Group actions, part 2!

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With many thanks to Matthew Macauley,
`http://www.math.clemson.edu/~macaule/`

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Overview

Intuitively, a group action occurs when a group G "naturally permutes" a set S *of states*.

Formal definition

A group G acts on a set S if there is a homomorphism $\phi\colon G \to \mathsf{Perm}(S)$.
We'll use right group actions,
and we'll write $s.\phi(g)$ to denote "where pushing the g-button sends state s."

Definition

A set S with a (right) action by G is called a (right) G-set.

Big ideas

- An action $\phi\colon G \to \mathsf{Perm}(S)$ endows S with an **algebraic structure**.
- *Action graphs are to G-sets, like how Cayley graphs are to groups.*

Five features of every group action

Every group action has **five fundamental features** that we will always try to understand:

	local (about an s or a g)	global (about the whole action ϕ)
subsets of S	orbit of s fixator of g	fixed points of the action
subgroups of G	stabilizer of s	kernel of the action

We will see parallels within and between these classes.

For example, two "local" features will be "dual" to each other, as will the global features.

Also, our global features can be expressed as intersections of our local features, either ranging over all $s \in S$, or over all $g \in G$.

We'll start by exploring the three local features.

Notation

Throughout, we'll denote identity elements by $1 \in G$ and $e \in \mathrm{Perm}(S)$.

Local: orbits, stabilizers, fixators

Two local features: orbits and stabilizers

Suppose G acts on set S, and pick some $s \in S$. We can ask two questions about it:

(i) What other states (in S) are reachable from s? (We call this the orbit of s.)

(ii) What group elements (in G) fix s? (We call this the stabilizer of s.)

Definition

Suppose that G acts on a set S (on the right) via $\phi \colon G \to \mathrm{Perm}(S)$.

(i) The orbit of $s \in S$ is the set

$$\mathrm{orb}(s) = \{ s.\phi(g) \mid g \in G \}.$$

(ii) The stabilizer of s in G is

$$\mathrm{stab}(s) = \{ g \in G \mid s.\phi(g) = s \}.$$

In terms of the action graph

(i) The orbit of $s \in S$ is the connected component containing s.

(ii) The stabilizer of $s \in S$ are the group elements whose paths start and end at s; “loops.”

The third local feature: fixators

Our first two local features were specific to a certain element $s \in S$.

Our last local feature is defined for each group element $g \in G$. A natural question to ask is:

(iii) What *states* (in S) does g fix?

Definition

Suppose that G acts on a set S (on the right) via $\phi\colon G \to \mathrm{Perm}(S)$.

(iii) The fixator of $g \in G$ are the elements $s \in S$ fixed by g:

$$\mathrm{fix}(g) = \{s \in S \mid s.\phi(g) = s\}.$$

In terms of the action graph

(iii) The fixator of $g \in G$ are the nodes from which the g-paths are loops.

In terms of the "group switchboard analogy"

(i) The orbit of $s \in S$ are the elements in S the can be obtained by pressing some combination of buttons.

(ii) The stabilizer of $s \in S$ consists of the buttons that have no effect on s.

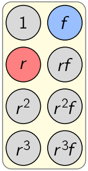
(iii) The fixator of $g \in G$ are the elements in S that don't move when we press the g-button.

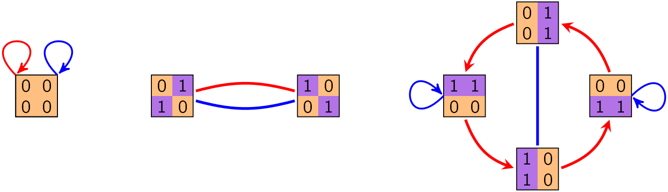
Three local features: orbits, stabilizers, and fixators

Here's the action graph of our running example of D_4 acting on S the set of binary squares.

Find the orbit and stabilizer of each binary square, and the fixator of each element of D_4.

"Group switchboard"

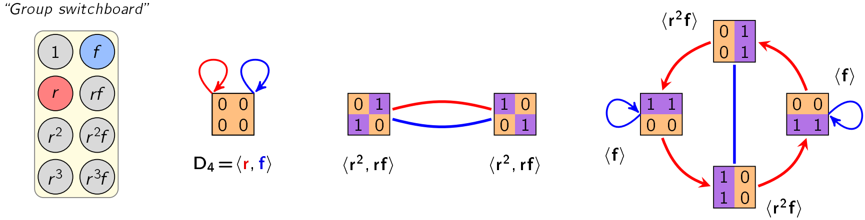




Three local features: orbits, stabilizers, and fixators

The orbits of our running example are the 3 connected components.

Each node is labeled by its stabilizer.



The fixators are $\mathrm{fix}(1) = S$, and

$$\mathrm{fix}(r) = \mathrm{fix}(r^3) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \qquad \mathrm{fix}(r^2) = \mathrm{fix}(rf) = \mathrm{fix}(r^3 f) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\mathrm{fix}(f) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\} \qquad \mathrm{fix}(r^2 f) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

Local duality: stabilizers vs. fixators

Consider the following table, where a checkmark at (g, s) means g fixes s.

