Homework #3 – Key

HW due Sunday 2/9 by pdf upload to Canvas; .tex source on the MATH 312 github repo.

Stuff about permutation groups and S_n

Problem 1. (Omitted from key bc any reasonable explanation is correct.)

Problem 2. Suppose that $g, h \in G$. We mentioned in class that the *conjugate* of h by g is the element ghg^{-1}. The *conjugacy class* of h is the set of all the possible conjugates of h: $\{ghg^{-1} \mid g \in G\}$.

- Find the conjugacy classes of all six elements of S_3. (If you use the permutation calculator, make sure you're multiplying left-to-right!)
  - Identity: $\{e\}$
  - Transpositions: $\{(1\ 2), (1\ 3), (2\ 3)\}$
  - 3-cycles: $\{(1\ 2\ 3), (1\ 3\ 2)\}$
- If you are having fun, find the conjugacy classes of all 24 elements of S_4.
  - Identity: $\{e\}$
  - Transpositions: $\{(1\ 2), (1\ 3), (1\ 4), (2\ 3), (2\ 4), (3\ 4)\}$
  - Double transpositions: $\{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$
  - 3-cycles: $\{(1\ 2\ 3), (1\ 3\ 2), (1\ 2\ 4), (1\ 4\ 2), (1\ 3\ 4), (1\ 4\ 3), (2\ 3\ 4), (2\ 4\ 3)\}$
  - 4-cycles: $\{(1\ 2\ 3\ 4), (1\ 2\ 4\ 3), (1\ 3\ 2\ 4), (1\ 3\ 4\ 2), (1\ 4\ 2\ 3), (1\ 4\ 3\ 2)\}$

Problem 3. Suppose $\sigma \in S_n$, and that $|\sigma| = k$.

- Explain why σ^k is an even permutation.

  Since $|\sigma| = k$, $\sigma^k = e$. The identity permutation e is an even permutation, as it takes 0 transpositions to write. Therefore, σ^k is an even permutation.
- Suppose that σ is an odd permutation. Is k even or odd? How do you know?

  Let t be the number of transpositions it takes to write σ. Since σ is an odd permutation, we know that t is odd.

  Now consider $\sigma^k = \underbrace{\sigma \cdot \sigma \cdot \ldots \cdot \sigma}_{k \text{ times}}$. How many total transpositions is that? Since each copy of σ represents t transpositions, σ^k represents $t \cdot k$ total transpositions.

  But $\sigma^k = e$ is an even transposition, so $t \cdot k$ must be even. Therefore, since t is odd, k must be even.

- Conclude that a cycle of odd length is an even permutation. Feel moderate annoyance.

  Let's remind ourselves what a "cycle of odd length" looks like. Here is a cycle of length 3: $(1\ 3\ 2)$. What is the order of this beast? Certainly it has order 3. Indeed, a general fact we learned earlier is that a cycle of length k must have order k.

  So say σ is a cycle of odd length. Therefore its order must also be odd – in the language we've been using, where $|\sigma| = k$, we have that k is odd.

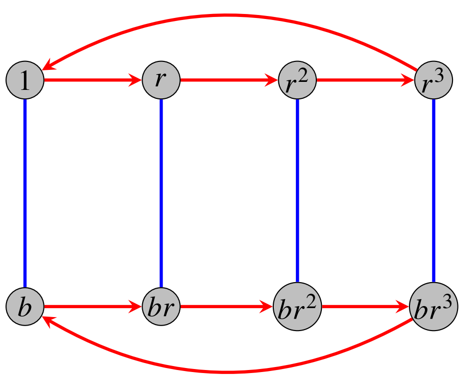
  By the previous bullet, if σ is an odd permutation, then its order k must be even. How shall we reach the conclusion of this bullet? Ah ha, with the contrapositive: if k is not odd, then σ is not an odd permutation. Or, reworded slightly: if k is even, then σ is an even permutation.

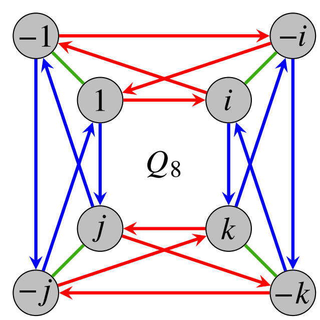
  We have reached our conclusion: a cycle of odd length (and therefore odd order) must be an even permutation.

Problem 4. Play a bit more with Cayley's theorem:

- Extract from the Cayley diagram of $C_4 \times C_2 = \langle r, b \rangle$ the two permutations that describe its arrows, and therefore describe $C_4 \times C_2$ as a subgroup of a symmetric group.

- Do the same for $Q_8 = \langle i, j, -1 \rangle$.





Stuff about direct products

Problem 5. I want to make the relationship between the inflate-the-Cayley-diagram description and the ordered-pairs description of a direct product a bit more evident. Say that $C_4 = \langle r \mid r^4 = 1 \rangle$ and $C_3 = \langle b \mid b^3 = 1 \rangle$.

- Write out all 12 elements of $C_4 \times C_3$ as ordered pairs. (They will all look like (r^k, b^j).)

- Use the inflation procedure to draw the Cayley diagram of $C_4 \times C_3$.

- Label each node in your Cayley diagram with the corresponding ordered pair.

- (Bonus problem: Is $C_4 \times C_3$ "secretly cyclic"?)

Problem 6. Explore $C_2 \times C_2 \times C_2$.

- Figure out how to repeat the inflation process to draw a Cayley diagram.

- Compute the orbits of each of the 8 elements and draw a cycle graph.
- Is this a new group? Groups of order 8 we already know are C_8, $C_4 \times C_2$, D_4, and Q_8.

Problem 7. Prove using algebra that $A \times B$ is abelian if and only if both A and B are abelian. *Hint:* Remember that an "if and only if" statement is actually looking for *two* proofs. I will provide you with the proof frames for each one:

($\Rightarrow$) Suppose that A and B are both abelian.
...
Therefore, $A \times B$ is abelian.

($\Leftarrow$) Suppose that $A \times B$ is abelian.
...
Therefore, A and B are both abelian.

Remember that filling in the second line and the second-to-last line, usually by unpacking a definition (here: what does "abelian" mean again?), is a good way to proceed after writing the proof frame.

Problem 8. Here we'll explore when the product of cyclic groups is "secretly cyclic".

(a) Say you have a generic group G such that $|G| = n$. Suppose further that you found an element $g \in G$ such that $|g| = n$. Prove that G is cyclic. (Hint: look at the orbit of g, and think about the definition of $|g|$.)

(b) Suppose that $a, b \in \mathbb{Z}$ are relatively prime. (Google it if you don't remember what this means.) Fact: $\operatorname{lcm}(a, b) = ab$. Optional challenge: prove it.

(c) Consider the direct product $\mathbb{Z}_a \times \mathbb{Z}_b$, with a and b relatively prime. What is $|(1, 1)|$?

(**NOTE TYPO FIX:** I had previously written $C_a \times C_b$, in which the element $(1, 1)$ would represent the identity. I could also have said, suppose that $C_a = \langle g \rangle$ and $C_b = \langle h \rangle$ and then think about the element $(g, h) \in C_a \times C_b$, but I think it's cleaner this way.)

(d) Conclude that $\mathbb{Z}_a \times \mathbb{Z}_b \cong \mathbb{Z}_{ab}$.