Normal subgroups!

Spencer Bagley

With many thanks to Matthew Macauley,
`http://www.math.clemson.edu/~macaule/`

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Goals for today:

1. Define what normal subgroups are
2. See some examples
3. Learn some properties of normal subgroups

Review of last time!

Cosets!

Definition

If $H \leq G$, then a left coset is a set

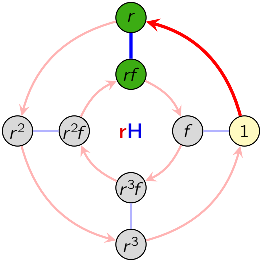
$$xH = \{xh \mid h \in H\},$$

for some fixed $x \in G$ called the representative.
Similarly, we can define a right coset as

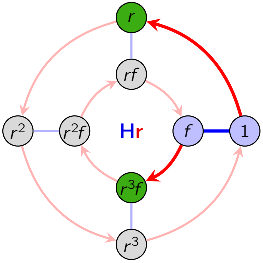
$$Hx = \{hx \mid h \in H\}.$$

Left vs. right cosets

- The **left coset** rH in D_4: first go to r, then traverse all "H-paths".
- The **right coset** Hr in D_4: first traverse all H-paths, then traverse the r-path.



$rH = r\{1, f\} = \{r, rf\} = rf\{f, 1\} = rfH$



$Hr = \{1, f\}r = \{r, r^3f\} = \{f, 1\}r^3f = Hr^3f$

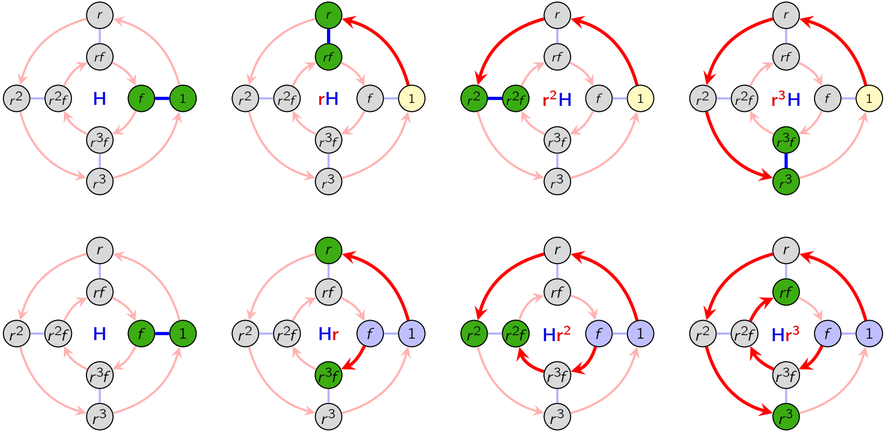
Because of our convention that arrows in a Cayley graph represent right multiplication:

- left cosets look like copies of the subgroup,
- right cosets are usually "scattered."

Key point

Left and right cosets are generally different.

Overview of left and right cosets of $\langle f \rangle$



- rH and Hr are different
- r^2H and Hr^2 are the same
- r^3H and Hr^3 are different

Properties of cosets

Proposition

For any subgroup $H \leq G$, the (left) cosets of H partition the group G:
every element $g \in G$ lives in exactly one (left) coset of H. ((Left) cosets never overlap.)

Proposition

For any subgroup $H \leq G$, the (left) cosets are all the same size, which is therefore $|H|$.

Proposition

For any subgroup $H \leq G$, there are always the same number of left cosets as there are of right cosets.

Definition

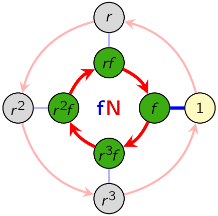
The index of a subgroup H in G, written $[G : H]$, is the number of cosets of H in G.

Lagrange's theorem

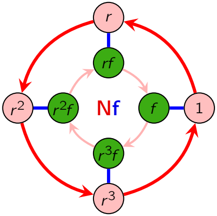
If H is a subgroup of a finite group G, then $|G| = [G : H] \cdot |H|$.

