Applications of group actions!

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With many thanks to Matthew Macauley,
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Overview

Intuitively, a group action occurs when a group G "naturally permutes" a set S *of states*.

Formal definition

A group G acts on a set S if there is a homomorphism $\phi\colon G \longrightarrow \mathrm{Perm}(S)$.
We'll use right group actions,
and we'll write $s.\phi(g)$ to denote "where pushing the g-button sends state s."

Definition

A set S with a (right) action by G is called a (right) G-set.

Big ideas

- An action $\phi\colon G \longrightarrow \mathrm{Perm}(S)$ endows S with an **algebraic structure**.
- *Action graphs are to G-sets, like how Cayley graphs are to groups.*

Notation

Throughout, we'll denote identity elements by $1 \in G$ and $e \in \mathrm{Perm}(S)$.

Five features of every group action

Every group action has **five fundamental features** that we will always try to understand.

	local (about an s or a g)	global (about the whole action ϕ)
subsets of S	$\mathrm{orb}(s)$ $\mathrm{fix}(g)$	$\mathrm{Fix}(\phi) = \bigcap_{g \in G} \mathrm{fix}(g)$
subgroups of G	$\mathrm{stab}(s)$	$\mathrm{Ker}(\phi) = \bigcap_{s \in S} \mathrm{stab}(s)$

"Duality:" columns vs. rows in the fixed-point table:

- the stablizers can be read off the columns: *group elements that fix $s \in S$*
- the kernel is the rows with a check in every column
- the fixators can be read off the rows: *set elements fixed by $g \in G$*
- the fixed points are the columns with a check in every row

Fixed-point tables

Here is the fixed-point table for $G = D_4$ acting on S the list of 7 "binary squares."

	$\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$	$\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$	$\begin{smallmatrix}0&0\\1&1\end{smallmatrix}$	$\begin{smallmatrix}0&1\\0&1\end{smallmatrix}$	$\begin{smallmatrix}1&1\\0&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\1&0\end{smallmatrix}$
1	✓	✓	✓	✓	✓	✓	✓
r	✓						
r^2	✓	✓	✓				
r^3	✓						
f	✓			✓		✓	
rf	✓	✓	✓				
r^2f	✓				✓		✓
r^3f	✓	✓	✓				

$\mathrm{Ker}(\phi) = \{1\}$ and $\mathrm{Fix}(\phi) = \{$ the 0 0 0 0 one$\}$.

Two big theorems

Orbit-stabilizer theorem

For any group action $\phi\colon G \to \mathrm{Perm}(S)$, and any $s \in S$,

$$|\mathrm{orb}(s)| \cdot |\mathrm{stab}(s)| = |G|.$$

Equivalently, *the size of the orbit containing s is* $|\mathrm{orb}(s)| = [G : \mathrm{stab}(s)]$.

Proof: Put elements $s.\phi(g)$ of $\mathrm{orb}(s)$ in correspondence with cosets of the stabilizer.

Orbit-counting theorem

Let a finite group G act on a set S via $\phi\colon G \to \mathrm{Perm}(S)$.
Then the number of orbits is the average size of the fixators:

$$|\mathrm{Orb}(\phi)| = \frac{1}{|G|} \sum_{g \in G} |\mathrm{fix}(g)|.$$

Equivalently, the number of orbits is the average size of the stabilizers:

$$|\mathrm{Orb}(\phi)| = \frac{1}{|G|} \sum_{s \in S} |\mathrm{stab}(s)|.$$

Proof: Count checkmarks in the fixed point table.

Groups acting on themselves!

Groups acting on “themselves”

It is frequently of interest to analyze the action of a group G on its elements, subgroups, or cosets of some fixed $H \leq G$.

Often, the orbits, stabilizers, and fixed points of these actions are familiar algebraic objects.

A number of deep theorems have a slick proof via a clever group action.

Here are common examples of group actions:

- G acts on itself (i.e., its set of elements) by multiplication.
- G acts on itself by conjugation.
- G acts on its subgroups by conjugation.
- G acts on the cosets of a fixed subgroup $H \leq G$ by multiplication.

(Please put the word “right” in a salt shaker and shake it all over those bullet points.)

Groups acting on subgroups by conjugation

Any group G acts on its set S of subgroups, $S = \{H \mid H \leq G\}$ by **right-conjugation**:

$$\phi\colon G \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } H \text{ to } g^{-1}Hg.$$

This is a **right action**, but there is an associated left action: $H \mapsto gHg^{-1}$.

Let $H \leq G$ be an element of S.

- The orbit of H consists of all conjugate subgroups:

$$\mathrm{orb}(H) = \left\{g^{-1}Hg \mid g \in G\right\} = \mathrm{cl}_G(H).$$

- The stabilizer of H is the normalizer of H in G:

$$\mathrm{stab}(H) = \left\{g \in G \mid g^{-1}Hg = H\right\} = N_G(H).$$

- The fixator of g are the subgroups that g normalizes:

$$\mathrm{fix}(g) = \left\{H \mid g^{-1}Hg = H\right\} = \left\{H \mid g \in N_G(H)\right\},$$

- The fixed points of ϕ are precisely the normal subgroups of G:

$$\mathrm{Fix}(\phi) = \left\{H \leq G \mid g^{-1}Hg = H \text{ for all } g \in G\right\}.$$

- The kernel of this action is the set of elements that normalize every subgroup:

$$\mathrm{Ker}(\phi) = \left\{g \in G \mid g^{-1}Hg = H \text{ for all } H \leq G\right\} = \bigcap_{H \leq G} N_G(H).$$

Groups acting on subgroups by conjugation

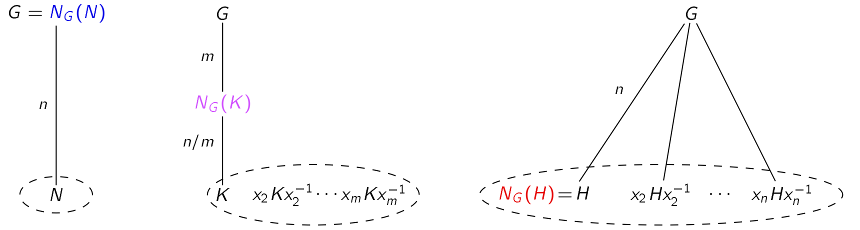
Let's apply our two theorems:

