Chapter 1

Variables

Believe it or not, algebra is useful. Really useful. It's useful in later courses you might take in mathematics, statistics, science, social science, or business. But it's also useful in real life. A lot of what happens in the world around us is easier to understand using algebra. That's what this course is all about: using algebra to answer questions.

The philosophy of this course is captured in this quote by Ethan Bolker from his text *Using algebra*.

> With common sense and some algebra you can understand the world better than you can with common sense alone.

In this first chapter we introduce the key concepts of variable and function that help us translate between problems stated in words and the mathematics explaining a situation. We explain the important tools of units, tables, and graphs. We also describe how functions change using the fundamental concept of rate of change. Throughout this chapter we keep a careful eye on evaluating the reasonableness of answers by connecting what we learn from algebra with our own life experience. After all, an answer to a real problem should make sense, right?

Some of our approach may feel strange to you. It is possibly very different from what you've seen in mathematics classes before. It might take you a little time to get used to, but it will be worth it.

Prelude: approximation

How tall is that Maple tree? If you think about it, it is not obvious how to measure the height of a tree. We could measure to the highest leaf, but it seems odd to say that the tree is shorter if a leaf falls off. Or we could measure to the top of a branch, but it might bend lower in the wind. Or we could measure to the top of a thick enough branch, whatever that means. The point is that we don't know how to measure the height of a tree that precisely. By the way, the word **precisely** refers to the number of decimal digits.

Could the Maple tree be 93.2 feet tall? No way. That's too precise. Is 93 feet tall correct? Maybe, but we could be off by a couple of feet depending on where we measure. Perhaps we can hedge slightly and call it 95 feet tall. Hopefully that's reasonable. Or maybe we should play it really safe and say it is not quite 100 feet tall. The point is: there is no such thing as "the" right answer. When we ask a real world question, we want a real world answer. *The answer depends on the question.*

While it is good to keep as many digits as possible during calculations, at the end of a problem you should approximate the answer by **rounding** – finding the closest number of a given precision. The height of 93.2 feet was likely rounded to the nearest tenth (one decimal place). We rounded to the nearest whole number to get 93 feet. The point is that 93.2 is closer to 93.0 than it is to 94.0, so our answer is 93.0 or 93 feet. Perhaps this is a good place to mention the notation. We write $93.2 \approx 93$ to indicate that we have rounded. The symbol $\approx$ means **approximately equal to**.

How much to round off the answer depends on the question. To begin you should apply your common sense. Your answer should definitely sound natural, something you might actually say to a friend or your boss. But there's also one more rule to know: *Your answer should not be more precise than the information in the problem.*

For example, suppose we read that the comprehensive fee at a local university is around \$23,000 and projected to increase by 12% per year. We want to calculate the comprehensive fee in 4 years. As we'll learn later in this course, the answer is

$$36{,}190.945\ldots$$

The dots indicate that we have not copied all the digits from the calculator. We could round to the nearest penny and say "around \$36,190.95." Or, we could round to the nearest dollar and say "around \$36,191." The numbers we are given (\$23,000 and 12%) have only two digits that matter, however, so we should actually round off and say "just over \$36,000."

By the way, when we refer to digits that matter, we are really referencing **significant digits**. That theory explains how combining numbers influences the number of digits in the answer that are accurate, which is why we wait until our final answer to round. In this text we do not follow those rules exactly, but you should be aware that some areas of study, such as Chemistry, do.

You might be surprised to learn that approximate answers are not only good enough; they are often best. For one thing, in practice we want a round number so it is easy to understand and work with our answer. A rounded answer is just approximate. Also often the numbers we are given in a problem were rounded or approximated – for the record, that fee was really \$23,058, not \$23,000. When we start with approximate numbers, then no matter how precise the mathematics we use, we can only get approximate answers. Also, in much of this course the methods we will use to calculate answers are, themselves, approximate. We might *suppose* that tuition increases exactly 12% each year, when we know in reality that the percent will likely vary. That is an example of using an **approximate model**. Last, we might have an actual model but use some numerical or graphical technique for solving. That is an example of using an **approximation technique.** In either case, if the model or technique we use is approximate, then our answer can only be either. There is an old saying we try to live by in this course.

