Isomorphism theorems!

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With many thanks to Matthew Macauley,
http://www.math.clemson.edu/~macaule/

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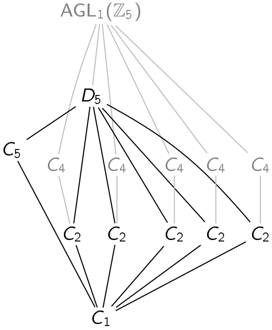
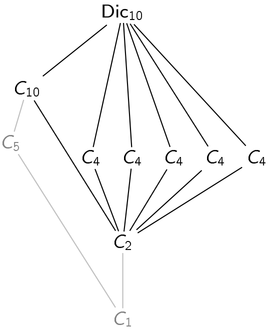
Preview: embeddings vs. quotients

The difference between embeddings and quotient maps can be seen in the subgroup lattice:





In one of these groups, D_5 is a subgroup, and it rises up from the floor.

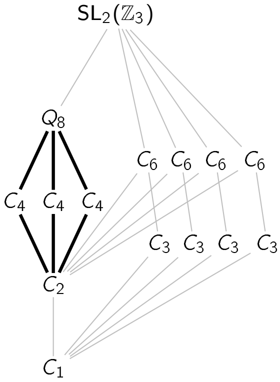
In the other, it arises as a quotient, and it descends from the ceiling.

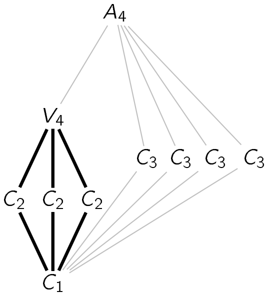
This, and much more, will be consequences of the celebrated **isomorphism theorems**.

Preview: subgroups, quotients, and subquotients

Often, we'll see familiar subgroup lattices in the middle of a larger lattice.

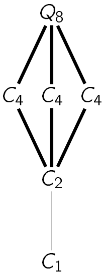
These are called **subquotients**.





subgroup of a quotient

quotient of a subgroup



The *isomorphism theorems* relates the structure of a group to that of its quotients and subquotients.

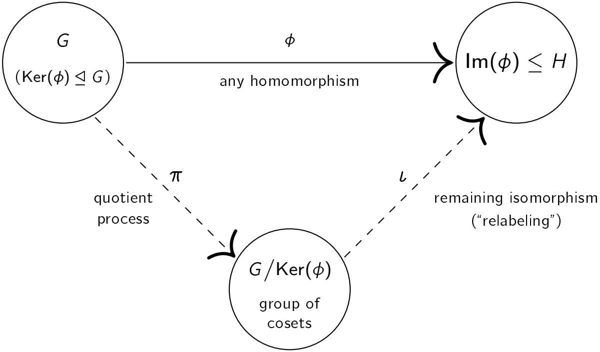
Every homomorphism image is a quotient

The following is one of the central results in group theory.

Fundamental homomorphism theorem (FHT)

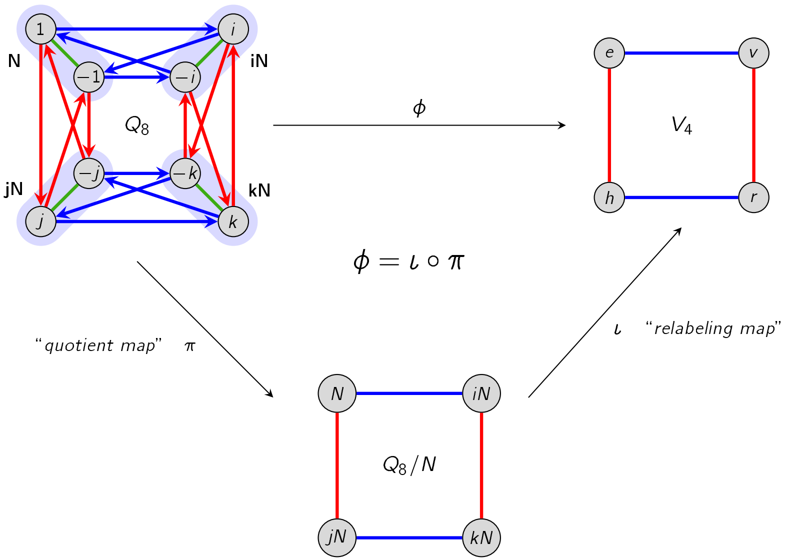
If $\phi\colon G \to H$ is a homomorphism, then $\mathrm{Im}(\phi) \cong G/\mathrm{Ker}(\phi)$.

The FHT says that every homomorphism can be decomposed into two steps: (i) quotient out by the kernel, and then (ii) relabel the nodes via ϕ.



Visualizing the FHT via Cayley graphs

(This is HW 8.14.)



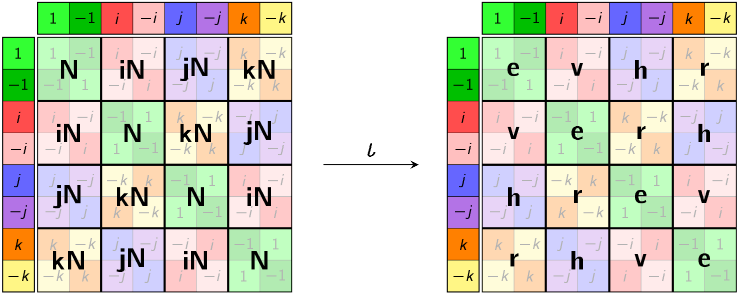
Visualizing the FHT via Cayley tables

Here's another way to think about the homomorphism

$$\phi\colon Q_8 \twoheadrightarrow V_4, \qquad \phi(i) = v, \;\; \phi(j) = h$$

as the composition of:

- a quotient by $N = \mathrm{Ker}(\phi) = \langle -1 \rangle = \{\pm 1\}$,
- a *relabeling map* $\iota\colon Q_8/N \to V_4$.



FHT preliminaries

Proposition (HW 8.9)

The kernel of any homomorphism $\phi\colon G \to H$, is a normal subgroup.

Proof

Let $N := \mathrm{Ker}(\phi)$. First, we'll show that it's a subgroup. Take any $a, b \in N$.

Identity: $\phi(e) = e$. ✓

Closure: $\phi(ab) = \phi(a)\,\phi(b) = e \cdot e = e$. ✓

Inverse: $\phi(a^{-1}) = \phi(a)^{-1} = e^{-1} = e$. ✓

Now we'll show it's normal. Take any $n \in N$. We'll show that $gng^{-1} \in N$ for all $g \in G$.

By the homomorphism property,

$$\phi(gng^{-1}) = \phi(g)\,\phi(n)\,\phi(g^{-1}) = \phi(g) \cdot e \cdot \phi(g)^{-1} = e.$$

Therefore, $gng^{-1} \in \mathrm{Ker}(\phi)$. $\square$

Key observation

Given any homomorphism $\phi\colon G \to H$, we can *always* form the quotient group $G/\,\mathrm{Ker}(\phi)$.

FHT preliminaries

Proposition (HW 8.10)

Let $\phi\colon G \to H$ be a homomorphism. Then each preimage $\phi^{-1}(h)$ is a coset of $\mathrm{Ker}(\phi)$.

Proof

Let $N = \mathrm{Ker}(\phi)$ and take any $g \in \phi^{-1}(h)$. (This means $\phi(g) = h$.)

We claim that $\phi^{-1}(h) = gN$. We need to verify both $\subseteq$ and $\supseteq$.

"$\subseteq$": Take $a \in \phi^{-1}(h)$, i.e., $\phi(a) = h$. We need to show that $a \in gN$.

From basic properties of cosets, we have the equivalences

$$a \in gN \iff aN = gN \iff g^{-1}aN = N \iff g^{-1}a \in N.$$

This last condition is true because

$$\phi(g^{-1}a) = \phi(g)^{-1}\phi(a) = h^{-1} \cdot h = 1_H. \qquad \checkmark$$

"$\supseteq$": Pick any $gn \in gN$. This is in $\phi^{-1}(h)$ because

$$\phi(gn) = \phi(g)\phi(n) = h \cdot 1_H = h. \qquad \checkmark$$

Proof of the FHT

Fundamental homomorphism theorem

If $\phi\colon G \to H$ is a homomorphism, then $\mathsf{Im}(\phi) \cong G/\,\mathsf{Ker}(\phi)$.

Proof

We'll construct an explicit map $\iota\colon G/\,\mathsf{Ker}(\phi) \longrightarrow \mathsf{Im}(\phi)$ and prove that it's an isomorphism.

Let $N = \mathsf{Ker}(\phi)$, and recall that $G/N = \{gN \mid g \in G\}$. Define

$$\iota\colon G/N \longrightarrow \mathsf{Im}(\phi)\,, \qquad \iota\colon gN \longmapsto \phi(g)\,.$$

• <u>*Show ι is well-defined*</u>: We must show that if $aN = bN$, then $\iota(aN) = \iota(bN)$.

Suppose $aN = bN$. We have

$$aN = bN \quad\Longrightarrow\quad b^{-1}aN = N \quad\Longrightarrow\quad b^{-1}a \in N\,.$$

By definition of $b^{-1}a \in \mathsf{Ker}(\phi)$,

$$1_H = \phi(b^{-1}a) = \phi(b^{-1})\,\phi(a) = \phi(b)^{-1}\,\phi(a) \quad\Longrightarrow\quad \phi(a) = \phi(b)\,.$$

By definition of ι: $\iota(aN) = \phi(a) = \phi(b) = \iota(bN)$. ✓

Proof of FHT (cont.) [Recall: $\iota\colon G/N \to \mathbf{Im}(\phi)$, $\iota\colon gN \mapsto \phi(g)$]

Proof (cont.)

- *Show ι is a homomorphism*: We must show that $\iota(aN \cdot bN) = \iota(aN)\,\iota(bN)$.

$$\begin{aligned}
\iota(aN \cdot bN) &= \iota(abN) && (aN \cdot bN := abN) \\
&= \phi(ab) && (\text{definition of } \iota) \\
&= \phi(a)\,\phi(b) && (\phi \text{ is a homomorphism}) \\
&= \iota(aN)\,\iota(bN) && (\text{definition of } \iota)
\end{aligned}$$

Thus, ι is a homomorphism. ✓

- *Show ι is surjective (onto)*:

Take any element in the codomain (here, $\mathbf{Im}(\phi)$). We need to find an element in the domain (here, G/N) that gets mapped to it by ι.

Pick any $\phi(a) \in \mathbf{Im}(\phi)$. By defintion, $\iota(aN) = \phi(a)$, hence ι is surjective. ✓

Proof (cont.)

- *Show ι is injective (1–1)*: We must show that $\iota(aN) = \iota(bN)$ implies $aN = bN$.

Suppose that $\iota(aN) = \iota(bN)$. Then

$$
\begin{aligned}
\iota(aN) = \iota(bN) \quad &\Longrightarrow \quad \phi(a) = \phi(b) && \text{(by definition)} \\
&\Longrightarrow \quad \phi(b)^{-1}\,\phi(a) = 1_H && \\
&\Longrightarrow \quad \phi(b^{-1}a) = 1_H && (\phi \text{ is a homom.)} \\
&\Longrightarrow \quad b^{-1}a \in N && (\text{definition of } \mathrm{Ker}(\phi)) \\
&\Longrightarrow \quad b^{-1}aN = N && (aH = H \;\Leftrightarrow\; a \in H) \\
&\Longrightarrow \quad aN = bN &&
\end{aligned}
$$

Thus, ι is injective. ✓

In summary, since $\iota\colon G/N \to \mathrm{Im}(\phi)$ is a well-defined homomorphism that is injective (1–1) and surjective (onto), it is an **isomorphism**.

Therefore, $G/N \cong \mathrm{Im}(\phi)$, and the FHT is proven. □

Consequences of the FHT

Corollary

If $\phi\colon G \to H$ is a homomorphism, then $\operatorname{Im}\phi \leq H$.

The two "extreme cases"

- If $\phi\colon G \hookrightarrow H$ is an embedding, then $\operatorname{Ker}(\phi) = \{1_G\}$. The FHT says that
$$\operatorname{Im}(\phi) \cong G/\{1_G\} \cong G.$$
- If $\phi\colon G \to H$ is the trivial map $\phi(g) = 1_H$ for all $h \in G$, then $\operatorname{Ker}(\phi) = G$. The FHT says that
$$\{1_H\} = \operatorname{Im}(\phi) \cong G/G.$$

Let's use the FHT to determine all homomorphisms $\phi\colon C_4 \to C_3$.

