Normal subgroups!

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With many thanks to Matthew Macauley,
`http://www.math.clemson.edu/~macaule/`

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Goals for today:

1. Define what quotient groups are
2. See some examples
3. Thus, see why we care so much about normal subgroupss

Some review!

Cosets!

Definition

If $H \leq G$, then a (left) coset is a set

$$xH = \{xh \mid h \in H\},$$

for some fixed $x \in G$ called the representative.
Similarly, we can define a right coset as

$$Hx = \{hx \mid h \in H\}.$$

Morally:

A coset of H is a shifted copy of H somewhere else in G.

A coset of H is always / sometimes / never:

- An element of G
- A subset of G
- Equal to H
- A subgroup of G

Conjugate subgroups!

Definition

For a fixed element $g \in G$, the conjugate of H by g is the set

$$gHg^{-1} = \{ghg^{-1} \mid h \in H\}.$$

A conjugate of H is always / sometimes / never:

- An element of G
- A subset of G
- Equal to H
- A subgroup of G

Definition

The conjugacy class of H in G is the set of all conjugates of H:

$$\mathrm{cl}_G(H) = \{gHg^{-1} \mid g \in G\}.$$

Morally

$\mathrm{cl}_G(H)$ is a list of all the subgroups of G that are "similar to" H.

Normal subgroups!

Formal definition

A subgroup H is a normal subgroup of G if $gH = Hg$ for *all* $g \in G$. We write $H \trianglelefteq G$.

Equivalent definition

... if $gHg^{-1} = H$ for *all* $g \in G$.

Equivalent definition #2

... if there is only one conjugate subgroup to H, ie., H itself.

Equivalent definition #3

... if $|\operatorname{cl}_G(H)| = 1$.

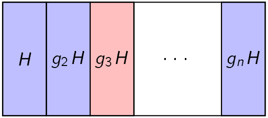
Morally

Normal subgroups are in some way unique in their group.

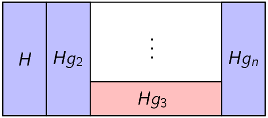
Normal-ish subgroups

Okay, well, if $H \leq G$ isn't normal, then a natural followup question is:

"*How* non-normal?" "*How many* left cosets of H are right cosets?"



Partition of G by the left cosets of H



Partition of G by the right cosets of H

- "Best case" scenario ($H \trianglelefteq G$): all of them
- "Worst case" scenario: only H (I mean for sure the identity coset $eH = He$)
- In general: somewhere between these two extremes

Normalizers!

Definition

The normalizer of H, denoted $N_G(H)$, is the set of elements $g \in G$ such that $gH = Hg$:

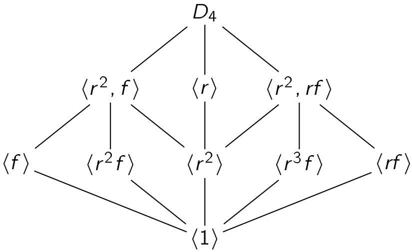
$$N_G(H) = \{g \in G \mid gH = Hg\}$$

Examples of normal sugroups

We've seen cases where we know a subgroup will be normal without having to check.

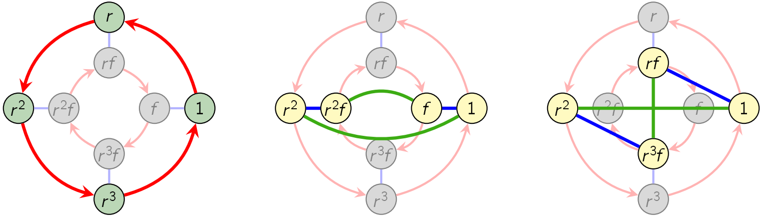
1. The subgroup $H = G$ is always normal in G. The only left coset is also the only right coset:
$$eG = G = Ge.$$
2. The subgroup $H = \{e\}$ is always normal. The left and right cosets are singleton sets:
$$gH = \{g\} = Hg.$$
3. Subgroups H of index 2 are normal. The two cosets (left or right) are H and $G - H$.
4. Subgroups of *abelian groups* are always normal, because for any $H \leq G$,
$$aH = \{ah \mid h \in H\} = \{ha \mid h \in H\} = Ha.$$
5. The center $Z(G)$ is always normal, for the same reason as above.
6. Relatedly, any subgroup of $Z(G)$ is always normal.