A different subgroup of D_4, $N = \langle r \rangle$

Since this subgroup is already half of the big group, every left coset has to be a right coset.



$fN = f\{1, r, r^2, r^3\} = \{f, fr, fr^2, fr^3\}$



$Nf = \{1, r, r^2, r^3\}f = \{f, rf, r^2f, r^3f\}$

Informal definition

A subgroup for which every left coset is also a right coset is called normal.

Caveat!

Equality of cosets $xK = Kx$ as sets is *different from* equality of elements $xk = kx$.
Example here: $fr \in fN$ is different from $rf \in Nf$, but that's okay because $fr = r^3f$ shows up later in Nf.

Normal subgroups!

Normal subgroups!

Formal definition

A subgroup H is a normal subgroup of G if $gH = Hg$ for *all* $g \in G$. We write $H \trianglelefteq G$.

Equivalent definition

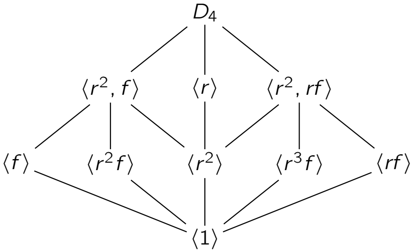
... if $gHg^{-1} = H$ for *all* $g \in G$. (More on this version later.)

Examples of normal sugroups

We've seen cases where we know a subgroup will be normal without having to check.

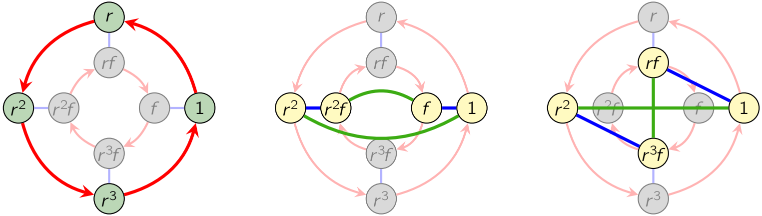
1. The subgroup $H = G$ is always normal in G. The only left coset is also the only right coset:
$$eG = G = Ge.$$
2. The subgroup $H = \{e\}$ is always normal. The left and right cosets are singleton sets:
$$gH = \{g\} = Hg.$$
3. Subgroups H of index 2 are normal. The two cosets (left or right) are H and $G - H$.
4. Subgroups of *abelian groups* are always normal, because for any $H \leq G$,
$$aH = \{ah \mid h \in H\} = \{ha \mid h \in H\} = Ha.$$
5. The center $Z(G)$ is always normal, for the same reason as above.
6. Relatedly, any subgroup of $Z(G)$ is always normal.

Normal subgroups in D_4



From our explorations, we found:

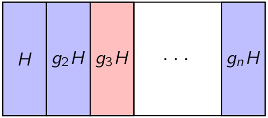
- $\langle r\rangle \lhd D_4$ (because it has index 2!)
- $\langle r^2, f\rangle \lhd D_4$ (index 2!)
- $\langle r^2, rf\rangle \lhd D_4$ (index 2!)
- $\langle r^2\rangle \lhd D_4$ (because it is $Z(D_4)$!)
  - (Also, it's the only guy who's a subgroup of 3 different groups)



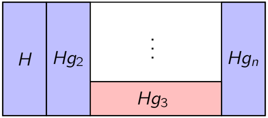
Normalizers

Okay, well, if $H \leq G$ isn't normal, then a natural followup question is:

"*How many left cosets of H are right cosets?*"



Partition of G by the left cosets of H



Partition of G by the right cosets of H

- "Best case" scenario ($H \trianglelefteq G$): all of them
- "Worst case" scenario: only H (I mean for sure the identity coset $eH = He$)
- In general: somewhere between these two extremes

Definition

The normalizer of H, denoted $N_G(H)$, is the set of elements $g \in G$ such that $gH = Hg$:

$$N_G(H) = \{g \in G \mid gH = Hg\},$$

i.e., the union of reps of left cosets that are also reps of right cosets.

