Group actions, part 2!

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With many thanks to Matthew Macauley,
http://www.math.clemson.edu/~macaule/

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- *Action graphs are to G-sets, like how Cayley graphs are to groups.*

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Notation

Throughout, we'll denote identity elements by $1 \in G$ and $e \in \mathrm{Perm}(S)$.

Local: orbits, stabilizers, fixators

Two local features: orbits and stabilizers

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(i) The orbit of $s \in S$ is the connected component containing s.

(ii) The stabilizer of $s \in S$ are the group elements whose paths start and end at s; "loops."

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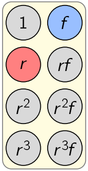
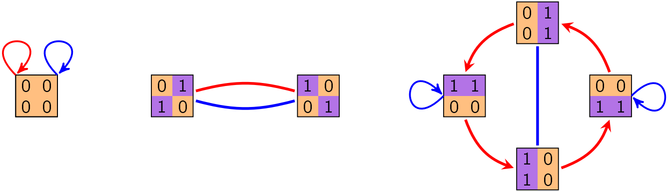
Three local features: orbits, stabilizers, and fixators

Here's the action graph of our running example of D_4 acting on S the set of binary squares.

Find the orbit and stabilizer of each binary square, and the fixator of each element of D_4.

"Group switchboard"





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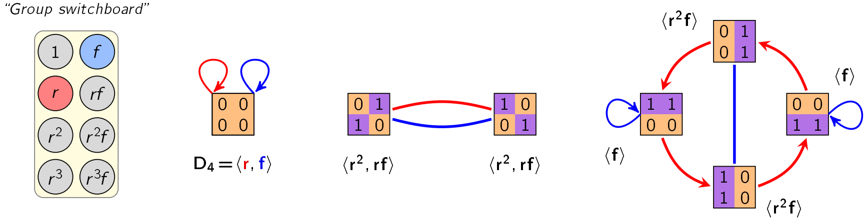
The orbits of our running example are the 3 connected components.

Each node is labeled by its stabilizer.



The fixators are $\text{fix}(1) = S$, and

$$\text{fix}(r) = \text{fix}(r^3) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \qquad \text{fix}(r^2) = \text{fix}(rf) = \text{fix}(r^3 f) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

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Local duality: stabilizers vs. fixators

Consider the following table, where a checkmark at (g, s) means g fixes s.

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1	✓	✓	✓	✓	✓	✓	✓
r	✓						
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- the stablizers can be read off the columns: *group elements that fix $s \in S$*
- the fixators can be read off the rows: *set elements fixed by $g \in G$*.

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All three of these are very intuitive in our our switchboard analogy.

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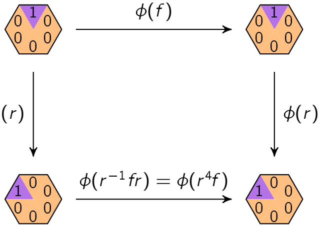
Here are several ways to visualize what this means and why.

$$\begin{tikzcd}
s \arrow[r, mapsto, "\phi(x)"] \arrow[d, mapsto, "\phi(g)"'] & s \arrow[d, mapsto, "\phi(g)"] \\
s' \arrow[r, mapsto, "\phi(g^{-1}xg)"] & s'
\end{tikzcd}$$



In other words, if x is a loop from s, and $s \xrightarrow{g} s'$, then $g^{-1}xg$ is a loop from s'.

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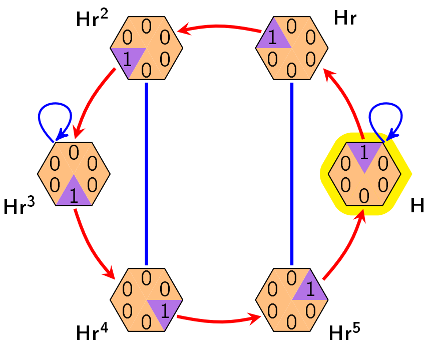
Here is another example of an action (or G-set), this time of $G = D_6$ acting on these six "Pacman hexagons."

Let s be the highlighted hexagon, and $H = \mathbf{stab}(s)$.