	$\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$	$\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$	$\begin{smallmatrix}0&0\\1&1\end{smallmatrix}$	$\begin{smallmatrix}0&1\\0&1\end{smallmatrix}$	$\begin{smallmatrix}1&1\\0&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\1&0\end{smallmatrix}$
1	✓	✓	✓	✓	✓	✓	✓
r	✓						
r^2	✓	✓	✓				
r^3	✓						
f	✓			✓		✓	
rf	✓	✓	✓				
r^2f	✓				✓		✓
r^3f	✓	✓	✓				

- the stablizers can be read off the columns: *group elements that fix* $s \in S$
- the fixators can be read off the rows: *set elements fixed by* $g \in G$.

The stabilizer subgroup

Notice how in our example, the stabilizer of each $s \in S$ is a subgroup.

This holds true for any action.

Proposition

For any $s \in S$, the set $\mathbf{stab}(s)$ is a subgroup of G.

Proof (outline)

To show $\mathbf{stab}(s)$ is a group, we need to show three things:

(i) **Identity**. That is, $s.\phi(1) = s$.

(ii) **Inverses**. That is, if $s.\phi(g) = s$, then $s.\phi(g^{-1}) = s$.

(iii) **Closure**. That is, if $s.\phi(g) = s$ and $s.\phi(h) = s$, then $s.\phi(gh) = s$.

Alternatively, it suffices to show that if $s.\phi(g) = s$ and $s.\phi(h) = s$, then $s.\phi(gh^{-1}) = s$,

You'll do this on the homework.

All three of these are very intuitive in our our switchboard analogy.

The stabilizer subgroup

As we've seen, elements in the same orbit can have different stabilizers.

Proposition (HW exercise)

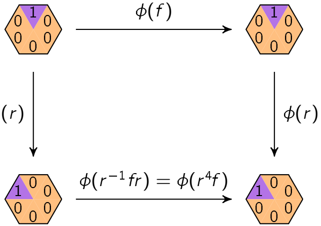
Set elements in the same orbit have conjugate stabilizers:

$$\operatorname{stab}(s.\phi(g)) = g^{-1}\operatorname{stab}(s)g, \quad \text{for all } g \in G \text{ and } s \in S.$$

In other words, if x stabilizes s, then $g^{-1}xg$ stabilizes $s.\phi(g)$.

Here are several ways to visualize what this means and why.

$$\begin{tikzcd} s \arrow[r, mapsto, "\phi(x)"] \arrow[d, mapsto, "\phi(g)"'] & s \arrow[d, mapsto, "\phi(g)"] \\ s' \arrow[r, mapsto, "\phi(g^{-1}xg)"] & s' \end{tikzcd}$$

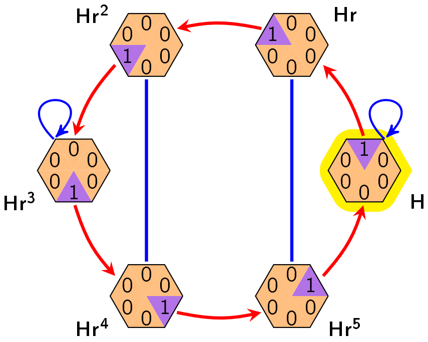


In other words, if x is a loop from s, and $s \xrightarrow{g} s'$, then $g^{-1}xg$ is a loop from s'.

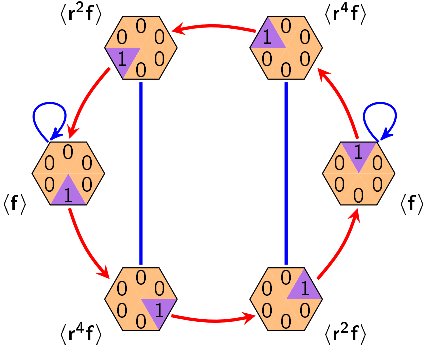
The stabilizer subgroup

Here is another example of an action (or G-set), this time of $G = D_6$ acting on these six "Pacman hexagons."

Let s be the highlighted hexagon, and $H = \mathbf{stab}(s)$.



labeled by destinations



labeled by stabilizers

Global: fixed points and the kernel

Two global features: fixed points and the kernel

Our last two features are properties of the action ϕ, rather than of specific elements.

The first definition is new, and the second is an familiar concept in this new setting.

Definition

Suppose that G acts on a set S via $\phi\colon G \to \mathsf{Perm}(S)$.

(iv) The kernel of the action is the set

$$\mathsf{Ker}(\phi) = \{k \in G \mid \phi(k) = e\} = \{k \in G \mid s.\phi(k) = s \text{ for all } s \in S\}.$$

(v) The fixed points of the action, denoted $\mathsf{Fix}(\phi)$, are the orbits of size 1:

$$\mathsf{Fix}(\phi) = \{s \in S \mid s.\phi(g) = s \text{ for all } g \in G\}.$$

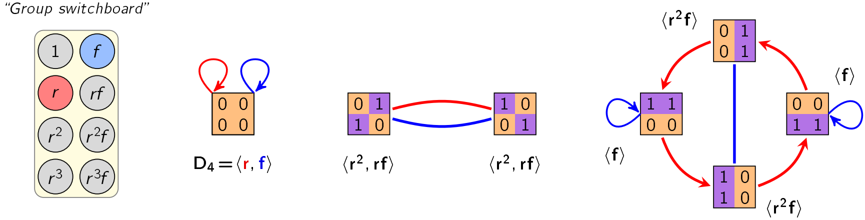
Proposition (global duality: fixed points vs. kernel)

Suppose that G acts on a set S via $\phi\colon G \to \mathsf{Perm}(S)$. Then

$$\mathsf{Ker}(\phi) = \bigcap_{s \in S} \mathsf{stab}(s), \qquad \text{and} \qquad \mathsf{Fix}(\phi) = \bigcap_{g \in G} \mathsf{fix}(g).$$

Let's also write $\mathsf{Orb}(\phi)$ for the set of orbits of ϕ.

