Homework #7 (due Mar 9)

Definition. Let $H \le G$. The set G/H (we say "G mod H") is the set of (left) cosets of H in G:

$$G/H = \{H, aH, bH, \ldots\}$$

If $H \trianglelefteq G$, then G/H becomes a group if we define the binary operation as

$$aH * bH := (a * b)H$$

Elements of quotient groups are cosets

Do a few of Problems 1 and 3. How many is "a few"? Idk. Make sure you do some that are written additively and some that are written multiplicatively. Part (j) is slightly harder but as a hint it is something we've seen before, perhapst in Homework 5? Hmmm

Problem 1. List out all the elements of the following quotient groups. (I promise that we're taking the quotient by a normal subgroup; no need to check.)

(a) $D_4/\langle r\rangle$

(b) $D_4/\langle r^2\rangle$

(c) $\mathbb{Z}/4\mathbb{Z}$

(d) $Q_8/\langle -1\rangle$

(e) $Q_8/\langle k\rangle$

(f) $\mathbb{Z}_6/\langle 3\rangle$

(g) $\mathbb{Z}_6/\langle 2\rangle$

(h) $(\mathbb{Z}_3 \times \mathbb{Z}_6)/\langle (1,1)\rangle$

(i) $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle (0,2)\rangle$

(j) $A_4/\langle (12)(34), (13)(24)\rangle$

Problem 2. While working on the previous problem, you may have conjectured this: Let G be a group and $H \trianglelefteq G$. Then $|G/H| = [G : H]$. In particular, if G is finite, then $|G/H| = |G|/|H|$.

Problem 3. Find the order of the given element in the quotient group. (Again, I promise we're taking the quotient by a normal subgroup.)

(a) $f\langle r\rangle \in D_4/\langle r\rangle$

(b) $r\langle r^2\rangle \in D_4/\langle r^2\rangle$

(c) $3 + 4\mathbb{Z} \in \mathbb{Z}/4\mathbb{Z}$

(d) $j\langle -1\rangle \in Q_8/\langle -1\rangle$

(e) $i\langle k\rangle \in Q_8/\langle k\rangle$

(f) $5 + \langle 3\rangle \in \mathbb{Z}_6/\langle 3\rangle$

(g) $5 + \langle 2\rangle \in \mathbb{Z}_6/\langle 2\rangle$

(h) $(2,1) + \langle (1,1)\rangle \in (\mathbb{Z}_3 \times \mathbb{Z}_6)/\langle (1,1)\rangle$

(i) $(1,3) + \langle (0,2)\rangle \in (\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle (0,2)\rangle$

(j) $(123)\langle (12)(34), (13)(24)\rangle$
$\in A_4/\langle (12)(34), (13)(24)\rangle$

Problem 4. $\mathbb{Q}$ is an abelian group under addition, so all its subgroups are normal. Describe the quotient group $\mathbb{Q}/\langle -1\rangle$. In particular, what do the elements look like?

Problem 5. If you take $\mathbb{Q}$ and throw away 0, then what's left is called $\mathbb{Q}^*$, and it is an abelian group under multiplication. Describe the quotient group $\mathbb{Q}^*/\langle -1\rangle$. In particular, what do the elements look like?

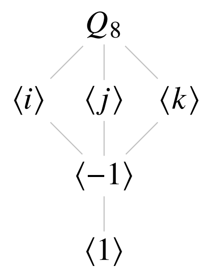
Quotient groups are visible in subgroup lattices

Problem 6. Consider $Q_8/\langle -1\rangle$, which I am tired of typing and will therefore just call "Q".

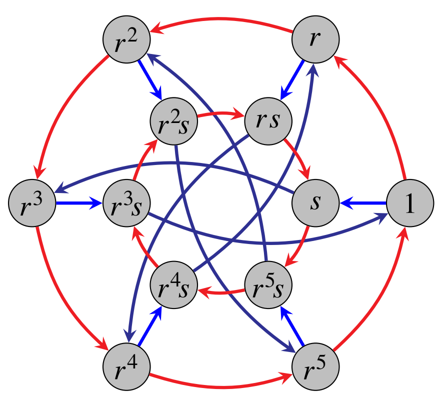
(a) In Problem 1, you found that Q has order 4. There are only two possible groups of order 4; is $Q \cong C_4$ or is $Q \cong V_4$? How do you know?

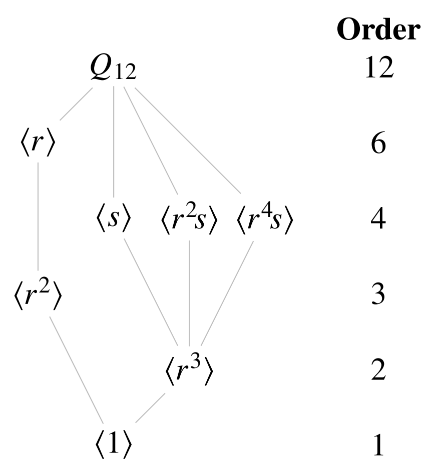
(b) Draw the subgroup lattice of Q.

(c) Where can you find the subgroup lattice of Q inside the subgroup lattice of Q_8?



Problem 7. Explore this phenomenon in a bigger and weirder group. LMFDB calls this group $C_3 : C_4$. Group Explorer calls it $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$. It's also called Q_{12} because it is a "generalized quaternion group." (If you look at the subgroup lattice I think you can kinda see why.)





(a) Looking at the subgroup lattice, which subgroups of Q_{12} are normal? How do you know?

(b) List the elements of $Q_{12}/\langle r^3\rangle$, draw a Cayley graph, and draw the subgroup lattice. Where do you see the subgroup lattice of $Q_{12}/\langle r^3\rangle$ inside the subgroup lattice of Q_{12}?

(c) (Bonus) Do the same for $Q_{12}/\langle r^2\rangle$.

(d) (Also bonus) Calculating with presentations is fun. Use the presentation

$$Q_{12} = \left\langle r, s \mid r^6 = 1, s^4 = 1, s^2 = r^3, rs = sr^{-1}\right\rangle$$

to show that all three of the order-4 subgroups are conjugate.

Interesting facts about quotient groups

Problem 8. Let G be a group and let $H \trianglelefteq G$. Show that if G is abelian, then so is G/H.

Problem 9. Let G be a group and let $H \trianglelefteq G$. Show that if G is cyclic, then so is G/H.

Problem 10. We previously proved that $Z(G) \trianglelefteq G$. Show that if $G/Z(G)$ is cyclic, then G is abelian.
Progressively hinty hints (decode at rot13.com):

- Gur pbfrgf bs n fhotebhc cnegvgvba gur tebhc.
- Jung qb gur pbfrgf bs M(T) ybbx yvxr?
- Fnl T/M(T) vf trarengrq ol tM(T). Nal ryrzrag k va T unf gb yvir va bar bs gur pbfrgf bs M(T).
- Fnl T/M(T) vf trarengrq ol tM(T). Fubj gung nal ryrzrag k va T vf fbzr cbjre bs t zhygvcyvrq ol fbzr ryrzrag bs M(T).

Problem 11 (Bonus)**.** Is the converse of Problem 8 true? Prove it, or find a specific counterexample.

Problem 12 (Bonus)**.** Is the converse of Problem 9 true? Prove it, or find a specific counterexample.

Problem 13 (Bonus)**.** Is the converse of Problem 10 true? Prove it, or find a specific counterexample.

Problem 14 (Andrew)**.** Prove that $\mathbb{Z}/n\mathbb{Z}$ is cyclic of order n.
Another way to say this is the following useful fact: $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$.
(This is actually a special case of Problem 9.)