1. **Orbit-stabilizer theorem**. "*the size of an orbit is the index of the stabilizer*":

$$\left|\mathrm{cl}_G(H)\right| = [G : N_G(H)] = \frac{|G|}{|N_G(H)|}.$$

2. **Orbit-counting theorem**. "*the number of orbits is the average number of elements fixed by a group element*":

$$\#\text{conjugacy classes of subgroups of } G = \mathbb{E}\big[\#\text{ subgroups } g \text{ normalizes}\big].$$



normal — $\left|\mathrm{cl}_G(N)\right| = 1$

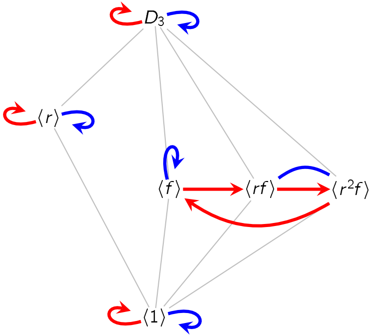
moderately unnormal — $1 < \left|\mathrm{cl}_G(K)\right| < [G : K]$

fully unnormal — $\left|\mathrm{cl}_G(H)\right| = [G : H]$; as large as possible

Groups acting on subgroups by conjugation

Here is an example of $G = D_3$ acting on its subgroups by a homomorphism $\tau : D_3 \longrightarrow \mathrm{Perm}(S) \cong S_6$.

$$\begin{array}{lccccccc}
\tau(1) & = & \langle 1\rangle & \langle r\rangle & \langle f\rangle & \langle rf\rangle & \langle r^2f\rangle & D_3 \\
\tau(r) & = & \langle 1\rangle & \langle r\rangle & \langle f\rangle & \langle rf\rangle & \langle r^2f\rangle & D_3 \\
\tau(r^2) & = & \langle 1\rangle & \langle r\rangle & \langle f\rangle & \langle rf\rangle & \langle r^2f\rangle & D_3 \\
\tau(f) & = & \langle 1\rangle & \langle r\rangle & \langle f\rangle & \langle rf\rangle & \langle r^2f\rangle & D_3 \\
\tau(rf) & = & \langle 1\rangle & \langle r\rangle & \langle f\rangle & \langle rf\rangle & \langle r^2f\rangle & D_3 \\
\tau(r^2f) & = & \langle 1\rangle & \langle r\rangle & \langle f\rangle & \langle rf\rangle & \langle r^2f\rangle & D_3
\end{array}$$



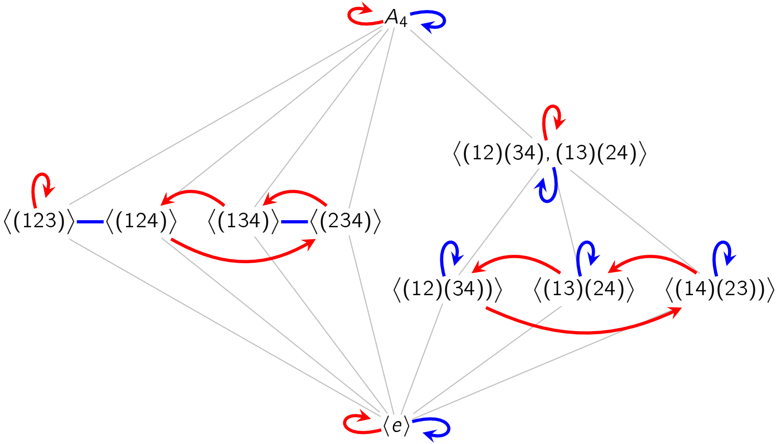
Observations

Do you see how to read stabilizers and fixed points off of the permutation diagram?

- $\mathrm{Ker}(\phi) = \langle 1\rangle$ consists of the row(s) with only fixed points.
- $\mathrm{Fix}(\phi) = \{\langle 1\rangle, \langle r\rangle, D_3\}$ consists of the column(s) with only fixed points.
- By the orbit-counting theorem, there are $|\mathrm{Orb}(\phi)| = 24/|D_3| = 4$ conjugacy classes.

Groups acting on subgroups by conjugation

Here is an example of $G = A_4 = \langle (123), (12)(34) \rangle$ acting on its subgroups.



Let's take a moment to revisit our "*three favorite examples*" from Chapter 3.

$$N = \langle (12)(34), (13)(24) \rangle, \qquad H = \langle (123) \rangle, \qquad K = \langle (12)(34) \rangle.$$

Groups acting on subgroups by conjugation

Here is the "*fixed point table*" of the action of A_4 on its subgroups.

	$\langle e \rangle$	$\langle (123) \rangle$	$\langle (124) \rangle$	$\langle (134) \rangle$	$\langle (234) \rangle$	$\langle (12)(34) \rangle$	$\langle (13)(24) \rangle$	$\langle (14)(23) \rangle$	$\langle (12)(34), (13)(24) \rangle$	A_4
e	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
(123)	✓	✓							✓	✓
(132)	✓	✓							✓	✓
(124)	✓		✓						✓	✓
(142)	✓		✓						✓	✓
(134)	✓			✓					✓	✓
(143)	✓			✓					✓	✓
(234)	✓				✓				✓	✓
(243)	✓				✓				✓	✓
$(12)(34)$	✓					✓	✓	✓	✓	✓
$(13)(24)$	✓					✓	✓	✓	✓	✓
$(14)(23)$	✓					✓	✓	✓	✓	✓

By the **orbit-counting theorem**, there are $|\operatorname{Orb}(\phi)| = 60/|A_4| = 5$ conjugacy classes.

A summary

Thus far, we have seen four important (right) actions of a group G, acting:

- on itself by multiplication
- on itself by conjugation.
- on its subgroups by conjugation.
- on the cosets of a fixed subgroup $H \leq G$ by multiplication.

set $S =$	G		subgroups of G	right cosets of H
operation	multiplication	conjugation	conjugation	right multiplication
orb(s)	G	$\mathrm{cl}_G(g)$	$\mathrm{cl}_G(H)$	all right cosets
stab(s)	$\langle 1 \rangle$	$C_G(g)$	$N_G(H)$	$x^{-1}Hx$
fix(g)	G or $\emptyset$	$C_G(g)$	$\{H \mid g \in N_G(H)\}$	$\{Hx \mid xgx^{-1} \in H\}$
Ker(ϕ)	$\langle 1 \rangle$	$Z(G)$	$\bigcap_{H \leq G} N_G(H)$	largest norm. subgp. $N \leq H$
Fix(ϕ)	$\emptyset$	$Z(G)$	normal subgroups	none

More applications of group actions!