Two global features: fixed points and the kernel



In terms of the action graph

(iv) The kernel of ϕ are the paths that are "loops from every $s \in S$."

(v) The fixed points of ϕ are the size-1 connected components.

In terms of the group switchboard analogy

(iv) The kernel of ϕ are the "broken buttons"; those $g \in G$ that have no effect on any s.

(v) The fixed points of ϕ are those $s \in S$ that are not moved by pressing any button.

Global duality: fixed points vs. kernel

Consider the following table, where a checkmark at (g, s) means g fixes s.

	$\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$	$\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$	$\begin{smallmatrix}0&0\\1&1\end{smallmatrix}$	$\begin{smallmatrix}0&1\\0&1\end{smallmatrix}$	$\begin{smallmatrix}1&1\\0&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\1&0\end{smallmatrix}$
1	✓	✓	✓	✓	✓	✓	✓
r	✓						
r^2	✓	✓	✓				
r^3	✓						
f	✓			✓		✓	
rf	✓	✓	✓				
$r^2 f$	✓				✓		✓
$r^3 f$	✓	✓	✓				

- the fixed points consist of columns with all checkmarks: *set elts fixed by everything*
- the kernel consists of the rows with all checkmarks: *group elements that fix everything.*

Theorems!

Two theorems on orbits, and their consequences

Our binary square example gives us some key intutition about group actions.

Qualitative observations

- elements in larger orbits tend to have smaller stabilizers, and vice-versa
- action tables with more "checkmarks" tend to have more orbits.

Both of these qualitative observations can be formalized into quantitative theorems.

Theorems

1. **Orbit-stabilizer theorem**: the size of an orbit is the index of the stabilizer.
2. **Orbit-counting theorem**: the number of orbits is the average number of things fixed by a group element.

If we set up our group actions correctly, the orbit-stabilizer theorem will imply:

- The size of the conjugacy class $\mathrm{cl}_G(H)$ is the index of the normalizer of $H \leq G$
- The size of the conjugacy class $\mathrm{cl}_G(x)$ is the index of the centralizer of $x \in G$

We can also determine the number of conjugacy classes from the orbit-counting theorem.

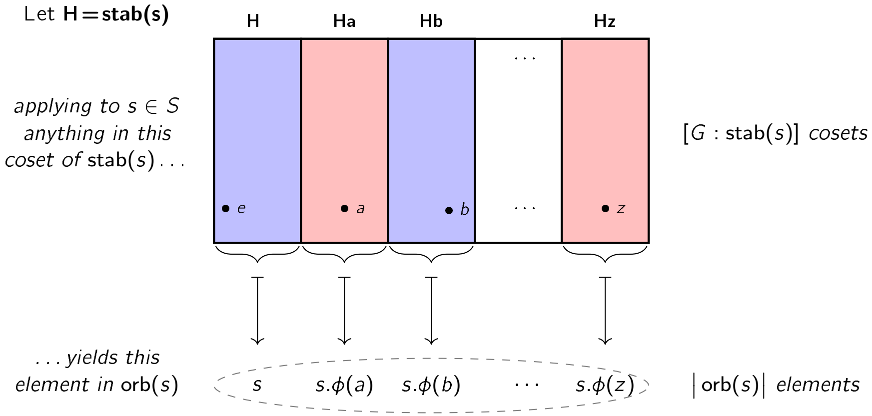
Our first theorem on orbits

Orbit-stabilizer theorem

For any group action $\phi\colon G \to \mathrm{Perm}(S)$, and any $s \in S$,

$$\left|\mathrm{orb}(s)\right| \cdot \left|\mathrm{stab}(s)\right| = \left|G\right|.$$

Equivalently, *the size of the orbit containing s is* $\left|\mathrm{orb}(s)\right| = [G : \mathrm{stab}(s)]$.



Our first theorem on orbits

Orbit-stabilizer theorem

For any group action $\phi\colon G \to \mathrm{Perm}(S)$, and any $s \in S$,

$$\lvert \operatorname{orb}(s) \rvert \cdot \lvert \operatorname{stab}(s) \rvert = \lvert G \rvert\,.$$

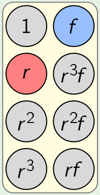
Equivalently, *the size of the orbit containing s is* $\lvert \operatorname{orb}(s) \rvert = [G : \operatorname{stab}(s)]$.

Proof

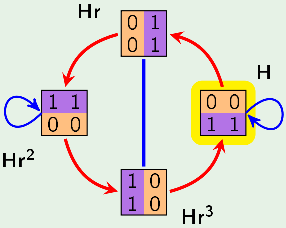
<u>**Goal**</u>: Exhibit a bijection between elements of $\operatorname{orb}(s)$, and right cosets of $\operatorname{stab}(s)$.

That is, "*two g-buttons send s to the same place iff they're in the same coset*".

