Here is my super-secret and extremely confidential collection of problems that I think would be good for oral exams. It would be so bad if this super-secret and extremely confidential list were to accidentally leak to students, because then students may be well-prepared to answer one or more of these basic yet revealing problems about group theory during their comprehensive conversations! Also, it would be *extra* bad if a github link were to leak, because then students would even be able to see if I added more stuff to the list!!

Define and prove things

Problem 1. Suppose G is a group, and let H be a sub**set** of G.

(a) Write a careful definition of what it means for H to be a sub**group** of G.

(b) Prove the "one-step subgroup test:" If $xy^{-1} \in H$ for all $x, y \in H$, then H is a subgroup of G.

Problem 2. Let H and K be subgroups of a group G, with $K \leq H \leq G$.

(a) Define $[G : H]$, the "index of H in G."

(b) State and prove Lagrange's theorem.

(c) Use Lagrange's theorem to prove the "tower law:" $[G : K] = [G : H] \cdot [H : K]$.

Problem 3. Let G be a group, N a normal subgroup, and $G/N = \{gN \mid g \in G\}$ be the set of left cosets of N.

(a) Write careful definitions for "coset" and "normal subgroup."

(b) Explain why G/N is equivalent to $N\backslash G = \{Ng \mid g \in G\}$, the set of right cosets of N.

(c) Prove that the binary operation on G/N defined by

$$aN \cdot bN := (ab)N$$

is "well-defined;" that is, it does not depend on choice of coset representative.

(d) Prove that G/N with this binary operation is a group.

Problem 4. Suppose that $\phi : G \to H$ is a homomorphism.

(a) Write down a careful definition of a homomorphism.

(b) Prove that $\phi(1_G) = 1_H$.

(c) Use this result to prove that $\phi(g^{-1}) = \phi(g)^{-1}$.

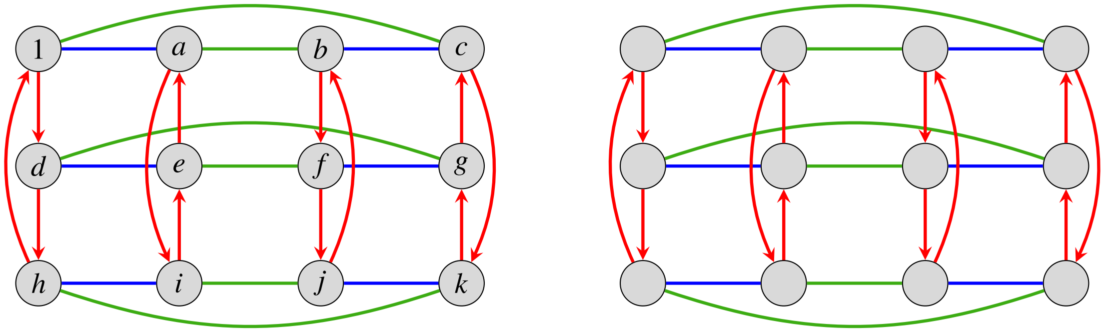
(d) Now you can show that $\mathrm{Im}(\phi) := \{\phi(g) \mid g \in G\}$ is a subgroup of __,

(e) and also that $\mathrm{Ker}(\phi) := \{g \in G \mid \phi(g) = 1\}$ is a subgroup of __.

(f) Indeed, you can prove that $\mathrm{Ker}(\phi)$ is a *normal* subgroup.

Calculate things

Problem 5. A Cayley graph of a mystery group G of order 12 is shown below; let 1 (not e) denote the identity element.



(a) Write the order of each element in the nodes of the blank Cayley graph on the right.

(b) The subgroup $H = \langle a, b\rangle \cong$ ________________, and $K = \langle d\rangle \cong$ ________________.

(c) The subgroup $\langle a, d\rangle$ has order ____________, and is isomorphic to ____________.

(d) The subgroup $\langle b, d\rangle$ has order ____________, and is isomorphic to ____________.

(e) The center of this group is $Z(G)$ = _____________. [Write it in terms of generator(s).]

