p-groups and the Sylow theorems!

Spencer Bagley

With many thanks to Matthew Macauley,
http://www.math.clemson.edu/~macaule/

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Overview

Intuitively, a group action occurs when a group G "naturally permutes" a set S *of states*.

Formal definition

A group G acts on a set S if there is a homomorphism $\phi\colon G \longrightarrow \mathrm{Perm}(S)$.
We'll use right group actions,
and we'll write $s.\phi(g)$ to denote "where pushing the g-button sends state s."

Definition

A set S with a (right) action by G is called a (right) G-set.

Big ideas

- An action $\phi\colon G \longrightarrow \mathrm{Perm}(S)$ endows S with an **algebraic structure**.
- *Action graphs are to G-sets, like how Cayley graphs are to groups.*

Notation

Throughout, we'll denote identity elements by $1 \in G$ and $e \in \mathrm{Perm}(S)$.

Five features of every group action

Every group action has **five fundamental features** that we will always try to understand.

	local (about an s or a g)	global (about the whole action ϕ)
subsets of S	$\mathrm{orb}(s)$ $\mathrm{fix}(g)$	$\mathrm{Fix}(\phi) = \bigcap_{g \in G} \mathrm{fix}(g)$
subgroups of G	$\mathrm{stab}(s)$	$\mathrm{Ker}(\phi) = \bigcap_{s \in S} \mathrm{stab}(s)$

"Duality:" columns vs. rows in the fixed-point table:

- the stablizers can be read off the columns: *group elements that fix $s \in S$*
- the kernel is the rows with a check in every column
- the fixators can be read off the rows: *set elements fixed by $g \in G$*
- the fixed points are the columns with a check in every row

More applications of group actions!

A creative application of a group action

Cauchy's theorem

If p is a prime dividing $|G|$, then G has an element (and hence a subgroup) of order p.

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Let P be the set of ordered p-tuples of elements from G whose product is e:

$$(x_1, x_2, \ldots, x_p) \in P \quad \text{iff} \quad x_1 x_2 \cdots x_p = e\,.$$

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Observe that $|P| = |G|^{p-1}$. (We can choose $x_1, \dots, x_{p-1}$ freely; then x_p is forced.)

The group $\mathbb{Z}_p$ acts on P by cyclic shift:

$$\phi \colon \mathbb{Z}_p \longrightarrow \mathrm{Perm}(P), \qquad (x_1, x_2, \dots, x_p) \xmapsto{\phi(1)} (x_2, x_3 \dots, x_p, x_1)\,.$$

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The only way that the orbit of $(x_1, x_2, \ldots, x_p)$ can have size 1 is if $x_1 = \cdots = x_p$.

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The $|G|^{p-1} - 1$ other elements in P sit in orbits of size 1 or p.

Since $p \nmid |G|^{p-1} - 1$, there must be other orbits of size 1. Thus, some $(x, \ldots, x) \in P$, with $x \neq e$ satisfies $x^p = e$. $\square$

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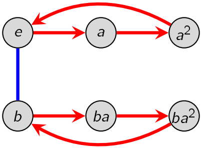

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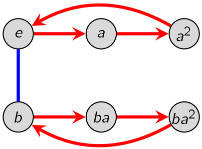

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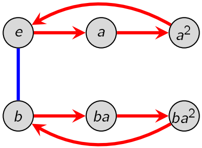

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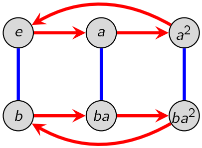

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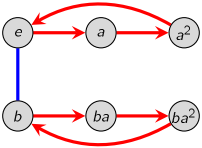

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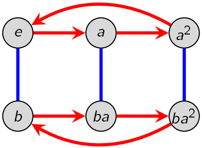

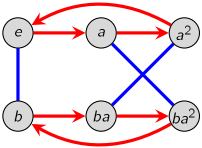

D_3

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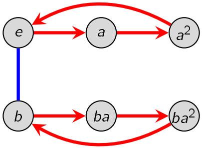

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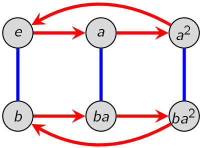

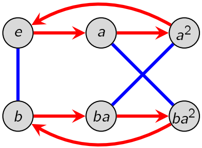

D_3

Exercise. Suppose that $|G| = pq$, where $p < q$ are primes and p doesn't divide $q - 1$. Prove that G is cyclic.

