Normal subgroups!

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With many thanks to Matthew Macauley,
http://www.math.clemson.edu/~macaule/

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Goals for today:

1. Define what normal subgroups are
2. See some examples
3. Learn some properties of normal subgroups

Review of last time!

Cosets!

Definition

If $H \leq G$, then a left coset is a set

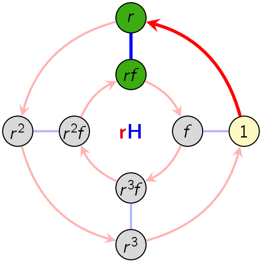
$$xH = \{xh \mid h \in H\},$$

for some fixed $x \in G$ called the representative.
Similarly, we can define a right coset as

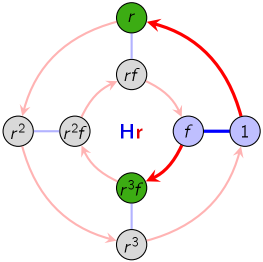
$$Hx = \{hx \mid h \in H\}.$$

Left vs. right cosets

- The **left coset** rH in D_4: first go to r, then traverse all "H-paths".
- The **right coset** Hr in D_4: first traverse all H-paths, then traverse the r-path.



$rH = r\{1, f\} = \{r, rf\} = rf\{f, 1\} = rfH$



$Hr = \{1, f\}r = \{r, r^3f\} = \{f, 1\}r^3f = Hr^3f$

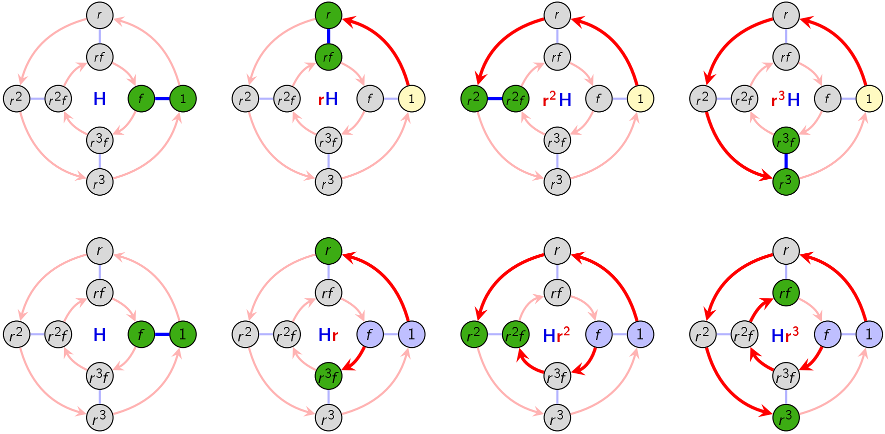
Because of our convention that arrows in a Cayley graph represent right multiplication:

- left cosets look like copies of the subgroup,
- right cosets are usually "scattered."

Key point

Left and right cosets are generally different.

Overview of left and right cosets of $\langle f \rangle$



- rH and Hr are different
- r^2H and Hr^2 are the same
- r^3H and Hr^3 are different

Properties of cosets

Proposition

For any subgroup $H \leq G$, the (left) cosets of H partition the group G:
every element $g \in G$ lives in exactly one (left) coset of H. ((Left) cosets never overlap.)

Proposition

For any subgroup $H \leq G$, the (left) cosets are all the same size, which is therefore $|H|$.

Proposition

For any subgroup $H \leq G$, there are always the same number of left cosets as there are of right cosets.

Definition

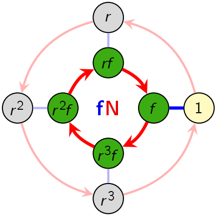
The index of a subgroup H in G, written $[G : H]$, is the number of cosets of H in G.

Lagrange's theorem

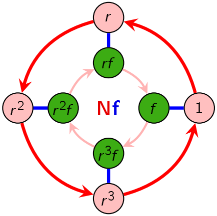
If H is a subgroup of a finite group G, then $|G| = [G : H] \cdot |H|$.

A different subgroup of D_4, $N = \langle r \rangle$

Since this subgroup is already half of the big group, every left coset has to be a right coset.



$fN = f\{1, r, r^2, r^3\} = \{f, fr, fr^2, fr^3\}$



$Nf = \{1, r, r^2, r^3\}f = \{f, rf, r^2f, r^3f\}$

Informal definition

A subgroup for which every left coset is also a right coset is called normal.

