Homework #11 (due Apr 13)

Definitions, for easy reference

All these definitions look really similar but have important differences.
Suppose that G acts on a set S (on the right) via $\phi : G \to \mathrm{Perm}(S)$.

Definition (or, really, notation)**.** For $s \in S$ and $g \in G$, we write $\mathbf{s}.\phi(\mathbf{g})$ to denote the new element of S that s is sent to when we push the g button. **Emphasizing:** $s.\phi(g)$ is a new element of S!

Definition (orbit)**.** The orbit of $s \in S$ is the set of all the new s's that s moves to:

$$\mathrm{orb}(s) = \{s.\phi(g) \mid g \in G\}.$$

Note $\mathrm{orb}(s) \subseteq S$.

Definition (stabilizer)**.** The stabilizer of $s \in S$ is the set of all the g's that don't move s:

$$\mathrm{stab}(s) = \{g \in G \mid s.\phi(g) = s\}.$$

Note $\mathrm{stab}(s) \leq G$.

Definition (fixator)**.** The fixator of $g \in G$ is the set of all the s's that don't get moved by g:

$$\mathrm{fix}(g) = \{s \in S \mid s.\phi(g) = s\}.$$

Note $\mathrm{fix}(g) \subseteq S$.

Definition (kernel)**.** The kernel of the action is the set of all the "broken buttons:"

$$\mathrm{Ker}(\phi) = \{k \in G \mid \phi(k) = e\} = \{k \in G \mid s.\phi(k) = s \textbf{ for all } s \in S\}.$$

Note $\mathrm{Ker}(\phi) \leq G$, and indeed, $\mathrm{Ker}(\phi) = \bigcap_{s \in S} \mathrm{stab}(s)$.

Definition (fixed points)**.** The fixed points of the action is the set of s's that never move:

$$\mathrm{Fix}(\phi) = \{s \in S \mid s.\phi(g) = s \textbf{ for all } g \in G\}.$$

Note $\mathrm{Fix}(\phi) \subseteq S$, and indeed, $\mathrm{Fix}(\phi) = \bigcap_{g \in G} \mathrm{fix}(g)$.

Problem 1. Write yourself a couple of good paragraphs discussing the similarities and differences between these five features of the group action.

Propositions from class

Problem 2. Prove that for any $s \in S$, the set stab(s) is a subgroup of G. (See outline on slide 10.)

Problem 3. Prove that any two elements in the same orbit have conjugate stabilizers. Specifically:

$$\operatorname{stab}(s.\phi(g)) = g^{-1} \operatorname{stab}(s) g, \text{ for all } g \in G \text{ and } s \in S.$$

In other words, if x stabilizes s, then $g^{-1}xg$ stabilizes $s.\phi(g)$.
(Parsing this out is the hardest part. See outline and intution on slide 11; note in particular that $s.\phi(g)$ is a generic element of the orbit of s.)

Proving the orbit-stabilizer theorem

Orbit-Stabilizer Theorem. Suppose G acts on a set S (on the right) by $\phi : G \to \operatorname{Perm}(S)$. Then for any $s \in S$, "the size of the orbit is the index of the stabilizer:"

$$|\operatorname{orb}(s)| = [G : \operatorname{stab}(s)].$$

Equivalently, $|\operatorname{orb}(s)| \cdot |\operatorname{stab}(s)| = |G|$.

Problem 4. In class we proved this for a **right** group action (written $s.\phi(g)$) by setting up a bijection (aka a correspondence) between orb(s) and **right** cosets of stab(s). Use similar logic to write down a proof of the orbit-stabilizer theorem for a **left** group action (written $\phi(g).s$).

Using the orbit-stabilizer theorem

The orbit-stabilizer theorem is a cheat code. There are a ton of cool theorems whose proofs are just filling in the blanks in this template:

> *Proof.* Let G act on ____ by ____. This defines a homomorphism
>
> $$\phi : G \to \operatorname{Perm}(\_\_\_\_)$$
>
> The orbit of __ is ____, and the stabilizer of __ is ____. Therefore, by the orbit-stabilizer theorem, . . . □

Problem 5. Use this template to prove that if $H \leq G$, then $|\operatorname{cl}_G(H)| = [G : N_G(H)]$ ("the size of the conjugacy class of H is the index of its normalizer"):

- Make G act on its set of subgroups, $S = \{H \mid H \leq G\}$, by (right) conjugation.
  (Your job in this step is to tell me specifically what is $H.\phi(g)$.)
- For some $H \in S$, what is stab(H)?
- For some $H \in S$, what is orb(H)?
- Apply the orbit-stabilizer theorem.

Problem 6. Here's a similar thing. We previously defined the conjugacy class of an element $x \in G$:

$$\mathrm{cl}_G(x) = \{g^{-1}xg \mid g \in G\}.$$

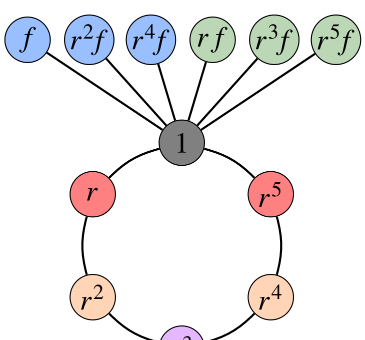
- Make G act on **itself**, $S = G$, by right conjugation.
- For some $x \in S$ (aka $x \in G$), what is $\mathrm{stab}(x)$?
  (This set is called the "centralizer" of x; do you see why?)
- For some $x \in S$ (aka $x \in G$), what is $\mathrm{orb}(x)$?
- Apply the orbit-stabilizer theorem to reach an interesting conclusion about how the sizes of these two things are related.