I'd rather be approximately right than precisely wrong.

One more subtlety. We have been rounding to the nearest number of a given precision. That process is also known as **rounding off**. There are times when we will need to **round up** – to the next highest number of a given precision, or **round down** – to the next lowest number of a given precision.

For example, during Happy Hour at a local restaurant, buffalo wings sell for 60¢ per wing. Your buddy only has \$7. After a quick calculation on his cell phone he decides to order a dozen wings. Your buddy probably calculated

$$7 \div .60 = 11.6666666\ldots \approx 12$$

Trouble is he cannot afford a dozen wings, because they would cost \$7.20. (Check $12 \times .60 =$ \$7.20.) Not to mention the tax, tip, and that beer he drank. Good thing you can point him to the bank machine so he can get cash and you won't have to pay his tab (again). What's the trouble here? Besides ignoring tax, tip, and that beer he rounded off when he should have rounded down

$$7 \div .60 = 11.6666666\ldots \approx 11$$

It should be clear form the story whether you will need to round off, round up, or round down. Again, our mantra is: the answer depends on the question.

1.1 Variables and functions

Things change, like the price of gasoline, and just about every day it seems. What does it mean when the price of a gallon of gas drops from \$3.999/gal to \$3.299/gal? The symbol / is short for "per" or "for each," so that means each gallon costs

$$\$3.999 - \$3.299 = \$.70 = 70¢$$

less. Does this 70¢ truly matter?

Before we answer that question, are you wondering why there's that extra 9 at the end of the price? We might think a gallon costs \$3.99 but there's really a small 9 following it. Sometimes that 9 is raised up slightly on the gas station sign. You have to read the fine print. What it means is an extra $\frac{9}{10}$¢ for each gallon. So the true price of a gallon gas would be \$3.999. Gas costs a tiny bit more than you thought. Good grief.

Back to our question. Does 70¢ truly matter to us? Probably not. Can't even buy a bag of potato chips for 70¢. But, how often do you buy just one gallon of gas? Typically you might put five, or ten, or even twenty gallons of gas into the tank. We want to understand how the price of gasoline influences what it really costs us at the pump. To do that let's compare our costs when we buy ten gallons of gas. There's no good reason for picking ten; it's just a nice number to work with.

If gas costs \$3.999/gal and we buy 10 gallons, it costs

$$10 \text{ gallons} * \frac{\$3.999}{\text{gallon}} = 10 \times 3.999 = \$39.99$$

See how we described the computation twice? First, with units, fractions, and $*$ for multiplication in what's sometimes called "algebraic notation." Then, with just numbers and $\times$ for multiplication – that's what you can type into a calculator.

If gas drops to \$3.299/gal and we buy 10 gallons, it costs

$$10 \text{ gallons} * \frac{\$3.299}{\text{gallon}} = 10 \times 3.299 = \$32.99$$

That's \$7 less. For \$7 savings on gas you could buy that bag of potato chips, and an iced tea to go with it, and still have change. That amount matters. I mean, especially since it's \$7 savings every time you put 10 gallons in the tank.

Gas prices have been changing wildly, and along with them, the price of 10 gallons of gas. In mathematics, things that change are called **variables**. The two variables we're focusing on in this story are

$$P = \text{price of gasoline (\$/gal)}$$
$$C = \text{total cost (\$)}$$

Notice that we gave each variable a letter name. It is helpful to just use a single letter chosen from the word it stands for. In our example, P stands for "price" and C stands for "cost". In this course we rarely use the letter X simply because so few words begin with X. Whenever we name a variable (P) we also describe in words what it represents (the price of gasoline), and we state what units it's measured in (\$/gal).

In talking about the relationship between these variables we might say "the cost depends on the price of gas," so C depends on P. That tells us that C is the **dependent variable** and P is the **independent variable**. In general, the variable we really care about is the dependent variable, in this case C the total amount of money it costs us. The concept of dependence is so important that there's yet another word for it. We say that C is a **function** of P, as in "cost is a function of price."