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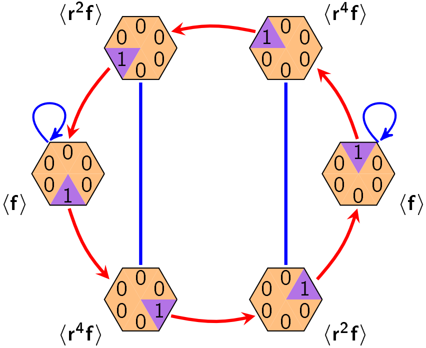

labeled by stabilizers

Global: fixed points and the kernel

Two global features: fixed points and the kernel

Our last two features are properties of the action ϕ, rather than of specific elements.

Two global features: fixed points and the kernel

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Let's also write $\mathsf{Orb}(\phi)$ for the set of orbits of ϕ.

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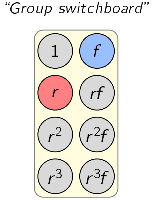

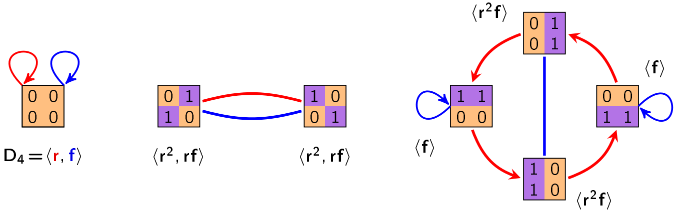

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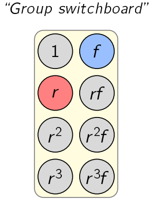

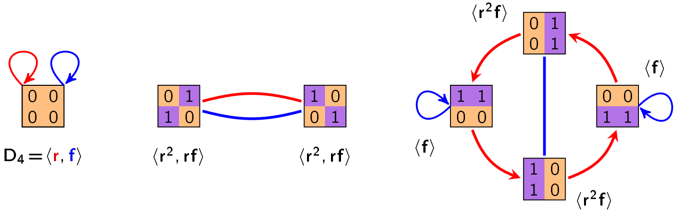

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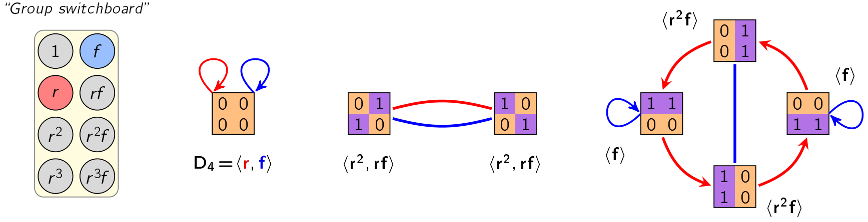

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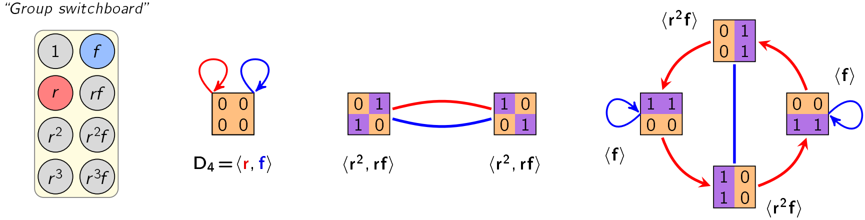

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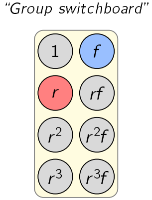

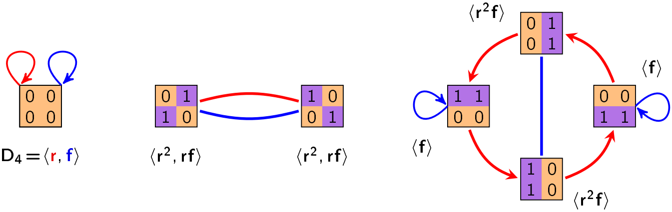

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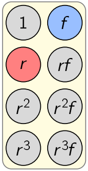

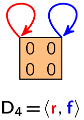

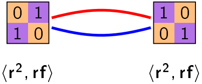

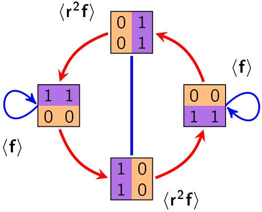

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1	✓	✓	✓	✓	✓	✓	✓
r	✓						
r^2	✓	✓	✓				
r^3	✓						
f	✓			✓		✓	
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r^2	✓	✓	✓				
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Theorems!