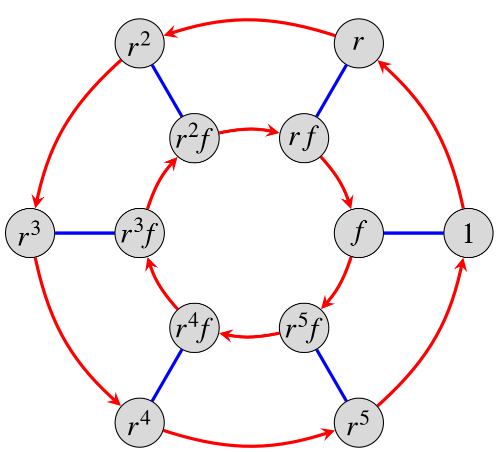
A specific example: D_6

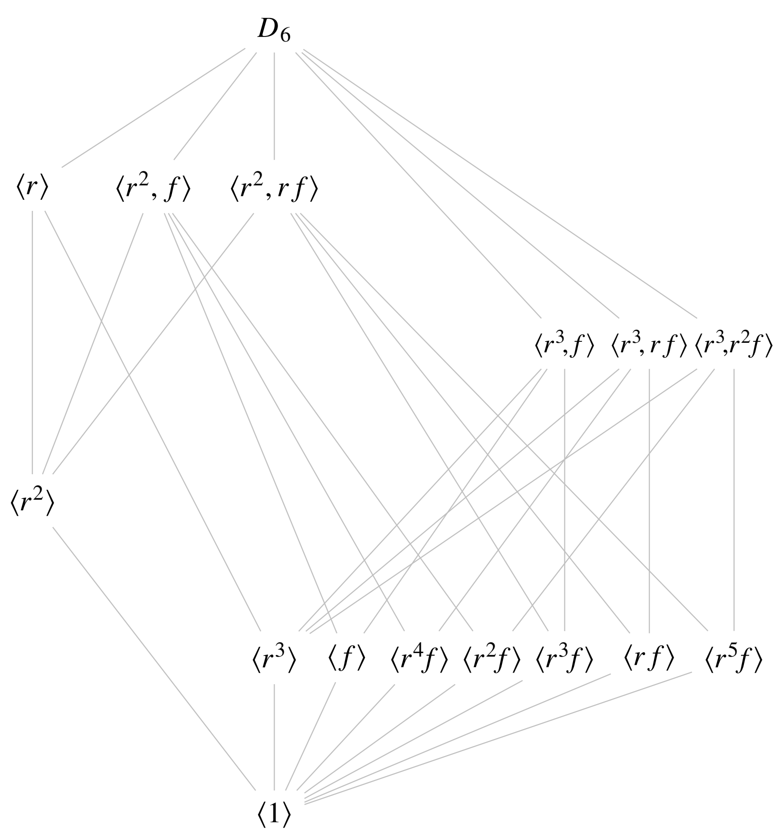
Let $G = D_6 = \langle r, f \rangle$ act on its set $S = \{H \leq D_6\}$ of subgroups by (right) conjugation, i.e.,

$$\phi\colon G \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } H \mapsto g^{-1}Hg.$$

A Cayley graph, cycle graph, and subgroup lattice for D_6 are shown below.







Problem 7. Draw the action graph superimposed on the subgroup lattice. For example, since $r^{-1}fr = r^4f$ (which you can read off the Cayley graph), the r-conjugate of $\langle f \rangle$ is $\langle f \rangle.\phi(r) = \langle r^4 f \rangle$, so I would draw a red arrow from $\langle f \rangle$ to $\langle r^4 f \rangle$ in the subgroup lattice.

Problem 8. Construct the fixed point table (y'know, the one with checkmarks).

Problem 9. Find $\mathrm{stab}(H)$ for each subgroup $H \leq D_6$, and $\mathrm{fix}(g)$ for each $g \in D_6$.

Problem 10. Find $\mathrm{Ker}(\phi)$ and $\mathrm{Fix}(\phi)$.

Problem 11. What do each of these things mean in this context?

- $\mathrm{orb}(H)$
- $\mathrm{stab}(H)$
- $[G : \mathrm{stab}(H)]$ (hint: your answer should sound like "the number of cosets of…")
- $\mathrm{Fix}(\phi)$
- $\mathrm{Ker}(\phi)$
- $\mathrm{fix}(g)$

Problem 12. Apply the orbit-counting theorem. What does the result mean in this context?

Bonus problems!

Problem 13. Use the results of several other problems on this homework set (including 3 and 5) to explore this question: Under what circumstances is $\mathrm{stab}(s)$ a **normal** subgroup of G?
(I don't know that there's one specific answer that I'm looking for, but you do have the tools to say several interesting things.)

Problem 14. Suppose a group G of order 55 acts on a set S of size 14, and pick some $s \in S$.

(a) What are the possible sizes of the orbit of s?

(b) What are the possible sizes of the stabilizer of s?

(c) Show that this action must have a fixed point.

(d) What is the fewest number of fixed points that this action can have? Justify your answer.

Problem 15. Prove these things we observed in class: if G acts on itself by right multiplication (so $S = G$ and $s.\phi(g) = sg$), then

- the action is *transitive*, i.e., there is only one orbit, and
- the action is *faithful*, i.e., $\mathrm{Ker}(\phi)$ is trivial.

Problem 16. Let G act on itself by conjugation, and derive the *class equation*:

$$|G| = |Z(G)| + \sum [G : C_G(x)],$$

where the sum is over one representative x from each conjugacy class that isn't in the center of the group, and $C_G(x)$ is the "centralizer" discussed in Problem 6.