Isomorphisms!

(but first, homomorphisms!)

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With many thanks to Matthew Macauley,
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Goals for today:

1. We have sure said the word "isomorphic" a lot
2. Let's figure out what that actually means
3. Lots of examples
4. Some problems to play with

First, something from the hw

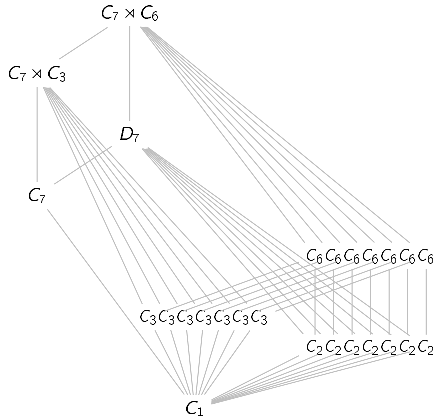
The quotient $G/Z(G)$ can *never* be a nontrivial cyclic subgroup

From homework:

If $G/Z(G)$ is cyclic, then G is abelian.

$\mathbf{G/Z(G)} = \langle \mathbf{gZ} \rangle$, *where* $Z = Z(G)$

• g^{n-1}	• $g^{n-1}z_1$	• $g^{n-1}z_2$	• $g^{n-1}z_3$	$\cdots$	$\mathbf{g^{n-1}Z}$
		$\vdots$			
• g^2	• g^2z_1	• g^2z_2	• g^2z_3	$\cdots$	$\mathbf{g^2Z}$
• g	• gz_1	• gz_2	• gz_3	$\cdots$	$\mathbf{gZ}$
• e	• z_1	• z_2	• z_3	$\cdots$	$\mathbf{Z}$



Note that if G is abelian, then $Z(G) = G$.

Definition and notation time!

Functions!

Nothing on this slide is specific to abstract algebra.

Extremely technical definition

Let A, B be two sets. A function f is a subset of the Cartesian product $A \times B$ such that:

- for all $a \in A$, there exists $b \in B$ such that $(a, b) \in f$ *(existence of images)*
- if $(a, b) \in f$ and $(a, b') \in f$, then $b = b'$ *(uniqueness of images)*

This definition sucks and I hate it.

Less technical but more useful definition

Let A, B be two sets. A function f is a map from A to B such that:

- for all $a \in A$, there exists $b \in B$ such that $f(a) = b$ *(existence of images)*
- if $f(a) = b$ and $f(a) = b'$, then $b = b'$ *(uniqueness of images)*

(Just don't ask me to formally explain what a "map" is.)

Moral definition

- f sends elements of A (inputs) to elements of B (outputs) *(existence of images)*
- and it does so reproducibly: the same input always gets sent to the same output. *(uniqueness of images)*

Notation and vocabulary!

Again, nothing on this slide is specific to abstract algebra.

Notation

- To say f is a function from A to B, we write $f : A \to B$ or $A \xrightarrow{f} B$
  - (We are specifying the *sets* that f plays with)
- To denote that $f(a) = b$, we also write $f : a \mapsto b$
  - or maybe even $a \mapsto b$ if it's clear what function we're talking about
  - (We are specifying the *elements* that f plays with)

Definitions

Let $f : A \to B$.

- The set A is called the domain of f.
- The set B is called the codomain of f.
- The image (or range) of f is the set of all actual outputs:

$$\mathrm{Im}(f) := \{b \in B \mid f(a) = b \text{ for some } a \in A\}.$$

"Isomorphic"

We can finally say what it means for two groups to be "isomorphic"!

Definition

Let G, H be groups. G is isomorphic to H ($G \cong H$) if there exists an isomorphism $\phi : G \longrightarrow H$.

Okay, smartass, what's an isomorphism?

Let G, H be groups. An isomorphism $\phi : G \longrightarrow H$ is a bijective homomorphism.

Istg if you don't tell me right now what a homomorphism is –

A homomorphism is a structure-preserving function between groups.

Homomorphisms!

Homomorphisms are structure-preserving functions

Since groups aren't just sets, they deserve maps that aren't just functions.