Caveat!

Equality of cosets $xK = Kx$ as sets is *different from* equality of elements $xk = kx$.
Example here: $fr \in fN$ is different from $rf \in Nf$, but that's okay because $fr = r^3f$ shows up later in Nf.

Normal subgroups!

Normal subgroups!

Formal definition

A subgroup H is a normal subgroup of G if $gH = Hg$ for *all* $g \in G$. We write $H \trianglelefteq G$.

Equivalent definition

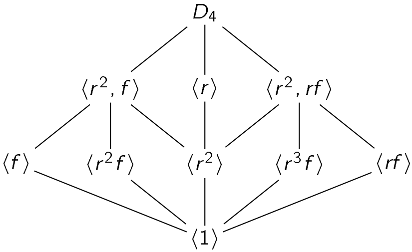
. . . if $gHg^{-1} = H$ for *all* $g \in G$. (More on this version later.)

Examples of normal sugroups

We've seen cases where we know a subgroup will be normal without having to check.

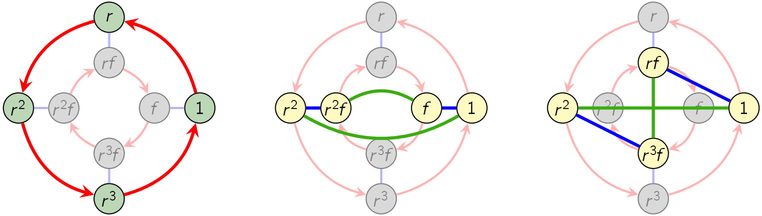
1. The subgroup $H = G$ is always normal in G. The only left coset is also the only right coset:
$$eG = G = Ge.$$
2. The subgroup $H = \{e\}$ is always normal. The left and right cosets are singleton sets:
$$gH = \{g\} = Hg.$$
3. Subgroups H of index 2 are normal. The two cosets (left or right) are H and $G - H$.
4. Subgroups of *abelian groups* are always normal, because for any $H \leq G$,
$$aH = \big\{ah \mid h \in H\big\} = \big\{ha \mid h \in H\big\} = Ha.$$
5. The center $Z(G)$ is always normal, for the same reason as above.
6. Relatedly, any subgroup of $Z(G)$ is always normal.

Normal subgroups in D_4



From our explorations, we found:

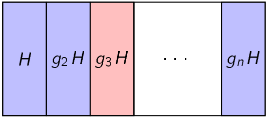
- $\langle r\rangle \lhd D_4$ (because it has index 2!)
- $\langle r^2, f\rangle \lhd D_4$ (index 2!)
- $\langle r^2, rf\rangle \lhd D_4$ (index 2!)
- $\langle r^2\rangle \lhd D_4$ (because it is $Z(D_4)$!)
  - (Also, it's the only guy who's a subgroup of 3 different groups)



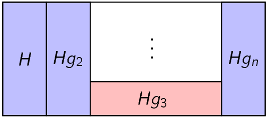
Normalizers

Okay, well, if $H \leq G$ isn't normal, then a natural followup question is:

"How many left cosets of H are right cosets?"



Partition of G by the left cosets of H



Partition of G by the right cosets of H

- "Best case" scenario ($H \trianglelefteq G$): all of them
- "Worst case" scenario: only H (I mean for sure the identity coset $eH = He$)
- In general: somewhere between these two extremes

Definition

The normalizer of H, denoted $N_G(H)$, is the set of elements $g \in G$ such that $gH = Hg$:

$$N_G(H) = \{g \in G \mid gH = Hg\},$$

i.e., the union of reps of left cosets that are also reps of right cosets.

Homework

Prove that $N_G(H) \leq G$, and also that $H \trianglelefteq N_G(H)$.

Tricks for spotting normal subgroups!

How to check if a subgroup is normal

If $gH = Hg$, then right-multiplying both sides by g^{-1} yields $gHg^{-1} = H$.

This gives us a new way to check whether a subgroup H is normal in G.

Useful remark

The following are equivalent ("TFAE") to a subgroup $H \leq G$ being normal:

(i) $gH = Hg$ for all $g \in G$; ("*left cosets are right cosets*")

(ii) $gHg^{-1} = H$ for all $g \in G$; ("*only one conjugate subgroup*")

(iii) $ghg^{-1} \in H$ for all $h \in H$, $g \in G$; ("*closed under conjugation*")

Proof

(i) $\Leftrightarrow$ (ii): Boringly obvious. (ii) $\Rightarrow$ (iii): Also boringly obvious.

