Haag-Kastler Nets in Isabelle/HOL

Formalising an Algebraic Approach to Quantum Field Theory

Richard Schmoetten

The University of Edinburgh

31st August 2022

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Testing the SM

- Fine structure constant $\alpha^{-1} = 137.035999084(21)$ CODATA
- Anomalous magnetic dipole of e^-
- Recoil frequency of Cesium-133

  compare
- In agreement...

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- In agreement...

1. related to the scale of EM interactions
   std uncertainty: 1.5×10^{-10}
   measuring polar diameter to within 0.1 hair's width!
2. Fine structure constant is calculated from $g - 2$ measurement and calculation of > 10000 diagrams! No more dark photons (2.5σ). Limited by $g_e - 2$ accuracy.
3. $< 1\sigma$ tension with prior recoil measurement, 2.5σ tension with gyromag measurement. Fermilab (4/2021) brings this to 4.2σ, on $g_\mu - 2$.

Quantum Field Theory

- Framework for the different parts of the standard model.
- Based on the idea of path integrals.
- Many theoretical breakthroughs: renormalisation, QCD asymptotic freedom, Higgs boson, ...

Difficulties

- What exactly are the common features of the theories?
- What exactly is the state space we're describing?
- Divergent integrals, asymptotic ($\neq$ converging) series, ...

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1. Diagrams:
   beta decay (mediated by weak force)
   Higgs production
2. van Howe -> different ways of quantising give different theories for the same model
3. what are states? different pictures and unitary inequivalence, Haag's theorem

Lie groups in Isabelle

```
locale lie_group =
  m_sqr: smooth_manifold_sqr charts +
  grp: group_on_with m_sqr.m1.carrier tms tms_one dvsn invs
  for charts::"('a::{t2_space, second_countable_topology},
               'e::euclidean_space) chart set"
    and tms tms_one dvsn invs +
  assumes smooth_mult:
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1. Some maths incoming - not so important for this talk.

General Linear Group

- $\mathrm{GL}(n, \mathbb{R})$ is the group of all invertible $n \times n$ matrices.
- Elements of $\mathrm{GL}(n, \mathbb{R})$ represent linear operators on the vector space $\mathbb{R}^n$.
- The homogeneous part of ¶ is a subgroup of $\mathrm{GL}(4, \mathbb{R})$.
- $\mathrm{GL}(n, \mathbb{R})$ is a Lie group! It is parametrised by real matrix elements.

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abbreviation "GL ≡
  {M::('a::comm_ring_1,'n::finite)square_matrix. invertible M}"
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abbreviation GL_R4::"(real,4)square_matrix set" where "GL_R4 ≡ GL"
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1.

Transfer

The *transfer* package transfers theorems between related types. Say we want to transfer a result about a type α to β.

$$\begin{tikzcd} \alpha \arrow[r, "\cong"] \arrow[d, "f"'] & \beta \arrow[d, "g"] \\ \alpha \arrow[r, "\cong"'] & \beta \end{tikzcd}$$

- α and β *related* by $\cong : \alpha \to \beta \to \texttt{bool}$
- prove *g* maps related *f*-inputs to related *f*-outputs
- prove similar *transfer rules* for all constants in the theorem to be transferred
- relations can be constructed from the type of a constant

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1. And then it works! It works both ways too.
2. This is the great contribution of transfer: generic construction for rules, and automation.
3.