# A zoo of examples of groups!

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With many thanks to Matthew Macauley,
http://www.math.clemson.edu/~macaule/

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# Families of groups

So far we've seen some examples of *individual* groups, but here we're going to see some examples of *families* of groups, because they'll be nice go-to examples:

1. **cyclic groups**: rotational symmetries
   - (Side quest: orbits and cycle graphs)
2. **dihedral groups**: rotational *and* reflective symmetries
3. **abelian groups**: where $ab = ba$ (always)
4. **matrix groups**: (actually, you get to think about this in the homework)
5. **permutation groups**: collections of rearrangements.

We'll show that *every* finite group is "isomorphic" to a permutation group.

Then, we'll see how to combine groups into bigger groups using

6. **direct products** and
7. **semidirect products** of groups.

We'll see a few other visualization techniques and surprises along the way.

# Some definitions

## Definition

A subgroup of $G$ is a subset $H \subseteq G$ that is also a group. We denote this by $H \leq G$.

(More on this soon.)

## Definition

The order of a group $G$ is its size as a set (how many distinct elements are in it), denoted by $|G|$.

## Example

$|\mathbf{Sq}| = 8$, and $|\mathbb{Z}| = \infty$.

## Definition

The order of an element $g \in G$ is $|g| := \big|\langle g \rangle\big|$, i.e., either

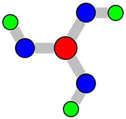
- the minimal $k \geq 1$ such that $g^k = e$, or
- $\infty$, if there is no such $k$.

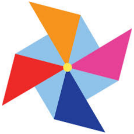
# Cyclic groups

**Definition**

A group is cyclic if it can be generated by a single element.

Finite cyclic groups describe the symmetries of objects that have *only* rotational symmetry.





**Remark**

You can make a cyclic group of any order you want.

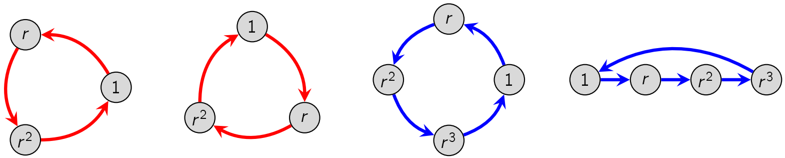
# Cyclic groups, multiplicatively

## Definition

For $n \geq 1$, the multiplicative cyclic group $C_n$ is the set

$$C_n = \{1, r, r^2, \ldots, r^{n-1}\},$$

where $r^i r^j = r^{i+j}$, and the exponents are taken modulo $n$. The identity is $r^0 = r^n = 1$.




It is clear that a presentation for this is

$$C_n = \langle r \mid r^n = 1 \rangle.$$

Note that $r^2$ generates $C_5$:

$$(r^2)^0 = 1, \qquad (r^2)^1 = r^2, \qquad (r^2)^2 = r^4, \qquad (r^2)^3 = r^6 = r, \qquad (r^2)^4 = r^8 = r^3.$$

*Do you have a conjecture about for which $k$ does $C_n = \langle r^k \rangle$?*

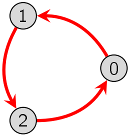
# Cyclic groups, additively

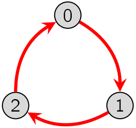
## Definition

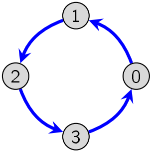
For $n \geq 1$, the additive cyclic group $\mathbb{Z}_n$ is the set

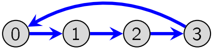
$$\mathbb{Z}_n = \big\{0, 1, \ldots, n-1\big\},$$

where the binary operation is addition modulo $n$. The identity is 0.










We can write a group presentation additvely:

$$\mathbb{Z}_n = \big\langle 1 \mid n \cdot 1 = 0 \big\rangle.$$

What else generates $\mathbb{Z}_5$?

## Remark

It is wrong to write $C_n = \mathbb{Z}_n$. (Why?)
Instead, we say $C_n$ is isomorphic to $\mathbb{Z}_n$, and we write $C_n \cong \mathbb{Z}_n$.

