Subgroups!

Spencer Bagley

With many thanks to Matthew Macauley,
http://www.math.clemson.edu/~macaule/

10 Feb 2025

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A proper subgroup $H < G$ is a subgroup that's not equal to the whole group.

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And in fact every subgroup looks like this.

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Writing $C_2 \leq D_3$ means *there is a copy of C_2 sitting inside of D_3 as a subgroup*.

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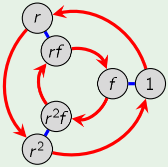
How many ways can you find C_2 sitting inside of D_3?



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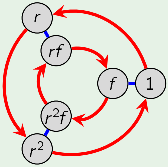

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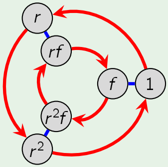

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$$C_2 \cong \langle f \rangle < D_3 \qquad C_2 \cong \langle rf \rangle < D_3 \qquad C_2 \cong \langle r^2 f \rangle < D_3$$

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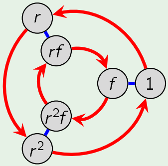

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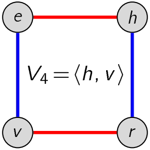
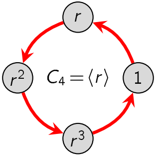
How about $C_3 \leq D_3$? There's only one!

Groups of order 4

The two groups of order 4

Let's start by considering the subgroups of the two groups of order 4.





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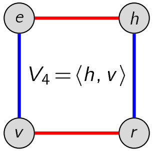

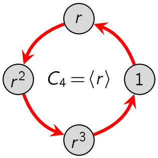

- Proper subgroups of V_4: $\langle h \rangle = \{e, h\}$, $\langle v \rangle = \{e, v\}$, $\langle r \rangle = \{e, r\}$, $\langle e \rangle = \{e\}$.
- Subgroups of C_4: $\langle r \rangle = \{1, r, r^2, r^3\} = \langle r^3 \rangle$, $\langle r^2 \rangle = \{1, r^2\}$, $\langle 1 \rangle = \{1\}$.

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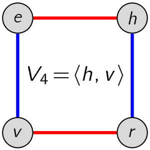

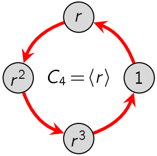

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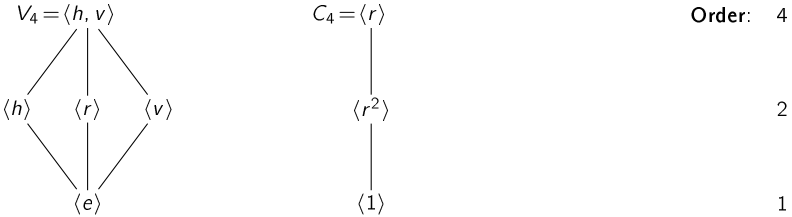
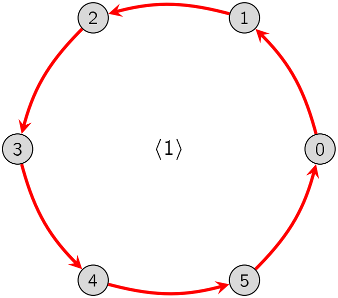
It is illustrative to arrange these in a subgroup lattice:



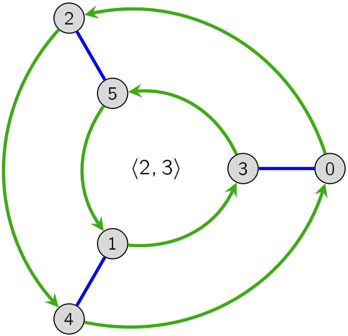
Groups of order 6

Subgroups of $\mathbb{Z}_6$

What subgroups can you find in $\mathbb{Z}_6$? I've drawn the Cayley diagram two different ways.