Knowing which variable is independent or dependent is helpful to us. To emphasize the dependence, we often make a notation next to the variable name.

$$P = \text{price of gasoline (\$/gal)} \sim \text{indep}$$
$$C = \text{total cost (\$)} \sim \text{dep}$$

This labeling is rarely used outside this textbook, so add it in for yourself if you need it. In some situations dependency can be viewed either way; there might not be one correct way to do it. Labeling the dependence is extra important then, so anyone reading your work knows which way you are thinking of it.

Given a choice, we usually assign dependence such that given a value of the independent variable, it is easy to calculate the corresponding value for the dependent variable. In our example it's easy to use the price per gallon, P, to figure out the total cost, C. We can work backwards – from C to P – but it's not as easy.

For example, suppose we buy 10 gallons of gas and it costs \$28.99. We can figure out that the price per gallon must be

$$P = \frac{\$28.99}{10 \text{ gallons}} = 28.99 \div 10 = \$2.899/\text{gal}$$

Notice that we use the fraction as part of the algebraic notation, but we use ÷ to indicate division on the calculator. Your calculator key for division may be / instead, which we reserve as a shorthand for "per."

From our experience we have a sense of what gas might cost. In my lifetime, I've seen gas prices as low as 35.9¢ /gallon in the 1960s to a high of \$4.099/gallon recently. This range of values sounds too specific, so it would sound better to say something general like

"Gas prices are (definitely) between \$0/gal and \$5/gal."

The mathematical shorthand for this sentence is

$$0 \le P \le 5$$

The inequality symbol $\le$ is pronounced "less than or equal to". Formally, the range of realistic values of the independent variable is called the **domain** of the function C. In this text, we rarely write the domain because it's usually clear from the story what realistic values would be. The exercises in this section ask you to do so for practice.

Be aware that there are often many different numbers in a story. Some numbers are examples of values the variables take on, such as \$3.999/gal or \$39.99 in our example. Other numbers are **constants**; they do not change (at least not during the story). The one constant in our story is that we are always buying 10 gallons of gas. Occasionally there are other numbers in a story that turn out not to be relevant at all, so be on the lookout.

Back to our story. A report says that the average price of gasoline in Minnesota was \$2.900/gal in 2010 and increased approximately 20% per year for the next several years. We would like to check what that says about the average price of gasoline in 2011 and 2012, say. (It is unlikely that the price increase continued much longer at that rate.)

To understand what that report is saying, we need to remember how percents work. Luckily, the word "percent" is very descriptive. The "cent" part means "hundred," like 100 cents in a dollar or 100 years in a century. And, as usual, "per" means "for each." Together, **percent** means "per hundred." The number 20% means 20 for each hundred. Written as a fraction it is $\frac{20}{100}$. Divide to get the decimal $20 \div 100 = .20$.

Think money: 20% is like 20¢, and .20 is like \$.20

Bottom line: 20%, $\frac{20}{100}$, and .20 mean exactly the same number.

$$20\% = \frac{20}{100} = 20 \div 100 = .20$$

To calculate the percent of a number we multiply by the decimal version. For example,

$$20\% \text{ of } \$2.900 = .20 \times 2.900 = \$.58$$

The report says the price increased by 20% each year, so by 2011 the price had increased an average of \$.58. That 58 cents is not what gas cost in 2011. It's how much *more* gas cost in 2011 compared to 2010. To see what the report projected for the 2011 cost we need to add that increase on to the original 2010 price.

$$\$2.099 + \$.58 = \$3.48 \text{ per gallon}$$

Sounds about right. Expensive, to be sure, but fairly accurate.

For 2012, the price increased by 20% again. That means 20% of what it was in 2011. We can't just add \$.58 again. That was 20% of the 2010 value, and we want 20% *of the 2011 value*. Going to have to calculate that.

$$20\% \text{ of } \$3.48 = .20 \times 3.48 = \$.696$$

so the projected 2012 value was

$$\$3.48 + \$.696 = \$4.176 \text{ per gallon}$$

One last note. The number 20% in the report sounds like a rough approximation. The report probably means the increase was around 20%, maybe a little less, maybe a little more. So our answers of \$3.48/gal and \$4.176/gal could be a little less or a little more too. But they sound so perfectly correct. To be safe, we really ought to round off these answers, to something more general like around \$3.50/gal in 2011 or approximately \$4.20/gal in 2012. Using our " approximately equal to" symbol we write $P \approx \$3.50$/gal in 2011 and $P \approx \$4.20$/gal in 2012.

