Subgroups!

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With many thanks to Matthew Macauley,
`http://www.math.clemson.edu/~macaule/`

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Definition time!

Here is the definition of a subgroup.

Definition

A subgroup of G is a subset $H \subseteq G$ that is also a group. We denote this by $H \leq G$.

Okay, but remind me what's the definition of a group?

Definition

A group $(G, \star)$ is a set of elements together with a binary operation $\star$ satisfying the following properties:

1. The operation is associative.
2. G contains the identity element.
3. Every element in G has an inverse element.
4. G is closed under the binary operation.

Trivial subgroups

Every group G has the following two boring subgroups: $G \leq G$, and $\{e\} \leq G$.

Definition

A proper subgroup $H < G$ is a subgroup that's not equal to the whole group.

Generating sets

We've previously looked at the orbit of an element:

Definition

The orbit of an element $g \in G$ is the cyclic subgroup that it generates,

$$\langle g \rangle = \{g^k \mid k \in \mathbb{Z}\},$$

and its order is $|g| := \big|\langle g \rangle\big|$.
In particular, if $|g| = n$ is finite, this is the set $\{g^0 = 1, g, g^2, \dots, g^{n-1}\}$.

This is a subgroup:

Cyclic subgroups are subgroups

For any element $g \in G$, $\langle g \rangle \leq G$.

But we need not restrain ourselves to generating by one element:

Definition

Let S be a subset of G. A word in S is a finite product of finite powers of elements of S or their inverses.

$\langle S \rangle = \{\text{words in } S\}$ is a subgroup of G, and it's called the subgroup generated by S.

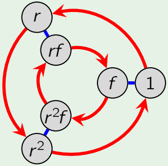
And in fact every subgroup looks like this.

Example: $C_2 \leq D_3$

Writing $C_2 \leq D_3$ means *there is a copy of C_2 sitting inside of D_3 as a subgroup*.

Question

How many ways can you find C_2 sitting inside of D_3?



Remark

It's more precise to express a subgroup by its generator(s).

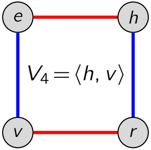
$$C_2 \cong \langle f \rangle < D_3 \qquad C_2 \cong \langle rf \rangle < D_3 \qquad C_2 \cong \langle r^2 f \rangle < D_3$$

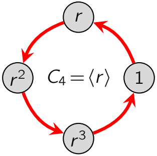
Question

How about $C_3 \leq D_3$? There's only one!

The two groups of order 4

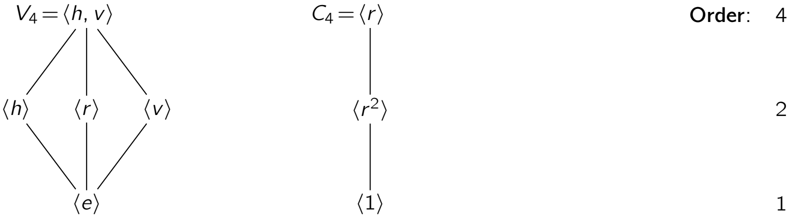
Let's start by considering the subgroups of the two groups of order 4.





- Proper subgroups of V_4: $\langle h \rangle = \{e, h\}$, $\langle v \rangle = \{e, v\}$, $\langle r \rangle = \{e, r\}$, $\langle e \rangle = \{e\}$.
- Subgroups of C_4: $\langle r \rangle = \{1, r, r^2, r^3\} = \langle r^3 \rangle$, $\langle r^2 \rangle = \{1, r^2\}$, $\langle 1 \rangle = \{1\}$.

