3.2 Solving linear inequalities

In the United States temperatures for everyday things like the weather or cooking are given in Fahrenheit, denoted °F. In this system, water freezes into ice at 32°F and boils into steam at 212°F. A common setting for room temperature is 68°F whereas average human body temperature is around 98.6°F. And, most importantly, chocolate brownies bake at 350°F.

In the sciences, medicine, and most other countries, temperatures are measured in Celsius, denoted °C. (For those of us who grew up in the 1960s or earlier, "Celsius" is the temperature scale formerly known as "centigrade.") For comparison's sake, it's useful to know that water freezes at 0°C and boils at 100°C. Not coincidentally – it was set up that way. Room temperature is 20°C whereas now average human body temperature is 37°C. And those brownies?

A common conversion is given by the equation

$$F = 1.8C + 32$$

where

$$\begin{aligned} F &= \text{Fahrenheit temperature } (^\circ\text{F}) \sim \text{dep} \\ C &= \text{Celsius temperature } (^\circ\text{C}) \sim \text{indep} \end{aligned}$$

You may have seen this equation before with fractions in it: $F = \frac{9}{5}C + 32$. Just another way to write the equation, since $\frac{9}{5} = 9 \div 5 = 1.8$. For example, when $C = 100$ we have

$$F = 1.8 * 100 + 32 = 1.8 \times \underline{100} + 32 = 212 \quad \checkmark$$

You can (and should check) the other examples in our story.

What about those chocolate brownies? We are looking for $F = 350$. That's the dependent variable, so you can practice your linear equation solving skills to find the independent variable, C. It turns out that chocolate brownies bake at around $177°C$.

But, actually, chocolate brownies just need to bake in a **moderate oven**, which means between $325°F$ and $375°F$. Let's first figure out when the oven temperature is under $375°F$. We want to know when

$$F \leq 375$$

so we have an inequality instead of an equation. Remember $\leq$ stands for **less than or equal to**. Using $F = 1.8C + 32$ we get

$$1.8C + 32 \leq 375$$

We're looking for values of C that make the left-hand side a number that's smaller than, or maybe as large as, 375, but no larger. Quick vocabulary: equations have equal signs (=). When we have inequality signs ($\leq$, $\geq$, $>$, or $<$), it's called an **inequality** instead.

To solve this inequality we begin the same way as we would if we were solving the equation, by subtracting 32 from each side to get

$$\begin{array}{rcl} 1.8C + \cancel{32} & \leq & 375 \\ -\cancel{32} & & -32 \end{array}$$

which simplifes to

$$1.8C \leq 375 - 32 = 343$$

To understand why the inequality stays the same when we subtract, think of the inequality as "little" $\leq$ "big." If one number is smaller than the other, the same will be true if we subtract the same amount from each number. For example, $18 \leq 21$. To make it real, suppose I have \$18 and you have \$21. Then imagine we each buy a movie ticket for \$12. I would have $\$18 - \$12 = \$6$ and you would have $\$21 - \$12 = \$9$. And still $6 \leq 9$.

Back to our example. We had $1.8C \leq 343$. Divide each side by 1.8 to get

$$\frac{\cancel{1.8}C}{\cancel{1.8}} \leq \frac{293}{1.8}$$

which simplifies to

$$C \leq \frac{293}{1.8} = 293 \div 1.8 = 190.555555\ldots \approx 190°C$$

The oven should be set at most $190°C$. We rounded down because we do not want the brownies to burn.

To understand why the inequality stays the same when we divide, again think of the inequality as "little" $\leq$ "big." If one number is littler than the other, the same will be true when we divide each number by the same divisor. For example, $6 \leq 9$, which we imagined as my having \$6 and your having \$9 after we each bought a movie ticket. While we're making up stories, suppose we each have three children who want some money from us for treats. We each divide our remaining cash among our three children, respectively. My kids each get $\$6 \div 3 = \$2$ and your kids each get $\$9 \div 3 = \$3$. And $2 \leq 3$ still.

There is a bit of caution when solving inequalities. When symbolically solving an equation, any operation you do to each side preserves the equality: start with equal amounts, do the same thing to each, end with equal amounts. But, when symbolically solving an inequality, only some operations you do to each side preserves the inequality: add or subtract from each side or multiply or divide each side by the same (positive) number. But other operations can reverse the inequality.

