☆ solutions ☆

5.3 Growth factors – Practice exercises

Formulas referenced in the worksheets:

PERCENT CHANGE FORMULA: (updated version)

- If a quantity changes by a percentage corresponding to growth rate r, then the growth factor is
$$g = 1 + r$$
- If the growth factor is g, then the growth rate is
$$r = g - 1$$

GROWTH FACTOR FORMULA

If a quantity is growing (or decaying) exponentially, then the growth (or decay) factor is
$$g = \sqrt[t]{\frac{a}{s}}$$
where s is the starting amount and a is the amount after t time periods.

1. In 1962, my grandfather had savings bonds that matured to $200. He gave those to my mother to keep for me. These bonds have continued to earn interest at a fixed, guaranteed rate so I have yet to cash them in. The table lists the value at various times since then.

$t=8$

year	1962	1970	1980	1990	2000	2010
Y	0	8	18	28	38	48
B	200.00	318.77	570.87	1,022.34	1,830.85	3,278.77

S a

Story also appears in 1.2 #1 and 4.1 #3

Remember $^x\sqrt{\ }$ might be MATH 5

by the Growth Factor formula

(a) Use the GROWTH FACTOR FORMULA to find the annual growth factor for the time period from 1962 to 1970.

$g=\sqrt[t]{\frac{a}{s}}=\sqrt[8]{\frac{318.77}{200.00}}=8\ {}^x\sqrt{\ }\,(318.77\div 200)=1.06000..$

$\boxed{g\approx 1.06}$

(b) Repeat for 1970 to 1980.

10 more years

$g=\sqrt[t]{\frac{a}{s}}=\sqrt[10]{\frac{570.87}{318.77}}=10\ {}^x\sqrt{\ }\,(570.87\div 318.77)=$
$=1.06000...$

$\boxed{g\approx 1.06}$

(c) What do you notice? What in the story told you that would happen?

same growth factor. Story told us it continued to earn interest at a fixed guaranteed rate.

(d) What is the corresponding interest rate?

$g=1.06 \quad r=g-1=1.06-1=.06 \xrightarrow{\times 100} \boxed{6\%}$

(e) Write an equation for the value of bonds over time.

$\boxed{B=200*1.06^Y}$

Variables in table:
B = value bonds ($)
Y = time (years since 1962)

(f) Use your equation to check the information for 1990, 2000, and 2010.

$200\times 1.06 \wedge 28 =$

1990
−1962

year	Y	B
1990	28	1,022.34 ✓
2000	38	1,830.85 ✓
2010	48	3,278.77 ✓

The problem continues . . .

(g) In what year will the bond be worth over $5,000? Set up and solve an equation to decide.

