Just Enough Algebra

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Contents

Chapter 2

Equations

For most of us the word "algebra" brings to mind equations, formulas, and all those symbols. One chapter into a book on algebra and we haven't seen any equations. What gives?

Remember, this course is all about using algebra to answer questions. Equations are going to be a very important part of that work for at least three reasons. First, equations provide a nice shorthand for describing a function. It's much quicker to write down an equation than to make a table or graph. Second, equations help us categorize problems which, in turn, helps us know what to expect in that type of situation. Lastly, there are lots of powerful "symbolic" techniques we can use to solve problems when we have an equation.

So why haven't we used equations yet? Why did the first chapter focus on describing functions using words (verbal), tables (numeric), and graphs (graphical)? It turns out that there's one thing equations can't do: it's hard to tell from an equation whether an answer makes sense in the real world. If we just worked with equations we might find an answer calling for us to produce a negative number of tables or wait 300 years for an investment to reach our payoff level, or similar nonsense.

Even as we add equations to our list of tools for describing and working with functions, we will rely on words, numbers, and graphs to help evaluate the reasonableness of our answer. Thus most problems will ask you to work with all of these modes.

In this chapter we introduce equations by taking a first look at the two most important types of equations – linear and exponential. Our emphasis will be on understanding where these equations arise and how to interpret them in context. Next, we work with a variety of equations, learning how to use equations and discovering general methods for approximating solutions to equations. In later chapters we will solve equations exactly (Chapter 3) and return to study linear and exponential equations each in greater depth (Chapters 4 and 5), so don't worry if we leave a few questions unresolved for now.

2.1 A first look at linear equations

You're expecting family for dinner in a few hours and, wouldn't you know it, but your kitchen sink is clogged. The bottle of drain opener didn't clear it out. Your brother-in-law has offered to help, but last time he tried he only made it worse. The plumber will charge you \$100 just to come to your house. In addition, there will be a charge of \$75 per hour for the service. If you decide to call the plumber, what will it cost?

For example, if the plumber takes one hour, then he'll charge you \$100 for showing up and \$75 for the one hour of work. So, the total plumber's bill will be

$$100 + 75 = \$175$$

For two hours, there's still the \$100 charge, but also \$75 for each of the two hours. That's an additional charge of

$$2 \text{ hours} * \frac{\$75}{\text{hour}} = 2 \times 75 = \$150$$

So, the total plumber's bill will be

$$\$100 + \$150 = \$250$$

Try this calculation all at once.

$$\$100 + 2 \text{ hours} * \frac{\$75}{\text{hour}} = 100 + 2 \times 75 = \$250$$

Let's hope it wouldn't take the plumber as long as three hours, but if it did, we can do a similar calculation. Add the fixed charge of \$100 to the additional charge of \$75 for each of the three hours. The plumber's bill would be

$$\$100 + 3 \text{ hours} * \frac{\$75}{\text{hour}} = 100 + 3 \times 75 = \$325$$

What would it cost if the plumber takes only $\frac{1}{2}$ hour? The plumber's bill would be

$$\$100 + \frac{1}{2} \text{ hours} * \frac{\$75}{\text{hour}} = 100 + .5 \times 75 = \$100 + \$37.50 = \$137.50$$

Notice we used $\frac{1}{2} = 1 \div 2 = .5$. Bet you knew that.

What would happen if the plumber was taking so long that before he got there you dumped another bottle of drain opener in the sink and that did the trick. But before you could call and cancel the plumber, wouldn't you know it, there he was. What do you owe him for that 0 hours of work? Probably \$100. Unless your plumber says to "forget it."

We see that the plumber's charge depends on the amount of time it takes to unclog the sink. We can name these variables.

$$T = \text{time plumber takes (hours)} \sim \text{indep}$$
$$P = \text{total plumber's charge (\$)} \sim \text{dep}$$

Look at the the relationship between T and P by making a table to describe how the plumber's bill is a function of the time.

T	0	.5	1	2	3
P	100	137.50	175	250	325

Each time we knew how long the plumber spent and calculated the plumber's bill P by starting with the trip charge of \$100 and adding in \$75 times the number of hours. For example, for 3 hours we calculated

$$\$100 + 3 \text{ hours} * \frac{\$75}{\text{hour}} = \$325$$

We have a name for the number of hours in general; it is T. So for T hours, we would calculate

$$\$100 + T \text{ hours} * \frac{\$75}{\text{hour}} = P$$

See how we just put the P in for \$325 and T where the 3 hours was? We're just generalizing from our example. Drop the units and we have our equation. If the plumber works for T hours, then the cost is \$$P$ where

$$P = 100 + T * 75$$

We started the equation "$P =$" because it is a convention to begin equations with the dependent variable, when possible.

An **equation** is a formula that shows how the value of the dependent variable (like P) depends on the value of the independent variable (like T). We usually write an equation in the form

$$\text{dep} = \text{formula involving indep}$$

The equation is another way to describe a function, and efficient one – an equation carries a lot of information in only a few symbols.

There is a mathematical convention that we write numbers before letters in an equation. So, instead of $T * 75$ we should write $75 * T$. There's a conventional shorthand for this product: when a number and letter are next to each other, it means that they are multiplied. So, instead of $75 * T$ we should write $75T$. Thus our equation is normally written as

$$P = 100 + 75T.$$

You'll have to remember the hidden multiplication when you're calculating.

If you wanted to write the equation as

$$P = 75T + 100,$$

that would be okay too. We can add the \$100 trip charge first, like we did in our examples, or at the end. Same answer.

Suppose the plumber shows up at your house and fixed the sink in 25 minutes. Whew! No sooner do you pay your bill than your first dinner guest arrives. How much do you owe the plumber? Notice that

$$25 \text{ minutes} * \frac{1 \text{ hour}}{60 \text{ minutes}} = 25 \div 60 = .4166666\ldots \text{ hours}$$

Therefore for 25 minutes we have $T \approx .4166$ hours

Using our equation we get

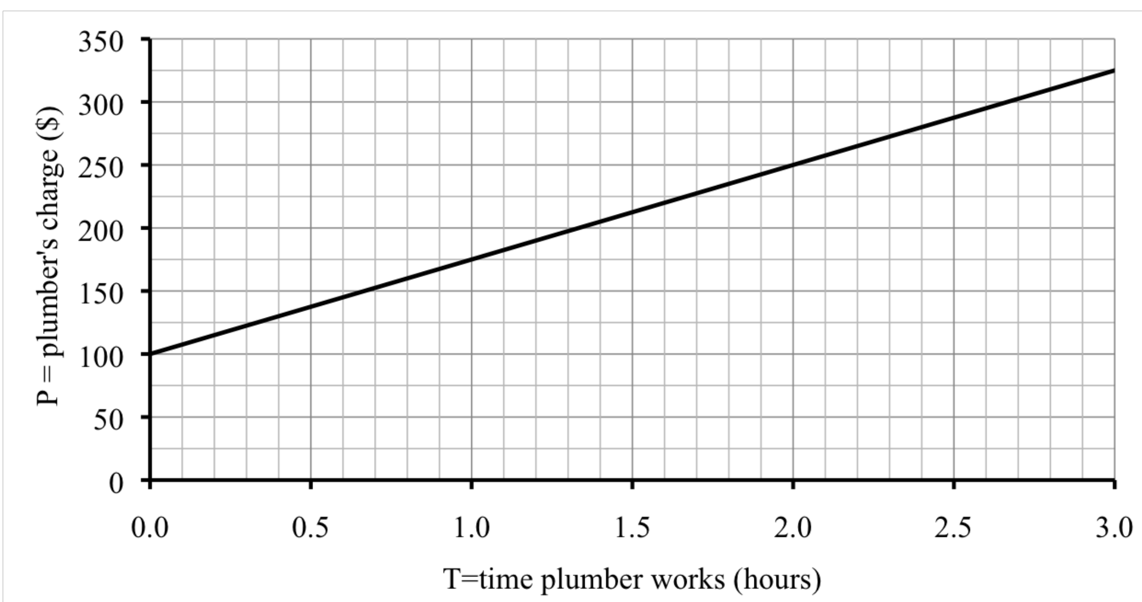
$$P = 100 + 75 * .4166 = 100 + 75 \times .4166 = 131.245 \approx \$131.25.$$

It was important that we rounded off our final answer because we had rounded off to get .4166 along the way. We could have done the entire calculation at once (avoiding the round off error) as

$$100 + 75 \times 25 \div 60 = 131.25$$

Either way, we owe the plumber \$131.25.

If we plot the points from the table of values in a graph, we see that the points lie on a line. In Chapter 1 we highlighted the points from our table on the graph. It is more common to just show the smooth curve or line.



