$$\sqrt[3]{8} = 2$$

The ***cube root*** of a number is whatever you would cube to get that number. Similarly

$$\sqrt[3]{1000} = 10$$

because

$$10^3 = 10 \wedge 3 = 1000$$

Some calculators have a key, or sequence of keys, that will take the cube root. For example, if your calculator has a key labeled $\sqrt[x]{y}$, then you can use it to find the cuberoot by doing

$$\sqrt[3]{8} = 3 \ \sqrt[x]{y} \ 8 = 2$$

Be careful if the key is just labelled $\sqrt{x}$ because that only takes square roots and can't be used for cube roots.

Otherwise you need to know that cube root is equivalent to raising to the 1/3rd power. So you can calculate

$$\sqrt[3]{8} = 8^{1/3} = 8 \wedge (1 \div 3) = 2$$

as we did in our example. Notice how the 1/3 is in parentheses. That's because we want the division before the exponent but the usual order of operations is the other way. If your calculator has a fraction entering key, you might be able to do

$$\sqrt[3]{8} = 8^{1/3} = 8 \wedge 1/3 = 2$$

instead, but don't forget to use the fraction key / not the divides key $\div$.

In general, the $n$***th root*** of a number is whatever number you would raise to the $n$ power to get the number. The formula is

THE ROOT FORMULA

$$\text{The equation } C^n = v \text{ has solution } C = \sqrt[n]{v} = v^{1/n}$$

SU – decide which comes first: solving expl or solving power, then have 2nd one refer to the 1st.

Back to our lemonade example. Let's practice solving this type of equation. If we wanted a lemon that yields 10 tablespoons of juice we would solve

$$J = 10$$

by putting in our equation we get

$$0.0185C^3 = 10$$

Next, divide each side by 0.0185 to get

$$\frac{\cancel{0.0185}C^3}{\cancel{0.0185}} = \frac{10}{0.0185}$$

so that

$$C^3 = 10 \div 0.0185 = 540.540\ldots$$

Then take cuberoots to get

$$C = \sqrt[3]{540.540\ldots} = 540.540\ldots \wedge (1 \div 3) = 8.14596\ldots \approx 8.1 \text{ inches}$$

As before we see that we solve in the reverse order of operations.

If evaluating, first do $\wedge$ (power), then do $\times, \div$

If solving, first do $\times, \div$, then do $\sqrt[n]{\phantom{x}}$ (root)

A few pages of calculations into our example and we have only half a cup of lemonade to show for it. How about if we're trying to make a 1/2 gallon pitcher of lemonade? Suppose the store sells lemons by the bag, where all the lemons in the bag are just about the same size (all small, all medium, etc.). How many lemons of a fixed size will it take to make 1/2 gallon of lemonade? We can write a new equation to describe this situation.

First, convert 1/2 gallon into tablespoons.

$$\frac{1}{2} \text{ gallon } * \frac{4 \text{ quarts}}{1 \text{ gallon}} * \frac{4 \text{ cups}}{1 \text{ quart}} * \frac{16 \text{ tablespoons}}{1 \text{ cup}} = 4 \times 4 \times 16 \div 2 = 128 \text{ tablespoons}$$

Let $L$ be the number of lemons (of a fixed size) and suppose each lemon yields $J$ tablespoons of juice. For example, if $J = 10$ tablespoons, then

$$\frac{128 \text{ tablespoons}}{10\frac{\text{tablespoons}}{\text{lemon}}} = 128 \div 10 = 12.8 \text{ lemons}$$

so we would need 13 lemons. In general the number of lemons we need is given by the equation

$$L = \frac{128}{J}$$

Look at a table of a few values and graph for this function. Notice that it's decreasing because the larger the lemons, the fewer we need to use. That makes sense, doesn't it?

| $J$ | 4 | 6 | 10 | 12 | 16 |
|---|---|---|---|---|---|
| $L$ | 32.0 | 21.3 | 12.8 | 10.7 | 8 |

SU NEED GRAPH

Last time I made a pitcher of lemonade it took 9 lemons. How much juice did each yield (assuming they're all about the same size)? Let's solve our equation to find the answer. Start with what we want

$$L = 9$$

and use our equation to get

$$\frac{128}{L} = 9$$

We haven't seen an equation like this before, where the independent variable in in the denominator (bottom) of the fraction, but not to worry. Remembering that $\frac{128}{L}$ means $128 \div L$, we can multiply both sides of the equation by $L$ to get

$$\cancel{L} * \frac{128}{\cancel{L}} = 9 * L$$

so that

$$9L = 128$$

(We switched the variable onto the left-hand side for convenience.) Now the equation looks much more familiar. Dividing both sides by 9 gives us

$$\frac{\cancel{9}L}{\cancel{9}} = \frac{128}{9}$$

so

$$L = \frac{128}{9} = 128 \div 9 = 14.22\ldots \approx 14.2$$

Each lemon must have yielded around 14.2 tablespoons of juice.

And what goes better with lemonade than lemon cake. For that we're going to need some grated lemon peel. As with juice, the amount of lemon peel depends on the size of the lemon. One equation is

$$P = 0.061C^2$$

where $C$ is the circumference of the lemon, measured in inches, as before and $P$ is the amount of lemon peel, measured in tablespoons. For example, a lemon of circumference 7 inches will produce about 3 tablespoons of grated lemon peel because

$$0.061 * 7^2 = 0.061 \times \underline{7} \wedge 2 = 2.989 \approx 3$$

As before, we can look at a table of select values (not all of which are realistic).

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $C$ | 0 | 2 | 4 | 6 | 7 | 8 | 9 | 10 |
| $P$ | 0 | 0.244 | 0.976 | 2.196 | 2.989 | 3.904 | 4.941 | 6.1 |

SU NEED GRAPH

What size lemon would give 4 tablespoons of grated lemon peel? From the graph it looks like just over 8 inches in circumference should do. A quick successive approximation shows it's around 8.1 inches in circumference.

| | | | |
|---|---|---|---|
| $C$ | 8 | 9 | 8.1 |
| $P$ | 3.904 | 4.941 | 4.00221 |
| vs. 4 | bit low | high | close |

To solve exactly we begin with what we want

$$P = 4$$

and use our equation to get

$$0.061C^2 = 4$$

Then divide both sides by 0.061 to get

$$\frac{\cancel{0.061}C^2}{\cancel{0.061}} = \frac{4}{0.061}$$

which simplifies to

$$C^2 = \frac{4}{0.061} = 4 \div 0.061 = 65.57\ldots$$

Now to undo the square we use regular square roots. There is most likely a special key on your calculator for square roots, but we'll do it the same as any other root here.

$$\sqrt{C^2} = \sqrt{65.57\ldots}$$

so

$$C = \sqrt{65.57\ldots} = 65.57\ldots^{1/2} = 65.57\ldots \wedge (1 \div 2) = 8.097\cdots \approx 8.1$$

as before.

## 3.4 Solving exponential equations (and logs)

Remember Jocelyn? She was asked to analyze information on rising health care costs. In 2007 the United States spent $2.26 trillion on health care and costs were projected to increase at an average of 6.7% annually for the subsequent decade. She found

$$H = 2.26 * 1.067^Y$$

where $H$ is health care costs (in $ trillions) and $Y$ is time (in years from 2007). Here are some sample years.

| year | 2007 | 2008 | 2009 | 2017 | 2027 |
|---|---|---|---|---|---|
| $Y$ | 0 | 1 | 2 | 10 | 20 |
| $H$ | 2.26 | 2.41 | 2.57 | 4.32 | 8.82 |

When did health care costs first pass $3 trillion? We can use successive approximation to find the answer. The answer is some time between 2009 and 2017. Let's split the difference and guess 2013. For that year, $Y = 2013 - 2007 = 6$ and so

$$H = 2.26 * 1.067^6 = 2.26 \times 1.067 \wedge \underline{6} = 3.334993223 \approx \$3.33 \text{ trillion}$$

which is already over $3 trillion. What about 2011? Then $Y = 4$ and so

$$H = 2.26 \times 1.067 \wedge \underline{4} = 2.929315279 \approx \$2.93 \text{ trillion}$$

almost, but not quite. Must be 2012 was the year. Let's check. $Y = 5$ and

$$H = 2.26 \times 1.067 \wedge \underline{5} = 3.125579403 \approx \$3.12 \text{ trillion}$$

That's it. Health care costs first passed $3 trillion in 2012. Well, at least according to our equation. As usual, let's display our work here in a table.

| year | 2009 | 2017 | 2013 | 2011 | 2012 |
|---|---|---|---|---|---|
| $Y$ | 2 | 10 | 6 | 4 | 5 |
| $H$ | 2.57 | 4.32 | 3.33 | 2.93 | 3.12 |
| | low | high | high | a bit low | just passed |

Successive approximation once again gives us the answer fairly quickly. But there is an even quicker way – solving the exponential equation.

Start with what we're looking for

$$H = 3$$

Next, use our equation $H = 2.26 * 1.067^Y$ to get

$$2.26 * 1.067^Y = 3$$

We want to find the value of $Y$, so we can divide both sides by 2.26 to get

$$\frac{\cancel{2.26} * 1.067^Y}{\cancel{2.26}} = \frac{3}{2.26} = 1.327433628\ldots$$

Thus we have

$$1.067^Y = 1.327433628\ldots$$

Hmm. How do we find $Y$ here? We saw how to use roots to solve power equations. In power equations, like $C^3 = 432.432\ldots$, we know the exponent (3) and want to find the number being raised to that power ($C$). That's when we take roots to get

$$C = \sqrt[3]{432.432\ldots} \approx 7.57$$

That approach is not going to work here because it's backwards now – we know the number being raised to a power (1.067) and are on the hunt for the exponent ($Y$).

Turns out there's a different formula here using something called ***logarithms*** (nickname: ***logs***). More about logs in a minute, but first let's write down the formula and practice working with it. The formula is

THE LOG-DIVIDES FORMULA

$$\text{The equation } g^Y = v \text{ has solution } Y = \frac{\log(v)}{\log(g)}.$$

Quick aside about the name. Some formulas have well-known names. Not this one. We call it the "Log-Divides Formula" because it has logs and divides in it. (Bet you already guessed that.) Other math books do not have an name for this formula, although it is related to something called the "change of base formula".

Okay. Back to solving our equation. We got stuck trying to solve

$$1.067^Y = 1.327433628\ldots$$

We have growth factor $g = 1.067$ and value $v = 1.327433628\ldots$. So the formula says

$$\begin{aligned} Y &= \frac{\log(v)}{\log(g)} \\ &= \frac{\log(1.327433628\ldots)}{\log(1.067)} \\ &= \log(1.327433628\ldots) \div \log(1.067) = \\ &= 4.367667365 \approx 4.37 \end{aligned}$$

Your calculator should have a key that says "log" or maybe "LOG". Try typing

$$\log(1.327433628) \div \log(1.067) = 4.367667365 \approx 4.37$$

A small note here about parentheses. Some calculators give the first parenthesis for free when you type log but you have to type the closing parenthesis in yourself. This answer of 4.37 means that costs are projected to exceed \$3 trillion just over 4 years after 2007. That's some time during 2011, or by 2012 for sure. Same answer as before. Whew!