Hello I am secretly also the cycle graph



Subgroups of D_3

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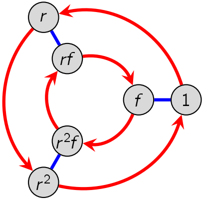

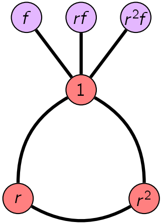

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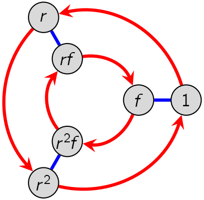

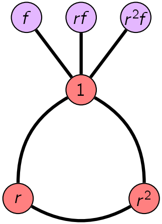

Here are the non-trivial proper subgroups of D_3:

$$\langle r\rangle = \{1, r, r^2\} = \langle r^2\rangle, \quad \langle f\rangle = \{1, f\}, \quad \langle rf\rangle = \{1, rf\}, \quad \langle r^2 f\rangle = \{1, r^2 f\}$$

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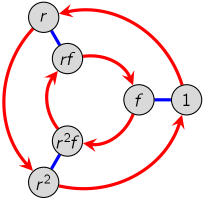

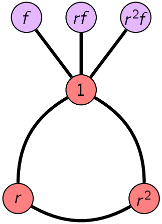

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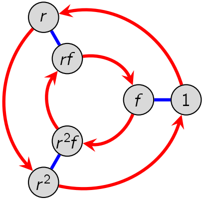

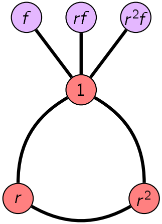

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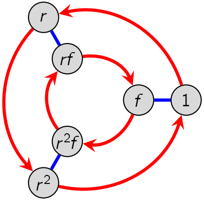

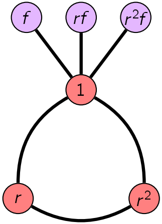

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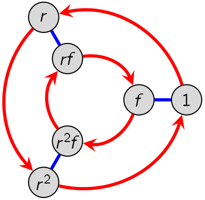

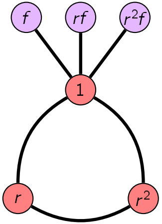

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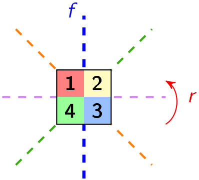
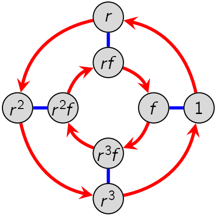
- The cycle graph helps us spot cyclic subgroups.
- For small groups like D_3, the cyclic subgroups may be the only proper subgroups.
- There might, however, be more complicated things that are harder to clock.

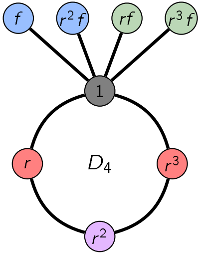
Groups of order 8

Subgroups of D_4

See if you can figure out all the subgroups of D_4.







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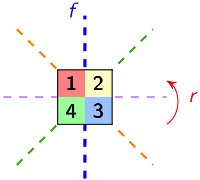

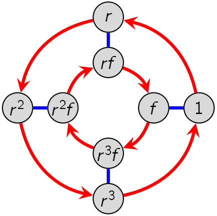

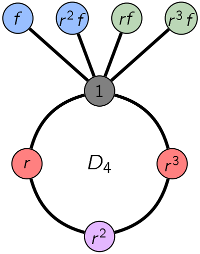

What do you think is a reasonable way to, like, arrange them?

Lattices

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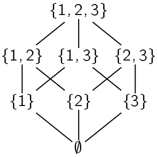
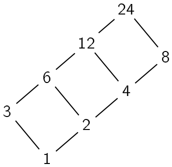
Examples you may have seen previously are subset lattices and divisor lattices.

$x \vee y = x \cup y$

$x \wedge y = x \cap y$





$x \vee y = \mathrm{lcm}(x, y)$

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This seems like a good way to organize subgroups, because:

Lattices

A lattice is a partially ordered set such that every pair of elements x, y has a unique:

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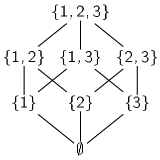

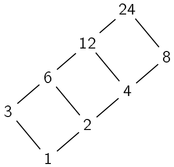

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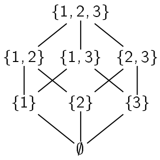

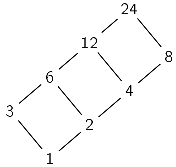

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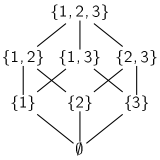

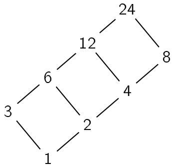

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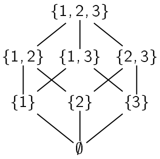

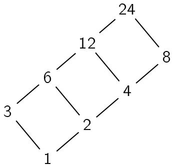

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(Note that $H \cup K$ is not in general a subgroup. Why not?)

