Subgroups!

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With many thanks to Matthew Macauley,
http://www.math.clemson.edu/~macaule/

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Definition time!

Here is the definition of a subgroup.

Definition

A subgroup of G is a subset $H \subseteq G$ that is also a group. We denote this by $H \leq G$.

Okay, but remind me what's the definition of a group?

Definition

A group $(G, \star)$ is a set of elements together with a binary operation $\star$ satisfying the following properties:

1. The operation is associative.
2. G contains the identity element.
3. Every element in G has an inverse element.
4. G is closed under the binary operation.

Trivial subgroups

Every group G has the following two boring subgroups: $G \leq G$, and $\{e\} \leq G$.

Definition

A proper subgroup $H < G$ is a subgroup that's not equal to the whole group.

Generating sets

We've previously looked at the orbit of an element:

Definition

The orbit of an element $g \in G$ is the cyclic subgroup that it generates,

$$\langle g \rangle = \{g^k \mid k \in \mathbb{Z}\},$$

and its order is $|g| := |\langle g \rangle|$.
In particular, if $|g| = n$ is finite, this is the set $\{g^0 = 1, g, g^2, \ldots, g^{n-1}\}$.

This is a subgroup:

Cyclic subgroups are subgroups

For any element $g \in G$, $\langle g \rangle \leq G$.

But we need not restrain ourselves to generating by one element:

Definition

Let S be a subset of G. A word in S is a finite product of finite powers of elements of S or their inverses.

$\langle S \rangle = \{\text{words in } S\}$ is a subgroup of G, and it's called the subgroup generated by S.

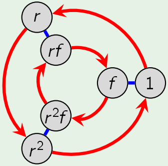
And in fact every subgroup looks like this.

Example: $C_2 \le D_3$

Writing $C_2 \le D_3$ means *there is a copy of C_2 sitting inside of D_3 as a subgroup*.

Question

How many ways can you find C_2 sitting inside of D_3?



Remark

It's more precise to express a subgroup by its generator(s).

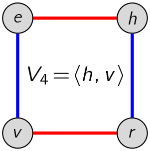
$C_2 \cong \langle f \rangle < D_3$ $\quad$ $C_2 \cong \langle rf \rangle < D_3$ $\quad$ $C_2 \cong \langle r^2 f \rangle < D_3$

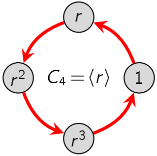
Question

How about $C_3 \le D_3$? There's only one!

The two groups of order 4

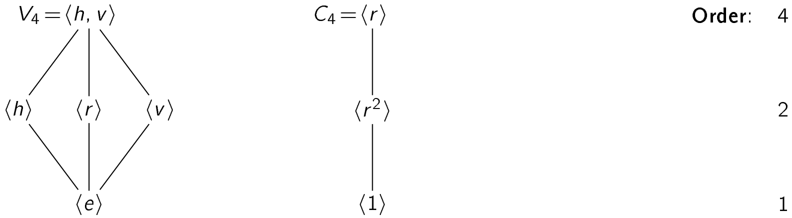
Let's start by considering the subgroups of the two groups of order 4.





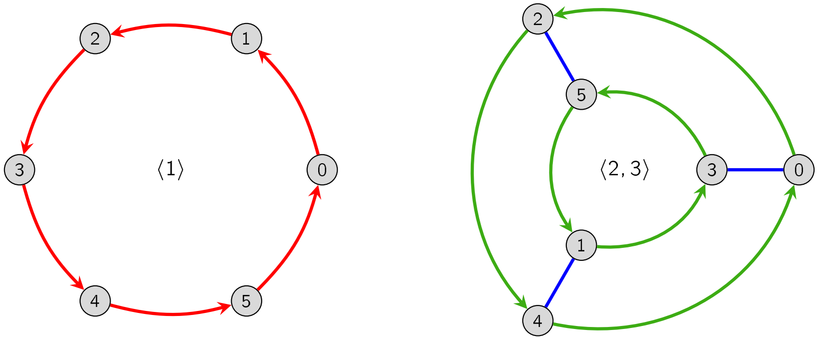
- Proper subgroups of V_4: $\langle h \rangle = \{e, h\}$, $\langle v \rangle = \{e, v\}$, $\langle r \rangle = \{e, r\}$, $\langle e \rangle = \{e\}$.
- Subgroups of C_4: $\langle r \rangle = \{1, r, r^2, r^3\} = \langle r^3 \rangle$, $\langle r^2 \rangle = \{1, r^2\}$, $\langle 1 \rangle = \{1\}$.

