Homework #4

HW due Sunday 2/16 by pdf upload to Canvas; .tex source on the MATH 312 github repo.

General facts about subgroups

Problem 1 (the one-step subgroup test)**.** A subset $H \subseteq G$ is a subgroup if and only if the following condition holds:

$$\text{If } x, y \in H, \text{ then } xy^{-1} \in H. \tag{1}$$

Prove it! Hint:

($\Rightarrow$) Suppose that $H \le G$.
...
Therefore, whenever $x, y \in H$, $xy^{-1} \in H$.

($\Leftarrow$) Suppose that whenever $x, y \in H$, $xy^{-1} \in H$.
...
Therefore, $H \le G$.

Problem 2. If $g \in G$, prove that $\langle g \rangle \le G$. ("Cyclic subgroups are subgroups.")

Problem 3. Prove that $Z(G) \le G$. ("The center of G is a subgroup of G.")

Problem 4. Consider the following (complete, correct) proof:

Theorem: If $S \subseteq G$, then $\langle S \rangle \le G$.

Proof. Remember that $\langle S \rangle$ was defined as the set of "words in S", ie., finite products of finite powers of letters in S and their inverses:

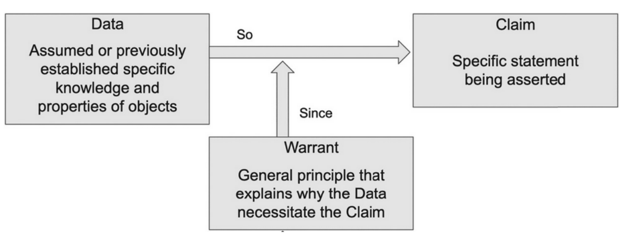
$$\langle S \rangle = \left\{ s_1^{p_1} \cdot s_2^{p_2} \cdot \ldots \cdot s_n^{p_n} \mid s_i \in S, p_i \in \mathbb{Z} \right\}.$$

Let $x = s_1^{p_1} \cdot s_2^{p_2} \cdot \ldots \cdot s_n^{p_n} \in \langle S \rangle$ and let $y = t_1^{q_1} \cdot t_2^{q_2} \cdot \ldots \cdot t_n^{q_n} \in \langle S \rangle$. By the shoes-and-socks theorem, $y^{-1} = t_n^{-q_n} \cdot \ldots \cdot t_2^{-q_2} \cdot t_1^{-q_1}$. Therefore,

$$\begin{aligned} xy^{-1} &= \left(s_1^{p_1} \cdot s_2^{p_2} \cdot \ldots \cdot s_n^{p_n} \right) \cdot \left(t_n^{-q_n} \cdot \ldots \cdot t_2^{-q_2} \cdot t_1^{-q_1} \right) \\ &= s_1^{p_1} \cdot s_2^{p_2} \cdot \ldots \cdot s_n^{p_n} \cdot t_n^{-q_n} \cdot \ldots \cdot t_2^{-q_2} \cdot t_1^{-q_1}. \end{aligned}$$

Since all the s_i's and all the t_i's are elements of S, and since all the p_i's and $-q_i$'s are integers, $xy^{-1} \in \langle S \rangle$. Therefore $\langle S \rangle \le G$, thanks to Problem 1. $\square$

Like the proof we saw in class that every subgroup of a cyclic group is cyclic, there are lots of things going on behind the scenes of this proof. See if you can break up this proof into data, claims, and (maybe implicit) warrants. (Click this small diagram for big.)



Helpful note: a warrant is a general principle and a data is a specific thing. For example, the group G that we're thinking about is a data, and the statement "group elements have inverses" is a warrant.

Subgroups of specific groups

Problem 5. Construct the subgroup lattice for A_4. Remember, this is the set of all the even permutations in S_4, so it consists of the following 12 elements:

$$\begin{aligned}
\text{0-cycles:}\ & ()\\
\text{3-cycles:}\ & (1\ 2\ 3), (1\ 3\ 2), (1\ 2\ 4), (1\ 4\ 2),\\
& (1\ 3\ 4), (1\ 4\ 3), (2\ 3\ 4), (2\ 4\ 3)\\
\text{Double transpositions:}\ & (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)
\end{aligned}$$

Notes:

- Why are 3-cycles even permutations? Note that for instance $(1\ 2\ 3) = (1\ 2)(1\ 3)$.
- Please use the permutation calculator or this will take forever. Make sure left-to-right multiplication is on.
- Start by finding all the cyclic subgroups. Then "build up."
- I guarantee that all the subgroups will have order 1, 2, 3, 4, 6, or 12. (But I won't guarantee that there's subgroups of all those orders.)