Group actions!

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With many thanks to Matthew Macauley,
`http://www.math.clemson.edu/~macaule/`

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Overview

Intuitively, a group action occurs when a group G "naturally permutes" a set S *of states*.

For example:

- The "Rubik's cube group" consists of the 4.3×10^{19} actions that *permute* the 4.3×10^{19} configurations of the cube.
- The group D_4 consists of the 8 symmetries of the square. These symmetries are *actions* that *permute* the 8 configurations of the square.

Group actions formalize the interplay between the actual group of actions and the sets of objects that they "rearrange."

There are many other examples of groups that "act on" sets of objects. We will see examples when the group and the set have different sizes.

The rich theory of group actions can be used to prove many deep results in group theory.

We have actually already seen many group actions, without knowing it, such as:

- groups acting on themselves by multiplication
- groups acting on themselves by conjugation
- groups acting on their subgroups by conjugation
- groups acting on cosets by multiplication
- automorphism groups acting on groups.

Actions vs. configurations

The group D_4 can be thought of as the 8 symmetries of the square:

1	2
4	3

There is a subtle but *important* distinction to make, between the actual 8 symmetries of the square, and the 8 configurations.

For example, the 8 symmetries (alternatively, “actions”) can be thought of as

$$1, \qquad r, \qquad r^2, \qquad r^3, \qquad f, \qquad rf, \qquad r^2 f, \qquad r^3 f\,.$$

The 8 configurations (or *states*) of the square are the following:

1	2	4	1	3	4	2	3	2	1	3	2	4	3	1	4
4	3	3	2	2	1	1	4	3	4	4	1	1	2	2	3

When we were just learning about groups, we made an action graph.

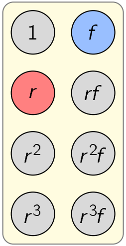
- The vertices corresponded to the states.
- The edges corresponded to generators.
- The paths corresponded to actions (group elements).

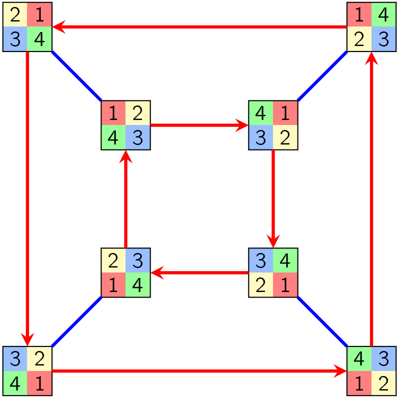
Action graphs!

Action graph of D_4

Here is the action graph of the group $D_4 = \langle r, f \rangle$:

"Group switchboard"





In the beginning of this course, we picked a configuration to be the "solved state," and this gave us a *bijection* between configurations and actions (group elements).

The resulting graph was a Cayley graph.

Action graphs

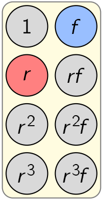
In all of the examples we saw in the beginning of the course, we had a bijective correspondence between actions and configurations. *This need not always happen!*

Suppose we have a size-7 set consisting of the following "binary squares."

$$S = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

Let's see what happens to these binary squares when we push different buttons on the D_4 "group switchboard."

"Group switchboard"



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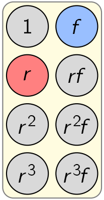
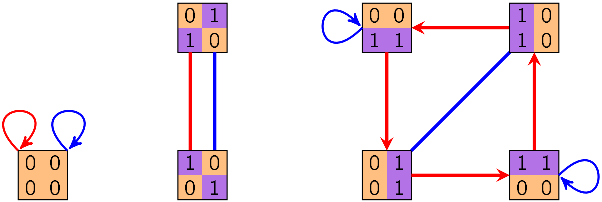
The group $D_4 = \langle r, f \rangle$ "acts on S" as follows:

"Group switchboard"





The action graph above has some properties of Cayley graphs, but there are some fundamental differences as well.

The "group switchboard" analogy

Suppose we have a "switchboard" for G, with every element $g \in G$ having a "button."

If $a \in G$, then pressing the a-button rearranges the objects in S—it is a permutation of S; call it $\phi(a)$.

If $b \in G$, then pressing the b-button also rearranges the objects in S. Call this permutation $\phi(b)$.