Subgroup lattices

$$\begin{tikzcd}[column sep=small]
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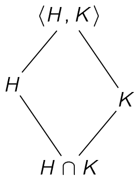

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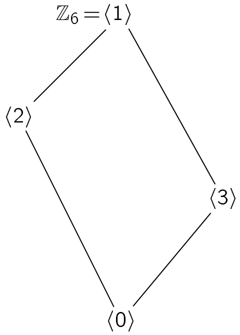

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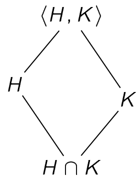

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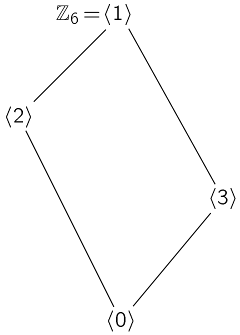

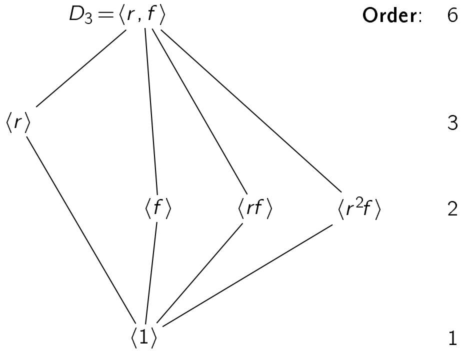

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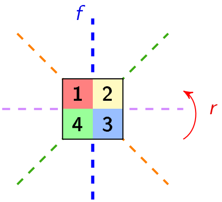

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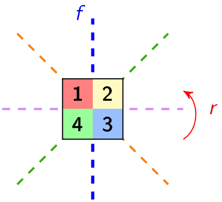

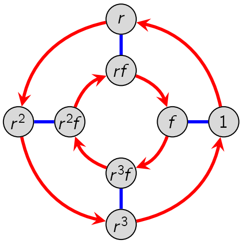

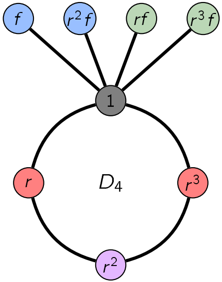

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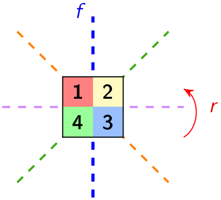

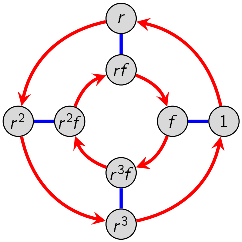

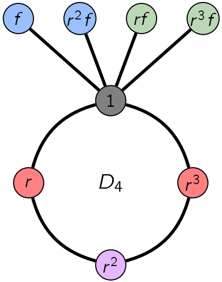

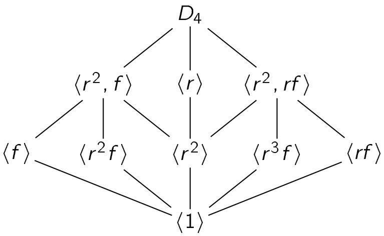

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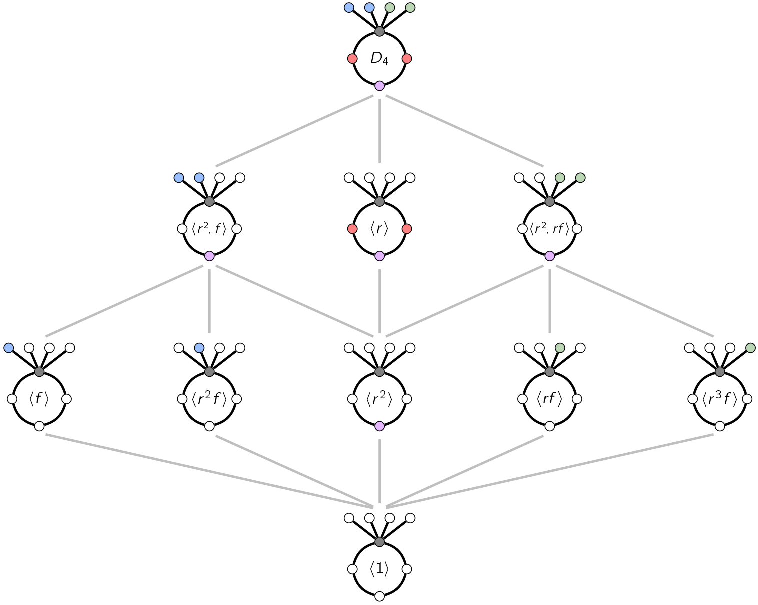