# Cayley tables of cyclic groups

Modular addition has a nice visual appearance in the Cayley tables for cyclic groups, if we order the elements $0, 1, \dots, n-1$.

Here are two different ways to write the Cayley table for $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$.

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 1 | 2 | 3 | 4 | 0 |
| 2 | 2 | 3 | 4 | 0 | 1 |
| 3 | 3 | 4 | 0 | 1 | 2 |
| 4 | 4 | 0 | 1 | 2 | 3 |

| | 0 | 1 | 3 | 2 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 3 | 2 | 4 |
| 1 | 1 | 2 | 4 | 3 | 0 |
| 3 | 3 | 4 | 1 | 0 | 2 |
| 2 | 2 | 3 | 0 | 4 | 1 |
| 4 | 4 | 0 | 2 | 1 | 3 |

(Hey, this looks kind of familiar, like the hilt of a sword)

# Infinite cyclic groups

## Definition

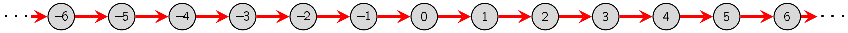
The additive infinite cyclic group is

$$\mathbb{Z} = \langle 1 \mid \quad \rangle,$$

the integers under addition. The multiplicative infinite cyclic group is

$$C_\infty := \langle r \mid \quad \rangle = \{ r^k \mid k \in \mathbb{Z} \}.$$

What does a Cayley graph of $\mathbb{Z}$ look like?



# Orbits and cycle graphs

## Definition

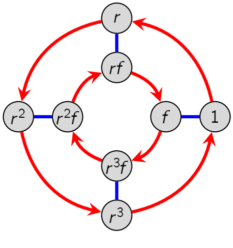
The orbit of an element $g \in G$ is the cyclic subgroup that it generates,

$$\langle g \rangle = \{g^k \mid k \in \mathbb{Z}\},$$

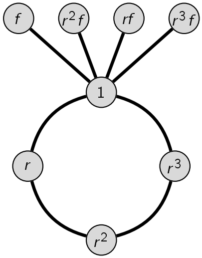
and its order is $|g| := |\langle g \rangle|$.

We can visualize the orbits by the (undirected) orbit graph, or cycle graph.

Let's think about this in the example of **Sq**. Use your Cayley graph to write down the orbits of each element.




| element | orbit |
|---|---|
| $1$ | $\{1\}$ |
| $r^2$ | $\{1, r^2\}$ |
| $r$<br>$r^3$ | $\{1, r, r^2, r^3\}$ |
| $f$ | $\{1, f\}$ |
| $rf$ | $\{1, rf\}$ |
| $r^2f$ | $\{1, r^2f\}$ |
| $r^3f$ | $\{1, r^3f\}$ |




By convention, we typically only draw maximal orbits.

# Dihedral groups

## Definition

The dihedral group $D_n$ or $\mathbf{Dih}_n$ is the group of symmetries of a regular $n$-gon.

## Examples

$\mathbf{Tri} = D_3$ and $\mathbf{Sq} = D_4$. :)

Conjecture time:

- What is the order of a generic $D_n$?
- What does the Cayley graph of a generic $D_n$ look like?
- Do you immediately see any subgroups of a generic $D_n$?
- What do you think is a presentation for a generic $D_n$?

# Dihedral groups

## Definition

The dihedral group $D_n$ is the group of symmetries of a regular $n$-gon. It has order $2n$.

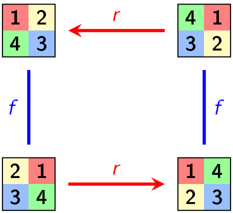
One possible choice of generators is

1. $r$ = counterclockwise rotation by $2\pi/n$ radians,
2. $f$ = flip across a fixed axis of symmetry.