Normal subgroups in D_4



From our explorations, we found:

- $\langle r\rangle \lhd D_4$ (because it has index 2!)
- $\langle r^2, f\rangle \lhd D_4$ (index 2!)
- $\langle r^2, rf\rangle \lhd D_4$ (index 2!)
- $\langle r^2\rangle \lhd D_4$ (because it is $Z(D_4)$!)
  - (Also, it's the only guy who's a subgroup of 3 different groups)



Tricks for spotting normal subgroups!

How to check if a subgroup is normal

If $gH = Hg$, then right-multiplying both sides by g^{-1} yields $gHg^{-1} = H$.

This gives us a new way to check whether a subgroup H is normal in G.

Useful remark

The following are equivalent ("TFAE") to a subgroup $H \leq G$ being normal:

(i) $gH = Hg$ for all $g \in G$; ("*left cosets are right cosets*")

(ii) $gHg^{-1} = H$ for all $g \in G$; ("*only one conjugate subgroup*")

(iii) $ghg^{-1} \in H$ for all $h \in H$, $g \in G$; ("*closed under conjugation*")

Proof

(i) $\Leftrightarrow$ (ii): Boringly obvious. (ii) $\Rightarrow$ (iii): Also boringly obvious.

(iii) $\Rightarrow$ (ii): Interesting; homework. :)

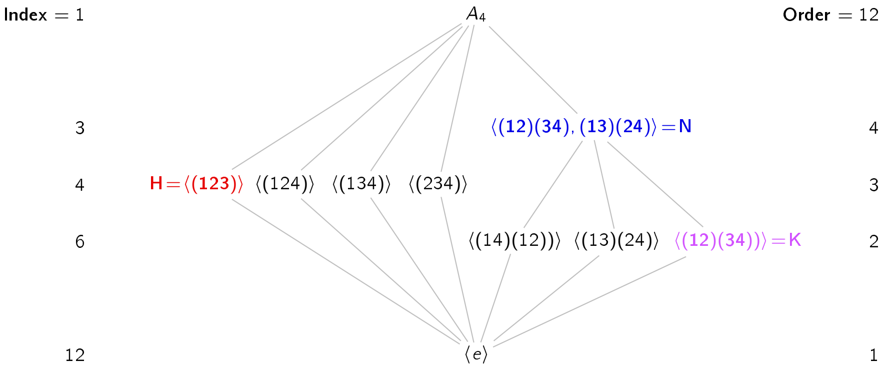
Sometimes, one of these methods is *much* easier than the others!

- to show $H \ntrianglelefteq G$, find *just one element* $h \in H$ for which $ghg^{-1} \notin H$ for some $g \in G$.
- if G has a unique subgroup of size $|H|$, then H *must* be normal. (Why?)

The subgroup lattice of A_4

I am highlighting the following three subgroups of A_4:

$$N = \langle (12)(34), (13)(24) \rangle = \{e,\ (12)(34),\ (13)(24),\ (14)(23)\} \cong V_4$$
$$H = \langle (123) \rangle = \{e,\ (123),\ (132)\} \cong C_3$$
$$K = \langle (12)(34) \rangle = \{e,\ (12)(34)\} \cong C_2.$$



Who could possibly be conjugate to N? to H? to K?
Who could possibly be $N_{A_4}(N)$? $N_{A_4}(H)$? $N_{A_4}(K)$?