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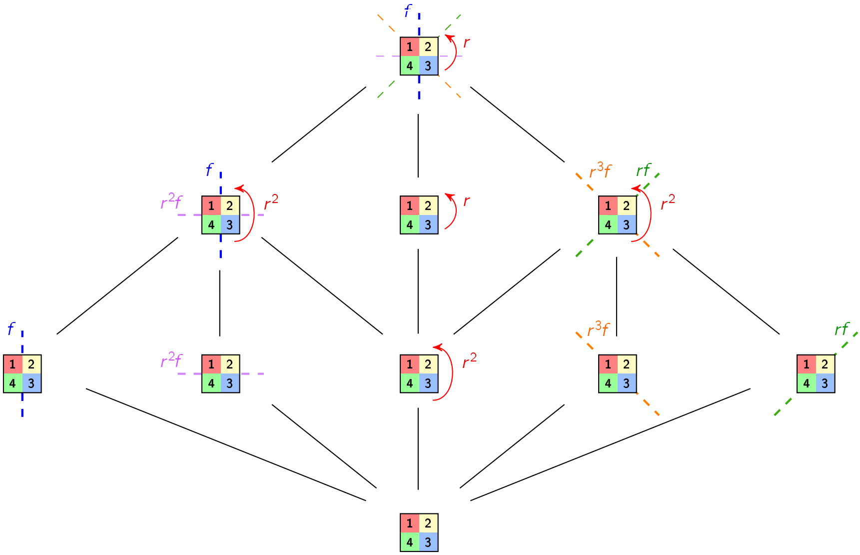

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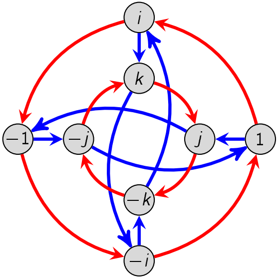

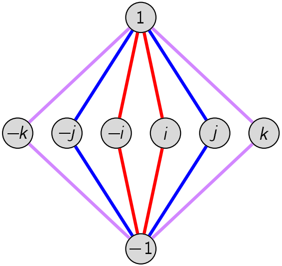

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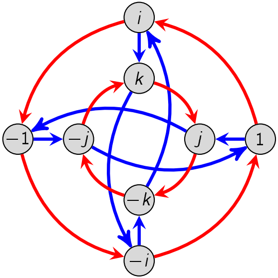

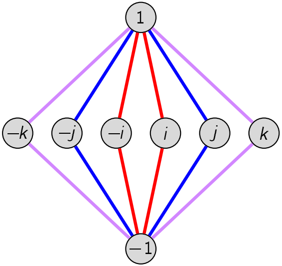

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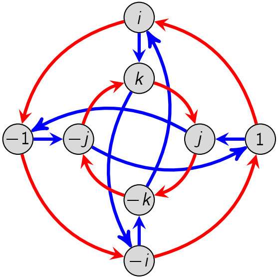

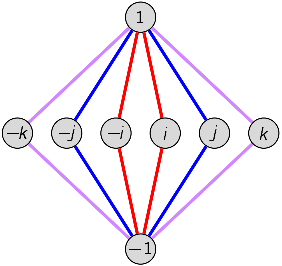

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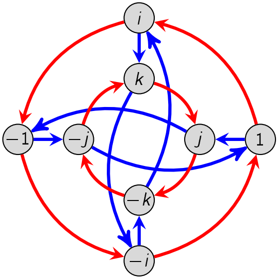

