Homework #5

HW due Sunday 2/23 by pdf upload to Canvas; .tex source on the MATH 312 github repo.
In all the following, let H be a generic subgroup of a generic group G; that is, let $H \leq G$.

Stuff from class slides that I promised would be on the homework

Problem 1. Prove that $N_G(H) \leq G$. (Here and elsewhere, hooray for the one-step subgroup test.)

Problem 2. A problem in two parts:

(a) Prove that $H \leq N_G(H)$.

(b) Prove that $H \trianglelefteq N_G(H)$. (Careful: you only have to coset by stuff in $N_G(H)$, not stuff in the wider universe of G.)

Remark. By stapling together the results of Problem 1 and Problem 2, we get a useful corollary: $N_G(H)$ has to be somewhere between H and G (inclusive) in the subgroup lattice of G.

Problem 3. Prove that if $ghg^{-1} \in H$ for all $h \in H$ and $g \in G$, then $gHg^{-1} = H$ for all $g \in G$.
Hint: Our favorite tool for proving that two sets A and B are equal is to show that $A \subseteq B$ and also that $A \supseteq B$. For $\subseteq$: Let $a \in A$. Blah blah blah, therefore $a \in B$. For $\supseteq$: vice versa.
(Our second favorite tool is to establish a bijection between A and B.)

Problem 4. Let $g \in G$. Prove that $gHg^{-1} \leq G$, and thus that gHg^{-1} deserves to be called a "conjugate subgroup."

Immediate corollaries of Lagrange's theorem

Problem 5. Show that if G is a finite group and $x \in G$, then the order of x divides the order of G.
(Moral: it is reasonable for us to use the same word "order" for these two ideas.)

Problem 6. Show that if $|G| = p$ a prime, then G is cyclic.

Problem 7. Show that if $[G : H] = p$ a prime, then either $N_G(H) = H$ or $H \triangleleft G$.
(Hint: use the tower law.)

Hint: All the problems on this page can be resolved in like two or three lines.

Exploring cosets

Problem 8. So far we have been looking exclusively at cosets in groups written multiplicatively, where they look like $gH = \{g \cdot h \mid h \in H\}$. Let's look at cosets in $(\mathbb{Z}, +)$, where we shift by *adding* instead of by *multiplying*.

(a) Consider the subgroup $4\mathbb{Z} = \{\dots -4, 0, 4, 8, \dots\} = \langle 4 \rangle$. Find all of its cosets. (Hint: you will be unsurprised about how many of them there are.)

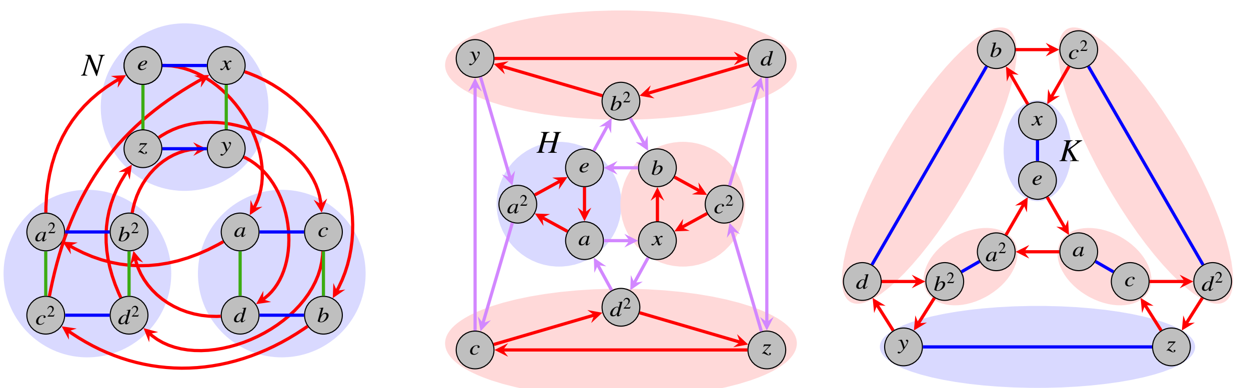
(b) What is $[\mathbb{Z} : 4\mathbb{Z}]$? (Hint: the answer is unsurprising and also it is not $\frac{\infty}{\infty}$.)

