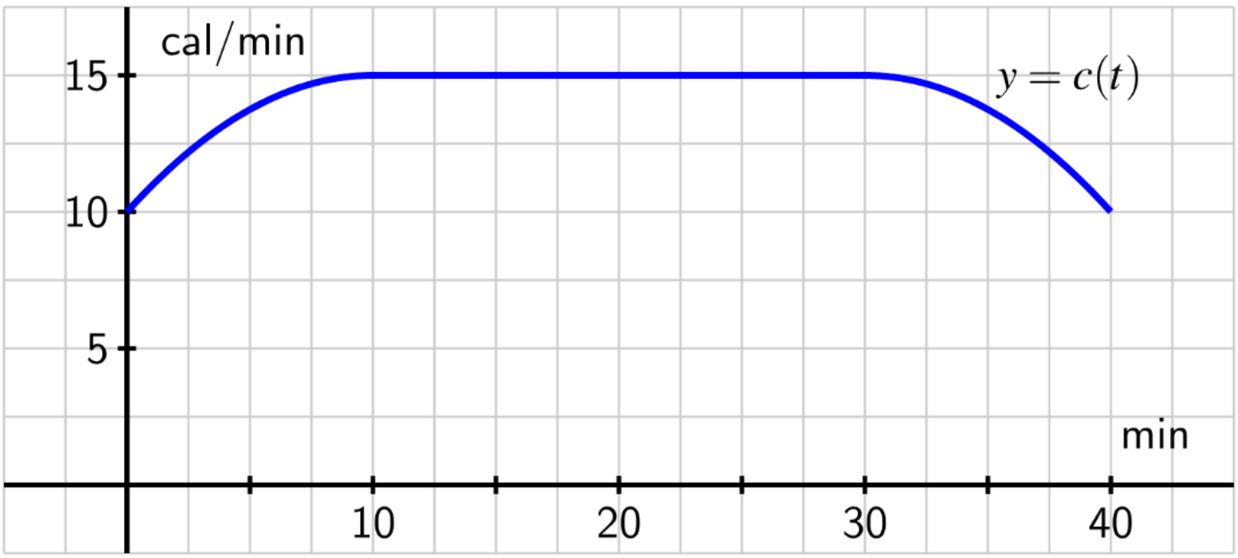
PS12: Definite integrals and the Fundamental Theorem of Calculus - Key

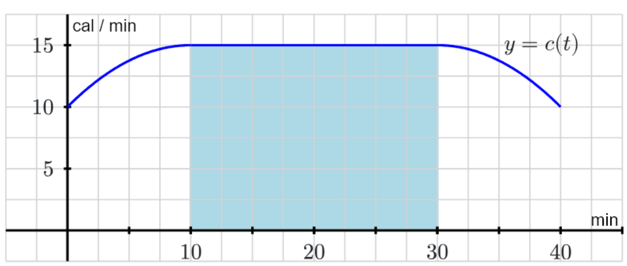
During a 40-minute workout, a person riding an exercise machine burns calories at a rate of $c(t)$ calories per minute, where the function $y = c(t)$ is given by the following information:

- On the interval $0 \le t \le 10$, the formula is $c(t) = -0.05t^2 + t + 10$ (warmup);
- on the interval $10 \le t \le 30$, the formula is $c(t) = 15$ (conditioning phase);
- on the interval $30 \le t \le 40$, the formula is $c(t) = -0.05t^2 + 3t - 30$ (cooldown).

Here's a graph of $c(t)$.



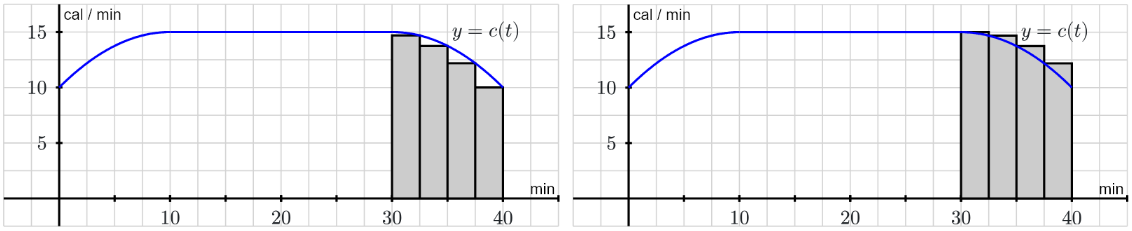
1. (IN1) Shade in the area under $c(t)$ between $t = 10$ and $t = 30$. Use some simple geometry to calculate this area. Give units.



2. (INx) Write a sentence explaining what the answer to part 1 *means* in the context of the person exercising.

   Between 10 minutes and 30 minutes, the person burned 300 total calories.

3. (IN2) Use a Riemann sum with 4 rectangles to approximate the area under $c(t)$ between $t = 30$ and $t = 40$:



Left sum overestimate comes out to 139.0625 calories.

Right sum underestimate comes out to 126.5625 calories.

Midpoint sum comes out to 133.59375 calories.

4. (IN3) Find a formula for an antiderivative $C(t)$ for the portion of $c(t)$ that's on the interval $0 \le t \le 10$.

$$C(t) = \frac{-0.05}{3}\,t^3 + \frac{1}{2}\,t^2 + 10\,t$$

5. (IN5) Use your antiderivative $C(t)$ to find the *exact* value of $\displaystyle\int_{t=0}^{t=10} c(t)\,dt$. Give units.

$$\begin{aligned}
\int_{t=0}^{t=10} c(t)\,dt &= \left[\frac{-0.05}{3}\,t^3 + \frac{1}{2}\,t^2 + 10t\right|_{t=0}^{10} \\
&= \left[\frac{-0.05}{3}\,(10)^3 + \frac{1}{2}\,(10)^2 + 10\,(10)\right] - \left[\frac{-0.05}{3}\,(0)^3 + \frac{1}{2}\,(0)^2 + 10\,(0)\right] \\
&= \left[\frac{400}{3}\right] - \left[0\right] \approx 133.3 \text{ calories}
\end{aligned}$$

6. (IN5) Now find the *exact* value of $\displaystyle\int_{t=30}^{t=40} c(t)\,dt$.

$$\begin{aligned}
\int_{t=30}^{t=40} c(t)\,dt &= \left[\frac{-0.05}{3}\,t^3 + \frac{3}{2}\,t^2 - 30\,t\right|_{t=30}^{40} \\
&= \left[\frac{-0.05}{3}\,(40)^3 + \frac{3}{2}\,(40)^2 - 30\,(40)\right] - \left[\frac{-0.05}{3}\,(30)^3 + \frac{3}{2}\,(30)^2 - 30\,(30)\right] \\
&= \left[\frac{400}{3}\right] - \left[0\right] \approx 133.3 \text{ calories}
\end{aligned}$$

7. Put it all together: Find the *exact* value of $\int_{t=0}^{t=40} c(t)\,dt$, give units, and explain what this number means in the context of the person exercising.

$$\begin{aligned}\int_{t=0}^{t=40} c(t)\,dt &= \int_{t=0}^{t=10} c(t)\,dt + \int_{t=10}^{t=30} c(t)\,dt + \int_{t=30}^{t=40} c(t)\,dt \\ &= \frac{400}{3} + 300 + \frac{400}{3} \\ &= \frac{1700}{3} \approx 566.67 \text{ calories}\end{aligned}$$