Subgroups of small index

Groups acting on cosets is a useful technique for establishing seemingly unrelated results.

Several of these involve showing that subgroups of "small index" are normal.

We've already seen that subgroups of index 2 are normal.

Of course, there are non-normal index-3 subgroups, like $\langle f \rangle \leq D_3$.

The following gives a sufficient condition for when index-3 subgroups are normal.

Proposition

If G has no subgroup of index 2, then any subgroup of index 3 is normal.

Proof

Let $H \leq G$ with $[G : H] = 3$.

Let G act on the cosets of H by multiplication, to get a nontrivial homomorphism

$$\phi \colon G \longrightarrow S_3.$$

$K := \mathrm{Ker}(\phi) \leq H$ is the largest normal subgroup of G contained in H. By the FHT,

$$G/K \cong \mathrm{Im}(\phi) \leq S_3.$$

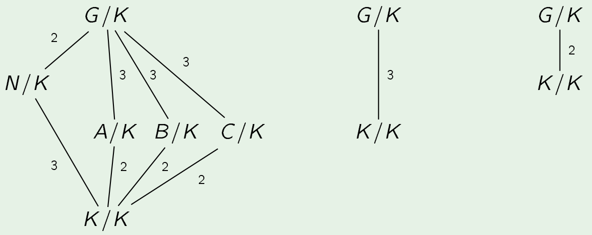
Subgroups of small index

Proof (contin.)

Thus, there are three cases for this quotient:

$$G/K \cong S_3, \qquad G/K \cong C_3, \qquad G/K \cong C_2.$$

Visually, this means that we have one of the following:



By the corrdespondence theorem, $K \leq H \lneq G$ implies $K/K \leq H/K \lneq G/K$.

Since G has no index-2 subgroup, only the middle case is possible (*Why?*).

This forces $K/K = H/K$, and so $K = H$ which is normal for multiple reasons. $\square$

Subgroups of small index

Proposition

Suppose $H \leq G$ and $[G : H] = p$, the smallest prime dividing $|G|$. Then $H \trianglelefteq G$.

Proof

Let G act on the cosets of H by multiplication, to get a non-trivial homomorphism

$$\phi \colon G \longrightarrow S_p.$$

The kernel $K = \mathrm{Ker}(\phi)$, is the largest normal subgroup of G such that $K \leq H \lneq G$.

We'll show that $H = K$, or equivalently, that $[H : K] = 1$. By the correspondence theorem:

$$\begin{tikzcd} G \arrow[d, dash, "p"] \\ H \arrow[d, dash, "q \text{ is not divisible by any prime} < p"] \\ K \end{tikzcd} \qquad\qquad \begin{tikzcd} G/K \cong S_p \arrow[d, dash, "p"] \\ H/K \arrow[d, dash, "q \text{ divides } (p-1)!"] \\ K/K \end{tikzcd}$$

Do you see why $q = 1$? $\square$

A creative application of a group action

Cauchy's theorem

If p is a prime dividing $|G|$, then G has an element (and hence a subgroup) of order p.

Proof

Let P be the set of ordered p-tuples of elements from G whose product is e:

$$(x_1, x_2, \dots, x_p) \in P \quad \text{iff} \quad x_1 x_2 \cdots x_p = e\,.$$

Observe that $|P| = |G|^{p-1}$. (We can choose $x_1, \dots, x_{p-1}$ freely; then x_p is forced.)

The group $\mathbb{Z}_p$ acts on P by cyclic shift:

$$\phi\colon \mathbb{Z}_p \longrightarrow \mathrm{Perm}(P), \qquad (x_1, x_2, \dots, x_p) \stackrel{\phi(1)}{\longmapsto} (x_2, x_3 \dots, x_p, x_1)\,.$$

The set P is partitioned into orbits, each of size $|\mathrm{orb}(s)| = [\mathbb{Z}_p : \mathrm{stab}(s)] = 1$ or p.

The only way that the orbit of $(x_1, x_2, \dots, x_p)$ can have size 1 is if $x_1 = \cdots = x_p$.

Clearly, $(e, \dots, e) \in P$ is a fixed point.

The $|G|^{p-1} - 1$ other elements in P sit in orbits of size 1 or p.

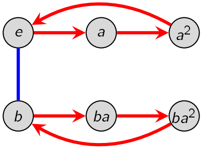
Since $p \nmid |G|^{p-1} - 1$, there must be other orbits of size 1. Thus, some $(x, \dots, x) \in P$, with $x \neq e$ satisfies $x^p = e$. $\square$

Classification of groups of order 6

By Cauchy's theorem, every group of order 6 must have:

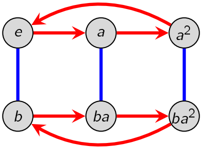
- an element a of order 3
- an element b of order 2.

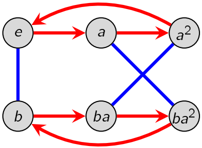
Clearly, $G = \langle a, b \rangle$, and so G must have the following "partial Cayley graph":



It is now easy to see that up to isomorphism, there are only 2 groups of order 6:

$C_6 \cong C_2 \times C_3$





D_3

Exercise. Suppose that $|G| = pq$, where $p < q$ are primes and p doesn't divide $q - 1$. Prove that G is cyclic.

p-groups and the Sylow theorems!

p-groups and the Sylow theorems

Definition

A *p-group* is a group whose order is a power of a prime p. A p-group that is a subgroup of a group G is a *p-subgroup* of G.

Can you tell me some examples of 2-groups?

Notational convention

Throughout, G will be a group of order $|G| = p^n \cdot m$, with $p \nmid m$.
That is, p^n is the *highest power* of p dividing $|G|$. (We are isolating all the p.)

