Cosets!

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With many thanks to Matthew Macauley,
http://www.math.clemson.edu/~macaule/

17 Feb 2025

Goals for today:

1. Define what cosets are
2. See some examples
3. Learn some properties of cosets
4. Prove Lagrange's theorem
5. Explore normal subgroups

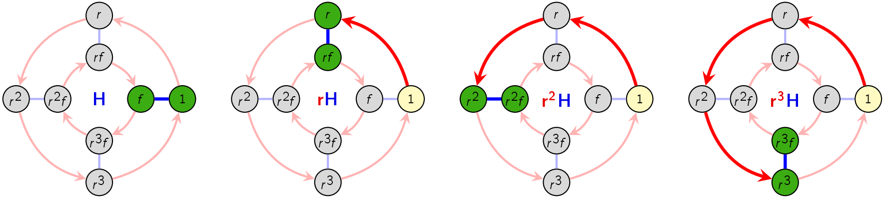
Definition time!

Definition time! (soon)

Vibes-based explanation (not a definition)

Let $H \leq G$. A coset of H is a shifted copy of H somewhere else in G.

Example: consider $H = \langle f \rangle \leq D_4$.



(Of course, only one of these is actually a subgroup; the others don't contain the identity.)

How do you shift to get from original H to each of these shifted copies?

Left cosets, and how to find them

To find the left coset xH in a Cayley graph, carry out the the following steps:

1. starting from the identity, follow a path to get to x ("shift by x")
2. from x, follow all "H-paths".

Definition time! (actually)

Definition

If $H \leq G$, then a left coset is a set

$$xH = \{xh \mid h \in H\},$$

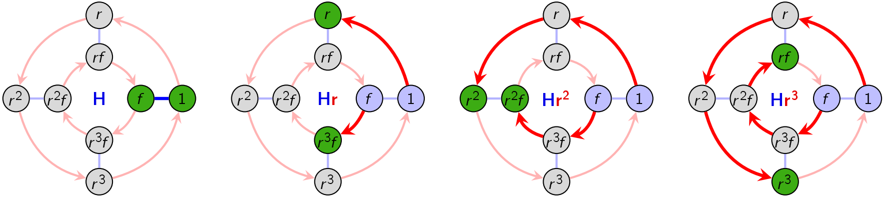
for some fixed $x \in G$ called the representative. Similarly, we can define a right coset as

$$Hx = \{hx \mid h \in H\}.$$

Left vs. right cosets

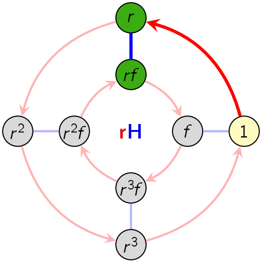
- The **left coset** rH in D_4: first shift by r, then traverse all "H-paths".
- The **right coset** Hr in D_4: first traverse all H-paths, then shift by r.

Let's look at the right cosets of $H = \langle f \rangle$ in D_4.

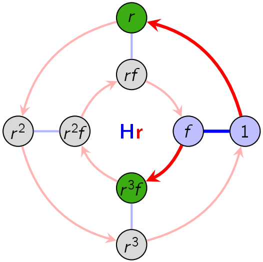


Left vs. right cosets

- The **left coset** rH in D_4: first go to r, then traverse all "H-paths".
- The **right coset** Hr in D_4: first traverse all H-paths, then traverse the r-path.



$rH = r\{1, f\} = \{r, rf\} = rf\{f, 1\} = rfH$



$Hr = \{1, f\}r = \{r, r^3f\} = \{f, 1\}r^3f = Hr^3f$

Because of our convention that arrows in a Cayley graph represent right multiplication:

- left cosets look like copies of the subgroup,
- right cosets are usually "scattered."