Using these generators, one (of many) ways to write the elements of $D_n = \langle r, f \rangle$ is

$$D_n = \{\underbrace{1, r, r^2, \dots, r^{n-1}}_{n \text{ rotations}}, \underbrace{f, rf, r^2f, \dots, r^{n-1}f}_{n \text{ reflections}}\}.$$

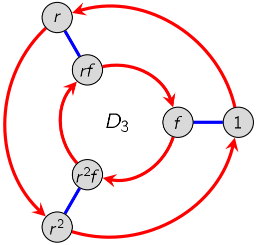
It is easy to check that $rf = fr^{-1}$:

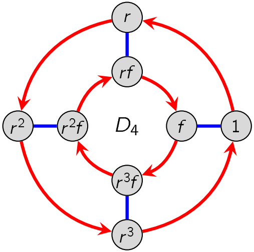


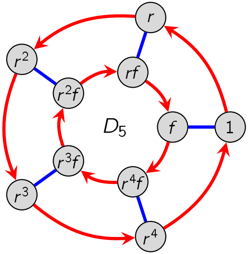
# Dihedral groups

Several different presentations for $D_n$ are:

$$D_n = \left\langle r, f \mid r^n = 1,\ f^2 = 1,\ rfr = f \right\rangle = \left\langle r, f \mid r^n = 1,\ f^2 = 1,\ rf = fr^{n-1} \right\rangle.$$




## Warning!

Many books denote the symmetries of the $n$-gon as $D_{2n}$.

A strong advantage to our convention is that we can write

$$C_n = \langle r \rangle = \left\{1, r, r^2, \ldots, r^{n-1}\right\} \leq \langle r, f \rangle = D_n.$$

(In the other convention, $C_n \leq D_{2n}$, which I find annoying.)

# Dihedral groups

Another canonical way to generate $D_n$ is with two reflections:

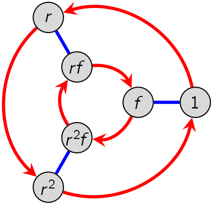
- $s := f$
- $t := fr = r^{n-1}f$ – Convince yourself that this is indeed a different reflection.

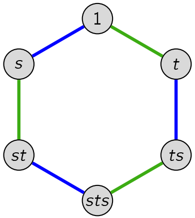
Composing these in either order is a rotation of $2\pi/n$ radians:

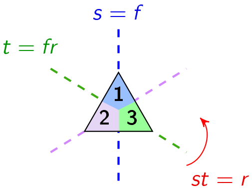
$$st = f(fr) = r, \qquad ts = (fr)f = (r^{n-1}f)f = r^{n-1}.$$

A group presentation with these generators is

$$D_n = \langle s, t \mid s^2 = 1,\ t^2 = 1,\ (st)^n = 1 \rangle = \{ \underbrace{1, st, ts, (st)^2, (ts)^2, \ldots,}_{\text{rotations}}\ \underbrace{s, t, sts, tst, \ldots}_{\text{reflections}} \}.$$



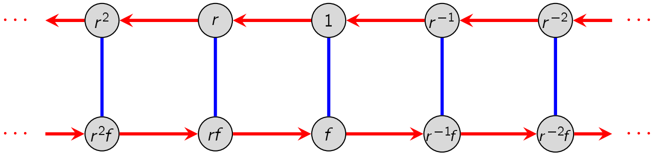



# Dihedral groups

## Definition

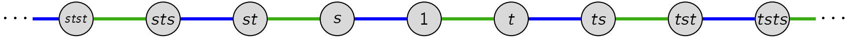
The infinite dihedral group, denoted $D_\infty$, has presentation

$$D_\infty = \langle r, f \mid f^2 = 1,\ rfr = f \rangle.$$




We can also generate $D_\infty$ with two reflections, $s := f$ and $t = fr$.

$$D_\infty = \langle s, t \mid s^2 = 1,\ t^2 = 1 \rangle = \{ \underbrace{1, st, ts, (st)^2, (ts)^2, \dots,}_{\text{“rotations”}}\ \underbrace{s, t, sts, tst, \dots}_{\text{“reflections”}} \}.$$