$\underbrace{\$5,000}_{B}$

$$\frac{200 * 1.06^Y}{200} = \frac{5000}{200}$$

$$1.06^Y = 25$$

Log-Divides Formula

$$Y = \frac{\log(25)}{\log(1.06)} = \log(25) \div \log(1.06) = 55.24\ldots$$

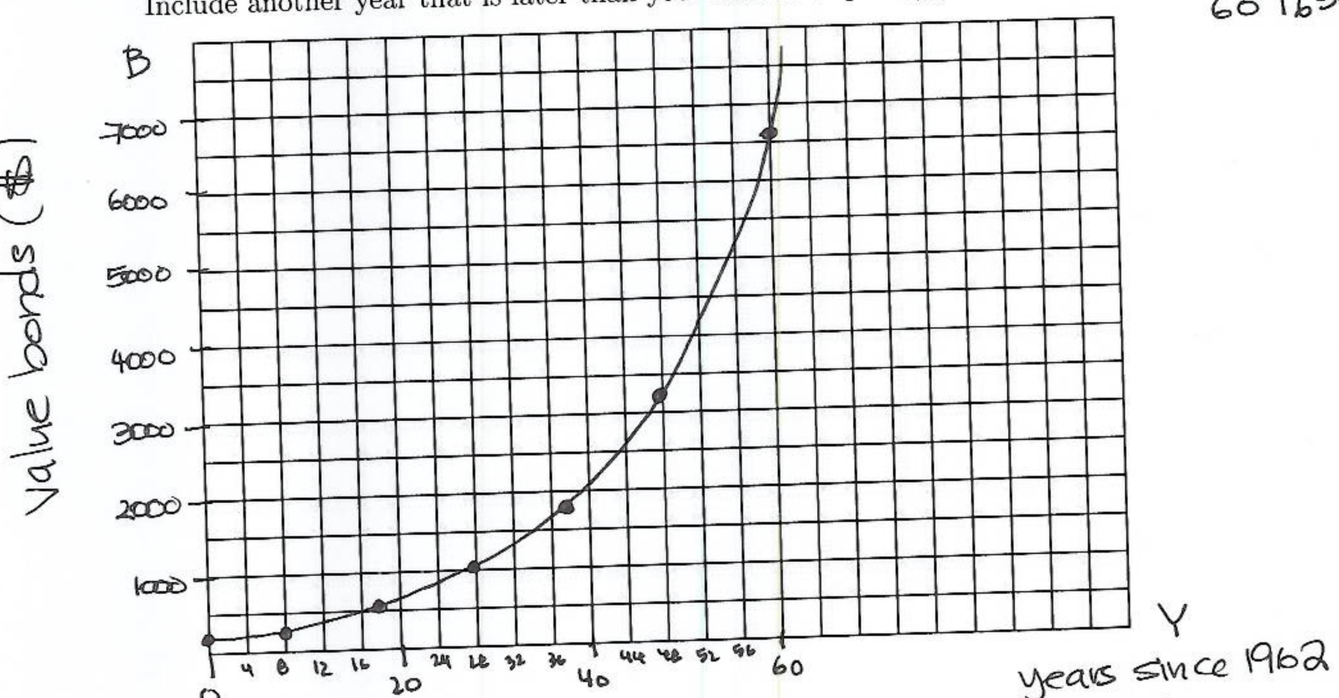
≈ 56 years

$+1962$

$= \boxed{2018}$ soon!

(h) Draw a graph using the data in the table, but not your answer to part (g). Include another year that is later than your answer to part (g).

Y	B
60	6597



(i) Does your answer to part (g) agree with your graph? If not, fix what's incorrect.

2. Have you read news stories about archaeological digs where a specimen (like a bone) is found that dates back thousands of years? How do scientists know how old something is? One method uses the radioactive decay of carbon. After an animal dies the carbon-14 in its body very slowly decays. By comparing how much carbon-14 remains in the bone to how much carbon-14 should have been in the bone when the animal was alive, scientist can estimate how long the animal has been dead. Clever, huh? Actually, it's so clever that Willard Libby won the Nobel Prize in Chemistry for it. The key information to know is that the half-life of carbon-14 is about 5,730 years. For this problem, suppose a bone is found that should have contained 300 milligrams of carbon-14 when the animal was alive. Source: Wikipedia (Radiocarbon Dating)

(a) Find the annual "growth" factor.

by the Growth Factor Formula →

$$g = \sqrt[t]{\frac{a}{s}} = \sqrt[5730]{\frac{150}{300}}$$

$$= 5730 \times\sqrt{\ } (150 \div 300) =$$

$$= \boxed{.999879039...}$$

age	C_{14} mg
0	300 = s
t = 5730	150 = a

÷2

(b) Name the variables and write an equation describing the dependence.

B = amount of Carbon 14 in bone (mg) ~ dep

A = age of bone (years) ~ indep

$$\boxed{B = 300 \times .999879^{A}}$$

(c) How many milligrams of carbon-14 should remain in this bone after 1,000 years? After 10,000 years? After 100,000 years?

A	1000	10,000	100,000
B	265.8	89.4	.0016

300 × .999879 ^ 100,000 =

(d) How many milligrams of carbon-14 should remain in this bone after 1 million years? Explain the answer your calculator gives you.