Let's practice. Suppose instead we want to know when health care costs would exceed \$10 trillion instead. (By the way – wow!) That means $H = 10$. Using our equation $H = 2.26 * 1.067^Y$ we get

$$2.26 * 1.067^Y = 10$$

Before we can use the Log-Divides Formula, we need to get rid of that 2.26. To do so, we can divide both sides by 2.26

$$\frac{\cancel{2.26} * 1.067^Y}{\cancel{2.26}} = \frac{10}{2.26} = 4.424778761\ldots$$

That means

$$1.067^Y = 4.424778761\ldots$$

Now our equation fits the format $g^Y = v$ for the Log-Divides formula with new value $v = 4.424778761\ldots$ (and the growth factor is $g = 1.067$ still). So the answer is

$$\begin{aligned} Y &= \frac{\log(v)}{\log(g)} \\ &= \frac{\log(4.424778761\ldots)}{\log(1.067)} \\ &= \log(4.424778761\ldots) \div \log(1.067) = \\ &= 22.932891 \approx 23 \end{aligned}$$

Want to avoid typing in the number 4.424778761...? Try this instead:

$$10 \div 2.26 = \log(\text{ANS}) \div \log(1.067) = 22.932891$$

where **ANS** stands for "answer,"

Again that means 23 years after 2007, or 2007 + 23 = 2030. Health care costs are projected to exceed \$10 million in the year 2030. Well, unless we do something about that. (Helps explain why government folks are often discussing how to contain health care costs.)

Log-Divides Formula – check. Time to fill you in a bit more about what these mysterious "logs" are. Look at these examples. Don't take my word for it – calculate them yourself.

$$\begin{aligned} \log(10) &= 1 \\ \log(100) &= 2 \\ \log(1{,}000) &= 3 \\ \log(10{,}000) &= 4 \end{aligned}$$

What do you see? In each case the logarithm is the number of zeros. For example, 10,000 has 4 zeros and $\log 10{,}000 = 4$. Another way to think of this connection is

$$10{,}000 = 10^4 \text{ and } \log 10{,}000 = 4$$

In other words, the logarithm is picking off the power of 10.

Wait a minute. The Log-Divides formula helped us find the value of $Y$ which was an exponent. And now we see that the log of a power of 10 is that exponent. So a logarithm is just an exponent. And logarithms help us find the exponent. Makes sense.

What about logs of numbers that aren't just powers of 10. Here are some examples to try.

$$\begin{aligned}
\log(25) &= 1.3979\ldots \\
\log(250) &= 2.3979\ldots \\
\log(2{,}500) &= 3.3979\ldots \\
\log(25{,}000) &= 4.3979\ldots
\end{aligned}$$

To see what's happening we want to involve powers of 10. Scientific notation will do that for us. Let's write these numbers in scientific notation and see what we learn. For example.

$$25{,}000 = 2.5 \times 10^4 \text{ and } \log 25{,}000 = 4.3979\ldots \approx 4$$

We are back to the power of 10. Well, approximately. Another example

$$250{=}2.5 \times 10^2 \quad \text{and} \quad \log(250) = 2.3979\ldots \approx 2$$

Before we write down a general rule, let's check out some other numbers including some really small numbers.

$$\begin{array}{lll}
7{,}420{,}000 = 7.42 \times 10^6 & \text{and} & \log(7{,}420{,}000) = 6.870403905\ldots \approx 6 \\
4 = 4 \times 10^0 & \text{and} & \log(4) = 0.602059991\ldots \approx 0 \\
.00917 = 9.17 \times 10^{-3} & \text{and} & \log(.00917) = -2.037630664\ldots \approx -3
\end{array}$$

In every case we are rounding down, but it's always the same.

$$\log(\text{number}) \approx \text{power of 10 in the scientific notation for that number}$$

## 3.5 Solving quadratic equations*

WHOLE BUNCH OF THIS SECTION IS MOVING BACK TO 2.4 WITH THIS STORY.

Libby likes to juggle. She throws one of her juggling balls high up into the air. The height of the ball changes over time as described by the equation

$$H = 3 + 15T - 16T^2$$

where

$$H = \text{height of ball (feet)}$$
$$T = \text{time (seconds)}$$

Let's make a table and draw a graph illustrating this function. For example, when $T = 0$ second, we have

$$H = 3 + 15(0) - 16(0)^2 = 3 + 15 \times \underline{0} - 16 \times \underline{0} \wedge 2 = 3 \text{ feet}$$

and when $T = 1$ second, we have

$$H = 3 + 15(1) - 16(1)^2 = 3 + 15 \times \underline{1} - 16 \times \underline{1} \wedge 2 = 2 \text{ feet}$$

Huh? I though the ball went up in the air. What's happening here? Oh, I know. it must be falling down by then. Let's look at $T = 0.1$ seconds. Then

$$H = 3 + 15(0.1) - 16(0.1)^2 = 3 + 15 \times \underline{0.1} - 16 \times \underline{0.1} \wedge 2 \approx 4.34 \text{ feet}$$

That makes more sense. As we fill in the table we see how Libby's juggling ball went up in the air and then back down.

| $T$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 | 1.1 | 1.2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $H$ | 3.0 | 4.34 | 5.36 | 6.06 | 6.44 | 6.50 | 6.34 | 5.66 | 4.76 | 3.54 | 2.00 | 0.14 | ~~-2.04~~ |

Notice as we evaluate at more times for $T = 1.2$ seconds we get

$$H = 3 + 15(0.1) - 16(0.1)^2 = 3 + 15 \times \underline{1.2} - 16 \times \underline{1.2} \wedge 2 \approx -2.04 \text{ feet}$$

We can't have negative feet. It means the ball must have already hit the ground by 1.2 seconds.

SU GRAPH – include Libby and label some key points with shapes to see. Perhaps add those shapes to the table to help codify it better?

A word of caution about this graph. It does not show where the ball travels, but just how it's height changes over time. The specially marked points on the graph correspond to the points in the table/picture. SU – Explain the shapes on the graphs and tables.

How long was the ball in the air? From our table and graph it looks like just over 1.1 seconds. We could refine our answer by successive approximations. The ball will hit the ground when it's height is 0 feet so that $H = 0$. Looks a little strange perhaps, but that's what we want.

| $T$ | 1.1 | 1.11 | 1.105 | 1.107 | 1.106 |
|---|---|---|---|---|---|
| $H$ | 0.14 | ~~-0.06~~ | 0.0368 | ~~-0.002~~ | 0.018 |
| vs. 0 | high | low | high | low | good |

So the ball hit the ground after approximately 1.106 seconds. That's probably way more precise than we need but there we have it.

In this chapter we've seen how to solve linear, power, and exponential equations. Is there some way to solve this type of equation too?

A little terminology. An equation like Libby's is called a ***quadratic equation***. It can be rewritten in the form

$$H = -16T^2 + 15T + 3$$

where the highest power of the independent variable ($T$ here) is listed first as is the custom. In general a quadratic equation can be written in the form

$$H = aT^2 + bT + c$$

where $a, b$, and $c$ are constants. For us those constants are

$$a = -16 \quad b = 15 \quad c = 3$$

If there were higher powers of our independent variable in the equation, then it would be called a ***polynomials***. We saw some polynomial equations in Section 2.3 SU CITE including

$$C = 0.036D^3 \text{ (snowball)}$$
$$V = 0.01W^3 - 0.95W^2 + 21W + 153 \text{ (investment stock)}$$
$$S = 10P^9 - 9P^{10} \text{ (orchids)}$$

Quadratics are just polynomials where the highest power is 2. Linears are officially polynomials as well, where the highest power is just 1.

Back to Libby. We are trying to figure out when $H = 0$. Putting in our equation we get

$$-16T^2 + 15T + 3 = 0$$

and we want to solve for $T$. Notice that because $T$ occurs twice in the equation, nothing we have seen to do to both sides of the equation can knock it down to just one $T$. That means none of our methods so far will work to solve this equation.

Sadly, there is no one-method-fits-all way of solving polynomial equations. In special cases there are various special methods, but no method works for all polynomials.

But it turns out that for quadratics there is a way to solve them. It's called the ***Quadratic Formula***. It tells us the answer, or perhaps I should say possible answer(s) to

$$aT^2 + bT + c = 0$$

are

$$T = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Oh my! First thing to understand in this complicated formula is that we actually get two possible answers

$$T = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad T = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

Let's figure this out for Libby. Remember we had

$$a = -16 \quad b = 15 \quad c = 3$$

First thing is

$$\frac{-b}{2a} = \frac{-(15)}{2(-16)} = (-)15 \div (2 \times (-)16) = 0.46875$$

Notice how we put parenthses around the denominator (bottom) of our fraction, as always, to preserve the order of operations. To get the negative numbers we hit the one key usually labelled (-). Remember that's the negation key, not the subtraction key, and what looks like parentheses are part of what's printed on the key.

Next part is

$$\frac{\sqrt{b^2 - 4ac}}{2a} = \frac{\sqrt{(15)^2 - 4(-16)(3)}}{2(-16)}$$

$$= \sqrt{\phantom{x}}((15) \wedge 2 - 4 \times (-)16 \times 3) \div (2 \times (-)16)$$

$$= -0.6381431\ldots \approx -0.63814$$

Notice how we have parentheses both around the number we are taking the square root of and, again, the denominator of our fraction. Also there's parentheses around the 15 before it was squared, which we don't really need in this problem, but we would need them if $b$ were negative. Lastly, we're using the squareroot key $\sqrt{\phantom{x}}$ on a calculator. If you don't have that key, then you'll need do $\wedge(1 \div 2)$ instead.

I'm guessing that you've noticed how complicated these calculations are. It is definitely worth having a calculator with a square root key here. Most students find it takes a lot of practice to get used to these calculations but luckily it's the same calculation for each problem.

Oh, and we're not done yet. Remember there are two possible answers. One is

$$0.46875 + (-0.63814) = -0.16939 \text{ secpmds}$$

which doesn't make any sense because time isn't negative. The other is

$$0.46875 - (-0.63814) = 1.10689 \text{ seconds}$$

We had guessed around 1.106 seconds, so that is definitely the right answer: Libby's ball will hit the ground after 1.10689 seconds. Yeah, too precise. But you get the idea.

Wait a minute! Any good juggler isn't about to let the ball fall on the ground. She's going to catch it again, probably at about 3 feet above ground (which is how high she threw it from). Looking back at our table and graph we see $H = 3$ at $T = 0$ when she threw the ball in the air, and again just after 0.9 seconds.

We could refine our guess with successive approximation. For example, when $T = 0.95$ We find

$$H = 3 + 15(0.95) - 16(0.95)^2 = 3 + 15 \times \underline{0.95} - 16 \times \underline{0.95} \wedge 2 \approx 2.81 \text{ feet}$$

| $T$ | 0.9 | 1.0 | 0.95 | 0.93 | 0.94 |
|---|---|---|---|---|---|
| $H$ | 3.54 | 2.00 | 2.81 | 3.1116 | 2.9624 |
| vs. 3 | high | low | low | high | close |

We estimate that Libby will catch the ball by 0.94 seconds.

Notice how the ball is falling during this time interval. That means to get a lower height we need to guess a bigger number for time and to get a higher height we need to guess a smaller number for time. That might seem backwards but decreasing functions always work that way. More on the rate of change later.

But first, there is a way to solve the equation to find this answer. We're looking for $H = 3$. Using our equation we get

$$3 + 15T - 16T^2 = 3$$

The Quadratic Formula only works if the equation has one side $= 0$, but we have $= 3$. It might seem that we're out of luck, but don't abandon hope because it's an easy fix. Just subtract 3 from both sides to get

$$\begin{aligned} 3 + 15T - 16T^2 &= \cancel{3} \\ &= \cancel{-3} \\ 15T - 16T^2 &= 0 \end{aligned}$$

So now we have $= 0$. Yes!