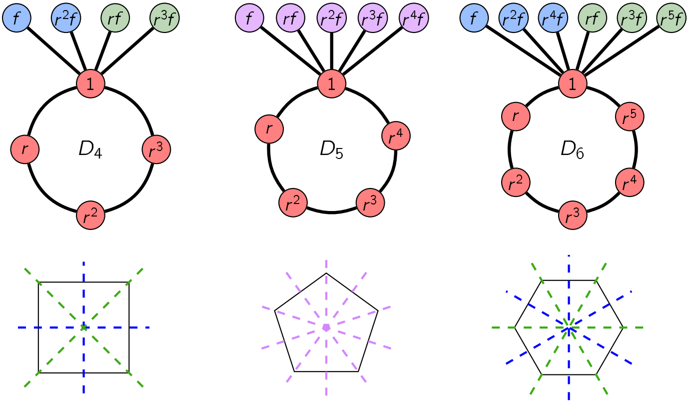


# Cycle graphs of dihedral groups

The maximal orbits of $D_n$ consist of

- 1 orbit of size $n$ containing $\{1, r, \ldots, r^{n-1}\}$;
- $n$ orbits of size 2 containing $\{1, r^k f\}$ for $k = 0, 1, \ldots, n-1$.

Unless $n$ is prime, the size-$n$ orbit will have smaller subsets that are orbits.



# Cayley tables of dihedral groups

The separation of $D_n$ into rotations and reflections is visible in its Cayley tables.

| | $1$ | $r$ | $r^2$ | $r^3$ | $f$ | $rf$ | $r^2f$ | $r^3f$ |
|---|---|---|---|---|---|---|---|---|
| $1$ | $1$ | $r$ | $r^2$ | $r^3$ | $f$ | $rf$ | $r^2f$ | $r^3f$ |
| $r$ | $r$ | $r^2$ | $r^3$ | $1$ | $rf$ | $r^2f$ | $r^3f$ | $f$ |
| $r^2$ | $r^2$ | $r^3$ | $1$ | $r$ | $r^2f$ | $r^3f$ | $f$ | $rf$ |
| $r^3$ | $r^3$ | $1$ | $r$ | $r^2$ | $r^3f$ | $f$ | $rf$ | $r^2f$ |
| $f$ | $f$ | $r^3f$ | $r^2f$ | $rf$ | $1$ | $r^3$ | $r^2$ | $r$ |
| $rf$ | $rf$ | $f$ | $r^3f$ | $r^2f$ | $r$ | $1$ | $r^3$ | $r^2$ |
| $r^2f$ | $r^2f$ | $rf$ | $f$ | $r^3f$ | $r^2$ | $r$ | $1$ | $r^3$ |
| $r^3f$ | $r^3f$ | $r^2f$ | $rf$ | $f$ | $r^3$ | $r^2$ | $r$ | $1$ |

| | $1$ | $r$ | $r^2$ | $r^3$ | $f$ | $rf$ | $r^2f$ | $r^3f$ |
|---|---|---|---|---|---|---|---|---|
| $1$ | $1$ | $r$ | $r^2$ | $r^3$ | $f$ | $rf$ | $r^2f$ | $r^3f$ |
| $r$ | $r$ | $r^2$ | $r^3$ | $1$ | $rf$ | $r^2f$ | $r^3f$ | $f$ |
| $r^2$ | $r^2$ | $r^3$ | $1$ | $r$ | $r^2f$ | $r^3f$ | $f$ | $rf$ |
| $r^3$ | $r^3$ | $1$ | $r$ | $r^2$ | $r^3f$ | $f$ | $rf$ | $r^2f$ |
| $f$ | $f$ | $r^3f$ | $r^2f$ | $rf$ | $1$ | $r^3$ | $r^2$ | $r$ |
| $rf$ | $rf$ | $f$ | $r^3f$ | $r^2f$ | $r$ | $1$ | $r^3$ | $r^2$ |
| $r^2f$ | $r^2f$ | $rf$ | $f$ | $r^3f$ | $r^2$ | $r$ | $1$ | $r^3$ |
| $r^3f$ | $r^3f$ | $r^2f$ | $rf$ | $f$ | $r^3$ | $r^2$ | $r$ | $1$ |

| | | |
|---|---|---|
| | **rotation** | **reflection** |
| | **reflection** | **rotation** |

The partition of $D_n$ as depicted above has the structure of group $C_2$.

