

Cosets!

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With many thanks to Matthew Macauley,
`http://www.math.clemson.edu/~macaule/`

17 Feb 2025

Goals for today:

1. Define what **cosets** are
2. See some examples
3. Learn some properties of cosets
4. Prove **Lagrange's theorem**
5. Explore **normal** subgroups

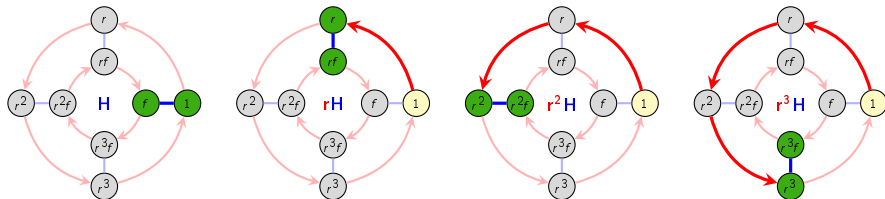
Definition time!

Definition time! (soon)

Vibes-based explanation (not a definition)

Let $H \leq G$. A coset of H is a **shifted copy** of H somewhere else in G .

Example: consider $H = \langle f \rangle \leq D_4$.



(Of course, only one of these is actually a subgroup; the others don't contain the identity.)

How do you shift to get from original H to each of these shifted copies?

Left cosets, and how to find them

To find the **left** coset xH in a Cayley graph, carry out the the following steps:

1. starting from the identity, follow a path to get to x ("shift by x ")
2. from x , follow all " H -paths".

Definition time! (actually)

Definition

If $H \leq G$, then a **left coset** is a set

$$xH = \{xh \mid h \in H\},$$

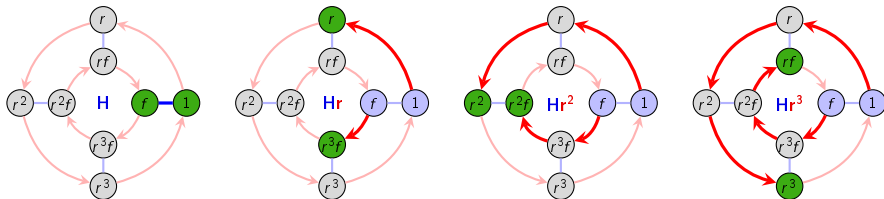
for some fixed $x \in G$ called the **representative**. Similarly, we can define a **right coset** as

$$Hx = \{hx \mid h \in H\}.$$

Left vs. right cosets

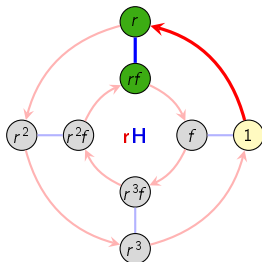
- The **left coset** rH in D_4 : first **shift by r** , then traverse all “ H -paths”.
- The **right coset** Hr in D_4 : first traverse all H -paths, then **shift by r** .

Let's look at the right cosets of $H = \langle f \rangle$ in D_4 .

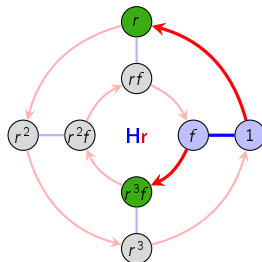


Left vs. right cosets

- The **left coset** rH in D_4 : first **go to r** , then traverse all “ H -paths”.
- The **right coset** Hr in D_4 : first traverse all H -paths, then traverse the r -path.



$$rH = r\{1, f\} = \{r, rf\} = rf\{f, 1\} = rfH$$



$$Hr = \{1, f\}r = \{r, r^3f\} = \{f, 1\}r^3f = Hr^3f$$

Because of our convention that arrows in a Cayley graph represent **right multiplication**:

- left cosets look like copies of the subgroup,
- right cosets are usually “scattered.”

Key point

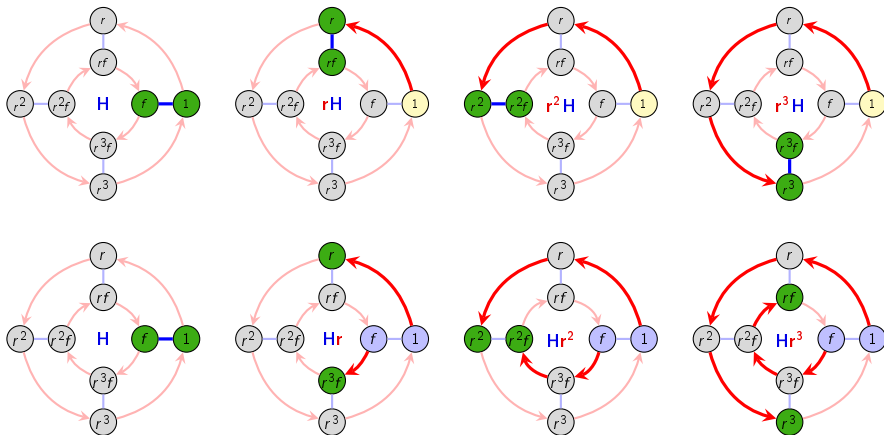
Left and right cosets are generally different.