As luck would have, the other 3 cancelled too. (That's just because we're looking for the same height that it started at.) We can write the new equation as

$$-16T^2 + 15T = 0$$

from which we see that

$$a = -16 \quad b = 15$$

What about $c$? Turns out $c = 0$. (Again, because we're looking for the starting height.) To see why, think of our equation as

$$-16T^2 + 15T + 0 = 0$$

.

Now we're set to use the Quadratic Formula. The first fraction is

$$\frac{-b}{2a} = \frac{-(15)}{2(-16)} = (-)15 \div (2 \times (-)16) = 0.46875$$

No surprise here. We used the same values of $a$ and $b$ as before, so we should have the same number here.

Next is the part with the square root.

$$\frac{\sqrt{b^2 - 4ac}}{2a} = \frac{\sqrt{(15)^2 - 4(-16)(0)}}{2(-16)}$$

$$= \sqrt{}((15) \wedge 2 - 4 \times (-)16 \times 0) \div (2 \times (-)16)$$

$$= -0.46875$$

That's the negative of the first number. What gives? Take a closer look at the part with the square root. We know 0 times any number is just 0 so that tells us

$$\frac{\sqrt{(15)^2 - 4(-16)(0)}}{2(-16)} = \frac{\sqrt{(15)^2 - 0}}{2(-16)} = \frac{\sqrt{(15)^2}}{2(-16)} = \frac{15}{2(-16)}$$

which is the same as before, except for the - sign.

Don't forget we need to put together these parts to find the possible answers. The sum gives us

$$0.46875 + -0.46875 = 0$$

and the difference gives us

$$0.46875 - -0.46875 = 0.9375$$

Both answers seem to make sense. The ball was 3 feet above ground both at the start of the problem ($T = 0$ seconds) and later one ($T = 0.9375$ seconds. Since we're interested in when Libby caught it on the way down, the answer we want is 0.9375 seconds. That agrees with our estimate from before.

This is a complicated process, but luckily we follow the same steps each time.

**Zero** Rewrite the equation in the form $H = aT^2 + bT + c = 0$ *If not already =0*

**ABC** Read off the values of $a, b$, and $c$

**Fraction** Evaluate $\frac{-b}{2a}$

**Squareroot** Evaluate $\frac{\sqrt{b^2-4ac}}{2a}$

**Add/Subtract** The possible answers are the sum or difference of those two numbers.

**Decide** Decide which answer (or none or both) makes sense in the story.

We noted that between 0.9 and 1.0 seconds the function was decreasing. For example we can calculate the rate of change over 0.1 second time intervals to get the speed of the juggling ball. From 0.9 to 1.0 seconds, we have

$$\text{speed} = \text{rate of change} = \frac{\text{change in height}}{\text{change in time}} = \frac{2 - 3.54}{1.0 - 0.9} = \frac{-1.54 \text{ feet}}{0.1 \text{ seconds}} = -15.4 \text{ feet/sec}$$

In general we have

| $T$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 | 1.1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $H$ | 3.0 | 4.34 | 5.36 | 6.06 | 6.44 | 6.50 | 6.34 | 5.66 | 4.76 | 3.54 | 2.00 | 0.14 |
| speed | – | 13.4 | 10.2 | 7.0 | 3.8 | 0.6 | -2.6 | -5.8 | -9.0 | -12.2 | -15.4 | -18.6 |

As usual, the positive rate of change correspond to increasing heights; in our story the ball is going up in the air. This happens until around 0.5 seconds or so. The negative rate of change correspond to decreasing heights; in our story the ball is falling back down. This happens starting just before 0.6 seconds.

What happens in the story at the point where the ball stops going up in the air and starts falling down? That must be when the ball is at its highest point. What is the speed at that highest point? Well, I guess 0. For a split second it's almost frozen in midair, neither rising nor falling. If we were able to compute the rate of change for an interval really, really small right then we would find the rate of change $\approx 0$.

Turns out it's easy to find that point for a quadratic equation, just plug in the first fraction from the Quadratic Formula! Check it out: when

$$T = \frac{-b}{2a} = \frac{-15}{2(-16)} = (-)15 \div (2 \times (-)16) = 0.46875 \text{ seconds}$$

we get

$$H = 3+15(0.46875)-16(0.46875)^2 = 3+15\times\underline{0.46875}-16\times\underline{0.46875}\wedge 2 = 6.515625\ldots \approx 6.516 \text{ feet}$$

| $T$ | 0.46875 |
|---|---|
| $H$ | 6.516 |

Libby's ball gets about 6.516 feet up. Converting to more normal units we get

$$.516 \text{ feet} * \frac{12 \text{ inches}}{\text{feet}} = .516 \times 12 = 6.192 \approx 6 \text{ inches}$$

So the juggling ball goes up to about 6'6". (Remember ' is short for feet and " is short for inches.) SU earlier??? Check 1.3 or 1.4

SU – picture showing the roots and vertex from old book. Need a sentence or two explaining it.

One last small note. In this problem the graph of the quadratic was $\cap$ shaped because it was increasing and then decreasing, and we found a maximum value. In other problems the graph of a quadratic might be $\cup$ shaped because it is the other way around – decreasing first and increasing later. In that case evaluating at $T = \frac{-b}{2a}$ would give the minimum value instead. In some problems the domain includes only part of the U shape so it might just increase, or just decrease, on the domain.

# Chapter 4

# A closer look at linear equations

Yada yada

## 4.1 Modeling with linear equations

More people need affordable housing than you might realize. It is recommended that a family spend no more than 30% of their income on housing costs, but in some cities clean, safe housing in that price range is hard to find. According to the U.S. Department of Housing and Urban Development, a family with one full-time worker earning minimum wage cannot afford the local fair-market rent for a two-bedroom apartment anywhere in the United States. There are various local, state, and federally funded programs as well as many volunteer, non-profit agencies working to increase the availability of affordable housing.

In our city there were an estimated 64,100 units of affordable housing. A "unit" is usually an apartment that a family can rent. Since then, the city partnered with local developers to build a target of 7,800 units per year. Since the amount of available housing and year are both changing in this problem, those are our variables.

$$H = \text{number of affordable housing units} \quad \text{and} \quad Y = \text{time (years from now)}$$

Assuming things proceed as planned, in one year, there would be

$$64{,}100 + 7{,}800 = 71{,}900 \text{ units}$$

By the end of the second year, there would be

$$71{,}900 + 7{,}800 = 79{,}700 \text{ units}$$

or, equivalently,

$$64{,}100 \text{ units } + 2 \text{ years} * \frac{7{,}800 \text{ units}}{\text{year}} = 64{,}000 + \underline{2} \times 7{,}800 = 79{,}700 \text{ units}$$

After 5 years, there would be

$$64{,}100 \text{ units } + 5 \text{ years} * \frac{7{,}800 \text{ units}}{\text{year}} = 64{,}000 + \underline{5} \times 7{,}800 = 103{,}100 \text{ units}$$

Generalizing, we get our equation

$$64{,}000 + Y * 7{,}800 = H$$

Rewriting in standard form we get

$$H = 64{,}100 + 7{,}800Y$$

This equation fits our template for a linear equation

$$\text{dep} \;=\; \text{start} \;+ \text{slope} * \text{indep}$$

It makes sense that the equation is linear because the rate of change of 7,800 affordable housing units per year is constant. Remember that for linear equations, the rate of change is also called the ***slope***. So the slope is 7,800 affordable housing units per year. Our starting amount was 64,100 affordable housing units, so that's the ***intercept***.

How many years will it take the city to reach 150,000 affordable housing units at this rate? We can make a table of values and use successive approximation to find the answer. We begin with the starting amount and number after five years. In ten years, there would be

$$64{,}000 + \underline{10} \times 7{,}800 = 142{,}100$$

which is still a bit low. Continuing we get

| $Y$ | 0 | 5 | 10 | 11 | 12 |
|---|---|---|---|---|---|
| $H$ | 64,100 | 103,100 | 142,100 | 149,900 | 157,700 |
| | low | low | low | almost | just over |

This city will reach 150,000 affordable housing units within 12 years.

As a review, we could have solved the linear equation

$$64{,}100 + 7{,}800Y = 150{,}000$$

instead. Since we want at least 150,000 affordable housing units, an inequality might be even better. Let's practice that.

$$64{,}100 + 7{,}800Y \geq 150{,}000$$

Subtract 64,100 from each side to get

$$\begin{array}{rcl} \cancel{64{,}100} + 7{,}800Y & \geq & 150{,}000 \\ -\cancel{64{,}100} & & -64{,}100 \end{array}$$

which simplifies to

$$7{,}800Y \geq 85{,}900$$

Divide each side by 7,800 to get

$$\frac{\cancel{7{,}800}Y}{\cancel{7{,}800}} \geq \frac{85{,}900}{7{,}800}$$

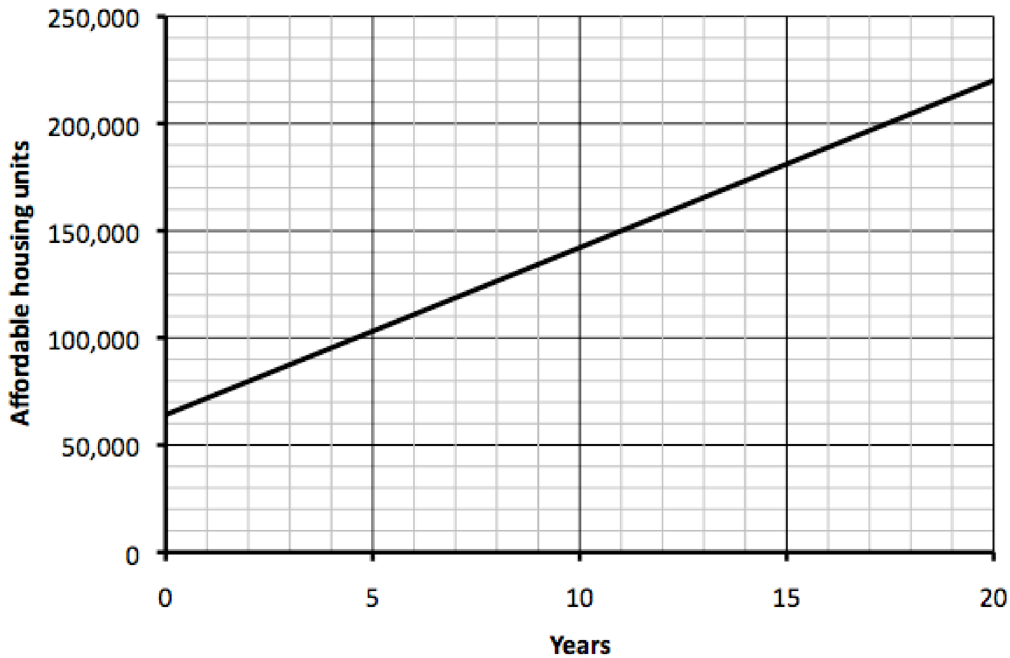
which simplifies to

$$Y \geq 85{,}900 \div 7{,}800 = 11.0128205...$$

To be sure $Y \geq 11.0128205...$ we need to round up to get

$$Y \geq 12$$

Let's confirm our findings on the graph.




Sure enough 150,00 units of affordable housing is after approximately 11 years. We also see the graph is a line, as expected for a linear function.

## 4.2 Systems of linear equations

A local factory produces small pulleys for industrial use. The old machine has seen better days and the manager, Mr. Sheaver, is shopping around for a new machine. He's narrowed it down to two options.

The purchase price, including installation, of a machine similar to the one they have now is \$3,200 and would run at a cost of \$1.25 per pulley. Another option is a machine that would cost \$5,400, including installation, but would run at a cost of only \$0.80 per pulley.

