What do isomorphisms do?

I keep saying that isomorphisms respect algebraic structure. This is a hugely-encompassing idea and I want to unpack what I mean and what some of the consequences are.

What is an isomorphism?

An isomorphism is a homomorphism that is also a bijection.

Okay, smartass, what is a homomorphism?

Suppose that $(G, \star)$ and $(H, \odot)$ are two groups. Then $\phi : G \to H$ is a homomorphism if

$$\phi(g_1 \star g_2) = \phi(g_1) \odot \phi(g_2).$$

Exercise. Circle three different things in that equation that are elements of H.

Morally what this means is that a homomorphism is a function that respects the groups' operations.
Another good maxim here is that a homomorphism sends products to products.
As a consequence of respecting the groups' operations, a homomorphism respects the groups' algebraic structures. Specifically:

Exercise. Prove each of the following statements:

- A homomorphism sends the identity to the identity.

  Proof. Say that e_G is the identity in G and e_H is the identity in H. Consider $\phi(e_G \star g)$. On the one hand, since $e_G \star g = g$, $\phi(e_G \star g) = \phi(g)$. On the other hand, using the homomorphism property, $\phi(e_G \star g) = \phi(e_G) \odot \phi(g)$. Therefore,

  $$\phi(g) = \phi(e_G) \odot \phi(g).$$

  Well, $\phi(g)$ is some element of H, so it has an inverse. Let's H-multiply both sides of this equation by the inverse on the right:

  $$\begin{aligned} \phi(g) \odot [\phi(g)]^{-1} &= \phi(e_G) \odot \phi(g) \odot [\phi(g)]^{-1} \\ e_H &= \phi(e_G) \odot \left(\phi(g) \odot [\phi(g)]^{-1}\right) \\ e_H &= \phi(e_G) \odot e_H \\ e_H &= \phi(e_G). \end{aligned}$$

  So: ϕ sends e_G to e_H. □

- A homomorphism sends inverses to inverses.

- A homomorphism sends G to a subgroup of H. (Vocabulary: the image of G under ϕ is the set $\mathrm{im}(\phi) = \{\phi(g) \mid g \in G\}$. Certainly this is a sub*set* of H, but is it a sub*group* of H?)

- A homomorphism sends powers to powers.
- A homomorphism sends orbits to orbits.
- A homomorphism sends conjugates to conjugates.
- A homomorphism sends conjugacy classes to conjugacy classes.

Here are some examples of homomorphisms.

Exercise. Prove that each of these "sends products to products":

- Squish everything in G down to the identity in H. (This is a rude homomorphism.)
  - Ponder: How does this map send orbits to orbits?
- Do nothing. (Define the "identity map" $\mathrm{id} : G \to G$ as $\mathrm{id}(g) = g$.)
- If $G \leq H$, define the "inclusion map" $\iota : G \to H$ as $\iota(g) = g$.
- Define the "exponential map" $\exp : (\mathbb{R}, +) \to (\mathbb{R}^+, *)$ by $\exp(x) = e^x$.
- $\ln : (\mathbb{R}^+, *) \to (\mathbb{R}, +)$.
  - This is, like, the best explanation for why the properties of logs are like that.
- Here is an interesting **non**-example: Let $s : D_n \to D_n$ be the "squaring map" $s(x) = x^2$. (Hint: Remember that D_n isn't abelian and compare $s(fr)$ to $s(f)s(r)$.)
- If G is an **abelian** group, then the squaring map $s : G \to G$ is indeed a homomorphism.
- Define $\phi : Q_8 \to V_4$ as follows: $\phi(\pm 1) = 1, \quad \phi(\pm i) = a, \quad \phi(\pm j) = b, \quad \phi(\pm k) = ab$.
- Define the "projection map" $\pi_A : A \times B \to A$ as $\pi_A(a, b) = a$. (Similar for π_B.)

What about isomorphisms?

A general theme in math is that if you make something more special, you get stronger results. By adding "bijection" to "homomorphism," you can thus expect to preserve even more structure.

Exercise. Let $\phi : G \to H$ be an isomorphism. Prove that:

- $|\phi(g)| = |g|$. ("ϕ preserves orders.")

  Proof. Say that $|g| = n$ – that is, $g^n = e$, but for any $k < n$, $g^k \neq e$. We need to show those two things are also true for $\phi(g)$. The first part is easy: since ϕ is a homomorphism, it sends powers to powers and the identity to the identity, so $\phi(g)^n = \phi(g^n) = \phi(e) = e$.

  For the second part, consider the orbit of g, $\langle g \rangle = \{g, g^2, \ldots, g^{n-1}, g^n = e\}$. All these powers of g are distinct. (Why?) So, since ϕ is a bijection (and in particular is 1-1), all their images $\{\phi(g), \phi(g^2), \ldots, \phi(g^{n-1}), \phi(g^n) = e\}$ are distinct. But since ϕ sends powers to powers, that list of distinct elements is $\{\phi(g), \phi(g)^2, \ldots \phi(g)^{n-1}, e\}$. Therefore, $\phi(g)^k \neq e$ for any $k < n$ – e is in that list of distinct elements at the end, so nobody else gets to be e. □

- Corollary: ϕ sends orbits to orbits *of the same size.*
- ϕ sends conjugacy classes to conjugacy classes *of the same size.*
- ϕ sends subgroups to subgroups *of the same size.*

If there is an isomorphism $\phi : G \to H$, we say that G and H are isomorphic and write $G \cong H$. Since an isomorphism ϕ preserves *so much* algebraic structure, this is why it's our formal version of the idea that G and H are "basically the same" but maybe just got relabeled or re-presented.

Exercise. Suppose that $G \cong H$. Prove that:

- G is abelian if and only if H is abelian.