There are three Sylow theorems, and loosely speaking, they describe the following about a group's p-subgroups:

1. **Existence**: In every group, p-subgroups of all possible sizes exist.
2. **Relationship**: All maximal p-subgroups are conjugate.
3. **Number**: Strong restrictions on the number of p-subgroups a group can have.

Together, these place strong restrictions on the structure of a group G with a fixed order.

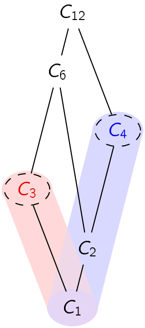
Groups of order $12 = 2^2 \cdot 3^1$

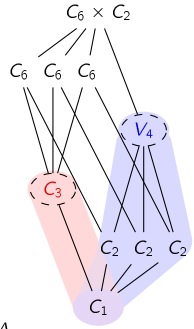
Sylow theorems:

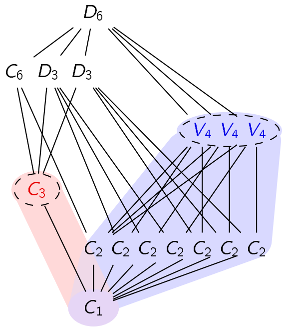
p-subgroups come in "towers."

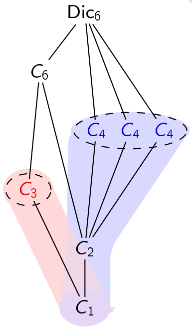
2-subgroups blue; 3-subgroups red.

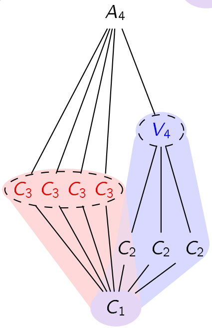
The tops of the towers are conjugate; there are restrictions on the size of their conjugacy classes.











p-groups

Before we introduce the Sylow theorems, we need to better understand p-groups.

p-group Lemma

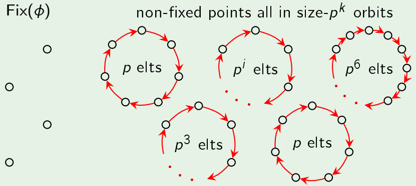
If a p-group G acts on a set S via $\phi\colon G \to \mathbf{Perm}(S)$, then

$$|\operatorname{Fix}(\phi)| \equiv_p |S|.$$

Proof (sketch)

Suppose $|G| = p^n$.

By the orbit-stabilizer theorem, the only possible orbit sizes are $1, p, p^2, \ldots, p^n$.



A lot of proofs about p-groups go like this: two things are equal mod p; set up some action of G on S; one of the things is the number of fixed points; the other thing is the size of S.

p-groups

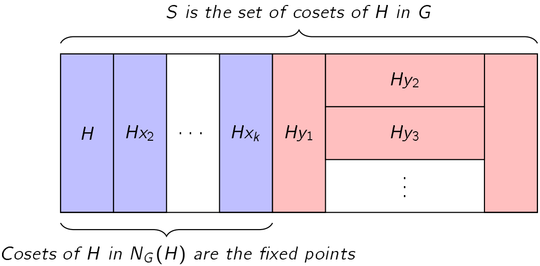
Normalizer lemma, Part 1

If H is a p-subgroup of G, then

$$[N_G(H)\colon H] \equiv_p [G\colon H].$$

Approach:

- Let H (not G!) act on the (right) cosets of H by (right) multiplication.



- Apply our lemma: $|\operatorname{Fix}(\phi)| \equiv_p |S|$.

Normalizer lemma, Part 1

If H is a p-subgroup of G, then

$$[N_G(H)\colon H] \equiv_p [G\colon H].$$

Proof

Let $S = H\backslash G = \{Hx \mid x \in G\}$. The group H acts on S by **right-multiplication**, via $\phi\colon H \to \mathrm{Perm}(S)$, where

$$\phi(h) = \text{the permutation sending each } Hx \text{ to } Hxh.$$

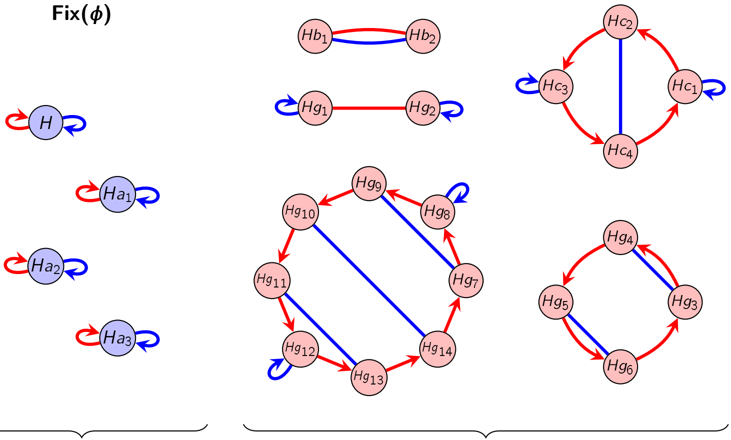
The fixed points of ϕ are the cosets Hx in the normalizer $N_G(H)$:

$$\begin{aligned} Hxh = Hx, \quad \forall h \in H \quad &\iff \quad Hxhx^{-1} = H, \quad \forall h \in H \\ &\iff \quad xhx^{-1} \in H, \quad \forall h \in H \\ &\iff \quad x \in N_G(H). \end{aligned}$$

Therefore, $|\mathrm{Fix}(\phi)| = [N_G(H)\colon H]$, and $|S| = [G : H]$. By our p-group Lemma,

$$|\mathrm{Fix}(\phi)| \equiv_p |S| \quad \Longrightarrow \quad [N_G(H)\colon H] \equiv_p [G\colon H]. \qquad \square$$

Here is a picture of the action of the p-subgroup H (for $p = 2$) on the set $S = H \backslash G$, from the proof of the normalizer lemma.



The fixed points are the cosets in $N_G(H)$

Cosets not in $N_G(H)$ are in orbits of order p^i, for various $i \geq 1$

p-subgroups

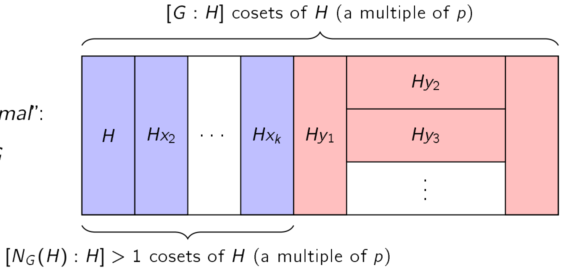
Recall that $H \leq N_G(H)$ (always), and H is fully unnormal if $H = N_G(H)$.