- By the FHT, $G/\operatorname{Ker}\phi \cong \operatorname{Im}\phi \leq C_3$, and so $|\operatorname{Im}\phi| = 1$ or 3.
- Since $\operatorname{Ker}\phi \leq C_4$, Lagrange's Theorem also tells us that $|\operatorname{Ker}\phi| \in \{1, 2, 4\}$, and hence $|\operatorname{Im}\phi| = |G/\operatorname{Ker}\phi| \in \{1, 2, 4\}$.

Thus, $|\operatorname{Im}\phi| = 1$, and so the *only* homomorphism $\phi\colon C_4 \to C_3$ is the trivial one.

Consequences of the FHT

Let's do a more complicated example: find all homomorphisms $\phi\colon \mathbb{Z}_{44} \to \mathbb{Z}_{16}$.

By the FHT,

$$\mathbb{Z}_{44}/\operatorname{Ker}(\phi) \cong \operatorname{Im}(\phi) \leq \mathbb{Z}_{16}.$$

This means that $44/|\operatorname{Ker}(\phi)|$ must be 1, 2, 4, ~~8~~, or ~~16~~.

Also, $|\operatorname{Ker}(\phi)|$ must divide 44. We are left with three cases: $|\operatorname{Ker}(\phi)| = 44$, 22, or 11.

Reminder

For each $d \mid n$, the group $\mathbb{Z}_n$ has a unique subgroup of order d, which is $\langle n/d \rangle$.

- **Case 1**: $|\operatorname{Ker}(\phi)| = 44$, which forces $|\operatorname{Im}(\phi)| = 1$, and so $\phi(1) = 0$ is the trivial homomorphism.
- **Case 2**: $|\operatorname{Ker}(\phi)| = 22$. By the FHT, $|\operatorname{Im}(\phi)| = 2$, which means $\operatorname{Im}(\phi) = \{0, 8\}$, and so $\phi(1) = 8$.
- **Case 3**: $|\operatorname{Ker}(\phi)| = 11$. By the FHT, $|\operatorname{Im}(\phi)| = 4$, which means $\operatorname{Im}(\phi) = \{0, 4, 8, 12\}$.

  There are two subcases: $\phi(1) = 4$ or $\phi(1) = 12$.

What does "well-defined" really mean?

Recall that we've seen the term "**well-defined**" arise in different contexts:

- a well-defined binary operation on a set G/N of cosets,
- a well-defined function $\iota\colon G/N \to H$ from a set (group) of cosets.

In both of these cases, well-defined means that:

"*our definition doesn't depend on our choice of coset representative.*"

Formally:

- If $N \trianglelefteq G$, then $aN \cdot bN := abN$ is a well-defined binary operation on the set G/N of cosets, because

$$\text{if } \ a_1N = a_2N \ \text{ and } \ b_1N = b_2N, \ \text{ then } \ a_1b_1N = a_2b_2N.$$

- The map $\iota\colon G/N \to H$, where $\iota(aN) = \phi(a)$, is a well-defined homomorphism, meaning that

$$\text{if } \ aN = bN, \ \text{ then } \ \iota(aN) = \iota(bN) \ \text{ (that is, } \phi(a) = \phi(b)\text{) holds.}$$

Remark

Whenever we define a map and the domain is a *quotient*, we must show it's well-defined.

What does "well-defined" really mean?

In some sense, well-defined and injective are "dual" concepts:

- f is well-defined if the same input cannot map to different outputs
- (that is, f is a function!)
- f is injective if different inputs cannot map to the same output.

$$\begin{tikzcd} aN \arrow[dr, mapsto] & \\ & f(aN)=f(bN) \\ bN \arrow[ur, mapsto] & \end{tikzcd}$$

not allowed if 1-to-1

$$\begin{tikzcd} & f(aN) \\ aH=bN \arrow[ur, mapsto] \arrow[dr, mapsto] & \\ & f(b(N)) \end{tikzcd}$$

not allowed if well-defined

Let's revisit the proof of the FHT, and the map

$$\iota \colon G/N \to H, \qquad \iota(aN) = \phi(a), \qquad \text{where } \; N = \mathrm{Ker}(\phi).$$

Showing ι is well-defined is done as follows:

$$aN=bN \;\Rightarrow\; b^{-1}aN=N \;\Rightarrow\; b^{-1}a \in N \;\Rightarrow\; \phi(b^{-1}a)=1 \;\Rightarrow\; \phi(a)=\phi(b) \;\Rightarrow\; \iota(aN)=\iota(bN).$$

Reversing each $\Rightarrow$ shows ι is 1-to-1.

How to show two groups are isomorphic

The standard way to show $G \cong H$ is to construct an isomorphism $\phi\colon G \to H$.

When the domain is a quotient, there is another method, due to the FHT.

Useful technique

Suppose we want to show that $G/N \cong H$. There are two approaches:

(i) Define a map $\phi\colon G/N \to H$ and prove that it is well-defined, a homomorphism, and a bijection.

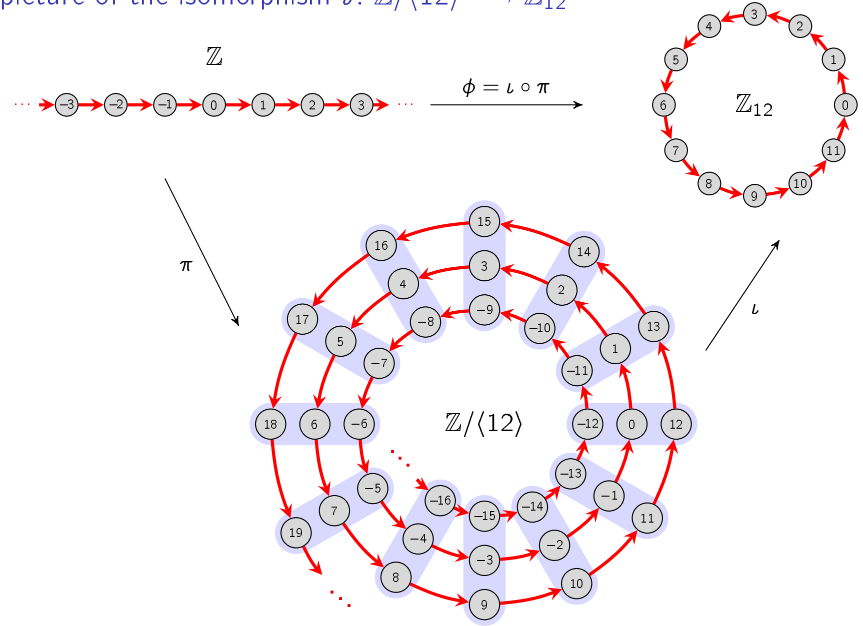
(ii) Define a map $\phi\colon G \to H$ and prove that it is a homomorphism, a surjection (onto), and that $\operatorname{Ker}\phi = N$.

Usually, Method (ii) is easier. Showing well-definedness and injectivity can be tricky.

For example, Method (ii) works quite well in showing the following:

- $\mathbb{Z}/\langle n\rangle \cong \mathbb{Z}_n$;
- $\mathbb{Q}^*/\langle -1\rangle \cong \mathbb{Q}^+$;
- $AB/B \cong A/(A \cap B)$
- $G/(A \cap B) \cong (G/A) \times (G/B) \quad$ (if $G = AB$).

A picture of the isomorphism $\iota\colon \mathbb{Z}/\langle 12\rangle \longrightarrow \mathbb{Z}_{12}$



The Isomorphism Theorems

The fundamental homomorphism theorem (FHT), or Noether's isomorphism theorem, is the first of four basic theorems about homomorphisms and their structure.

These are commonly called "The Isomorphism Theorems."

- Fundamental homomorphism theorem: "*All homomorphic images are quotients*"
- Correspondence theorem or lattice theorem: Characterizes "*subgroups of quotients*"
- Fraction theorem: Characterizes "*quotients of quotients*"
- Diamond theorem: "*Duality of subquotients.*"

These all have analogues for other algebraic structures, e.g., rings, vector spaces, modules, Lie algebras.

All of these theorems can look messy and unmotivated algebraically.

However, they all have beautiful visual interpretations, especially involving subgroup lattices.

The correspondence theorem: subgroups of quotients

Given $N \trianglelefteq G$, the quotient G/N has a group structure, via $aN \cdot bN = abN$.

Moreover, by the FHT theorem, *every* homomorphism image is a quotient.

Natural question

What are the subgroups of a quotient?

Fortunately, this has a simple answer that is easy to remember.

Correspondence theorem (informal)

The subgroups of the quotient G/N are quotients of the subgroups $H \leq G$ that contain N.

Moreover, "most properties" of $H/N \leq G/N$ are inherited from $H \leq G$.

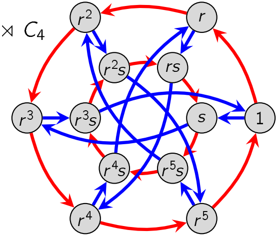
This is best understood by interpreting the subgroup lattices of G and G/N.

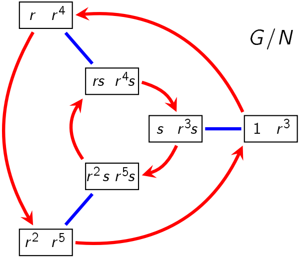
Let's do some examples for intuition, and then state the correspondence theorem formally.

The correspondence theorem: subgroups of quotients

Compare $G = C_3 \rtimes C_4$ with the quotient by $N = \langle r^3 \rangle$. (This is HW 7.7.)

$G = C_3 \rtimes C_4$





We know the subgroups structure of $G/N = \{N,\ rN,\ r^2N,\ sN,\ rsN,\ r^2sN\} \cong D_3$.

"*The subgroups of the quotient G/N are the quotients of the subgroups that contain N.*"

"*shoes out of the box*"

r^2	r^5	r^2s	r^5s
r	r^4	rs	r^4s
1	r^3	s	r^3s

$\langle r \rangle \leq G$

"*shoeboxes; lids off*"

r^2	r^5	r^2s	r^5s
r	r^4	rs	r^4s
1	r^3	s	r^3s

$\langle r \rangle / N \leq G/N$

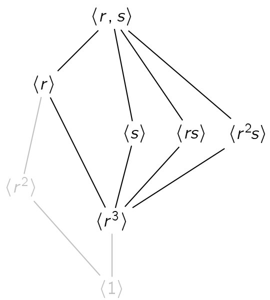
"*shoeboxes; lids on*"

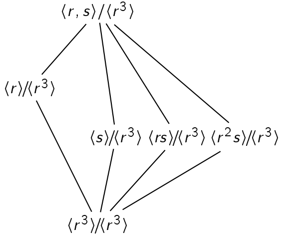
r^2N	r^2sN
rN	rsN
N	sN

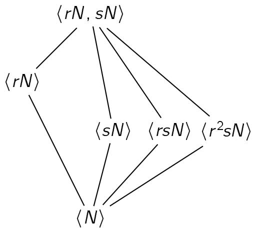
$\langle rN \rangle \leq G/N$

The correspondence theorem: subgroups of quotients

Here is the subgroup lattice of $G = C_3 \rtimes C_4$, and of the quotient G/N, where $N = \langle r^3 \rangle$.







"*The subgroups of the quotient G/N are the quotients of the subgroups that contain N.*"

"*shoes out of the box*"

r^2	r^5	$r^2 s$	$r^5 s$
r	r^4	rs	$r^4 s$
1	r^3	s	$r^3 s$

$\langle s\rangle \leq G$

"*shoeboxes; lids off*"

r^2	r^5	$r^2 s$	$r^5 s$
r	r^4	rs	$r^4 s$
1	r^3	s	$r^3 s$

$\langle s\rangle / N \leq G/N$

"*shoeboxes; lids on*"

$r^2 N$	$r^2 sN$
rN	rsN
N	sN

$\langle sN\rangle \leq G/N$

The correspondence theorem: subgroups of quotients

Correspondence theorem (informally)

There is a bijection between subgroups of G/N and subgroups of G that contain N.

"Everything that we want to be true" about the subgroup lattice of G/N is inherited from the subgroup lattice of G.