Mr. Sheaver is comparing the costs of each options. Since Machine #1 is less expensive, he knows it is the right choice if the factory only produces a small number of pulleys. But since Machine #2 costs less per pulley to run, he knows it will pay off if the factory makes a large number of pulleys. He would like to understand the costs better, particularly the number of pulleys at which it would be worth it to invest in the more expensive machine.

He recognizes that each machine has a linear function for the total cost. The variables are

$$P = \text{amount produced (pulleys)} \quad \text{and} \quad M = \text{total cost (\$)}$$

The equations are

$$\textbf{Machine \#1:} \quad M = 3{,}200 + 1.25P$$

$$\textbf{Machine \#2:} \quad M = 5{,}400 + 0.80P$$

As a first step he makes a table. For example, for 2,000 pulleys Mr. Sheaver calculates

$$\textbf{Machine \#1:} \quad M = 3{,}200 + 1.25 \times \underline{2{,}000} = \$5{,}700$$

$$\textbf{Machine \#2:} \quad M = 5{,}400 + 0.80 \times \underline{2{,}000} = \$7{,}000$$

If the factory were only going to make 2,000 pulleys, then Machine #1 would be the most affordable. Here are more examples he calculated.

| $P$ | 2,000 | 4,000 | 6,000 | 8,000 | 10,000 |
|---|---|---|---|---|---|
| **Machine #1:** M | 5,700 | 8,200 | 10,700 | 13,200 | 15,700 |
| **Machine #2:** M | 7,000 | 8,600 | 10,200 | 11,800 | 13,400 |

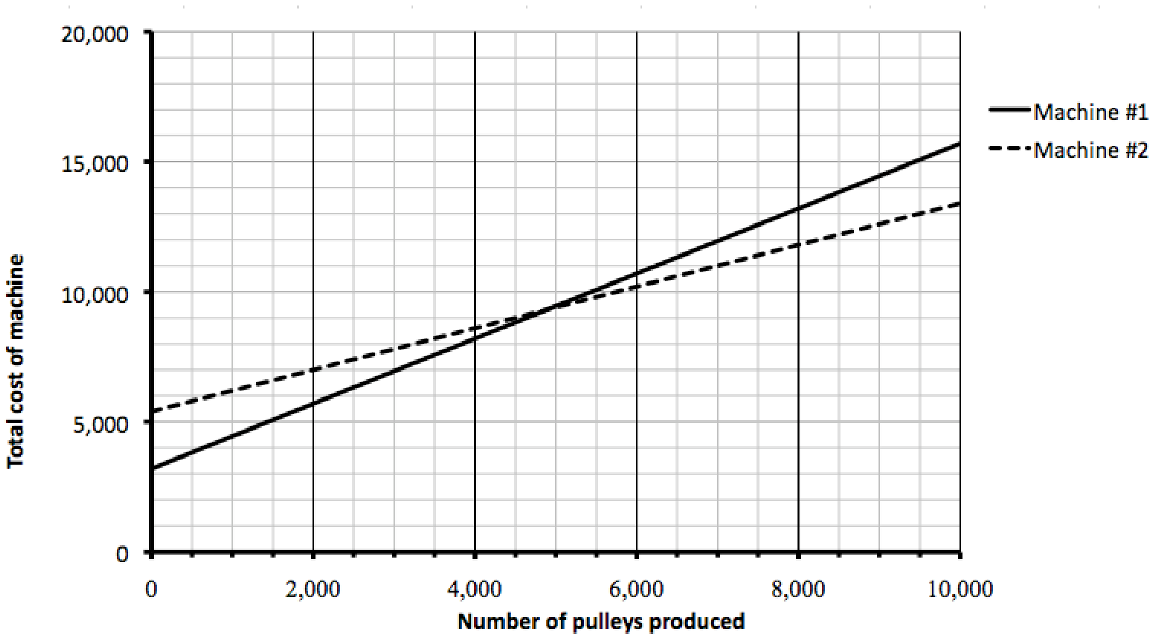
Up until 4,000 pulleys, Machine #1 is the better deal. After 6,00 pulleys, Machine #2 becomes the better deal. Somewhere in between, it switches.

Mr. Sheaver makes a quick graph to see what's going on. (It's on the next page.) On the graph whichever line is lower corresonds to the less expensive machine and whichever line is higher corresponds to the more expensive machine. As suspected, for a smaller number of pulleys, the line for Machine #1 is lower on the graph, since that machine is less expensive. For a larger number of pulleys, the line for Machine #2 is lower on the graph, since then that machine is less expensive. Where they switch corresponds to the point on the graph where the two lines cross. Looks like around 5,000 pulleys.

Time to extend the table for a little successive approximation.

| $P$ | 4,000 | 6,000 | 5,000 | 4,500 | 4,800 | 4,900 |
|---|---|---|---|---|---|---|
| **Machine #1:** M | 8,200 | 10,700 | 9,450 | 8,825 | 9,200 | 9,325 |
| **Machine #2:** M | 8,600 | 10,200 | 9,400 | 9,000 | 9,240 | 9,320 |
| Less expensive option | #1 | #2 | #2 | #1 | #1 | #2 |

So the choice changes somewhere between 4,800 and 4,900 pulleys.




As you might suspect, there is a way for Mr. Sheaver to solve the problem symbolically. Since there are two linear equations, we refer to this process as "solving the system of linear equations." We want to determine the number of pulleys, $P$, where

$$\text{cost of Machine \#1} = \text{cost of Machine \#2}$$

Using our equations $M = 3{,}200 + 1.25P$ for Machine #1 and $M = 5{,}400 + 0.80P$ for Machine #2 we have

$$3{,}200 + 1.25P = 5{,}400 + 0.80P$$

We start by subtracting 3,200, the smaller of the two purchase prices, from each side to get

$$\begin{array}{rcl} \cancel{3{,}200} + 1.25P & = & 5{,}400 + 0.80P \\ -\cancel{3{,}200} & & -3{,}200 \end{array}$$

which simplifies to

$$1.25P = 2{,}200 + 0.80P$$

Pause for a minute. What does that \$2,200 mean in our story? It's the extra cost of buying Machine #2 because \$5,400 − \$3,200 = \$2,200.

What next? This equation has the variable $P$ on each side. We need to combine them somehow. Here's how to do that. Subtract $0.80P$ from each side. Look closely. We are subtracting $0.80P$, not just 0.80. We get

$$\begin{array}{rcl} 1.25P & = & 2{,}200 + \cancel{0.80P} \\ -0.80P & & -\cancel{0.80P} \end{array}$$

How do we simplify $1.25P - 0.80P$? Think about what these numbers represent in the story. The cost was \$1.25 per pulley versus \$0.80 per pulley. The difference is \$1.25-\$0.80 = \$0.45 per pulley. So that means

$$1.25P - 0.80P = 0.45P$$

Think: 125 apples − 80 apples = 45 apples. We can now simplify our equation to get

$$0.45P = 2{,}200$$

Ah, we can solve this equation just by dividing each side by 0.45 to get

$$\frac{\cancel{0.45}P}{\cancel{0.45}} = \frac{2{,}200}{0.45}$$

which simplifies to

$$P = \frac{2{,}200}{0.45} = 2{,}200 \div 0.45 = 4888.88888... \approx 4,889 \text{ pulleys}$$

If they plan to produce 4,889 pulleys or more, Mr. Sheaver should go ahead and buy the more expensive machine, Machine #2. Yeah, that's what we guessed – just under 4,900 pulleys is the payoff.

We solved an equation here, but really we wanted to know when

$$\text{Machine \#1} \geq \text{Machine \#2}$$

We can solve again, this time using inequality notation. Since it's the same steps, we just list them here.

$$\begin{array}{rcl} \cancel{3{,}200} + 1.25P & \geq & 5{,}400 + 0.80P \\ -\cancel{3{,}200} & & -3{,}200 \end{array}$$

$$\begin{array}{rcl} 1.25P & \geq & 2{,}200 + \cancel{0.80P} \\ -0.80P & & -\cancel{0.80P} \end{array}$$

$$\frac{\cancel{0.45}P}{\cancel{0.45}} = \frac{2{,}200}{0.45}$$

$$P \geq \frac{2{,}200}{0.45} = 2{,}200 \div 0.45 = 4888.88888... \approx 4,889 \text{ pulleys}$$

## 4.3 Intercepts (and direct proportionality)

Kaleb runs $8\frac{1}{2}$ minute miles, which means it takes him around 8.5 minutes to run each mile. Yesterday he was out for 30 minutes and ran the 2.8 mile loop by our house. Not that I care, but it is curious. I mean if he ran 2.8 miles at 8.5 minutes per mile that should take

$$2.8 \text{ miles} * \frac{8.5 \text{ minutes}}{\text{mile}} = 2.8 \times 8.5 = 23.8 \approx 24 \text{ minutes}$$

He took 30 minutes. That's 6 minutes longer than expected. What's up?

Oh, I bet I know what it is. Ever since Kaleb turned fifty years old, he's been having trouble with his knees. I bet he's finally stretching like his doctor ordered. Must be 6 minutes of stretches after each run.

Our variables are

$$D = \text{distance (miles)} \quad \text{and} \quad T = \text{total time (minutes)}$$

Notice that we're determining how the time depends on the distance, so the time $T$ is our dependent variable. Often time is the independent variable, but not so here. The distance $D$ is the independent variable. Assuming Kaleb runs a steady 8.5 minutes per mile, the equation is linear with slope of 8.5 minutes per mile. The 6 minutes Kaleb spends stretching is the intercept. The equation is

$$\textbf{Kaleb:} \quad T = 6 + 8.5D$$

By the way, there's a shorter way to find the intercept.

$$\text{intercept } = 30 - 8.5 \times 2.8 = 6.2 \approx 6 \text{ minutes}$$

In general,

$$\text{intercept} = \text{dep} - \text{slope} * \text{indep}$$

Kaleb's daughter Muna runs considerably faster, 7 minute miles, and she's not into stretching at all. For her to run the 2.8 mile loop by our house, it would take

$$\frac{7 \text{ minutes}}{\text{mile}} * 2.8 \text{ miles} = 7 \times 2.8 = 19.6 \text{ minutes}$$

While her dad would take 30 minutes to run the loop and do his stretches, Muna can run it in just under 20 minutes.

The equation for Muna is

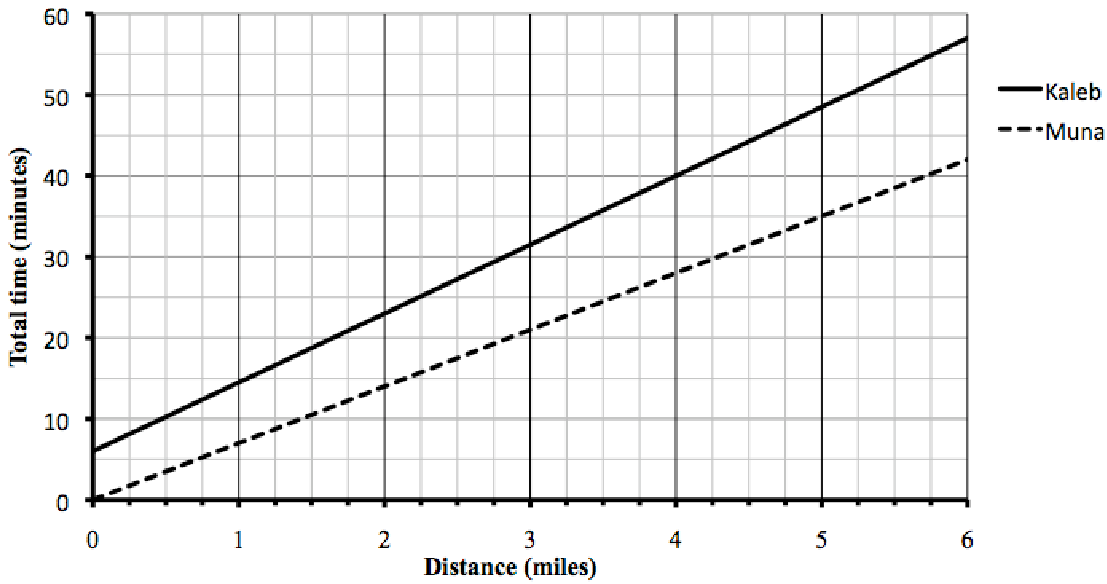
$$\textbf{Muna:} \quad T = 7D$$

The slope is 7 minutes per mile. What's the intercept for this equation? There's no time for stretching in her equation, so it's like

$$T = 0 + 7D$$

The intercept is 0 minutes.

