Chapter 2

Equations

For most of us the word "algebra" brings to mind equations, formulas, and all those symbols. One chapter into a book on algebra and we haven't seen any equations. What gives?

Remember, this course is all about using algebra to answer questions. Equations are going to be a very important part of that work for at least three reasons. First, equations provide a nice shorthand for describing a function. It's much quicker to write down an equation than to make a table or graph. Second, equations help us categorize problems which, in turn, helps us know what to expect in that type of situation. Lastly, there are lots of powerful "symbolic" techniques we can use to solve problems when we have an equation.

So why haven't we used equations yet? Why did the first chapter focus on describing functions using words (verbal), tables (numeric), and graphs (graphical)? It turns out that there's one thing equations can't do: it's hard to tell from an equation whether an answer makes sense in the real world. If we just worked with equations we might find an answer calling for us to produce a negative number of tables or wait 300 years for an investment to reach our payoff level, or similar nonsense.

Even as we add equations to our list of tools for describing and working with functions, we will rely on words, numbers, and graphs to help evaluate the reasonableness of our answer. Thus most problems will ask you to work with all of these modes.

In this chapter we introduce equations by taking a first look at the two most important types of equations – linear and exponential. Our emphasis will be on understanding where these equations arise and how to interpret them in context. Next, we work with a variety of equations, learning how to use equations and discovering general methods for approximating solutions to equations. In later chapters we will solve equations exactly (Chapter 3) and return to study linear and exponential equations each in greater depth (Chapters 4 and 5), so don't worry if we leave a few questions unresolved for now.

2.1 A first look at linear equations

You're expecting family for dinner in a few hours and, wouldn't you know it, but your kitchen sink is clogged. The bottle of drain opener didn't clear it out. Your brother-in-law has offered to help, but last time he tried he only made it worse. The plumber will charge you \$100 just to come to your house. In addition, there will be a charge of \$75 per hour for the service. If you decide to call the plumber, what will it cost?

For example, if the plumber takes one hour, then he'll charge you \$100 for showing up and \$75 for the one hour of work. So, the total plumber's bill will be

$$100 + 75 = \$175$$

For two hours, there's still the \$100 charge, but also \$75 for each of the two hours. That's an additional charge of

$$2 \text{ hours} * \frac{\$75}{\text{hour}} = 2 \times 75 = \$150$$

So, the total plumber's bill will be

$$\$100 + \$150 = \$250$$

Try this calculation all at once.

$$\$100 + 2 \text{ hours} * \frac{\$75}{\text{hour}} = 100 + 2 \times 75 = \$250$$

Let's hope it wouldn't take the plumber as long as three hours, but if it did, we can do a similar calculation. Add the fixed charge of \$100 to the additional charge of \$75 for each of the three hours. The plumber's bill would be

$$\$100 + 3 \text{ hours} * \frac{\$75}{\text{hour}} = 100 + 3 \times 75 = \$325$$

What would it cost if the plumber takes only $\frac{1}{2}$ hour? The plumber's bill would be

$$\$100 + \frac{1}{2} \text{ hours} * \frac{\$75}{\text{hour}} = 100 + .5 \times 75 = \$100 + \$37.50 = \$137.50$$

Notice we used $\frac{1}{2} = 1 \div 2 = .5$. Bet you knew that.

What would happen if the plumber was taking so long that before he got there you dumped another bottle of drain opener in the sink and that did the trick. But before you could call and cancel the plumber, wouldn't you know it, there he was. What do you owe him for that 0 hours of work? Probably \$100. Unless your plumber says to "forget it."

We see that the plumber's charge depends on the amount of time it takes to unclog the sink. We can name these variables.

$$T = \text{time plumber takes (hours)} \sim \text{indep}$$
$$P = \text{total plumber's charge (\$)} \sim \text{dep}$$

Look at the the relationship between T and P by making a table to describe how the plumber's bill is a function of the time.

T	0	.5	1	2	3
P	100	137.50	175	250	325

Each time we knew how long the plumber spent and calculated the plumber's bill P by starting with the trip charge of \$100 and adding in \$75 times the number of hours. For example, for 3 hours we calculated

$$\$100 + 3 \text{ hours} * \frac{\$75}{\text{hour}} = \$325$$

We have a name for the number of hours in general; it is T. So for T hours, we would calculate

$$\$100 + T \text{ hours} * \frac{\$75}{\text{hour}} = P$$

See how we just put the P in for \$325 and T where the 3 hours was? We're just generalizing from our example. Drop the units and we have our equation. If the plumber works for T hours, then the cost is $\$P$ where

$$P = 100 + T * 75$$

We started the equation "$P =$" because it is a convention to begin equations with the dependent variable, when possible.

An **equation** is a formula that shows how the value of the dependent variable (like P) depends on the value of the independent variable (like T). We usually write an equation in the form

$$\text{dep} = \text{formula involving indep}$$

The equation is another way to describe a function, and efficient one – an equation carries a lot of information in only a few symbols.