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Together, these place strong restrictions on the structure of a group G with a fixed order.

Groups of order $12 = 2^2 \cdot 3^1$

Head to LMFDB and look at subgroup lattices of each of the groups of order 12. What do you notice about the p-subgroups?

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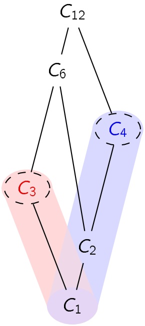
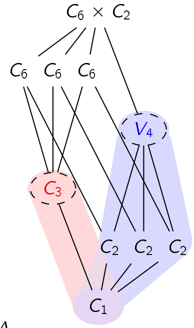
Sylow theorems:

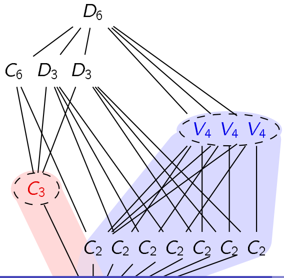
p-subgroups come in "towers."

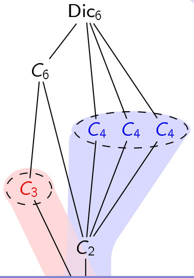
2-subgroups blue; 3-subgroups red.

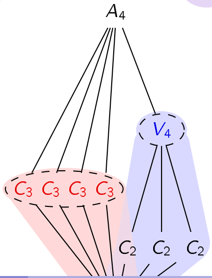
The tops of the towers are conjugate; there are restrictions on the size of their conjugacy classes.











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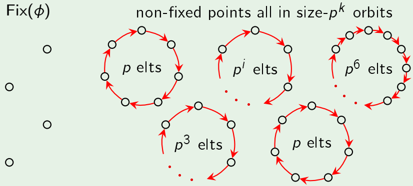

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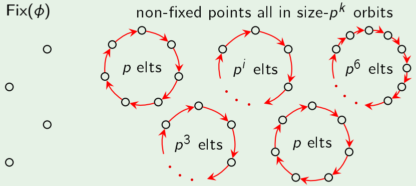

A lot of proofs about p-groups go like this: two things are equal mod p; set up some action of G on S; one of the things is the number of fixed points; the other thing is the size of S.

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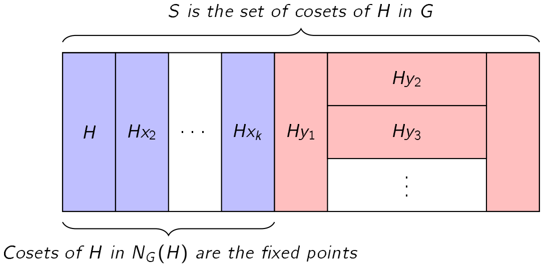

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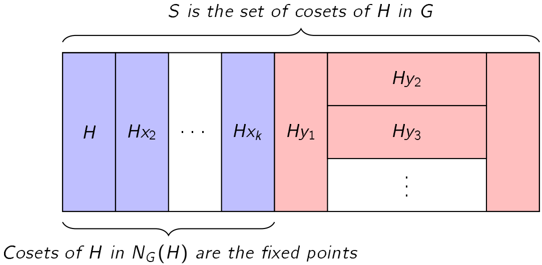

- Apply our lemma: $|\,\mathrm{Fix}(\phi)| \equiv_p |S|$.