Most of these can be summarized as:

"The __________ of the quotient is just the quotient of the __________"

Correspondence theorem (formally)

Let $N \leq H \leq G$ and $N \leq K \leq G$ be chains of subgroups and $N \trianglelefteq G$. Then

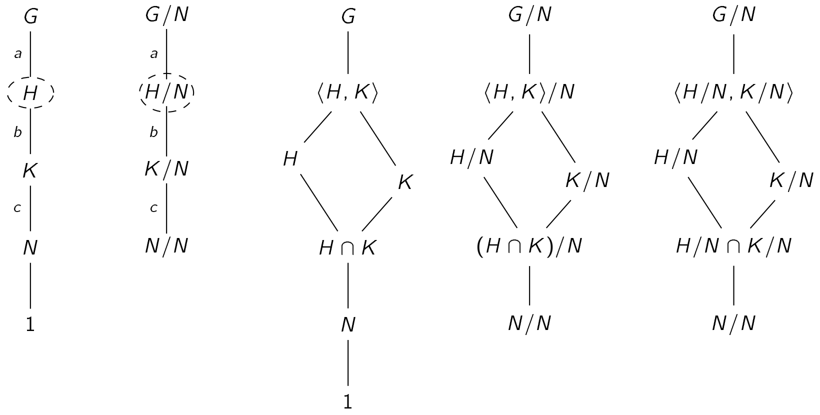
1. Subgroups of the quotient G/N are quotients of the subgroup $H \leq G$ that contain N.
2. $H/N \trianglelefteq G/N$ if and only if $H \trianglelefteq G$
3. $[G/N : H/N] = [G : H]$
4. $H/N \cap K/N = (H \cap K)/N$
5. $\langle H/N, K/N \rangle = \langle H, K \rangle / N$
6. H/N is conjugate to K/N in G/N iff H is conjugate to K in G.

The correspondence theorem: subgroups of quotients

All parts of the correspondence theorem have nice subgroup lattice interpretations.

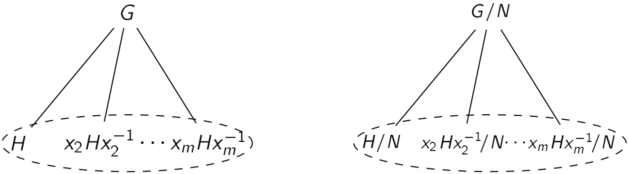
We've already interpreted the the first part.

Here's what the next four parts say.

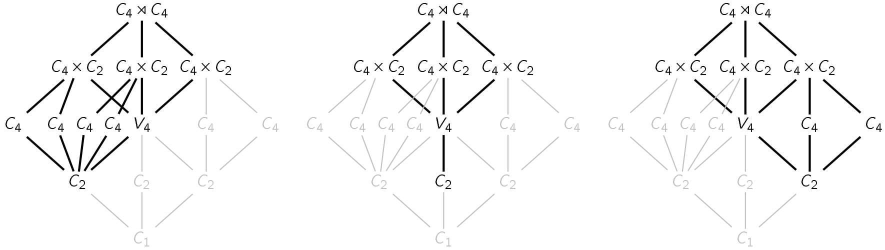


The correspondence theorem: subgroups of quotients

The last part says that we can characterize the conjugacy classes of G/N from those of G.



Let's apply that to find the conjugacy classes of $C_4 \rtimes C_4$ by inspection alone.



The correspondence theorem: subgroups of quotients

Let's prove the first (main) part of the correspondence theorem.

Correspondence theorem (first part)

The subgroups of the quotient G/N are quotients of the subgroup $H \leq G$ that contain N.

Proof

Let S be a subgroup of G/N. Then S is a collection of cosets, i.e.,

$$S = \{hN \mid h \in H\},$$

for some subset $H \subseteq G$. We just need to show that H is a subgroup.

We'll use the one-step subgroup test: take h_1N, $h_2N \in S$. Then S must also contain

$$(h_1N)(h_2N)^{-1} = (h_1N)(h_2^{-1}N) = (h_1h_2^{-1})N. \tag{1}$$

That is, $h_1h_2^{-1} \in H$, which means that H is a subgroup. ✓

Conversely, suppose that $N \leq H \leq G$. The one-step subgroup test shows that $H/N \leq G/N$; see Eq. (1). □

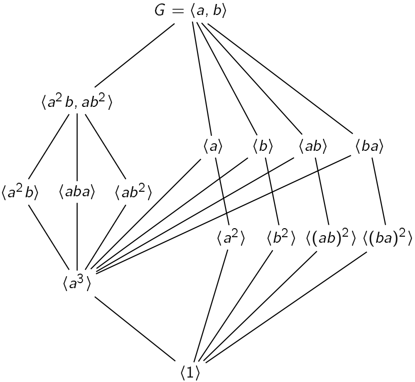
The other parts are straightforward and will be left as exercises.

The "subgroup" and "quotient" operations commute

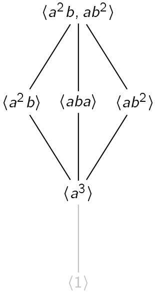
Key idea

The quotient of a subgroup is just the subgroup of the quotient.

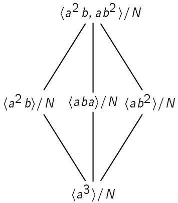
Example: Consider the group $G = \mathrm{SL}_2(\mathbb{Z}_3)$.



subgroup $\mathbf{H \cong Q_8}$



$\mathbf{H/N \cong V_4}$



"*quotient of the subgroup*"

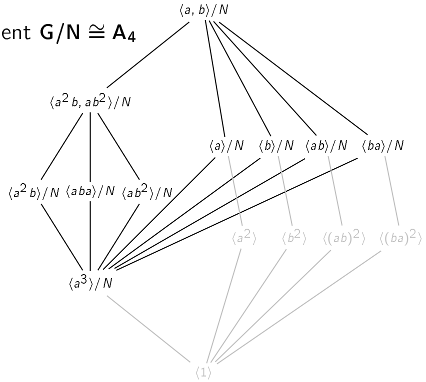
The “subgroup” and “quotient” operations commute

Key idea

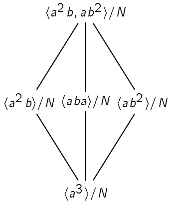
The quotient of a subgroup is just the subgroup of the quotient.

Example: Consider the group $G = \mathrm{SL}_2(\mathbb{Z}_3)$.

quotient $\mathbf{G/N \cong A_4}$



$\mathbf{V_4 \cong H/N \leq G/N}$



“*subgroup of the quotient*”

The fraction theorem: quotients of quotients

The correspondence theorem characterizes the subgroup structure of the quotient G/N.

Every subgroup of G/N is of the form H/N, where $N \leq H \leq G$.

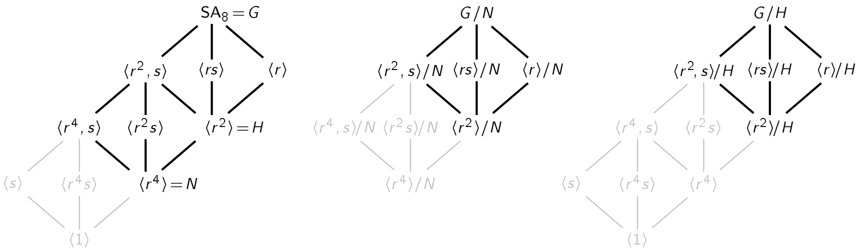
Moreover, if $H \trianglelefteq G$, then $H/N \trianglelefteq G/N$. In this case, we can ask:

What is the quotient group $(G/N)/(H/N)$ isomorphic to?

Fraction theorem

Given a chain $N \leq H \leq G$ of normal subgroups of G,

$$(G/N)/(H/N) \cong G/H.$$

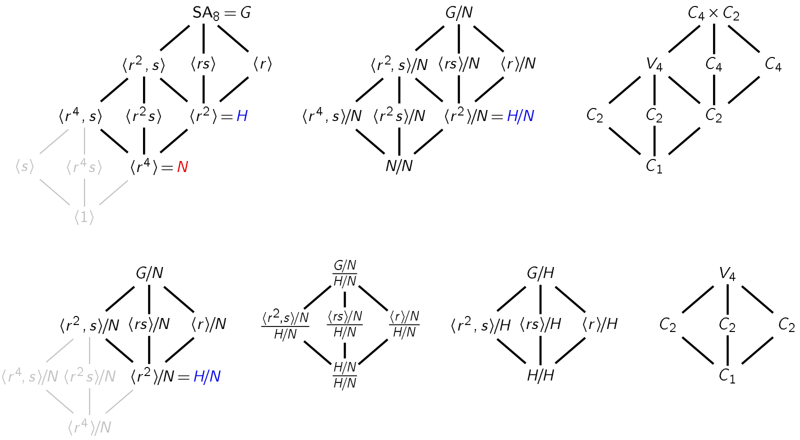


The fraction theorem: quotients of quotients

Fraction theorem

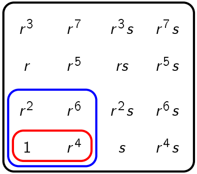
Given a chain $N \leq H \leq G$ of normal subgroups of G,

$$(G/N)/(H/N) \cong G/H.$$

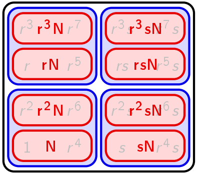


The fraction theorem: quotients of quotients

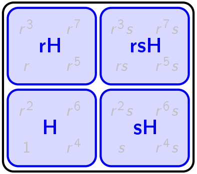
Let's continue our example of the semiabelian group $G = \mathrm{SA}_8 = \langle r, s \rangle$.



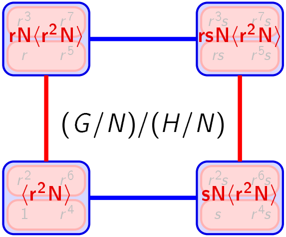
$N \leq H \leq G$

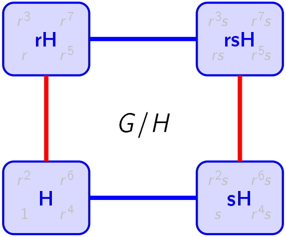


$G/N = \langle rN, sN \rangle \cong C_4 \times C_2$
$H/N = \langle r^2N \rangle = \{N, r^2N\} \cong C_2$



$G/H = \langle rH, sH \rangle \cong V_4$
$(G/N)/(H/N) \cong G/H$





The fraction theorem: quotients of quotients

Fraction theorem

Given a chain $N \leq H \leq G$ of normal subgroups of G,

$$(G/N)/(H/N) \cong G/H.$$

Proof

This is tailor-made for the FHT. Define the map

$$\phi\colon G/N \longrightarrow G/H, \qquad \phi\colon gN \longmapsto gH.$$

- *Show ϕ is well-defined*: Suppose $g_1 N = g_2 N$. Then $g_1 = g_2 n$ for some $n \in N$. But $n \in H$ because $N \leq H$. Thus, $g_1 H = g_2 H$, i.e., $\phi(g_1 N) = \phi(g_2 N)$. ✓
- *ϕ is clearly onto and a homomorphism*. ✓
- *Apply the FHT*:

$$\begin{aligned} \mathrm{Ker}(\phi) &= \{gN \in G/N \mid \phi(gN) = H\} \\ &= \{gN \in G/N \mid gH = H\} \\ &= \{gN \in G/N \mid g \in H\} = H/N \end{aligned}$$

By the FHT, $(G/N)/\,\mathrm{Ker}(\phi) = (G/N)/(H/N) \cong \mathrm{Im}(\phi) = G/H$. □

The fraction theorem: quotients of quotients

For another visualization, consider $G = \mathbb{Z}_6 \times \mathbb{Z}_4$ and write elements as strings.

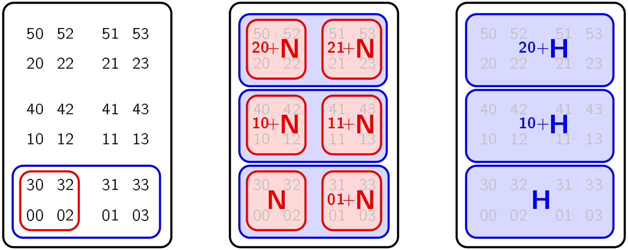
Consider the subgroups $N = \langle 30, 02 \rangle \cong V_4$ and $H = \langle 30, 01 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

Notice that $N \leq H \leq G$, and $H = N \cup (01{+}N)$, and

$$G/N = \big\{N,\ 01{+}N,\ 10{+}N,\ 11{+}N,\ 20{+}N,\ 21{+}N\big\}, \qquad H/N = \{N,\ 01{+}N\}$$

$$G/H = \Big\{N \cup (01{+}N),\ (10{+}N) \cup (11{+}N),\ (20{+}N) \cup (21{+}N)\Big\}$$

$$(G/N)/(H/N) = \Big\{\{N,\ 01{+}N\},\ \{10{+}N,\ 11{+}N\},\ \{20{+}N,\ 21{+}N\}\Big\}.$$



$N \leq H \leq G$

G/N consists of 6 cosets
$H/N = \{N,\ 01{+}N\}$

G/H consists of 3 cosets
$(G/N)/(H/N) \cong G/H$

The diamond theorem: duality of subquotients

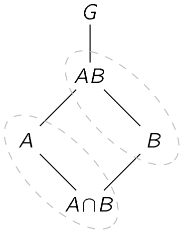
Diamond theorem

Suppose $A, B \leq G$, and that A normalizes B. Then

(i) $A \cap B \trianglelefteq A$ and $B \trianglelefteq AB$.