For example, we can swap sides of an equation, but if we swap sides of an inequality then the direction of the sign reverses. In this brownie example, we want

$$F \geq 325$$

Remember $\geq$ stands for **greater than or equal to**. That's like "big" $\geq$ "little." We can rewrite that inequality as "little" $\leq$ "big," or equivalently

$$325 \leq F$$

In each case, 325 is "little" and F is "big". Make sense?

Multiplying or dividing each side of an inequality by a negative number switches the inequality sign as well. Watch out for that with decreasing functions because that's where the slope is negative. And the number we're dividing by is actually the slope.

Remember that the recipe for chocolate brownies says to bake in a moderate oven, between $325°F$ and $375°F$. We just figured out that $F \leq 375$ corresponds to $C \leq 190$. But that's only half of the story. We also wanted $F \geq 325$. While we could solve that inequality separately, it turns out there's an easier way.

Inequalities are a very useful notation for indicating "between". We want between $325°F$ and $375°F$ to bake the brownies. We can write

$$325 \leq F \leq 375$$

which is read

"F is between 325 and 375 (inclusive)"

The word **inclusive** indicates that we're allowing $F = 325$ or $F = 375$.

The good news is that we can solve this chain of inequalities all at once using the same steps as before but now being sure to do the same thing to all *three* sides. "*Three* sides?" you ask. Yes, "three," I confirm. Watch how this works. Start with

$$325 \leq F \leq 375$$

Using $F = 1.8C + 32$ we get

$$325 \leq 1.8C + 32 \leq 375$$

Subtract 32 from each of the three sides to get

$$\begin{array}{ccccc} 325 & \leq & 1.8\text{C} + \cancel{32} & \leq & 375 \\ \text{-32} & & \text{-}\cancel{32} & & \text{-32} \end{array}$$

which simplifies to

$$293 \leq 1.8C \leq 343$$

Next, divide all three sides by 1.8 to get

$$\frac{293}{1.8} \leq \frac{\cancel{1.8}C}{\cancel{1.8}} \leq \frac{343}{1.8}$$

which simplifies to

$$293 \div 1.8 \leq C \leq 343 \div 1.8$$

so,

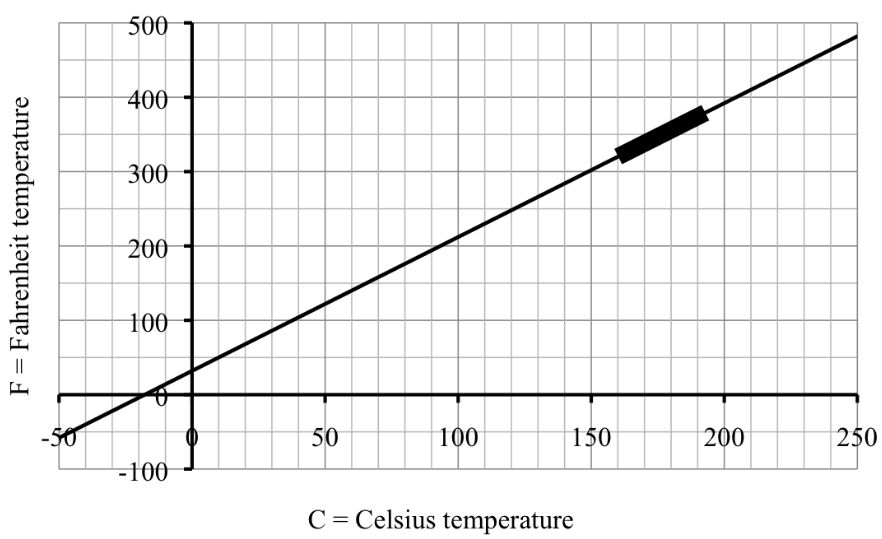
$$162.777777... \leq C \leq 190.555555...$$

Probably best to say

$$163 \leq C \leq 190$$

Chocolate brownies bake between $163°C$ and $190°C$. Oven actually aren't that precise, so somewhere between $170°C$ and $190°C$ should do the job.

If we graph our linear function $F = 1.8C + 32$, we can check our answer for the right temperature range for our brownies. Since we want F between 325 and 375 we start on the vertical axis and then use the graph to find the right range on the horizontal axis. You can see from the highlighted region that our answer is reasonable. Now, who wants brownies?



Homework

Start by doing Practice exercises #1-4 in the workbook.

Do you know ...

- Common phrases that indicate an inequality?