(f) Find all left cosets of $H = \langle a, b\rangle$. Then find all right cosets. Write your answers as subsets of G, or describe them in words (e.g., "the rows" or "the columns").

(g) Find all left cosets of $K = \langle d\rangle$. Then find all right cosets. Write your answers as subsets of G, or describe them in words.

(h) The normalizers are $N_G(H) = \langle \qquad\qquad \rangle$ and $N_G(K) = \langle \qquad\qquad \rangle$.

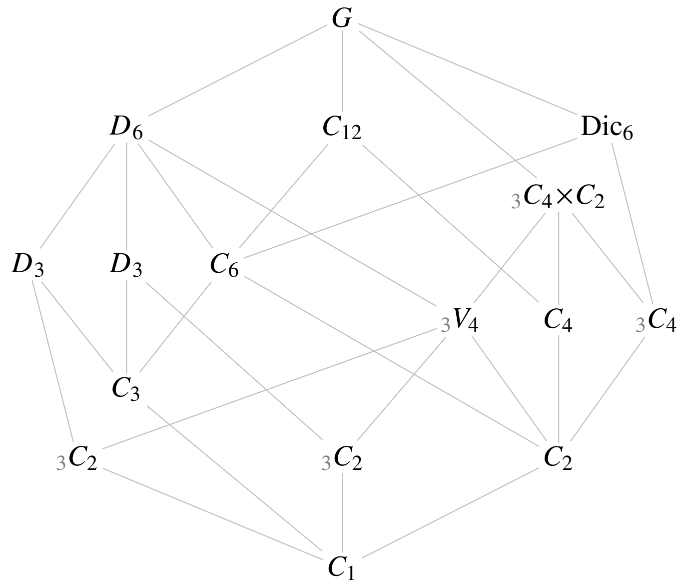
(i) Find all conjugate subgroups to H and to K. Write each subgroup only once.

(j) Is H normal? Is K normal?

(k) Draw the subgroup lattice of G.

Problem 6. Answer questions about the following group, whose subgroup lattice is shown below.

Order = 24 **Index**



(a) Determine the order and the index of each row of subgroups in the lattice.

(b) G has ________________ subgroup, which fall into ________________ conjugacy classes.

(c) G has exactly ____ normal subgroups.

(d) G has ______________ subgroup(s) of order 2 and ______________ element(s) of order 2.

(e) G has ______________ subgroup(s) of order 3 and ______________ element(s) of order 3.

(f) G has ________________ subgroup(s) of order 4, of which ________________ are cyclic.

(g) Find a normal subgroup $N \trianglelefteq G$ such that $G/N \cong V_4$.

(h) Each non-normal order-2 subgroup has a normalizer isomorphic to ____________________.

(i) Each D_3 subgroup has a normalizer isomorphic to __________________________________.

(j) This group has a quotient G/C_4 isomorphic to _____. [*Hint*: Determine the order, then count the index-2 subgroups.]

(k) This group has a quotient G/C_2 isomorphic to _____. [*Hint*: Same as above.]

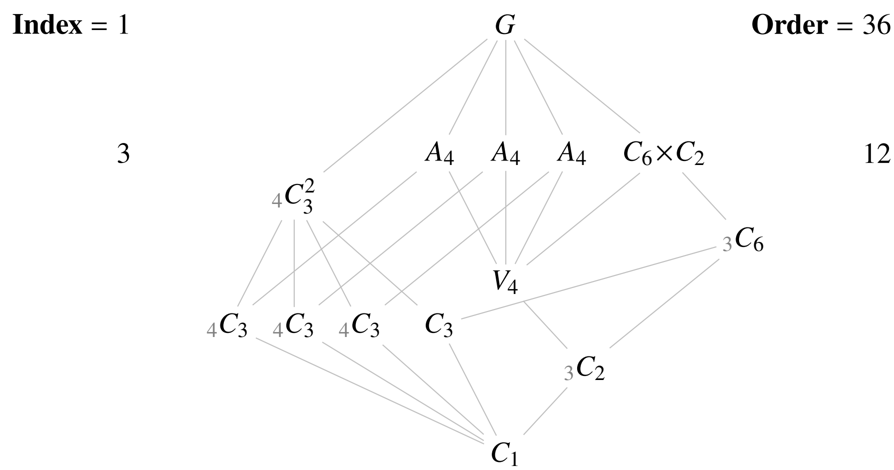
(l) The quotient G/C_3 is isomorphic to _____. [*Hint*: Determine the order. Which lattice do you see?]