Two theorems on orbits, and their consequences

Our binary square example gives us some key intutition about group actions.

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We can also determine the number of conjugacy classes from the orbit-counting theorem.

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Orbit-stabilizer theorem

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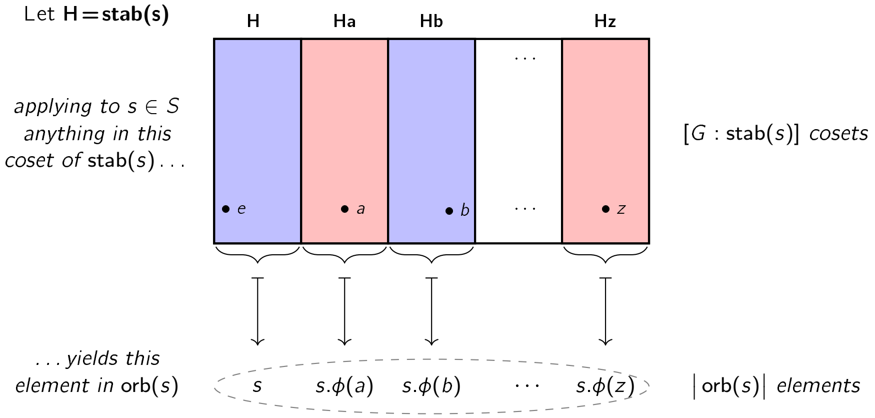
$$\left|\mathrm{orb}(s)\right| \cdot \left|\mathrm{stab}(s)\right| = \left|G\right| .$$

Equivalently, *the size of the orbit containing s is* $\left|\mathrm{orb}(s)\right| = [G : \mathrm{stab}(s)]$.



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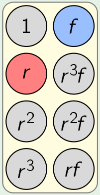
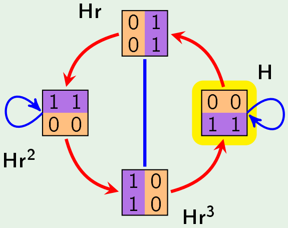


1	f	$\mathbf{H = stab(s)}$
r	fr	$\mathbf{Hr}$
r^2	fr^2	$\mathbf{Hr^2}$
r^3	fr^3	$\mathbf{Hr^3}$



Note that $s.\phi(g) = s.\phi(k)$ iff g and k are in the same right coset of H in G.

The orbit-stabilizer theorem: $|\operatorname{orb}(s)| \cdot |\operatorname{stab}(s)| = |G|$

Proof (cont.)

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If we have instead, a left group action, the proof carries through but using left cosets.

Our second theorem on orbits

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Let a finite group G act on a set S via $\phi\colon G \longrightarrow \mathsf{Perm}(S)$. Then

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This says that the "*average number of checkmarks per row*" is the number of orbits:

	$\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$	$\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$	$\begin{smallmatrix}0&0\\1&1\end{smallmatrix}$	$\begin{smallmatrix}0&1\\0&1\end{smallmatrix}$	$\begin{smallmatrix}1&1\\0&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\1&0\end{smallmatrix}$
1	✓	✓	✓	✓	✓	✓	✓
r	✓						
r^2	✓	✓	✓				
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f	✓			✓		✓	
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Equating this last term with the first term gives the desired result. □

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Theorems that we have observed but haven't been able to prove yet will fall in our lap!

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Assume $|G| > 2$. The group G acts on itself (that is, $S = G$) by **right-multiplication**:

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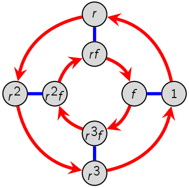

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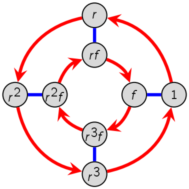

Cayley's theorem

If $|G| = n$, then there is an embedding $G \hookrightarrow S_n$.