Formal definition

Let $(G, \star)$ and $(H, \odot)$ be two groups. A homomorphism is a function $\phi : G \longrightarrow H$ that respects the operations:

$$\phi(g_1 \star g_2) = \phi(g_1) \odot \phi(g_2)$$

Hey, c'mere

- Circle everything in that definition that is an element of G.
- Box everything in that definition that is an element of H.

Why this?

A common theme in various maths is that we study objects and then maps between objects.

When the objects are special in some way, we want the maps to be nice to that specialness.

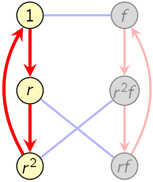
Example: in topology, we study open sets, so we use continuous functions because they are nice to open sets.

Morally:

- Homomorphisms preserve structure – specifically, the structure of a group.
- Homomoprhisms respect group operations.
- Homomorphisms send products to products.

An example homomorphism

Here is D_3 but I'm highlighting a subgroup that "looks like" $\mathbb{Z}_3$:



This can be formalized by a homomorphism $\phi\colon \mathbb{Z}_3 \longrightarrow D_3$, defined by $\phi\colon n \mapsto r^n$.

Let's check that ϕ meets the definition of being a homomorphism,

$$\phi(g_1 \star g_2) = \phi(g_1) \odot \phi(g_2)$$

What is the operation in $\mathbb{Z}_3$? in D_3?

$$\phi(n_1 + n_2) = r^{n_1+n_2} = r^{n_1} \cdot r^{n_2} = \phi(n_1) \cdot r^{n_2} = \phi(n_1) \cdot \phi(n_2)$$

Some more fun examples

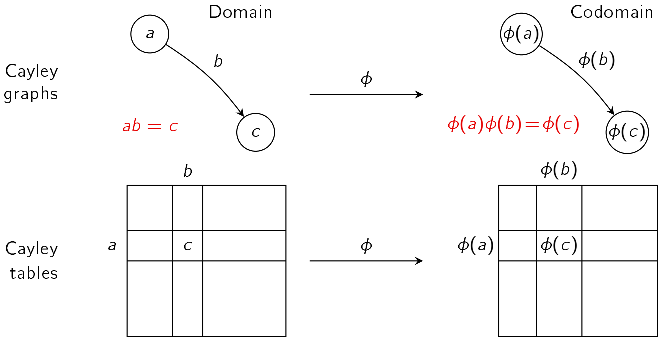
- Define a map $G \longrightarrow H$ that just squishes everything down to the identity in H.
- Define the "exponential map" $\mathbf{exp} : (\mathbb{R}, +) \longrightarrow (\mathbb{R}^*, *)$ by $\mathbf{exp}(x) = e^x$.
  ($\mathbb{R}^*$ means $\mathbb{R} - \{0\}$.)
- $\ln : (\mathbb{R}^+, *) \longrightarrow (\mathbb{R}, +)$.
- The domain and the codomain can be the same:
  consider the "squaring map" $s : C_6 \longrightarrow C_6$ defined by $s : g \mapsto g^2$.
- What about the same squaring map, but in D_4?

Important caveat:

Not every function between groups is a homomorphism!

Preserving structure

The $\phi(ab) = \phi(a)\phi(b)$ condition has visual interpretations on the level of Cayley graphs and Cayley tables.



Note that in the Cayley graphs, b and $\phi(b)$ are paths; they need not just be edges.

An example

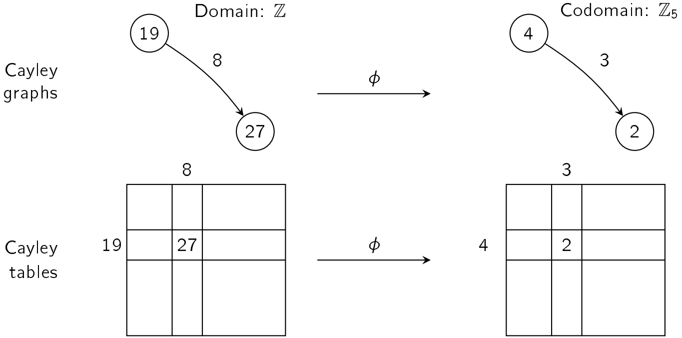
Consider the function ϕ that reduces an integer modulo 5:

$$\phi\colon \mathbb{Z} \longrightarrow \mathbb{Z}_5\,, \qquad \phi(n) = n \pmod 5.$$