(m) There are n_2 = _____ Sylow 2-subgroups, which are isomorphic to _______.

(n) There are n_3 = _____ Sylow 3-subgroups, which are isomorphic to _______.

(o) The largest order of an element in G is _____, and there are _____ element(s) of that order.

Problem 7. The subgroup diagram of a group G is shown below.



(a) Determine the order and the index of each row of subgroups in the lattice.

(b) The group G has ____________ subgroups, which fall into ____________ conjugacy classes.

(c) The quotient of G by its unique normal subgroup N of order 3 has order _____, and G/N is isomorphic to the familiar group ____.

(d) There are $n_2 =$________________ Sylow 2-subgroup(s), isomorphic to ________________,

and $n_3 =$___________________ Sylow 3-subgroup(s), isomorphic to ___________________.

(e) Find all distinct ways that G can be written as a direct or semidirect product of two of its proper subgroups.

(f) Find the center $Z(G)$, and justify your answer. Though this cannot always be done by inspection, it can in this case, using a result from the previous part.

(g) Let G act on its subgroups by conjugation. This action has ____________________ orbit(s)

and ____________________ fixed point(s).

(h) Let G act on itself by multiplication. This action has __________________________ orbit(s)

and ____________________ fixed point(s).

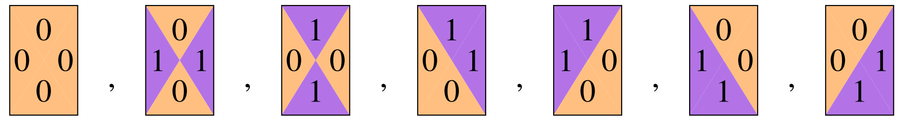
(i) Let H be a subgroup of order 9, and let G act on the right cosets of H by right multiplication. This action has ____________________ orbit(s) and ____________________ fixed point(s).

(j) Still assuming that $|H| = 9$, let G act on the *left* cosets of H by *left* multiplication. This action has ______________________ orbit(s) and ______________________ fixed point(s).

(k) Let $g \in G$ be an element of order 2. Then g commutes with exactly ___ element(s), and the centralizer of $\langle g \rangle$ is isomorphic to ____________________________. The centralizer of g is

(bigger than)(smaller than)(equal to) [$\longleftarrow$ *circle one*] its normalizer.

Problem 8. Consider the following set of "binary rectangles":

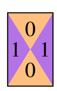
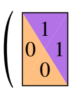
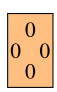
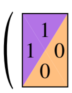
$S = \Big\{$  $\Big\}$

The Klein-4 group $V = \{1, v, h, vh\}$ acts on S via $\phi\colon V \longrightarrow \mathrm{Perm}(S)$, where

$$\begin{aligned} \phi(v) &= \text{flips each tile vertically} \\ \phi(h) &= \text{flips each tile horizontally} \\ \phi(vh) &= \text{rotates each tile by } 180^\circ \end{aligned}$$

(a) Pick a minimal generating set and then draw the *action graph*. (Feel free to label the rectangles above A,B,C,D,E,F,G to save time.)

(b) Find the following:

- stab $\Big($  $\Big) =$
- stab $\Big($  $\Big) =$
- stab $\Big($  $\Big) =$
- stab $\Big($  $\Big) =$
- $\mathrm{fix}(1) =$
- $\mathrm{fix}(h) =$
- $\mathrm{fix}(v) =$
- $\mathrm{fix}(vh) =$
- This action has _________ orbits, which by the orbit-counting theorem, is also equal to the average ____.
- $\mathrm{Fix}(\phi) =$
- $\mathrm{Ker}(\phi) =$