Physics applications of definite integrals

Here are a bunch of situations where we want to calculate something by multiplying some stuff together, but it doesn't work because one of those things isn't constant – it's some function of something else – so we have to slice-approximate-integrate.

For example, the leaky bucket problem: If a constant force F measured in Newtons is applied to move an object a distance d measured in meters, then the work done is $W = F \cdot d$, measured in Joules, $\mathrm{J} = \mathrm{N} \cdot \mathrm{m}$. (Or, in imperial units, force is measured in pounds, distance is measured in feet, and work is measured in foot-pounds.) However, if you're hauling a leaky bucket up out of a well, the force is nonconstant, depending in some way on the height: $F(h)$. Therefore we must *slice* into pieces of approximately constant height, on which the work is *approximately* $\Delta W = F(h) \cdot \Delta h$, and therefore the total work is given by the definite *integral*

$$W = \int_{h=a}^{h=b} F(h)\,dh.$$

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1. The kinetic energy of an object with mass m and constant speed v is $K = \frac{1}{2}mv^2$ (measured in Joules, $\mathrm{J} = \mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}^2$). Picture the second hand of a clock: it's a rod that is 0.1 m long with a total mass of 0.03 kg, and it is rotating around one of its ends at a rate of one revolution per minute. I'd like to know the total kinetic energy of this object.

   (a) Draw a picture.

   (b) Which quantity in the formula for kinetic energy isn't constant throughout your picture? What quantity does it depend on? Label that quantity in your picture.

   (c) Draw a representative small slice where that quantity is approximately constant. Label that slice with a Δ___.

   (d) Write an equation for ΔK for that slice.

   (e) Write a definite integral that gives the total kinetic energy K of the entire second hand.

   (f) Evaluate the integral, assign it the proper units, and explain the meaning of your result.

   (g) Describe the meaning of each factor of your integral, and give its units.

2. Picture an oil slick on the surface of the ocean. It's probably roughly a circle, and the oil is probably denser in the center than it is at the edges. I'd like to know the total mass of oil in this oil slick; let's say that the density of oil at a distance r meters from the center of the slick is given by
$$\rho(r) = \frac{50}{1+r}\,\mathrm{kg/m^2}.$$
   (a) Draw a picture.
   (b) Which quantity in the formula for mass isn't constant throughout your picture? What quantity does it depend on? Label that quantity in your picture.
   (c) Draw a representative small slice where that quantity is approximately constant. Label that slice with a Δ___.
   (d) Write an equation for ΔM for that slice.
   (e) Write a definite integral that gives the total mass M of the entire oil slick; let's say that the oil slick is 20 km across.
   (f) Evaluate the integral, assign it the proper units, and explain the meaning of your result. Hint: $\frac{r}{1+r} = \frac{r+1-1}{1+r} = 1 - \frac{1}{1+r}$.
   (g) Describe the meaning of each factor of your integral, and give its units.

3. The force of gravity that the earth exerts on an object diminishes as the object gets further away from the earth. Specifically, according to Newton, the force of gravity is governed by an "inverse square law:" for some constant k that depends on the masses of the earth and the object, the gravitational force at distance r from the center of the earth is given by
$$F(r) = \frac{k}{r^2}\,\mathrm{N}.$$
The energy required to move an object a distance d is $E = F \cdot d$, measured in Joules, $\mathrm{J} = \mathrm{N} \cdot \mathrm{m}$, if the force is constant over the distance d. I'd like to know the total energy necessary to launch a CubeSat with mass 2 kg all the way to the moon.
   (a) Draw a picture.
   (b) Which quantity in the formula for energy isn't constant throughout your picture? What quantity does it depend on? Label that quantity in your picture.
   (c) Draw a representative small slice where that quantity is approximately constant. Label that slice with a Δ___.
   (d) Write an equation for ΔE for that slice.
   (e) Write a definite integral that gives the total energy E needed for the entire launch. The distance from the surface of the earth to the surface of the moon is about 362,000km.
   (f) Let's figure out the right value for k. Our CubeSat has mass 2 kg, and at the surface of the earth ($r = 6371\,\mathrm{km}$), the force of gravity on the CubeSat is $2\,\mathrm{kg} \cdot 9.8\,\mathrm{m/s^2} = 19.6\,\mathrm{N}$.
   (g) Evaluate the integral, assign it the proper units, and explain the meaning of your result.
   (h) Describe the meaning of each factor of your integral, and give its units.

4. The energy required to move an object a distance x while exerting a constant force F is $E = F \cdot x$. Suppose that you have two magnets and a wire. One magnet is attached to the end of the wire and the other can slide along the wire. If the magnets are arranged so that they repel each other, then it will require force to push the movable magnet toward the fixed magnet. The amount of force needed to move the magnet increases as the two get closer together. In fact, the force at a distance x is proportional to $1/x^2$; that is, for some constant k, the force at a distance x is $k \cdot 1/x^2$. I'd like to know the total energy needed to push the movable magnet a certain distance toward the fixed magnet.

   (a) Draw a picture.

   (b) Which quantity in the formula for energy isn't constant throughout your picture? What quantity does it depend on? Label that quantity in your picture.

   (c) Draw a representative small slice where that quantity is approximately constant. Label that slice with a Δ___.

   (d) Write an equation for ΔE for that slice.

   (e) Write a definite integral that gives the total energy E needed to push the magnet from 5 cm away to 3 cm away.

   (f) Evaluate the integral, assign it the proper units, and explain the meaning of your result. Your answer will involve the constant k, which we don't have enough information to figure out. (What information would we need?)

   (g) Describe the meaning of each factor of your integral, and give its units.

5. Consider the atmosphere: it's quite thick at ground level, but thins out as you get higher and higher, until you get to space and it's basically nothing. I'd like to know the total mass of the atmosphere; it turns out that a good model for the density of the atmosphere as a function of height h (measured in meters up from sea level) is

$$\rho(h) = 1.28e^{-0.000124h}\,\mathrm{kg/m^3}.$$

   (a) Draw a picture.

   (b) Which quantity in the formula for mass isn't constant throughout your picture? What quantity does it depend on? Label that quantity in your picture.

   (c) Draw a representative small slice where that quantity is approximately constant. Label that slice with a Δ___.

   (d) Write an equation for ΔM for that slice. **Be careful:** your equation will probably depend on r, which is not the same as h.

   (e) Write a definite integral that gives the total mass M of the atmosphere all the way out to the "Kármán line" of 83.8 km above sea level, which is the highest height theoretically achievable by an aircraft. (Note: the earth's radius is about 6300 km.)

   (f) Evaluate the integral (use Wolfram Alpha or something), assign it the proper units, and explain the meaning of your result.

   (g) Describe the meaning of each factor of your integral, and give its units.

6. The gravitational attraction between two **particles** of mass m_1 and m_2 at a distance r apart is

$$F(r) = \frac{Gm_1 m_2}{r^2}.$$

Picture a thin uniform rod with mass M and length ℓ, and then a particle of mass m that's a meters away from the end of the rod. (I am imagining a pool cue hitting a pool ball.) I'd like to know the total gravitational force between the rod and the particle; **the problem is that the rod is not a particle.**

(a) Draw a picture.

(b) Which quantity in the formula for force isn't constant throughout your picture? What quantity does it depend on? Label that quantity in your picture.

(c) Draw a representative small slice where that quantity is approximately constant. Label that slice with a Δ___.

(d) Write an equation for ΔF for that slice.

(e) Write a definite integral that gives the total force F between the rod and the particle.

(f) Evaluate the integral (your answer will involve lots of letters), assign it the proper units, and explain the meaning of your result.