Hey, y'know, just some problems to think about during class

I think that thinking your way through some problems is the best way to get comfortable with the idea of quotient groups. Have fun!

Definition. Let $H \leq G$. The set G/H (we say "G mod H") is the set of (left) cosets of H in G:

$$G/H = \{H, aH, bH, \ldots\}$$

If $H \trianglelefteq G$, then G/H becomes a group if we define the binary operation as

$$aH * bH := (a * b)H$$

Elements of quotient groups are cosets

It may help to look at the Cayley diagrams and/or subgroup lattices of the groups below.

Problem 1. List out all the elements of the following quotient groups. (I promise that we're taking the quotient by a normal subgroup; no need to check.)

(a) $D_4/\langle r\rangle$

(b) $D_4/\langle r^2\rangle$

(c) $\mathbb{Z}/4\mathbb{Z}$

(d) $Q_8/\langle -1\rangle$

(e) $Q_8/\langle k\rangle$

(f) $\mathbb{Z}_6/\langle 3\rangle$

(g) $\mathbb{Z}_6/\langle 2\rangle$

(h) $(\mathbb{Z}_3 \times \mathbb{Z}_6)/\langle (1,1)\rangle$

(i) $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle (0,2)\rangle$

(j) $A_4/\langle (12)(34), (13)(24)\rangle$

Problem 2. While working on the previous problem, you may have conjectured this: Let G be a group and $H \trianglelefteq G$. Then $|G/H| = [G : H]$. In particular, if G is finite, then $|G/H| = |G|/|H|$.

Problem 3. Find the order of the given element in the quotient group. (Again, I promise we're taking the quotient by a normal subgroup.)

(a) $f\langle r\rangle \in D_4/\langle r\rangle$

(b) $r\langle r^2\rangle \in D_4/\langle r^2\rangle$

(c) $3 + 4\mathbb{Z} \in \mathbb{Z}/4\mathbb{Z}$

(d) $j\langle -1\rangle \in Q_8/\langle -1\rangle$

(e) $i\langle k\rangle \in Q_8/\langle k\rangle$

(f) $5 + \langle 3\rangle \in \mathbb{Z}_6/\langle 3\rangle$

(g) $5 + \langle 2\rangle \in \mathbb{Z}_6/\langle 2\rangle$

(h) $(2,1) + \langle (1,1)\rangle \in (\mathbb{Z}_3 \times \mathbb{Z}_6)/\langle (1,1)\rangle$

(i) $(1,3) + \langle (0,2)\rangle \in (\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle (0,2)\rangle$

(j) $(123)\langle (12)(34), (13)(24)\rangle$
$\in A_4/\langle (12)(34), (13)(24)\rangle$

Problem 4. $\mathbb{Q}$ is an abelian group under addition, so all its subgroups are normal. Describe the quotient group $\mathbb{Q}/\langle -1\rangle$. In particular, what do the elements look like?

Problem 5. If you take $\mathbb{Q}$ and throw away 0, then what's left is called $\mathbb{Q}^*$, and it is an abelian group under multiplication. Describe the quotient group $\mathbb{Q}^*/\langle -1\rangle$. In particular, what do the elements look like?

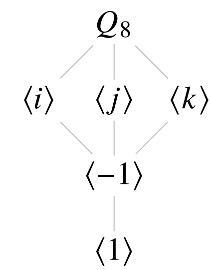
Quotient groups are visible in subgroup lattices

Problem 6. Consider $Q_8/\langle -1\rangle$, which I am tired of typing and will therefore just call "Q".

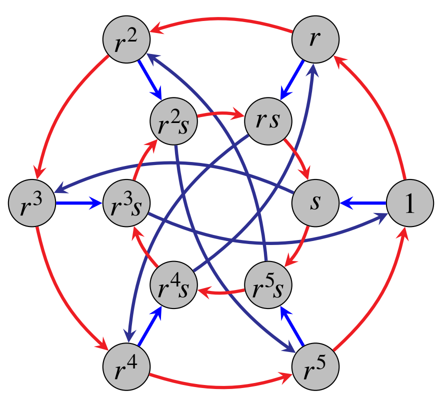
(a) In Problem 1, you found that Q has order 4. There are only two possible groups of order 4; is $Q \cong C_4$ or is $Q \cong V_4$? How do you know?

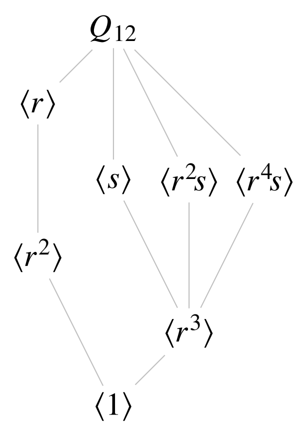
(b) Draw the subgroup lattice of Q.

(c) Where can you find the subgroup lattice of Q inside the subgroup lattice of Q_8?



Problem 7. Explore this phenomenon in a bigger and weirder group. LMFDB calls this group $C_3 : C_4$. It's also called Q_{12} because it is a "generalized quaternion group." (If you look at the subgroup lattice I think you can kinda see why.)





(a) Looking at the subgroup lattice, which subgroups of Q_{12} are normal? How do you know?

(b) List the elements of $Q_{12}/\langle r^3\rangle$, draw a Cayley graph, and draw the subgroup lattice. Where do you see the subgroup lattice of $Q_{12}/\langle r^3\rangle$ inside the subgroup lattice of Q_{12}?

(c) (Bonus) Do the same for $Q_{12}/\langle r^2\rangle$.

Interesting facts about quotient groups

Problem 8. Let G be a group and let $H \trianglelefteq G$. Show that if G is abelian, then so is G/H.

Problem 9. Let G be a group and let $H \trianglelefteq G$. Show that if G is cyclic, then so is G/H.

Problem 10. We previously proved that $Z(G) \trianglelefteq G$. Show that if $G/Z(G)$ is cyclic, then G is abelian.

Problem 11. Is the converse of Problem 8 true? Prove it, or provide a specific counterexample.

Problem 12. Is the converse of Problem 9 true? Prove it, or provide a specific counterexample.

Problem 13. Is the converse of Problem 10 true? Prove it, or provide a specific counterexample.