There is a mathematical convention that we write numbers before letters in an equation. So, instead of $T * 75$ we should write $75 * T$. There's a conventional shorthand for this product: when a number and letter are next to each other, it means that they are multiplied. So, instead of $75 * T$ we should write $75T$. Thus our equation is normally written as

$$P = 100 + 75T.$$

You'll have to remember the hidden multiplication when you're calculating.

If you wanted to write the equation as

$$P = 75T + 100,$$

that would be okay too. We can add the \$100 trip charge first, like we did in our examples, or at the end. Same answer.

Suppose the plumber shows up at your house and fixed the sink in 25 minutes. Whew! No sooner do you pay your bill than your first dinner guest arrives. How much do you owe the plumber? Notice that

$$25 \text{ minutes} * \frac{1 \text{ hour}}{60 \text{ minutes}} = 25 \div 60 = .4166666\ldots \text{ hours}$$

Therefore for 25 minutes we have $T \approx .4166$ hours

Using our equation we get

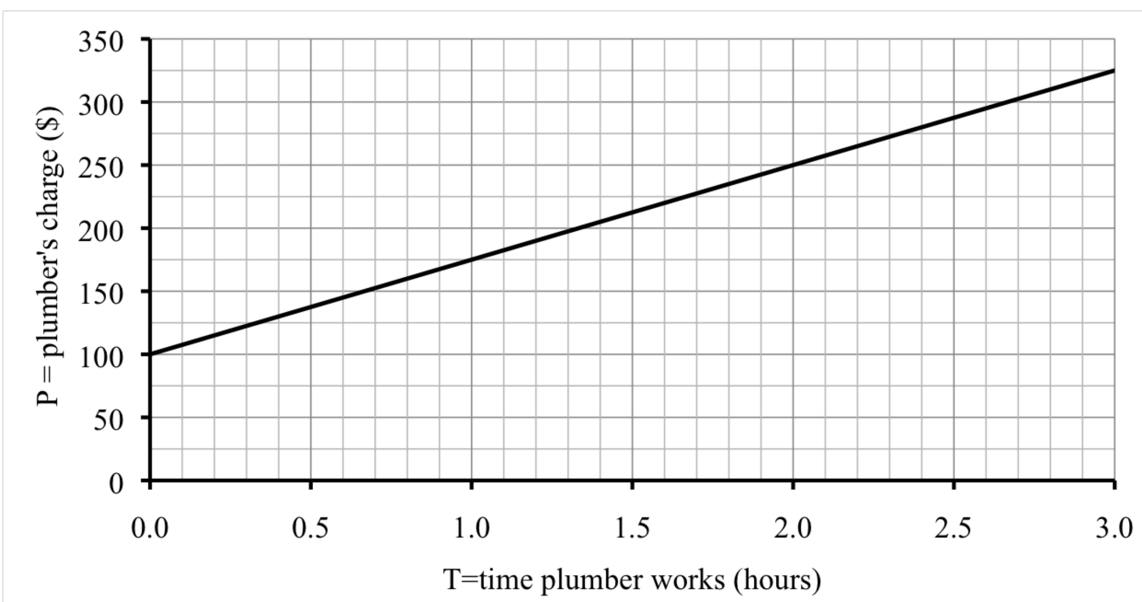
$$P = 100 + 75 * .4166 = 100 + 75 \times .4166 = 131.245 \approx \$131.25.$$

It was important that we rounded off our final answer because we had rounded off to get .4166 along the way. We could have done the entire calculation at once (avoiding the round off error) as

$$100 + 75 \times 25 \div 60 = 131.25$$

Either way, we owe the plumber \$131.25.

If we plot the points from the table of values in a graph, we see that the points lie on a line. In Chapter 1 we highlighted the points from our table on the graph. It is more common to just show the smooth curve or line.



Why is the graph a line? Remember that the rate of change tells us how steep the graph is. For example, let's find the rate of change between 1 hour and 2 hours.

$$\text{rate of change} = \frac{\text{change dep}}{\text{change indep}} = \frac{\$250 - \$175}{2 \text{ hours} - 1 \text{ hour}} = \frac{\$75}{1 \text{ hour}} = \$75 \text{ per hour}$$

Sure! We knew that. The plumber charges an extra $75 for each extra hour he works. The rate of change is precisely $75/hour, no matter where we calculate it. Since the rate of change is constant, the graph is the same steepness everywhere. So, the graph is a line, and the function is **linear**. Another way to say this is a function with constant rate of change is **linear**. The plumber's total charge is a linear function of time.

Look back at our equation.

$$P = 100 + 75T$$

Any linear equation fits this template.