Normalizer lemma, Part 2

Suppose $|G| = p^n m$, and $H \leq G$ with $|H| = p^i < p^n$. Then $H \lneq N_G(H)$, and the index $[N_G(H) : H]$ is a multiple of p.

H is not "*fully unnormal*":

$$H \lneq N_G(H) \leq G$$



Important corollaries

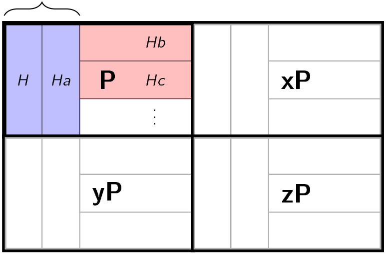
- p-groups cannot have any fully unnormal subgroups (i.e., $H \lneq N_G(H)$).
- In *any* finite group, the only fully unnormal p-subgroups are maximal.

Normalizers of p-subgroups

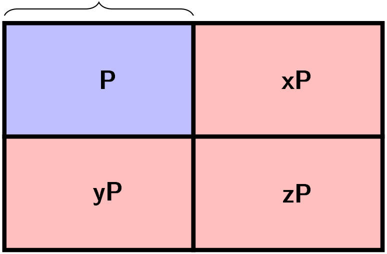
Let H be properly contained in a maximal p-subgroup $P \lneq G$.

- The normalizer of H *must* grow in P (and hence in G)
- The normalizer of P *need not* grow in G.

$H \lneq N_P(H) \leq N_G(H)$



it may happen that $P = N_G(P)$



Proof of the normalizer lemma

Normalizer lemma, Part 2

Suppose $|G| = p^n m$, and $H \leq G$ with $|H| = p^i < p^n$. Then $H \lneq N_G(H)$, and the index $[N_G(H) : H]$ is a multiple of p.

Proof

Since $H \trianglelefteq N_G(H)$, we can create the quotient map

$$\pi\colon N_G(H) \longrightarrow N_G(H)/H\,, \qquad\qquad \pi\colon g \longmapsto gH\,.$$

The size of the quotient group is $[N_G(H)\colon H]$, the number of cosets of H in $N_G(H)$.

By the normalizer lemma Part 1, $[N_G(H)\colon H] \equiv_p [G\colon H]$. By Lagrange's theorem,

$$[N_G(H)\colon H] \equiv_p [G\colon H] = \frac{|G|}{|H|} = \frac{p^n m}{p^i} = p^{n-i} m \equiv_p 0.$$

Therefore, $[N_G(H)\colon H]$ is a multiple of p, so $N_G(H)$ must be strictly larger than H. $\square$

The Sylow theorems

Recall the following question that we asked earlier in this course.

Open-ended question

What group structural properties are possible, what are impossible, and how does this depend on $|G|$?

One approach is to decompose large groups into "building block subgroups." For example:

> *given a group of order* $72 = 2^3 \cdot 3^2$, *what can we say about its* 2*-subgroups and* 3*-subgroups?.*

This is the idea behind the Sylow theorems, developed by Norwegian mathematician Peter Sylow (1832–1918).

The Sylow theorems address the following questions of a finite group G:

1. How big are its p-subgroups?
2. How are the p-subgroups related?
3. How many p-subgroups are there?
4. Are any of them normal?

The Sylow theorems

Notational convention

Througout, G will be a group of order $|G| = p^n \cdot m$, with $p \nmid m$.

That is, p^n is the *highest power* of p dividing $|G|$.

A subgroup of order p^n is called a Sylow p-subgroup.

Let $\mathrm{Syl}_p(G)$ denote the set of Sylow p-subgroups, and $n_p := \left|\mathrm{Syl}_p(G)\right|$.

There are three Sylow theorems, and loosely speaking, they describe the following about a group's p-subgroups:

1. **Existence**: In every group, p-subgroups of all possible sizes exist, and they're "*nested*".
2. **Relationship**: All maximal p-subgroups are conjugate.
3. **Number**: There are strong restrictions on n_p, the number of Sylow p-subgroups.

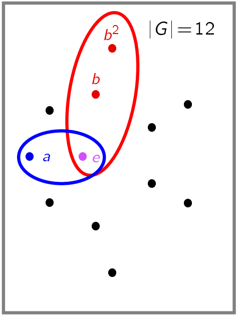
Together, these place strong restrictions on the structure of a group G with a fixed order.

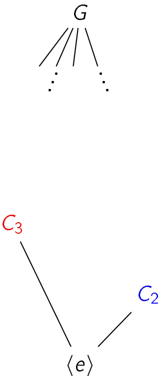
Our unknown group of order 12

Throughout, we will have a running example, a "mystery group" G of order $12 = 2^2 \cdot 3$.

We already know a little bit about G. By Cauchy's theorem, it must have:

- an element a of order 2, and
- an element b of order 3.





Using *only* the fact that $|G| = 12$, we will unconver as much about its structure as we can.

The 1^{st} Sylow theorem: existence of p-subgroups

First Sylow theorem

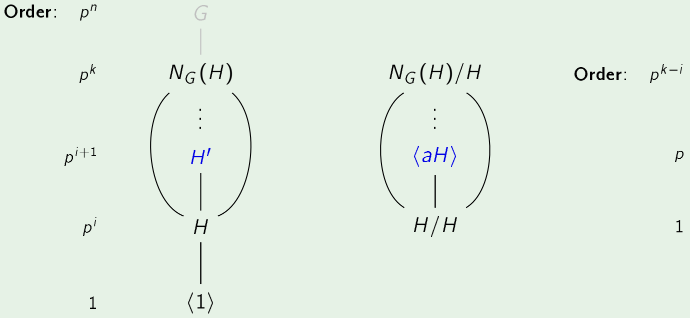
G has a subgroup of order p^k, for each p^k dividing $|G|$.

Also, every non-Sylow p-subgroup sits inside a larger p-subgroup.