(c) Suppose that $p > 2$ is a prime number. (2 is an annoying exception to *many* theorems about prime numbers.) Which cosets is it possible for p to live in?

Problem 9. Here's an extended problem where you can explore the relationship between left cosets, right cosets, conjugate subgroups, and normalizers.

Below are three Cayley diagrams of A_4, each highlighting the left cosets of a different subgroup. These are the subgroups N, H, and K from slide 17 of the normal-subgroups slides from class on Wednesday. To make the notation suck less and the Cayley diagrams more readable, we can take $a = (123)$, $b = (134)$, $x = (12)(34)$, and $z = (13)(24)$; arrows in the Cayley diagrams are color-coded appropriately. Then:

$$N = \langle x, z \rangle; \qquad H = \langle a \rangle; \qquad K = \langle x \rangle.$$



(a) Label each of the "coset bubbles" in each diagram above with which left coset it is. For instance, $\{a, c, b, d\}$ is certainly aN.

(b) For each subgroup shown above, partition A_4 into its right cosets. (Work smarter not harder: think about which elements you actually need to bother shifting by!) Write the right cosets as subsets of A_4, consisting of permutations in cycle notation. Also, highlight them by colors on a fresh copy of the Cayley diagrams – see the next page.

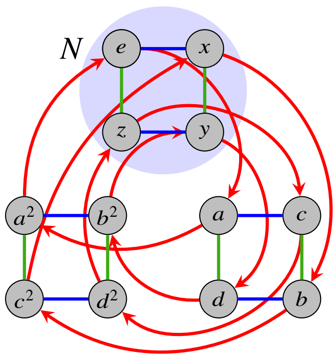
(c) Conjecture as to why I made some of the bubbles blue and some of them red. Relatedly, find $N_{A_4}(N)$, $N_{A_4}(H)$, and $N_{A_4}(K)$.

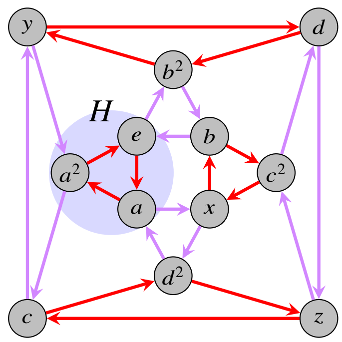
(d) For each (non-identity) left coset gH, illustrate the construction of the conjugate subgroup gHg^{-1} on a fresh copy of the Cayley diagram – see next page. Repeat this for N and K.

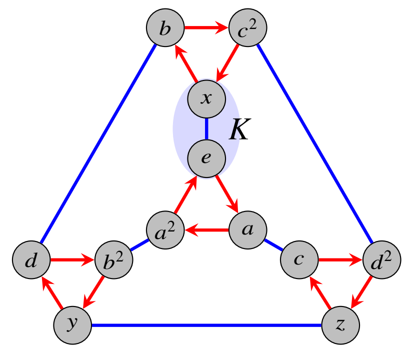
Fresh Cayley diagrams for Problem 9

Please please *please* print this out and draw your coset bubbles by hand (or by marking up a pdf on a tablet). I promise that it would suck *so much* to do this in tikz.

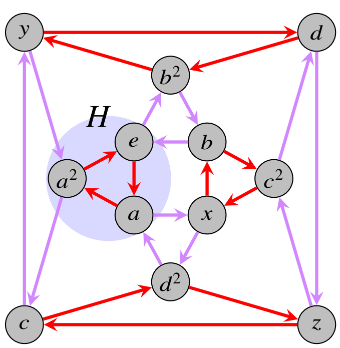
For part (b):