(ii) The following quotient groups are isomorphic:

$$AB/B \cong A/(A \cap B)$$



Proof (sketch)

Define the following map

If we can show: $\phi\colon A \longrightarrow AB/B\,, \qquad \phi\colon a \longmapsto aB\,.$

1. ϕ is a homomorphism,
2. ϕ is surjective (onto),
3. $\mathrm{Ker}(\phi) = A \cap B$,

then the result will follow *immediately* from the FHT. The details are left as HW.

Corollary

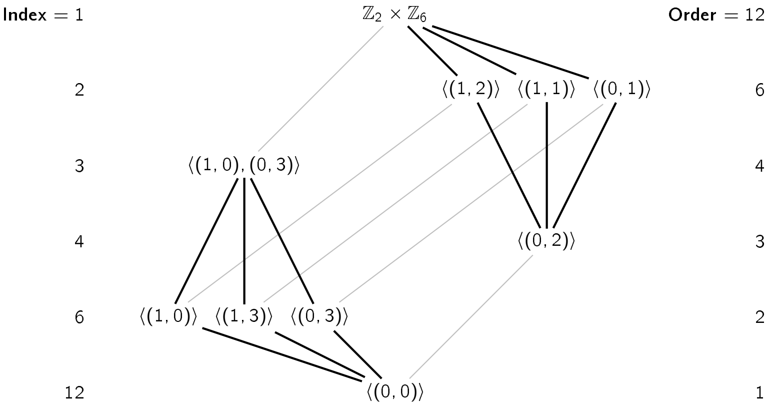
Let $A, B \leq G$, with one of them normalizing the other. Then $|AB| = \dfrac{|A| \cdot |B|}{|A \cap B|}$.

The diamond theorem: duality of subquotients

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_6$, and consider subgroups $A = \langle (1,0), (0,3) \rangle$, and $B = \langle (0,2) \rangle$.

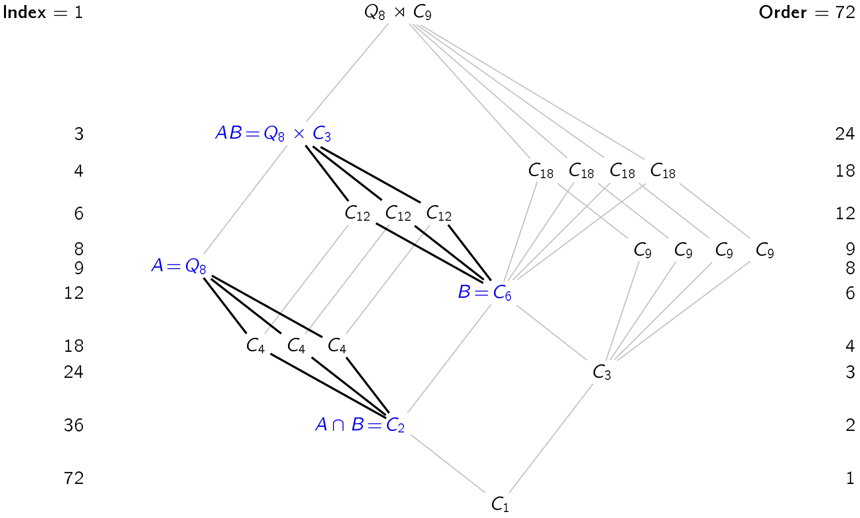
Then $G = AB$, and $A \cap B = \langle (0,0) \rangle$.

Let's interpret the diamond theorem $AB/B \cong A/A \cap B$ in terms of the subgroup lattice.



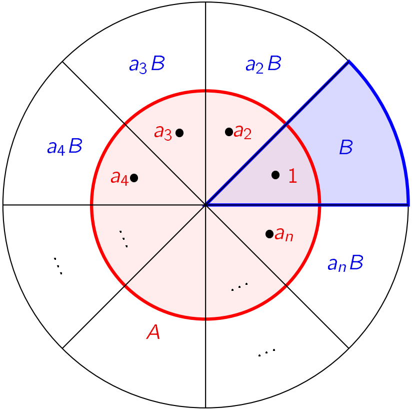
The fact that the subgroup lattice of V_4 is diamond shaped is coincidental.

The diamond theorem: duality of subquotients



The diamond theorem illustrated by a "pizza diagram"

The following analogy is due to Douglas Hofstadter:



AB = large pizza

A = small pizza

B = large pizza slice

$A \cap B$ = small pizza slice

$AB/B = \{\text{large pizza slices}\}$

$A/(A \cap B) = \{\text{small pizza slices}\}$

Diamond theorem: $AB/B \cong A/(A \cap B)$

The diamond theorem: duality of subquotients

Proposition

Suppose H is a subgroup of S_n that is not contained in A_n. Then exactly half of the permutations in H are even.

$$\begin{tikzcd}[column sep=small]
 & S_n \arrow[dl, dash, "m"'] \arrow[dr, dash, "2"] & \\
H \arrow[dr, dash, "2"'] & & A_n \arrow[dl, dash, "m"] \\
 & H \cap A_n &
\end{tikzcd}$$

Index		**Order**
$= 1$	S_n	$= n!$
2	A_n	$n!/2$
m	H	$n!/m$
$2m$	$H \cap A_n$	$n!/2m$

Proof

It suffices to show that $[H : H \cap A_n] = 2$, or equivalently, that $H/(H \cap A_n) \cong C_2$.

Since $H \nleq A_n$, the product HA_n must be strictly larger, and so $HA_n = S_n$.

By the diamond theorem,

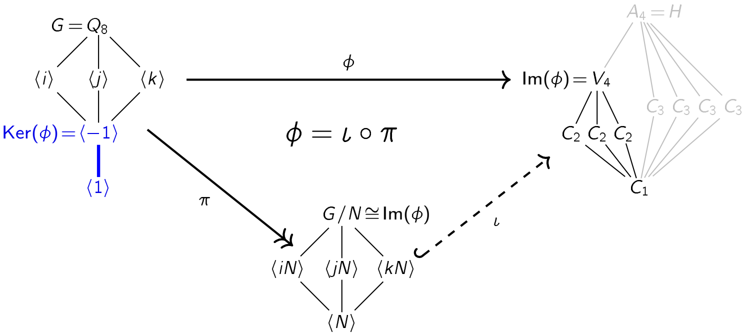
$$H/(H \cap A_n) = HA_n/A_n = S_n/A_n \cong C_2. \qquad \square$$

A generalization of the FHT

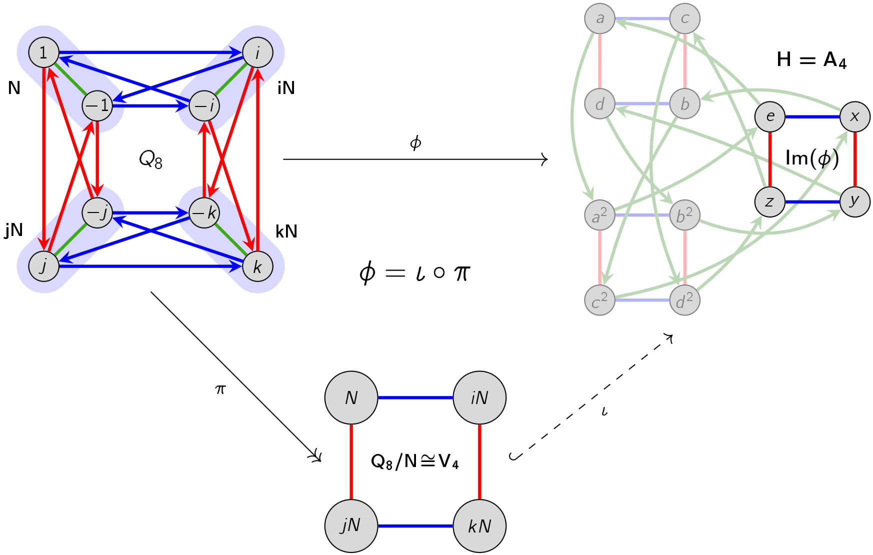
Theorem (exercise)

Every homomorphism $\phi\colon G \to H$ can be factored as a quotient and embedding:

$$\begin{tikzcd} G \arrow[rr, "\phi"] \arrow[dr, two heads, "\pi"'] & & H \\ & G/N \arrow[ur, hook, dashed, "\iota"'] & \end{tikzcd} \qquad \begin{tikzcd} g \arrow[rr, mapsto, "\phi"] \arrow[dr, mapsto, two heads, "\pi"'] & & \phi(g) \\ & gN \arrow[ur, mapsto, dashed, "\iota"'] & \end{tikzcd}$$



A generalization of the FHT



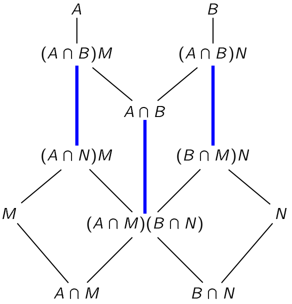
A theorem of Hans Zassenhaus

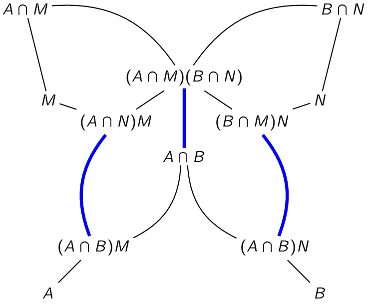
Butterfly lemma (see book for proof)

Let A, B be subgroups of a group, that contain $M \trianglelefteq A$ and $N \trianglelefteq B$. Then

1. $(A \cap N)M \trianglelefteq (A \cap B)M$,
2. $(B \cap M)N \trianglelefteq (A \cap B)N$,
3. The following quotient groups are isomorphic:

$$\frac{(A \cap B)M}{(A \cap N)M} \cong \frac{(A \cap B)N}{(B \cap M)N}.$$





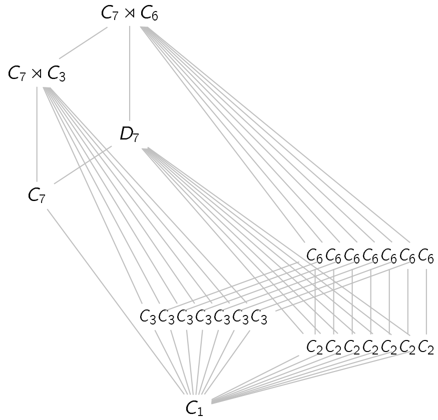
The quotient $G/Z(G)$ can *never* be a nontrivial cyclic subgroup

Lemma (exercise; see images below)

If $G/Z(G)$ is cyclic, then G is abelian.

$\mathbf{G/Z(G)} = \langle \mathbf{gZ} \rangle$, *where* $Z = Z(G)$

• g^{n-1}	• $g^{n-1}z_1$	• $g^{n-1}z_2$	• $g^{n-1}z_3$	$\cdots$	$\mathbf{g^{n-1}Z}$
		$\vdots$			
• g^2	• g^2z_1	• g^2z_2	• g^2z_3	$\cdots$	$\mathbf{g^2Z}$
• g	• gz_1	• gz_2	• gz_3	$\cdots$	$\mathbf{gZ}$
• e	• z_1	• z_2	• z_3	$\cdots$	$\mathbf{Z}$



Note that if G is abelian, then $Z(G) = G$.

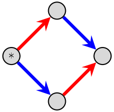
Commutators

We've seen how to divide $\mathbb{Z}$ by $\langle 12\rangle$, thereby "forcing" all multiples of 12 to be zero. This is one way to construct the integers modulo 12: $\mathbb{Z}_{12} \cong \mathbb{Z}/\langle 12\rangle$.

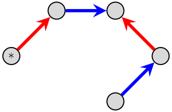
Now, suppose G is nonabelian. We'd like to divide G by its "non-abelian parts," making them zero and leaving only "abelian parts" in the resulting quotient.