Homework

Prove that $N_G(H) \leq G$, and also that $H \trianglelefteq N_G(H)$.

Tricks for spotting normal subgroups!

How to check if a subgroup is normal

If $gH = Hg$, then right-multiplying both sides by g^{-1} yields $gHg^{-1} = H$.

This gives us a new way to check whether a subgroup H is normal in G.

Useful remark

The following are equivalent ("TFAE") to a subgroup $H \leq G$ being normal:

(i) $gH = Hg$ for all $g \in G$; ("*left cosets are right cosets*")

(ii) $gHg^{-1} = H$ for all $g \in G$; ("*only one conjugate subgroup*")

(iii) $ghg^{-1} \in H$ for all $h \in H$, $g \in G$; ("*closed under conjugation*")

Proof

(i) $\Leftrightarrow$ (ii): Boringly obvious. (ii) $\Rightarrow$ (iii): Also boringly obvious.

(iii) $\Rightarrow$ (ii): Interesting; homework. :)

Sometimes, one of these methods is *much* easier than the others!

- to show $H \ntrianglelefteq G$, find *just one element* $h \in H$ for which $ghg^{-1} \notin H$ for some $g \in G$.
- if G has a unique subgroup of size $|H|$, then H *must* be normal. (Why?)

Conjugate subgroups

For a fixed element $g \in G$, the set

$$gHg^{-1} = \{ghg^{-1} \mid h \in H\}$$

is called the conjugate of H by g.

Homework

For any $g \in G$, the conjugate gHg^{-1} is a subgroup of G.

Observation

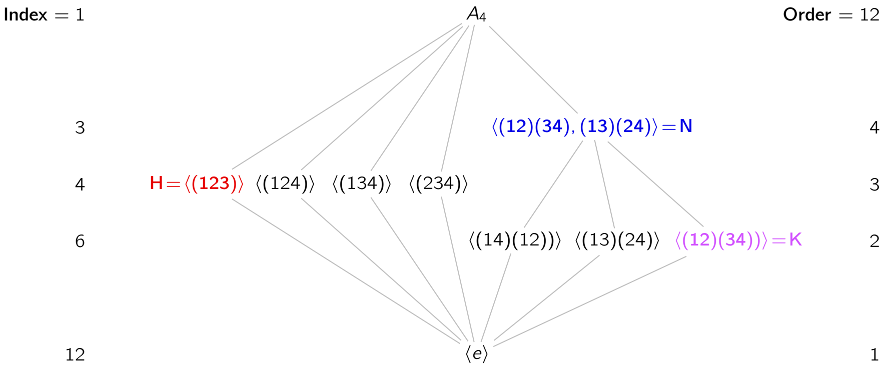
$|gHg^{-1}| = |H|$. (Proof: Look at the definition.)

Later, we'll prove that H and gHg^{-1} are isomorphic subgroups.

The subgroup lattice of A_4

I am highlighting the following three subgroups of A_4:

$$N = \langle (12)(34), (13)(24) \rangle = \{e,\ (12)(34),\ (13)(24),\ (14)(23)\} \cong V_4$$
$$H = \langle (123) \rangle = \{e,\ (123),\ (132)\} \cong C_3$$
$$K = \langle (12)(34) \rangle = \{e,\ (12)(34)\} \cong C_2.$$



Who could possibly be conjugate to N? to H? to K?
Who could possibly be $N_{A_4}(N)$? $N_{A_4}(H)$? $N_{A_4}(K)$?