Overview of left and right cosets of $\langle f \rangle$

Definition

Let $H \leq G$. Given $x \in G$, its **left coset** xH and **right coset** Hx are:

$$xH = \{xh \mid h \in H\}, \quad Hx = \{hx \mid h \in H\}.$$



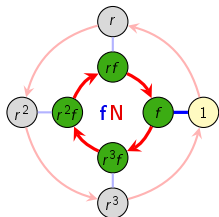
Your turn!

Your turn!

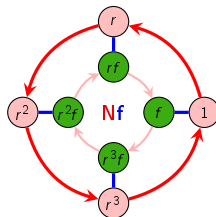
Find all the left and right cosets of a different subgroup, $N = \langle r \rangle$.

Reminder: finding left vs right cosets

- Left coset xN : first **shift by x** , then traverse all N -paths
- Right coset Nx : first traverse all N -paths, then **shift by x**



$$fN = f\{1, r, r^2, r^3\} = \{f, fr, fr^2, fr^3\}$$



$$Nf = \{1, r, r^2, r^3\}f = \{f, rf, r^2f, r^3f\}$$

Observations?

- There are multiple representatives for the same coset:

$$fN = (rf)N = (r^2f)N = (r^3f)N, \quad Nf = N(rf) = N(r^2f) = N(r^3f).$$

- For this subgroup, each left coset is a right coset. Such a subgroup is called **normal**.

Your turn!

Now try:

- The other cyclic subgroups of D_4
- $K = \langle r^2, f \rangle \leq D_4$

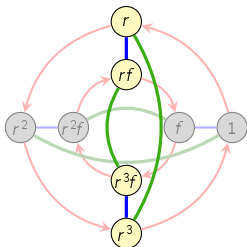
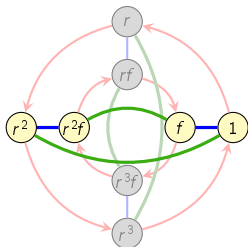
Observations?

- Cosets never overlap
- All the cosets are always the same size as the original subgroup
- Cosets always cover the whole group
- Any time there are just two cosets, each left coset is a right coset

Equality of sets vs. equality of elements

Caveat!

An equality of cosets $xK = Kx$ as sets *does not* imply (or require) an equality of elements $xk = kx$.



rK	r	r^3	rf	r^3f
K	1	r^2	f	r^2f

Kr	r	r^3	fr	fr^3
K	1	r^2	f	fr^2

rK and Kr are the same **as sets**, even though the elements occur in a different order.

Properties of cosets!

Basic properties of cosets

The following results are “visually clear” from the Cayley graphs, but let’s now prove them:

Proposition

Each (left) coset can have multiple representatives: if $b \in aH$, then $aH = bH$.

Proof

Since $b \in aH$, we can write $b = ah$, for some $h \in H$. That is, $h = a^{-1}b$ and $a = bh^{-1}$.

To show that $aH = bH$, we need to verify both $aH \subseteq bH$ and $aH \supseteq bH$.

“ \subseteq ”: Take $ah_1 \in aH$. We need to write it as bh_2 , for some $h_2 \in H$. By substitution,

$$ah_1 = (bh^{-1})h_1 = b(h^{-1}h_1) \in bH.$$

“ \supseteq ”: Pick $bh_3 \in bH$. We need to write it as ah_4 for some $h_4 \in H$. By substitution,

$$bh_3 = (ah)h_3 = a(hh_3) \in aH.$$

Therefore, $aH = bH$, as claimed. □

Corollary (boring but useful)

The equality $xH = H$ holds if and only if $x \in H$. (And analogously, for $Hx = H$.)

Basic properties of cosets

Proposition

For any subgroup $H \leq G$, the (left) cosets of H **partition** the group G : every element $g \in G$ lives in **exactly one** coset of H .

Proof

We know that the element $g \in G$ lies in a (left) coset of H , namely gH . Uniqueness follows because if $g \in kH$, then $gH = kH$. \square

Proposition

All (left) cosets of $H \leq G$ have the same size. \square

Proof

It suffices to show that $|xH| = |H|$, for any $x \in H$.

Define a map

$$\phi: H \longrightarrow xH, \quad h \longmapsto xh.$$

It is elementary to show that this is a bijection. \square

Lagrange's theorem!

Lagrange's theorem

Remark

For any subgroup $H \leq G$, the left cosets of H partition G into subsets of equal size.

The right cosets also partition G into subsets of equal size, but *they may be different*.

Let's compare these two partitions for the subgroup $H = \langle f \rangle$ of $G = D_4$.