It is illustrative to arrange these in a subgroup lattice:



$V_4 = \langle h, v \rangle$

$\langle h \rangle$ $\langle r \rangle$ $\langle v \rangle$

$\langle e \rangle$

$C_4 = \langle r \rangle$

$\langle r^2 \rangle$

$\langle 1 \rangle$

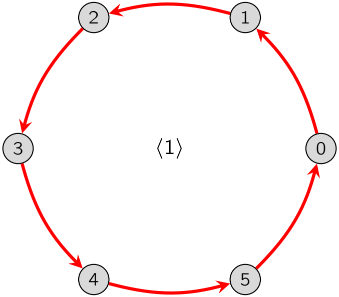
Order: 4

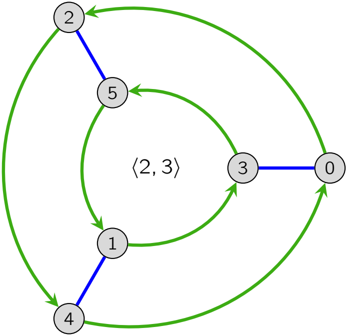
2

1

Subgroups of $\mathbb{Z}_6$

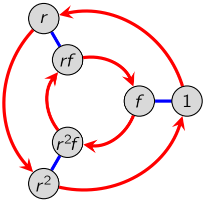
What subgroups can you find in $\mathbb{Z}_6$? I've drawn the Cayley diagram two different ways.

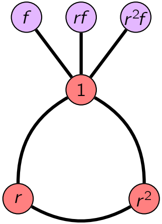




Subgroups of D_3

Let's figure out all the subgroups of D_3.





Here are the non-trivial proper subgroups of D_3:

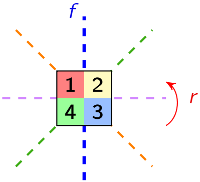
$$\langle r\rangle = \{1, r, r^2\} = \langle r^2\rangle, \quad \langle f\rangle = \{1, f\}, \quad \langle rf\rangle = \{1, rf\}, \quad \langle r^2f\rangle = \{1, r^2f\}, \quad \langle 1\rangle = \{1\}.$$

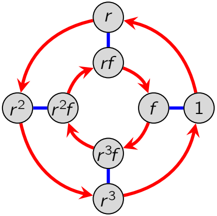
Observations:

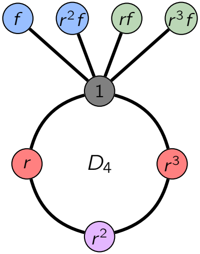
- The cycle graph helps us spot cyclic subgroups.
- For small groups like D_3, the cyclic subgroups may be the only proper subgroups.
- There might, however, be more complicated things that are harder to clock.

Subgroups of D_4

See if you can figure out all the subgroups of D_4.







What do you think is a reasonable way to, like, arrange them?

Lattices

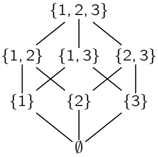
A lattice is a partially ordered set such that every pair of elements x, y has a unique:

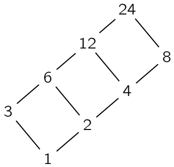
- join, or sup, or least upper bound $x \vee y$
- meet, or inf, or greatest lower bound $x \wedge y$.

Examples you may have seen previously are subset lattices and divisor lattices.

$x \vee y = x \cup y$

$x \wedge y = x \cap y$





$x \vee y = \mathrm{lcm}(x, y)$

$x \wedge y = \gcd(x, y)$

This seems like a good way to organize subgroups, because:

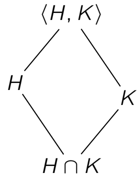
Theorem

If $H \leq G$ and $K \leq G$ are two subgroups, then $H \cap K$ is a subgroup.
(Indeed, it's the largest subgroup that's contained in both H and K.)

Theorem

$\langle H, K \rangle$ is the smallest subgroup containing both H and K.
(Note that $H \cup K$ is not in general a subgroup. Why not?)