Two pretty good reasons why N is normal

Useful remark

The following are equivalent ("TFAE") to a subgroup $H \leq G$ being normal:

(i) $gH = Hg$ for all $g \in G$; ("*left cosets are right cosets*")

(ii) $gHg^{-1} = H$ for all $g \in G$; ("*only one conjugate subgroup*")

(iii) $ghg^{-1} \in H$ for all $h \in H$, $g \in G$; ("*closed under conjugation*")

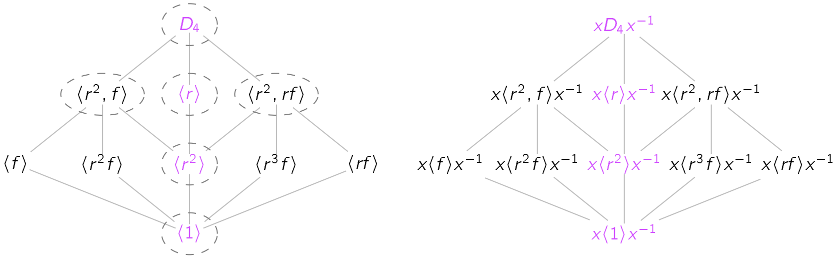
1. N is the only subgroup of its size in the subgroup lattice of A_4, so definitely $gNg^{-1} = N$
2. $N_{A_4}(N)$ has to be between N and A_4 in the lattice, so it's either N itself or all of A_4.
   - So, pick something outside of N and see if it normalizes N.

Unicorn subgroups

Suppose we conjugate $G = D_4$ by some element $x \in D_4$.

Very useful idea

Conjugating a normal subgroup $N \leq G$ by $x \in G$ shuffles its elements and subgroups.
In particular, this includes conjugating all of G by some $x \in G$.



Subgroups at a unique "lattice neighborhood" are called unicorns, and must be normal.

For example, $\langle r^2\rangle = x\langle r^2\rangle x^{-1}$ is the only size-2 subgroup "*with 3 parents*."

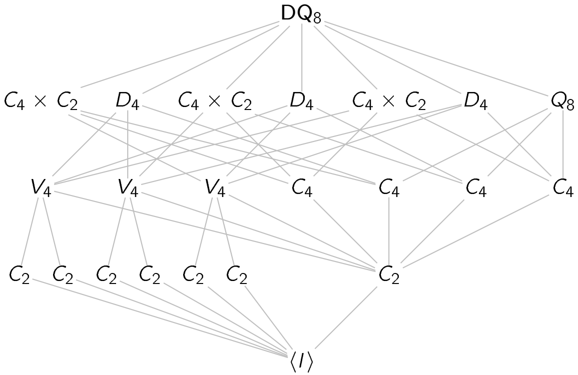
The groups G and $\langle 1\rangle$ are always unicorns, and hence normal.

The index-2 subgroups $\langle r^2, f\rangle$, $\langle r\rangle$, and $\langle r^2, rf\rangle$ must be normal.

Unicorns in the diquaternion group

Our definition of unicorn could be strengthened, but we want to keep things simple.

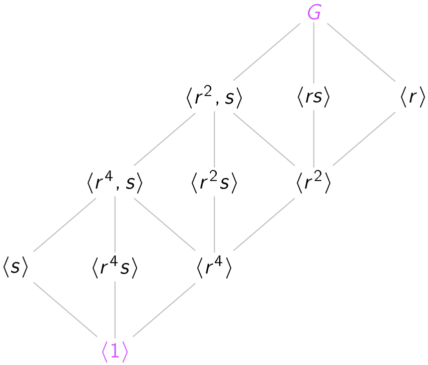
Here's the lattice for a group called DQ_8, which has order 16.
Are any of the C_4 subgroups of DQ_8 unicorns, i.e., "not like the others"?



(Preview: What can we say about the conjugacy classes of the subgroups of DQ_8 just from the lattice?)

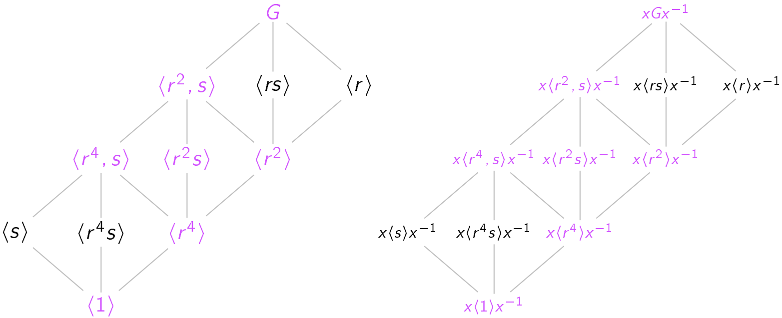
A mystery group of order 16

Here is the subgroup lattice for some actual group of order 16 but I'm not telling you which one it is. Find as many unicorns as you can.



