Homework #8 (due Mar 16 or 23, whichever you prefer)

Definition. Let G and H be two groups. A homomorphism $\phi : G \to H$ is a map from G to H that sends products to products:

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2).$$

(Note: An output of ϕ is an element of H. So, $\phi(g_1)$ and $\phi(g_2)$ and $\phi(g_1 g_2)$ are all elements of H.)

Properties of homomorphisms

The following are straightforward to verify, but writing out the symbols is good practice for "sending products to products." In each part, suppose that $\phi : G \to H$ is a homomorphism.

Problem 1. Figure out how to write each of these in homomorphism language, and prove them.

(a) ϕ sends the identity to the identity. ($\phi(1_G) = 1_H$)

(b) ϕ sends inverses to inverses.

(c) ϕ sends powers to powers.

(d) ϕ sends orbits to orbits.

(e) ϕ sends conjugates to conjugates.

Problem 2. Prove that if $|g|$ is finite, $|\phi(g)|$ divides $|g|$. Hint: Use 1(c) and (a).

Problem 3. Prove that $\mathrm{Im}(\phi) \leq H$ ("the image of ϕ is a subgroup of H").

Examples and non-examples

Problem 4. Show that there is no embedding $\phi\colon \mathbb{Z}_n \hookrightarrow \mathbb{Z}$, for $n \geq 2$.
Hint: ϕ is determined by what it does to generators. $\mathbb{Z}_n = \langle 1 \rangle$; what can possibly be $\phi(1)$?

Problem 5. Decide whether or not each of these is a homomorphism.

(a) The "projection map" $\pi_A : A \times B \to A$ defined by $\pi_A : (a, b) \mapsto a$. (Similar for π_B, btw.)

(b) The ol' switcheroo: define $\gamma : A \times B \to B \times A$ by $\gamma : (a, b) \mapsto (b, a)$.

(c) Conjugation by a fixed element: choose $x \in G$ and define $\phi_x : G \to G$ by $\phi_x(g) = xgx^{-1}$.

(d) The tripling map: $\theta : \mathbb{Z}_6 \to \mathbb{Z}_6$ defined by $\theta(m) = 3m$. (Careful: the operation is +.)

(e) More generally, $\theta : \mathbb{Z}_n \to \mathbb{Z}_n$ defined by $\theta(m) = km$ for some fixed integer k.

(f) Define the "squaring map" $s : D_4 \to D_4$ by $s(x) = x^2$.

(g) For an abelian group G, define the "squaring map" $s : G \to G$ by $s(x) = x^2$.

(h) Under what circumstances is $\phi : \mathbb{Z}_n \to \mathbb{Z}_m$ defined by $\phi(1) = 1$ a homomorphism?
(Hint: Does this work for $\mathbb{Z}_3 \to \mathbb{Z}_4$? How about for $\mathbb{Z}_3 \to \mathbb{Z}_6$?)

Problem 6. For each of (a), (b), (c), (d), and (e) in Problem 5, decide whether the map is an embedding (injective), a quotient map (surjective), an isomorphism (bijective).
For number (e), your answer will be "it depends:" under what circumstances is this map injective? surjective? bijective?

Problem 7. Prove that each of these are automorphisms (isomorphisms from a group to itself).

(a) The "identity map" $\iota : G \to G$ defined by $\iota(g) = g$.

(b) It has probably been a minute since you thought about the complex numbers $\mathbb{C}$, so here's a bit of review. A complex number $z \in \mathbb{C}$ looks like $z = a + bi$, where $i = \sqrt{-1}$, and we like to think of this number living at the coordinate (a, b) on the complex plane; the real axis is horizontal and the imaginary axis is vertical. You can also use polar form $z = re^{i\theta}$ to specify this location, because $e^{i\theta} = \cos\theta + i \sin\theta$.

The "complex conjugate" $\overline{z}$ is the reflection of z across the real axis: so, define $\overline{} : \mathbb{C} \to \mathbb{C}$ by $\overline{a + bi} = a - bi$. Or, in polar form $re^{i\theta}$, then $\overline{re^{i\theta}} = re^{i(-\theta)}$.

(c) (You have already proved that conjugation by a fixed element $\phi_x(g) = xgx^{-1}$ is an automorphism.)

(d) Under what circumstances is the "inversion map" $\phi(g) = g^{-1}$ an automorphism?

Problem 8. Automorphisms of cyclic groups:

(a) Find every possible automorphism of $\mathbb{Z}_5$. (There are four; where can the generator go?)

(b) Find every possible automorphism of $\mathbb{Z}_6$. (There are two; why aren't there five?)

(c) (Bonus!) Conjecture how many automorphisms there are of $\mathbb{Z}_n$ for a general n.

Kernels and preimages

Problem 9. As promised in class:

(a) Prove that $\operatorname{Ker}\phi$ is a subgroup of G.

(b) Prove that $\operatorname{Ker}\phi$ is normal in G. (Hint: for $k \in \operatorname{Ker}\phi$, calculate $\phi(xkx^{-1})$.)