A commutator is an element of the form $aba^{-1}b^{-1}$. Since G is nonabelian, *there are non-identity commutators:* $aba^{-1}b^{-1} \neq e$ *in* G.

$ab = ba$



$ab \neq ba$



In this case, the set $C := \{aba^{-1}b^{-1} \mid a, b \in G\}$ contains *more* than the identity.

Definition

The commutator subgroup G' of G is

$$G' := \left\langle aba^{-1}b^{-1} \mid a, b \in G \right\rangle .$$

The commutator subgroup is normal in G, and G/G' is abelian (homework).

The abelianization of a group

Definition

The abelianization of G is the quotient group G/G'.

The commutator subgroup G' is the smallest normal subgroup N of G such that G/N is abelian. [Note that G would be the "largest" such subgroup.]

Equivalently, the quotient G/G' is the largest abelian quotient of G. [Note that $G/G \cong \langle e \rangle$ would be the "smallest" such quotient.]

Universal property of commutator subgroups

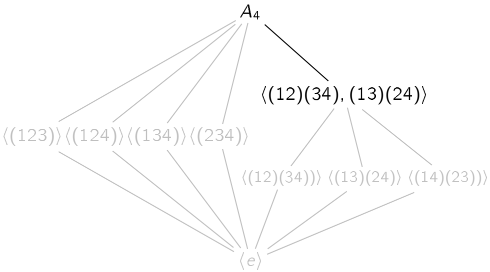
Suppose $f\colon G \to A$ is a homomorphism to an abelian group A. Then there is a unique homomorphism $h\colon G/G' \to A$ such that $f = h \circ \pi$:

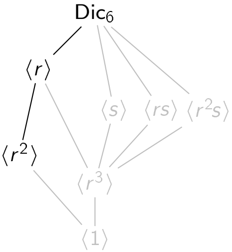
$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "\pi"'] & & A \\ & G/G' \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

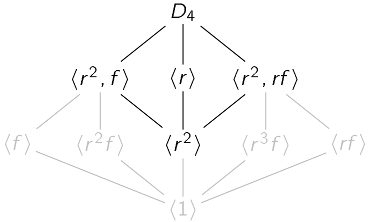
We say that f "factors through" the abelianization, G/G'.

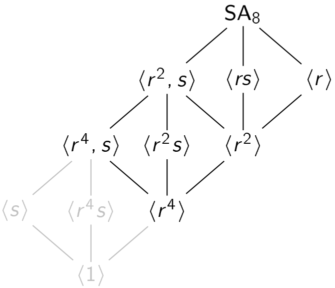
Some examples of abelianizations

By the isormophism theorems, we can usually identitfy the commutator subgroup G and abelianation by inspection, from the subgroup lattice.









Automorphisms

We have already seen automorphisms of cyclic groups: "*structure-preserving rewirings.*"

For a general group G, an automorphism is a isomorphism $\phi \colon G \to G$.

The set of automorphisms of G defines the automorphism group of G, denoted $\mathrm{Aut}(G)$.

Proposition

The automorphism group of $\mathbb{Z}_n$ is $\mathrm{Aut}(\mathbb{Z}_n) = \{\sigma_a \mid a \in U_n\} \cong U_n$, where

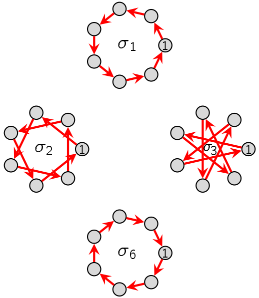
$$\sigma_a \colon \mathbb{Z}_n \longrightarrow \mathbb{Z}_n\,, \qquad \sigma_a(1) = a\,.$$

$\mathbf{U_7} = \langle \mathbf{3} \rangle \cong \mathbf{C_6}$

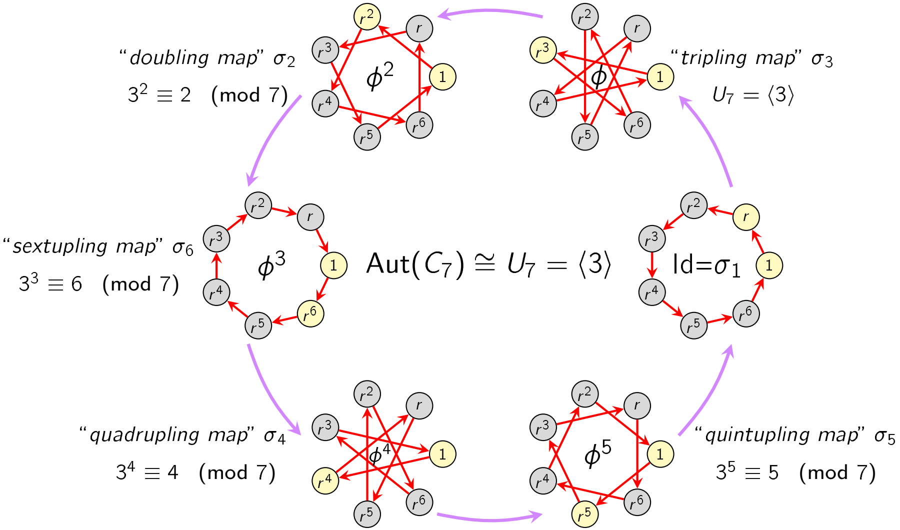
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

$\mathrm{Aut}(\mathbf{C_7}) = \langle \sigma_3 \rangle \cong \mathbf{U_7}$

	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
σ_1	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
σ_2	σ_2	σ_4	σ_6	σ_1	σ_3	σ_5
σ_3	σ_3	σ_6	σ_2	σ_5	σ_1	σ_4
σ_4	σ_4	σ_1	σ_5	σ_2	σ_6	σ_3
σ_5	σ_5	σ_3	σ_1	σ_6	σ_4	σ_2
σ_6	σ_6	σ_5	σ_4	σ_3	σ_2	σ_1



An example: the automorphism group of C_7



Automorphisms of noncyclic groups

An automorphism is determined by where it sends the generators.

Examples

1. An automorphism ϕ of $V_4 = \langle h, v \rangle$ is determined by the image of h and v.

   There are 3 choices for $\phi(h)$, then 2 choices for $\phi(v)$, thus $|\mathrm{Aut}(V_4)| = 6$.

   Every permutation of $\{h, v, r\}$ is an automorphism, and so $\mathrm{Aut}(V_4) \cong S_3$.

2. Every $\phi \in \mathrm{Aut}(D_3)$ is determined by $\phi(r)$ and $\phi(f)$.

   Since automorphisms preserve order, if $\phi \in \mathrm{Aut}(D_3)$, then

$$\phi(1) = 1\,, \qquad \phi(r) = \underbrace{r \text{ or } r^2}_{\text{2 choices}}, \qquad \phi(f) = \underbrace{f,\ rf,\ \text{or } r^2 f}_{\text{3 choices}}\,.$$

   Thus, $|\mathrm{Aut}(D_3)| \leq 6$. Both of the following define automorphisms of D_3:

$$\begin{cases} \alpha(r) = r \\ \alpha(f) = rf \end{cases} \qquad\qquad \begin{cases} \beta(r) = r^2 \\ \beta(f) = f \end{cases}$$

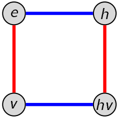
It is elementary to check that $\alpha\beta = \beta\alpha^2$, and so $\mathrm{Aut}(D_3) \cong D_3 \cong S_3$.

Automorphisms of $V_4 = \langle h, v \rangle$

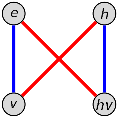
The following permutations are both automorphisms:

$$\boldsymbol{\alpha}: \quad \begin{tikzcd} h \arrow[r, bend left] & v \arrow[r, bend left] & hv \arrow[ll, bend left] \end{tikzcd} \qquad \textit{and} \qquad \alpha\beta: \quad \begin{tikzcd} h \arrow[r, bend left] & v \arrow[l, bend left] & hv \end{tikzcd}$$

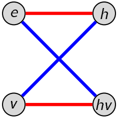
$$\begin{array}{rcl} h & \stackrel{id}{\longmapsto} & h \\ v & \longmapsto & v \\ hv & \longmapsto & hv \end{array}$$



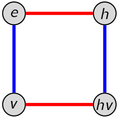
$$\begin{array}{rcl} h & \stackrel{\alpha}{\longmapsto} & v \\ v & \longmapsto & hv \\ hv & \longmapsto & h \end{array}$$



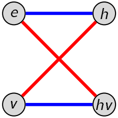
$$\begin{array}{rcl} h & \stackrel{\alpha^2}{\longmapsto} & hv \\ v & \longmapsto & h \\ hv & \longmapsto & v \end{array}$$



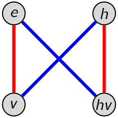
$$\begin{array}{rcl} h & \stackrel{\beta}{\longmapsto} & v \\ v & \longmapsto & h \\ hv & \longmapsto & hv \end{array}$$



$$\begin{array}{rcl} h & \stackrel{\alpha\beta}{\longmapsto} & h \\ v & \longmapsto & hv \\ hv & \longmapsto & v \end{array}$$



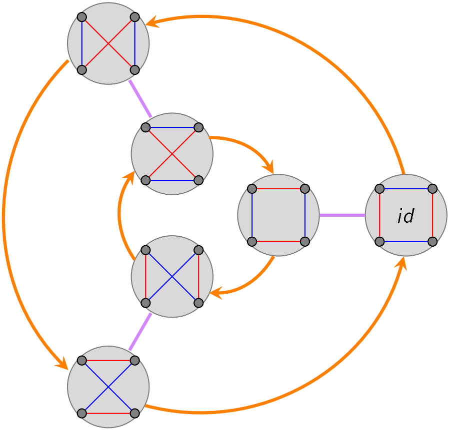
$$\begin{array}{rcl} h & \stackrel{\alpha^2\beta}{\longmapsto} & hv \\ v & \longmapsto & v \\ hv & \longmapsto & h \end{array}$$



Automorphisms of $V_4 = \langle h, v\rangle$

Here is the Cayley table and Cayley graph of $\textbf{Aut}(V_4) = \langle \alpha, \beta\rangle \cong S_3 \cong D_3$.

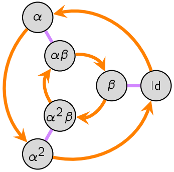
	id	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
id	id	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
α	α	α^2	id	$\alpha\beta$	$\alpha^2\beta$	β
α^2	α^2	id	α	$\alpha^2\beta$	β	$\alpha\beta$
β	β	$\alpha^2\beta$	$\alpha\beta$	id	α^2	α
$\alpha\beta$	$\alpha\beta$	β	$\alpha^2\beta$	α	id	α^2
$\alpha^2\beta$	$\alpha^2\beta$	$\alpha\beta$	β	α^2	α	id

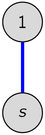


Recall that α and β can be thought of as the permutations h v hv and h v hv and so $\textbf{Aut}(G) \hookrightarrow \textbf{Perm}(G) \cong S_n$ always holds.

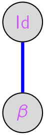
The construction of $V_4 \rtimes C_2$

A labeling map $\theta_i \colon C_2 \longrightarrow \mathrm{Aut}(V_4) \cong D_3$ is just a homomorphism. There are four:

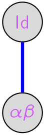




$s \overset{\theta}{\longmapsto} \mathrm{Id}$



$s \overset{\theta_0}{\longmapsto} \beta$

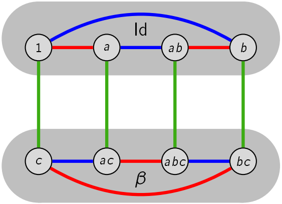


$s \overset{\theta_1}{\longmapsto} \alpha\beta$

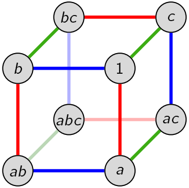
$s \overset{\theta_2}{\longmapsto} \alpha^2\beta$

Let's now carry out our "inflation method" to construct $V_4 \rtimes C_2$.

Start with a copy of $B = C_2$



Inflate each node, insert rewired versions of $A = V_4$, and connect corresponding nodes



rearrange the Cayley graph

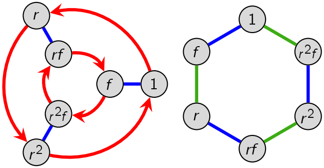
What familiar group is $V_4 \rtimes C_2$?