Why is the graph a line? Remember that the rate of change tells us how steep the graph is. For example, let's find the rate of change between 1 hour and 2 hours.

$$\text{rate of change} = \frac{\text{change dep}}{\text{change indep}} = \frac{\$250 - \$175}{2\text{ hours} - 1\text{ hour}} = \frac{\$75}{1\text{ hour}} = \$75\text{ per hour}$$

Sure! We knew that. The plumber charges an extra \$75 for each extra hour he works. The rate of change is precisely \$75/hour, no matter where we calculate it. Since the rate of change is constant, the graph is the same steepness everywhere. So, the graph is a line, and the function is **linear**. Another way to say this is a function with constant rate of change is **linear**. The plumber's total charge is a linear function of time.

Look back at our equation.

$$P = 100 + 75T$$

Any linear equation fits this template.

LINEAR EQUATION TEMPLATE: dep = start + slope ∗ indep

Notice our two variables are in our equation and there are two constants. Each constant has its own meaning. The first constant is 100 and it is measured in dollars. It is the trip charge, the fixed amount we would owe the plumber even if he does 0 hours work. In our standard form we refer to this quantity as the **starting value** (or **start** for short), but it's official name is **intercept**. On the graph it's where the line crosses the vertical axis. Think of a football player (running along the vertical axis) intercepting a pass (coming in the line). We can find the intercept from our equation by plugging in $T = 0$ so that

$$P = 100 + 75 \times 0 = 100$$

The second constant is 75 and though its tempting to say it is measured in dollars, it is really measured in \$ per hour. This number is the rate of change and in the context of linear equations it gets its own name too. Its called the **slope**. Since the rate of change measures the steepness of any curve or line, the word "slope", like mountain slope, makes sense. In our plumber example the intercept was \$100 and the slope was \$75/hour.

Homework

Start by doing Practice exercises #1-4 in the workbook.

Do you know . . .

- How to generalize from an example to find an equation?
- Where the dependent variable usually is in an equation?
- What the slope of a linear function means in the story and what it tells us about the graph?
- What the intercept of a linear function means in the story and what it tells us about the graph?
- The template for a linear equation? *Ask your instructor if you need to remember the template or if it will be provided during the exam.*
- Where the slope and intercept appear in the template for a linear equation?
- What makes a function linear?
- How to plot negative numbers on a graph?
- What the graph of a linear function looks like?

  If you're not sure, work the rest of exercises and then return to these questions. Or, ask your instructor or a classmate for help.

Exercises

5. Plumbers are really expensive, so I'm comparing their rates. Write an equation for each possibility, using the same variables as our example: T for the time the plumber takes (in hours) and P for the plumber's total charge (in dollars).

   Story also appears in 4.1 #4

   (a) James charges \$50 to show up plus \$120 per hour.

   (b) Jo the plumber is just getting started in the business. She charges \$45 to show up plus \$55 per hour.

   (c) Mario advertises "no trip charge" but his hourly rate is \$90 per hour.

   (d) Not to be outdone, Luigi offers to unclog any drain for \$150, no matter how long it takes. ("Wake up, Luigi! The only time plumbers sleep on the job is when we're working by the hour," says Mario.)

6. Abduwali has just opened a restaurant. He spent \$82,500 to get started but hopes to earn back \$6,300 each month. *Story also appears in 3.1 Exercises*

   (a) If all goes according to plan, will he have made money 10 months from now?

   (b) Name the variables and write an equation relating them.

   (c) Identify the slope and intercept, along with their units, and explain what each means in terms of the story.

   (d) Make a small tables of values and use it to draw a graph showing Abduwali's profit.

7. When Gretchen walks on her treadmill, she burns 125 calories per mile.

   Story also appears in 3.1 and 3.2 Exercises

   (a) How many calories will Gretchen burn if she walks 2.3 miles?

   (b) Name the variables and write an equation relating them.

   (c) Identify the slope and intercept, along with their units, and explain what each means in terms of the story.

   (d) Make a table showing the calories she burns walking 0, 1, 2, 3, or 4 miles.

8. The local burger restaurant had a promotion this summer. Typically a bacon double cheeseburger costs \$7.16. They reduced the price by 2¢ for each degree in the daily high temperature. For example, if the high temperature was 80°F, they would decrease the price by $.02 \times 80 = \$1.60$, so the double cheeseburger would cost $7.16 - 1.60 = \$5.56$. Mmmm. *Story also appears in 3.1 #4*

   (a) Name the variables in the story and write a linear equation relating them.

   (b) Is the function increasing or decreasing?

   (c) Make a table showing the price of a bacon cheeseburger when the daily high temperature is 65°F, 75°F, and 90°F.

   (d) Draw a graph illustrating how the price of a bacon double cheeseburger depends on the temperature. Start the temperature on your graph at 60°F.

9. A report on health care back in 1975 stated that the U.S. had around 1,466,000 hospital beds and since then the number of beds has declined by around 16,000 beds per year. Source: Center for Disease Control and Prevention

   (a) Name the variables, including units and dependence.

   (b) Write an equation illustrating the function.

(c) Is the function increasing or decreasing?

(d) Make a table showing the number of hospital beds projected for 1980, 1990, 2000, 2010, and 2020.

(e) At this rate of decline, in what year will we have only 1/2 million beds? First estimate the answer from your table. Then figure it out, to the nearest year.

10. The stretch of interstate highway through downtown averages 1,450 cars per hour during the morning rush hour. The economy is improving (finally), but with that the county manager predicts traffic levels with increase around 130 cars per hour more each week for the next couple of years. *Story also appears in 3.1 Exercises*

(a) Name the variables and write an equation relating them.

(b) Make a table showing the number of cars per hour anticipated now and in 2 years, 4 years, 6 years, 8 years, and 10 years.

(c) Significant slowdown are expected if traffic levels exceed 2,000 cars per hour. When do they expect that will happen? Estimate your answer from your table. (Or, figure it out.)

(d) If traffic levels exceed 2,500 the county plans to install control lights at the on ramps. When is that expected to happen?

2.2 A first look at exponential equations

Jocelyn got a job right out of college, as an administrative assistant earning \$28,000 a year. The position turned out to be a great fit for her, and after one year she was promoted to data analyst with a 15% raise. The next year Joceyln was promoted again, to senior data analyst along with a 21% raise. "Not bad," her friend Audun said, "a 36% raise in two years." But Jocelyn quickly corrected him. "Audun, it's even better than that! It's over 39%"

After the first year, Jocelyn's salary of \$28,000 was increased by 15%. Remember that means 15% of \$28,000 more. To calculate 15% of \$28,000 we multiply using the decimal form

$$15\% = \frac{15}{100} = 15 \div 100 = .15$$

to get

$$15\% \text{ of } 28{,}000 = .15 \times 28{,}000 = 4{,}200$$

That's how much Jocelyn's raise was that first year. By adding that amount to the original salary we get

$$28{,}000 + 4{,}200 = 32{,}200$$

After one year Jocelyn's salary was \$32,200.

After the second year, Jocelyn got a 21% raise. This means her rose by 21% from what it was just before the raise, that is, from the \$32,200. (The 21% does not refer back to the original \$28,000 value.) So, to calculate the increase, we take 21% of \$32,200, which is

$$21\% \text{ of } \$32{,}200 = .21 \times 32{,}200 = \$6{,}762$$

By adding on this raise we get

$$32{,}200 + 6{,}762 = 38{,}962$$

After the second year Jocelyn was earning \$38,962.

Since Jocelyn's original salary was \$28,000, the net increase in her salary is the difference

$$\$38{,}962 - \$28{,}000 = \$10{,}962$$

The corresponding percentage increase was

$$\frac{10{,}962}{28{,}000} = 10{,}962 \div 28{,}000 = .3915 = .3915 \times 100\% = 39.15\%$$

As Jocelyn said, that's over 39% increase.

What's going on here? Audun thought that 15% and 21% would be 36% because $15 + 21 = 36$. The reason it doesn't work that was is that while the 15% is of the original \$28,000, the 21% was actually calculated on the \$32,200. So, we can't just combine percentages by adding.

Each time we figured out Jocelyn's salary, we did a two-step process. First, we calculated the amount of the increase and second, we found the new value by adding on. There's actually an easier way.

Jocelyn's salary was \$28,000 and then went up by 15%. For her new salary we want to add her old salary (all of it) plus 15%. So we want 100% plus 15%, or 115% of her old salary. That works in general. When we increase a number by 15%, we end up with 100% of what we started with plus 15% more, for a grand total of 115% of what we started with. So we can just multiply by 1.15, which is 115% written in decimal, since

$$115\% = \frac{115}{100} = 115 \div 100 = 1.15$$

Looks weird, works great.

That means all we have to do to find Jocelyn's salary is

$$28{,}000 \times 1.15 = 32{,}200$$

We can do the same thing for the next calculation

$$32{,}200 \times 1.21 = 38{,}962$$

Here we multiplied by 1.21 because after a 21% increase you have 121% of what you started with. And 121% in decimal form is just 1.21.