The element $ab \in G$ also has a button. We require that pressing the ab-button does the same as pressing the a-button, followed by the b-button. That is,

$$\phi(ab) = \phi(a)\phi(b)\,, \qquad \text{for all } a, b \in G\,.$$

Let $\mathbf{Perm}(S)$ be the group of permutations of S. Thus, if $|S| = n$, then $\mathbf{Perm}(S) \cong S_n$. (We typically think of S_n as the permutations of $\{1, 2, \ldots, n\}$.)

Definition

A group G acts on a set S if there is a homomorphism $\phi\colon G \to \mathbf{Perm}(S)$.

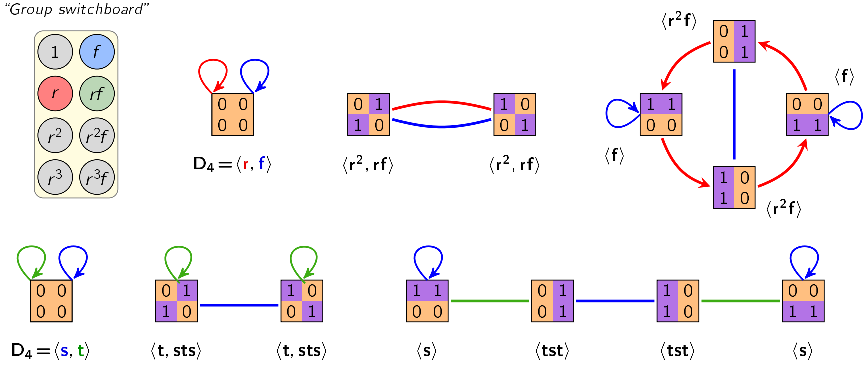
Action graphs and G-sets

Definition

A set S with an action by G is called a (right) G-set.

Big ideas

- An action $\phi\colon G \to \mathrm{Perm}(S)$ endows S with an **algebraic structure**.
- *Action graphs are to G-sets, like how Cayley graphs are to groups.*

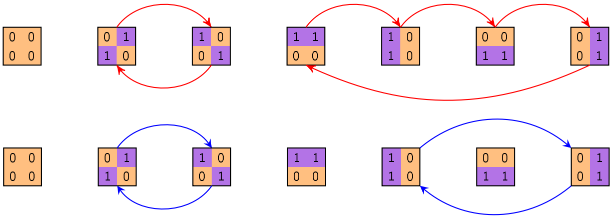


The “group switchboard” analogy

In our binary square example, pressing the r-button and f-button permutes S as follows:

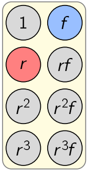
$\phi(r)$:

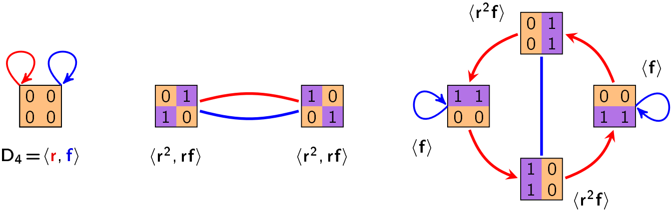
$\phi(f)$:



Observe how these permutations are encoded in the action graph. (Next to each $s \in S$ is the subgroup that fixes it.)

“Group switchboard”



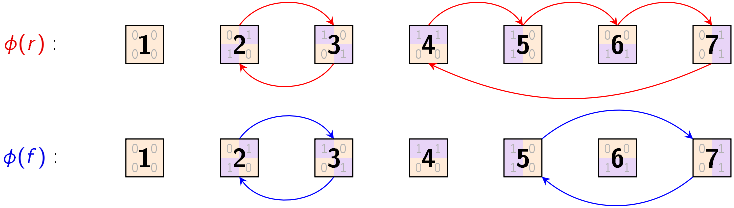


The "group switchboard" analogy

This action is an embedding $\phi\colon D_4 \hookrightarrow \mathbf{Perm}(S) \cong S_7$. How?

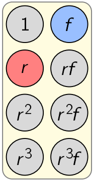
On your handout, label your binary squares 1 through 7.

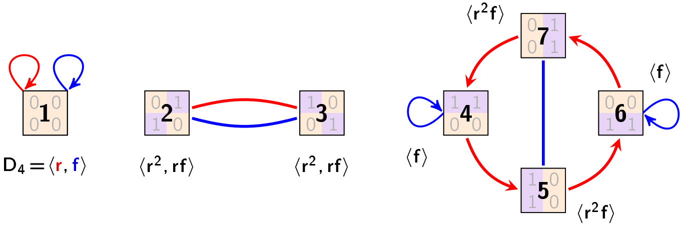
Then write your arrows in permutation notation.



