

# Cosets!

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With many thanks to Matthew Macauley,  
`http://www.math.clemson.edu/~macaule/`

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## Goals for today:

1. Define what cosets are
2. See some examples

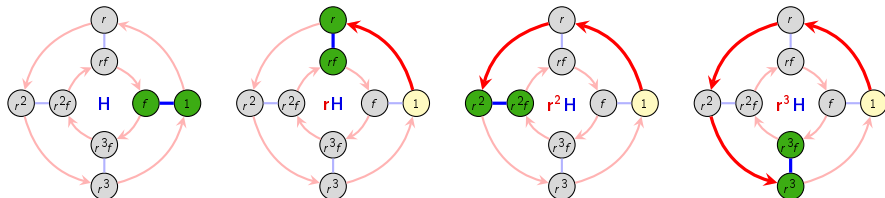
Definition time!

## Definition time! (soon)

### Vibes-based explanation (not a definition)

Let  $H \leq G$ . A coset of  $H$  is a **shifted copy** of  $H$  somewhere else in  $G$ .

Example: consider  $H = \langle f \rangle \leq D_4$ .



(Of course, only one of these is actually a subgroup; the others don't contain the identity.)

How do you shift to get from original  $H$  to each of these shifted copies?

### Left cosets, and how to find them

To find the **left** coset  $xH$  in a Cayley graph, carry out the the following steps:

1. starting from the identity, follow a path to get to  $x$  ("shift by  $x$ ")
2. from  $x$ , follow all " $H$ -paths".

# Definition time! (actually)

## Definition

If  $H \leq G$ , then a **left coset** is a set

$$xH = \{xh \mid h \in H\},$$

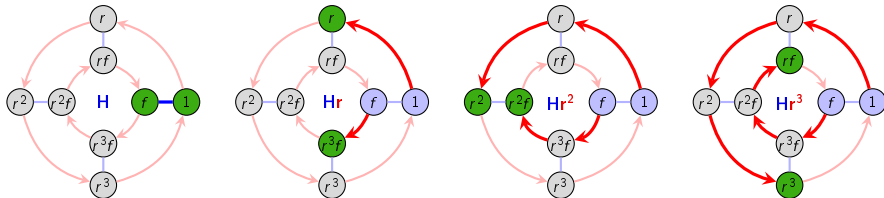
for some fixed  $x \in G$  called the **representative**. Similarly, we can define a **right coset** as

$$Hx = \{hx \mid h \in H\}.$$

## Left vs. right cosets

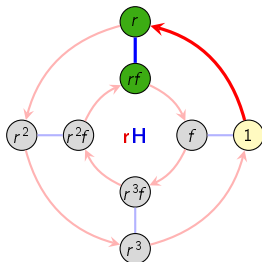
- The **left coset**  $rH$  in  $D_4$ : first **shift by  $r$** , then traverse all “ $H$ -paths”.
- The **right coset**  $Hr$  in  $D_4$ : first traverse all  $H$ -paths, then **shift by  $r$** .

Let's look at the right cosets of  $H = \langle f \rangle$  in  $D_4$ .

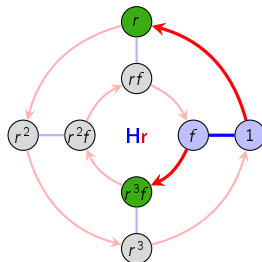


## Left vs. right cosets

- The **left coset**  $rH$  in  $D_4$ : first **go to  $r$** , then traverse all “ $H$ -paths”.
- The **right coset**  $Hr$  in  $D_4$ : first traverse all  $H$ -paths, then traverse the  $r$ -path.



$$rH = r\{1, f\} = \{r, rf\} = rf\{f, 1\} = rfH$$



$$Hr = \{1, f\}r = \{r, r^3f\} = \{f, 1\}r^3f = Hr^3f$$

Because of our convention that arrows in a Cayley graph represent **right multiplication**:

- left cosets look like copies of the subgroup,
- right cosets are usually “scattered.”