Key point

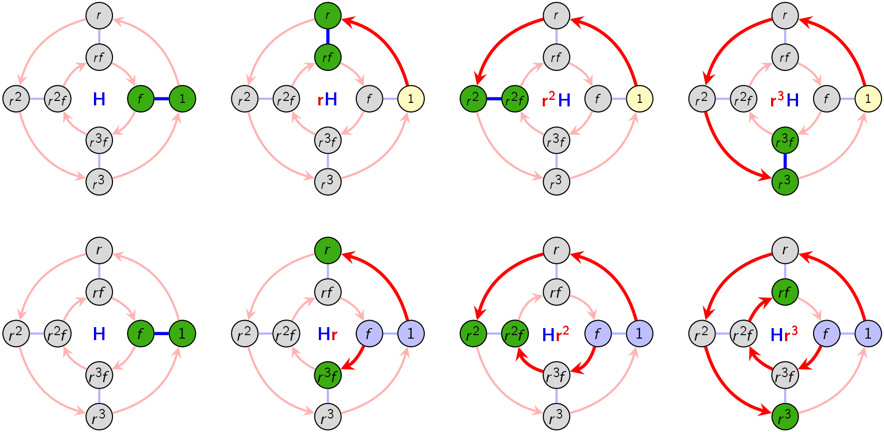
Left and right cosets are generally different.

Overview of left and right cosets of $\langle f \rangle$

Definition

Let $H \leq G$. Given $x \in G$, its left coset xH and right coset Hx are:

$$xH = \{xh \mid h \in H\}, \qquad Hx = \{hx \mid h \in H\}.$$



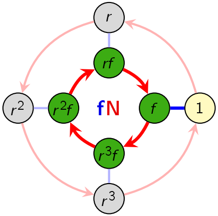
Your turn!

Your turn!

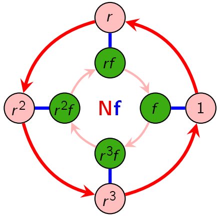
Find all the left and right cosets of a different subgroup, $N = \langle r \rangle$.

Reminder: finding left vs right cosets

- Left coset xN: first shift by x, then traverse all N-paths
- Right coset Nx: first traverse all N-paths, then shift by x



$fN = f\{1, r, r^2, r^3\} = \{f, fr, fr^2, fr^3\}$



$Nf = \{1, r, r^2, r^3\}f = \{f, rf, r^2f, r^3f\}$

Observations?

- There are multiple representatives for the same coset:

$$fN = (rf)N = (r^2f)N = (r^3f)N, \qquad Nf = N(rf) = N(r^2f) = N(r^3f).$$

- For this subgroup, each left coset is a right coset. Such a subgroup is called normal.

Your turn!

Now try:

- The other cyclic subgroups of D_4
- $K = \langle r^2, f \rangle \leq D_4$

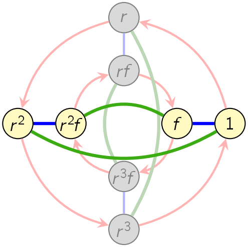
Observations?

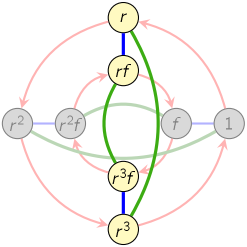
- Cosets never overlap
- All the cosets are always the same size as the original subgroup
- Cosets always cover the whole group
- Any time there are just two cosets, each left coset is a right coset

Equality of sets vs. equality of elements

Caveat!

An equality of cosets $xK = Kx$ as sets *does not* imply (or require) an equality of elements $xk = kx$.





rK	r	r^3	rf	r^3f
K	1	r^2	f	r^2f

r	r^3	fr	fr^3	**Kr**
1	r^2	f	fr^2	**K**

rK and Kr are the same as sets, even though the elements occur in a different order.

Properties of cosets!

Basic properties of cosets

The following results are "visually clear" from the Cayley graphs, but let's now prove them:

Proposition

Each (left) coset can have multiple representatives: if $b \in aH$, then $aH = bH$.

Proof

Since $b \in aH$, we can write $b = ah$, for some $h \in H$. That is, $h = a^{-1}b$ and $a = bh^{-1}$.

To show that $aH = bH$, we need to verify both $aH \subseteq bH$ and $aH \supseteq bH$.