Now hang on to your hat, because we can combine these parts together. In our example, we started with \$28,000. Then we multiplied by 1.15, which gave us \$32,200. And then we multiplied that answer by 1.21, to get our final answer of \$38,962. So really we just did

$$28{,}000 \times 1.15 \times 1.21 = 38{,}962$$

Same answer. A lot less effort. And, check it out

$$1.15 \times 1.21 = 1.3915 = 139.15\%$$

That's where the 39.15% increase is hidden. Cool.

A little terminology before we move on. A percentage increase is known as the **growth rate** and the number we multiply in the one-step method is called the **growth factor**. For example, in calculating 15% increase, the growth rate was $15\% = .15$ in decimal, and so the growth factor was

$$115\% = 1.15$$

If you're into formulas here it is.

> PERCENT CHANGE FORMULA:
>
> If a quantity increases by a percentage corresponding to growth rate r, then the growth factor is
>
> $$g = 1 + r$$

We had $r = .15$ and so

$$g = 1 + r = 1 + .15 = 1.15$$

Jocelyn's most recent assignment has been analyzing information on rising health care costs. In 2007 the United States spent \$2.26 trillion on health care. Written out with all its zeros that's

$$\$2.26 \text{ trillion} = \$2{,}260{,}000{,}000{,}000$$

Health care costs were projected to increase at an average of 6.7% annually. That means we have

$$r = 6.7\% = .067$$

and

$$g = 1 + r = 1 + .067 = 1.067$$

So, to find the effect of a 6.7% increase, we can just multiply by 1.067. Again, that's the 100% of what we started with plus 6.7% more for a grand total of $106.7\% = 1.067$.

We are ready to do some examples. In 2007 the United States spent \$2.26 trillion on health care. The projection for one year later is

$$2.26 \times 1.067 = 2.41142 \approx \$2.41 \text{ trillion}$$

Another year later, projected health care costs are

$$2.41 \times 1.067 = 2.57147 \approx \$2.57 \text{ trillion}$$

And so on. For each year we multiply by another 1.067.

For example, by 2017 (ten years later), health care costs are projected to be

$$2.26 \times 1.067 \times 1.067 \times 1.067 \times 1.067 \times 1.067 \times 1.067 \times 1.067 \times 1.067 \times 1.067 \times 1.067$$

I don't know about you, but I would rather not type that all into a calculator. Luckily, multiplying by 1.067 ten times is the same as multiplying by 1.067^{10}. Recall that the **base** 1.067 is the number we multiply by and the **exponent** (or **power**) 10 tells us how many times. So we can calculate

$$2.26 * 1.067^{10} = 2.26 \times 1.067 \wedge 10 = 4.322675488 \approx \$4.32 \text{ trillion}$$

Notice that the order of operations is exactly what we wanted here: first raise 1.067 to the 10^{th} power, then multiply by 2.26. So we can enter it all at once without needing parentheses. Bottom line: health care costs are expected to be around \$4.32 trillion by the year 2017. Oh my!

We're so close to the equation now, we can smell it. Our variables are

$$\begin{aligned} H &= \text{health care costs (\$ trillions)} \sim \text{dep} \\ Y &= \text{time (years since 2007)} \sim \text{indep} \end{aligned}$$

We just found the cost after 10 years was

$$\$2.26 * 1.067^{10} \approx \$4.32 \text{ trillion}$$

We can generalize to get the equation by putting in Y (instead of 10) and H for the (instead of \$4.32 trillion). When we do we get

$$2.26 * 1.067^{Y} = H$$

Rewriting the equation to begin with the dependent variable we get

$$H = 2.26 * 1.067^{Y}$$

By the way, there are two other standard ways of writing this equation

$$H = 2.26(1.067)^{Y} \quad \text{and} \quad H = 2.26\left(1.067^{Y}\right)$$

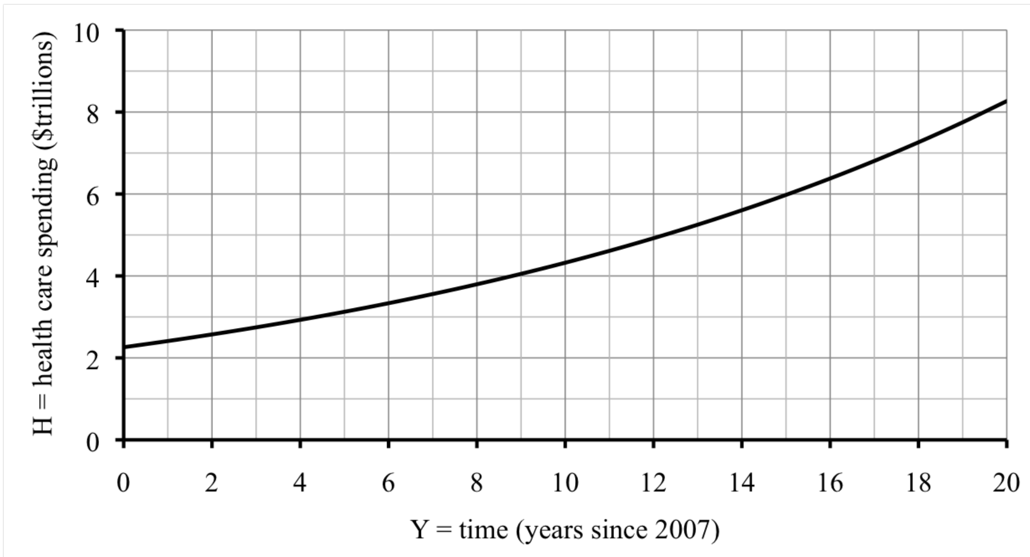
For example, in 2027 we have $Y = 2027 - 2007 = 20$ years and our equation projects health care cost at

$$2.26 * 1.067^{20} = 2.26 \times 1.067 \wedge 20 = 8.821882053 \approx \$8.82 \text{ trillion}$$

Not sure it's realistic to expect that steady an increase for 20 years, but that is what the equation projects, so let's work with it for now.

What does the graph of this function look like? Here are the values we found so far. That should be enough.

year	2007	2008	2009	2017	2027
Y	0	1	2	10	20
H	2.26	2.41	2.57	4.32	8.82



Can you see that the graph curves slightly? It's not a line. That's because this function isn't linear.

Look back at our equation

$$H = 2.26 * 1.067^Y$$

This type of equation is called an **exponential equation** because the independent variable is in the exponent. Any exponential equation fits this template.

Exponential equation template: $\quad$ dep $=$ start $*$ growth factor$^{\text{indep}}$

Notice our two variables are in our equation and there are two constants. Each constant has its own meaning. The first constant is 2.26 and it is measured in trillions of dollars. It is the amount spent on health care in the starting year of 2007. In our standard form we refer to this quantity as the **starting value** (or **start** for short). As with linear equations, it's official name is **intercept** and it's the value where the curve crosses the vertical axis on the graph.

The second constant is the growth factor 1.067, and is the decimal equivalent of the 106.7%. The growth factor for an exponential equation is similar to the slope of a linear equation because both tell us how fast the function is increasing. But the slope measures the rate of change – how much is added at each step, while the growth factor corresponds to the percent increase. Another way to say that is linear functions correspond to situations where we are adding the same amount each time and exponential functions correspond to situations where we are adding the same percentage each time (or, equivalently, multiplying by the same amount each time).

Sometimes the graph of an exponential equation looks a lot like a line, especially if you only plot a few points. So, be sure to plot five or more points to see the curve in the graph of an exponential equation.

Homework

Start by doing Practice exercises #1-4 in the workbook.

Do you know . . .

- How to find the growth factor if you know the percent increase?
- How to calculate percent increase in one step?
- What makes a function exponential?
- The template for an exponential equation? *Ask your instructor if you need to remember the template or if it will be provided during the exam.*
- Where the starting value and growth factor appear in the template for an exponential equation?
- What the graph of an exponential function looks like?

If you're not sure, work the rest of exercises and then return to these questions. Or, ask your instructor or a classmate for help.

Exercises

5. Mai's salary was \$78,000 before she got a 6% raise. Now the economy was not doing as well and she got only a 1.5% raise this year.

   (a) What was her salary after the second raise?

   (b) Her colleague Tomáš started with a salary of \$78,000 but did not get a raise the first year like Mai did. What percentage raise would Tomáš need now in order to have the same final salary as Mai?

(c) Would Mai's salary have been the more than, less than, or the same as now if she had received the 1.5% raise first and then the 6% raise?

(d) Which order would you rather have: 6% then 1.5% or 1.5% then 6%? Why?

6. The number of school children in the district from a single parent household has been on the rise. In one district there were 1,290 children from single parent households in 2010 and that number was expected to increase about 3% per year.

Story also appears in 3.4 and 5.3 Exercises

(a) Calculate the annual growth factor.

(b) How many children from single parent households are expected in that district by 2015?

(c) Name the variables and write an equation relating them.

(d) Make a table showing the number of school children in the district from a single parent household in 2010, 2015, 2020, and 2030.