Notice that $\mathbf{Im}(\phi) = \big\langle (23)(4567), (23)(57) \big\rangle \cong D_4 \leq S_7$.

"Group switchboard"





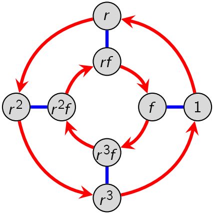
G-sets generalize groups. Action graphs generalize Cayley graphs

The group $G = D_4 = \langle r, f \rangle$ can act on itself ($S = D_4$), or on its subgroups,

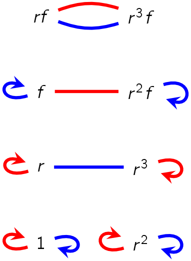
$$S = \big\{ D_4, \langle r \rangle, \langle r^2, f \rangle, \langle r^2, rf \rangle, \langle f \rangle, \langle rf \rangle, \langle r^2 f \rangle, \langle r^3 f \rangle, \langle r^2 \rangle, \langle 1 \rangle \big\}.$$

There are several ways to define the result of *"pressing the g-button on our switchboard"*.

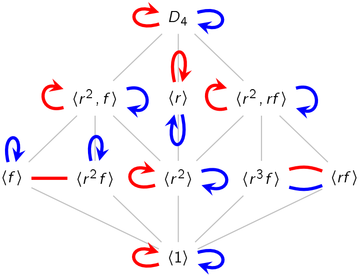
We say that: "*G acts on...*"



"*...itself by right-multiplication*"



"*...itself by conjugation*"



"*...its subgroups by conjugation*"

Big idea

Every Cayley graph is the action graph of a particular group action.

Left actions vs. right actions (an annoyance we can deal with)

As we've defined group actions, "*pressing the a-button followed by the b-button should be the same as pressing the ab-button*."

However, sometimes it appears like it's the same as "*pressing the ba-button*."

This is best seen by an example. Suppose our action is conjugation:

"Left group action"

$$\begin{tikzcd} H \arrow[r, "\substack{\text{conjugate}\\ \text{by } a}"] \arrow[rr, bend right, "\text{conjugate by } ba"'] & aHa^{-1} \arrow[r, "\substack{\text{conjugate}\\ \text{by } b}"] & baHa^{-1}b^{-1} \end{tikzcd}$$

$$\phi(a)\phi(b) = \phi(ba)$$

"Right group action"

$$\begin{tikzcd} H \arrow[r, "\substack{\text{conjugate}\\ \text{by } a}"] \arrow[rr, bend right, "\text{conjugate by } ab"'] & a^{-1}Ha \arrow[r, "\substack{\text{conjugate}\\ \text{by } b}"] & b^{-1}a^{-1}Hab \end{tikzcd}$$

$$\phi(a)\phi(b) = \phi(ab)$$

We'll call aHa^{-1} the left conjugate of H by a, and $a^{-1}Ha$ the right conjugate.

Some books forgo our "ϕ-notation" and use the following notation to distinguish left vs. right group actions:

$$g.(h.s) = (gh).s, \qquad (s.g).h = s.(gh).$$

We'll usually keep the ϕ, and write $\phi(g)\phi(h)s = \phi(gh)s$ and $s.\phi(g)\phi(h) = s.\phi(gh)$. As with groups, the "dot" will be optional.

Left actions vs. right actions (an annoyance we can deal with)

Alternative definition (other textbooks)

A right group action is a mapping

$$G \times S \longrightarrow S\,, \qquad (a, s) \longmapsto s.a$$

such that

- $s.(ab) = (s.a).b$, for all $a, b \in G$ and $s \in S$
- $s.e = s$, for all $s \in S$.

A left group action is defined similarly. Theorems for left actions have analogues for right actions.

Each left action has a related right action, and vice-versa. We'll use right actions, and write

$$s.\phi(g)$$

for "*the element of S that the permutation $\phi(g)$ sends s to*," i.e., where pressing the g-button sends s.

If we have a left action, we'll write $\phi(g).s$.

If needed, we can distinguish left G-sets with right G-sets.