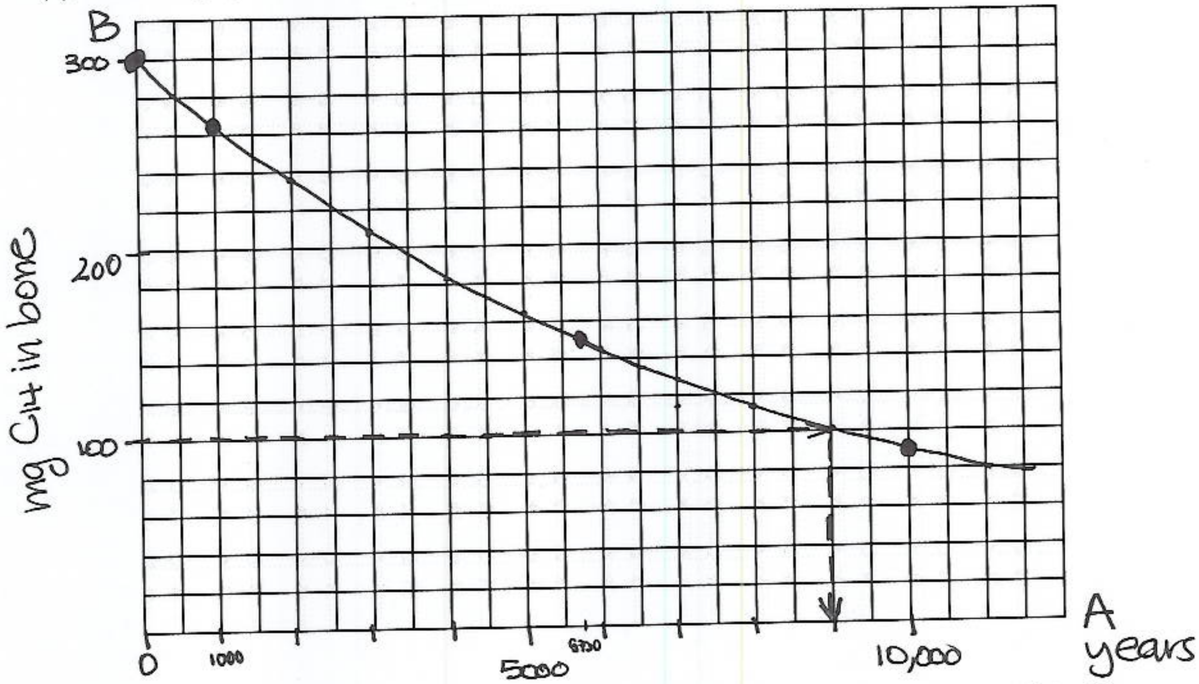
$$B = 300 \times .999879 \wedge 1,000,000 =$$

$$= 8.400583155 \times 10^{-51}$$

$$\approx \underbrace{.000000....000}_{50 \text{ zeros}}84 \text{ mg} \quad \boxed{\approx 0 \text{ mg}}$$

The problem continues ...

(e) Draw a graph that shows up to 10,000 years.



(f) If the bone is determined to have 100 milligrams of carbon-14, how old is it? That is, approximately how long ago did it die? Start by estimating the answer from your graph. Then revise your estimate using successive approximation. Display your work in a table. $\boxed{\approx 9{,}000 \text{ years old}}$ ⇐ GRAPH

$300 \times .999879 \wedge 9000 =$

A	9000	9100	9050	9070	9080
B	100.9 ✓	99.7	100.3	100.1	99.9
vs 100	high	low	high		

$\boxed{\approx 9{,}080 \text{ years old}}$

(g) Solve the equation exactly.

$$\frac{300 * .999879^A}{300} = \frac{100}{300}$$

$$.999879^A = .333333\ldots$$

$$A = \frac{\log(.333333)}{\log(.999879)} = \log(\text{ANS}) \div \log(.999879) =$$

$$= 9078.89\ldots \quad \boxed{\approx 9079 \text{ years old}}$$

(well, something over 9000 years)