### Key point

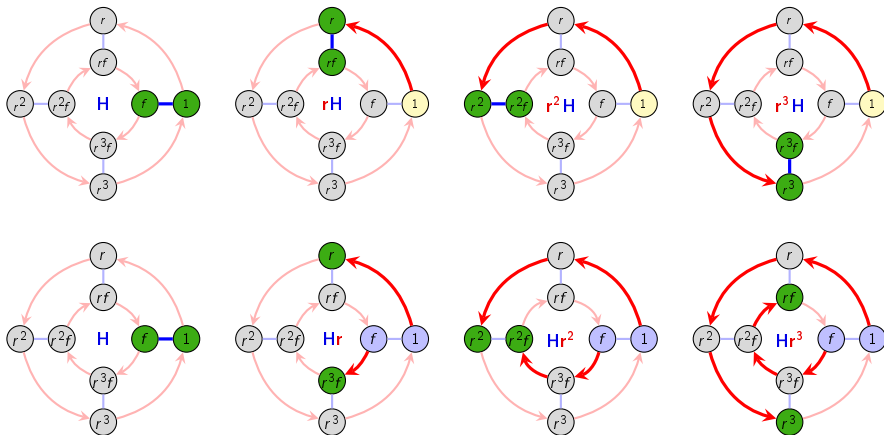
Left and right cosets are generally different.

# Overview of left and right cosets of $\langle f \rangle$

## Definition

Let  $H \leq G$ . Given  $x \in G$ , its **left coset**  $xH$  and **right coset**  $Hx$  are:

$$xH = \{xh \mid h \in H\}, \quad Hx = \{hx \mid h \in H\}.$$



Your turn!

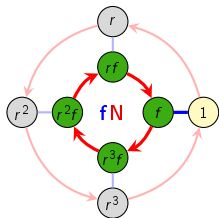


## Your turn!

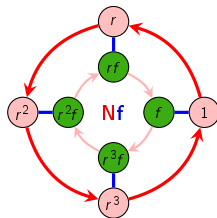
Find all the left and right cosets of a different subgroup,  $N = \langle r \rangle$ .

### Reminder: finding left vs right cosets

- Left coset  $xN$ : first **shift by  $x$** , then traverse all  $N$ -paths
- Right coset  $Nx$ : first traverse all  $N$ -paths, then **shift by  $x$**



$$fN = f\{1, r, r^2, r^3\} = \{f, fr, fr^2, fr^3\}$$



$$Nf = \{1, r, r^2, r^3\}f = \{f, rf, r^2f, r^3f\}$$

### Observations?

- There are multiple representatives for the same coset:

$$fN = (rf)N = (r^2f)N = (r^3f)N, \quad Nf = N(rf) = N(r^2f) = N(r^3f).$$

- For this subgroup, each left coset is a right coset. Such a subgroup is called **normal**.

## Your turn!

Now try:

- The other cyclic subgroups of  $D_4$
- $K = \langle r^2, f \rangle \leq D_4$

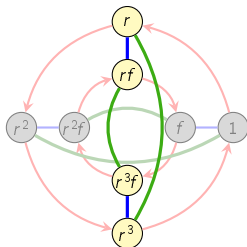
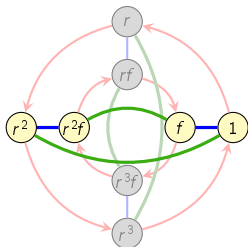
### Observations?

- Cosets never overlap
- All the cosets are always the same size as the original subgroup
- Cosets always cover the whole group
- Any time there are just two cosets, each left coset is a right coset

# Equality of sets vs. equality of elements

## Caveat!

An equality of cosets  $xK = Kx$  as sets *does not* imply (or require) an equality of elements  $xk = kx$ .



$rK$	$r$	$r^3$	$rf$	$r^3f$
$K$	$1$	$r^2$	$f$	$r^2f$

$Kr$	$r$	$r^3$	$fr$	$fr^3$
$K$	$1$	$r^2$	$f$	$fr^2$

$rK$  and  $Kr$  are the same **as sets**, even though the elements occur in a different order.

## Properties of cosets!

## Basic properties of cosets

The following results are “visually clear” from the Cayley graphs, but let’s now prove them:

### Proposition

Each (left) coset can have multiple representatives: if  $b \in aH$ , then  $aH = bH$ .

### Proof

Since  $b \in aH$ , we can write  $b = ah$ , for some  $h \in H$ . That is,  $h = a^{-1}b$  and  $a = bh^{-1}$ .

To show that  $aH = bH$ , we need to verify both  $aH \subseteq bH$  and  $aH \supseteq bH$ .

