Here is my super-secret and extremely confidential collection of problems that I think would be good for oral exams. It would be so bad if this super-secret and extremely confidential list were to accidentally leak to students, because then students may be well-prepared to answer one or more of these basic yet revealing problems about group theory during their comprehensive conversations! Also, it would be *extra* bad if a github link were to leak, because then students would even be able to see if I added more stuff to the list!!

Define and prove things

Problem 1. Suppose G is a group, and let H be a sub**set** of G.

(a) Write a careful definition of what it means for H to be a sub**group** of G.

(b) Prove the "one-step subgroup test:" If $xy^{-1} \in H$ for all $x, y \in H$, then H is a subgroup of G.

Problem 2. Let H and K be subgroups of a group G, with $K \leq H \leq G$.

(a) Define $[G : H]$, the "index of H in G."

(b) State and prove Lagrange's theorem.

(c) Use Lagrange's theorem to prove the "tower law:" $[G : K] = [G : H] \cdot [H : K]$.

Problem 3. Let G be a group, N a normal subgroup, and $G/N = \{gN \mid g \in G\}$ be the set of left cosets of N.

(a) Write careful definitions for "coset" and "normal subgroup."

(b) Explain why G/N is equivalent to $N\backslash G = \{Ng \mid g \in G\}$, the set of right cosets of N.

(c) Prove that the binary operation on G/N defined by

$$aN \cdot bN := (ab)N$$

is "well-defined;" that is, it does not depend on choice of coset representative.

(d) Prove that G/N with this binary operation is a group.

Problem 4. Suppose that $\phi : G \to H$ is a homomorphism.

(a) Write down a careful definition of a homomorphism.

(b) Prove that $\phi(1_G) = 1_H$.

(c) Use this result to prove that $\phi(g^{-1}) = \phi(g)^{-1}$.

(d) Now you can show that $\mathrm{Im}(\phi) := \{\phi(g) \mid g \in G\}$ is a subgroup of __,

(e) and also that $\mathrm{Ker}(\phi) := \{g \in G \mid \phi(g) = 1\}$ is a subgroup of __.

(f) Indeed, you can prove that $\mathrm{Ker}(\phi)$ is a *normal* subgroup.

Calculate things

Problem 5. Macauley S24 midterm 1, problem 3, but also, draw the subgroup lattice

Problem 6. Macauley S24 midterm 1, problem 4

Problem 7. Macauley F22 final, problem 7