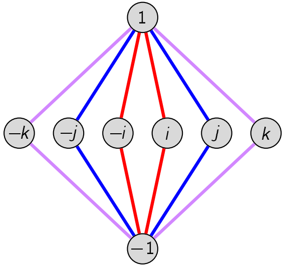

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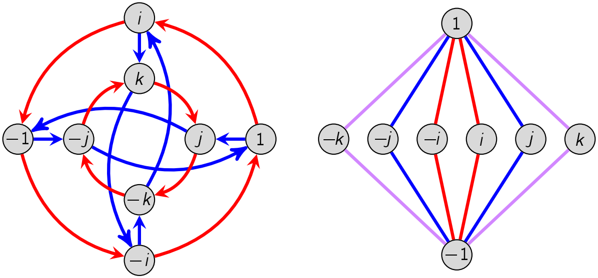

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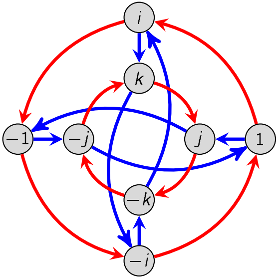

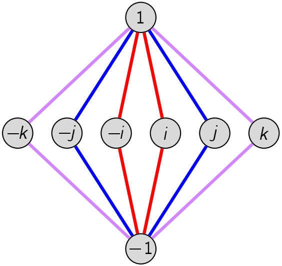

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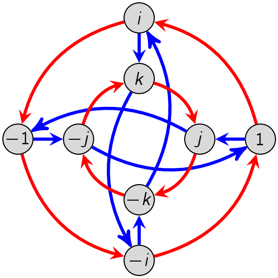

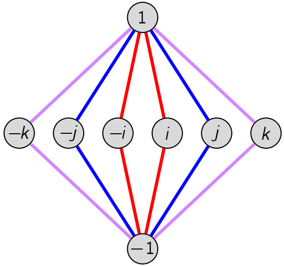

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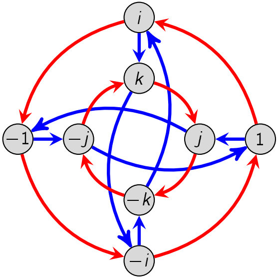

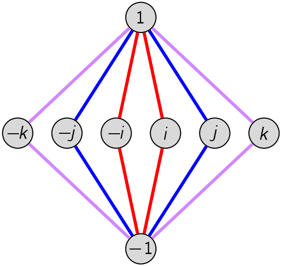

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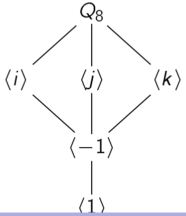

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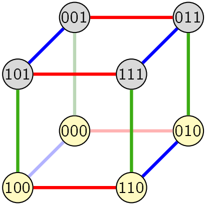

$\langle 100, 010 \rangle$

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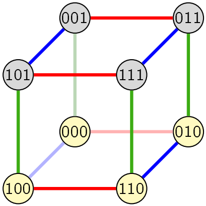

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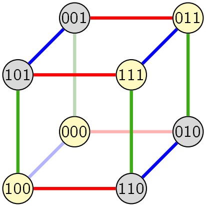

$\langle 100, 011 \rangle$

Subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

We've seen the subgroup lattices of two groups of order 8:

- D_4 has five elements of order 2, and 10 subgroups.
- Q_8 has one element of order 2, and 6 subgroups.
- $\mathbb{Z}_2^3$ has seven *elements* of order 2.

Rule of thumb

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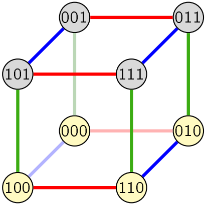

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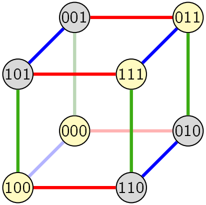

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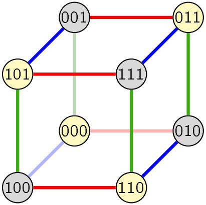

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The subgroup lattice of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

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All $\binom{7}{2} = 21$ pairs of non-identity elements generate a subgroup isomorphic to V_4.

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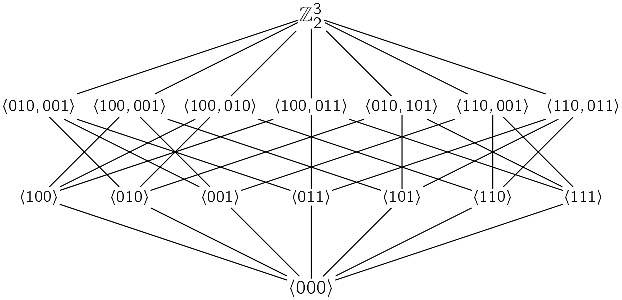

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Draw the Cayley diagram of Z_8 and find all its subgroups.
Arrange them in a lattice.