(e) Graph the function.

7. Um Archivo data consultant group reported earnings of $42.7 billion in 2012. At that time executives projected 17% increase in earnings annually.

Story also appears in 5.1 Exercises

(a) Name the variables and find an equation relating them.

(b) According to your equation, what would Um Archivo's earnings be in 2020.

(c) If Um Archivo reports earnings of $78.1 billion in 2020, would you say the projected rate of 17% was too high or too low? Explain.

(d) Draw a graph showing how Um Archivo's profits are expected to increase.

8. In 2005, poultry production was 78 million tons estimated to be growing at a rate of around 1.6% per year. Source: Worldwatch Institute

Story also appears in 3.4 Exercises

(a) Write an equation showing how poultry production is expected to rise. Don't forget to name the variables.

(b) Make a table showing the production in 2005, 2010, 2020, and 2050 (at least according to the equation.)

9. Back in January 2008, e-book sales were averaging $5.1 million per month and were increasing approximately 6.3% per month. (We are ignoring seasonal variation in this problem.) Source: ReadWriteWeb

(a) Name the variables including units.

(b) Calculate the monthly growth factor.

(c) Write an exponential equation illustrating this dependence.

(d) By January 2010, monthly sales averaged 21.9 million. How does that compare to your equation's estimate?

(e) What does your equation project for average monthly ebook sales in January 2014?

2.3 Using equations

The Cadillac Escalade is a cross between a sports utility vehicle (SUV) and luxury car. Either way, it's a big car. And it takes awhile to stop. One study showed that the 2010 Escalade traveling at 60 miles per hour takes about 144 feet to come to a complete stop from when the driver first hits the brakes. In fact, the braking distance of any car depends on how fast it is going. If someone is driving slowly they can stop in shorter distance than if they are driving fast. Which is why you should drive slowly on residential streets.

We would like to be able to calculate the braking distances at other speeds, so our two variables are

$$S = \text{speed (mph)} \sim \text{indep}$$
$$B = \text{braking distance (feet)} \sim \text{dep}$$

Using the data and equations from physics, automobile analysts were able to determine that the equation relating these two variables is

$$B = .04S^2$$

Remember that the .04 written next to the S^2 means they are multiplied. We might equally well have written

$$B = .04 * S^2$$

You may be a little surprised to see the variable S **squared** (raised to the 2nd power) or wonder what the number .04 means. This equation is not something we can figure out because it relies both on the data and knowledge of the physics involved. But, we can still work with this equation to find the braking distances at any speed. (If you must know, this equation is only approximate since things like tire and road conditions are a factor, but for what we want it is good enough.)

Although in the last couple of sections we were able to find equations by generalizing examples, there are actually many different mathematical and statistical techniques for finding equations. A scientist might use lab experiments and some theory to figure it out. An economist might recognize that the equation fits a certain template because of the underlying economics. A store manager might know from years of experience that a certain equation works well to predict sales. It can be comforting to know where an equation comes from but whether we find an equation for ourselves or get it from an expert, we can use it to answer our questions and make predictions.

Now that we have an equation we should check the reported stopping distance at 60 mph. We have $S = 60$ so we substitute 60 in place of S in the equation to get

$$B = .04 * 60^2 = .04 \times \underline{60} \wedge 2 = 144 \text{ feet} \quad \checkmark$$

Quick bit of terminology. When we know the independent variable, like $S = 60$ and we substitute into the equation to find the dependent variable, like B, we say we **evaluate** the function B at $S = 60$. You might have noticed that the 60 was underlined in the calculation above. In this book we underline the value of the independent variable when we are evaluating. That way it's easier to see which numbers come from the equation and which number we're plugging in. Feel free to do the same. Or not.

Let's use our equation to calculate the braking distance for a Cadillac Escalade traveling 30, 50, 70 or 90 miles per hour. For 30 miles per hour, we have $S = 30$. So, we evaluate at $S = 30$ to get

$$B = .04 * 30^2 = .04 \times \underline{30} \wedge 2 = 36 \text{ feet}$$

At 30 mph, it takes the Cadillac Escalade 36 feet to stop. As we expected, it doesn't take nearly as far to stop as it did at 60 mph. For the other speeds we do the same thing: evaluate at the appropriate value of S. When $S = 50$ mph we get

$$B = .04 * 50^2 = .04 \times \underline{50} \wedge 2 = 100 \text{ feet}$$

When $S = 70$ mph we get

$$B = .04 * 70^2 = .04 \times \underline{70} \wedge 2 = 196 \text{ feet}$$

When $S = 90$ mph we get

$$B = .04 * 90^2 = .04 \times \underline{90} \wedge 2 = 324 \text{ feet}$$

And, what does our equation tell us when the speed is 0 mph? We evaluate at $S = 0$ mph to get

$$B = .04 * 0^2 = .04 \times \underline{0} \wedge 2 = 0 \text{ feet}$$

Well, sure! If the car isn't moving, then it won't need any distance to stop. Here's what we've found so far, displayed in a table.

S	0	30	50	60	70	90
B	0	36	100	144	196	324

My neighbor Jeff happens to drive a 2010 Cadillac Escalade. The other day he almost was in an accident on the highway. Luckily no one was hurt, but he had to slam on the brakes to stop. The police report mentioned they believe it took his car 183 feet to stop. Jeff says he was not driving over the posted speed limit of 65 mph. Should we believe him?

We can see from the table that braking distance of 183 feet falls in between the 144 and 196 on our table which leads us to believe that Jeff was traveling faster than 60 mph and slower than 70 mph. We can figure out if Jeff were driving at 65 mph, then his braking distance would have been

$$B = .04 * 65^2 = .04 \times \underline{65} \wedge 2 = 169 \text{ feet}$$

That's less than the 183 feet Jeff took to stop. So, it appears that Jeff was driving faster than 65 mph.

But wait a minute. The braking distance is just the time it takes from when the driver's foot hits the brake until the car stops. That distance doesn't take into account the driver's reaction time – how long between when the driver thinks to stop and when the driver's foot actually hits the brake. We have a new dependent variable

$$\begin{aligned} D &= \text{total stopping distance (feet)} \sim \text{dep} \\ S &= \text{speed (mph)} \sim \text{indep} \quad \textit{as before} \end{aligned}$$

How can we include this reaction time into an equation? Suppose it takes 1 second to react. We would like to know how many feet that adds to the equation. This is something we can figure out. We know the speed and the time, so multiply them to get the distance, right? One small snag: the speed is in mph (miles per hour). We need to convert units.

$$1\ \cancel{\text{sec}} * \frac{1\ \cancel{\text{min}}}{60\ \cancel{\text{sec}}} * \frac{1 \text{ hour}}{60\ \cancel{\text{min}}} * \frac{5{,}280 \text{ feet}}{1 \text{ mile}} = 1 \div 60 \div 60 \times 5{,}280 = 1.4666\ldots \approx 1.47 \text{ feet per mph}$$

added to the stopping distance. Notice the fancy fraction work with the units?

$$\frac{\text{hour} * \text{feet}}{\text{mile}} = \frac{\text{feet}}{\frac{\text{miles}}{\text{hour}}} = \frac{\text{feet}}{\text{mph}}$$

What all this mess means is that we should add 1.47 feet for every mph of speed. So S mph adds $1.47 * S$ feet to the stopping distance. Long story short, our new equation is

$$D = .04S^2 + 1.47S$$

Something interesting about this equation. The independent variable S appears twice; first for the braking distance and again because of the reaction time. When we evaluate the equation we need to plug in the value of S in two places. Check it out. When $S = 30$ mph we have

$$D = .04 * 30^2 + 1.47 * 30 = .04 \times \underline{30} \wedge 2 + 1.47 \times \underline{30} = 80.10 \approx 80 \text{ feet}$$

That's a lot further than just the braking distance of 36 feet. When $S = 50$ mph we have

$$D = .04 * 50^2 + 1.47 * 50 = .04 \times \underline{50} \wedge 2 + 1.47 \times \underline{50} = 173.50 \approx 174 \text{ feet}$$

again, much more than the braking distance of 100 feet. Here's the revised table of values.