"Group switchboard"



1	f	$\mathbf{H = stab(s)}$
r	fr	$\mathbf{Hr}$
r^2	fr^2	$\mathbf{Hr^2}$
r^3	fr^3	$\mathbf{Hr^3}$



Note that $s.\phi(g) = s.\phi(k)$ iff g and k are in the same right coset of H in G.

The orbit-stabilizer theorem: $|\mathrm{orb}(s)| \cdot |\mathrm{stab}(s)| = |G|$

Proof (cont.)

Throughout, let $H = \mathrm{stab}(s)$.

"$\Rightarrow$" *If two elements send s to the same place, then they are in the same coset.*

Suppose $g, k \in G$ both send s to the same element of S. This means:

$$\begin{aligned} s.\phi(g) = s.\phi(k) \quad & \Longrightarrow \quad s.\phi(g)\phi(k)^{-1} = s \\ & \Longrightarrow \quad s.\phi(g)\phi(k^{-1}) = s \\ & \Longrightarrow \quad s.\phi(gk^{-1}) = s \qquad \text{(i.e., } gk^{-1} \text{ stabilizes } s) \\ & \Longrightarrow \quad gk^{-1} \in H \qquad \text{(recall that } H = \mathrm{stab}(s)) \\ & \Longrightarrow \quad Hgk^{-1} = H \\ & \Longrightarrow \quad Hg = Hk \end{aligned}$$

"$\Leftarrow$" *If two elements are in the same coset, then they send s to the same place.*

Take two elements $g, k \in G$ in the same right coset of H. This means $Hg = Hk$.

This is the last line of the proof of the forward direction, above. We can change each $\Longrightarrow$ into $\Longleftrightarrow$, and thus conclude that $s.\phi(g) = s.\phi(k)$. $\square$

If we have instead, a left group action, the proof carries through but using left cosets.

Our second theorem on orbits

Orbit-counting theorem

Let a finite group G act on a set S via $\phi\colon G \to \mathrm{Perm}(S)$. Then

$$\left|\mathrm{Orb}(\phi)\right| = \frac{1}{|G|}\sum_{g\in G}\left|\mathrm{fix}(g)\right|.$$

This says that the "*average number of checkmarks per row*" is the number of orbits:

	$\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$	$\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$	$\begin{smallmatrix}0&0\\1&1\end{smallmatrix}$	$\begin{smallmatrix}0&1\\0&1\end{smallmatrix}$	$\begin{smallmatrix}1&1\\0&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\1&0\end{smallmatrix}$
1	✓	✓	✓	✓	✓	✓	✓
r	✓						
r^2	✓	✓	✓				
r^3	✓						
f	✓			✓		✓	
rf	✓	✓	✓				
r^2f	✓				✓		✓
r^3f	✓	✓	✓				

Orbit-counting theorem: $|\text{Orb}(\phi)| = \dfrac{1}{|G|}\displaystyle\sum_{g\in G} |\text{fix}(g)|$.

Proof

Let's first count the number of checkmarks in the action table, three ways:

$$\underbrace{\sum_{g\in G} |\text{fix}(g)|}_{\text{count by rows}} = \Big|\{(g,s)\in G\times S \mid s.\phi(g)=s\}\Big| = \underbrace{\sum_{s\in S} |\text{stab}(s)|}_{\text{count by columns}}.$$

By the orbit-stabilizer theorem, we can replace each $|\text{stab}(s)|$ with $|G|/|\text{orb}(s)|$:

$$\sum_{s\in S} |\text{stab}(s)| = \sum_{s\in S} \frac{|G|}{|\text{orb}(s)|} = |G|\sum_{s\in S}\frac{1}{|\text{orb}(s)|}.$$

Let's express this sum over all disjoint orbits $S = \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_k$ separately:

$$|G|\sum_{s\in S}\frac{1}{|\text{orb}(s)|} = |G|\sum_{\mathcal{O}\in\text{Orb}(\phi)}\Bigg(\underbrace{\sum_{s\in\mathcal{O}}\frac{1}{|\text{orb}(s)|}}_{=1 \quad (why?)}\Bigg) = |G|\sum_{\mathcal{O}\in\text{Orb}(\phi)} 1 = |G|\cdot|\text{Orb}(\phi)|.$$

Equating this last term with the first term gives the desired result. □

Groups acting on themselves!

Groups acting on elements, subgroups, and cosets

It is frequently of interest to analyze the action of a group G on its elements, subgroups, or cosets of some fixed $H \leq G$.

Often, the orbits, stabilizers, and fixed points of these actions are familiar algebraic objects.

A number of deep theorems have a slick proof via a clever group action.

Here are common examples of group actions:

- G acts on itself by multiplication.
- G acts on itself by conjugation.
- G acts on its subgroups by conjugation.
- G acts on the cosets of a fixed subgroup $H \leq G$ by multiplication.

For each of these, we'll characterize the orbits, stabilizers, fixators, fixed points, and kernel.

We'll encounter familiar objects such as conjugacy classes, normalizers, stabilziers, and normal subgroups, as some of our "five fundamental features".

Theorems that we have observed but haven't been able to prove yet will fall in our lap!