Compare the graphs. Remember that the intercept shows where the line meets the vertical axis. Kaleb's line crosses at 6 minutes, but Muna's crosses at 0 minutes, where the two axes meet (at the ***origin***).




SU Then do twice as long, define proportional (direct)

## 4.4 Slopes

Last week our supplier delivered 13 cases of paper for the office for a total charge of \$534.87. This week, they delivered 20 cases of paper for \$814.80. Each price includes a fixed delivery fee and per case cost, so the dependence is linear. We would like to write a linear equation.

What to do? We can name the variables and put the information we are given into a table. That's a start. The variables must be

$$T = \text{ total cost (\$)} \quad \text{and} \quad P = \text{ amount of paper delivered (cases)}$$

We know

| $P$ | 13 | 20 |
|---|---|---|
| $T$ | 534.87 | 814.80 |

Let's see. The fixed delivery fee in \$ is the starting amount, or intercept. The per case cost in \$/case is the slope. To write a linear equation we need to know both amounts. The story does not tell us either one so we're going to have to figure them both out.

The slope is just the rate of change, so we can figure out the slope just from the information in our table.

$$\text{rate of change} = \frac{\text{change dep}}{\text{change indep}} = \frac{\$814.80 - \$534.87}{20 - 13 \text{ cases}} = \frac{\$279.93}{7 \text{ cases}} = \$39.99 \text{ per case}$$

Excellent.

Now that we know the slope, we can find the intercept. At \$39.99 per case we would expect 13 cases to cost

$$13 \text{ cases } * \frac{\$39.99}{\text{case}} = 13 \times 39.99 = \$519.87$$

But the story tells us that 13 cases cost \$534.87. The difference \$534.87 − \$519.87 = \$15 must be the delivery fee which is the starting amount or intercept. Remember we can find the intercept in one calculation as

$$\text{intercept} = \text{dep} - \text{slope} * \text{indep} = 534.87 - 39.99 \times 13 = 15$$

Why did we use 13 cases at \$534.87? No good reason. Look what happens if we use 20 cases at \$814.80 instead.

$$\text{intercept} = \text{dep} - \text{slope} * \text{indep} = 814.80 - 39.99 \times 20 = 15$$

Yup. Still \$15 delivery fee.

The equation is linear so it fits our template

$$\text{dep} = \text{intercept} + \text{slope} * \text{indep}$$

We have

$$\textbf{Delivered:} \quad T = 15 + 39.99P$$

Let's check. When $P = 13$ we get

$$T = 15 + 39.99 \times \underline{13} = 534.87$$

and when $P = 20$ we get

$$T = 15 + 39.99 \times \underline{20} = 814.80$$

Just saw an advertisement for the same paper we use for \$4.25 per ream at a supply store. Ream? Yes. 500 sheets of paper, usually wrapped in paper. Anyway, I'd like to know if that's a good deal or not. Since the office buys by the case, we need to know that there are 10 reams in a case. So the advertised price would be

$$\frac{\$4.25}{\text{ream}} * \frac{10 \text{ reams}}{\text{case}} = 4.25 \times 10 = \$42.50/\text{case}$$

The office pays \$39.99 per case so the advertised price of \$42.50 per case is more expensive.

But, at the office we pay a delivery charge. For one case, I would only pay \$42.50 at the supply store, but if the office has it delivered it would cost

$$T = 15 + 39.99 \times \underline{1} = \$54.99$$

For just one case it is definitely less expensive to buy a case of paper at the supply store. I'm not sure I want to go get it myself, because a case of paper is pretty heavy to lift (see exercises), but it would be cheaper.

If we use the same variables, we can write a new equation for the cost of the paper at the supply store. Since there is no delivery charge, the starting amount or intercept is \$0. Remember what that means? We have a direct proportionality. The slope is now \$42.50/case. Our equation is

$$\textbf{Store:} \quad T = 42.50P$$

For example, 13 cases would cost

$$T = 42.50 \times \underline{13} = \$552.50$$

and 20 cases would cost

$$T = 42.50 \times \underline{20} = \$850$$

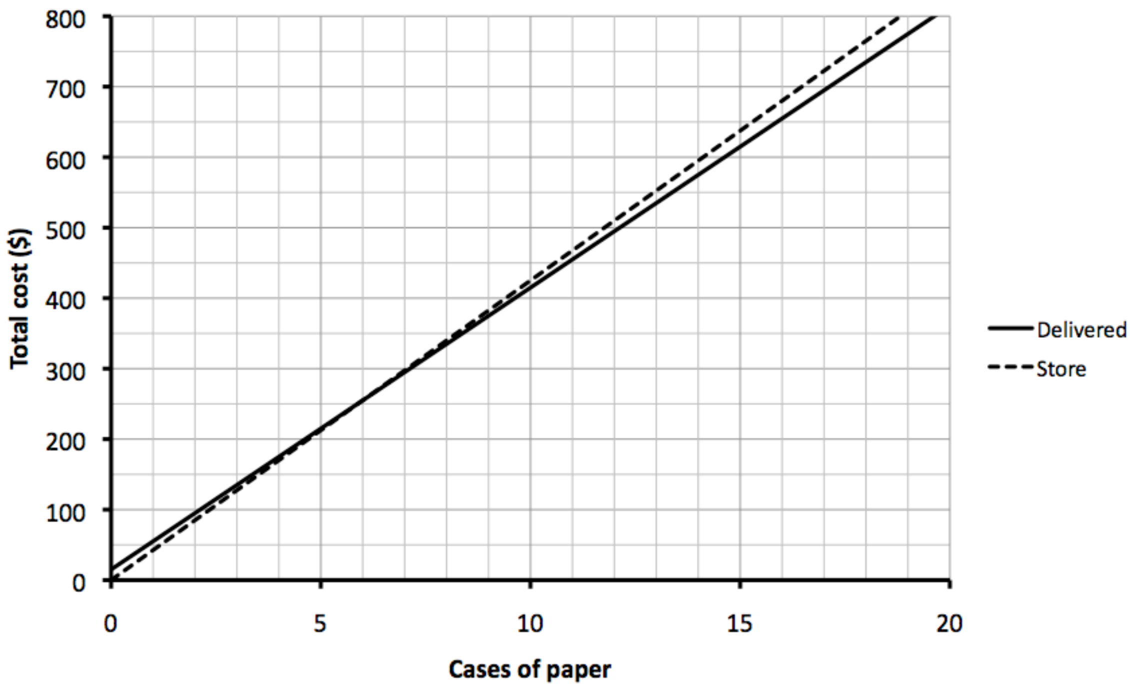
We can make a table to compare prices. Looks like a by 13 cases it's worth paying the delivery fee, maybe even fewer cases is enough.

| | $P$ | 1 | 13 | 20 |
|---|---|---|---|---|
| **Delivered:** | $T$ | 26 | 534.87 | 814.80 |
| **Store:** | $T$ | 42.50 | 552.50 | 850 |

The graph suggests that perhaps as few as 6 cases is worth paying the delivery fee. Let's compare.

$$\begin{array}{ll} \textbf{Delivered:} & T = 15 + 39.99 \times \underline{6} = \$254.94 \\ \textbf{Store:} & T = 42.50 \times \underline{6} = \$255.00 \end{array}$$

Just about equal! For 6 cases or more we should just pay the delivery fee.




Remember we know how to solve this system of equation too. Let's practice. To get

**Delivered = Store**

we solve

$$15 + 39.99P = 42.50P$$

Because only the left-hand side of our equation has a constant (15), we don't need to adjust the constants. We subtract $39.99P$ from both sides to get

$$\begin{array}{rcl} 15 + 39.99P & = & 42.50P \\ -\cancel{39.99P} & & -39.99 \end{array}$$

Since $42.50 - 39.99 = 2.51$, our equation simplifies to

$$15 = 2.51P$$

Swapping sides and dividing each side by 2.51 we get

$$\frac{\cancel{2.51}P}{\cancel{2.51}} = \frac{15}{2.51}$$

which simplifies to

$$P = \frac{15}{2.51} = 15 \div 2.51 = 5.97609561... \approx 6$$

As suspected, at 6 cases or more it makes sense to pay for delivery

## 4.5 Fitting lines to data*

INTRODUCTORY EXAMPLE

Su: now ;use technology to find the equation of best fit line. Maybe add drawing residuals and correlation.

Perhaps introduce secant lines to curves and the idea of convexity (+ is over, - is under) estimate using linear interpolation.

# Chapter 5

# A closer look at exponential equations

We have seen how exponential equations help us understand all sorts of things from investments and loans, to rising health care costs, to population growth, or any quantity increasing at a fixed percentage. But that's not all. In this chapter we explore other situations in which exponential equations help us understand any type of exponential growth (increasing) as well as exponential decay (decreasing).

In this chapter we begin by reviewing how to write an exponential equation knowing the percent increase (from §2.2 = FIRSTLOOK); how to evaluate, make a table, graph, and approximate solutions of an exponential equation (from §2.3-2.4 =USING/APPROX); and how to solve an exponential equation using logs (from §3.4 = SOLVEEXP). We also look at how rate of change works with exponential equations. Next, we study exponential growth and exponential decay in more detail, including writing an equation when we are not told the percent change. After that we compare how linear or exponential equations model situations where it's not clear which to use (if either). We close with a second on fancier types of equations that use exponentials, particularly logistic growth.

## 5.1 Modeling with exponential equations

My grandmother was born in eastern Europe at the end of the 1800s to a poor family. Her community was treated very badly by the government. When she was eight years old her parents brought her and your younger sister and brother to the United States. They were poor and so both her parents had to work. My grandmother took care of the children, which now included another brother and sister, and so my grandmother dropped out of school after eighth grade.

Time passed and she married a handsome young veteran of World War I, who had also immigrated to the country as a young child. He was poor too, but had just enough to set up shop selling men's clothing. Things were always tight financially, but for her wedding dowry his parents bought my grandmother a set of sterling silverware, valued at \$800 in 1920.

Over the years, the sterling has increased in value, let's say by around 3% per year. In 1957, my grandmother handed it down to my mother as a wedding present. In 1990, my mother handed it down to me. What was it was it worth at those times, and how much should it be insured for in 2015?

Let's write the equation to answer these questions. The variables should be

$$S = \text{value of the sterling (\$) and } Y = \text{time (years since 1920)}$$

We're saying that the sterling increased 3% per year in value. For example, in 1921, the sterling was worth

$$\$800 + 3\% \text{ of } \$800 = 800 + .03 \times 800 = 800 + 24 = \$824$$

Remember the shortcut here? Check it out.

$$800 \times 1.03 = 824$$

The idea is after one year we have the original \$800 plus 3% more for a grand total of 103% of what we had before. And 103% = 1.03. Yup. We say 3% is ***growth rate*** and 1.03 is ***growth factor***.