Two pretty good reasons why N is normal

Useful remark

The following are equivalent ("TFAE") to a subgroup $H \leq G$ being normal:

(i) $gH = Hg$ for all $g \in G$; ("*left cosets are right cosets*")

(ii) $gHg^{-1} = H$ for all $g \in G$; ("*only one conjugate subgroup*")

(iii) $ghg^{-1} \in H$ for all $h \in H$, $g \in G$; ("*closed under conjugation*")

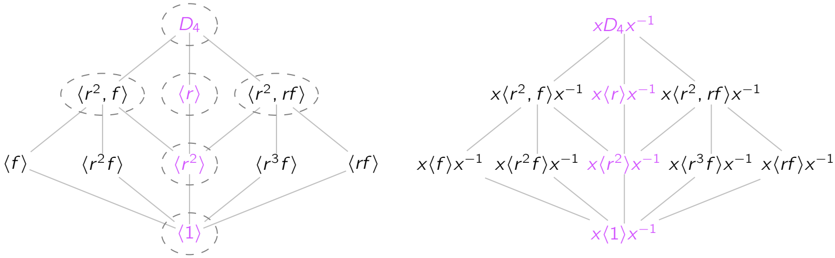
1. N is the only subgroup of its size in the subgroup lattice of A_4, so definitely $gNg^{-1} = N$
2. $N_{A_4}(N)$ has to be between N and A_4 in the lattice, so it's either N itself or all of A_4.
   - So, pick something outside of N and see if it normalizes N.

Unicorn subgroups

Suppose we conjugate $G = D_4$ by some element $x \in D_4$.

Very useful idea

Conjugating a normal subgroup $N \leq G$ by $x \in G$ shuffles its elements and subgroups.
In particular, this includes conjugating all of G by some $x \in G$.



Subgroups at a unique "lattice neighborhood" are called unicorns, and must be normal.

For example, $\langle r^2\rangle = x\langle r^2\rangle x^{-1}$ is the only size-2 subgroup "*with 3 parents*."

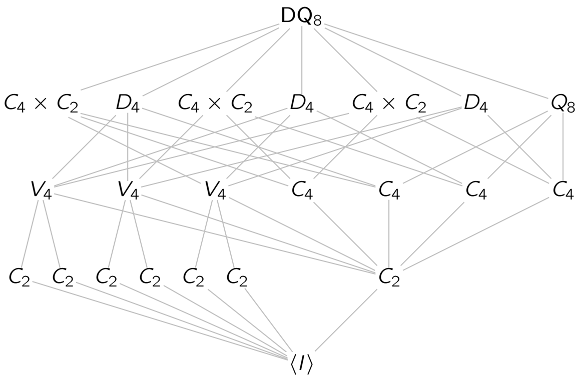
The groups G and $\langle 1\rangle$ are always unicorns, and hence normal.

The index-2 subgroups $\langle r^2, f\rangle$, $\langle r\rangle$, and $\langle r^2, rf\rangle$ must be normal.

Unicorns in the diquaternion group

Our definition of unicorn could be strengthened, but we want to keep things simple.

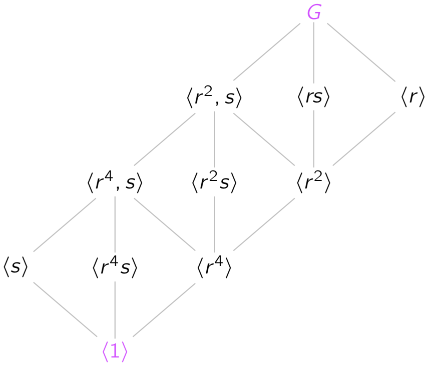
Here's the lattice for a group called DQ_8, which has order 16.
Are any of the C_4 subgroups of DQ_8 unicorns, i.e., "not like the others"?



(Preview: What can we say about the conjugacy classes of the subgroups of DQ_8 just from the lattice?)

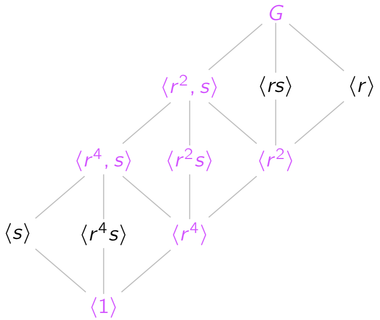
A mystery group of order 16

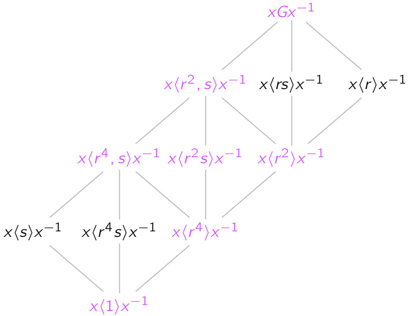
Here is the subgroup lattice for some actual group of order 16 but I'm not telling you which one it is. Find as many unicorns as you can.