“ $\subseteq$ ”: Take  $ah_1 \in aH$ . We need to write it as  $bh_2$ , for some  $h_2 \in H$ . By substitution,

$$ah_1 = (bh^{-1})h_1 = b(h^{-1}h_1) \in bH.$$

“ $\supseteq$ ”: Pick  $bh_3 \in bH$ . We need to write it as  $ah_4$  for some  $h_4 \in H$ . By substitution,

$$bh_3 = (ah)h_3 = a(hh_3) \in aH.$$

Therefore,  $aH = bH$ , as claimed. □

### Corollary (boring but useful)

The equality  $xH = H$  holds if and only if  $x \in H$ . (And analogously, for  $Hx = H$ .)

# Basic properties of cosets

## Proposition

For any subgroup  $H \leq G$ , the (left) cosets of  $H$  **partition** the group  $G$ : every element  $g \in G$  lives in **exactly one** coset of  $H$ .

## Proof

We know that the element  $g \in G$  lies in a (left) coset of  $H$ , namely  $gH$ . Uniqueness follows because if  $g \in kH$ , then  $gH = kH$ .  $\square$

## Proposition

All (left) cosets of  $H \leq G$  have the same size.  $\square$

## Proof

It suffices to show that  $|xH| = |H|$ , for any  $x \in H$ .

Define a map

$$\phi: H \longrightarrow xH, \quad h \longmapsto xh.$$

It is elementary to show that this is a bijection.  $\square$

Lagrange's theorem!

# Lagrange's theorem

## Remark

For any subgroup  $H \leq G$ , the left cosets of  $H$  partition  $G$  into subsets of equal size.

The right cosets also partition  $G$  into subsets of equal size, but *they may be different*.

Let's compare these two partitions for the subgroup  $H = \langle f \rangle$  of  $G = D_4$ .

H	$r^2H$	$rH$	$r^3H$
$f$	$r^2f$	$rf$	$r^3$
1	$r^2$	$r$	$r^3f$

H	$Hr^2$			
$f$	$fr^2$	$fr^3$	$r^3$	$Hr^3$
1	$r^2$	$r$	$fr$	$Hr$

## Definition

The **index** of a subgroup  $H$  of  $G$ , written  $[G : H]$ , is the number of distinct left (or equivalently, right) cosets of  $H$  in  $G$ .

## Lagrange's theorem

If  $H$  is a subgroup of finite group  $G$ , then  $|G| = [G : H] \cdot |H|$ .

□



## Funny historical aside

Guess who proved Lagrange's theorem. **Not Lagrange!**

- Lagrange, 1771: if a polynomial in  $n$  variables has its variables permuted in all  $n!$  ways, the number of different polynomials that are obtained is always a factor of  $n!$ .
  - What does this have to do with cosets?
  - Take  $H \leq S_n$  to be the set of permutations that **fix** the polynomial.  $n! = |S_n| = [S_n : H] \cdot |H|$ .
  - The number of different polynomials is the number of cosets of  $H$ , aka the index  $[S_n : H]$ .
  - So: true for **special subgroups** of  $S_n$ .
- Gauss, 1801: the **special case** of subgroups of  $(\mathbb{Z}/p\mathbb{Z})^*$
- Cauchy, 1844: true for **any** subgroup  $H \leq S_n$
- Jordan, 1861: true for any subgroup  $H$  of a **permutation group**
  - (so, now it's true for subgroups of **subgroups of  $S_n$** )
- Cayley, 1854: every group is a permutation group

Importantly: Lagrange would not have had the words “index,” “coset,” “group.”

# The tower law

## Proposition

Let  $G$  be a finite group and  $K \leq H \leq G$  be a chain of subgroups. Then

$$[G : K] = [G : H][H : K].$$

Here is a “proof by picture”:

$[G : H] = \#$  of cosets of  $H$  in  $G$

$[H : K] = \#$  of cosets of  $K$  in  $H$

$[G : K] = \#$  of cosets of  $K$  in  $G$

<b>zH</b>	$z_1 K$	$z_2 K$	$z_3 K$	$\dots$	$z_n K$
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
<b>aH</b>	$a_1 K$	$a_2 K$	$a_3 K$	$\dots$	$a_n K$
<b>H</b>	$K$	$h_2 K$	$h_3 K$	$\dots$	$h_n K$

## Proof

By Lagrange's theorem,

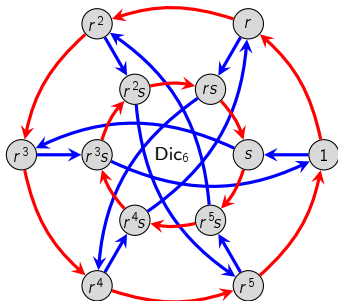
$$[G : H][H : K] = \frac{|G|}{|H|} \cdot \frac{|H|}{|K|} = \frac{|G|}{|K|} = [G : K].$$