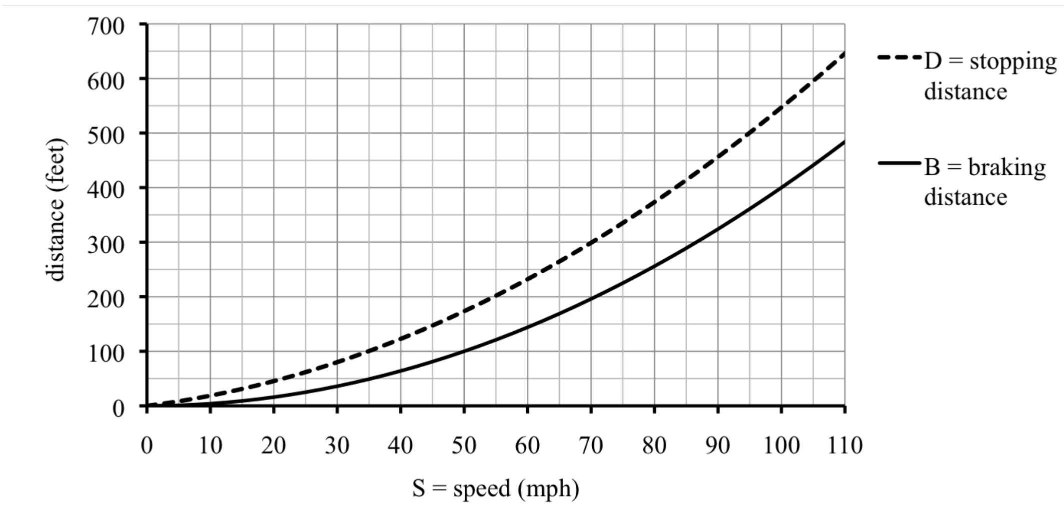
S	0	30	50	60	70	90
D	0	80	174	232	299	456

These numbers make us rethink Jeff's assertion. Given that he stopped in 183 feet, which is much less than the 232 feet it takes to stop at 60 mph, it looks like Jeff was driving less than 60 mph. To be sure, calculate that at 65 mph, it would have taken Jeff

$$D = .04 * 65^2 + 1.47 * 65 = .04 \times \underline{65} \wedge 2 + 1.47 \times \underline{65} = 264.55 \approx 265 \text{ feet}$$

Again, we should believe Jeff. And, be glad nobody was hurt.

A quick glance at the graph confirms our findings.



The distance of 183 feet falls just above the unlabeled gridline for 175 and below the gridline for 200. Looking at the corresponding point on the braking distance curve, it looks like it falls around 67 mph, but looking at the corresponding point on the stopping distance curve, it looks like just over 50 mph.

By the way, our first equation

$$B = .04 * S^2$$

is a **power equation** because the independent variable is being raised to a power, $n = 2$, and then scaled by a **proportionality constant**, $k = .04$. Any power equation fits this template.

POWER EQUATION TEMPLATE: $\text{dep} = k * \text{indep}^n$

Our second equation

$$D = .04S^2 + 1.47S$$

is a **polynomial equation** because includes both a linear and powers. The exercises introduce more polynomial equations. Polynomials can have any powers, but in this equation the highest power happens to be 2. This type of polynomial equation has a special name. It is a **quadratic equation**. Any quadratic equation fits this template.

QUADRATIC EQUATION TEMPLATE: $\text{dep} = a * \text{indep}^2 + b * \text{indep} + c$

For our equation $a = .04$, $b = 1.47$, and the mysteriously missing $c = 0$.

Homework

Start by doing Practice exercises #1-4 in the workbook.

Do you know . . .

- What it means to evaluate a function?
- Why some numbers are underlined in our calculation?
- How to evaluate an function when the independent variable occurs more than once?
- How to generate a table or graph from an equation?
- What graphs of different types of functions look like?
- What a power, polynomial, or quadratic equation look like?

If you're not sure, work the rest of exercises and then return to these questions. Or, ask your instructor or a classmate for help.

Exercises

5. The 2002 Chevrolet Tahoe 4WD will take about 158.1 feet to stop when traveling at 60 mph in normal highway conditions. Let S be the speed at which the vehicle is traveling, in miles per hour (mph), and D the distance it takes to stop, in feet. This information together with a little physics gives the equation

$$D = .0439S^2 + 1.47S$$

(a) According to this equation, how many feet does it take to stop the Tahoe when traveling 80 miles per hour?

(b) In driver's training classes they teach the "two-second rule" for safety: you should follow no closer than two seconds behind the car in front of you. If you are traveling 80 miles per hour, how many feet can you travel in two seconds? *Hint: convert to feet per second, then multiply by two seconds.*

(c) Compare your results from parts (a) and (b) to decide if the "two-second rule" is adequate for safety at 80 miles per hour. That is, if you are following two seconds behind the car in front of you and calamity strikes that car, will you be able to stop before hitting it?

(d) Is the "two-second rule" adequate at 50 mph?

6. Mom always said to sit close to the lamp when I was reading. The intensity of light L, measured in percentage (%) that you see from a lamp depends on your distance from the lamp, F feet as described by the formula

$$L = \frac{100}{F^2}$$

Story also appears in 1.1 and 3.3 Exercises

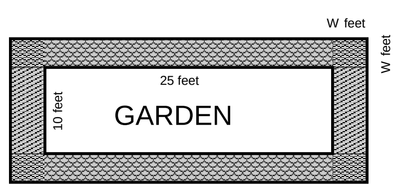
(a) Calculate the intensity when I sit 1 foot, 2 feet, or 3 feet away.

(b) The other day I was sitting 27 inches from the lamp. What was the intensity of the light there?

(c) Calculate the rate of change of light intensity between 2 feet and 27 inches. What does that tell you in terms of the story?

(d) Draw a graph illustrating the function.

7. Urban community gardens are catching on. What was once an abandoned lot down the block is now a thriving 10'×25' vegetable and berry garden for the neighborhood. (Remember ' stands for "feet," so the garden is 10 feet wide and 25 feet long.) One neighbor volunteered to donate gravel to make a path around the garden. The path will be 3 inches deep and the same width all around.



Story also appears in 2.4 Exercises and 3.5 #4

(a) The other measurements are in feet, so convert the depth of the path (3 inches) into feet also.

(b) Suppose for the moment that the path will be 4 feet wide. Calculate the area of the path. Here's one way to do it: first, find the area of the outer rectangle, then subtract the area of the garden itself.

Hint: the length of that outer rectangle includes the 25 feet of garden plus the width of the path on each side. Same for the width of that outer rectangle.

(c) Figure out how much gravel they would need (in cubic feet) for a 4 foot wide path by multiplying your answers to (a) and (b).

(d) Actually, they aren't sure how wide the path should be and how much gravel they can get. Let's write W = width of path (feet) and G = amount of gravel (cubic feet). It turns out that

$$G = W^2 + 17.5W$$

Check that when you evaluate at $W = 4$ you get the same answer as in (c).

(e) How many cubic feet of gravel would they need to make the path 2 feet wide? 3 feet wide? 42 inches?

8. One measure of the diversity of our news source is the count of the number of different daily newspapers in circulation. A reasonable equation estimating this count over the past century is

$$N = -.0021T^3 + .34T^2 - 20T + 2{,}226$$

where N is the number of daily newspapers in circulation in the United States T years after 1900.

(a) Based on this equation, how many daily newspapers were in circulation in 1920? In 1955? In 1995? In 2010?

(b) During which period were the number of newspapers in circulation dropping faster: 1900 to 1920 or 1995 to 2010?

(c) Draw a graph illustrating the dependence.

9. Mrs. Weber's cooking class came up with the equation

$$M = 1.2F^2 + 4F + 7$$

to approximate the grilling time of a piece of fish depending on its thickness. Here M is the number of minutes to grill the fish and F is the thickness of the fish (in inches).

Story also appears in 1.1 and 3.5 Exercises

(a) Evaluate the equation at $F = .25, 1, 1.5$, and 2.

(b) What do the answers you found say about cooking fish?

(c) Draw a graph showing how the cooking time depends on the thickness of the fish.

(d) Your graph should show that the function is increasing. Explain how that makes sense in terms of the story.

10. Wynter has a pretty decent job. He is paid a salary of \$780 per week but his hours vary week-to-week. Even though Wynter is not paid by the hour, he can figure out what his hourly wage would be depending on the number of hours he works. For example, in a week where Wynter works 40 hours, he's earning the equivalent of \$19.50/hr because

$$\frac{\$780}{40 \text{ hours}} = 780 \div 40 = \$19.50/\text{hour}$$

Story also appears in 3.3 Exercises

(a) What's Wynter's equivalent hourly wage in a week when he works 50 hours? 60 hours?

(b) Name the variables and write an equation relating them.

(c) Explain why this function is decreasing.

2.4 Approximating solutions of equations

Which country on Earth has the most people? If you guess China and India, in that order, you'd be right. And by a lot compared to other countries. A very distant third is the United States, with several countries close on our heels. Here are the population and growth rates estimates made in 2011. Source: CIA Factbook

1.	China	population 1.343 billion	growth rate 0.48%
2.	India	population 1.205 billion	growth rate 1.31%
3.	United States	population .313 billion	growth rate 0.90%

India's population is growing fastest of these top three, so let's take a closer look. In the exercises you can explore China and other countries. Let's tackle this question: when is India's population projected to pass 1.5 billion?