Homework

Start by doing Practice exercises #1-4 in the workbook.

Do you know ...

- The difference between a variable and a constant?
- The information needed to "name" a variable?
- Which variable is dependent and which variable is independent?
- What "domain" means?
- How to calculate percent increase?

⋆ The symbol for "approximately equal to"?

⋆ Why an approximate answer is often as good as we can get?

⋆ When to round your answer up or down instead of off?

⋆ What the term "precisely" refers to?

⋆ How to decide how precisely to round your answer?

⋆ indicates question based on *Prelude: approximation*

If you're not sure, work the rest of exercises and then return to these questions. Or, ask your instructor or a classmate for help.

Exercises

5. It's about time! For each story, try to figure out the answer to the question(s).

   (a) The Nussbaums planted a walnut tree years ago when they first bought their house. The tree was 5 feet tall then and has grown around 2 feet a year. The tree is now 40 feet tall. How long ago did the Nussbaums plant their walnut tree?

   (b) After his first beer, Stephen's blood alcohol content (BAC) was already .04 and as he continued to drink, his BAC level rose 45% per hour. Note that
   $$45\% = \frac{45}{100} = 45 \div 100 = .45$$
   What was Stephen's BAC after 1 hour? After 2 hours?

   Story also appears in 2.4 Exercises and 3.4 #1

   (c) When McKenna drives 60 mph (miles per hour) it takes her 20 minutes on the highway to get between exits, but when traffic is bad it can take her an hour. How slow is McKenna driving when traffic is bad? *Hint: can you figure out the distance between exits?*

   (d) The sun set at 6:00 p.m. today and I heard on the radio that it sets about 2 minutes earlier each day this time of year. In how many days will the sun set at 4:30 p.m.? *Bonus question: in what month is the story set?*

   Stories also appear in 1.1 #4

6. The temperature was 40°F at noon yesterday downtown Minneapolis but it dropped 3°F an hour in the afternoon. *Story also appears in 1.2 and 4.1 Exercises*

   (a) Which number is a constant in this story: the temperature (40) or the rate at which the temperature dropped (3)?

   (b) Name the variables, including units and dependence.

   (c) When did the temperature drop below freezing (32°F)?

7. Mrs. Nystrom's Social Security benefit was $746.17/month when she retired from teaching in 2009. She had taught in elementary school since I was a girl. Benefits have increased by 4% per year. *Story also appears in 1.2 and 5.1 Exercises*

(a) Name the variables, including units and dependence.

(b) What was her benefit in 2012?

(c) When will her benefit pass $900/month? A reasonable guess is fine.

8. Between e-mail, automatic bill pay, and online banking, it seems like I hardly ever actually mail something. But for those times, I need postage stamps. The corner store sells as many (or few) stamps as I want for 44¢ each but they charge a 75¢ convenience fee for the whole purchase. *Story also appears in 3.1 Exercises*

   (a) Identify and name the variables, including the units.

   (b) Which variable is dependent and which is independent?

   (c) How many stamps could I buy for $10? Try to figure it out from the story.

9. Sofía bought her car new for $22,500. Now the car is fairly old and just passed 109,000 miles. Sofía looked online and estimates the car is still worth $5,700.

   Story also appears in Section 5.4

   (a) Identify and name the variables, including the units

   (b) Explain the dependence using a sentence of the form "___ is a function of ___"

   (c) What is a realistic number of miles for a car to drive? Express the domain as an inequality.

   (d) Sofía wonders when the car would be practically worthless, meaning under $500. Make a reasonable guess.

10. For each story, name the variables including units and dependence.

   (a) The closer you sit to a lamp, the brighter the light is.

   Story also appears in 2.3 and 3.3 Exercises.

   (b) The thicker the piece of fish, the longer it takes to grill it.

   Story also appears in 2.3 and 3.5 Exercises.

   (c) Wind turbines are used to generate electricity. The faster the wind, the more power they generate. *Story also appears in 1.3, 2.4, and 3.3 Exercises.*