(iii) $\Rightarrow$ (ii): Interesting; homework. :)

Sometimes, one of these methods is *much* easier than the others!

- to show $H \ntrianglelefteq G$, find *just one element* $h \in H$ for which $ghg^{-1} \notin H$ for some $g \in G$.
- if G has a unique subgroup of size $|H|$, then H *must* be normal. (Why?)

Conjugate subgroups

For a fixed element $g \in G$, the set

$$gHg^{-1} = \{ghg^{-1} \mid h \in H\}$$

is called the conjugate of H by g.

Homework

For any $g \in G$, the conjugate gHg^{-1} is a subgroup of G.

Observation

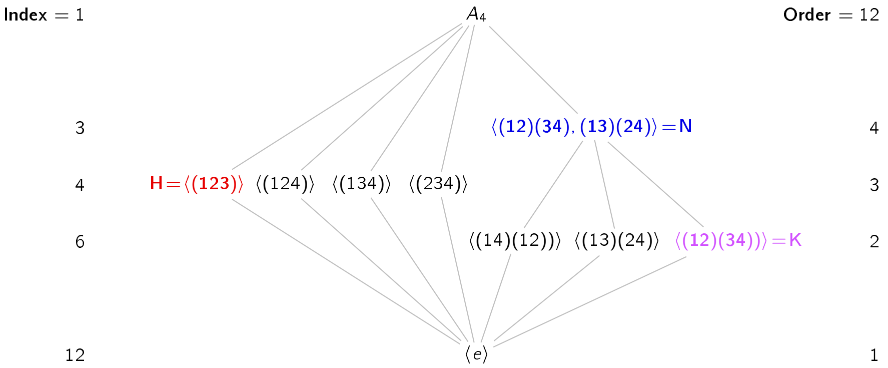
$|gHg^{-1}| = |H|$. (Proof: Look at the definition.)

Later, we'll prove that H and gHg^{-1} are isomorphic subgroups.

The subgroup lattice of A_4

I am highlighting the following three subgroups of A_4:

$$N = \langle (12)(34), (13)(24) \rangle = \{e,\ (12)(34),\ (13)(24),\ (14)(23)\} \cong V_4$$
$$H = \langle (123) \rangle = \{e,\ (123),\ (132)\} \cong C_3$$
$$K = \langle (12)(34) \rangle = \{e,\ (12)(34)\} \cong C_2.$$



Who could possibly be conjugate to N? to H? to K?
Who could possibly be $N_{A_4}(N)$? $N_{A_4}(H)$? $N_{A_4}(K)$?

Two pretty good reasons why N is normal

Useful remark

The following are equivalent ("TFAE") to a subgroup $H \leq G$ being normal:

(i) $gH = Hg$ for all $g \in G$; ("*left cosets are right cosets*")

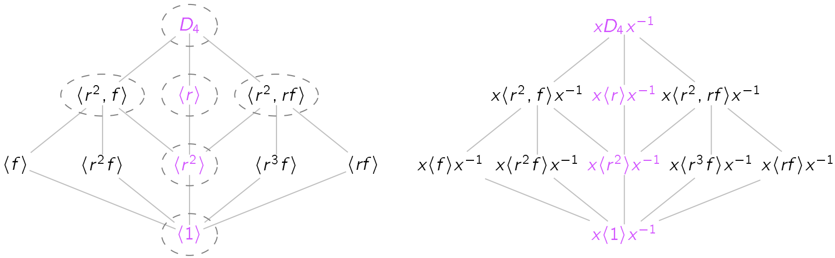
(ii) $gHg^{-1} = H$ for all $g \in G$; ("*only one conjugate subgroup*")

(iii) $ghg^{-1} \in H$ for all $h \in H$, $g \in G$; ("*closed under conjugation*")

1. N is the only subgroup of its size in the subgroup lattice of A_4, so definitely $gNg^{-1} = N$
2. $N_{A_4}(N)$ has to be between N and A_4 in the lattice, so it's either N itself or all of A_4.
   - So, pick something outside of N and see if it normalizes N.

Unicorn subgroups

Suppose we conjugate $G = D_4$ by some element $x \in D_4$.