Start by writing an equation. The variables are

$$P = \text{population of India (billion people)} \sim \text{dep}$$
$$Y = \text{time (years since 2011)} \sim \text{indep}$$

Since there is a fixed percentage growth, or at least that's what we're assuming, the population grows exponentially. The template for an exponential equation is

$$\text{dep} = \text{start} * \text{growth factor}^{\text{indep}}$$

For India's population, we know that the growth rate is

$$r = 1.31\% = \frac{1.31}{100} = 1.31 \div 100 = .0131$$

so the corresponding growth factor is

$$g = 1 + r = 1 + .0131 = 1.0131$$

We also know that the starting population is 1.205 billion in 2011. We're good to go. The equation is

$$P = 1.205 * 1.0131^{Y}$$

We want to know when India's population will pass 1.5 billion people. That means we are looking for the value of Y that corresponds to $P = 1.5$. That answer is called the **solution** of our equation. In other words, a solution to an equation is the value of the independent variable we plug in to get the desired value of the dependent variable. It's not always easy to *find* a solution; but it's easy to *check* – just evaluate.

Since we can easily check and see if any number is a solution, one way to find the solution is to **guess and check**. Let's try that to determine when India's population will pass 1.5 billion people. Since we're not sure where to start, let's see what the equation projects for 2012, when $Y = 1$.

$$P = 1.205 * 1.0131^1 = 1.205 \times 1.0131 \wedge \underline{1} = 1.2208096... \approx 1.221 \text{ billion}$$

Hardly budged. Well, comparatively.

What about in 5 years? The corresponding population is

$$P = 1.205 * 1.0131^5 = 1.205 \times 1.0131 \wedge \underline{5} = 1.28614961 \approx 1.286 \text{ billion}$$

Less than 1.5 billion. Let's try $Y = 10$. The equation gives us

$$P = 1.205 * 1.0131^{10} = 1.205 \times 1.0131 \wedge \underline{10} = 1.37276417 \approx 1.373 \text{ billion}$$

Still much less than 1.5 billion

This is going slowly. We would really like to find a point at which the equation gives us more than 1.5 billion. Then we can work backwards from there to narrow things down. How about 50 years?

$$P = 1.205 * 1.0131^{50} = 1.205 \times 1.0131 \wedge \underline{50} = 2.31223244 \approx 2.312 \text{ billion}$$

That's too much, but the good news is now we know the solution is between 10 years and 50 years.

Let's summarize what we have so far in a table. Notice how we've added a third row to keep track of our progress for our goal.

Y	0	1	5	10	50				
P	1.205	1.221	1.287	1.373	2.312				
vs. 1.5	low	low	low	low	high				

We know the solution is between 10 and 50 years, and it seems closer to 10, so let's guess 20 years. In 20 years, the population should be around 1.564 billion. Where's that 1.564 from? Just our equation again. It would be good practice for you to evaluate at $Y = 20$ to check.

The 10 year estimate is too low and the 20 year estimate is too high. That means the solution is between 10 years and 20 years, so let's split the difference and guess 15 years which gives 1.465 billion. (Check again, for practice.) Ooooh, we're getting close. The population should pass 1.5 billion some time between 15 and 20 years, and likely closer to 15 so let's guess 17 years. Estimate is 1.504 billion. Would 16 years have been enough? That gives 1.484 billion, not quite enough. Let's add these numbers to our table.

Y	0	1	5	10	50	20	15	17	16
P	1.205	1.221	1.287	1.373	2.312	1.564	1.465	1.504	1.484
vs. 1.5	low	low	low	low	high	high	low	high	low

According to our equation, the population of the India should pass 1.5 billion after 17 years, which would be in the year 2028. By the way, it works to add the year and number.

$$2011 + 17 \text{ years } = 2028$$

The strategy we used to find the solution of our equation is **successive approximation**. Essentially it's just the guess and check method, but it's called "successive" because we're trying to get a closer guess each time. Typically once we have a value that's too big and one that's too small, we guess a value in between (for example, their average). This sort of splitting the difference method of guessing is a rough version of the **bisection method**. Now you know.

You might be surprised that you're supposed to guess the solution at this point in the course. I mean, in the beginning of the course we didn't have equations, just tables and graphs, and so guessing was all we had to work with. But now we have actual equations, right? In previous courses your instructor or textbook might have emphasized getting the "exact" solution.

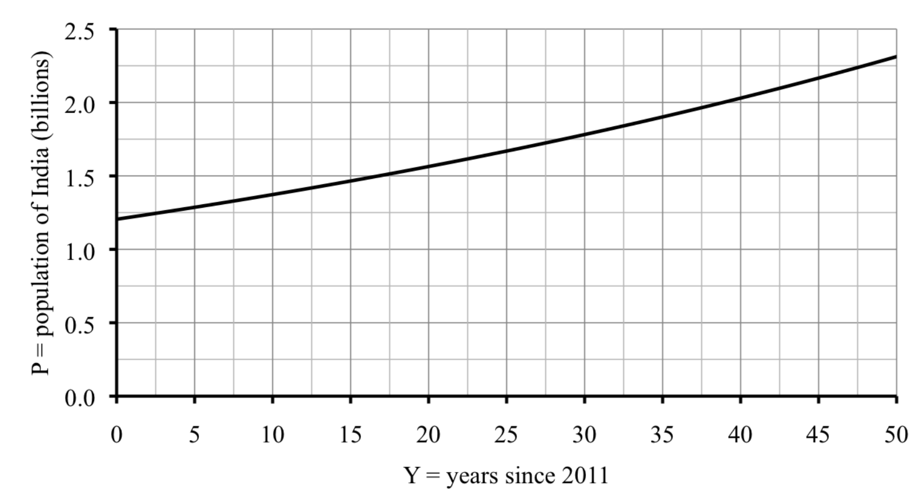
Here's why it's different in this course. First, in almost every story in this book the numbers in the problem are approximations, or at least rounded off. If you start with approximations, no matter how exact your mathematics is, the solutions will still be approximate. Second, even if our numbers started out precisely exact, chances are that the equation is only approximating reality. Do we really know what the population growth rate will be in India over the next twenty years? And, if the equation is just approximate, then no matter how exact the numbers or the mathematics, the solution will again still be approximate. Last, and this is good news – we really just want approximations. Do you really need to know that a sandwich has 427.2889 calories? Isn't 430 calories close enough? (Sound familiar? These ideas were discussed in more detail in the prelude on approximation, before Section 1.1.)

In previous mathematics courses you may have seen ways to solve equations "exactly," and we will talk about those methods in the next chapters of this text. It is true that successive approximations can taking a long time and, because of that, is a bit annoying. Solving techniques we'll learn later are much, much quicker.

There are two important reasons for using successive approximations, even if you know quicker solving techniques. First, the method of successive approximations works in most situations for any type of equation. Solving methods that we will see later on just work for one type of an equation or another – one technique for linear equations, a different technique for exponential equations, etc. That's a lot of different methods to know. Second, even if you're going to use a formal equation-solving technique to solve a problem it's a good habit to guess-and-check a bit first to make sure your solution is reasonable. It is easy to make mistakes when using those formal techniques. Remember,

I'd rather be approximately right than precisely wrong.

Okay, enough digression. Let's check our answer using the graph.



It looks like 1.5 billion corresponds to just before the unlabeled gridline halfway between 15 and 20. That line would be 17.5, so the answer of 17 (which was year 2028) looks perfect.

Homework

Start by doing Practice exercises #1-4 in the workbook.

Do you know . . .

- What a solution to an equation is?
- When you approximate a solution of an equation, as opposed to just evaluating?
- How to use successive approximation, including organizing your work in a table?
- How to get a reasonable first guess from a graph?
- What to do if you do not have a reasonable first guess?

- How precise your answer should be?
- How to find numbers between given numbers, for example between .3 and .4?

  If you're not sure, work the rest of exercises and then return to these questions. Or, ask your instructor or a classmate for help.

Exercises

5. The population of China in 2011 was approximately 1.343 billion and growing at around .48% per year. An equation estimating the population of China is

   $$P = 1.343 * 1.0048^Y$$

   where P is the population of China (measured in billions) and Y is the years since 2011. Source: CIA Factbook

   (a) In what year is China's population projected to reach 1.5 billion?

   (b) In what year is India's population expected to pass China's? Remember that we discussed India's population in this section.

   (c) Explain how we got the equation for China.

   (d) Draw graph showing both equations.

6. A company who makes electronics was doing great business in 1996, but sales quickly slid after 2000. Their sales M in millions of \$ Y years from 1996 is given by the following equation

   $$M = 104.4 + 11.5Y - 1.4Y^2$$

   Story appears in 3.5 Exercises

   (a) What were the company's sales in 1996, 2000, 2005?

   (b) The company decided to declare bankruptcy when sales fell below \$20 billion. In what year was that? Show how to use successive approximations to estimate the answer to the nearest year.

   (c) An analyst had suggested that they close down shop earlier, once sales were below \$50 billion. In what year did sales fall that low? Again, use successive approximation.