Since the group operation is additive, the "homomorphism property" becomes

$$\phi(a+b) = \phi(a) + \phi(b)\,.$$

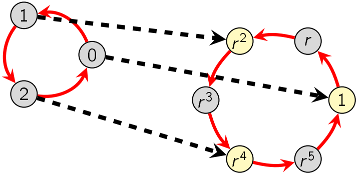
In plain English, this just says that one can "first add and then reduce modulo 5," OR "first reduce modulo 5 and then add."

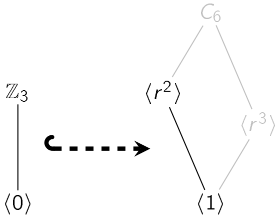


Types of homomorphisms!

Injective homomorphisms aka embeddings

Consider the following homomorphism $\theta\colon \mathbb{Z}_3 \to C_6$, defined by $\theta(n) = r^{2n}$:





Note that $\theta(a+b) = \theta(a)\theta(b)$. The red arrow in $\mathbb{Z}_3$ gets mapped to the 2-step path in C_6.

A homomorphism $\phi\colon G \to H$ that is one-to-one or injective is an embedding: the group G "embeds" into H as a subgroup. Optional: write $\phi\colon G \hookrightarrow H$.

Formally:

A homomorphism $\phi\colon G \to H$ is 1-1 or injective if "every output comes from only one input":

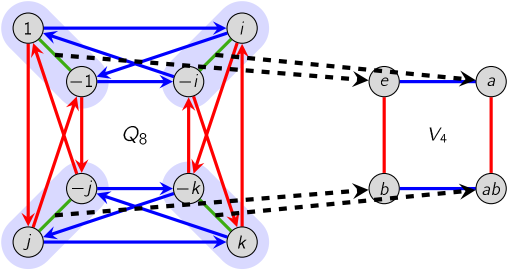
$$\text{if } \phi(g_1) = \phi(g_2), \text{ then } g_1 = g_2.$$

"If two outputs are the same, then actually the two inputs were the same."

Surjective homomorphisms

Consider the homomorphism $\alpha : Q_8 \longrightarrow V_4 = \langle a, b \rangle$, defined by $\alpha(i) = a$ and $\alpha(j) = b$.

Where does α send everything else in Q_8?



If $\phi(G) = H$ ("the image of ϕ is the entire codomain"), then ϕ is onto, or surjective.
We call ϕ a quotient map (yes, it's related!). Optional: write $\phi \colon G \twoheadrightarrow H$.

Formal definition

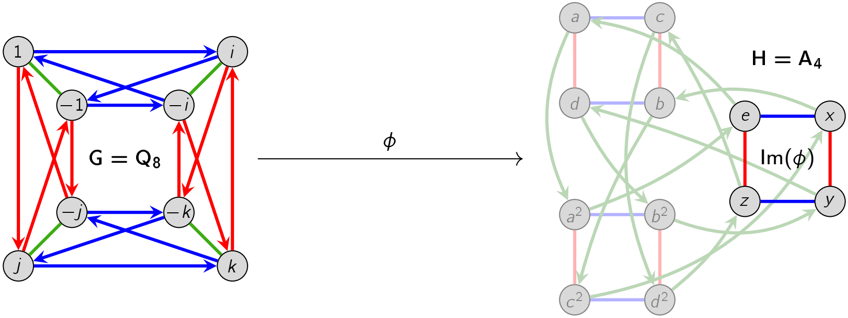
The homomorphism $\phi : G \longrightarrow H$ is onto, or surjective, if for all $h \in H$, there is some $g \in G$ such that $\phi(g) = h$.