A mystery group of order 16

Unicorns are purple.



We can deduce that every subgroup is normal, except possibly $\langle s\rangle$ and $\langle r^4 s\rangle$.

There are two cases:

- $\langle s\rangle$ and $\langle r^4 s\rangle$ are normal $\Rightarrow s \in Z(G) \Rightarrow G$ is abelian.
- $\langle s\rangle$ and $\langle r^4 s\rangle$ are not normal $\Rightarrow \mathrm{cl}_G(\langle s\rangle) = \{\langle s\rangle, \langle r^4 s\rangle\} \Rightarrow G$ is nonabelian.

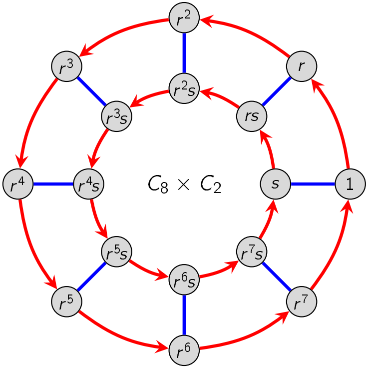
This doesn't necessarily mean that both of these are actually possible. . .

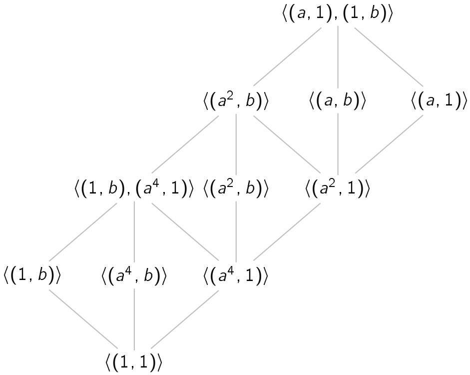
A mystery group of order 16

It's straightforward to check that this is the subgroup lattice of

$$C_8 \times C_2 = \left\langle r, s \mid r^8 = s^2 = 1, srs = r \right\rangle.$$

Let $r = (a, 1)$ and $s = (1, b)$, and so $C_8 \times C_2 = \langle r, s \rangle = \left\langle (a, 1), (1, b) \right\rangle$.

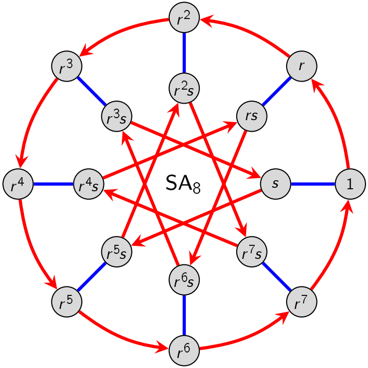


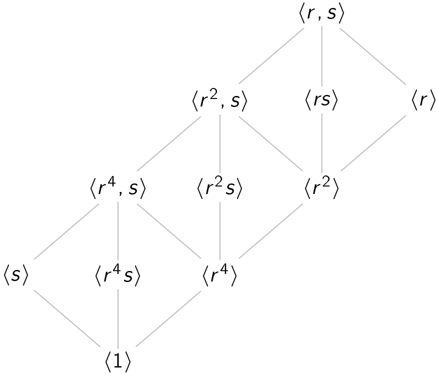


A mystery group of order 16

However, the nonabelian case is possible as well! The following also works:

$$\mathrm{SA}_8 = \langle r, s \mid r^8 = s^2 = 1, srs = r^5 \rangle.$$





Conjugacy classes!

The conjugacy class of a subgroup

Proposition

Conjugation is an equivalence relation on the set of subgroups of G.