A mystery group of order 16

Unicorns are purple.





We can deduce that every subgroup is normal, except possibly $\langle s\rangle$ and $\langle r^4 s\rangle$.

There are two cases:

- $\langle s\rangle$ and $\langle r^4 s\rangle$ are normal $\Rightarrow s \in Z(G) \Rightarrow G$ is abelian.
- $\langle s\rangle$ and $\langle r^4 s\rangle$ are not normal $\Rightarrow \mathrm{cl}_G(\langle s\rangle) = \{\langle s\rangle, \langle r^4 s\rangle\} \Rightarrow G$ is nonabelian.

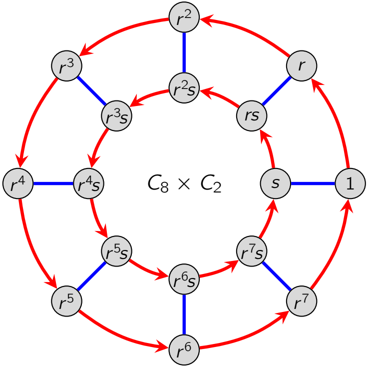
This doesn't necessarily mean that both of these are actually possible. . .

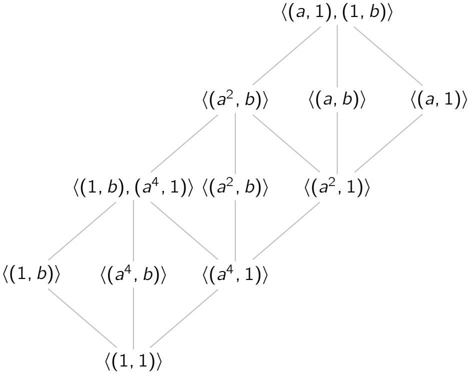
A mystery group of order 16

It's straightforward to check that this is the subgroup lattice of

$$C_8 \times C_2 = \left\langle r, s \mid r^8 = s^2 = 1, srs = r \right\rangle.$$

Let $r = (a, 1)$ and $s = (1, b)$, and so $C_8 \times C_2 = \langle r, s \rangle = \left\langle (a, 1), (1, b) \right\rangle$.

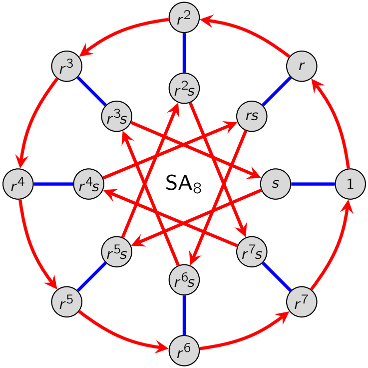


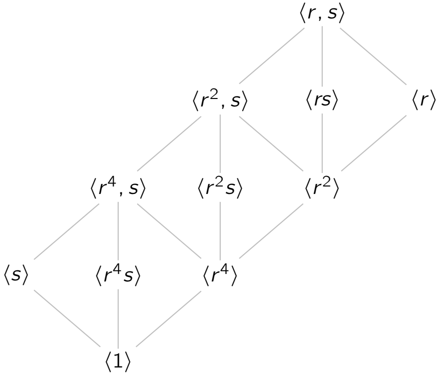


A mystery group of order 16

However, the nonabelian case is possible as well! The following also works:

$$\mathrm{SA}_8 = \left\langle r, s \mid r^8 = s^2 = 1, srs = r^5 \right\rangle.$$





Conjugacy classes!

The conjugacy class of a subgroup

Proposition

Conjugation is an equivalence relation on the set of subgroups of G.