Automorphisms of D_3

$\boldsymbol{\alpha}$: $r \quad r^2 \quad f \quad rf \quad r^2 f$ *and* $\boldsymbol{\beta}$: $r \quad r^2 \quad f \quad rf \quad r^2 f$

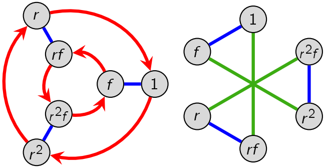
$r \xrightarrow{id} r$

$f \longrightarrow f$



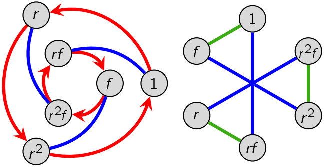
$r \xrightarrow{\beta} r^2$

$f \longrightarrow f$



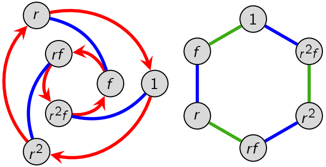
$r \xrightarrow{\alpha} r$

$f \longrightarrow rf$



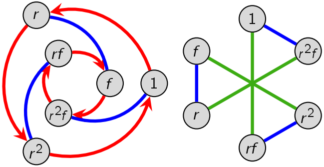
$r \xrightarrow{\alpha\beta} r^2$

$f \longrightarrow r^2 f$



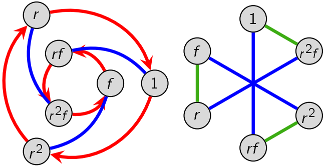
$r \xrightarrow{\alpha^2} r$

$f \longrightarrow r^2 f$



$r \xrightarrow{\alpha^2\beta} r^2$

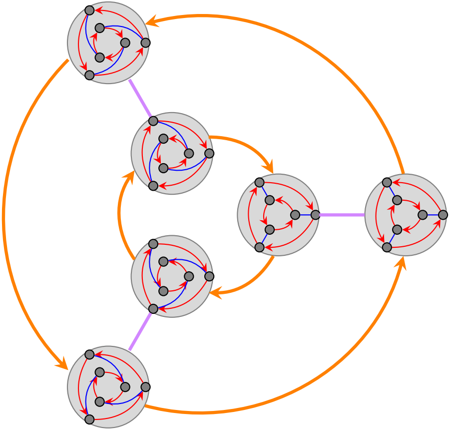
$f \longrightarrow rf$



Automorphisms of D_3

Here is the Cayley table and Cayley graph of $\mathbf{Aut}(D_3) = \langle \alpha, \beta \rangle$.

	id	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
id	id	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
α	α	α^2	id	$\alpha\beta$	$\alpha^2\beta$	β
α^2	α^2	id	α	$\alpha^2\beta$	β	$\alpha\beta$
β	β	$\alpha^2\beta$	$\alpha\beta$	id	α^2	α
$\alpha\beta$	$\alpha\beta$	β	$\alpha^2\beta$	α	id	α^2
$\alpha^2\beta$	$\alpha^2\beta$	$\alpha\beta$	β	α^2	α	id

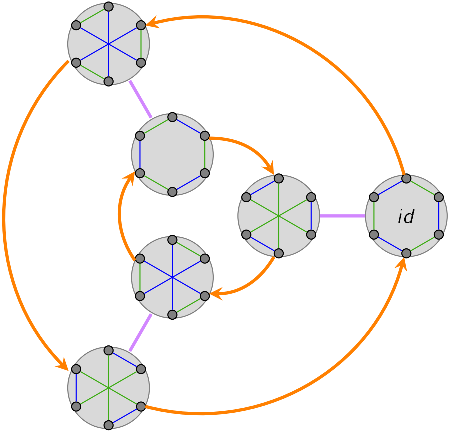


$\boldsymbol{\alpha}: \quad r \quad r^2 \quad f \quad rf \quad r^2 f$ and $\boldsymbol{\beta}: \quad r \quad r^2 \quad f \quad rf \quad r^2 f$

Automorphisms of D_3

Here is the Cayley table and Cayley graph of $\mathbf{Aut}(D_3) = \langle \alpha, \beta \rangle$.

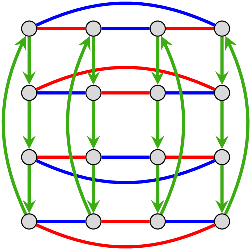
	id	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
id	id	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
α	α	α^2	id	$\alpha\beta$	$\alpha^2\beta$	β
α^2	α^2	id	α	$\alpha^2\beta$	β	$\alpha\beta$
β	β	$\alpha^2\beta$	$\alpha\beta$	id	α^2	α
$\alpha\beta$	$\alpha\beta$	β	$\alpha^2\beta$	α	id	α^2
$\alpha^2\beta$	$\alpha^2\beta$	$\alpha\beta$	β	α^2	α	id

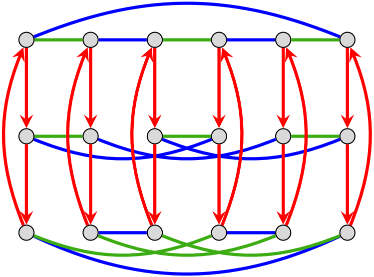


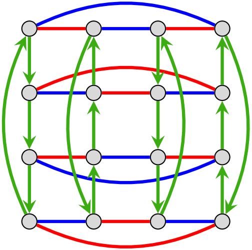
$\boldsymbol{\alpha}$: $r \quad r^2 \quad f \quad rf \quad r^2 f$ *and* $\boldsymbol{\beta}$: $r \quad r^2 \quad f \quad rf \quad r^2 f$

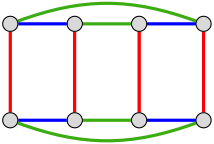
A few more examples of semidirect products

What groups are these?





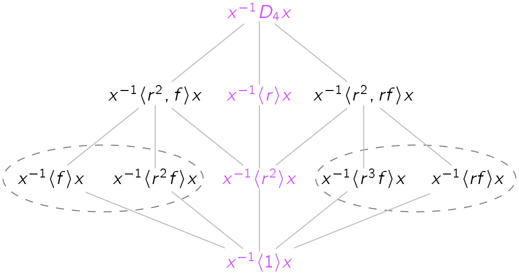


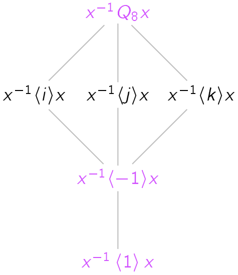


Inner and outer automorphisms

Earlier in this class, we conjugated an entire group G by a fixed element $x \in G$.

This is an example of an inner automorphism. Here are two examples:





This permutes subgroups *within a conjugacy class*: $r^{-1}\langle f\rangle r = \langle rf\rangle$.

Every subgroup of Q_8 is normal, thus any inner automorphism fixes every subgroup.

However, there is an automorphism of Q_8 that permutes subgroups, defined by

$$\phi\colon Q_8 \longrightarrow Q_8, \qquad \phi(i) = j, \quad \phi(j) = k \quad \Rightarrow \quad \phi(k) = \phi(ij) = \phi(i)\phi(j) = jk = i.$$

This is called an outer automorphism.

The inner automorphism group

Definition

An inner automorphism of G is an automorphism $\varphi_x \in \mathrm{Aut}(G)$ defined by

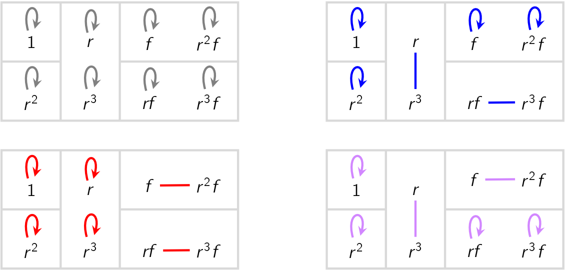
$$\varphi_x(g) := x^{-1}gx, \qquad \text{for some } x \in G.$$

The inner automorphisms of G form a group, denoted $\mathrm{Inn}(G)$. (Exercise)

There are four inner automorphisms of D_4:

$\mathrm{Id} = \varphi_1 = \varphi_{r^2}$ $\qquad$ $\varphi_f = \varphi_{r^2 f}$

$\varphi_r = \varphi_{r^3}$ $\qquad$ $\varphi_{rf} = \varphi_{r^3 f}$



Since $\varphi_x^2 = \mathrm{Id}$ for all of these, $\mathrm{Inn}(D_4) = \langle \varphi_r, \varphi_f \rangle \cong V_4$.

Are there any other automorphisms of D_4?

The inner automorphism group

Proposition (exercise)

$\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

Remarks

- Many books define $\varphi_x(g) = xgx^{-1}$. Our choice is so $\varphi_{xy} = \varphi_x\varphi_y$ (reading L-to-R).
- If $z \in Z(G)$, then $\varphi_z \in \mathrm{Inn}(G)$ is trivial.
- If $x = yz$ for some $z \in Z(G)$, then $\varphi_x = \varphi_y$ in $\mathrm{Inn}(G)$:

$$\varphi_x(g) = x^{-1}gx = (yz)^{-1}g(yz) = z^{-1}(y^{-1}gy)z = y^{-1}gy = \varphi_y(g).$$

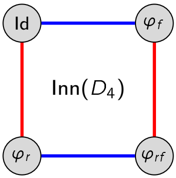
That is, if x and y are in the same coset of $Z(G)$, then $\varphi_x = \varphi_y$. (And conversely.)

Z	rZ	fZ	rfZ
1	r	f	rf
r^2	r^3	r^2f	r^3f

cosets of $Z(D_4)$ are in bijection with inner automorphisms of D_4

$\mathrm{cl}(1)$	1	r	f	rf
$\mathrm{cl}(r^2)$	r^2	r^3	r^2f	r^3f
		$\mathrm{cl}(r)$	$\mathrm{cl}(f)$	$\mathrm{cl}(rf)$

inner automorphisms of D_4 permute elements within conjugacy classes



The inner automorphism group

Key point

Two elements $x, y \in G$ are in the same coset of $Z(G)$ if and only if $\varphi_x = \varphi_y$ in $\mathrm{Inn}(G)$.

Proposition

In any group G, we have $G/Z(G) \cong \mathrm{Inn}(G)$.

Proof

Consider the map

$$f : G \longrightarrow \mathrm{Inn}(G), \qquad x \longmapsto \varphi_x,$$

It is straightfoward to check this this is (i) a homomorphism, (ii) onto, and (iii) that $\mathrm{Ker}(f) = Z(G)$.

The result is now immediate from the FHT. $\square$

We just saw that $\mathrm{Aut}(D_3) \cong D_3$, and we know that $Z(D_3) = \langle 1 \rangle$. Therefore,

$$\mathrm{Inn}(D_3) \cong D_3/Z(D_3) \cong D_3 \cong \mathrm{Aut}(D_3),$$

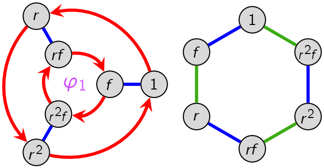
i.e., every automorphism is inner.

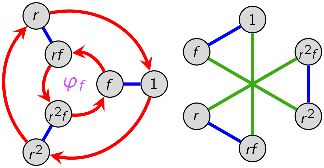
Inner automorphisms of D_3

Let's label each $\phi \in \mathrm{Aut}(D_3)$ with the corresponding inner automorphism.

$r \xrightarrow{Id} r$

$f \longrightarrow f$



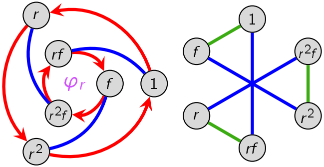


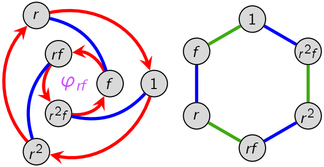
$r \xrightarrow{\beta} r^2$

$f \longrightarrow f$

$r \xrightarrow{\alpha} r$

$f \longrightarrow rf$



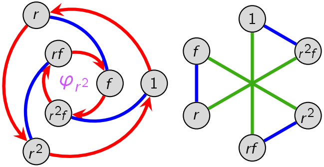


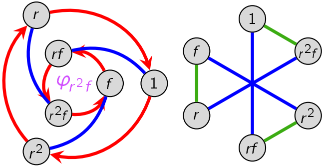
$r \xrightarrow{\alpha\beta} r^2$

$f \longrightarrow r^2 f$

$r \xrightarrow{\alpha^2} r$

$f \longrightarrow r^2 f$





$r \xrightarrow{\alpha^2\beta} r^2$

$f \longrightarrow rf$

Automorphisms of D_4

Every automorphism of $D_4 = \langle r, f \rangle$ is determined by where it sends the generators:

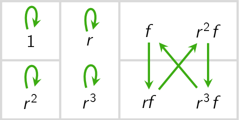
$$\phi(r) = \underbrace{r \text{ or } r^3}_{\text{2 choices}}, \qquad \phi(f) = \underbrace{f,\ rf,\ r^2 f,\ r^3 f,\ \text{or } r^2}_{\text{5 choices}}.$$

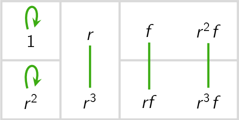
Thus $|\mathrm{Aut}(D_4)| \leq 10$. But $\mathrm{Inn}(D_4) \leq \mathrm{Aut}(D_4)$, forces $|\mathrm{Aut}(D_4)| = 4$ or 8. Moreover,

$$\omega \colon D_4 \longrightarrow D_4, \qquad \omega(r) = r, \quad \omega(f) = rf$$

is an (outer) automorphism, which swaps the "two types" of reflections of the square.

