$$\begin{bmatrix} \sim & \sim & \sim \\ \sim & \sim & \sim \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \sim \\ \sim \\ 1 \end{bmatrix}$$

Now try

$$\begin{bmatrix} \sim & \sim & \sim \\ \sim & \sim & \sim \\ \sim & \sim & \sim \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \sim \\ \sim \\ \sim \end{bmatrix}$$

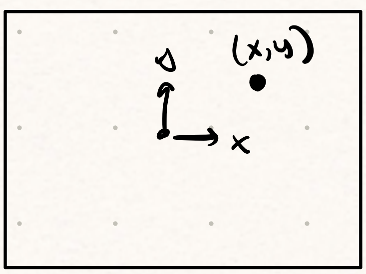
not a 1 anymore?

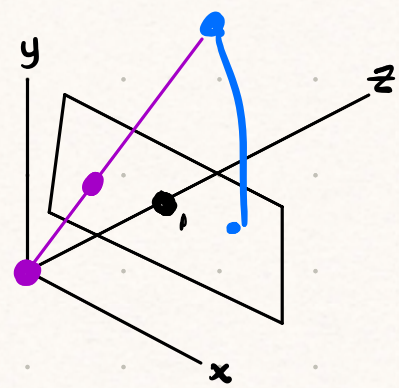
what does this mean?

2D

$\mathbb{R}^2 \qquad \mathbb{R}^3$

$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$





$\begin{bmatrix} 3\times 3 \end{bmatrix}$ maps $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ to $\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix}$

any 3D point $\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$ write as $\begin{bmatrix} wx' \\ wy' \\ w \end{bmatrix}$ (as long as $w \neq 0$)

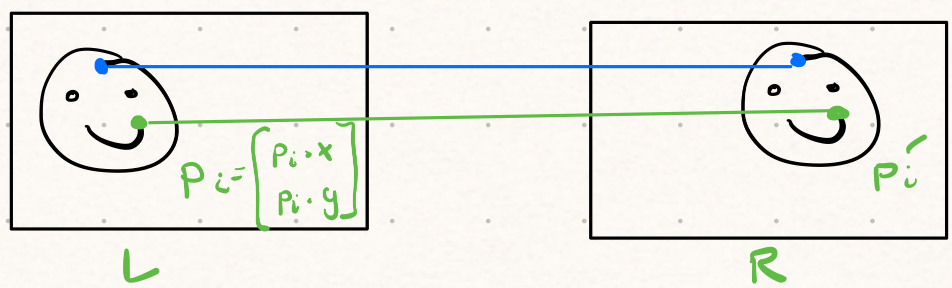
the "ray" of $\begin{bmatrix} x \\ y \end{bmatrix}$ is all points $\begin{bmatrix} wx \\ wy \\ w \end{bmatrix}$ with $w>0$

Given $\begin{bmatrix} x \\ y \end{bmatrix}$

1. move into 3D $\rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$
2. apply linear transformation: $\begin{bmatrix} T \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} X \\ Y \\ w \end{bmatrix}$
3. divide by w : $\begin{bmatrix} X \\ Y \\ w \end{bmatrix} \rightarrow \begin{bmatrix} X/w \\ Y/w \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

Let's fit the best transformations.

① Easiest case: fit a translation model



case: only one match $\quad T = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$

where $t_x = p_i'.x - p_i.x$
$t_y = p_i'.y - p_i.y$

more matches: $\rightarrow$ average!

$$t_x = \frac{1}{n}\sum_i p_i'.x - p_i x$$

$$t_y = \frac{1}{n}\sum_i p_i'.y - p_i.y$$

$$t = \frac{1}{n}\sum_i \left(p_i' - p_i\right)$$

Harder: Affine

$$T = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$$

need at least 3 matches

Digression

Linear Algebra: field of math concerned with $\underbrace{A \cdot x}_{\text{matrix vector product}}$

3 problems

- $Ax = b$
  - ↳ usually no solution
  - ↳ "best solution"

$$\min_x \| Ax - b \|$$

- $Ax = 0$

  $x = 0$ always a solution

  other solutions?

- $Ax = \lambda x$

Tool: $\|Ax - b\|$ ↳ minimize

which A? which b?

which x?

$$x = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix}$$

matched pair of points

$$p_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix} \qquad p_i' = \begin{bmatrix} x_i' \\ y_i' \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \begin{bmatrix} x_i' \\ y_i' \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a x_i + b y_i + c \\ d x_i + e y_i + f \\ \cancel{1} \end{bmatrix} \approx \begin{bmatrix} x_i' \\ y_i' \\ \cancel{1} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} x_i & y_i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_i & y_i & 1 \end{bmatrix}}_{\text{known}} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix}}_{\substack{\text{unknown} \\ \text{want to estimate}}} \approx \underbrace{\begin{bmatrix} x_i' \\ y_i' \end{bmatrix}}_{\text{known}}$$

$$\min \left\| \underbrace{\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 \\ \hdashline x_2 & y_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & y_2 & 1 \\ \hdashline & & \vdots & & & \\ \hdashline x_n & y_n & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_n & y_n & 1 \end{bmatrix}}_{2n \times 6} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix}}_{6 \times 1} - \underbrace{\begin{bmatrix} x_1' \\ y_1' \\ \hdashline x_2' \\ y_2' \\ \hdashline \vdots \\ \hdashline x_n' \\ y_n' \end{bmatrix}}_{\substack{2n \times 1 \\ b}} \right\|^2$$