Proof

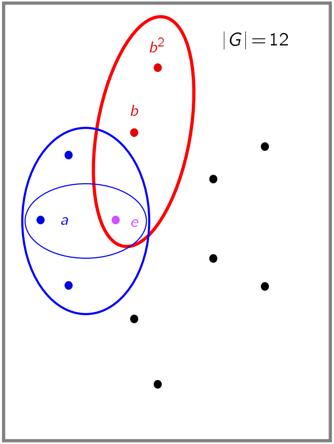
Take any $H \leq G$ with $|H| = p^i < p^n$. We know $H \trianglelefteq N_G(H)$ and p divides $|N_G(H)/H|$.

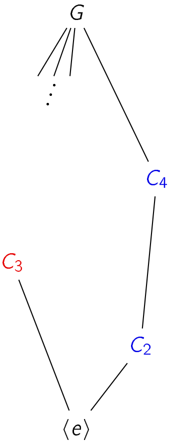
Find an element aH of order p. The union of cosets in $\langle aH \rangle$ is a subgroup of order p^{i+1}.

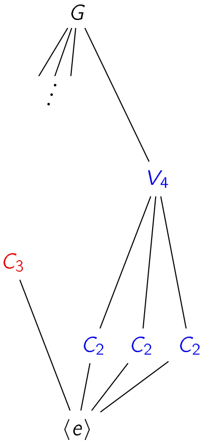


Our unknown group of order 12

By the first Sylow theorem, $\langle a \rangle$ is contained in a subgroup of order 4, which could be V_4 or C_4, or possibly both.







The 2nd Sylow theorem: relationship among p-subgroups

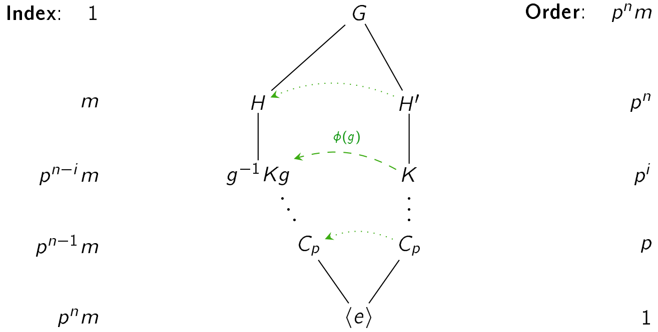
Second Sylow theorem

Any two Sylow p-subgroups are conjugate (and hence isomorphic).

We'll actually prove a stronger version, which easily implies the 2nd Sylow theorem.

Strong second Sylow theorem

Let $H \in \mathrm{Syl}(G)$, and $K \leq G$ any p-subgroup. Then K is conjugate to a subgroup of H.



The 2nd Sylow theorem: All Sylow p-subgroups are conjugate

Strong second Sylow theorem

Let H be a Sylow p-subgroup, and $K \leq G$ any p-subgroup. Then K is conjugate to some subgroup of H.

Proof

Let $S = H \backslash G = \{Hg \mid g \in G\}$, the set of right cosets of H.

The group K acts on S by **right-multiplication**, via $\phi \colon K \to \mathrm{Perm}(S)$, where

$$\phi(k) = \text{the permutation sending each } Hg \text{ to } Hgk.$$

A fixed point of ϕ is a coset $Hg \in S$ such that

$$\begin{aligned} Hgk = Hg\,, \quad \forall k \in K \qquad &\Longleftrightarrow \qquad Hgkg^{-1} = H\,, \quad \forall k \in K \\ &\Longleftrightarrow \qquad gkg^{-1} \in H\,, \quad \forall k \in K \\ &\Longleftrightarrow \qquad gKg^{-1} \subseteq H\,. \end{aligned}$$

Thus, *if we can show that ϕ has a fixed point Hg, we're done!*

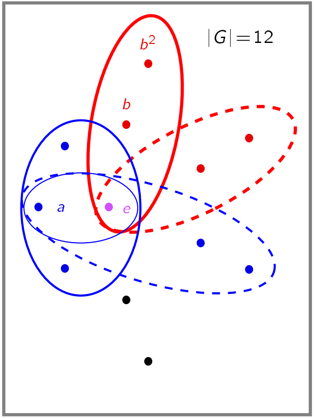
All we need to do is show that $|\mathrm{Fix}(\phi)| \not\equiv_p 0$. By the p-group Lemma,

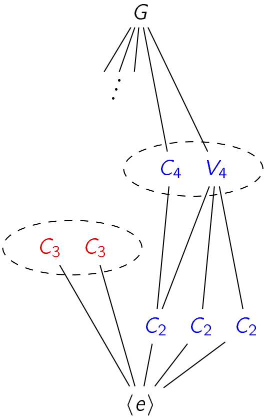
$$|\mathrm{Fix}(\phi)| \equiv_p |S| = [G : H] = m \not\equiv_p 0. \qquad \square$$

Our unknown group of order 12

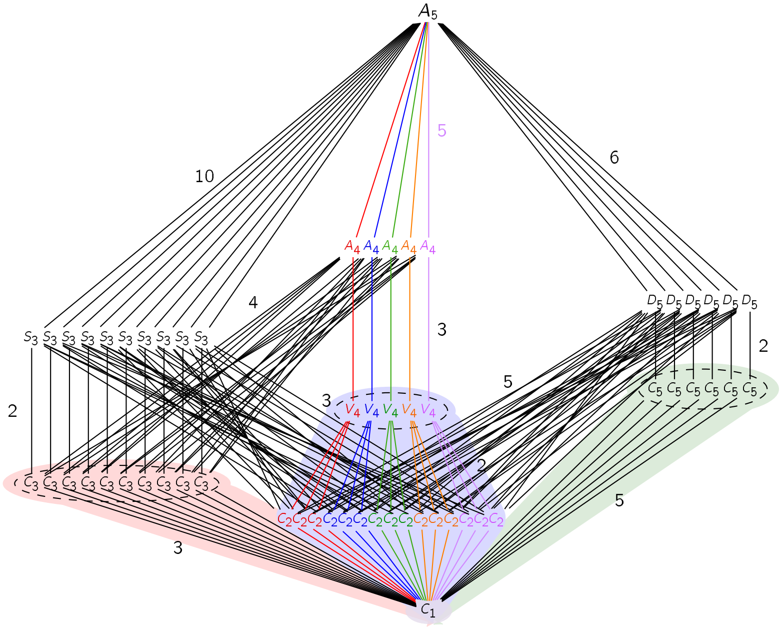
By the second Sylow theorem, all Sylow p-subgroups are conjugate, and hence isomorphic.