Continuing that reasoning, after 5 years the sterling was worth

$$800 * 1.03 * 1.03 * 1.03 * 1.03 * 1.03 = 800 * 1.03^5$$

since multiplying by 1.03 five times is the same as multiplying by $1.03^5$ On the calculator we do

$$800 \times 1.03 \wedge 5 = 927.4192594 \approx \$927$$

Ready for the equation? Replacing 5 by $Y$ and 927 by $S$ we get

$$800 \times 1.03 \wedge Y = S$$

Rewriting our calculation in algebraic notation, and putting the dependent variable first we get

$$S = 800 * 1.03^Y$$

And that fits our template for an exponential equation:

$$\text{dep} \;=\; \text{start (growth factor)}^{\text{indep}}$$

Let's answer those questions. In 1957, we had $Y = 1957 - 1920 = 37$ years and so

$$S = 800 * 1.03^{37} = 800 \times 1.03 \wedge \underline{37} = 2388.181342 \approx \$2{,}388$$

By 1990, we had $Y = 1990 - 1920 = 70$ years and so

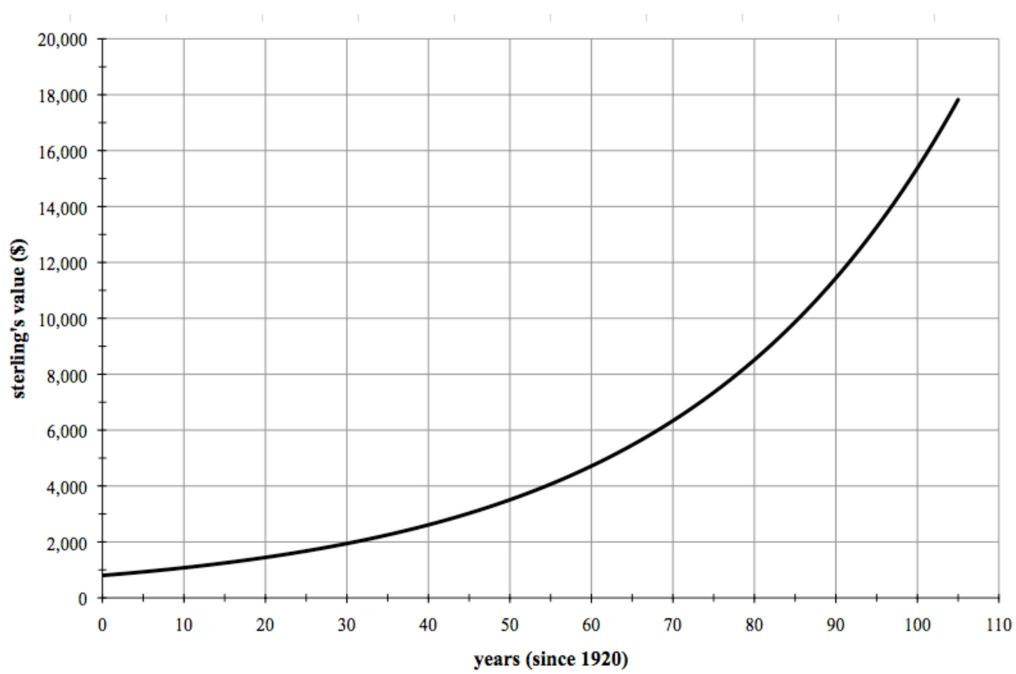
$$S = 800 * 1.03^{70} = 800 \times 1.03 \wedge \underline{70} = 6334.25753 \approx \$6{,}334$$

By 2015, we have $Y = 2015 - 1920 = 95$ years and so

$$S = 800 * 1.03^{95} = 800 \times 1.03 \wedge \underline{95} = 13262.52862 \approx \$13{,}262$$

We can summarize our findings in a table and draw a graph. Notice that we have extended our graph a bit beyond the values in the table in anticipation of doing further projections.

| year | 1920 | 1921 | 1925 | 1957 | 1990 | 2015 |
|---|---|---|---|---|---|---|
| $Y$ | 0 | 1 | 5 | 37 | 70 | 95 |
| $S$ | 800 | 824 | 927 | 2,388 | 6,334 | 13,262 |




It turns out that the insurance policy allows for up to \$20,000. According to our graph and table, when would the sterling reach that value? The curve we drew in the graph does not quite extend to \$20,000. Guess we should have extended it further. Looks like it won't cross \$20,000 until past 100 years, probably somewhere around 110 years. Since $1920 + 110 = 2030$, let's say the value will reach \$20,000 some time around 2030.

We can use successive approximation to improve our answer. First, at 100 years (in 2020) we have

$$S = 800 * 1.03^{100} = 800 \times 1.03 \wedge \underline{100} = 15374.90558 \approx 15{,}375$$

By 2030 we have $Y = 2030 - 1920 = 110$ as expected and so

$$S = 800 * 1.03^{110} = 800 \times 1.03 \wedge \underline{110} = 20662.58745 \approx 20{,}663$$

Aha, just before 2030 should do it. By 2029 we can find $S = 20{,}061$ and, just be sure we can find by 2028 we have $S = 19{,}476$ Easier to see in a table, but it seems that the value with reach \$20,000 during 2029.

| year | 2020 | 2030 | 2029 | 2028 |
|---|---|---|---|---|
| $Y$ | 100 | 110 | 109 | 108 |
| $S$ | 15,375 | 20,663 | 20,061 | 19,476 |
| | low | high | a bit high | not quite |

As a reminder, we actually know how to solve this equation. To find when $S = 20{,}000$ we use our equation $S = 800 * 1.03^Y$ to get

$$20{,}000 = 800 * 1.03^Y$$

Divide both sides by 800 to get

$$\frac{20{,}000}{800} = \frac{\cancel{800} * 1.03^Y}{\cancel{800}}$$

and so

$$1.03^Y = 25$$

Since we want to solve for the exponent, we use LOG-DIVIDES FORMULA with growth factor $g = 1.03$ and the value $v = 25$ to get

$$Y = \frac{\log(v)}{\log(g)} = \frac{\log(25)}{\log(1.03)} = \log(25) \div \log(1.03) = 108.89737 \approx 109$$

We rounded up to make sure it would reach the full \$20,000. Since $1920 + 109 = 2029$, we see (again) that the value should reach \$20,000 in the year 2029.

Here's the formula again, so we have it handy.

THE LOG-DIVIDES FORMULA

The equation $g^Y = v$ has solution $Y = \dfrac{\log(v)}{\log(g)}$.

Look what happens when we calculate the rate of change for this function. For example, from 1920 to 1925 the value rose from \$800 to around \$927.

$$\text{Rate of change} = \frac{\text{change in value}}{\text{change in time}} = \frac{\$927 - \$800}{1925 - 1920} = \frac{\$27}{5 \text{ years}} = 25.4 \approx \$25/\text{year}$$

That means from 1920 to 1925, the value of the sterling increased an average of around \$25 per year. During the next time period, from 1925 to 1957, the value rose from \$927 to \$2,388

$$\text{Rate of change} = \frac{\text{change in value}}{\text{change in time}} = \frac{\$2{,}388 - \$927}{1957 - 1925} = \frac{\$1{,}461}{32 \text{ years}} =\approx \$45.6/\text{year}$$

So, from 1925 to 1957 it increased an average of about \$46 per year.

Were we supposed to get different numbers here? Well, the graph's not a line and it's not a linear equation. That tells us the rate of change isn't going to be constant. So, sure, different numbers are fine. Does it make sense that the rate of change would itself increase? That the value increases at an increasing rate? Yes. Although we are always just adding on 3%, we're taking 3% of larger numbers each year. So more is added each year. You can check that the rate of change keeps increasing.

| year | 1925 | | 1930 | | 1957 | | 1990 | | 2015 | | 2030 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $S$ | 800 | | 927 | | 2,388 | | 6,334 | | 13,262 | | 20,663 |
| rate of change | | 25 | | 46 | | 120 | | 277 | | 493 | |

## 5.2 Exponential growth and decay

It is 2:00 a.m. and Joe is still up studying. The dorm has quieted down, but Joe's feeling mighty jittery. He drank 5 large mugs of coffee in the past few hours and all that caffeine is peaking in his system now. At around 200 mg per mug, Joe wonders when his caffeine levels will drop down to where he can sleep a little.

First things first: staying up that late to study is probably a bad idea. I mean, who can think properly at 2:00 in the morning? And, how tired is Joe going to be by the time his test rolls around?

Plus, we know that

$$5 \text{ mugs} * \frac{200 \text{ mg}}{\text{mug}} = 5 \times 200 = 1{,}000 \text{ mg}$$

By the way – quick metric prefix review. Milli was one-in-a-thousand. So, 1,000 mg $=$ 1 gram. Either way, that's a lot of caffeine.

I guess at this point Joe is stuck so let's help him. Let's say that at 2:00 a.m. he has 1,000 mg of caffeine in his blood. Joe searches online and discovers that 13% of the caffeine should leave his body each hour and below 300 mg he should be fine. When will that happen?

We know how percent increase works, but here the caffiene is leaving his body according to a percent decrease. I guess we need to figure it out one step at a time. After one hour (by 3:00 a.m.), Joe will have

$$1000 \text{ mg} - 13\% \text{ of } 1000 \text{ mg} = 1000 - .13 \times 1000 = 1000 - 130 = 870 \text{ mg}$$

By 4:00 a.m. (after 2 hours), Joe will have

$$870 \text{ mg} - 13\% \text{ of } 870 \text{ mg} = 870 - .13 \times 870 = 870 - 113.1 = 756.9 \text{ mg}$$

Not even close.

Wait a minute. When we calculated 13% decrease on 1000 mg we got 870 mg. That's 87% of 1,000. Yeah, that's right, take off 13% and you should be left with 87% of what you started with because

$$100\% - 13\% = 87\%$$

So that means we could have calculated at 3:00 a.m.,

$$1000 \text{ mg} - 13\% \text{ of } 1000 \text{ mg} = 87\% \text{ of } 1{,}000 = .87 \times 1{,}000 = 870$$

and then

$$.87 \times 870 \text{ mg} = 756.9$$

So, to find the amount after a 13% decrease we just multiply by

$$1 - .13 = .87$$

Using this shortcut we find after 3 hours, there will be

$$.87 * 756.9 \text{ mg} \approx 658.5$$

and after 4 hours there will be

$$.87 * 658.5 \approx 572.9 \text{ mg}$$

So by 6:00 a.m., Joe will still not be down to the 300 mg cutoff where he hopes he can fall asleep.

Mutliplying by .87 each time is not necessary. For example, after 4 hours we really calculated

$$1000 * .87 * .87 * .87 * .87 = 1000 * .87^4$$

where we use a power to abbreviate repeatedly multiplying. Check it out.

$$1000 \times .87 \wedge 4 = 572.89761 \approx 572.9 \text{ mg}$$

Aha! We can write the equation. Say

$$J = \text{ Joe's caffeine level (mg) and } H = \text{ time (hours since 2:00 a.m.)}$$

Then our equation is

$$J = 1000 * .87^H$$

Notice this equation fits our template for an exponential equation.

$$\text{dep} = \text{start (growth factor)}^{\text{indep}}$$

A little terminology here. When something decreases exponentially, the situation is called ***exponential decay***. It sounds a little odd to say "growth factor" if the quantity is getting smaller so we sometimes call that number, our .87, the ***decay factor*** instead.