"$\subseteq$": Take $ah_1 \in aH$. We need to write it as bh_2, for some $h_2 \in H$. By substitution,

$$ah_1 = (bh^{-1})h_1 = b(h^{-1}h_1) \in bH.$$

"$\supseteq$": Pick $bh_3 \in bH$. We need to write it as ah_4 for some $h_4 \in H$. By substitution,

$$bh_3 = (ah)h_3 = a(hh_3) \in aH.$$

Therefore, $aH = bH$, as claimed. $\square$

Corollary (boring but useful)

The equality $xH = H$ holds if and only if $x \in H$. (And analogously, for $Hx = H$.)

Basic properties of cosets

Proposition

For any subgroup $H \leq G$, the (left) cosets of H partition the group G:
every element $g \in G$ lives in exactly one coset of H.

Proof

We know that the element $g \in G$ lies in a (left) coset of H, namely gH. Uniqueness follows because if $g \in kH$, then $gH = kH$. $\square$

Proposition

All (left) cosets of $H \leq G$ have the same size. $\square$

Proof

It suffices to show that $|xH| = |H|$, for any $x \in H$.

Define a map

$$\phi \colon H \longrightarrow xH, \qquad h \longmapsto xh.$$

It is elementary to show that this is a bijection. $\square$

Lagrange's theorem!

Lagrange's theorem

Remark

For any subgroup $H \leq G$, the left cosets of H partition G into subsets of equal size.

The right cosets also partition G into subsets of equal size, but *they may be different.*

Let's compare these two partitions for the subgroup $H = \langle f \rangle$ of $G = D_4$.

H	**r^2H**	**rH**	**r^3H**
f	$r^2 f$	rf	r^3
1	r^2	r	$r^3 f$

H	**Hr^2**			
f	fr^2	fr^3	r^3	**Hr^3**
1	r^2	r	fr	**Hr**

Definition

The index of a subgroup H of G, written $[G : H]$, is the number of distinct left (or equivalently, right) cosets of H in G.

Lagrange's theorem

If H is a subgroup of finite group G, then $|G| = [G : H] \cdot |H|$. $\square$

Funny historical aside

Guess who proved Lagrange's theorem. Not Lagrange!

- Lagrange, 1771: if a polynomial in n variables has its variables permuted in all $n!$ ways, the number of different polynomials that are obtained is always a factor of $n!$.
  - What does this have to do with cosets?
  - Take $H \leq S_n$ to be the set of permutations that fix the polynomial. $n! = |S_n| = [S_n : H] \cdot |H|$.
  - The number of different polynomials is the number of cosets of H, aka the index $[S_n : H]$.
  - So: true for special subgroups of S_n.
- Gauss, 1801: the special case of subgroups of $(\mathbb{Z}/p\mathbb{Z})^*$
- Cauchy, 1844: true for any subgroup $H \leq S_n$
- Jordan, 1861: true for any subgroup H of a permutation group
  - (so, now it's true for subgroups of subgroups of S_n)
- Cayley, 1854: every group is a permutation group

Importantly: Lagrange would not have had the words "index," "coset," "group."

The tower law

Proposition

Let G be a finite group and $K \leq H \leq G$ be a chain of subgroups. Then

$$[G : K] = [G : H][H : K].$$

Here is a "proof by picture":

$[G : H] = \#$ of cosets of H in G

$[H : K] = \#$ of cosets of K in H

$[G : K] = \#$ of cosets of K in G

zH	$z_1 K$	$z_2 K$	$z_3 K$	$\cdots$	$z_n K$
	$\vdots$	$\vdots$	$\vdots$	$\iddots$	$\vdots$
aH	$a_1 K$	$a_2 K$	$a_3 K$	$\cdots$	$a_n K$
H	K	$h_2 K$	$h_3 K$	$\cdots$	$h_n K$