Subgroup lattices



$\langle H, K\rangle$

H

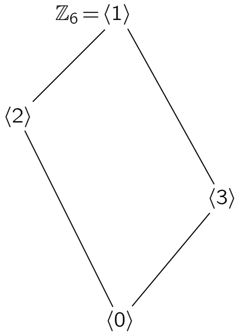
K

$H \cap K$

$H \vee K$: "*smallest subgroup above both H and K*"

$H \wedge K$: "*largest subgroup below both H and K*"

Examples:

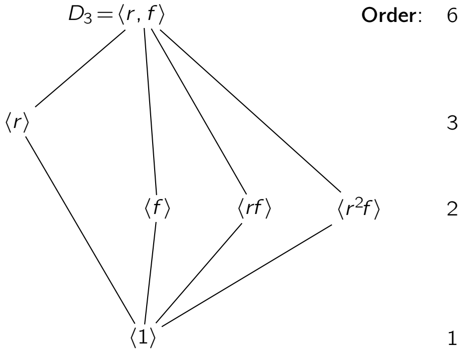


$\mathbb{Z}_6 = \langle 1\rangle$

$\langle 2\rangle$

$\langle 3\rangle$

$\langle 0\rangle$



$D_3 = \langle r, f\rangle$

Order: 6

$\langle r\rangle$

3

$\langle f\rangle$ $\langle rf\rangle$ $\langle r^2 f\rangle$

2

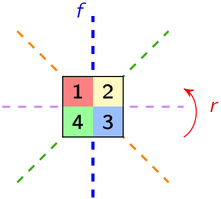
$\langle 1\rangle$

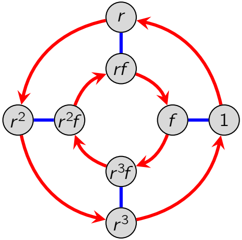
1

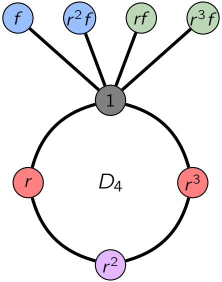
The subgroup lattice of D_4

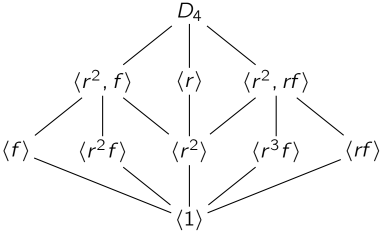
The subgroups of D_4 are:

- The entire group D_4, and the trivial group $\langle 1 \rangle$
- 4 subgroups generated by reflections: $\langle f \rangle$, $\langle rf \rangle$, $\langle r^2 f \rangle$, $\langle r^3 f \rangle$.
- 1 subgroup generated by a 180° rotation, $\langle r^2 \rangle \cong C_2$
- 1 subgroup generated by a 90° rotation, $\langle r \rangle \cong C_4$
- 2 subgroups isomorphic to V_4: $\langle r^2, f \rangle$, $\langle r^2, rf \rangle$.