"Shrinking" a group in this way is called a quotient.

It yields a group of order 2 with the following Cayley table:

| | $1$ | $f$ |
|---|---|---|
| $1$ | $1$ | $f$ |
| $f$ | $f$ | $1$ |

# Abelian groups

## Definition

A group $G$ is abelian if $ab = ba$ for all $a, b \in G$.

## Claim

Every cyclic group is abelian.

## Remark

To check that $G$ is abelian, it suffices to only check that $ab = ba$ for all pairs of generators.

## Jokes

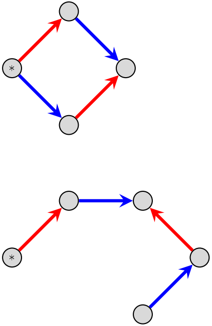
- What's purple and commutes?
- What's warm, nourishing, delicious, and commutative?

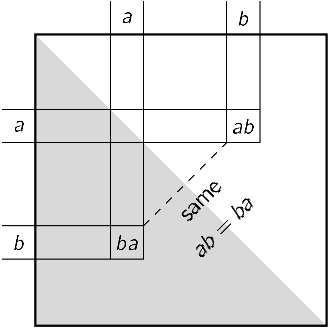
# Abelian groups

It is easy to check whether a group is abelian from either its Cayley graph or Cayley table.

$ab = ba$

$ab \neq ba$





# Abelian groups

One way to build abelian groups is to "glue together" cyclic groups using direct products.

## Fundamental Theorem of Finite Abelian Groups

Every finite abelian group $A$ is isomorphic to a direct product of cyclic groups

$$A \cong \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \cdots \times \mathbb{Z}_{k_m}, \qquad \text{for some } k_1, k_2, \ldots, k_m \in \mathbb{N}.$$

(More on this later.)

What *infinite* abelian groups might there be?

- The *rational numbers*, $\mathbb{Q}$, under addition
- The *real numbers*, $\mathbb{R}$, under addition
- The *complex numbers*, $\mathbb{C}$, under addition
- all of these (with 0 removed) under multiplication: $\mathbb{Q}^*$, $\mathbb{R}^*$, and $\mathbb{C}^*$.
- the positive versions of these under multiplication: $\mathbb{Q}^+$, $\mathbb{R}^+$ (but not $\mathbb{C}^+$).

# Other abelian groups

It is clear that $\mathbb{Z} \le \mathbb{Q} \le \mathbb{R} \le \mathbb{C}$. However, there are many more subgroups of $\mathbb{C}$ than these.

Most of the following are actually rings: additive groups also closed under multiplication. We'll study these more later.

## Definition

The Gaussian integers are the complex numbers of the form

$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}.$$

We'll see $\mathbb{Z}[\sqrt{-m}]$ and others when we encounter rings of algebraic integers.

The set of polynomials in $x$ "*over the integers*" is a group under addition, denoted

$$\mathbb{Z}[x] = \{a_n x^n + \cdots + a_1 x + a_0 \mid a_i \in \mathbb{Z}\}.$$

We can also look at certain subgroups, like the polynomials of degree $\le n$.

Polynomials can be defined in multiple variables, like

$$\mathbb{Z}[x, y] = \Big\{ \sum a_{ij} x^i y^j \mid a_{ij} \in \mathbb{Z}, \ \text{ all but finitely many } a_{ij} = 0 \Big\},$$

or over a finite ring such as $\mathbb{Z}_n$.

# Groups of permutations

Loosely speaking, a permutation is an action that rearranges a set of objects.

## Definition

Let $X$ be a set. A permutation of $X$ is a bijection $\pi\colon X \to X$.

## Definition

The permutations of a set $X$ form a group that we denote $S_X$. The special case when $X = \{1, \ldots, n\}$ is called the symmetric group, and denoted $S_n$.

If $|X| = |Y|$, then $S_X \cong S_Y$, so we'll usually work with $S_n$, which has order $n! = n(n-1)\cdots 2 \cdot 1$.

There are several notations for permutations, each with their strengths and weaknesses.