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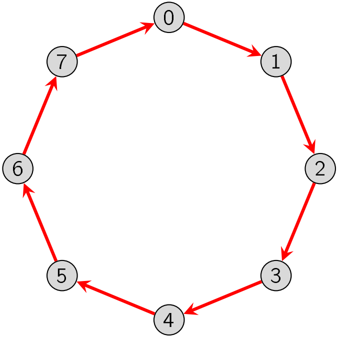

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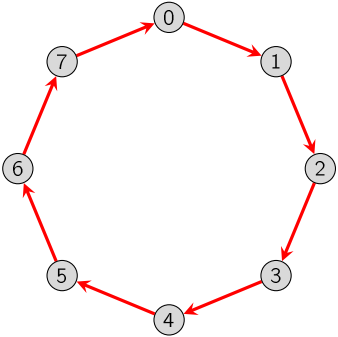

$$\mathbb{Z}_8 = \langle 1 \rangle$$

$$\langle 2 \rangle$$

$$\langle 4 \rangle$$

$$\langle 0 \rangle$$

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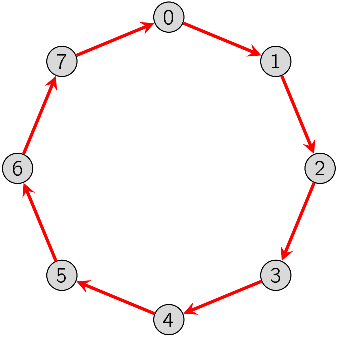

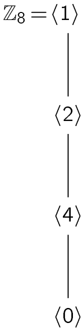

Theorem

Every subgroup of a cyclic group is cyclic.

Groups of order 8

There is one more group of order 8, which is $\mathbb{Z}_4 \times \mathbb{Z}_2$.

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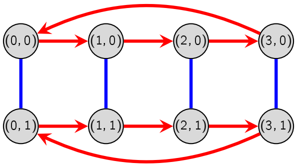

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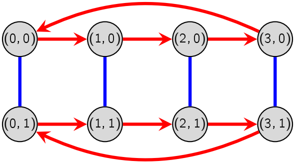

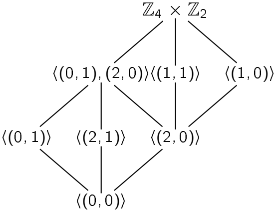

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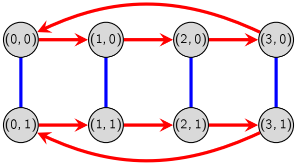

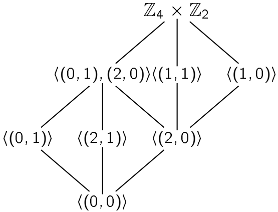

Let's summarize the sizes of the subgroups of the groups of order 8 that we have seen.

	C_8	Q_8	$C_4 \times C_2$	D_4	C_2^3
# elts. of order 8	4	0	0	0	0
# elts. of order 4	2	6	4	2	0
# elts. of order 2	1	1	3	5	7
# elts. of order 1	1	1	1	1	1
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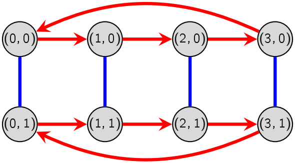

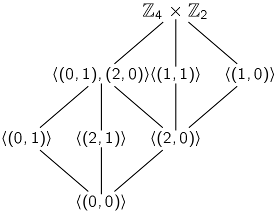

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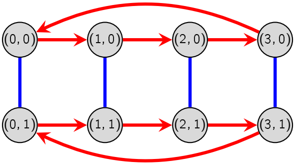

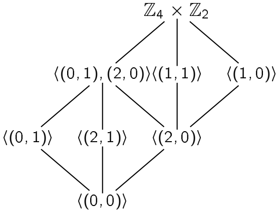

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Observations?

- Groups that have more elements of small order tend to have more subgroups.
- In all of these cases, the order of each subgroup divides $|G|$.

The end!