3. For each equation, find the growth rate and state its units. For example, something might "grow 2% per year" while something else might "drop 7% per hour"

   (a) The number of households watching reality television R (in millions) was estimated by the equation

   $$R = 2.5 * 1.072^Y$$

   where Y is the years since 1990. *Story also appears in 5.1 Exercises*

   $g = 1.072 \quad r = g - 1 = 1.072 - 1 = \underbrace{.072 = 7.2\%}_{\times 100}$

   $\boxed{\text{\# households watching reality tv grew 7.2\%/year}}$

   (b) Chlorine is often used to disinfect water in swimming pools, but the concentration of chlorine C (in ppm) drops as the swimming pool is used for H hours according to the equation

   $$C = 2.5 * .975^H$$

   Story also appears in 3.4 #2

   $g = .975$
   $r = g - 1 = .975 - 1 = \underbrace{-.025 = -2.5\%}_{\times 100}$

   $\boxed{\text{the concentration dropped 2.5\%/hour}}$

   (c) The number of players of a wildly popular mobile app drawing game has been growing exponentially according to the equation

   $$N = 2 * 1.57^W$$

   where N is the number of players (in millions) and W is the number of weeks since people started playing the game. *Story also appears in 5.1 Exercises*

   $g = 1.57 \quad r = g - 1 = 1.57 - 1 = \underbrace{.57 = 57\%}_{\times 100}$

   $\boxed{\text{\# players increased 57\%/week}}$

4. Find the annual growth factor g and annual growth rate r for each story. Don't forget to include the negative sign for decay rates.

(a) Donations to the food shelf have increased 35% per year for the past few years.

$g = 1 + r = 1 + .35 = 1.35$

$r = 35\% \xrightarrow{\div 100} .35$

$g = 1.35$
$r = 35\%$

(b) People picking up food at the food shelf has increased exponentially too, from 120 per week in 2005 to 630 per week in 2011.

year	people
2005	120 = s
2011	630 = a

$t = 2011 - 2005 = 6$

$g = \sqrt[t]{\frac{a}{s}} = \sqrt[6]{\frac{630}{120}} = 6 \times\sqrt{}(630 \div 120) = 1.318\ldots$

$r = g - 1 = 1.318 - 1 = .318 \xrightarrow{\times 100} 31.8\%$

$g = 1.318$
$r = 31.8\%$

(c) The crime rate has dropped 3% each year recently.

$r = -3\% \xrightarrow{\div 100} -.03$

$g = 1 + r = 1 - .03 = .97$

$g = .97$
$r = -3\%$

(d) The creeping vine taking over my lawn doubles in area each year.

year	area
0	S
1	2S

$\times 2$

$g = 2$ (just 1 year)

$r = g - 1 = 2 - 1 = 1 \xrightarrow{\times 100} 100\%$

$g = 2$
$r = 100\%$

(e) Attendance at parent volunteer night has done so well it has doubled every 3 years.

by Growth Factor Formula

year	att.
0	S
t = 3	2S = a

$\times 2$

$g = \sqrt[t]{\frac{a}{s}} = \sqrt[3]{\frac{2\cancel{s}}{\cancel{s}}} = \sqrt[3]{2} = 3 \times\sqrt{}2 = 1.2599\ldots \approx 1.26$

$r = g - 1 = 1.26 - 1 = .26 \xrightarrow{\times 100} 26\%$

$g = 1.26$
$r = 26\%$

(f) The new stop sign has decreased accidents exponentially, from 40 in 2008 to 17 in 2013.

year	acc.
2008	40 = s
2013	17 = a

$t = 2013 - 2008 = 5$

$g = \sqrt[t]{\frac{a}{s}} = \sqrt[5]{\frac{17}{40}} = 5 \times\sqrt{}(17 \div 40) = .84270\ldots \approx .843$

$r = g - 1 = .843 - 1 = -.157 \xrightarrow{\times 100} -15.7\%$

$g = .843$
$r = -15.7\%$