An example that is neither an embedding nor a quotient

Consider the homomorphism $\phi\colon Q_8 \to A_4$ defined by

$$\phi(i) = (12)(34), \qquad \phi(j) = (13)(24).$$

Using the property of homomorphisms, compute ϕ of every other element of Q_8.



Note that the words injective and surjective aren't only used in abstract algebra.

Definition

If a function is both injective and surjective, then it is called bijective (or a bijection).

Okay, smartass, what's an isomorphism?

Let G, H be groups. An **isomorphism** $\phi : G \longrightarrow H$ is a bijective homomorphism.

G is **isomorphic** to H, written $G \cong H$, if there is an isomorphism between G and H.

Definition

An **automorphism** is an isomorphism from a group to itself.

An example of an isomorphism

We have already seen that D_3 is isomorphic to S_3.

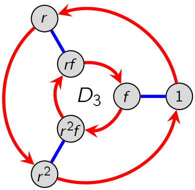
This means that there's a bijective correspondence $f : D_3 \to S_3$.

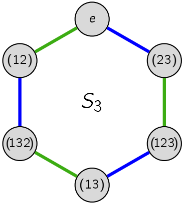
But not just any bijection will do. Intuitively (but also for order reasons),

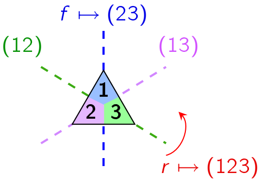
- (123) and (132) should be the rotations
- (12), (13), and (23) should be the reflections
- The identity permutation must be the identity symmetry.

It is easy to verify that the following is an isomorphism:

$$\phi \colon D_3 \longrightarrow S_3, \qquad \phi(r) = (123), \quad \phi(f) = (23).$$







However, there are other isomorphisms between these groups.

Properties of homomorphisms!

Some basic properties of homomorphisms

Proposition

For any homomorphism $\phi\colon G \to H$:

(i) "ϕ sends the identity to the identity" $\phi(1_G) = 1_H$

(ii) "ϕ sends inverses to inverses" $\phi(g^{-1}) = \phi(g)^{-1}$

(iii) "ϕ sends powers to powers"

(iv) "ϕ sends orbits to orbits"

(v) "ϕ sends conjugates to conjugates"

(vi) "ϕ is determined by what it does to generators"

What other properties along these lines can you conjecture?

Homework

If $|g|$ is finite, then $|\phi(g)|$ must divide $|g|$.

A word of caution

Just because a homomorphism $\phi\colon G \to H$ is determined by the image of its generators does *not* mean that every such image will work.

For example, let's try to define a homomorphism $\phi\colon \mathbb{Z}_3 \to \mathbb{Z}_4$ by $\phi(1) = 1$. Then we get

$$\phi(2) = \phi(1+1) = \phi(1) + \phi(1) = 2,$$

$$\phi(0) = \phi(1+1+1) = \phi(1) + \phi(1) + \phi(1) = 3 \neq 0.$$

This is *impossible*, because $\phi(0)$ must be $0 \in \mathbb{Z}_4$.

That's not to say that there isn't a homomorphism $\phi\colon \mathbb{Z}_3 \to \mathbb{Z}_4$; note that there is always the trivial homomorphism between two groups:

$$\phi\colon G \longrightarrow H\,, \qquad \phi(g) = 1_H \quad \text{for all } g \in G.$$

Exercise

Show that there is no embedding $\phi\colon \mathbb{Z}_n \hookrightarrow \mathbb{Z}$, for $n \geq 2$. That is, *any* such homomorphism must satisfy $\phi(1) = 0$.

Kernels!

Kernels

Definition

Let $\phi : G \to H$. The kernel of ϕ is "everybody who gets squished down to the identity:"

$$\ker(\phi) := \{x \in \quad \mid \phi(x) = 1\}.$$

(I am just going to quickly say the word "nullspace" from linear algebra.)

Properties of the kernel

(i) $\ker(\phi) \leq G$.

(ii) In fact, $\ker(\phi) \trianglelefteq G$!