7. Suppose a special kind of window glass is 1 inch thick and lets through only 75% of the light. If two thicknesses of this glass are used, the product is 2 inches thick and lets in 56.25% since

   $$75\% \text{ of } 75\% = (.75)(.75) = .5625 = 56.25\%$$

Story also appears in 3.4 and 5.4 Exercises

(a) If three thickness of this glass is used, explain why the product is 3 inches think and lets in about 42.19% of the light.

(b) If four thickness of this glass is used, how thick will the product be and what percentage (%) of the light will be let through?

(c) Identify and name the variables, including their units, and write an equation relating them.

(d) Use successive approximation to figure out what thickness glass should be used to let through less than 10% of the light. Display your work in a table.

(e) Graph the function.

8. Wind turbines are used to generate electricity. For a particular wind turbine, the equation

$$W = 2.4S^3$$

can be used to calculate the amount of electricity generated (W watts) for a given wind speed (S mph), over a fixed period of time.

Story also appears in 1.1, 1.3, and 3.3 Exercises

(a) Make a table showing the amount of electricity produced when the wind speed is 10 mph, 25 mph, and 40 mph.

(b) Draw a graph illustrating this equation.

(c) Approximate the wind speed that will generate 12,500 watts of electricity.

9. After his first beer, Stephen's blood alcohol content (BAC) was already .04 and as he continued to drink, his BAC level rose 45% per hour. The equation is

$$S = .04 * 1.45^H$$

where S is Stephen's BAC and H is the time, measured in hours.

Story also appears in 1.1 #4 and 3.4 #1

(a) Make a table showing Stephen's BAC at the start of the problem and each of the next four hours.

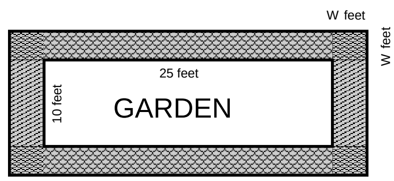
(b) Draw a graph showing how Stephen's BAC changed over time.

(c) At a BAC of .10 it is illegal for Stephen to drive. Approximately when does that happen?

10. Urban community gardens are catching on. What was once an abandoned lot down the block is now a thriving 10’×25’ vegetable and berry garden for the neighborhood. One neighbor volunteered to donate gravel to make a path around the garden. The path will be 3 inches deep and the same width all around. The amount of gravel they need (G cubic feet) is given by the equation

$$G = W^2 + 17.5W$$

where W is the width of the path in feet.



Story also appears in 2.3 Exercises and 3.5 #4

(a) If the neighbor donates 60 cubic feet of gravel, how wide a path can they build? Report your answer to two decimal places.

(b) Convert your answer to feet and inches. Do you think that’s a wide enough path?

2.5 Finance formulas

Hector is trying to figure out his finances – finding a good investment for his tax refund, saving for a down payment on a house, and dealing with his student loans. While there are various online tools that will "do the math" for him, Hector would really like to work out the formulas for himself.

First that tax refund. What a relief: \$1,040 back this year. Much as Hector is tempted to spend the money on something fun, he knows he should save it. His local bank offers him two choices: a savings account paying 1.2% interest compounded monthly or a 3-year certificate of deposit paying 3% interest compounded monthly.

The description **compounded monthly** means that the bank will pay him 1/12 of the annual interest each month, and then use that new balance in computing his interest in the month that follows. For example, that savings account would pay 1.2% interest compounded monthly, so that's

$$\frac{1.2\%}{12} = 1.2\% \div 12 = .1\%$$

interest each month. Rather than figuring out Hector's balance one month at a time, we can use this formula.

COMPOUND INTEREST FORMULA: $$a = p\left(1 + \frac{r}{12}\right)^{12y}$$
where

a = account balance (\$)
y = time invested (years)
p = initial deposit or "principal" (\$)
r = interest rate compounded monthly (as a decimal)

If you're curious, this equation fits the template for an exponential equation.

$$\text{dep} = \text{start} * \text{growth factor}^{\text{indep}}$$

The starting amount is p. The annual growth rate is r, which means the monthly growth rate is $\frac{r}{12}$, and so the monthly growth factor is

$$g = 1 + \frac{r}{12}$$

Since the interest is added each month, it would make sense to measure time in months. It is customary, however, to measure time in years instead. No big deal. After 3 years, for example, we have

$$3 \cancel{\text{years}} * \frac{12 \text{ months}}{1 \cancel{\text{year}}} = 3 \times 12 = 36 \text{ months}$$

Yup, just multiply the years by 12 to get the months. In the general formula, the number of years is y and so the number of months is $12y$.

Let's figure out how much Hector would have in his account for each choice. For the savings account we have $p = \$1{,}040$, $r = 1.2\% = 1.2 \div 100 = .012$, and $y = 3$ years so we use the COMPOUND INTEREST FORMULA to get

$$\begin{aligned}
a &= p\left(1 + \frac{r}{12}\right)^{12y} \\
&= 1{,}040\left(1 + \frac{.012}{12}\right)^{12*3} \\
&= 1{,}040 \times (1 + .012 \div 12) \wedge (12 \times 3) \\
&= 1078.10269\ldots \approx \$1{,}078.10
\end{aligned}$$

See the parentheses? The parentheses around the base were already in the equation. They make sure that the inside quantity gets calculated before it's raised to the power. We inserted the parentheses around the exponent to override the order of operations again. This time we wanted the product $(12 \times 3 = 36)$ calculated first.

Of course, Hector might want to choose that certificate of deposit instead. That pays 3% interest compounded monthly, so the only change is $r = 3\% = 3 \div 100 = .03$. Your turn. Check that using the COMPOUND INTEREST FORMULA you get \$1,137.81.

It looks like the certificate of deposit is a clear winner, but there is a catch. If Hector wants his money before the three year term is up, he loses all (or most) of the interest earned. Ouch. So Hector should decide not only based on what the accounts pay: \$1,078.10 in the savings account versus \$1,137.81 in the certificate of deposit, but also on whether he is comfortable leaving the money alone for three years or not.

Unimpressed by the \$59.71 difference and uncomfortable locking his money in for that long, Hector decides on the savings account. When he reads the account information carefully he is surprised to see the account pays "1.207% APR." What does that mean?

The acronymn **APR** stands for **annual percentage rate**. It means that 1.2% interest compounded monthly has the same net effect as paying 1.207% at the end of each year. where does that number come from? Imagine \$1 in the account $(p = 1)$ for one year $(y = 1)$ with at 1.2% interest (so $r = .012$ again). Silly, yes, but watch what we learn. The balance from the COMPOUND INTEREST FORMULA would be around \$1.01207, as you can (and should) check. That means the annual growth factor is $g = 1.01207$ which corresponds to the equivalent of annual growth rate of

$$r = g - 1 = 1.01207 - 1 = .01207 = 1.207\% \text{ APR}$$

There's a formula for this too.

> EQUIVALENT APR FORMULA: $\quad \text{APR} = \left(1 + \dfrac{r}{12}\right)^{12} - 1$
>
> where
>
> r = interest rate compounded monthly (as a decimal)

For example, the CD pays 3% compounded monthly, so $r = 3\% = 3 \div 100 = .03$. Using the EQUIVALENT APR FORMULA we get

$$\begin{aligned}
\text{APR} &= \left(1 + \frac{r}{12}\right)^{12} - 1 \\
&= \left(1 + \frac{.03}{12}\right)^{12} - 1 \\
&= (1 + .03 \div 12) \wedge 12 - 1 \\
&= .030416\ldots \approx .0304 \\
&= .0304 \times 100\% = 3.04\% \text{ APR}
\end{aligned}$$

All this thinking about savings reminds Hector that he wants to own his own place someday. He promised himself that he would start putting away some money each month to save for a down payment on a condo, or maybe even a house. Sharing an apartment with three friends, postponing buying his first car, and bringing lunch from home most days leaves Hector nearly \$1,000 per month to save. His bank offers a special savings account paying 4.5% compounded monthly if he commits to making a deposit every month for at least two years.

Suppose Hector deposits \$1,000 to the account at the end of every month. Two things happen at the end of each month – first, the interest is added to the account and, second, Hector puts another \$1,000 into the account. Let's do a couple quick examples. At the end of the first month he just has the \$1,000. What is his balance at the end of the second month? From the COMPOUND INTEREST FORMULA (with $p = \$1{,}000$, $r = 4.5\% = 4.5 \div 100 = .045$, and $12y = 1$) he has \$1,003.75 to which he adds another \$1,000 for a grand total of

$$\$1{,}003.75 + \$1{,}000 = \$2{,}003.75$$

At the end of the third month from the Compound Interest Formula (now with $p =$ \$2,003.75 but still $r = .045$ and $12y = 1$) he has \$2,011.26 to which he adds another \$1,000 for a grand total of

$$\$2{,}011.26 + \$1{,}000 = \$3{,}011.26$$

It would take too long to keep calculating one month at a time. Any sequence of regular deposits is an **annuity**. Luckily there's a formula for the (future) value of an annuity.