This eliminates the following subgroup lattice of a group of order 12.





Example: A_5 has no nontrival proper normal subgroups



The normalizer of the normalizer

Notice how in A_5:

- all Sylow p-subgroups are moderately unnormal
- the normalizer of each Sylow p-subgroup is fully unnormal. That is:

$$N_G(N_G(P)) = N_G(P)$$

Proposition

Let P be a non-normal Sylow p-subgroup of G. Then its normalizer is fully unnormal.

Proof

We'll verify the equivalent statement of $N_G(N_G(P)) = N_G(P)$.

Note that P is a normal Sylow p-subgroup of $N_G(P)$.

By the 2nd Sylow theorem, P is the unique Sylow p-subgroup of $N_G(P)$.

Take an element x that normalizes $N_G(P)$ (i.e., $x \in N_G(N_G(P))$. We'll show that it also normalizes P. By definition, $xN_G(P)x^{-1} = N_G(P)$, and so

$$P \leq N_G(P) \qquad \Longrightarrow \qquad xPx^{-1} \leq xN_G(P)x^{-1} = N_G(P).$$

But xPx^{-1} is also a Sylow p-subgroup of $N_G(P)$, and by uniqueness, $xPx^{-1} = P$. $\square$

The 3rd Sylow theorem: number of p-subgroups

Third Sylow theorem

Let n_p be the number of Sylow p-subgroups of G. Then

$$n_p \text{ divides } |G| \qquad \text{and} \qquad n_p \equiv_p 1\,.$$

(Note that together, these imply that $n_p \mid m$, where $|G| = p^n \cdot m$.)

Proof

Take $H \in \mathrm{Syl}_p(G)$. By the 2nd Sylow theorem, $n_p = |\mathrm{cl}_G(H)| = [G : N_G(H)] \,\big|\, |G|$. ✓

The subgroup H acts on $S = \mathrm{Syl}_p(G)$ by conjugation, via $\phi \colon G \to \mathrm{Perm}(S)$, where

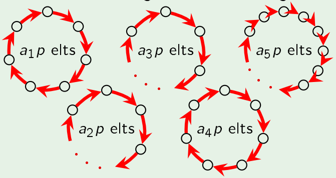
$$\phi(h) = \text{the permutation sending each } K \text{ to } h^{-1}Kh.$$

Goal: *show that H is the unique fixed point.*

$|\mathrm{Fix}(\phi)| = 1$

H

other Sylow p-subgroups are in larger orbits



total # Sylow p-subgroups
$= n_p = |S| \equiv_p |\mathrm{Fix}(\phi)|$

The 3rd Sylow theorem: number of p-subgroups

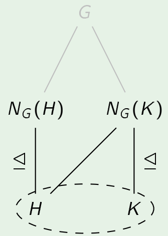
Proof (cont.)

Goal: *show that H is the unique fixed point.*

Let $K \in \mathrm{Fix}(\phi)$. Then $K \leq G$ is a Sylow p-subgroup satisfying

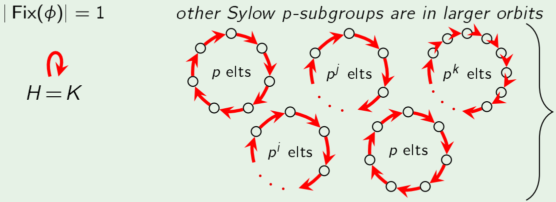
$$h^{-1}Kh = K\,, \quad \forall h \in H \quad \iff \quad H \leq N_G(K) \leq G\,.$$

- H and K are p-Sylow in G, and in $N_G(K)$.
- H and K are conjugate in $N_G(K)$. (2nd Sylow thm.)
- $K \trianglelefteq N_G(K)$, thus is only conjugate to itself in $N_G(K)$.



Thus, $K = H$. That is, $\mathrm{Fix}(\phi) = \{H\}$.

By the p-group Lemma, $n_p := |S| \equiv_p |\mathrm{Fix}(\phi)| = 1$. $\square$



total # Sylow p-subgroups $= n_p = |S| \equiv_p |\mathrm{Fix}(\phi)| = 1$

Summary of the proofs of the Sylow theorems

For the 1st Sylow theorem, we started with $H = \{e\}$, and inductively created larger subgroups of size $p, p^2, \ldots, p^n$.

For the 2^{nd} and 3^{rd} Sylow theorems, we used a clever group action and then applied one or both of the following:

(i) *orbit-stabilizer theorem*. If G acts on S, then $|\mathrm{orb}(s)| \cdot |\mathrm{stab}(s)| = |G|$.

(ii) *p-group lemma*. If a p-group acts on S, then $|S| \equiv_p |\mathrm{Fix}(\phi)|$.

To summarize, we used:

S2 The action of $K \in \mathrm{Syl}_p(G)$ on $S = H \backslash G$ by right multiplication for some other $H \in \mathrm{Syl}_p(G)$.

S3a The action of G on $S = \mathrm{Syl}_p(G)$, by conjugation.

S3b The action of $H \in \mathrm{Syl}_p(G)$ on $S = \mathrm{Syl}_p(G)$, by conjugation.

Our mystery group order 12

By the 3rd Sylow theorem, every group G of order $12 = 2^2 \cdot 3$ must have:

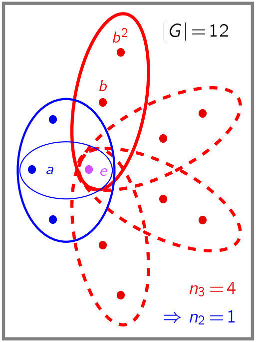
- n_3 Sylow 3-subgroups, each of order 3.