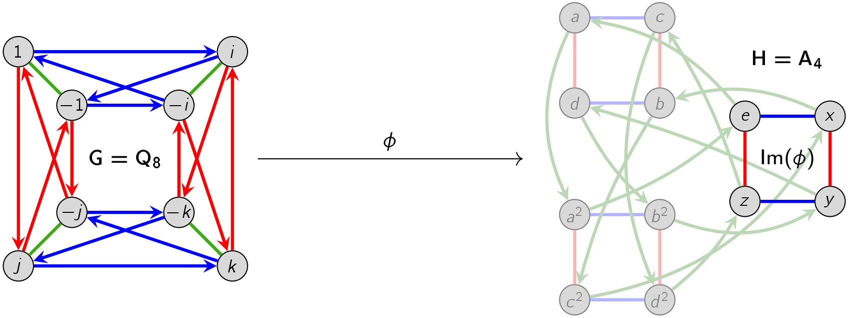
(iii) $\ker(\phi)$ is trivial iff ϕ is injective.

Example: Find the kernel!

Consider the homomorphism $\phi \colon Q_8 \to A_4$ defined by

$$\phi(i) = (12)(34), \qquad \phi(j) = (13)(24).$$

Who all is in $\ker \phi$?



Preimages

Here's a slightly more general version of the idea of the kernel:

Definition

Let $\phi : G \longrightarrow H$ and choose a fixed element $h \in H$.
The preimage of h is "everybody who gets sent to h:"

$$\phi^{-1}(h) := \{g \in G \mid \phi(g) = h\}.$$

Alternative names: fiber above h, pullback of h

Let's go back and look at our example again.

A word of caution:

ϕ^{-1} is in general not a function! (Unless. . .)

Theorem (homework)

(i) The kernel of ϕ is the fiber above 1.

(ii) For every element $h \in H$, the fiber above h is a coset of $\ker(\phi)$.

An example of a quotient

Let's write $C_2 = \langle -1 \rangle = \{1, -1\}$. Consider the following quotient map:

$$\phi \colon D_4 \twoheadrightarrow C_2\,, \qquad \text{defined by } \phi(r) = 1 \text{ and } \phi(f) = -1\,.$$

(Check: Is this a homomorphism?) Note that:

$$\phi(r^k) = \phi(r)^k = 1^k = 1, \qquad \phi(r^k f) = \phi(r^k)\phi(f) = \phi(r)^k\phi(f) = 1^k(-1) = -1.$$

	1	r	r^2	r^3	f	rf	r^2f	r^3f
1	1	r	r^2	r^3	f	rf	r^2f	r^3f
r	r	r^2	r^3	1	rf	r^2f	r^3f	f
r^2	r^2	r^3	1	r	r^2f	r^3f	f	rf
r^3	r^3	1	r	r^2	r^3f	f	rf	r^2f
f	f	r^3f	r^2f	rf	1	r^3	r^2	r
rf	rf	f	r^3f	r^2f	r	1	r^3	r^2
r^2f	r^2f	rf	f	r^3f	r^2	r	1	r^3
r^3f	r^3f	r^2f	rf	f	r^3	r^2	r	1

	1	r	r^2	r^3	f	rf	r^2f	r^3f
1	1	r	r^2	r^3	f	rf	r^2f	r^3f
r	r	r^2	r^3	1	rf	r^2f	r^3f	f
r^2	r^2	r^3	1	r	r^2f	r^3f	f	rf
r^3	r^3	1	r	r^2	r^3f	f	rf	r^2f
f	f	r^3f	r^2f	rf	1	r^3	r^2	r
rf	rf	f	r^3f	r^2f	r	1	r^3	r^2
r^2f	r^2f	rf	f	r^3f	r^2	r	1	r^3
r^3f	r^3f	r^2f	rf	f	r^3	r^2	r	1

1	**−1**
−1	**1**

$\mathrm{Ker}(\phi) = \phi^{-1}(1) = \langle r \rangle$ ("*rotations*"), $\qquad \phi^{-1}(-1) = f\langle r \rangle$ ("*reflections*").

The end!