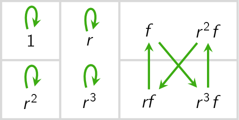
ω

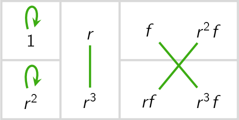




$\varphi_f \omega$

$\varphi_r \omega$

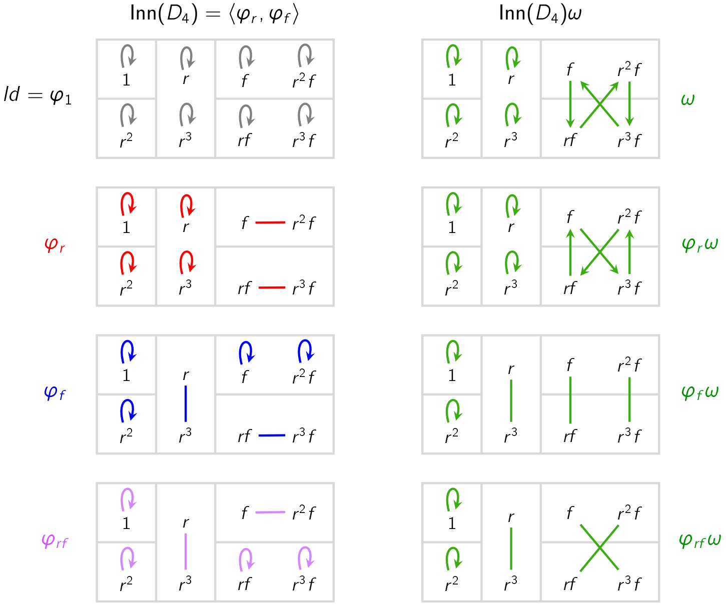




$\varphi_{rf} \omega$

$$\mathrm{Aut}(D_4) = \{Id,\ \varphi_r,\ \varphi_f,\ \varphi_{rf},\ \omega,\ \varphi_r\omega,\ \varphi_f\omega,\ \varphi_{rf}\omega\} = \mathrm{Inn}(D_4) \cup \mathrm{Inn}(D_4)\omega \cong D_4.$$

The full automorphism group of D_4

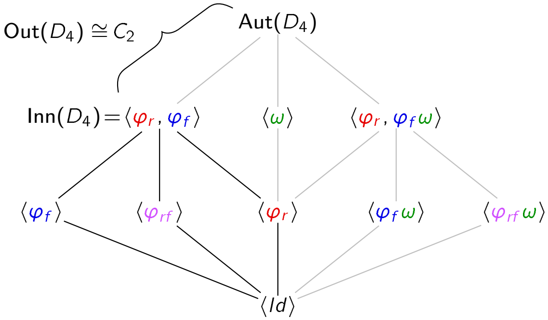


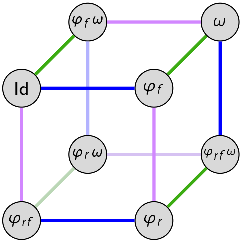
The outer automorphism group

Definition

An outer automorphism of G is any automorphism that is not inner.

The outer automorphism group of G is the quotient $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$.





$\mathrm{Aut}(D_4) \cong \mathrm{Inn}(D_4) \rtimes \mathrm{Out}(D_4)$

Note that there are four outer automorphisms, but $|\mathrm{Out}(D_4)| = 2$.

We have seen: $\mathrm{Out}(V_4) \cong D_3$, $\mathrm{Out}(D_3) \cong \{\mathrm{Id}\}$, $\mathrm{Out}(D_4) \cong C_2$, $\mathrm{Out}(Q_8) \cong S_3$.

Class automorphisms

Proposition (exercise)

Automorphisms permute conjugacy classes. That is, $g, h \in G$ are conjugate if and only if $\phi(g)$ and $\phi(h)$ are conjugate.

It is natural to ask if an automorphism being inner is equivalent to being the identity permutation on conjugacy classes.

In other words:

"*if* $\phi \in \mathrm{Aut}(G)$ *sends every element to a conjugate, must* $\phi \in \mathrm{Inn}(G)$*?*"

The answer is "no". Burnside found examples of groups of order at least 729 that admit such an automorphism.

Definition

A class automorphism is an automorphism that sends every element to another in its conjugacy class.

In 1947, G.E. Wall found a group of order 32 with a class automorphism that is outer.

Semidirect products, algebraically

Thus far, we've see how to construct $A \rtimes_\theta B$ with our "inflation method."

Given A (for "*automorphism*") and B (for "*balloon*"), we label each inflated node $b \in B$ with $\phi \in \mathrm{Aut}(A)$ via some labeling map

$$\theta \colon B \longrightarrow \mathrm{Aut}(A).$$

Of course can all be defined algebraically. Denote multiplication in $A \times B$ by

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, b_1 b_2).$$

Definition

The (external) **semidirect product** $A \rtimes_\theta B$ of A and B, with respect to the homomorphism

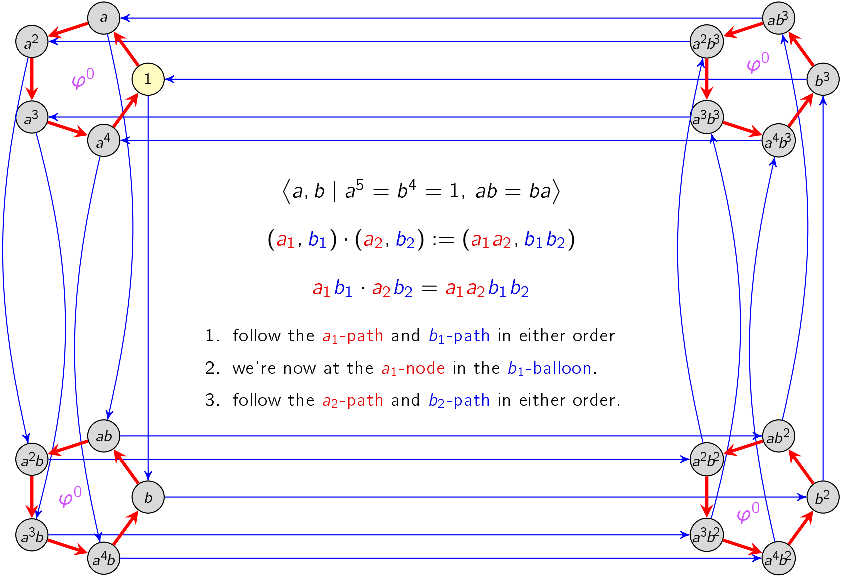
$$\theta \colon B \longrightarrow \mathrm{Aut}(A),$$

is on the underlying set $A \times B$, where the binary operation $*$ is defined as

$$(a_1, b_1) * (a_2, b_2) := (a_1, b_1) \cdot (\theta(b_1) a_2, b_2) = (a_1 \theta(b_1) a_2, b_1 b_2).$$

The isomorphic group on $B \times A$ by swapping the coordinates above is written $B \ltimes_\theta A$.

An example: the direct product $C_5 \times C_4$



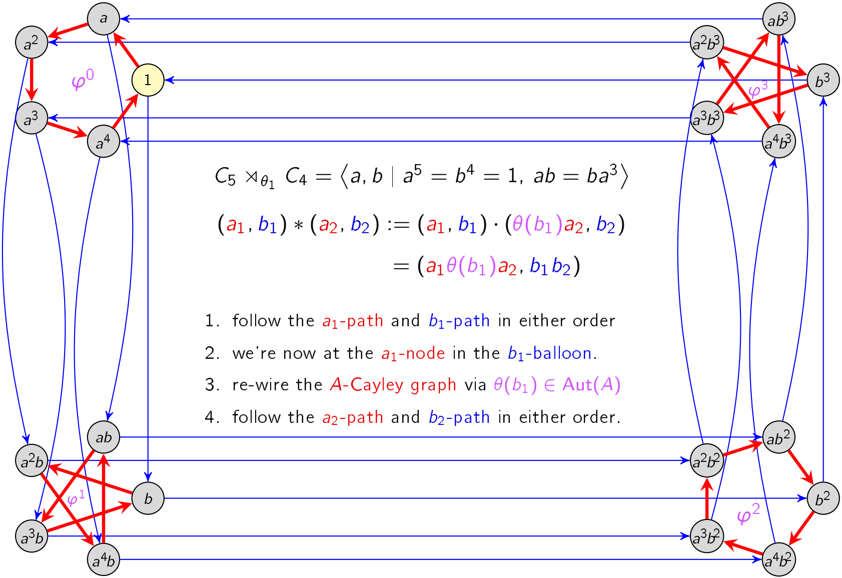
$$\langle a, b \mid a^5 = b^4 = 1,\ ab = ba \rangle$$

$$(a_1, b_1) \cdot (a_2, b_2) := (a_1 a_2, b_1 b_2)$$

$$a_1 b_1 \cdot a_2 b_2 = a_1 a_2 b_1 b_2$$

1. follow the a_1-path and b_1-path in either order
2. we're now at the a_1-node in the b_1-balloon.
3. follow the a_2-path and b_2-path in either order.

An example: the semidirect product $C_5 \rtimes_\theta C_4$



$$C_5 \rtimes_{\theta_1} C_4 = \langle a, b \mid a^5 = b^4 = 1,\ ab = ba^3 \rangle$$

$$(a_1, b_1) * (a_2, b_2) := (a_1, b_1) \cdot (\theta(b_1)a_2, b_2)$$

$$= (a_1\theta(b_1)a_2, b_1 b_2)$$

1. follow the a_1-path and b_1-path in either order
2. we're now at the a_1-node in the b_1-balloon.
3. re-wire the A-Cayley graph via $\theta(b_1) \in \mathrm{Aut}(A)$
4. follow the a_2-path and b_2-path in either order.

Revisiting semidirect products

Recall how to multipy in $A \rtimes_\theta B$:

$$(a_1, b_1) * (a_2, b_2) := (a_1, b_1) \cdot (\theta(b_1)a_2, b_2) = (a_1\theta(b_1)a_2, b_1 b_2).$$

Lemma

The subgroup $A \times \{1\}$ is normal in $A \rtimes_\theta B$.

Proof

Let's conjugate an arbitrary element $(g, 1) \in A \times \{1\}$ by an element $(a, b) \in A \rtimes_\theta B$.

$$(a, b)(x, 1)(a, b)^{-1} = (a\,\theta(b)g, b)(a^{-1}, b^{-1}) = (\underbrace{a\,\theta(b)g\,\theta(b)a^{-1}}_{\in A}, 1) \in A \times \{1\}.$$

Not all books use the same notation for semidirect product. Ours is motivated by:

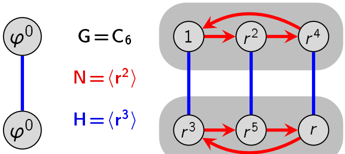
- In $A \times B$, both factors are normal (technically, $A \times \{1\}$ and $\{1\} \times B$).
- In $A \rtimes B$, the group on the "open" side of $\rtimes$ is normal.

Internal products

Previously, we've looked at outer products: taking two unrelated groups and constructing a direct or semidirect product.

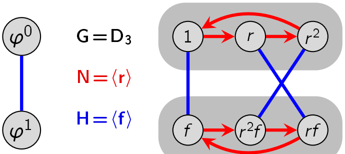
Now, we'll explore when a group $G = NH$ is isomorphic to a direct or semidirect product.

