Homework #13 (due Apr 27)

Wiki updates

Problem 1. The Wiki of theorems never got updated after we first made it. During one of your study groups, take some time to add some Important Theorems to this list. Certainly not every theorem is Important; which ones do you want to make sure are there for easy reference?

There exist group actions with no fixed points!

Problem 2. Here's the "fixed point table" for the action $D_4 \xrightarrow{\phi} \text{Perm}(S)$, where S is the set of 7 binary squares that are listed across the top of the table:

	$\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$	$\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$	$\begin{smallmatrix}0&0\\1&1\end{smallmatrix}$	$\begin{smallmatrix}0&1\\0&1\end{smallmatrix}$	$\begin{smallmatrix}1&1\\0&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\1&0\end{smallmatrix}$
1	✓	✓	✓	✓	✓	✓	✓
r	✓						
r^2	✓	✓	✓				
r^3	✓						
f	✓			✓		✓	
rf	✓	✓	✓				
r^2f	✓				✓		✓
r^3f	✓	✓	✓				

The fixed points of the action ϕ are columns with **all** checkmarks:
set elements fixed by **every** group element.
The kernel of the action ϕ consists of the rows with **all** checkmarks:
group elements who fix **every** set element.

(a) Find a binary square that is **not** a fixed point of ϕ.

(b) Identify all the fixed points of ϕ.

(c) Identify all group elements in the kernel of ϕ.

(d) The kernel of any group action is always nonempty. Why?

(e) Find or make up an example of a group action that has no fixed points.

Hint: maybe look at ones where G acts on its own elements, its subgroups, or its cosets.

The Sylow theorems

Handy hint: ask Wolfram Alpha "divisors of $5 \cdot 7 \cdot 11$," etc., to save lots of time.

Problem 3. Prove that there is no simple group of order $312 = 2^3 \cdot 3 \cdot 13$.

Problem 4. Suppose $p < q$ are primes such that p doesn't divide $q-1$, and then suppose $|G| = pq$. One example of such a setup is $p = 3$, $q = 11$ (so then $q - 1 = 10$ which isn't divisible by 3), and $pq = 33$.

(a) Think of a couple more examples of p and q that would work.

(b) Prove that G is not simple.

(c) What goes wrong if p *does* divide $q - 1$?

Problem 5. Prove that there is no simple group of order $56 = 2^3 \cdot 7$.

Problem 6. Prove that there is no simple group of order $30 = 2 \cdot 3 \cdot 5$.

Problem 7. Prove that there is no simple group of order $6545 = 5 \cdot 7 \cdot 11 \cdot 17$.

Problem 8 (easier than you think)**.** Suppose $|G| = pqr$, where $p < q < r$ are all primes. Prove that G has a normal Sylow subgroup for either p, q, or r. (Hint: 30 is like this; think about Problem 6.)

Problem 9 (secretly not a Sylow theorem problem)**.** Prove that if $|G| = p^n$, then G is not simple. Hint: let G act on itself by conjugation, write down the class equation (from HW 12!), and decide that $|Z(G)| > 1$. Why does that mean G is not simple?

Further thoughts (optional)

Problem 10. Make up your own "there is no simple group of order __" problem.

Problem 11. Following on from Problem 4: If p *does* divide $q - 1$, construct a non-abelian group of order pq (hint: $\rtimes$). Explain why this group you made has a normal subgroup.

Problem 12. A group of order $p^2 q$, where p and q are distinct primes, can't be simple. (There are two cases you'll have to think about separately: $p > q$ and $p < q$.)

Problem 13. If $|G| = 30$, then G has a normal subgroup isomorphic to $\mathbb{Z}_{15}$. (Use Problem 6.)

Problem 14. In class I skipped over a "side quest" about the normalizer of the normalizer. Watch Matt Macauley's lecture 5.11, starting in particular at about 24:25, and prove the following theorem: if P is a (non-normal) Sylow p-subgroup of G, then $N_G(N_G(P)) = N_G(P)$.

Problem 15. Lots of fun stuff on Matt Macauley's homework from last week, lol.

Problem 16. Write a program that finds "potential bad Sylow numbers:"

(a) gives each odd number $n < 10{,}000$ that is not a power of a prime and that has some prime divisor p such that n_p is not forced to be 1

(b) for each such n, shows the factorization of n and gives the list of all permissible values of n_p for each p dividing n

(c) does the same for even numbers below 1000; explain the relative lengths of the two lists!