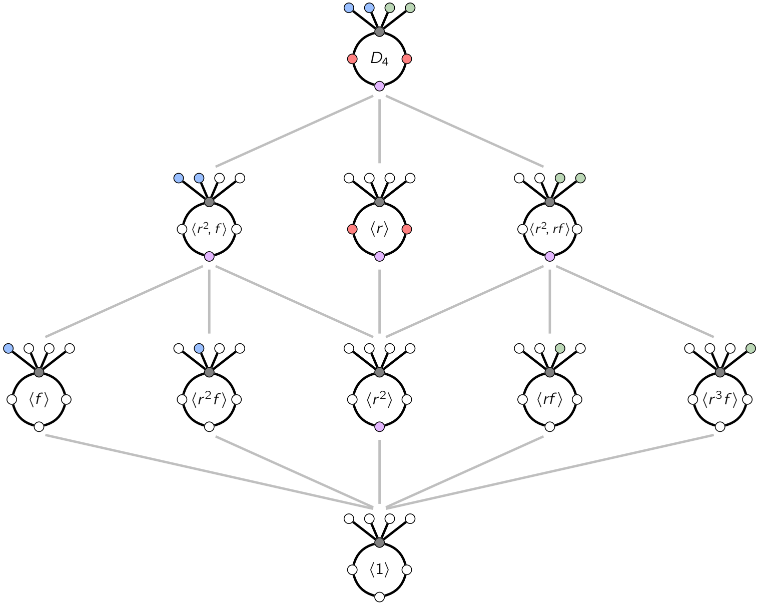




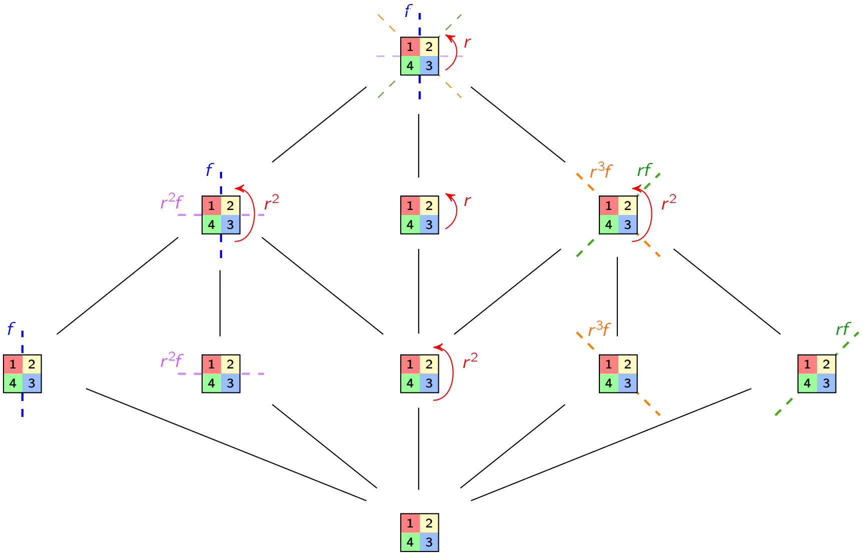




The subgroup lattice of D_4



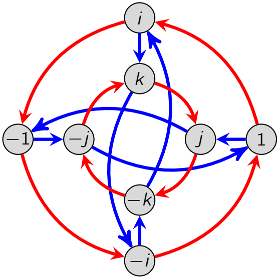
The subgroup lattice of D_4

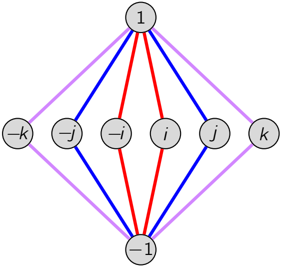


The subgroup lattice of Q_8

Let's determine all subgroups of the quaternion group

$$Q_8 = \left\langle i, j, k \mid i^2 = j^2 = k^2 = ijk = -1 \right\rangle.$$



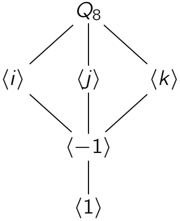


Every element generates a cyclic subgroup:

$\langle 1 \rangle = \{1\}$, $\langle -1 \rangle = \{\pm 1\}$, $\langle i \rangle = \langle -i \rangle = \{\pm 1, \pm i\}$,

$\langle j \rangle = \langle -j \rangle = \{\pm 1, \pm j\}$, $\langle k \rangle = \langle k \rangle = \{\pm 1, \pm k\}$.

Are there any other proper subgroups?



Subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

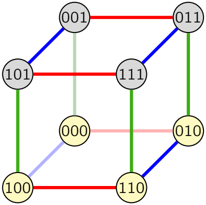
We've seen the subgroup lattices of two groups of order 8:

- D_4 has five elements of order 2, and 10 subgroups.
- Q_8 has one element of order 2, and 6 subgroups.
- $\mathbb{Z}_2^3$ has seven *elements* of order 2.