It is illustrative to arrange these in a subgroup lattice:



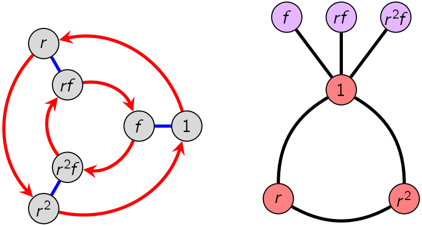
Subgroups of $\mathbb{Z}_6$

What subgroups can you find in $\mathbb{Z}_6$? I've drawn the Cayley diagram two different ways.



Subgroups of D_3

Let's figure out all the subgroups of D_3.



Here are the non-trivial proper subgroups of D_3:

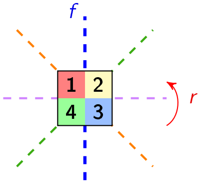
$$\langle r \rangle = \{1, r, r^2\} = \langle r^2 \rangle, \quad \langle f \rangle = \{1, f\}, \quad \langle rf \rangle = \{1, rf\}, \quad \langle r^2 f \rangle = \{1, r^2 f\}, \quad \langle 1 \rangle = \{1\}.$$

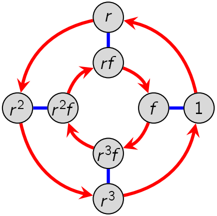
Observations:

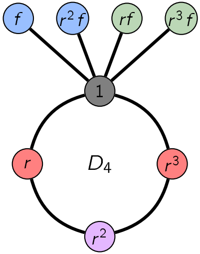
- The cycle graph helps us spot cyclic subgroups.
- For small groups like D_3, the cyclic subgroups may be the only proper subgroups.
- There might, however, be more complicated things that are harder to clock.

Subgroups of D_4

See if you can figure out all the subgroups of D_4.







What do you think is a reasonable way to, like, arrange them?

Lattices

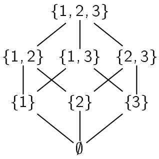
A lattice is a partially ordered set such that every pair of elements x, y has a unique:

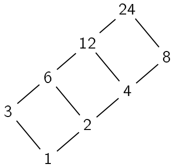
- join, or sup, or least upper bound $x \vee y$
- meet, or inf, or greatest lower bound $x \wedge y$.

Examples that we're familiar with are subset lattices and divisor lattices.

$x \vee y = x \cup y$

$x \wedge y = x \cap y$





$x \vee y = \mathrm{lcm}(x, y)$

$x \wedge y = \gcd(x, y)$

This seems like a good way to organize subgroups, because:

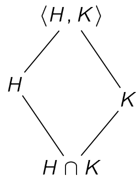
Theorem

If $H \leq G$ and $K \leq G$ are two subgroups, then $H \cap K$ is a subgroup.
(Indeed, it's the largest subgroup that's contained in both H and K.)

Theorem

$\langle H, K \rangle$ is the smallest subgroup containing both H and K.
(Note that $H \cup K$ is not in general a subgroup.)

