Isomorphisms!

(but first, homomorphisms!)

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With many thanks to Matthew Macauley,
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Goals for today:

1. We have sure said the word "isomorphic" a lot
2. Let's figure out what that actually means
3. Lots of examples
4. Some problems to play with

Definition and notation time!

Functions!

Nothing on this slide is specific to abstract algebra.

Extremely technical definition

Let A, B be two sets. A function f is a subset of the Cartesian product $A \times B$ such that:

- for all $a \in A$, there exists $b \in B$ such that $(a, b) \in f$ *(existence of images)*
- if $(a, b) \in f$ and $(a, b') \in f$, then $b = b'$ *(uniqueness of images)*

This definition sucks and I hate it.

Less technical but more useful definition

Let A, B be two sets. A function f is a map from A to B such that:

- for all $a \in A$, there exists $b \in B$ such that $f(a) = b$ *(existence of images)*
- if $f(a) = b$ and $f(a) = b'$, then $b = b'$ *(uniqueness of images)*

(Just don't ask me to formally explain what a "map" is.)

Moral definition

- f sends elements of A (inputs) to elements of B (outputs) *(existence of images)*
- and it does so reproducibly: the same input always gets sent to the same output. *(uniqueness of images)*

Notation and vocabulary!

Again, nothing on this slide is specific to abstract algebra.

Notation

- To say f is a function from A to B, we write $f : A \longrightarrow B$ or $A \xrightarrow{f} B$
  - (We are specifying the *sets* that f plays with)
- To denote that $f(a) = b$, we also write $f : a \longmapsto b$
  - or maybe even $a \longmapsto b$ if it's clear what function we're talking about
  - (We are specifying the *elements* that f plays with)

Definitions

Let $f : A \longrightarrow B$.

- The set A is called the domain of f.
- The set B is called the codomain of f.
- The image (or range) of f is the set of all actual outputs:

$$\mathrm{Im}(f) := \{b \in B \mid f(a) = b \text{ for some } a \in A\}.$$

“Isomorphic”

We can finally say what it means for two groups to be “isomorphic”!

Definition

Let G, H be groups. G is isomorphic to H ($G \cong H$) if there exists an isomorphism $\phi : G \to H$.

Okay, smartass, what’s an isomorphism?

Let G, H be groups. An isomorphism $\phi : G \to H$ is a bijective homomorphism.

Istg if you don’t tell me right now what a homomorphism is –

A homomorphism is a structure-preserving function between groups.

Homomorphisms!

Homomorphisms are structure-preserving functions

Since groups aren't just sets, they deserve maps that aren't just functions.

Formal definition

Let $(G, \cdot)$ and $(H, \star)$ be two groups. A homomorphism is a function $\phi : G \to H$ that respects the operations:

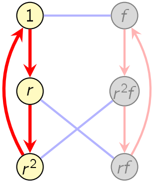
$$\phi(g_1 \cdot g_2) = \phi(g_1) \star \phi(g_2)$$

Hey, c'mere

- Circle everything in that definition that is an element of G.
- Box everything in that definition that is an element of H.

An example homomorphism

Here is D_3 but I'm highlighting a subgroup $\mathbb{Z}_3 \leq D_3$:



When we say $\mathbb{Z}_3 \leq D_3$, we really mean that *the structure of* $\mathbb{Z}_3$ *appears in* D_3.

This can be formalized by a homomorphism $\phi\colon \mathbb{Z}_3 \to D_3$, defined by $\phi\colon n \mapsto r^n$.

Let's check that ϕ meets the definition of being a homomorphism,
$\phi(g_1 \cdot g_2) = \phi(g_1) \star \phi(g_2)$:

$$\phi(n_1 + n_2) = r^{n_1+n_2} = r^{n_1} \cdot r^{n_2} = \phi(n_1) \cdot r^{n_2} = \phi(n_1) \cdot \phi(n_2)$$

The end!