Haag-Kastler Nets in Isabelle/HOL

Formalising an Algebraic Approach to Quantum Field Theory

Richard Schmoetten

The University of Edinburgh

31st August 2022

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Testing the SM

- Fine structure constant $\alpha^{-1} = 137.035999084(21)$ CODATA
- Anomalous magnetic dipole of e^-
- Recoil frequency of Cesium-133

  compare
- In agreement...

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- $\left.\begin{array}{l}\text{Anomalous magnetic dipole of } e^- \\ \text{Recoil frequency of Cesium-133}\end{array}\right\}$ compare
- In agreement...

1. related to the scale of EM interactions
   std uncertainty: 1.5×10^{-10}
   measuring polar diameter to within 0.1 hair's width!
2. Fine structure constant is calculated from $g - 2$ measurement and calculation of > 10000 diagrams! No more dark photons (2.5σ). Limited by $g_e - 2$ accuracy.
3. $< 1\sigma$ tension with prior recoil measurement, 2.5σ tension with gyromag measurement. Fermilab (4/2021) brings this to 4.2σ, on $g_\mu - 2$.

Quantum Field Theory

- Framework for the different parts of the standard model.
- Based on the idea of path integrals.
- Many theoretical breakthroughs: renormalisation, QCD asymptotic freedom, Higgs boson, ...

Difficulties

- What exactly are the common features of the theories?
- What exactly is the state space we're describing?
- Divergent integrals, asymptotic ($\neq$ converging) series, ...

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1. Diagrams:
   beta decay (mediated by weak force)
   Higgs production
2. van Howe -> different ways of quantising give different theories for the same model
3. what are states? different pictures and unitary inequivalence, Haag's theorem

Lie groups in Isabelle

```
locale lie_group =
  m_sqr: smooth_manifold_sqr charts +
  grp: group_on_with m_sqr.m1.carrier tms tms_one dvsn invs
  for charts::"('a::{t2_space, second_countable_topology},
              'e::euclidean_space) chart set"
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  assumes smooth_mult:
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1. Some maths incoming - not so important for this talk.

General Linear Group

- $GL(n, \mathbb{R})$ is the group of all invertible $n \times n$ matrices.
- Elements of $GL(n, \mathbb{R})$ represent linear operators on the vector space $\mathbb{R}^n$.
- The homogeneous part of ¶ is a subgroup of $GL(4, \mathbb{R})$.
- $GL(n, \mathbb{R})$ is a Lie group! It is parametrised by real matrix elements.

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abbreviation "GL ≡
  {M::('a::comm_ring_1,'n::finite)square_matrix. invertible M}"
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Transfer

The *transfer* package transfers theorems between related types. Say we want to transfer a result about a type α to β.

$$\begin{tikzcd} \alpha \arrow[r, "\cong"] \arrow[d, "f"'] & \beta \arrow[d, "g"] \\ \alpha \arrow[r, "\cong"'] & \beta \end{tikzcd}$$

- α and β *related* by $\cong : \alpha \to \beta \to \texttt{bool}$
- prove g maps related f-inputs to related f-outputs
- prove similar *transfer rules* for all constants in the theorem to be transferred
- relations can be constructed from the type of a constant

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1. And then it works! It works both ways too.
2. This is the great contribution of transfer: generic construction for rules, and automation.
3.

WLOG for interval endpoints

Proofs mirror the structure of the preceding lemma.

1. State the desired result
2. Split up the proof into essentially distinct cases with fixed events

```
let ?prop = "λ I J. is_int (I∩J) ∨ (I∩J) = {}"
{ fix I J a b c d
  assume "I = interval a b" "J = interval c d"
  { assume "betw4 a b c d"
    have "I∩J = {}" ...
  } { assume "betw4 a c b d"
    have "I∩J = interval c b" ...
  } { assume "betw4 a c d b"
    have "I∩J = interval c d" ...
  } }
then show "is_int (I1∩I2)"
  using wlog_interval_endpoints_distinct symmetry assms
  by simp
```