Future Value Annuity Formula: $$a = p * \frac{\left(1 + \frac{r}{12}\right)^{12y} - 1}{\frac{r}{12}}$$

where

$$\begin{aligned} a &= \text{account balance (\$)} \\ y &= \text{time invested (years)} \\ p &= \text{regular (monthly) deposits (\$)} \\ r &= \text{interest rate compounded monthly (as a decimal)} \end{aligned}$$

Notice that p now represents the regular, monthly deposit instead of the initial deposit. In either case think: p stands for "put in." In Hector's situation, we have $p = \$1{,}000$ and $r = 4.5\% = 4.5 \div 100 = .045$. To check his balance after 3 months, we need to convert to years. Here goes.

$$3 \cancel{\text{months}} \frac{1 \text{ year}}{12 \cancel{\text{months}}} = 3 \div 12 = .25 \text{ years}$$

so $y = .25$. From the Future Value Annuity Formula, his balance is

$$\begin{aligned} a &= p * \frac{\left(1 + \frac{r}{12}\right)^{12y} - 1}{\frac{r}{12}} \\ &= 1{,}000 * \frac{\left(1 + \frac{.045}{12}\right)^{12*.25} - 1}{\frac{.045}{12}} \\ &= 1{,}000 \times ((1 + .045 \div 12) \wedge (12 \times .25) - 1) \div (.045 \div 12) \\ &= 3{,}011.2640625\ldots \approx \$3{,}011.26 \end{aligned}$$

as we expected.

Notice how we need parentheses not only where they appear in the formula, but also around the entire numerator (top) of the fraction, around the entire denominator (bottom) of the fraction, and around the exponent. That's going to take some practice to get used to. Especially since there are actually two open parentheses in a row.

And how much will Hector have if he continues to save for a full 2 years? Use the Future Value Annuity Formula (now with $y = 2$) to get the answer of \$25,064. Wow.

He'll be buying his own house in no time. By the way, if he just stuck that \$1,000 in a shoebox under his bed (meaning earning no interest) he'd have

$$\frac{\$1{,}000}{\cancel{\text{month}}} * \frac{12\ \cancel{\text{months}}}{\cancel{\text{years}}} * 2\ \cancel{\text{years}} = 1{,}000 \times 12 \times 2 = \$24{,}000$$

The \$25,064 in his account represents a total of \$1,064 in interest. The bank is better than the shoebox.

Oh, but wait, there's those looming student loans. Hector currently owes \$16,700 at 5.75% interest compounded monthly. He's ready to start paying some of that loan off every month, which means this loan repayment is another example of an annuity. Again, two things happen at the end of each month – first, the interest is added to the account and, second, the payment is subtracted. Instead of trying examples by hand, let's fast forward to the formula. The formula that gives the (monthly) payment for a loan (or any annuity) is

LOAN PAYMENT FORMULA: $p = \dfrac{a * \frac{r}{12}}{1 - \left(1 + \frac{r}{12}\right)^{-12y}}$

where

$a =$ loan amount (\$)
$y =$ time invested (years)
$p =$ regular (monthly) payment (\$)
$r =$ interest rate compounded monthly (as a decimal)

Notice that p now represents the regular, monthly payment and a is the loan amount. In Hector's situation $a = \$16{,}700$ and $r = 5.75\% = 5.75 \div 100 = .0575$. he would like to pay off the student loan in the next two years before he buys that house, so $y = 2$ years. The monthly payment from the LOAN PAYMENT FORMULA would be

$$\begin{aligned} p &= \frac{a * \frac{r}{12}}{1 - \left(1 + \frac{r}{12}\right)^{-12y}} \\ &= \frac{16{,}700 * \frac{.0575}{12}}{1 - \left(1 + \frac{.0575}{12}\right)^{-12*2}} \\ &= (16{,}700 \times .0575 \div 12) \div (1 - (1 + .0575 \div 12) \wedge ((\text{-})12 \times 2)) \\ &= 738.2743896\ldots \approx \$738.28 \end{aligned}$$

Oh boy. If Hector has to pay \$738.28 per month for his student loan, that's really going to cut into how much he can save for that down payment on a condo. (And forget about a house.) Know why we round up? Banks always do, and then the very last payment is a tiny bit less to make up for all that round up.

This calculation is our most complicated so far. See the negative in the exponent? Look closely at where we added parentheses – top of fraction, bottom of fraction, exponent. Same as before. It's going to take practice but once you get the hang of it, it is the same steps. One suggestion: write down what you plan to enter into your calculator. That helps you get it correct and, in case you mess up, someone else can at least see what your plan was.

Back to poor Hector. Luckily the student loan is not due in 2 years. He's allowed 10 years to pay it back. Let's recalculate his monthly payment assuming he takes the full 10 years instead. From the Loan Payment Formula (now with $y = 10$ instead), we find his new monthly payment would be \$183.32. (Check!) Much more realistic. That means he would have

$$1{,}000 - \$183.32 = \$816.68$$

to save each month towards that condo. Two years probably isn't going to be enough time, so suppose he saves for three years instead. Notice that now we're saving money so we went back to the Future Value Annuity Formula with the regular deposit of $p = \$816.58$, monthly interest rate of $4.5\% = 4.5 \div 100 = .045$, and $y = 3$ years. Go for it. Did you get $\approx \$31{,}410$? Good. And, great news for Hector.

There are four different formulas in this section. Each has a different purpose. The exercises say which formula to use, but in subsequent courses you would have to choose for yourself.

A short disclaimer is in order. These formulas only work if the interest is compounded *monthly*. Can you guess how to change the formulas if the interest is paid at some other interval? Just trade out the 12s for monthly and use how ever many times per year the interest is paid instead, usually called n in other textbooks.

Homework

Start by doing Practice exercises #1-4 in the workbook.

Do you know . . .

- How to determine which formula to use? *Ask your instructor if you will be told which formula to use during the exam.*

- What the quantities a, p, y, and r from the formulas mean in the story?

- How to evaluate the formulas on your calculator? *Ask your instructor which formulas you need to remember, and whether any formulas will be provided during the exam.*
- Why parentheses are needed around the exponent, numerator, and denominator in most of the formulas?
- What APR means, and why it is different from the (nominal) interest rate?

If you're not sure, work the rest of exercises and then return to these questions. Or, ask your instructor or a classmate for help.

Exercises

5. As we have seen, Hector is trying to figure out his finances.

   (a) Check that if Hector deposits \$1,040 into a certificate of deposit earning 3% interest compounded monthly, then at the end of three years he will have \$1,137.81. Use the COMPOUND INTEREST FORMULA.

   (b) Check that if Hector takes 10 years to pay back his student loan of \$16,700 at 5.75% interest compounded monthly, then his monthly payment will be \$183.32. Use the LOAN PAYMENT FORMULA.

   (c) Check that if Hector deposits \$816.58 each month into an account earning 4.5% interest compounded monthly for 3 years, then his balance will be \$31,410. Use the FUTURE VALUE ANNUITY FORMULA.

   (d) What is the equivalent APR of 4.5% interest compounded monthly? Use the EQUIVALENT APR FORMULA.

6. (a) If Ayah invests \$35,000 for three years, how much will she have if her money earns each of the following rates compounded monthly? Use the COMPOUND INTEREST FORMULA.

   i. 6% ii. 11% iii. 1.9%

   (b) Name the variables, make a table, and draw a graph showing how her balance after three years is a function of the interest rate. Include 0% interest on your graph.

7. Lue's family bought a house three years ago and owes \$192,000 on their mortgage. In reality, their monthly payment will include taxes, insurance, and money for escrow but let's ignore those amounts for this problem. In each part of this problem, use the LOAN PAYMENT FORMULA.

(a) They currently owe \$192,000 on their mortgage for the remaining 27 years at 4.5% compounded monthly. Calculate their monthly payment.

(b) Lue's family can refinance at 3.5% compounded monthly on a 30-year mortgage loan. Rolling in closing costs, their new loan would be for \$195,000. Calculate their monthly payment if they refinance.

(c) Or, they can refinance to a 15-year mortgage at 3.25% compounded monthly. With closing costs, their new loan would be again be for \$195,000. Calculate their monthly payment if they refinance this way instead.

8. (a) Make a table showing the balance now, after 1 year, after 5 years, and after 12 years if Kurt invests \$50,000 in a certificate of deposit earning 4.77% interest compounded monthly. Use the COMPOUND INTEREST FORMULA.

   (b) Name the variables and draw a graph showing how the balance is a function of the time.

9. Soo Jin is borrowing more money for college. Compare the APR for each choice, using the EQUIVALENT APR FORMULA.

   (a) A nationally subsidized loan at 3.4% compounded monthly.

   (b) Her bank's "college loan" at 7.9% compounded monthly.

   (c) Paying her tuition on her credit card that charges of 19.8% compounded monthly.