This is best seen with an example:

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\pi(i)$ | 2 | 3 | 1 | 6 | 5 | 4 |

"*one-line notation*"


"*permutation diagram*"

$$\pi = (1\,2\,3)\,(4\,6)$$

"*cycle notation*"

## Permutation notations

**One-line notation**: $\pi = 231654, \qquad \sigma = 564123$

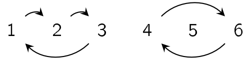
Pros:

- concise
- nice visualization of rearrangement

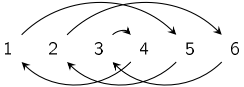
Cons:

- bad for combining permutations
- not clear where elements get mapped
- hard to compute the inverse

**Permutation diagram**: $\pi$:




$\sigma$:




Pros:

- can see where elements get mapped
- easy to compute inverses
- convenient for combining permutations

Cons:

- cumbersome to write
- can get tangled

**Cycle notation**: $\pi = (1\,2\,3)\,(4\,6), \qquad \sigma = (1\,5\,2\,6\,3\,4)$;

Pros:

- short and concise
- easy to see the disjoint cycles
- convenient for combining permutations

Cons:

- representation isn't unique
- not clear what $n$ is

# Cycle notation

The cycle $(1\,4\,6\,5)$ means

*"1 goes to 4, which goes to 6, which does to 5, which goes back to 1."*

Thus, we can write $(1\,4\,6\,5) = (4\,6\,5\,1) = (6\,5\,1\,4) = (5\,1\,4\,6)$.

To find the inverse of a cycle, write it backwards:

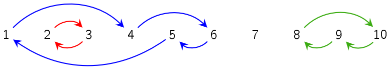
$$(1\,4\,6\,5)^{-1} = (5\,6\,4\,1) = (1\,5\,6\,4) = \cdots$$

Though it's not necessary, we usually prefer to begin a cycle with its smallest number.

## Remark

Every permutation in $S_n$ can be written in cycle notation as a product of disjoint cycles, and this is unique up to commuting and cyclically shifting cycles.

For example, consider the following permutation in $S_{10}$:




as $(1\ 4\ 6\ 5)\,(2\ 3)\,(8\ 10\ 9)$.

This is a product of four disjoint cycles. Since they are disjoint, they commute:

$$(1465)\,(23)\,(8\ 10\ 9) = (23)\,(8\ 10\ 9)\,(1465) = (23)\,(8\ 10\ 9)\,(1465) = \cdots$$

## Composing permutations

**Remark**

The order of a permutation is the least common multiple of the sizes of its disjoint cycles.

For example, $(1\ 3\ 8\ 6)(2\ 9\ 7\ 4\ 10\ 5) \in S_{10}$ has order 12; this should be intuitive.

When cycles are not disjoint, order matters.

Many books compose permutations from right-to-left, due to function composition.

Since we have been using right Cayley graphs, we will compose them from left-to-right.

**Notational convention**

Composition of permutations will be done left-to-right. That is, given $\pi, \sigma \in S_n$,

$$\pi\sigma \quad \text{means "}\textit{do } \pi\textit{, then do } \sigma\text{".}$$

The main drawback about our convention is that it does not work well with function notation applied to elements, like $\pi(i)$.

For example, notice that

$$(\pi\sigma)(i) = \sigma(\pi(i)) \neq \pi(\sigma(i)).$$

However, we will hardly ever use this notation, so that drawback is minimal.

# Composing permutations

Here are two ways illustrating how permutations are composed, with the example

*First do*

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\pi(i)$ | 4 | 3 | 1 | 2 | 6 | 5 |

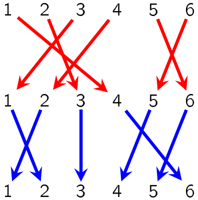
*then do*

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\sigma(i)$ | 2 | 1 | 3 | 6 | 4 | 5 |

- "By stacking:"

*first* (1423)(56)

*then* (12)(465)