H	r^2H	rH	r^3H
f	r^2f	rf	r^3
1	r^2	r	r^3f

H	Hr^2			
f	fr^2	fr^3	r^3	Hr^3
1	r^2	r	fr	Hr

Definition

The **index** of a subgroup H of G , written $[G : H]$, is the number of distinct left (or equivalently, right) cosets of H in G .

Lagrange's theorem

If H is a subgroup of finite group G , then $|G| = [G : H] \cdot |H|$.



Funny historical aside

Guess who proved Lagrange's theorem. **Not Lagrange!**

- Lagrange, 1771: if a polynomial in n variables has its variables permuted in all $n!$ ways, the number of different polynomials that are obtained is always a factor of $n!$.
 - What does this have to do with cosets?
 - Take $H \leq S_n$ to be the set of permutations that **fix** the polynomial. $n! = |S_n| = [S_n : H] \cdot |H|$.
 - The number of different polynomials is the number of cosets of H , aka the index $[S_n : H]$.
 - So: true for **special subgroups** of S_n .
- Gauss, 1801: the **special case** of subgroups of $(\mathbb{Z}/p\mathbb{Z})^*$
- Cauchy, 1844: true for **any** subgroup $H \leq S_n$
- Jordan, 1861: true for any subgroup H of a **permutation group**
 - (so, now it's true for subgroups of **subgroups of S_n**)
- Cayley, 1854: every group is a permutation group

Importantly: Lagrange would not have had the words “index,” “coset,” “group.”

The tower law

Proposition

Let G be a finite group and $K \leq H \leq G$ be a chain of subgroups. Then

$$[G : K] = [G : H][H : K].$$

Here is a “proof by picture”:

$[G : H] = \#$ of cosets of H in G

$[H : K] = \#$ of cosets of K in H

$[G : K] = \#$ of cosets of K in G

zH	$z_1 K$	$z_2 K$	$z_3 K$...	$z_n K$
	\vdots	\vdots	\vdots	\ddots	\vdots
aH	$a_1 K$	$a_2 K$	$a_3 K$...	$a_n K$
H	K	$h_2 K$	$h_3 K$...	$h_n K$

Proof

By Lagrange's theorem,

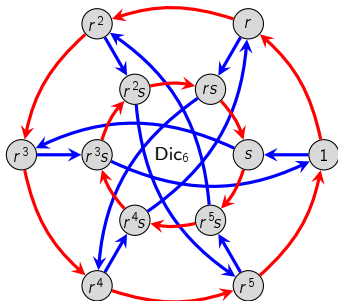
$$[G : H][H : K] = \frac{|G|}{|H|} \cdot \frac{|H|}{|K|} = \frac{|G|}{|K|} = [G : K].$$

□

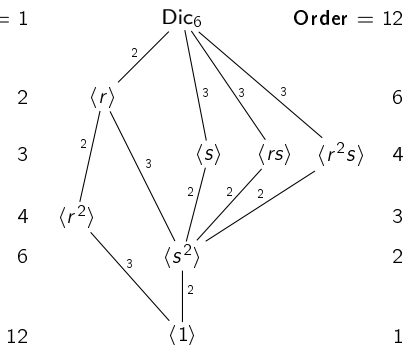
The tower law

Another way to visualize the tower law involves subgroup lattices.

It is often helpful to label the edge from H to K in a subgroup lattice with the index $[H : K]$.



Index = 1



The tower law and subgroup lattices

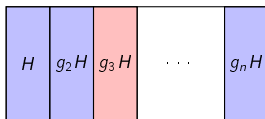
For any two subgroups $K \leq H$ of G , the index of K in H is just the *products of the edge labels* of any path from H to K .

Normal subgroups!

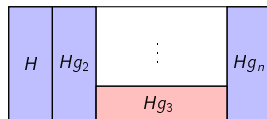
Normal subgroups and normalizers

Given a subgroup H of G , it is natural to ask the following question:

“How many left cosets of H are right cosets?”



*Partition of G by the
left cosets of H*



*Partition of G by the
right cosets of H*

- “Best case” scenario: all of them
- “Worst case” scenario: only H
- In general: somewhere between these two extremes

Definition

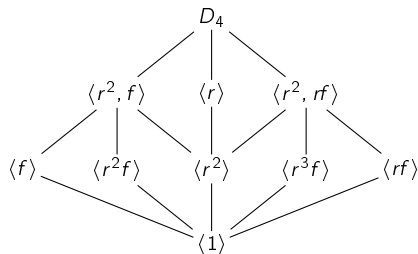
A subgroup H is a **normal subgroup** of G if $gH = Hg$ for all $g \in G$. We write $H \trianglelefteq G$.

The **normalizer** of H , denoted $N_G(H)$, is the set of elements $g \in G$ such that $gH = Hg$:

$$N_G(H) = \{g \in G \mid gH = Hg\},$$

i.e., the **union of left cosets that are also right cosets**.

Normal subgroups in D_4



Which of these subgroups are normal in D_4 ?

The end!