Proof

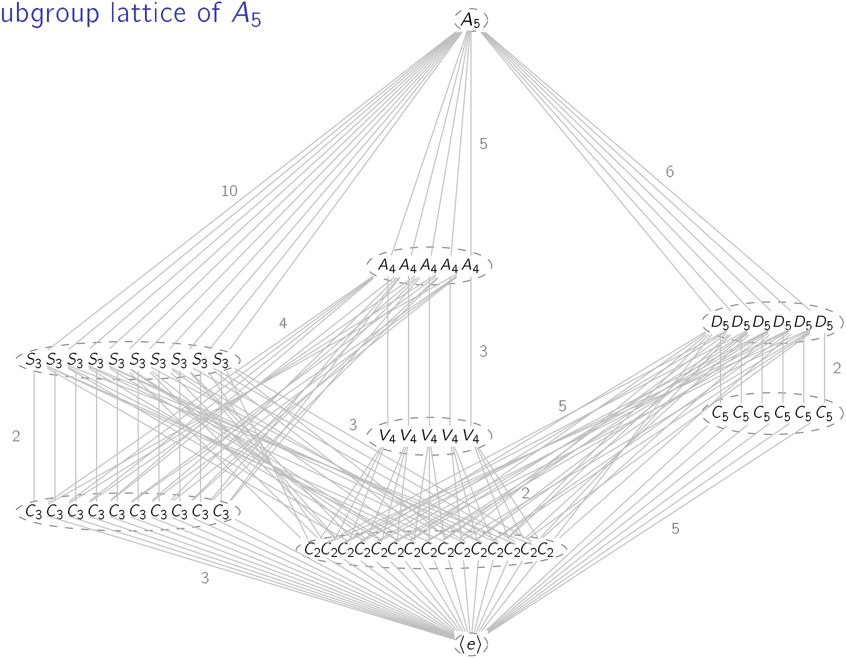
We need to show that conjugacy is reflexive, symmetric, and transitive.

- **Reflexive**: $eHe^{-1} = H$. ✓
- **Symmetric**: Suppose H is conjugate to K, by $aHa^{-1} = K$. Then K is conjugate to H:
$$a^{-1}Ka = a^{-1}(aHa^{-1})a = H. \qquad \checkmark$$
- **Transitive**: Suppose $aHa^{-1} = K$ and $bKb^{-1} = L$. Then H is conjugate to L:
$$(ba)H(ba)^{-1} = b(aHa^{-1})b^{-1} = bKb^{-1} = L. \qquad \checkmark$$

Definition

The set of all subgroups conjugate to H is its conjugacy class, denoted
$$\mathrm{cl}_G(H) = \{gHg^{-1} \mid g \in G\}.$$

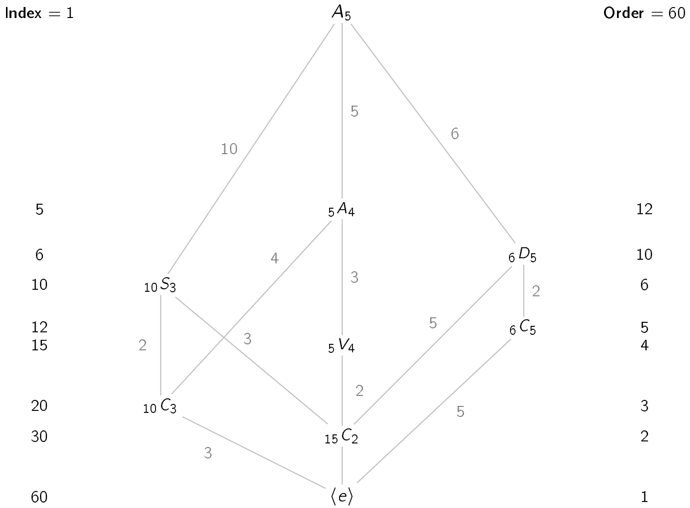
The subgroup lattice of A_5



"Reducing" subgroup lattices

Sometimes it is convenient to collapse conjugacy classes into single nodes in the lattice.

Left-subscripts denote the size of the conjugacy class. We call this a subgroup diagram.
(In some circumstances it might not actually be a lattice.)



The end!