Proof

Groups acting on themselves by multiplication

Assume $|G| > 2$. The group G acts on itself (that is, $S = G$) by **right-multiplication**:

$$\phi\colon G \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } x \mapsto xg.$$

- there is only one orbit: $\mathrm{orb}(x) = G$, for all $x \in G$
- the stabilizer of each $x \in G$ is $\mathrm{stab}(x) = \langle 1 \rangle$
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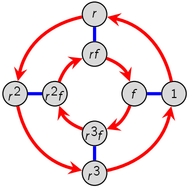
- there are no fixed points, and the kernel is trivial:

$$\mathrm{Fix}(\phi) = \bigcap_{g \in G} \mathrm{fix}(g) = \emptyset, \qquad \text{and} \qquad \mathrm{Ker}(\phi) = \bigcap_{s \in S} \mathrm{stab}(s) = \langle 1 \rangle.$$



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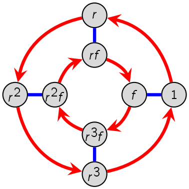

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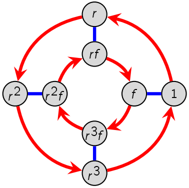

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Since $\mathrm{Ker}(\phi) = \langle 1 \rangle$, it is an embedding. $\square$

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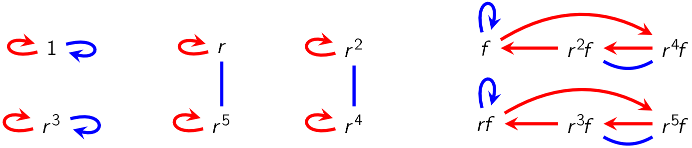

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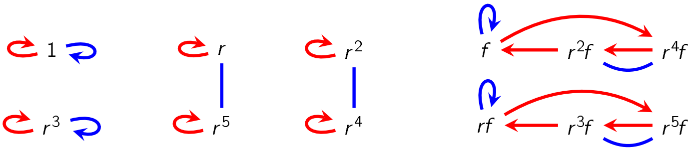

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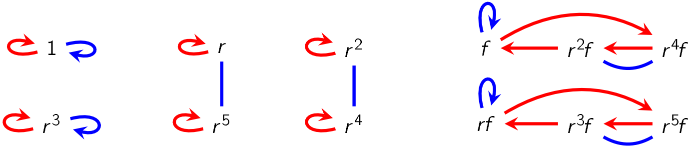

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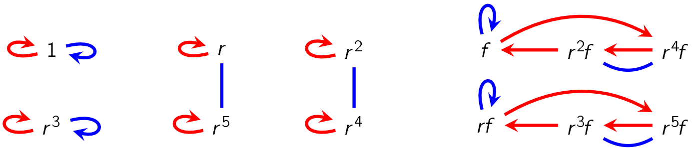

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Groups acting on themselves by conjugation

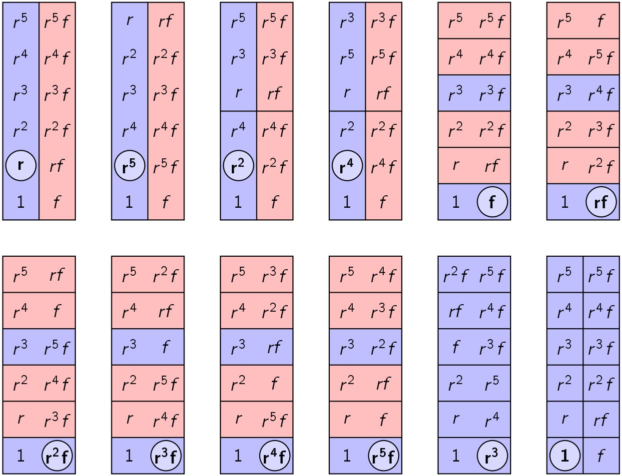
Here is the "*fixed point table*". Note that $\mathrm{Ker}(\phi) = \mathrm{Fix}(\phi) = \langle r^3 \rangle$.

	1	r	r^2	r^3	r^4	r^5	f	rf	r^2f	r^3f	r^4f	r^5f
1	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
r	✓	✓	✓	✓	✓	✓						
r^2	✓	✓	✓	✓	✓	✓						
r^3	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
r^4	✓	✓	✓	✓	✓	✓						
r^5	✓	✓	✓	✓	✓	✓						
f	✓			✓			✓			✓		
rf	✓			✓				✓			✓	
r^2f	✓			✓					✓			✓
r^3f	✓			✓			✓			✓		
r^4f	✓			✓				✓			✓	
r^5f	✓			✓					✓			✓

By the **orbit-counting theorem**, there are $|\mathrm{Orb}(\phi)| = 72/|D_6| = 6$ conjugacy classes.