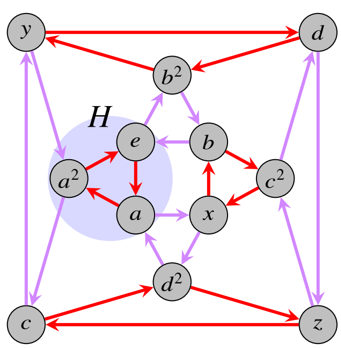


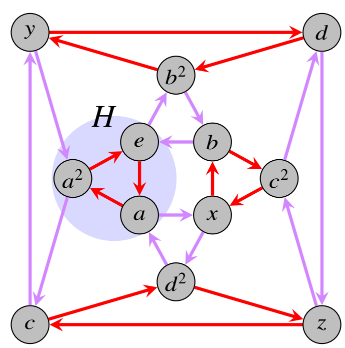




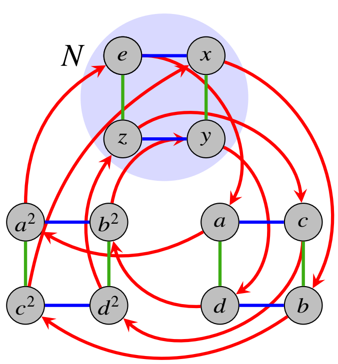
For part (d), subgroup H (3 copies bc $[A_4 : H] = 4$ and I don't care about one of 'em):

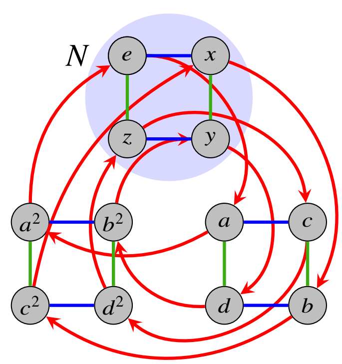






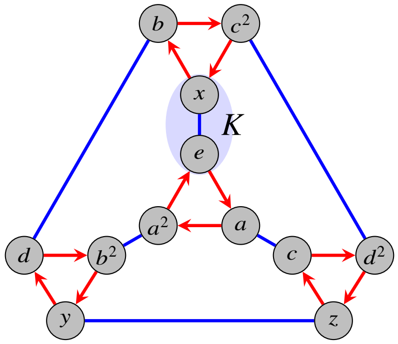
For part (d), subgroup N (note $[A_4 : N] = 3$):

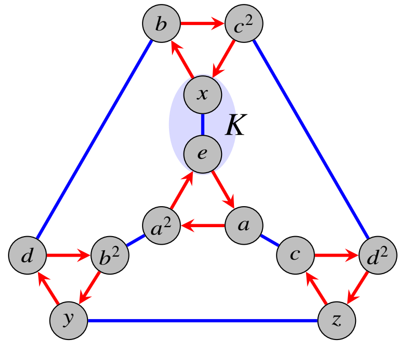


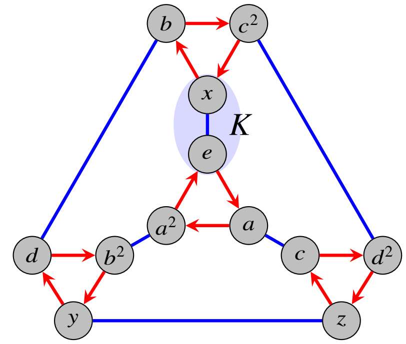


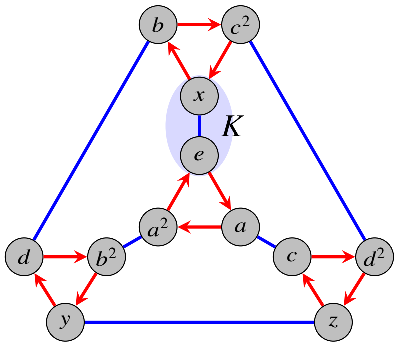
Ran outta room, see next page

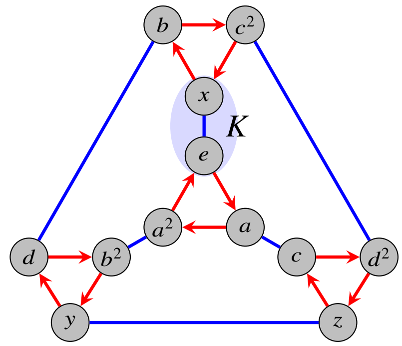
For part (d), subgroup K (why am I giving you 5 copies?):