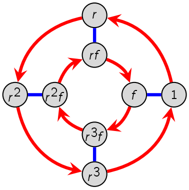
Groups acting on themselves by multiplication

Assume $|G| > 2$. The group G acts on itself (that is, $S = G$) by **right-multiplication**:

$$\phi\colon G \longrightarrow \mathrm{Perm}(S), \qquad \phi(g) = \text{the permutation that sends each } x \mapsto xg.$$

- there is only one orbit: $\mathrm{orb}(x) = G$, for all $x \in G$
- the stabilizer of each $x \in G$ is $\mathrm{stab}(x) = \langle 1 \rangle$
- the fixator of $g \neq 1$ is $\mathrm{fix}(g) = \emptyset$.
- there are no fixed points, and the kernel is trivial:

$$\mathrm{Fix}(\phi) = \bigcap_{g \in G} \mathrm{fix}(g) = \emptyset, \qquad \text{and} \qquad \mathrm{Ker}(\phi) = \bigcap_{s \in S} \mathrm{stab}(s) = \langle 1 \rangle.$$



Cayley's theorem

If $|G| = n$, then there is an embedding $G \hookrightarrow S_n$.

Proof

Let G act on itself by right multiplication. This defines a homomorphism

$$\phi\colon G \longrightarrow \mathrm{Perm}(S) \cong S_n.$$

Since $\mathrm{Ker}(\phi) = \langle 1 \rangle$, it is an embedding. $\square$

Groups acting on themselves by conjugation

Another way a group G can act on itself (that is, $S = G$) is by **right-conjugation**:

$$\phi\colon G \longrightarrow \mathrm{Perm}(S), \qquad \phi(g) = \text{the permutation that sends each } x \mapsto g^{-1}xg.$$

- The orbit of $x \in G$ is its conjugacy class:

$$\mathrm{orb}(x) = \{x.\phi(g) \mid g \in G\} = \{g^{-1}xg \mid g \in G\} = \mathrm{cl}_G(x).$$

- The stabilizer of x is its centralizer:

$$\mathrm{stab}(x) = \{g \in G \mid g^{-1}xg = x\} = \{g \in G \mid xg = gx\} := C_G(x)$$

- The fixator of $g \in G$ is also its centralizer, because

$$\mathrm{fix}(g) = \{x \in S \mid x.\phi(g) = x\} = \{x \in G \mid g^{-1}xg = x\} = C_G(g).$$

- The fixed points and kernel are the center, because

$$\mathrm{Fix}(\phi) = \bigcap_{g\in G} \mathrm{fix}(g) = \bigcap_{g\in G} C_G(g) = Z(G) = \bigcap_{x\in G} C_G(x) = \bigcap_{x\in G} \mathrm{stab}(x) = \mathrm{Ker}(\phi).$$

Groups acting on themselves by conjugation

Let's apply our two theorems:

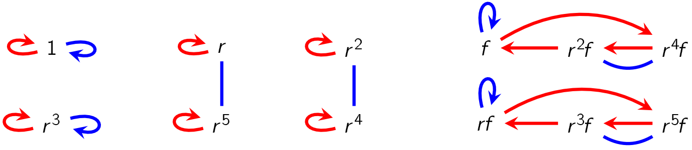
1. **Orbit-stabilizer theorem**. "*the size of an orbit is the index of the stabilizer*":

$$\left|\mathrm{cl}_G(x)\right| = [G : C_G(x)] = \frac{|G|}{|C_G(x)|}.$$

2. **Orbit-counting theorem**. "*the number of orbits is the average number of elements fixed by a group element*":

$$\#\text{conjugacy classes of } G = \text{average size of a centralizer}.$$

Let's revisit our old example of conjugacy classes in $D_6 = \langle r, f \rangle$:



Notice that the stabilizers are $\mathrm{stab}(r) = \mathrm{stab}(r^2) = \mathrm{stab}(r^4) = \mathrm{stab}(r^5) = \langle r \rangle$,

$$\mathrm{stab}(1) = \mathrm{stab}(r^3) = D_6, \qquad \mathrm{stab}(r^i f) = \langle r^3, r^i f \rangle.$$

Groups acting on themselves by conjugation

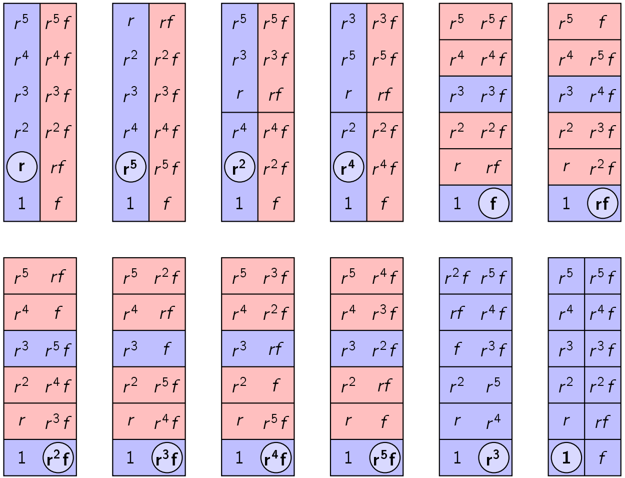
Here is the "*fixed point table*". Note that $\text{Ker}(\phi) = \text{Fix}(\phi) = \langle r^3 \rangle$.