Groups acting on themselves by conjugation

Here are the cosets of all 12 cyclic subgroups in D_6 (some coincide).



Do you see how to deduce from the orbit-counting theorem that there are 6 conjugacy classes?

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Any group G acts on its set S of subgroups by **right-conjugation**:

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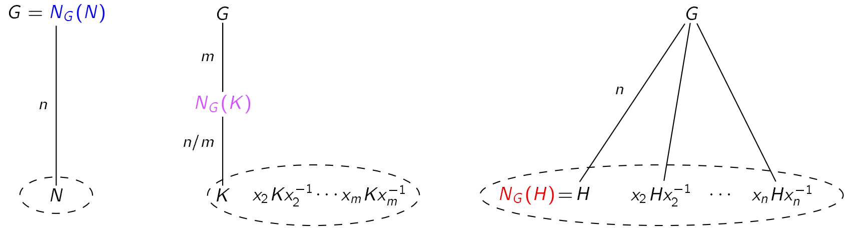

normal $\left|\operatorname{cl}_G(N)\right| = 1$

moderately unnormal $1 < \left|\operatorname{cl}_G(K)\right| < [G : K]$

fully unnormal $\left|\operatorname{cl}_G(H)\right| = [G : H]$; as large as possible

Groups acting on subgroups by conjugation

Here is an example of $G = D_3$ acting on its subgroups.

$\tau(1)$	$=$	$\langle 1\rangle$	$\langle r\rangle$	$\langle f\rangle$	$\langle rf\rangle$	$\langle r^2f\rangle$	D_3
$\tau(r)$	$=$	$\langle 1\rangle$	$\langle r\rangle$	$\langle f\rangle$	$\langle rf\rangle$	$\langle r^2f\rangle$	D_3
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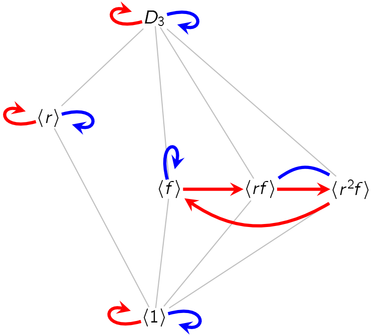

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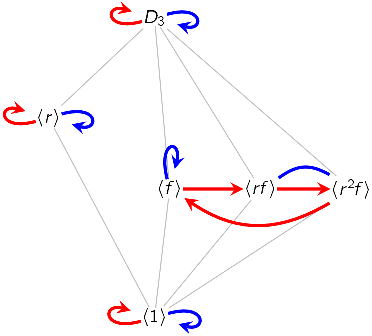

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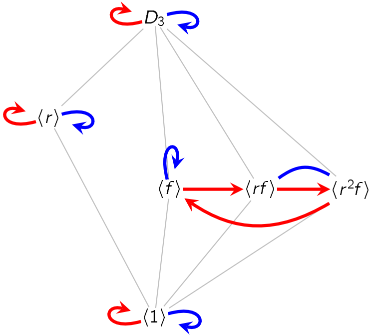

Observations

Do you see how to read stabilizers and fixed points off of the permutation diagram?

- $\mathrm{Ker}(\phi) = \langle 1\rangle$ consists of the row(s) with only fixed points.
- $\mathrm{Fix}(\phi) = \{\langle 1\rangle, \langle r\rangle, D_3\}$ consists of the column(s) with only fixed points.

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\begin{array}{lclllllll}
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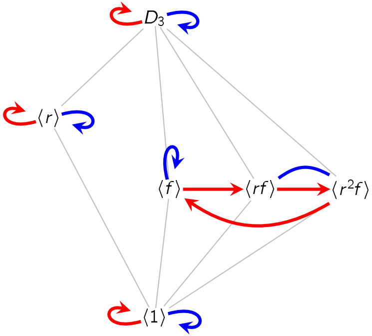

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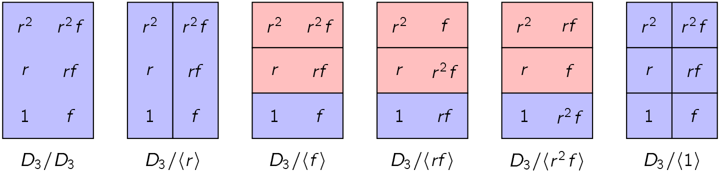
- $\mathrm{Fix}(\phi) = \{\langle 1 \rangle, \langle r \rangle, D_3\}$ consists of the column(s) with only fixed points.
- By the orbit-counting theorem, there are $|\mathrm{Orb}(\phi)| = 24/|D_3| = 4$ conjugacy classes.