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If H is a p-subgroup of G, then

$$[N_G(H)\colon H] \equiv_p [G\colon H]\,.$$

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Let $S = H\backslash G = \{Hx \mid x \in G\}$. The group H acts on S by **right-multiplication**, via $\phi\colon H \to \mathrm{Perm}(S)$, where

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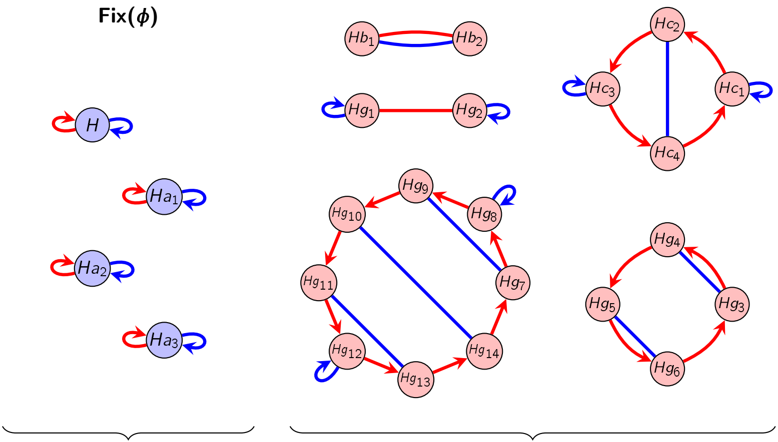
p-groups

Here is a picture of the action of the p-subgroup H (for $p = 2$) on the set $S = H\backslash G$, from the proof of the normalizer lemma.



The fixed points are the cosets in $N_G(H)$

Cosets not in $N_G(H)$ are in orbits of order p^i, for various $i \geq 1$

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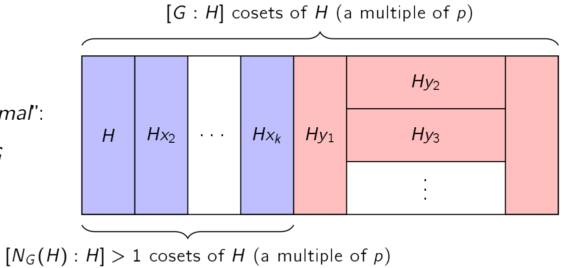

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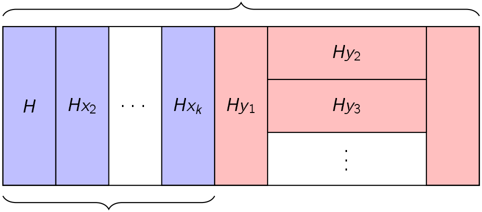


$[N_G(H) : H] > 1$ cosets of H (a multiple of p)

Important corollaries

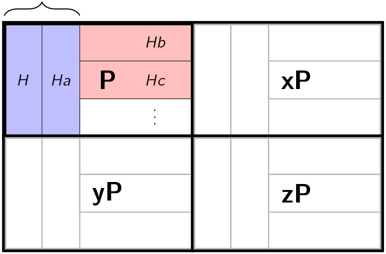
- p-groups cannot have any fully unnormal subgroups (i.e., $H \lneq N_G(H)$).
- In *any* finite group, the only fully unnormal p-subgroups are maximal.

Normalizers of p-subgroups

Let H be properly contained in a maximal p-subgroup $P \lneq G$.

- The normalizer of H *must* grow in P (and hence in G)
- The normalizer of P *need not* grow in G.

$\overbrace{H \lneq N_P(H)} \leq N_G(H)$



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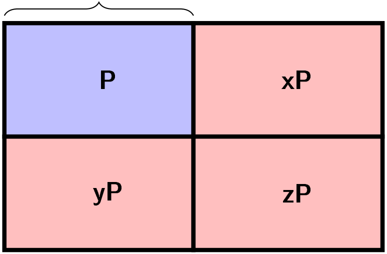

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Therefore, $[N_G(H) \colon H]$ is a multiple of p, so $N_G(H)$ must be strictly larger than H. $\square$

The Sylow theorems!

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Together, these place strong restrictions on the structure of a group G with a fixed order.