	1	r	r^2	r^3	r^4	r^5	f	rf	r^2f	r^3f	r^4f	r^5f
1	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
r	✓	✓	✓	✓	✓	✓						
r^2	✓	✓	✓	✓	✓	✓						
r^3	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
r^4	✓	✓	✓	✓	✓	✓						
r^5	✓	✓	✓	✓	✓	✓						
f	✓			✓			✓			✓		
rf	✓			✓				✓			✓	
r^2f	✓			✓					✓			✓
r^3f	✓			✓			✓			✓		
r^4f	✓			✓				✓			✓	
r^5f	✓			✓					✓			✓

By the **orbit-counting theorem**, there are $|\text{Orb}(\phi)| = 72/|D_6| = 6$ conjugacy classes.

Groups acting on themselves by conjugation

Here are the cosets of all 12 cyclic subgroups in D_6 (some coincide).



Do you see how to deduce from the orbit-counting theorem that there are 6 conjugacy classes?

Groups acting on subgroups by conjugation

Any group G acts on its set S of subgroups by **right-conjugation**:

$$\phi\colon G \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } H \text{ to } g^{-1}Hg.$$

This is a **right action**, but there is an associated left action: $H \mapsto gHg^{-1}$.

Let $H \leq G$ be an element of S.

- The orbit of H consists of all conjugate subgroups:

$$\mathrm{orb}(H) = \{g^{-1}Hg \mid g \in G\} = \mathrm{cl}_G(H).$$

- The stabilizer of H is the normalizer of H in G:

$$\mathrm{stab}(H) = \{g \in G \mid g^{-1}Hg = H\} = N_G(H).$$

- The fixator of g are the subgroups that g normalizes:

$$\mathrm{fix}(g) = \{H \mid g^{-1}Hg = H\} = \{H \mid g \in N_G(H)\},$$

- The fixed points of ϕ are precisely the normal subgroups of G:

$$\mathrm{Fix}(\phi) = \{H \leq G \mid g^{-1}Hg = H \text{ for all } g \in G\}.$$

- The kernel of this action is the set of elements that normalize every subgroup:

$$\mathrm{Ker}(\phi) = \{g \in G \mid g^{-1}Hg = H \text{ for all } H \leq G\} = \bigcap_{H \leq G} N_G(H).$$

Groups acting on subgroups by conjugation

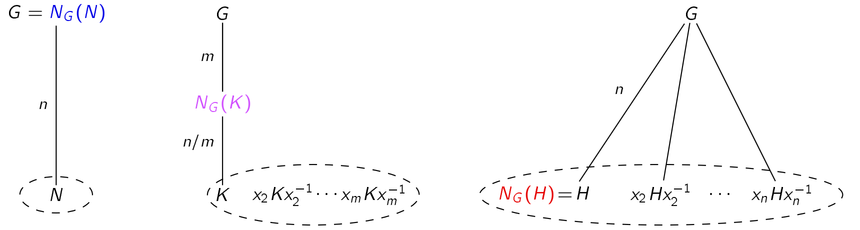
Let's apply our two theorems:

1. **Orbit-stabilizer theorem**. "*the size of an orbit is the index of the stabilizer*":

$$\left|\operatorname{cl}_G(H)\right| = [G : N_G(H)] = \frac{|G|}{|N_G(H)|}.$$

2. **Orbit-counting theorem**. "*the number of orbits is the average number of elements fixed by a group element*":

$$\#\text{conjugacy classes of subgroups of } G = \mathbb{E}\big[\#\text{ subgroups } g \text{ normalizes}\big].$$



normal $\quad$ $\left|\operatorname{cl}_G(N)\right| = 1$

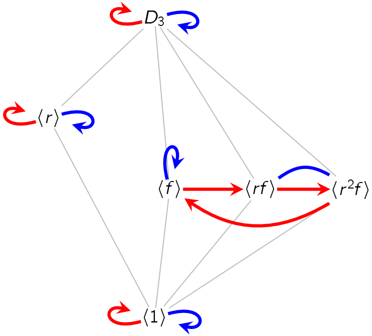
moderately unnormal $\quad$ $1 < \left|\operatorname{cl}_G(K)\right| < [G : K]$

fully unnormal $\quad$ $\left|\operatorname{cl}_G(H)\right| = [G : H]$; as large as possible

Groups acting on subgroups by conjugation

Here is an example of $G = D_3$ acting on its subgroups.

$$\begin{array}{lcllllll}
\tau(1) & = & \langle 1\rangle & \langle r\rangle & \langle f\rangle & \langle rf\rangle & \langle r^2f\rangle & D_3 \\
\tau(r) & = & \langle 1\rangle & \langle r\rangle & \langle f\rangle & \langle rf\rangle & \langle r^2f\rangle & D_3 \\
\tau(r^2) & = & \langle 1\rangle & \langle r\rangle & \langle f\rangle & \langle rf\rangle & \langle r^2f\rangle & D_3 \\
\tau(f) & = & \langle 1\rangle & \langle r\rangle & \langle f\rangle & \langle rf\rangle & \langle r^2f\rangle & D_3 \\
\tau(rf) & = & \langle 1\rangle & \langle r\rangle & \langle f\rangle & \langle rf\rangle & \langle r^2f\rangle & D_3 \\
\tau(r^2f) & = & \langle 1\rangle & \langle r\rangle & \langle f\rangle & \langle rf\rangle & \langle r^2f\rangle & D_3
\end{array}$$



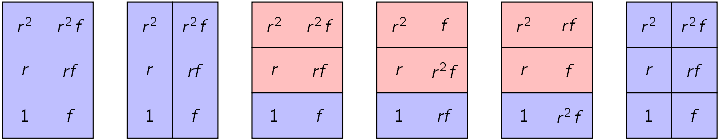
Observations

Do you see how to read stabilizers and fixed points off of the permutation diagram?