Groups acting on subgroups by conjugation

Consider the partitions of D_3 by the left cosets of its six subgroups:



- fix(g) are the subgroups H for which "*g appears in a blue coset of H*"
- Ker(ϕ) are elements that "*only appear in blue cosets*"
- By the orbit-counting theorem, the subgroups fall into

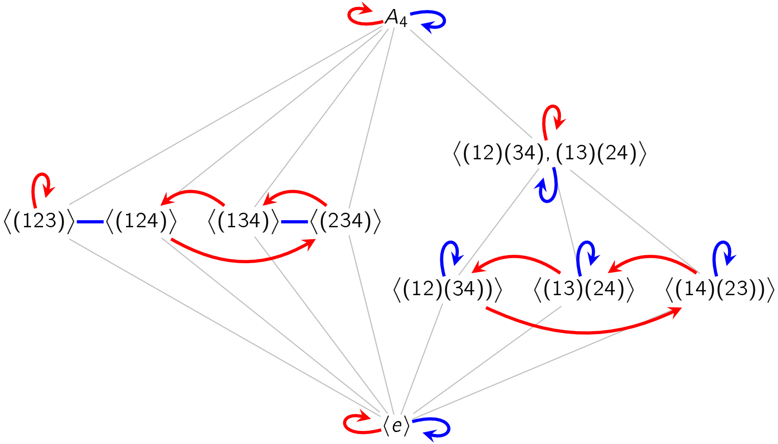
$$|\operatorname{Orb}(\phi)| = \text{average \# checkmarks per row} = \frac{\text{total \# of blue entries}}{|G|}$$

conjugacy classes.

Equivalently: *how many full "G-boxes" the blue cosets can be rearranged to fill up.*

Groups acting on subgroups by conjugation

Here is an example of $G = A_4 = \langle (123), (12)(34) \rangle$ acting on its subgroups.



Let's take a moment to revisit our "*three favorite examples*" from Chapter 3.

$$N = \langle (12)(34), (13)(24) \rangle, \qquad H = \langle (123) \rangle, \qquad K = \langle (12)(34) \rangle.$$

Groups acting on subgroups by conjugation

Here is the "*fixed point table*" of the action of A_4 on its subgroups.

	$\langle e \rangle$	$\langle (123) \rangle$	$\langle (124) \rangle$	$\langle (134) \rangle$	$\langle (234) \rangle$	$\langle (12)(34) \rangle$	$\langle (13)(24) \rangle$	$\langle (14)(23) \rangle$	$\langle (12)(34), (13)(24) \rangle$	A_4
e	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
(123)	✓	✓							✓	✓
(132)	✓	✓							✓	✓
(124)	✓		✓						✓	✓
(142)	✓		✓						✓	✓
(134)	✓			✓					✓	✓
(143)	✓			✓					✓	✓
(234)	✓				✓				✓	✓
(243)	✓				✓				✓	✓
$(12)(34)$	✓					✓	✓	✓	✓	✓
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By the **orbit-counting theorem**, there are $|\operatorname{Orb}(\phi)| = 60/|A_4| = 5$ conjugacy classes.

Groups acting on cosets of H by multiplication

Fix a subgroup $H \leq G$. Then G acts on its **right cosets** by **right-multiplication**:

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  Notice that $\langle 1\rangle \leq \mathrm{Ker}\,\phi \leq H$, and $\mathrm{Ker}(\phi) = H$ iff $H \trianglelefteq G$.

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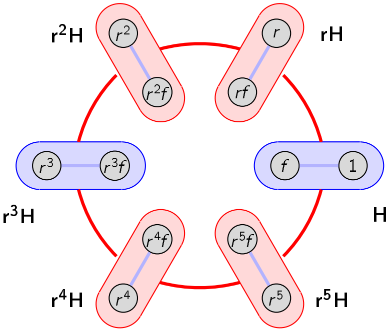
The quotient process is done by collapsing the Cayley graph by the left cosets of H.