- How to represent the idea of "between" using a double-sided inequality?
- Why we "do the same thing to each side" of an inequality when solving?
- How to solve a linear inequality? A chain of inequalities?
- Why the inequality sign is reversed if we switch sides of the equation?
- When to solve an inequality, as opposed to solving an equation?

If you're not sure, work the rest of exercises and then return to these questions. Or, ask your instructor or a classmate for help.

Exercises

5. Recall that the conversion between Fahrenheit (F) and Celsius (C) temperatures is given by the equation
$$F = 1.8C + 32$$
   (a) Evaluate the equation at the appropriate values to check that 0°C= 32°F, 20°C= 68°F, and 37°C= 98.6°F.
   (b) Set up and solve an equation to find the Celsius equivalent of brownie-baking temperature (350°F).
   (c) You're planning a trip to Norway over Christmas and have heard it's will be around 10°C. What sort of jacket will you need? Convert to Fahrenheit to decide.
   (d) You want to explain to your Norwegian hosts that back in Minnesota this time of year temperatures can range between -20°F and 40°F. Express this range in Celsius instead. Set up and solve a chain of inequalities.
   (e) Your Norwegian hosts ask about the temperature in Minnesota during the summer. You explain that summer temperatures typically range from 55°F and 105°F. Express this range in Celsius instead. Set up and solve a chain of inequalities.

6. After borrowing some money through a line of credit on my bank account, I started paying off the interest plus \$250 a month. Even once the loan is paid off I plan to continue to deposit \$250 each month to start savins. That means my balance is given by the equation
$$A = 250M - 2{,}189.57$$
where M is the number of months since the loan and A is the account balance, in dollars. *Story also appears in 2.1 #3*

(a) Set up and solve an inequality to determine when I will have paid off my line of credit. That means the account balance will be \$0 or more.

(b) Set up and solve an inequality to determine when I will have saved at least \$2,000.

7. When Gretchen walks on her treadmill, the number of calories per mile that she burns is given by the equation

$$C = 125M$$

where C is the number of calories Gretchen burns by walking M miles.

Story also appears in 2.1 and 3.1 Exercises

(a) Draw a graph showing the number of calories Gretchen burns if she walks 0, 1, 2, or 3 miles.

(b) How far does she need to walk to burn at least 200 calories? Set up and solve an inequality.

(c) Highlight the part of your graph where she burns at least 200 calories.

8. The water in the local reservoir has been dropping steadily. In fact,

$$D = 47 - 1.5W$$

where D is the depth of the water (in feet) after W weeks. Any depth below 20 feet is considered dangerously low. When will that happen, assuming no change in the weather? Set up and solve an inequality. And, check your answer.

Story also appears in 2.1 #2 and 4.1 #3

9. A manufacturer makes family-sized bags of potato chips, advertised as containing 200 grams each. In fact it's difficult to control the exact weight of a bag of potato chips, so it varies. The standard deviation is rather high, about 3.8 grams per bag. The company would rather have bags too heavy than too light, lest they be accused of false advertising, so their average bag actually weighs 207 grams. It turns out that approximately 97% of all bags of chips weigh 200 grams or more. We can compute the standard Z-score of a given bag of chips weighing B grams using the equation

$$Z = \frac{B - 207}{3.8}$$

(a) What is the Z-score for a bag of potato chips weighing the advertised 200 grams? Remember above average Z-scores are positive and below average Z-scores are negative, so your answer should be negative.

(b) About 3/4 of all bags of chips will have $Z \geq -.67$. What weight bag has Z-score of $-.67$? Set up and solve an inequality.

(c) A standard Z-score between -1 and +1 is considered ordinary. What weight bags are considered ordinary?

(d) Oh, and if a serving size is 28 grams (approximately 1 ounce), how many servings are in a bag that weights 207 grams?

10. The cost of vacation to Cork, Ireland from the Minneapolis/St. Paul airport for two people is given by the equation formula

$$C = 2{,}828 + 310N$$

where C is the total cost in U.S. dollars and N is the number of days. Ciara wants to take her boyfriend Seamus to Cork to meet Ciara's grandmother.

(a) What would it cost Ciara to go to Ireland with Seamus for six days?

(b) What might the number 2,828 mean in terms of the story, and what are its units?

(c) What might the number 310 mean in terms of the story, and what are its units?

(d) If Ciara has budgeted up to \$10,000, how many days can they afford to spend in Ireland? Set up and solve an inequality to find the answer.