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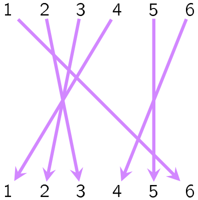
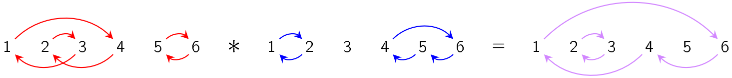



(164)(23)

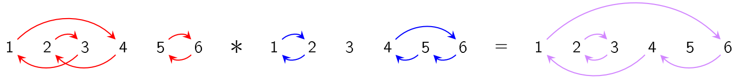
- "By cycles:"




*first* (1423)(56) *then* (12)(465) (164)(23)

# Composing permutations in cycle notation

Let's practice composing two permutations:




*first* (1423)(56) *then* (12)(465) (164)(23)

Let's now do that in slow motion.

In the example above, we start with 1 and then read off:

- "**1** goes to 4, then 4 goes to 6"; Write: (1 6
- "6 goes to 5, then 5 goes to 4"; Write: (1 6 4
- "4 goes to 2, then 2 goes to **1**"; Write: (1 6 4), and start a new cycle.
- "**2** goes to 3, then 3 is fixed"; Write: (1 6 4) (2 3
- "3 goes to 1, then 1 goes to **2**"; Write: (1 6 4) (2 3), and start a new cycle.
- "**5** goes to 6, then 6 goes to **5**"; Write: (1 6 4) (2 3) (5); now we're done.

We typically omit 1-cycles (fixed points), so the permutation above is just (1 6 4) (2 3).

# Cayley's theorem

A set of permutations that forms a group is called a permutation group.

A fundamental theorem by British mathematician Arthur Cayley (1821–1895) says that every finite group can be thought of as a collection of permutations.

This is clear for groups of symmetries like $V_4$, $C_n$, or $D_n$, but less so for groups like $Q_8$.

## Cayley's theorem

Every finite group is "isomorphic to" a collection of permutations, i.e., some subgroup of $S_n$.

We don't have the mathematical tools to prove this formally, but we'll get a 1-line proof when we study group actions.

Let's make an intuitive argument, though.

# Constructing permutations from a Cayley graph

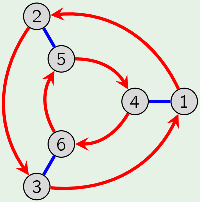
Here is an algorithm given a Cayley graph with $n$ nodes:

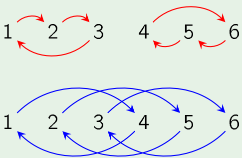
1. number the nodes 1 through $n$,
2. interpret each arrow type in the Cayley graph as a permutation.

Take the permutations corresponding to the generators.

## Example

Let's try this with $D_3 = \langle r, f \rangle$.




We see that $D_3$ is isomorphic to the subgroup $\langle (123)(456),\ (14)(25)(36) \rangle$ of $S_6$.

## Question:

Would this have worked if we had chosen a different numbering?

# Constructing permutations from a Cayley table

Here is an algorithm given a Cayley table with $n$ elements:

1. replace the table headings with 1 through $n$,
2. make the appropriate replacements throughout the rest of the table,
3. interpret each row (or column) as a permutation.

Take the permutations corresponding to *any* generating set.

## Example

Let's try this with the Cayley table for $D_3 = \langle r, f \rangle$.

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 2 | 3 | 1 | 5 | 6 | 4 |
| 3 | 3 | 1 | 2 | 6 | 4 | 5 |
| 4 | 4 | 6 | 5 | 1 | 3 | 2 |
| 5 | 5 | 4 | 6 | 2 | 1 | 3 |
| 6 | 6 | 5 | 4 | 3 | 2 | 1 |

**Row 1** $(1)$: 1 2 3 4 5 6

**Row 2** $(r)$: 1 2 3 4 5 6

**Row 3** $(r^2)$: 1 2 3 4 5 6

**Row 4** $(f)$: 1 2 3 4 5 6

**Row 5** $(rf)$: 1 2 3 4 5 6

**Row 6** $(r^2f)$: 1 2 3 4 5 6

We see that $D_3$ is isomorphic to the subgroup $\langle (123)(456), (14)(26)(35) \rangle$ of $S_6$.