(c) Prove that $\operatorname{Ker}\phi$ is trivial iff ϕ is injective.

Problem 10. Prove that for any $h \in H$, $\phi^{-1}(h)$ is a coset of $\operatorname{Ker}\phi$.

Problem 11. Find the kernel of each of these homomorphisms (hint: none of them are trivial):

(a) $\pi_A : A \times B \to A$ given by $\pi_A(a, b) = a$ (and how about π_B?)

(b) $\phi : \mathbb{Z} \to \mathbb{Z}_5$ where $\phi(n)$ is the remainder of n mod 5. (For instance, $\phi(17) = 2$.)

(c) The very rude "squishing map" $\phi : G \to H$ defined by $\phi(g) = 1_H$.

(d) $\phi : (\mathbb{R}^2, +) \to (\mathbb{R}, +)$ defined by $\phi(x, y) = x + y$. (The domain is the xy-plane, basically; what does the kernel look like in the xy-plane?)

Lastly, a fun problem that previews something very cool

In class we saw these two homomorphisms:

$$\alpha : Q_8 \to V_4 = \langle a, b \rangle \text{ defined by } \alpha(i) = a, \quad \alpha(j) = b$$

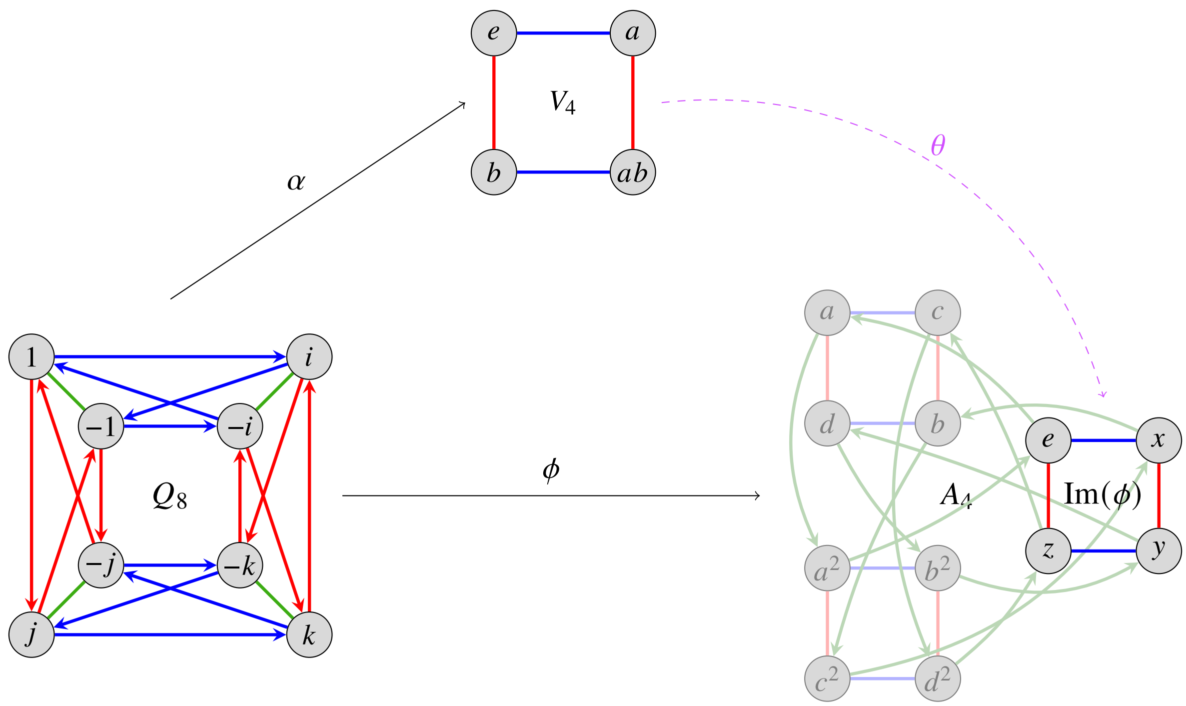
$$\phi : Q_8 \to A_4 \text{ defined by } \phi(i) = (12)(34), \quad \phi(j) = (13)(24)$$

They are related in an important way.

Problem 12. Compute α for all the rest of the elements of Q_8.

Problem 13. Compute ϕ for all the rest of the elements of Q_8. (This was the warmup on Wednesday.)

Problem 14. Here are the Cayley diagram pictures of these homomorphisms. Note that Q_8 is the domain of both, so both the α arrow to V_4 and the ϕ arrow to A_4 are leaving the same place Q_8.



It really seems like there should just be a homomorphism θ from V_4 to $\mathrm{Im}(\phi)$, as I've indicated with a dashed purple arrow.

- What properties would you like this map to have? Injective, surjective, both?
- Create an explicit homomorphism $\theta : V_4 \to A_4$.
  (That is, tell me what element of A_4 you want $\theta(a)$ and $\theta(b)$ to be, kinda like ϕ above.)
- Show that your map has the properties you want.