Subgroups at a unique "lattice neighborhood" are called unicorns, and must be normal.

For example, $\langle r^2\rangle = x\langle r^2\rangle x^{-1}$ is the only size-2 subgroup "*with 3 parents*."

The groups G and $\langle 1\rangle$ are always unicorns, and hence normal.

The index-2 subgroups $\langle r^2, f\rangle$, $\langle r\rangle$, and $\langle r^2, rf\rangle$ must be normal.

Remark

Conjugating a normal subgroup $N \leq G$ by $x \in G$ shuffles its elements and subgroups.
In particular, this includes conjugating all of G by some $x \in G$.

The conjugacy class of a subgroup

Proposition

Conjugation is an equivalence relation on the set of subgroups of G.

Proof

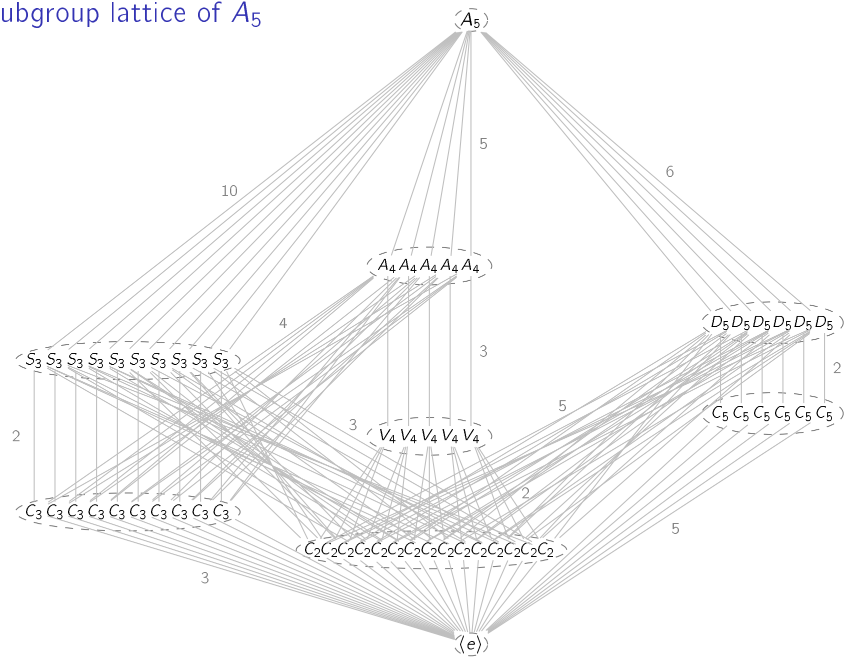
We need to show that conjugacy is reflexive, symmetric, and transitive.

- **Reflexive**: $eHe^{-1} = H$. ✓
- **Symmetric**: Suppose H is conjugate to K, by $aHa^{-1} = K$. Then K is conjugate to H:
$$a^{-1}Ka = a^{-1}(aHa^{-1})a = H. \quad ✓$$
- **Transitive**: Suppose $aHa^{-1} = K$ and $bKb^{-1} = L$. Then H is conjugate to L:
$$(ba)H(ba)^{-1} = b(aHa^{-1})b^{-1} = bKb^{-1} = L. \quad ✓$$

Definition

The set of all subgroups conjugate to H is its conjugacy class, denoted
$$\mathrm{cl}_G(H) = \{gHg^{-1} \mid g \in G\}.$$

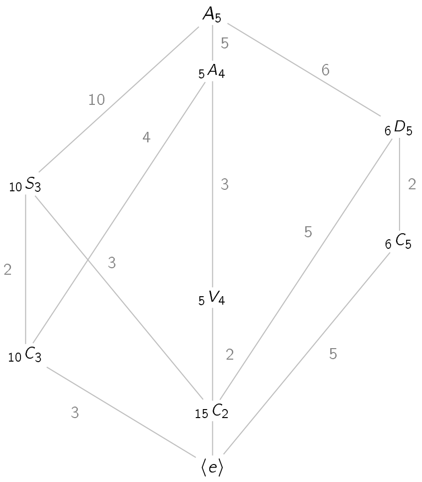
The subgroup lattice of A_5



“Reducing” subgroup lattices

Sometimes it is convenient to collapse conjugacy classes into single nodes in the lattice.

Left-subscripts denote size. We call this a subgroup diagram (bc it might not actually be a lattice.)



The end!