Proof

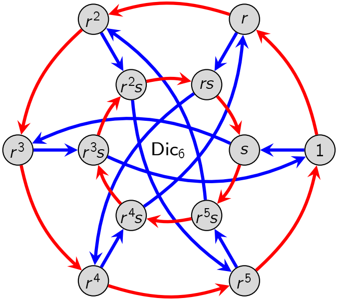
By Lagrange's theorem,

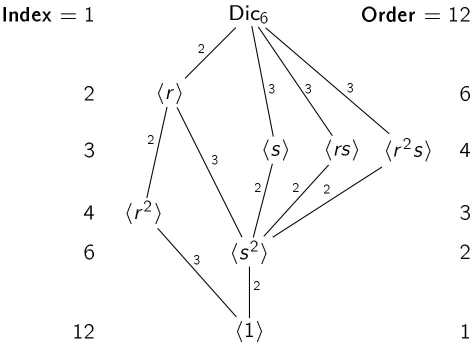
$$[G : H][H : K] = \frac{|G|}{|H|} \cdot \frac{|H|}{|K|} = \frac{|G|}{|K|} = [G : K]. \qquad \square$$

The tower law

Another way to visualize the tower law involves subgroup lattices.

It is often helpful to label the edge from H to K in a subgroup lattice with the index $[H : K]$.





The tower law and subgroup lattices

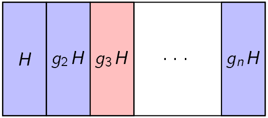
For any two subgroups $K \leq H$ of G, the index of K in H is just the *products of the edge labels* of any path from H to K.

Normal subgroups!

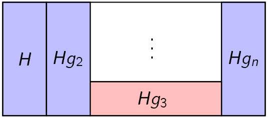
Normal subgroups and normalizers

Given a subgroup H of G, it is natural to ask the following question:

"How many left cosets of H are right cosets?"



Partition of G by the left cosets of H



Partition of G by the right cosets of H

- "Best case" scenario: all of them
- "Worst case" scenario: only H
- In general: somewhere between these two extremes

Definition

A subgroup H is a normal subgroup of G if $gH = Hg$ for *all* $g \in G$. We write $H \trianglelefteq G$.

The normalizer of H, denoted $N_G(H)$, is the set of elements $g \in G$ such that $gH = Hg$:

$$N_G(H) = \{g \in G \mid gH = Hg\},$$

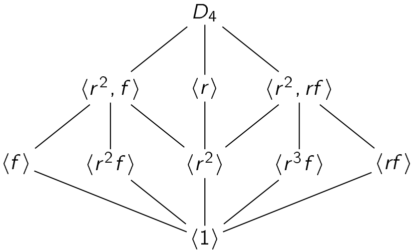
i.e., the union of left cosets that are also right cosets.

Examples of normal sugroups

We've seen cases where we know a subgroup will be normal without having to check.

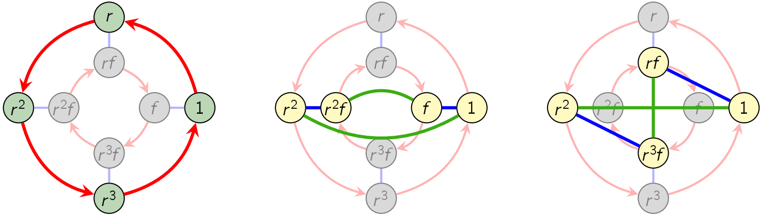
1. The subgroup $H = G$ is always normal in G. The only left coset is also the only right coset:
$$eG = G = Ge.$$
2. The subgroup $H = \{e\}$ is always normal. The left and right cosets are singleton sets:
$$gH = \{g\} = Hg.$$
3. Subgroups H of index 2 are normal. The two cosets (left or right) are H and $G - H$.
4. Subgroups of *abelian groups* are always normal, because for any $H \leq G$,
$$aH = \{ah \mid h \in H\} = \{ha \mid h \in H\} = Ha.$$
5. The center $Z(G)$ is always normal, for the same reason as above.
6. Relatedly, any subgroup of $Z(G)$ is always normal.

Normal subgroups in D_4



From our explorations, we found:

- $\langle r \rangle \lhd D_4$ (because it has index 2!)
- $\langle r^2, f \rangle \lhd D_4$ (index 2!)
- $\langle r^2, rf \rangle \lhd D_4$ (index 2!)
- $\langle r^2 \rangle \lhd D_4$ (because it is $Z(D_4)$!)



The end!