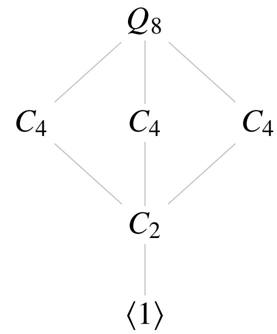






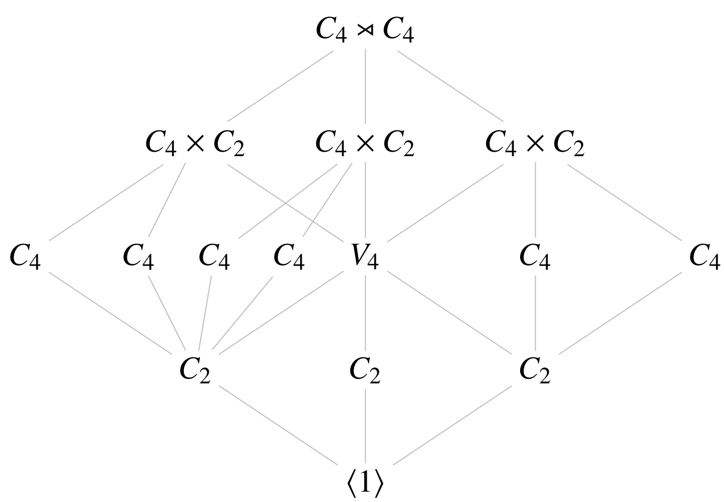
Finding unicorns and normal subgroups

Problem 10. Here is the subgroup lattice for Q_8:



Without doing any quaternion calculations, prove that every subgroup of Q_8 is normal. (!!)

Problem 11. The subgroup lattice of some weird group called $C_4 \rtimes C_4$ is shown below. This is a "semidirect product" – it is kinda like a direct product but the copies of C_4 that went in the inflated nodes of C_4 are "rewired." All you need to know about it is that, like the direct product, elements are ordered pairs, so $|C_4 \rtimes C_4| = 16$.



(a) Identify as many unicorns as you can. Write a sentence or two explaining how you know each one is a unicorn.

(b) Identify as many normal subgroups as you can. Write a sentence or two about each one.

(c) See if that adds any more unicorns to your initial list of unicorns.

Challenges

All problems in this section are optional.

Open question. Can you make / find a group that has an *interesting* subgroup lattice in which every subgroup is a unicorn? (It is easy to do this in a *boring* way: for instance, consider the subgroup lattice of C_6.)

Problem 12. Here we shall track down the details from our discussion of the mystery group of order 16 from class on Wednesday.

(a) Let $g \in G$ and suppose that $\langle g \rangle$ is a normal subgroup of order 2. Prove that $g \in Z(G)$.

(b) Suppose that G is generated by two generators, say $G = \langle g, h \mid \ldots \rangle$. Prove that if $g \in Z(G)$, then $h \in Z(G)$.

(c) Let G be a finitely generated group, say $G = \langle g_1, \ldots, g_n \mid \ldots \rangle$. (Note that G doesn't have to be finite – the integers, for example, are finitely generated.) Prove that if all the generators $g_i \in Z(G)$, then G is abelian.

(d) Now, getting more specific: in the mystery group, we knew that $s^2 = r^8 = 1$. How did we know those two things?

(e) Suppose that $\langle s \rangle$ and $\langle r^4 s \rangle$ *aren't* normal; therefore they must be conjugate. Prove that $srs = r^5$. (Hint: conjugate by r.)

Problem 13. Screw around with the LMFDB database of abstract groups linked on Canvas. Find something fun / interesting and tell me about it.

Problem 14. Write down a full proof of Lagrange's theorem:

$$\text{if } H \leq G, \text{ then } |H| \text{ divides } |G|, \text{ and further, } |G| = [G : H] \cdot |H|.$$

(This just entails stringing together the arguments we made on the slides before the Lagrange's theorem slide, but I think it's moderately nice to see it all written out.)

Problem 15. State the converse of (the first sentence of) Lagrange's theorem. Prove it or find a counterexample.

Problem 16. Repeat Problem 8, but for the subgroup $6\mathbb{Z}$, and increase p to be $p > 3$.

Problem 17. Repeat again but for $8\mathbb{Z}$, and tell me what the correct lower bound on p is.

Problem 18. Prove that $|\operatorname{cl}_G(H)| = [G : N_G(H)]$.
(This is easier than you think; translate this sentence into human words and you are like 71% there.)

Problem 19. Come up with a reasonable definition for how to quantify "how normal" slash "how unnormal" a subgroup is. For instance, in Problem 9, N should be yes 100% normal, K should be moderately un-normal, and H should be as un-normal as possible.