Procedure:

make matrix A

make vector b

solve $\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \text{linsolve}(A, b)$

(this works ✓)

Hardest case: $\underbrace{\text{general } 3\times 3 \text{ transformation}}$ ↳ homography

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} \approx \begin{bmatrix} w x_i' \\ w y_i' \\ w \end{bmatrix}$$

$$\begin{bmatrix} h_{11}x + h_{12}y + h_{13} \\ h_{21}x + h_{22}y + h_{23} \\ x h_{31} + y h_{32} + h_{33} \end{bmatrix} = \begin{bmatrix} w x' \\ w y' \\ w \end{bmatrix}$$

$$x h_{31} + y h_{32} + h_{33} = w$$

$$\begin{bmatrix} h_{11}x + h_{12}y + h_{13} \\ h_{21}x + h_{22}y + h_{23} \end{bmatrix} = \begin{bmatrix} x'(x h_{31} + y h_{32} + h_{33}) \\ y'(x h_{31} + y h_{32} + h_{33}) \end{bmatrix}$$

$$\begin{bmatrix} h_{11}x + h_{12}y + h_{13} - h_{31}xx' - h_{32}yx' - h_{33}x' \\ h_{21}x + h_{22}y + h_{23} - h_{31}xy' - h_{32}yy' - h_{33}y' \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x & y & 1 & 0 & 0 & 0 & -xx' & -yx' & -x' \\ 0 & 0 & 0 & x & y & 1 & -yx' & -yy' & -y' \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

mistake made in class I'm so sorry.

$-xy'$

now get lots of matches

2 rows per match.

$$\underbrace{A}_{2n\times 9} \underbrace{\vec{h}}_{9\times 1} = \underbrace{0}_{2n\times 1}$$

Solve! $\rightarrow$ $\vec{h} = 0$ ...

not the solution we want

solve: $\min \|Ah\|$ with $\|h\| = 1$

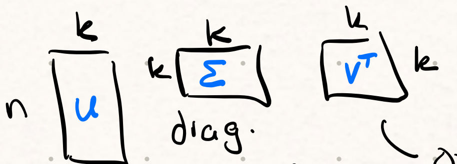
known solution involving eigenvalues

$$\min \|Ah\|^2$$
st $\|h\|=1$

$$\|Ah\|^2 = (Ah)^T(Ah) = h^T \underset{\sim}{A^TA}\, h$$

Singular Vale Decompostion

$$A = U\Sigma V^T$$



orthogonal

orthogonal

diag: $\Sigma = \Sigma^T$

orthog: $UU^T = I$

$$h^TA^TAh = h^T(V\Sigma \cancel{U}^T)(\cancel{U}\Sigma V^T)h$$

$$= h^T V \underbrace{\Sigma^2}_{\text{diag.}} V^T h$$

If $A = U\Sigma V^T$
then $A^T = (U\Sigma V^T)^T$
$= V^{TT}\Sigma^T U^T$
$= V\Sigma U^T$

$$\Sigma V^T h : \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_q \end{bmatrix} \begin{bmatrix} - & v_1 & - \\ - & v_2 & - \\ & \vdots & \\ - & v_q & - \end{bmatrix} \begin{bmatrix} \\ h \\ \\ \end{bmatrix}$$

sort these big to small

$$= \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_q \end{bmatrix} \begin{bmatrix} v_1 \cdot h \\ \vdots \\ v_q \cdot h \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1 & v_1 & h \\ & \vdots & \\ \sigma_q & v_q & h \end{bmatrix}$$

h is a unit vector

pos #s in decreasing order

all unit vecs

want to be small

takeaway:

$\Sigma V^T h$ minimized when $h = v_q$

so $h^T V \Sigma^2 V^T h = h^T A^T A h = \| A h \|$

minimized when $h = v_q$

Procedure:

Form matrix A.

Decompose with svd:

$$A = \underbrace{\begin{bmatrix} \text{———} \\ \text{———} \\ \text{———} \\ \text{———} \\ \text{———} \end{bmatrix}}_{U} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_q \end{bmatrix} \underbrace{\begin{bmatrix} \text{———} \\ \text{———} \\ \text{———} \\ \text{———} \\ \text{———} \\ \text{———} \end{bmatrix}}_{V^T}$$

take v_9 : The last row of V^T

That's our h!

orthogonal:

$UU^T = I$

and $U^T U = I$