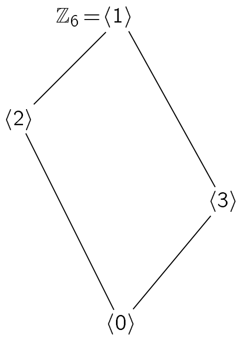
Subgroup lattices

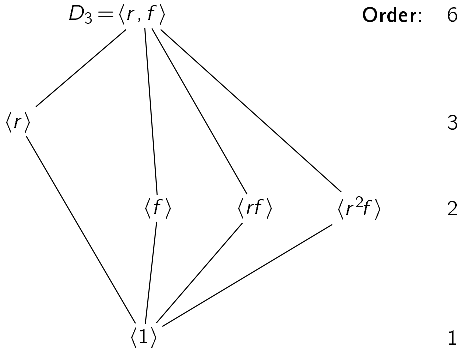


$H \vee K$: *"smallest subgroup above both H and K"*

$H \wedge K$: *"largest subgroup below both H and K"*

Examples:

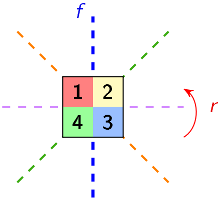


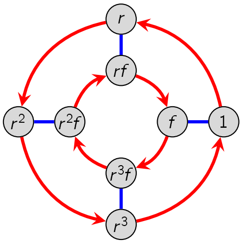


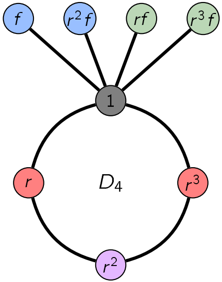
The subgroup lattice of D_4

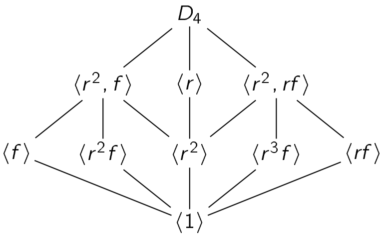
The subgroups of D_4 are:

- The entire group D_4, and the trivial group $\langle 1 \rangle$
- 4 subgroups generated by reflections: $\langle f \rangle$, $\langle rf \rangle$, $\langle r^2 f \rangle$, $\langle r^3 f \rangle$.
- 1 subgroup generated by a 180° rotation, $\langle r^2 \rangle \cong C_2$
- 1 subgroup generated by a 90° rotation, $\langle r \rangle \cong C_4$
- 2 subgroups isomorphic to V_4: $\langle r^2, f \rangle$, $\langle r^2, rf \rangle$.

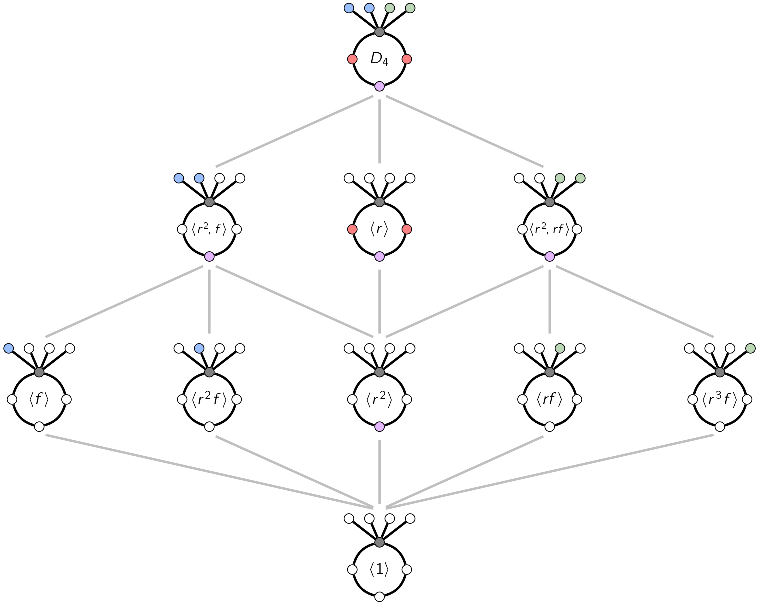








The subgroup lattice of D_4



The subgroup lattice of D_4