Back to jittery Joe. Let's summarize what we've found in our equation and add a few more times so we can see when Joe's caffeine level should fall below 300 mg. For example

| time | 2:00 | 3:00 | 4:00 | 5:00 | 6:00 | 7:00 | 8:00 | 9:00 | 10:00 | 11:00 |
|---|---|---|---|---|---|---|---|---|---|---|
| $H$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| $J$ | 1,000 | 870 | 756.9 | 658.5 | 572.9 | 498.4 | 433.6 | 377.3 | 328.2 | 285.5 |

That means Joe should be able to fall asleep by around 11:00 a.m. Exactly when his exam starts. Not good news for Joe.

We could have solved the equation instead of successive approximation. Let's do that for practice. We were looking for $J = 300$. Plugging into our equation $J = 1{,}000 * .87^H$ we get

$$300 = 1{,}000 * .87^H$$

Divide both sides by the starting amount of 1,000 mg to get

$$\frac{300}{1{,}000} = \frac{\cancel{1{,}000} * .87^H}{\cancel{1{,}000}}$$

That simplifies to

$$.87^H = .3$$

We find ourselves in the familiar situation – solving to find the exponent. Logs to the rescue! By the THE LOG-DIVIDES FORMULA with growth factor $g = 1.87$ and the value $v = .3$ we get

$$Y = \frac{\log(v)}{\log(g)} = \frac{\log(.3)}{\log(.87)} = \log(.3) \div \log(.87) = 9$$

Solving tells us that Joe's caffeine levels will drop under 300 mg after about 9 hours, or at 11:00 a.m. Same answer. Much quicker.

Let's calculate the rate of change and think about what it means. During the first hour,

$$\text{rate of change} = \frac{\text{change in caffeine}}{\text{change in time}} = \frac{870 - 1{,}000 \text{ mg}}{3{:}00 \text{ a.m.} - 2{:}00 \text{ a.m.}} = \frac{-130 \text{ mg}}{1 \text{ hour}} = -130 \text{ mg/hour}$$

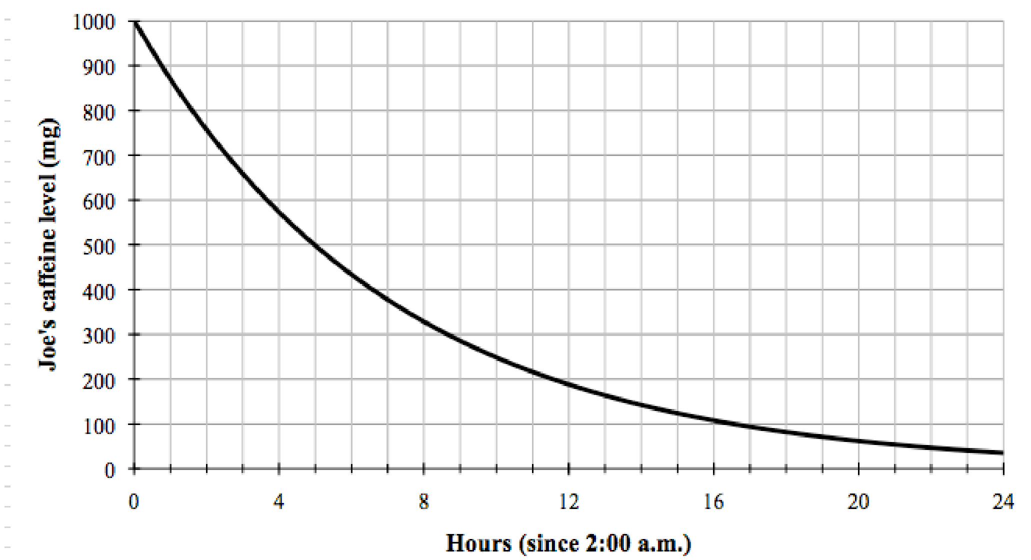
Was the rate of change supposed to be negative? Sure. Joe's caffeine level is dropping. Functions that decrease like this one have negative rates of change.

And, as exam approaches,

$$\text{rate of change} = \frac{\text{change in caffeine}}{\text{change in time}} = \frac{285.5 - 328.2}{11{:}00 \text{ a.m.} - 10{:}00 \text{ a.m.}} = \frac{-42.7 \text{ mg}}{1 \text{ hour}} \approx -43 \text{ mg/hour}$$

Joe's caffeine level was dropping faster at first and is not dropping as fast now.

A glance at the graph confirms our findings.




One last thing. There's a useful vocabulary here. When our story began Joe's caffeine level was around 1,000 mg and after 5 hours is was at 498.4 mg. That's just about 500 mg, or half of what he started with. We say the ***half-life*** of caffeine is around 5 hours.

Doesn't sound very important but check this out. Start with 1000 mg. After 5 hours, there's 500 mg left. (Okay, approximately.) Now go another 5 hours, which means 10 hours total. Evaluate our equation $J = 1{,}000 * .87^H$ when $H = 10$ to get

$$1000 * .87 \wedge \underline{10} = 248.4234142 \approx 250$$

That means half of what was left is now gone. Go another 5 hours. Lose another half. Check for yourself:

$$1000 * .87 \wedge \underline{15} = 123.8194268 \approx 125$$

And so on. Cool.

SU – look at water lily story

## 5.3 Growth factors

Obesity among children ages 6-11 continues to increase. From 1994 to 2010, the proportion of children classified as obese (meaning weighing more than 95% of the other children of their age and gender) rose from an average of 1.1 out of every ten children in 1994 to around 2 out of every ten children in 2010. Assuming that the prevalence of childhood obesity increases exponentially, what is the annual percent increase and what does the equation project for the year 2020? Well, unless we are able to make drastic improvements in how children eat and how much they exercise.

Because we are told to use an exponential equation we can use our template.

$$\text{dep} \;=\; \text{start (growth factor)}^{\text{indep}}$$

The variables are

$$C = \text{ children ages 6-11 who are obese (out of every ten children)}$$

$$Y = \text{ year (years since 1994)}$$

We know the starting amount in 1994 is 1.1 children out of every ten. The equation is of the form

$$C = 1.1 * g^Y$$

Trouble is we don't actually know what the growth factor $g$ is. Yet.

We do know that in 2010 we have $Y = 2010 - 1994 = 16$ years and $C = 2$. We can put those values into our equation

$$2 = 1.1 * g^{16}$$

That's supposed to be true but we don't know what number $g$ is so we can't check. Argh.

Oh, wait a minute. The only unknown in that equation is the growth factor $g$. What if we solve for $g$? Let's go for it. First, divide both sides of our equation by the starting amount 1.1 to get

$$\frac{2}{1.1} = \frac{\cancel{1.1} * g^{16}}{\cancel{1.1}}$$

Simplify to get

$$g^{16} = \frac{2}{1.1} = 2 \div 1.1 = 1.818181818$$

Since we want to solve for the base being raised to a power, we have a power equation. We use the ROOT FORMULA with power $n = 16$ and value $v = 1.818181818$ to get

$$g = \sqrt[n]{v} = \sqrt[16]{1.818181818} = 16 \sqrt[x]{\ } 1.818181818 = 1.038071653 \approx 1.0381$$

Want a quicker way to find the growth factor? Here's what we did in one combined formula.

GROWTH FACTOR FORMULA

If a quantity is growing (or decaying) exponentially, then the growth (or decay) factor is

$$g = \sqrt[t]{\frac{a}{s}}$$

where $s$ is the starting amount and $a$ is the amount after $t$ time periods.

In our example, the number of obese children grew from the starting amount of $s = 1.1$ to the amount of $a = 2$ after $t = 16$ years. By the GROWTH FACTOR FORMULA we have

$$\begin{aligned} g &= \sqrt[t]{\frac{a}{s}} \\ &= \sqrt[16]{\frac{2}{1.1}} \\ &= 16 \sqrt[x]{\ } (2 \div 1.1) \\ &= 1.038071653 \approx 1.0381 \end{aligned}$$

Much quicker! Notice we added parentheses because the normal order of operations would do the root first and division second. We wanted the division calculated before the root.

We knew the equation was in the form $C = 1.1 * g^Y$. Now that we found the growth factor $g \approx 1.0381$ we get our final equation

$$C = 1.1 * 1.0381^Y$$

For example, we can check that in 2010, we have $Y = 16$ still and so

$$C = 1.1 * 1.0381^{16} = 1.1 \times 1.0381 \wedge \underline{16} = 2.000874004 \approx 2 \checkmark$$

You might wonder why we didn't just round off and get

$$C = 1.1 * 1.04^Y$$

See what happens when we evaluate at $Y = 16$ if we used the rounded off equation instead. We would get

$$C = 1.1 * 1.04^{16} = 1.1 \times 1.04 \wedge \underline{16} = 2.06027937 \approx 2.1$$

Not a big difference (2.1 vs. 2.0) but enough to encourage us to keep extra digits in the growth factor in our equation.

Back to the more reliable equation

$$C = 1.1 * 1.0381^Y$$

We can now answer the two questions. First, in 2020 we have $Y = 2020 - 1994 = 26$ and so

$$C = 1.1 * 1.0381^{26} = 1.1 \times 1.0381 \wedge \underline{26} = 2.908115507 \approx 2.9$$

According to our equation, by 2020 there would be approximately 2.9 obese children for every ten children.

The other question was what the annual percent increase is. Think back to some of the examples we have seen. For example, health care costs began at $2.26 million and grew 6.7% per year so we had the equation

$$H = 2.26 * 1.067^Y$$

Our equation modeling childhood obesity is

$$C = 1.1 * 1.0381^Y$$

That must correspond to a 3.81% increase. Think of it as converting to percent

$$1.0381 = 103.81\%$$

and then ignoring the 100% to see the 3.81% increase. Childhood obesity has increased around 3.81% each year. Well, on average.

We have done just fine figuring out the growth factor from the percent change and vice versa, but if you're itching for a formula, here it is

PERCENT CHANGE FORMULA

If a quantity changes by $r\%$, then the growth (or decay) factor is

$$g = 1 + \frac{r}{100}$$

If the growth (or decay) factor is $g$, then the percent change is

$$r = 100g - 100$$

Let's check. For health care we had $r = 6.7\%$ so

$$g = 1 + \frac{r}{100} = 1 + \frac{6.7}{100} = 1 + .067 = 1.067$$

In our childhood obesity example we had $g = 1.0381$ and so the growth rate is

$$r = 100g - 100 = 100 * 1.0381 - 100 = 100 \times 1.0381 - 100 = 103.81 - 100 = 3.81\%$$

Not sure we really need these formulas, but there you have it.

These formula works just fine if a quantity decreases by a fixed percent. One example we saw was Joe, who drank too much coffee. He peaked at 1,000 mg of caffeine and it left at the rate of 13% per hour. The equation was

$$J = 1000 * .87^H$$

To apply the Percent Change Formula here, use $r = -13\%$. Yup, negative. For increasing we use a positive percent for $r$ and for decreasing we use a negative percent. Here goes

$$g = 1 + \frac{r}{100} = 1 + \frac{-13}{100} = 1 + -.13 = 1 - .13 = .87$$

If we had started knowing the decay factor of $g = .87$, we could use the Percent Change Formula to get

$$r = 100g - 100 = 100 * .87 - 100 = 100 \times .87 - 100 = 87 - 100 = -13\%$$

Again, the negative means that we have a percent decrease.