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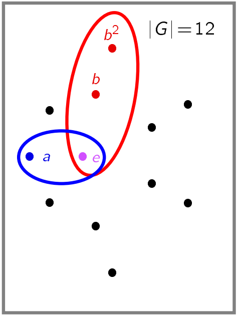

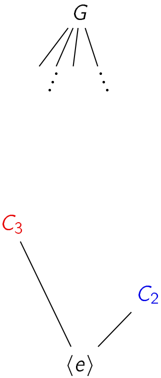

Using *only* the fact that $|G| = 12$, we will unconver as much about its structure as we can.

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First Sylow theorem

G has a subgroup of order p^k, for each p^k dividing $|G|$.

Also, every non-Sylow p-subgroup sits inside a larger p-subgroup.

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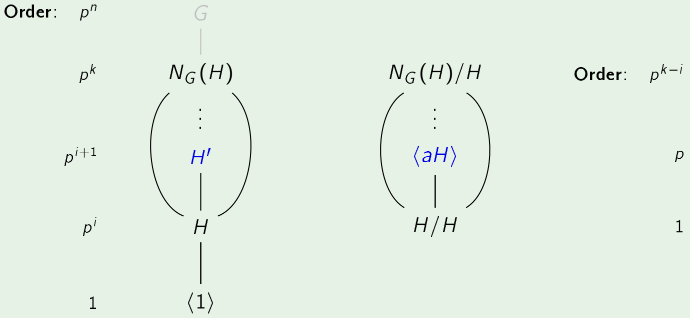
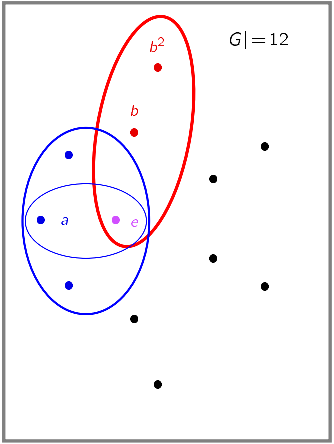
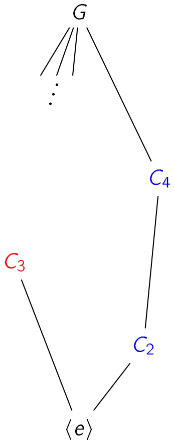
Find an element aH of order p. The union of cosets in $\langle aH \rangle$ is a subgroup of order p^{i+1}.

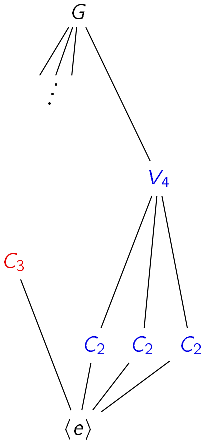


Our unknown group of order 12

By the first Sylow theorem, $\langle a \rangle$ is contained in a subgroup of order 4, which could be V_4 or C_4, or possibly both.







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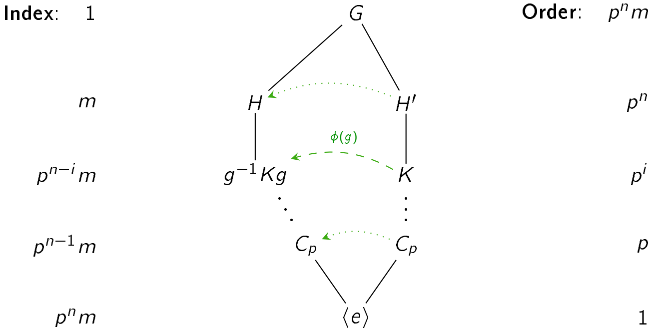

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Let H be a Sylow p-subgroup, and $K \leq G$ any p-subgroup. Then K is conjugate to some subgroup of H.