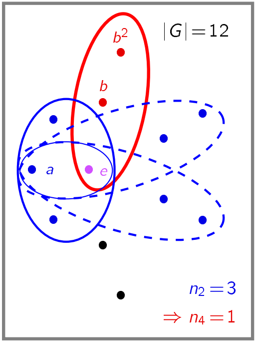
$$n_3 \mid 4, \qquad n_3 \equiv 1 \pmod 3 \qquad \Longrightarrow \qquad n_3 = 1 \text{ or } 4.$$

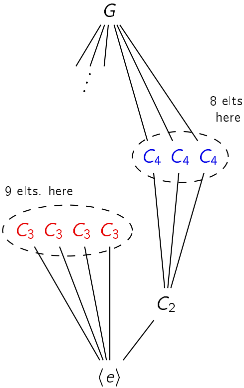
- n_2 Sylow 2-subgroups of order $2^2 = 4$.

$$n_2 \mid 3, \qquad n_2 \equiv 1 \pmod 2 \qquad \Longrightarrow \qquad n_2 = 1 \text{ or } 3.$$

But both are not possible! (There aren't enough elements.)





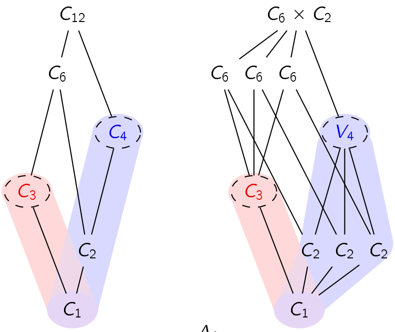


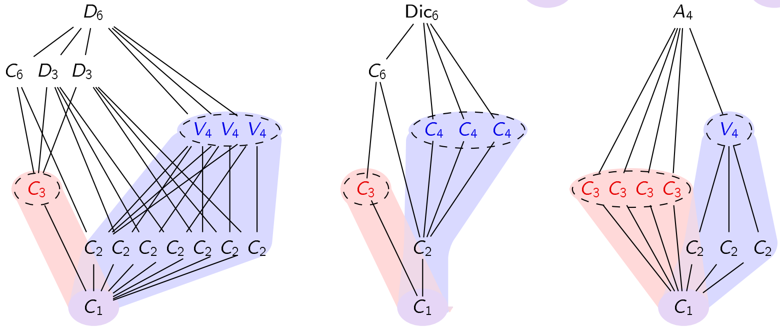
The five groups of order 12

With a little work and the Sylow theorems, we can classify all groups of order 12.

We've already seen them all. Here are their subgroup lattices.

Note that *all* of these decompose as a direct or semidirect product of Sylow subgroups.





Simple groups and the Sylow theorems

Definition

A group G is simple if its only normal subgroups are G and $\langle e \rangle$.

Simple groups are to groups what primes are to integers, and are essential to understand.

The Sylow theorems are very useful for establishing statements like:

"There are no simple groups of order k (for some k)."

Since all Sylow p-subgroups are conjugate, the following result is immediate.

Remark

A Sylow p-subgroup is normal in G iff it's the unique Sylow p-subgroup (that is, if $n_p = 1$).

Thus, if we can show that $n_p = 1$ for some p dividing $|G|$, then G cannot be simple.

For some $|G|$, this is harder than for others, and sometimes it's not possible.

Tip

When trying to show that $n_p = 1$, it's usually helpful to analyze the largest primes first.

An easy example

We'll see three examples of showing that groups of a certain size cannot be simple, in successive order of difficulty.

Proposition

There are no simple groups of order 84.

Proof

Since $|G| = 84 = 2^2 \cdot 3 \cdot 7$, the third Sylow theorem tells us:

- n_7 divides $2^2 \cdot 3 = 12$ (so $n_7 \in \{1, 2, 3, 4, 6, 12\}$)
- $n_7 \equiv_7 1$.

The only possibility is that $n_7 = 1$, so the Sylow 7-subgroup must be normal. $\square$

Observe why it is beneficial to use the largest prime first:

- n_3 divides $2^2 \cdot 7 = 28$ and $n_3 \equiv_3 1$. Thus $n_3 \in \{1, \cancel{2}, 4, 7, \cancel{14}, 28\}$.
- n_2 divides $3 \cdot 7 = 21$ and $n_2 \equiv_2 1$. Thus $n_2 \in \{1, 3, 7, 21\}$.

A harder example

Proposition

There are no simple groups of order 351.

Proof

Since $|G| = 351 = 3^3 \cdot 13$, the third Sylow theorem tells us:

- n_{13} divides $3^3 = 27$ (so $n_{13} \in \{1, 3, 9, 27\}$)
- $n_{13} \equiv_{13} 1$.

The only possibilies are $n_{13} = 1$ or 27.

A Sylow 13-subgroup P has order 13, and a Sylow 3-subgroup Q has order $3^3 = 27$. Therefore, $P \cap Q = \{e\}$.

Suppose $n_{13} = 27$. Every Sylow 13-subgroup contains 12 non-identity elements, and so G must contain $27 \cdot 12 = 324$ elements of order 13.

This leaves $351 - 324 = 27$ elements in G not of order 13. Thus, G contains only one Sylow 3-subgroup (i.e., $n_3 = 1$) and so G cannot be simple. $\square$

The hardest example

Proposition

There are no simple groups of order $24 = 2^3 \cdot 3$.

From the 3rd Sylow theorem, we can only conclude that $n_2 \in \{1, 3\}$ and $n_3 = \{1, 4\}$.

Let H be a Sylow 2-subgroup, which has relatively "small" index: $[G : H] = 3$.

Lemma

If G has a subgroup of index $[G : H] = n$, and $|G|$ does not divide $n!$, then G is not simple.

Proof

Let G act on the **right cosets** of H (i.e., $S = H \backslash G$) by **right-multiplication**:

$$\phi \colon G \longrightarrow \mathrm{Perm}(S) \cong S_n\,, \qquad \phi(g) = \text{the permutation that sends each } Hx \text{ to } Hxg.$$

Recall that $\mathrm{Ker}(\phi) \trianglelefteq G$, and is the intersection of all conjugate subgroups of H:

$$\langle e \rangle \leq \mathrm{Ker}(\phi) = \bigcap_{x \in G} x^{-1} H x \lneq G$$

If $\mathrm{Ker}(\phi) = \langle e \rangle$ then $\phi \colon G \hookrightarrow S_n$ is an embedding, which is impossible because $|G| \nmid n!$. $\square$