- $\mathrm{Ker}(\phi) = \langle 1\rangle$ consists of the row(s) with only fixed points.
- $\mathrm{Fix}(\phi) = \{\langle 1\rangle, \langle r\rangle, D_3\}$ consists of the column(s) with only fixed points.
- By the orbit-counting theorem, there are $|\mathrm{Orb}(\phi)| = 24/|D_3| = 4$ conjugacy classes.

Groups acting on subgroups by conjugation

Consider the partitions of D_3 by the left cosets of its six subgroups:



D_3/D_3 $\quad$ $D_3/\langle r\rangle$ $\quad$ $D_3/\langle f\rangle$ $\quad$ $D_3/\langle rf\rangle$ $\quad$ $D_3/\langle r^2f\rangle$ $\quad$ $D_3/\langle 1\rangle$

- $\mathrm{fix}(g)$ are the subgroups H for which "*g appears in a blue coset of H*"
- $\mathrm{Ker}(\phi)$ are elements that "*only appear in blue cosets*"
- By the orbit-counting theorem, the subgroups fall into

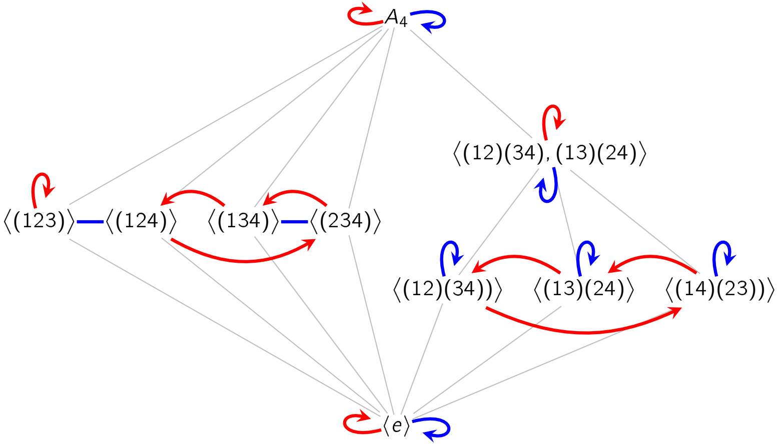
$$|\operatorname{Orb}(\phi)| = \text{average \# checkmarks per row} = \frac{\text{total \# of blue entries}}{|G|}$$

conjugacy classes.

Equivalently: *how many full "G-boxes" the blue cosets can be rearranged to fill up.*

Groups acting on subgroups by conjugation

Here is an example of $G = A_4 = \langle (123), (12)(34) \rangle$ acting on its subgroups.



Let's take a moment to revisit our "*three favorite examples*" from Chapter 3.

$$N = \langle (12)(34), (13)(24) \rangle, \qquad H = \langle (123) \rangle, \qquad K = \langle (12)(34) \rangle.$$

Groups acting on subgroups by conjugation

Here is the "*fixed point table*" of the action of A_4 on its subgroups.

	$\langle e\rangle$	$\langle(123)\rangle$	$\langle(124)\rangle$	$\langle(134)\rangle$	$\langle(234)\rangle$	$\langle(12)(34)\rangle$	$\langle(13)(24)\rangle$	$\langle(14)(23)\rangle$	$\langle(12)(34),(13)(24)\rangle$	A_4
e	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
(123)	✓	✓							✓	✓
(132)	✓	✓							✓	✓
(124)	✓		✓						✓	✓
(142)	✓		✓						✓	✓
(134)	✓			✓					✓	✓
(143)	✓			✓					✓	✓
(234)	✓				✓				✓	✓
(243)	✓				✓				✓	✓
$(12)(34)$	✓					✓	✓	✓	✓	✓
$(13)(24)$	✓					✓	✓	✓	✓	✓
$(14)(23)$	✓					✓	✓	✓	✓	✓

By the **orbit-counting theorem**, there are $|\operatorname{Orb}(\phi)| = 60/|A_4| = 5$ conjugacy classes.

Groups acting on cosets of H by multiplication

Fix a subgroup $H \leq G$. Then G acts on its **right cosets** by **right-multiplication**:

$$\phi \colon G \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } Hx \text{ to } Hxg.$$

Let Hx be an element of $S = H\backslash G$ (the right cosets of H).

- There is only one orbit. For example, given two cosets Hx and Hy,

$$\phi(x^{-1}y) \;\text{ sends }\; Hx \longmapsto Hx(x^{-1}y) = Hy.$$

- The stabilizer of Hx is the conjugate subgroup $x^{-1}Hx$:

$$\mathrm{stab}(Hx) = \{g \in G \mid Hxg = Hx\} = \{g \in G \mid Hxgx^{-1} = H\} = x^{-1}Hx.$$

- There doesn't seem to be a standard term for the fixator of g:

$$\mathrm{fix}(g) = \{Hx \mid Hxg = Hx\} = \{Hx \mid xgx^{-1} \in H\}.$$

- Assuming $H \neq G$, there are no fixed points of ϕ.
- The kernel of this action is the intersection of all conjugate subgroups of H:

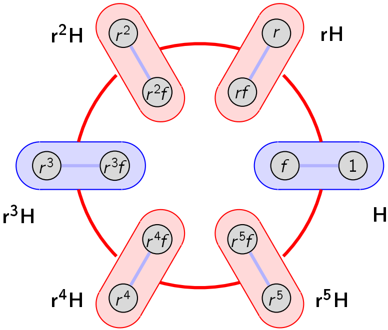
$$\mathrm{Ker}(\phi) = \bigcap_{x \in G} \mathrm{stab}(x) = \bigcap_{x \in G} x^{-1}Hx.$$

Notice that $\langle 1 \rangle \leq \mathrm{Ker}\,\phi \leq H$, and $\mathrm{Ker}(\phi) = H$ iff $H \trianglelefteq G$.