LINEAR EQUATION TEMPLATE: $\quad$ dep = start + slope ∗ indep

Notice our two variables are in our equation and there are two constants. Each constant has its own meaning. The first constant is 100 and it is measured in dollars. It is the trip charge, the fixed amount we would owe the plumber even if he does 0 hours work. In our standard form we refer to this quantity as the **starting value** (or **start** for short), but it's official name is **intercept**. On the graph it's where the line crosses the vertical axis. Think of a football player (running along the vertical axis) intercepting a pass (coming in the line). We can find the intercept from our equation by plugging in $T = 0$ so that

$$P = 100 + 75 \times 0 = 100$$

The second constant is 75 and though its tempting to say it is measured in dollars, it is really measured in $ per hour. This number is the rate of change and in the context of linear equations it gets its own name too. Its called the **slope**. Since the rate of change measures the steepness of any curve or line, the word "slope", like mountain slope, makes sense. In our plumber example the intercept was $100 and the slope was $75/hour.

Homework

Start by doing Practice exercises #1-4 in the workbook.

Do you know . . .

- How to generalize from an example to find an equation?
- Where the dependent variable usually is in an equation?
- What the slope of a linear function means in the story and what it tells us about the graph?
- What the intercept of a linear function means in the story and what it tells us about the graph?
- The template for a linear equation? *Ask your instructor if you need to remember the template or if it will be provided during the exam.*
- Where the slope and intercept appear in the template for a linear equation?
- What makes a function linear?
- How to plot negative numbers on a graph?
- What the graph of a linear function looks like?

If you're not sure, work the rest of exercises and then return to these questions. Or, ask your instructor or a classmate for help.

Exercises

5. Plumbers are really expensive, so I'm comparing their rates. Write an equation for each possibility, using the same variables as our example: T for the time the plumber takes (in hours) and P for the plumber's total charge (in dollars).

   Story also appears in 4.1 #4

   (a) James charges $50 to show up plus $120 per hour.

   (b) Jo the plumber is just getting started in the business. She charges $45 to show up plus $55 per hour.

   (c) Mario advertises "no trip charge" but his hourly rate is $90 per hour.

   (d) Not to be outdone, Luigi offers to unclog any drain for $150, no matter how long it takes. ("Wake up, Luigi! The only time plumbers sleep on the job is when we're working by the hour," says Mario.)

6. Abduwali has just opened a restaurant. He spent \$82,500 to get started but hopes to earn back \$6,300 each month. *Story also appears in 3.1 Exercises*

   (a) If all goes according to plan, will he have made money 10 months from now?

   (b) Name the variables and write an equation relating them.

   (c) Identify the slope and intercept, along with their units, and explain what each means in terms of the story.

   (d) Make a small tables of values and use it to draw a graph showing Abduwali's profit.

7. When Gretchen walks on her treadmill, she burns 125 calories per mile.

   Story also appears in 3.1 and 3.2 Exercises

   (a) How many calories will Gretchen burn if she walks 2.3 miles?

   (b) Name the variables and write an equation relating them.

   (c) Identify the slope and intercept, along with their units, and explain what each means in terms of the story.

   (d) Make a table showing the calories she burns walking 0, 1, 2, 3, or 4 miles.

8. The local burger restaurant had a promotion this summer. Typically a bacon double cheeseburger costs \$7.16. They reduced the price by 2¢ for each degree in the daily high temperature. For example, if the high temperature was 80°F, they would decrease the price by $.02 \times 80 = \$1.60$, so the double cheeseburger would cost $7.16 - 1.60 = \$5.56$. Mmmm. *Story also appears in 3.1 #4*

   (a) Name the variables in the story and write a linear equation relating them.

   (b) Is the function increasing or decreasing?

   (c) Make a table showing the price of a bacon cheeseburger when the daily high temperature is 65°F, 75°F, and 90°F.

   (d) Draw a graph illustrating how the price of a bacon double cheeseburger depends on the temperature. Start the temperature on your graph at 60°F.

9. A report on health care back in 1975 stated that the U.S. had around 1,466,000 hospital beds and since then the number of beds has declined by around 16,000 beds per year. Source: Center for Disease Control and Prevention

   (a) Name the variables, including units and dependence.

   (b) Write an equation illustrating the function.

(c) Is the function increasing or decreasing?

(d) Make a table showing the number of hospital beds projected for 1980, 1990, 2000, 2010, and 2020.

(e) At this rate of decline, in what year will we have only $1/2$ million beds? First estimate the answer from your table. Then figure it out, to the nearest year.

10. The stretch of interstate highway through downtown averages 1,450 cars per hour during the morning rush hour. The economy is improving (finally), but with that the county manager predicts traffic levels with increase around 130 cars per hour more each week for the next couple of years. *Story also appears in 3.1 Exercises*

    (a) Name the variables and write an equation relating them.

    (b) Make a table showing the number of cars per hour anticipated now and in 2 years, 4 years, 6 years, 8 years, and 10 years.

    (c) Significant slowdown are expected if traffic levels exceed 2,000 cars per hour. When do they expect that will happen? Estimate your answer from your table. (Or, figure it out.)

    (d) If traffic levels exceed 2,500 the county plans to install control lights at the on ramps. When is that expected to happen?