□

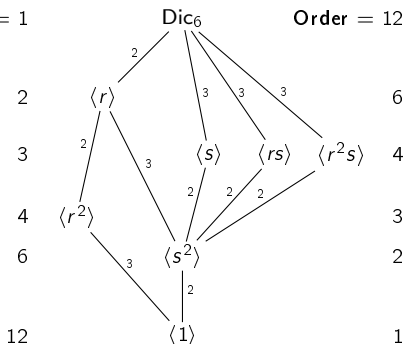
## The tower law

Another way to visualize the tower law involves subgroup lattices.

It is often helpful to label the edge from  $H$  to  $K$  in a subgroup lattice with the index  $[H : K]$ .



Index = 1



## The tower law and subgroup lattices

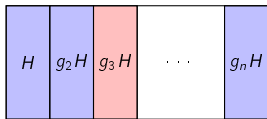
For any two subgroups  $K \leq H$  of  $G$ , the index of  $K$  in  $H$  is just the *products of the edge labels* of any path from  $H$  to  $K$ .

Normal subgroups!

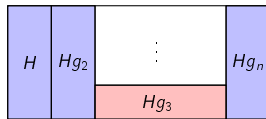
## Normal subgroups and normalizers

Given a subgroup  $H$  of  $G$ , it is natural to ask the following question:

*“How many left cosets of  $H$  are right cosets?”*



Partition of  $G$  by the  
left cosets of  $H$



Partition of  $G$  by the  
right cosets of  $H$

- “Best case” scenario: all of them
- “Worst case” scenario: only  $H$
- In general: somewhere between these two extremes

### Definition

A subgroup  $H$  is a **normal subgroup** of  $G$  if  $gH = Hg$  for all  $g \in G$ . We write  $H \trianglelefteq G$ .

The **normalizer** of  $H$ , denoted  $N_G(H)$ , is the set of elements  $g \in G$  such that  $gH = Hg$ :

$$N_G(H) = \{g \in G \mid gH = Hg\},$$

i.e., the **union of left cosets that are also right cosets**.

## Examples of normal subgroups

We've seen cases where we know a subgroup will be normal without having to check.

1. The subgroup  $H = G$  is always normal in  $G$ . The only left coset is also the only right coset:

$$eG = G = Ge.$$

2. The subgroup  $H = \{e\}$  is always normal. The left and right cosets are singleton sets:

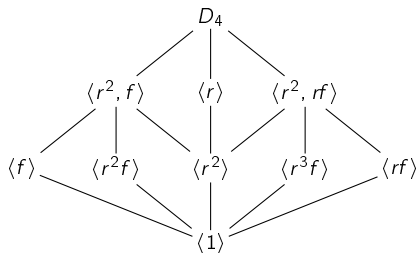
$$gH = \{g\} = Hg.$$

3. Subgroups  $H$  of index 2 are normal. The two cosets (left or right) are  $H$  and  $G - H$ .
4. Subgroups of *abelian groups* are always normal, because for any  $H \leq G$ ,

$$aH = \{ah \mid h \in H\} = \{ha \mid h \in H\} = Ha.$$

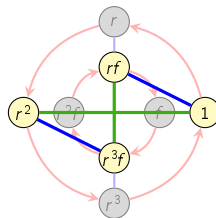
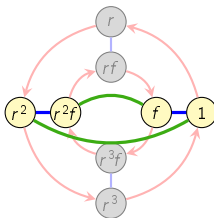
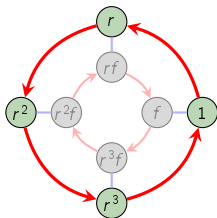
5. The center  $Z(G)$  is always normal, for the same reason as above.
6. Relatedly, any subgroup of  $Z(G)$  is always normal.

# Normal subgroups in $D_4$



From our explorations, we found:

- $\langle r \rangle \triangleleft D_4$  (because it has index 2!)
- $\langle r^2, f \rangle \triangleleft D_4$  (index 2!)
- $\langle r^2, rf \rangle \triangleleft D_4$  (index 2!)
- $\langle r^2 \rangle \triangleleft D_4$  (because it is  $Z(D_4)$ !)



The end!