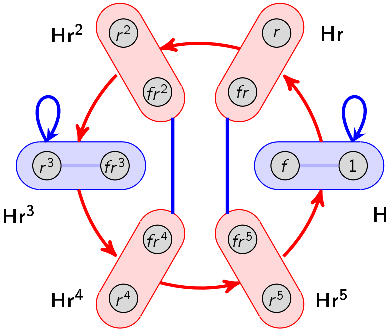
Groups acting on cosets of H by multiplication

The quotient process is done by collapsing the Cayley graph by the left cosets of H.

In contrast, this action is the result of collapsing the Cayley graph by the right cosets.



not a valid action graph



action graph of ϕ

Subgroups of small index

Groups acting on cosets is a useful technique for establishing seemingly unrelated results.

Several of these involving showing that subgroups of "small index" are normal.

We've already seen that subgroups of index 2 are normal.

Of course, there are non-normal index-3 subgroups, like $\langle f \rangle \leq D_3$.

The following gives a sufficient condition for when index-3 subgroups are normal.

Proposition

If G has no subgroup of index 2, then any subgroup of index 3 is normal.

Proof

Let $H \leq G$ with $[G : H] = 3$.

Let G act on the cosets of H by multiplication, to get a nontrivial homomorphism

$$\phi \colon G \longrightarrow S_3.$$

$K := \mathrm{Ker}(\phi) \leq H$ is the largest normal subgroup of G contained in H. By the FHT,

$$G/K \cong \mathrm{Im}(\phi) \leq S_3.$$

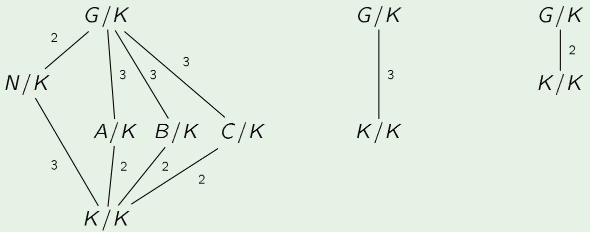
Subgroups of small index

Proof (contin.)

Thus, there are three cases for this quotient:

$$G/K \cong S_3, \qquad G/K \cong C_3, \qquad G/K \cong C_2.$$

Visually, this means that we have one of the following:



By the corrdespondence theorem, $K \leq H \lneq G$ implies $K/K \leq H/K \lneq G/K$.

Since G has no index-2 subgroup, only the middle case is possible (*Why?*).

This forces $K/K = H/K$, and so $K = H$ which is normal for multiple reasons. $\square$

Subgroups of small index

Proposition

Suppose $H \le G$ and $[G : H] = p$, the smallest prime dividing $|G|$. Then $H \trianglelefteq G$.

Proof

Let G act on the cosets of H by multiplication, to get a non-trivial homomorphism

$$\phi \colon G \longrightarrow S_p.$$

The kernel $K = \mathrm{Ker}(\phi)$, is the largest normal subgroup of G such that $K \le H \lneq G$.

We'll show that $H = K$, or equivalently, that $[H : K] = 1$. By the correspondence theorem:

$$\begin{tikzcd} G \arrow[d, dash, "p"] \\ H \arrow[d, dash, "q \text{ is not divisible by any prime } < p"] \\ K \end{tikzcd} \qquad\qquad \begin{tikzcd} G/K \cong S_p \arrow[d, dash, "p"] \\ H/K \arrow[d, dash, "q \text{ divides } (p-1)!"] \\ K/K \end{tikzcd}$$

Do you see why $q = 1$? $\square$

A summary of our four actions

Thus far, we have seen four important (right) actions of a group G, acting:

- on itself by multiplication
- on itself by conjugation.
- on its subgroups by conjugation.
- on the cosets of a fixed subgroup $H \leq G$ by multiplication.

set $S =$	G		subgroups of G	right cosets of H
operation	multiplication	conjugation	conjugation	right multiplication
$\mathrm{orb}(s)$	G	$\mathrm{cl}_G(g)$	$\mathrm{cl}_G(H)$	all right cosets
$\mathrm{stab}(s)$	$\langle 1 \rangle$	$C_G(g)$	$N_G(H)$	$x^{-1}Hx$
$\mathrm{fix}(g)$	G or $\emptyset$	$C_G(g)$	$\{H \mid g \in N_G(H)\}$	$\{Hx \mid xgx^{-1} \in H\}$
$\mathrm{Ker}(\phi)$	$\langle 1 \rangle$	$Z(G)$	$\bigcap_{H \leq G} N_G(H)$	largest norm. subgp. $N \leq H$
$\mathrm{Fix}(\phi)$	$\emptyset$	$Z(G)$	normal subgroups	none

The end!