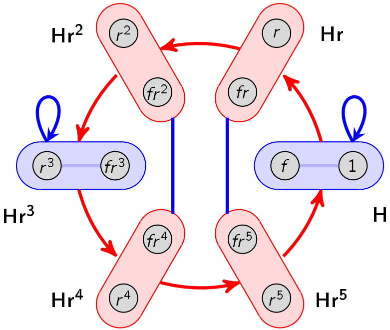
Groups acting on cosets of H by multiplication

The quotient process is done by collapsing the Cayley graph by the left cosets of H.

In contrast, this action is the result of collapsing the Cayley graph by the right cosets.



not a valid action graph



action graph of ϕ

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$$G/K \cong \mathrm{Im}(\phi) \leq S_3.$$

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Proof (contin.)

Thus, there are three cases for this quotient:

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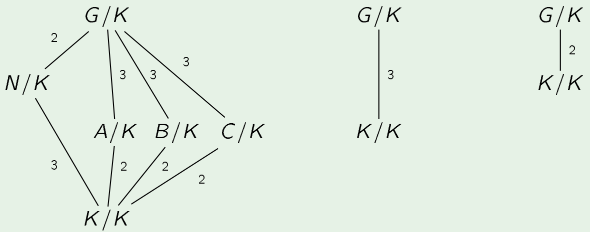

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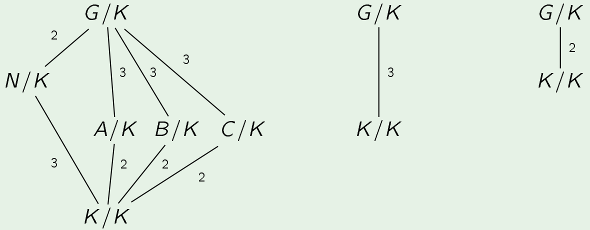

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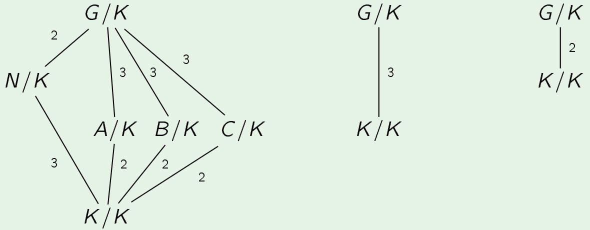

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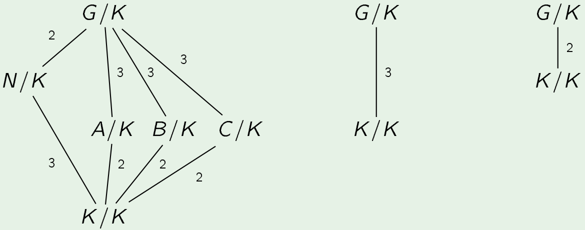

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Since G has no index-2 subgroup, only the middle case is possible (*Why?*).

This forces $K/K = H/K$, and so $K = H$ which is normal for multiple reasons. $\square$

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Suppose $H \leq G$ and $[G : H] = p$, the smallest prime dividing $|G|$. Then $H \trianglelefteq G$.

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$$\begin{tikzcd} G \arrow[d, dash, "p"] \\ H \arrow[d, dash, "q \text{ is not divisible by any prime} < p"] \\ K \end{tikzcd} \qquad \begin{tikzcd} G/K \cong S_p \arrow[d, dash, "p"] \\ H/K \arrow[d, dash, "q \text{ divides } (p-1)!"] \\ K/K \end{tikzcd}$$

Do you see why $q = 1$? □

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Thus far, we have seen four important (right) actions of a group G, acting:

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set $S =$	G		subgroups of G	right cosets of H
operation	multiplication	conjugation	conjugation	right multiplication
orb(s)	G	$\mathrm{cl}_G(g)$	$\mathrm{cl}_G(H)$	all right cosets
stab(s)	$\langle 1 \rangle$	$C_G(g)$	$N_G(H)$	$x^{-1}Hx$
fix(g)	G or $\emptyset$	$C_G(g)$	$\{H \mid g \in N_G(H)\}$	$\{Hx \mid xgx^{-1} \in H\}$
Ker(ϕ)	$\langle 1 \rangle$	$Z(G)$	$\bigcap_{H \leq G} N_G(H)$	largest norm. subgp. $N \leq H$
Fix(ϕ)	$\emptyset$	$Z(G)$	normal subgroups	none

The end!