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$$|\mathrm{Fix}(\phi)| \equiv_p |S| = [G : H] = m \not\equiv_p 0. \qquad \square$$

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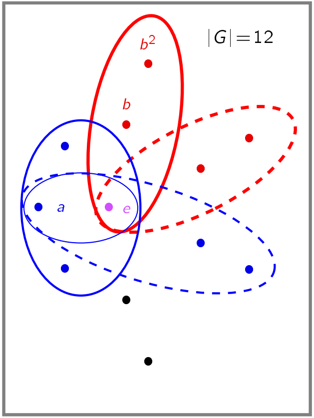
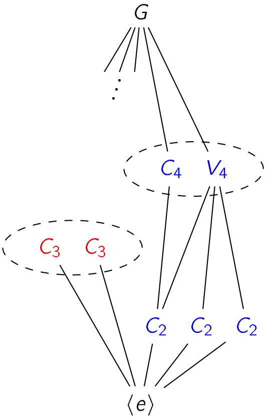
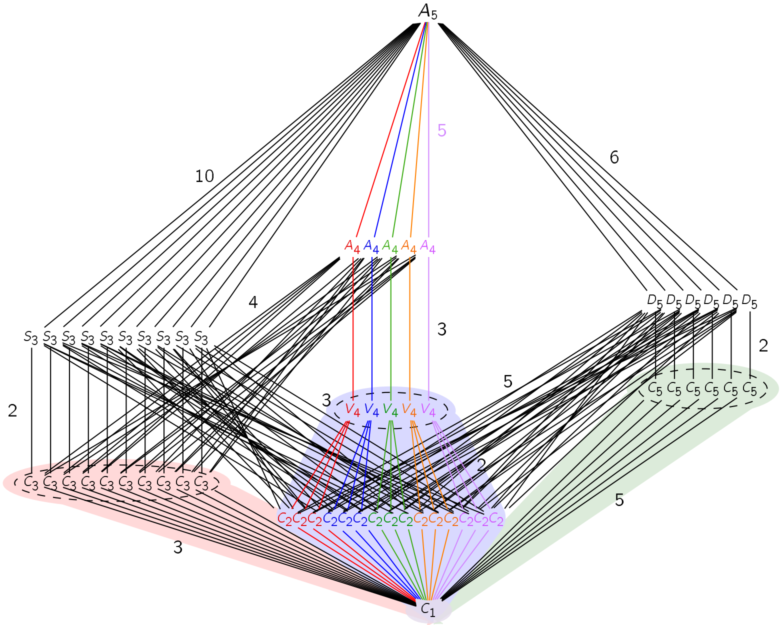
This eliminates the following subgroup lattice of a group of order 12.





Example: A_5 has no nontrival proper normal subgroups



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But xPx^{-1} is also a Sylow p-subgroup of $N_G(P)$, and by uniqueness, $xPx^{-1} = P$. $\square$

The 3rd Sylow theorem: number of p-subgroups

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Let n_p be the number of Sylow p-subgroups of G. Then

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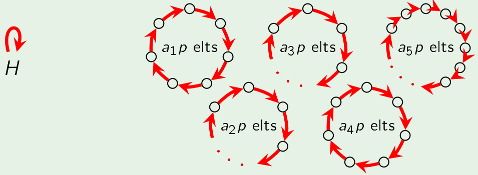

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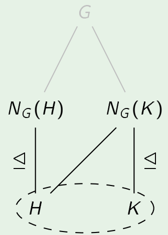

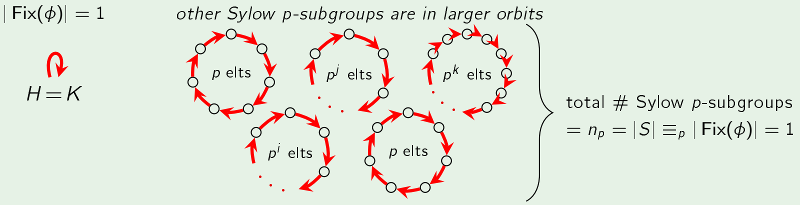

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To summarize, we used:

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S2 The action of $K \in \mathrm{Syl}_p(G)$ on $S = H \backslash G$ by right multiplication for some other $H \in \mathrm{Syl}_p(G)$.

S3a The action of G on $S = \mathrm{Syl}_p(G)$, by conjugation.