These are called internal products. Let's see two examples:



$\theta_0 : r \mapsto \varphi^0$

$C_6 = NH \cong N \times H$



$\theta_1 : r \mapsto \varphi$

$D_3 = NH \cong N \rtimes_\theta H$

Questions

- Can we characterize when $NH \cong N \times H$ and/or $NH \cong N \rtimes_\theta H$?
- If $NH \cong N \rtimes_\theta H$, then what is the map $\theta \colon H \longrightarrow \mathrm{Aut}(N)$?

Internal direct products

When $G = NH$ is isomorphic to $N \times H$, we have an isomorphism

$$i \colon N \times H \longrightarrow NH, \qquad i \colon (n, h) \longmapsto nh.$$

Since $N \times \{1\}$ and $\{1\} \times H$ are normal in $N \times H$, the subgroups N and H are normal in NH.

Recall that earlier, we showed that

$$|NH| = \frac{|N| \cdot |H|}{|N \cap H|},$$

and so it follows that if $NH \cong N \times H$, then $N \cap H = \{e\}$.

Theorem

Let $N, H \leq G$. Then $G \cong N \times H$ iff the following conditions hold:

(i) N and H are normal in G

(ii) $N \cap H = \{e\}$

(iii) $G = NH$.

Remark

This has a very nice interpretation in terms of subgroup lattices! Groups for which (ii) and (iii) hold are called lattice complements.

Internal semidirect products

When $G = NH$ is isomorphic to $N \rtimes_\theta H$, we have an isomorphism

$$i\colon N \rtimes_\theta H \longrightarrow NH, \qquad i\colon (n,h) \longmapsto nh.$$

This time, only $N \times \{1\}$ needs to be normal in $N \times H$, and so $N \trianglelefteq NH$.

As before, from

$$|NH| = \frac{|N| \cdot |H|}{|N \cap H|},$$

we conclude that if $NH \cong N \rtimes_\theta H$, then $N \cap H = \{e\}$.

Theorem

Let $N, H \leq G$. Then $G \cong N \rtimes H$ iff the following conditions hold:

(i) N is normal in G

(ii) $N \cap H = \{e\}$

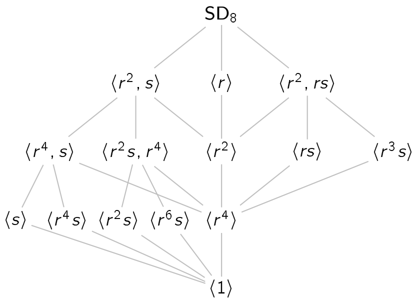
(iii) $G = NH$,

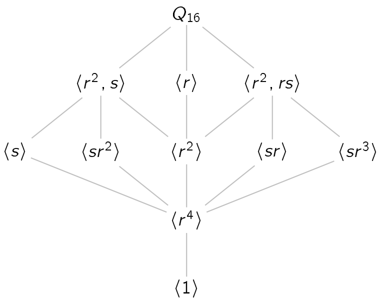
and the homomorphism θ sends h to the inner automorphism $\varphi_{h^{-1}}$:

$$\theta\colon H \longrightarrow \mathrm{Aut}(N), \qquad \theta\colon h \longmapsto \big(n \overset{\varphi_{h^{-1}}}{\longmapsto} h^{-1}nh\big).$$

Let's do several examples for intution, before proving this.

Examples of internal semidirect products





Observations

- The group SD_8 decomposes as a semidirect product several ways:

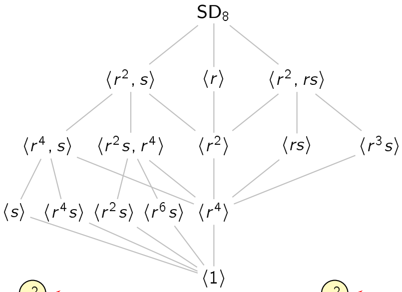
$$N = \langle r\rangle \cong C_8, \quad H = \langle s\rangle \cong C_2, \qquad \mathrm{SD}_8 = NH \cong C_8 \rtimes_{\theta_3} C_2.$$

  or alternatively,

$$N = \langle r^2, rs\rangle \cong Q_8, \quad H = \langle s\rangle \cong C_2, \qquad \mathrm{SD}_8 = NH \cong Q_8 \rtimes_{\theta'} C_2.$$

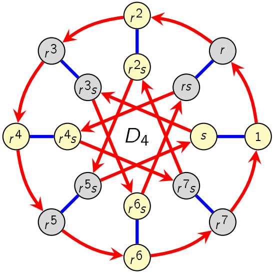
- The group Q_{16} does *not* decompose as a semidirect product!

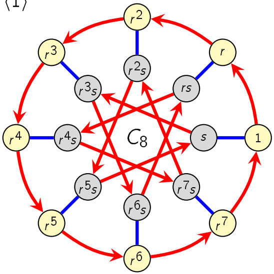
Semidihedral groups as semidirect products

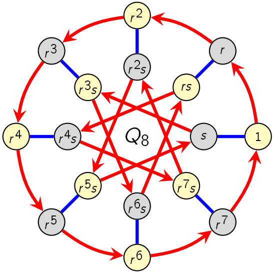


$$\mathrm{SD}_8 \cong \langle r\rangle \rtimes \langle s\rangle \cong C_8 \rtimes C_2$$

$$\mathrm{SD}_8 \cong \langle r^2, rs\rangle \rtimes \langle s\rangle \cong Q_8 \rtimes C_2$$



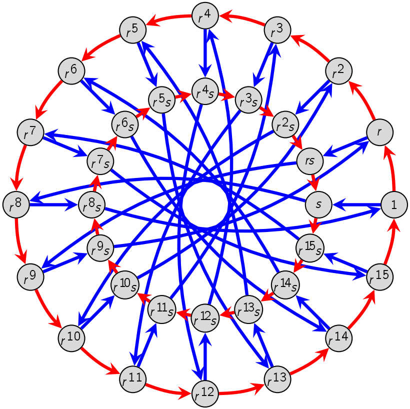


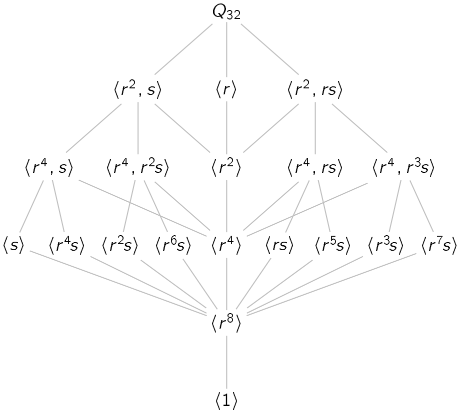


Generalized quaternion groups

Recall that a generalized quaternion group is a dicyclic group whose order is a power of 2.

It's not hard to see that $r^8 = s^2 = -1$ is contained in every cyclic subgroup.





Therefore, $Q_{2^n} \not\cong N \rtimes H$ for any of its nontrivial subgroups.

Internal semidirect products and inner automorphisms

Theorem

Let $N, H \leq G$. Then $G \cong N \rtimes H$ iff the following conditions hold:

(i) N is normal in G

(ii) $N \cap H = \{e\}$

(iii) $G = NH$,

and the homomorphism θ sends h to the inner automorphism φ_h:

$$\theta\colon H \longrightarrow \mathbf{Aut}(N), \qquad \theta\colon h \longmapsto \left(n \overset{\varphi_{h^{-1}}}{\longmapsto} h^{-1}nh\right).$$

Proof

We only need to establish that θ sends $h \mapsto \varphi_{h^{-1}}$.

Take n_1h_1 and n_2h_2 in NH. Their product is

$$(n_1h_1) * (n_2h_2) = n_1\theta(h_1)n_2h_1h_2$$

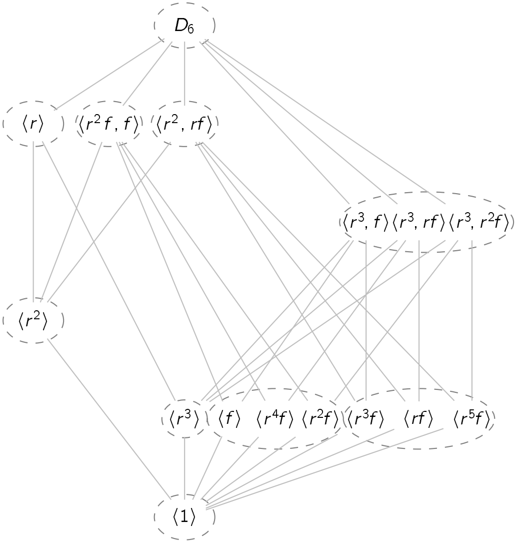
for some $\theta(h_1) \in \mathbf{Aut}(N)$.

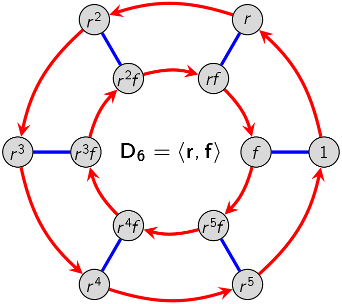
To see why $\theta(h_1)$ is the inner automorphism φ_{h_1}, note that

$$n_1\varphi_{h_1^{-1}}(n_2)h_1h_2 = n_1(h_1^{-1}n_2h_1)h_1h_2 = (n_1h_1) * (n_2h_2). \qquad \square$$

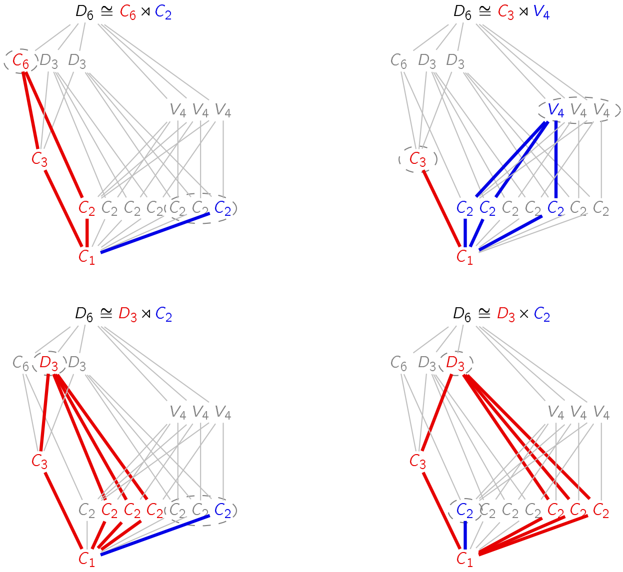
Internal direct and semidirect products

How many ways does D_6 decompose as an direct or semidirect product of its subgroups?



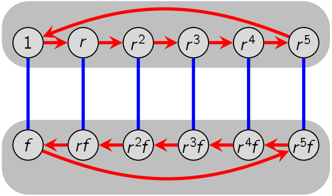


Decompositions of D_6 into direct and semdirect products

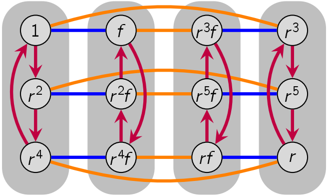


Decompositions of D_6 into direct and semdirect products

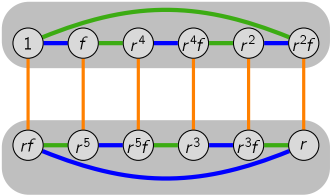
$C_6 \rtimes C_2$



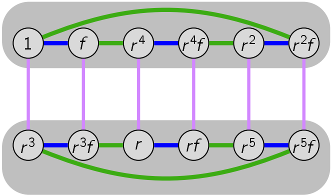
$C_3 \rtimes V_4$



$D_3 \rtimes C_2$



$D_3 \times C_2$



Central products

The following 3 conditions characterize when $G = NH \cong N \times H$.

1. H and N are normal,
2. $G = \langle H, N \rangle$,
3. $H \cap N = \langle 1 \rangle$.

If weaken the first to only N being normal, we get $G = NH \cong N \rtimes H$.

Alernatively, we can keep the first two but weaken the third.

Definition

Suppose H and N are subgroups of G satisfying:

1. H and N are normal,
2. $G = \langle H, N \rangle$,
3. $H \cap N \leq Z(G)$.

The G is an internal central product of H and K, denoted $G \cong H \circ K$.

We can also define an *external central product* of A and B, but we won't do that here.

Central products

The diquaternion group DQ_8 is a central product two nontrivial ways:

- $\mathrm{DQ}_8 \cong C_4 \circ Q_8$
- $\mathrm{DQ}_8 \cong C_4 \circ D_4$.

Recall that $Z(\mathrm{DQ}_8) = N \cong C_4$.