Is that too many formulas all at once? For the purpose of this section, you can forget about the Root Formula and just use Growth Factor Formula instead. Both the Growth Factor Formula and the Percent Change Formula tell us the growth (or decay) factor but they apply in separate situations.

We use the Growth Factor Formula when we know the starting and ending amount and are told the equation is exponential. It's a good question how we know it's exponential in those situations, but don't forget there's often a combination of science and data behind the scenes. We use the (much easier) Percent Change Formula when we are told (or are looking for) the percent increase or decrease.

Don't forget that sometimes the story tells us the growth or decay factor in the story. Like where three of you make a pact to each invite 10 friends to join your online group, and they each invite 10 friends, and so on. There the equation was

$$F = 3 * 10^N$$

which means the growth factor was $g = 10$. An example where the story told us the decay factor was with sheets of glass filtering out the light, only letting 75% through. There the equation was

$$L = 100 * .75^S$$

which means the decay factor was $g = .75$.

## 5.4 Linear vs. exponential models

Remeber Gilberto? He bought his car new for \$22,500 but now the car is ten years old and worth only \$7,500. We were wondering when the would be practically worthless, meaning worth under \$500.

We can describe the variables in this story.

C = value of Gilberto's car (\$)
A = time since Gilberto bought the car (years)

But what's the equation? Hmm. Don't know for sure what type of equation might work here. Tell you what, let's compare what a linear and exponential model would tell us about the value of the car.

Been awhile since we've thought about linear equations, hasn't it? The template is

**Linear equation:** dep = start + slope ∗ indep

The start of our story was ten years ago, when the car was new. The starting value of the car was \$22,500. We need to figure out the slope, which is the same as the rate of change.

$$\text{slope} = \frac{\text{change dep}}{\text{change indep}} = \frac{\$7{,}500 - \$22{,}500}{10 \text{ years}} = \frac{-\$15{,}000}{10 \text{ years}} = -\$1{,}500/\text{year}$$

Since his car is worth less every year, we should not be surprised to have a negative slope.

We now have all the pieces to write the linear equation. The dependent variable is $C$, the independent variable is $A$, the start (or intercept) is \$22,500, and now we know the slope is -\$1,500/year. Our equation is

$$C = 22{,}500 - 1{,}500A$$

As usual we wrote subtract, instead of adding the negative slope.

Let's check. The car is now ten years old and worth only \$7,500. We evaluate when $A = 10$ to get

$$C = 22{,}500 - 1{,}500 * 10 = 22{,}500 - 1{,}500 \times \underline{10} = 7{,}500 \quad \checkmark$$

When will Gilberto's car be worth under \$500? Let's solve the equation to find out. Plugging in $C = 500$ we get

$$500 = 22{,}500 - 1{,}500A$$

Subtract 22,500 from each side to get

$$\begin{array}{rcl} 500 & = & \cancel{22{,}500} - 1{,}500A \\ -22{,}500 & & -\cancel{22{,}500} \end{array}$$

Simplify to get

$$-22{,}000 = -1{,}500A$$

Two negatives too negative for you? Not to worry. Just go ahead and divide each side by -1,500 to get

$$\frac{-22{,}000}{1{,}500} = \frac{-\cancel{1{,}500}A}{-\cancel{1{,}500}}$$

Swapping $A$ to the left we get

$$A = \frac{-22{,}000}{-1{,}500} = (-)22{,}000 \div (-)1{,}500 = 14.66666... \approx 15$$

Okay, if you really don't like having two negatives, you can cancel them to get

$$\frac{22{,}000}{1{,}500} = 14.66666... \approx 15$$

Same answer, of course.

There's a subtlety here: $A = 15$ corresponds to when the car is fifteen years old. Right now the car is ten years old. So $A = 15$ means in another five years because $15 - 10 = 5$. According to the linear equation we found, Gilberto's car will be worth under \$500 in about five more years.

Done with the linear model. Let's take a look at the exponential model and then compare the two more carefully.

Here goes. The template should look familiar. It is

$$\textbf{Exponential equation:} \qquad \text{dep} \ = \ \text{start (growth factor)}^{\text{indep}}$$

We know everything except the growth factor. (Or should I say decay factor here since it's decreasing?) To find it we use the GROWTH FACTOR FORMULA. The starting amount is $s = 22{,}500$ and the ending amount is $a = 7{,}500$ after $t = 10$ years. By the formula we have

$$g = \sqrt[t]{\frac{a}{s}} = \sqrt[10]{\frac{7{,}500}{22{,}500}} = 10 \sqrt[x]{\ }(7{,}500 \div 22{,}500) = .89595846... \approx .8959$$

Since his car is worth less every year, we should not be surprised to have a growth factor less than 1. In case you're curious, the corresponding percent decrease from the PERCENT CHANGE FORMULA is

$$r = 100g - 100 = 100 \times .8959 - 100 = 89.59 - 100 = -10.41\%$$

The car is dropping by just over 10% in value each year.

We now have all the pieces to write the exponential equation. As before, the dependent variable is $C$, the independent variable is $A$, and the starting amount is \$22,500. Now we know the growth factor is .8959. Our equation is

$$C = 22{,}500 * .8959^A$$

Let's check. The car is now ten years old and worth only \$7,500. We evaluate when $A = 10$ to get

$$C = 22{,}500 * .8959^{10} = 22{,}500 \times .8959 \wedge \underline{10} = 7495.1078... \approx 7{,}500 \quad \checkmark$$

Remember we don't expect the exact answer because we rounded off the growth factor.

When will Gilberto's car be worth under $500 according to this equation? Let's solve the equation to find out. Plugging in $C = 500$ we get

$$500 = 22{,}500 * .8959^A$$

Divide both sides by 22,500 to get

$$\frac{500}{22{,}500} = \frac{\cancel{22{,}500} * .8959^A}{\cancel{22{,}500}}$$

Simplify to get

$$.8959^A = \frac{500}{22{,}500} = 500 \div 22{,}500 = .022222222...$$

By the THE LOG-DIVIDES FORMULA with growth factor $g = .8959$ and the value $v = .022222222...$ we get

$$A = \frac{\log(v)}{\log(g)} = \frac{\log(.022222222...)}{\log(.8959)} = \log(.022222222) \div \log(.8959) = 34.6291678 \approx 35$$

Remember $A = 35$ corresponds to when the car is 35 years old. Right now the car is ten years old. So $A = 35$ means in another 25 years. According to the exponential equation we found, Gilberto's car won't be worth under $500 for another 25 more years.

Done with the exponential model. Which one makes more sense? (Or is neither great?)

First things first, the car is already ten years old. Don't know what make or model the car is, but another five years seems a reasonable time until is worth under $500. That's what the linear equation projects. The exponential equation projects that the car will still be worth something for another 25 years. That doesn't seem realistic, does it? Would say linear looks better.

On the other hand, look at the values after six years from now. The car will be 16 years old then and so $A = 16$. Evaluating we get

| | |
|---|---|
| **linear:** | $C = 22{,}500 - 1{,}500 * 16 = 22{,}500 - 1{,}500 \times \underline{16} = \text{-}1{,}500$ |
| **exponential:** | $C = 22{,}500 * .8959^{16} = 22{,}500 \times .8959 \wedge \underline{16} = 3875.5657... \approx 3{,}875$ |

A sixteen year old car ought to still be worth something. Would say exponential looks better now.

Another think to think about is whether it reasonable that the car loses the same amount of value each year. That's what it means to be linear. Each year, same decrease. Nah, that's not right either. Once you drive that new car off the lot, once that strange vinyl smell of a new car wears off, once it's officially "used" the car is worth a lot less, even if it's not old at all. What would each model say the car was worth when it was almost new, say 1 year old?

The linear equation is

$$C = 22{,}500 - 1{,}500A$$

When $A = 1$ we get

$$C = 22{,}500 - 1{,}500 * 1 = 22{,}500 - 1{,}500 \times \underline{1} = \$21{,}000$$

That's nearly the original price.

The exponential equation is

$$C = 22{,}500 * .8959^A$$

When $A = 1$ we get

$$C = 22{,}500 * .8959^1 = 22{,}500 \times .8959 \wedge \underline{1} = 20157.7500... \approx \$20{,}158$$

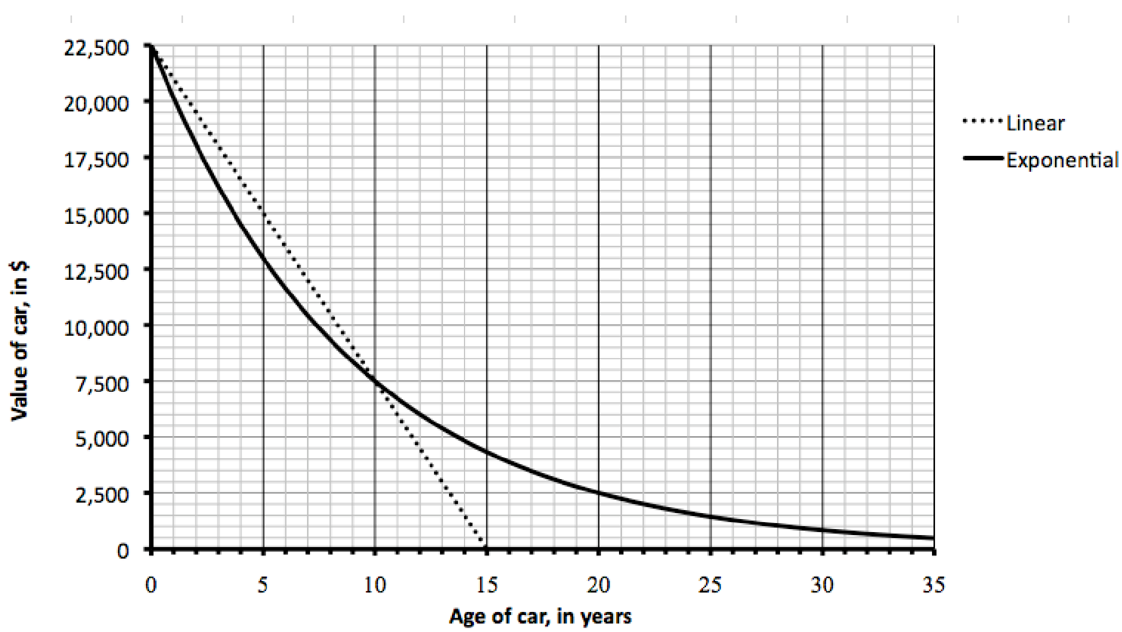
That seems more realistic. Would say exponential looks better here too.

To see what's happening, we can draw a graph. For that we need a few more values.

| $A$ | 0 | 1 | 5 | 10 | 15 | 25 | 35 |
|---|---|---|---|---|---|---|---|
| $C$ (if linear) | 22,500 | 21,000 | 15,000 | 7,500 | 0 | ~~- 15,000~~ | ~~-30,000~~ |
| $C$ (if exponential) | 22,500 | 20,158 | 12,986 | 7,500 | 4,326 | 1,441 | 480 |

Crossed out the negative values because that doesn't make sense.

The graph is




Notice that both graphs start at \$25,000 and after 10 years both are at \$7,500. Those are the two sure points we know. There's no correct answer here as to whether the model ought to be linear or exponential (or even something completely different), but a reasonable guess might be that the exponential makes more sense, but only up to maybe 20 years.

## 5.5 Logistic growth*

INTRO EXAMPLE = logistic growth

Also do the saturation models. Need idea from John.

# Solutions to practice exercises and practice exams

SU: handwrite and scan in here

# Answers to exercises

*Solutions to the practice exercises appear in an earlier section.*

SU: type in here – preferably with links to the actual problems so don't have to hand enter?