Summary of the proofs of the Sylow theorems

For the 1st Sylow theorem, we started with $H = \{e\}$, and inductively created larger subgroups of size $p, p^2, \dots, p^n$.

For the 2^{nd} and 3^{rd} Sylow theorems, we used a clever group action and then applied one or both of the following:

(i) *orbit-stabilizer theorem*. If G acts on S, then $|\,\mathrm{orb}(s)| \cdot |\,\mathrm{stab}(s)| = |G|$.

(ii) *p-group lemma*. If a p-group acts on S, then $|S| \equiv_p |\,\mathrm{Fix}(\phi)|$.

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Our mystery group order 12

By the 3rd Sylow theorem, every group G of order $12 = 2^2 \cdot 3$ must have:

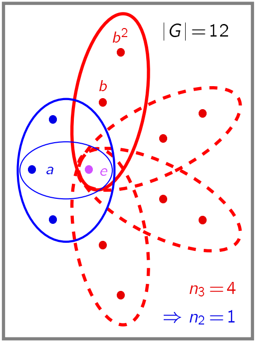
- n_3 Sylow 3-subgroups, each of order 3.

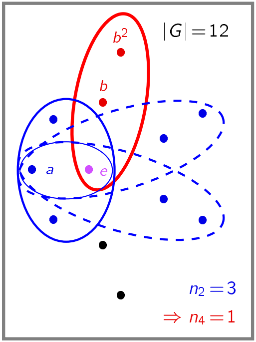
$$n_3 \mid 4, \qquad n_3 \equiv 1 \pmod 3 \qquad \Longrightarrow \qquad n_3 = 1 \text{ or } 4.$$

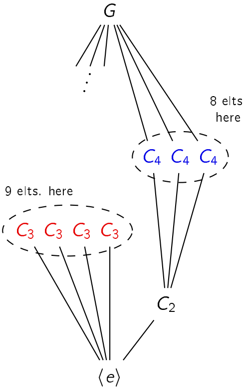
- n_2 Sylow 2-subgroups of order $2^2 = 4$.

$$n_2 \mid 3, \qquad n_2 \equiv 1 \pmod 2 \qquad \Longrightarrow \qquad n_2 = 1 \text{ or } 3.$$

But both are not possible! (There aren't enough elements.)





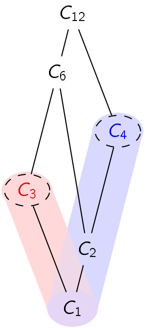


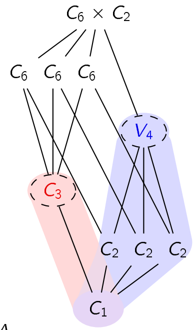
The five groups of order 12

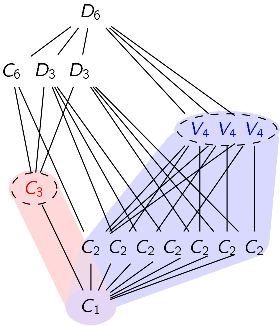
With a little work and the Sylow theorems, we can classify all groups of order 12.

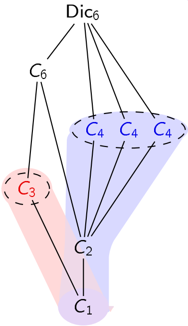
We've already seen them all. Here are their subgroup lattices.

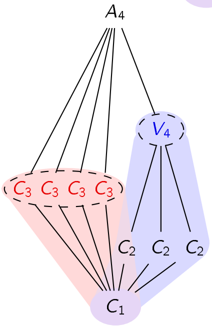
Note that *all* of these decompose as a direct or semidirect product of Sylow subgroups.











Simple groups and the Sylow theorems

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Tip

When trying to show that $n_p = 1$, it's usually helpful to analyze the largest primes first.

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If $\mathrm{Ker}(\phi) = \langle e \rangle$ then $\phi \colon G \hookrightarrow S_n$ is an embedding, which is impossible because $|G| \nmid n!$. $\square$