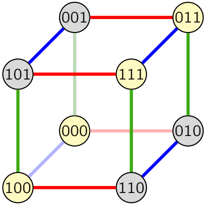
Rule of thumb

Groups with elements of small order tend to have more subgroups than those with elements of large order.

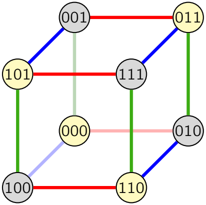
The following Cayley graphs show three different subgroups of order 4 in $\mathbb{Z}_2^3$.



$\langle 100, 010 \rangle$



$\langle 100, 011 \rangle$



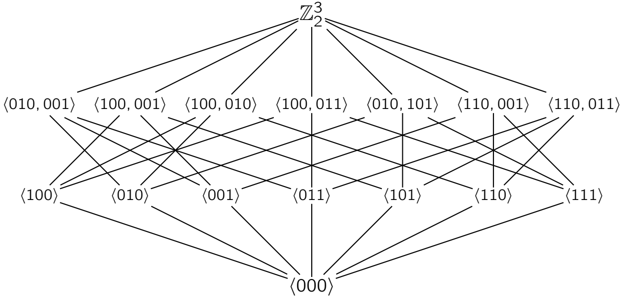
$\langle 110, 011 \rangle$

The subgroup lattice of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

All $\binom{7}{2} = 21$ pairs of non-identity element elements generate a subgroup isomorphic to V_4.

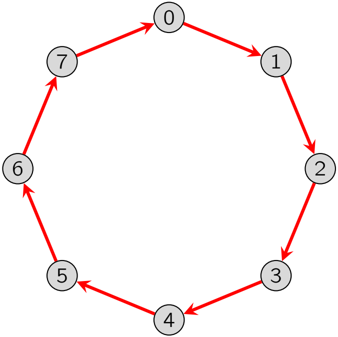
But this triple-counts all such subgroups. In summary, the subgroups of $\mathbb{Z}_2^3$ are:

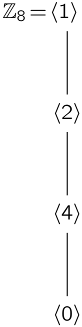
- The subgroups G and $\{000\}$,
- 7 subgroups isomorphic to C_2,
- 7 subgroups isomorphic to V_4.



The subgroup lattice of $\mathbb{Z}_8$

Draw the Cayley diagram of Z_8 and find all its subgroups.
Arrange them in a lattice.





$\mathbb{Z}_8 = \langle 1 \rangle$

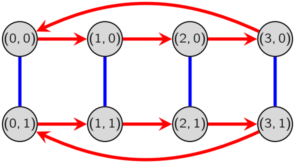
$\langle 2 \rangle$

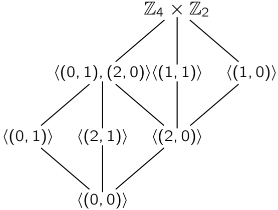
$\langle 4 \rangle$

$\langle 0 \rangle$

Groups of order 8

There is one more group of order 8, which is $\mathbb{Z}_4 \times \mathbb{Z}_2$.





$\mathbb{Z}_4 \times \mathbb{Z}_2$

$\langle (0,1),(2,0) \rangle$ $\langle (1,1) \rangle$ $\langle (1,0) \rangle$

$\langle (0,1) \rangle$ $\langle (2,1) \rangle$ $\langle (2,0) \rangle$

$\langle (0,0) \rangle$

Let's summarize the sizes of the subgroups of the groups of order 8 that we have seen.

	C_8	Q_8	$C_4 \times C_2$	D_4	C_2^3
# elts. of order 8	4	0	0	0	0
# elts. of order 4	2	6	4	2	0
# elts. of order 2	1	1	3	5	7
# elts. of order 1	1	1	1	1	1
# subgroups	4	6	8	10	16

Observations?

- Groups that have more elements of small order tend to have more subgroups.
- In all of these cases, the order of each subgroup divides $|G|$.