Proof

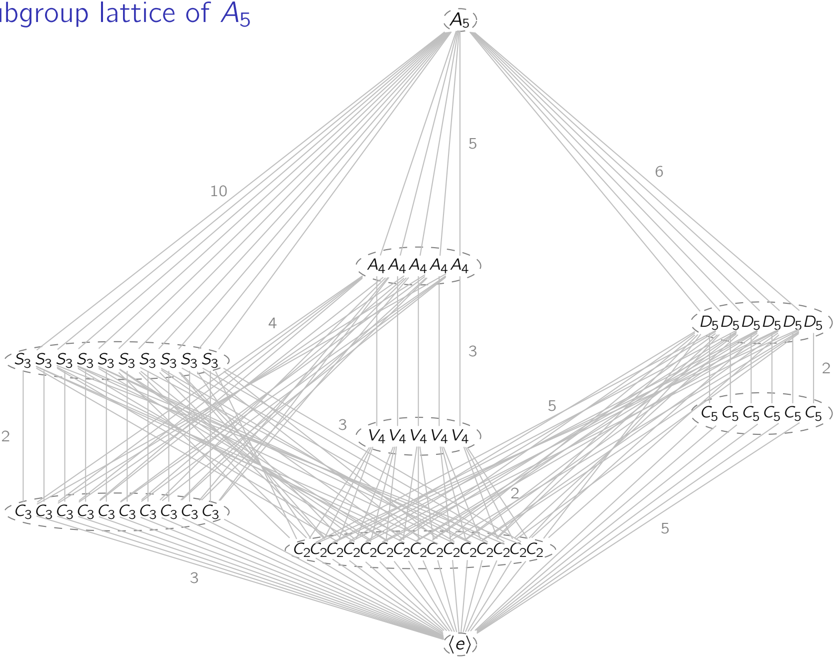
We need to show that conjugacy is reflexive, symmetric, and transitive.

- **Reflexive**: $eHe^{-1} = H$. ✓
- **Symmetric**: Suppose H is conjugate to K, by $aHa^{-1} = K$. Then K is conjugate to H:
$$a^{-1}Ka = a^{-1}(aHa^{-1})a = H. \quad ✓$$
- **Transitive**: Suppose $aHa^{-1} = K$ and $bKb^{-1} = L$. Then H is conjugate to L:
$$(ba)H(ba)^{-1} = b(aHa^{-1})b^{-1} = bKb^{-1} = L. \quad ✓$$

Definition

The set of all subgroups conjugate to H is its conjugacy class, denoted
$$\mathrm{cl}_G(H) = \{gHg^{-1} \mid g \in G\}.$$

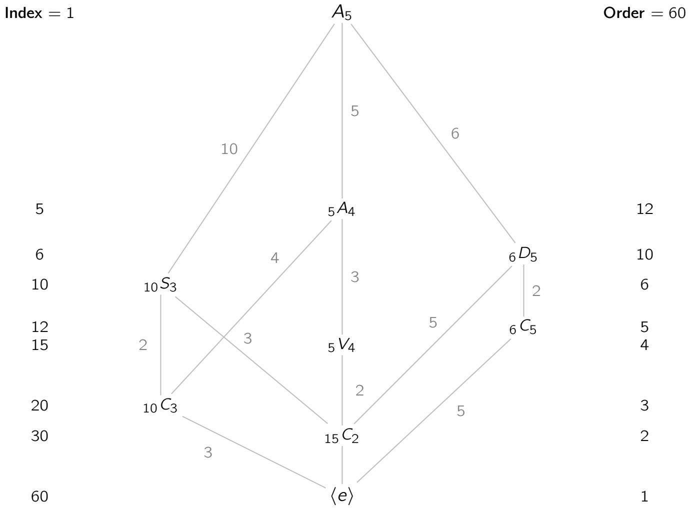
The subgroup lattice of A_5



"Reducing" subgroup lattices

Sometimes it is convenient to collapse conjugacy classes into single nodes in the lattice.

Left-subscripts denote the size of the conjugacy class. We call this a subgroup diagram. (In some circumstances it might not actually be a lattice.)



The end!