DIFFERENTIAL CALCULUS & CO-ORDINATE GEOMETRY

Concepts of Exam

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Differential Calculus

01 Function

Definition: Function, Domain, Range, Inverse function, Odd function, Even Function etc.

02 Limit

Definition of Limit, $(\delta - \epsilon)$ definition,

By $(\delta - \epsilon)$ definition prove

i. $\lim_{x\to 4}(2x-2) = 6$

ii. $\lim_{x\to 3}(2x^3 - 3x^2 - 18x + 29) = 2$

iii. $\lim_{x\to -2}\frac{x^2-4}{x+2} = -4$

etc.

Definition of Left-Hand Limit and Right-Hand Limit.

Prove the followings:

i. $\lim_{x\to 0}\frac{x^2 \sin\frac{1}{x}}{\sin x} = 0$

ii. $\lim_{x\to \pi}\frac{\sin x}{\pi - x} = 1$

iii. $\lim_{x\to \frac{\pi}{4}}(\sec 2x - \tan 2x) = 0$

Find the following limits:

i. $\lim_{x\to 0}\frac{1-\cos x}{x}$

ii. $\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{\tan x}\right)$

iii. $\lim_{x\to 0}\frac{x(\cos x + \cos 2x)}{\sin x}$

iv. $\lim_{x\to \frac{\pi}{4}}\frac{2\cos x - \sin 2x}{1+\cos 2x}$

etc.

L'Hospital's rule: Prove the followings

i. $\lim_{x\to 0}\frac{e^x - e^{-x} - 2x}{x - \sin x} = 2$

ii. $\lim_{x\to 0}\frac{a^x - 1 - x\ln a}{x^2} = \frac{1}{2}(\ln a)^2$

iii. $\lim_{x\to 0}\frac{e^x - e^{-x} - 2\ln(1+x)}{x \sin x} = 0$

iv. $\lim_{x\to 0} \frac{3\tan x - 3x - x^3}{x^5} = \frac{2}{5}$

v. $\lim_{x\to 0} (\sin x)^x = 1$

vi. $\lim_{x\to 0} (\cos x)^{\cot x} = \frac{1}{\sqrt{e}}$

03 Continuity

Definition of continuous function.

Questions:

i. Discuss the discontinuity of $\tan x$ at $x = \frac{\pi}{2}$.

ii. The function f is defined as follows:

$$f(x) = \begin{cases} -x & when \ \ x \le 0 \\ x & when \ \ 0 < x < 1 \\ 2 - x & when \ \ x \ge 1 \end{cases}$$

show that it is continuous at $x = 0$ and $x = 1$.

iii. Discuss the continuity of the following functions at the points indicated:

a. $f(x) = \begin{cases} x & when \ \ 0 < x < 1 \\ 2 - x & when \ \ 1 \le x \le 2 \\ x - \frac{1}{2}x^2 & when \ \ 0 < x < 1 \end{cases}$ at $x = 2$

b. $f(x) = \begin{cases} \frac{\tan^2 x}{3x} & when \ \ x \ne 0 \\ \frac{2}{3} & when \ \ x = 0 \end{cases}$ at $x = 0$

c. $f(x) = \begin{cases} x^2 + x & when \ \ 0 \le x < 1 \\ 2 & when \ \ x = 1 \\ 2x^3 - x + 1 & when \ \ 1 < x < 2 \end{cases}$ at $x = 1$

iv. The function f is defined as follows:

$$f(x) = \begin{cases} -2\sin x & when \ \ -\pi \le x \le -\frac{\pi}{2} \\ a\sin x + b & when \ -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \cos x & when \ \ \frac{\pi}{2} \le x \le \pi \end{cases}$$

if $f(x)$ is continuous in the interval $-\pi \le x \le \pi$, find the values of a and b .

04 Differentiability

Define derivative of a function. Define differentiable function, define Left-Hand Derivative and Right-Hand Derivative.

05 Differentiation

Find, from the definition of differentiation (or first principle), derivatives (or differential coefficients) of the following functions:

i. e^x

ii. $\sin x$

iii. $\cos x$

iv. $\sqrt{x}$

etc.

Find the derivatives of the following functions with respect to $\boldsymbol{x}$:

i. e^{x^x}

ii. $e^{\tan x}$

iii. $e^{\sin^{-1} x}$

iv. $\log_{\sin x} \sec x$

v. $\cos 2x \cos 3x$

vi. $\sin 2x \cos x$

vii. $\tan^{-1} \sqrt{x}$

viii. $\tan \sin^{-1} x$

ix. $x^3 \sqrt{\frac{x^2+4}{x^2+3}}$

x. $\sqrt{\frac{1+x}{1-x}}$

xi. e^{e^x}

xii. $\tan \ln \sin e^{x2}$

xiii. x^{x^x}

xiv. $\cos^{-1} \frac{1-x^2}{1+x^2}$

xv. $\sin^{-1} \frac{2x}{1+x^2}$

xvi. $\tan^{-1} \frac{2x}{1+x^2}$

xvii. $\tan^{-1} \frac{x}{\sqrt{1-x^2}}$

Find $\frac{dy}{dx}$ from the followings:

i. $x = y \ln(xy)$

ii. $y = x^y$

iii. $(\cos x)^y = (\sin y)^x$

iv. $\ln(xy) = x^2 + y^2$

v. $x^y = y^x$

vi. $x^y y^x = 1$

vii. $e^{xy} - 4xy = 2$

If $y = x + \cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+to\ \infty}}}}$, show that $\frac{dy}{dx} = \cfrac{1}{2-\cfrac{x}{x+\cfrac{1}{x+\cfrac{1}{x+to\ \infty}}}}$.

06 Successive differentiation

State and prove Leibnitz's theorem.

Find the followings:

i.	$\frac{d^n}{dx^n}\sin x$	iv.	$\frac{d^n}{dx^n}x^n$
ii.	$\frac{d^n}{dx^n}\cos x$	v.	$\frac{d^n}{dx^n}e^{mx}$
iii.	$\frac{d^n}{dx^n}\ln x$	vi.	$\frac{d^n}{dx^n}\log_a x$

Answer the followings:

i. If $y = \cos(m \sin^{-1} x)$, show that

   a. $(1 - x^2)y_2 - x\, y_1 + m^2 y = 0$

   b. $(1 - x^2)y_{n+2} - (2n + 1)x\, y_{n+1} + (m^2 - n^2)y_n = 0$

ii. If $y = e^{\cos^{-1} x}$, show that

$$(1 - x^2)y_2 - x\, y_1 - y = 0$$

iii. If $y = (\sin^{-1} x)^2$, show that

   a. $(1 - x^2)y_2 - x\, y_1 - 2 = 0$

   b. $(1 - x^2)y_{n+2} - (2n + 1)x\, y_{n+1} - n^2 y_n = 0$

etc.

07 Expansion of function

Find the expansion of the following functions:

i. $\sin x$

ii. $\cos x$

iii. e^x

iv. $\ln(1 + x)$

v.

08 Maxima and Minima

i. Find for what values of x, the following expression is maximum and minimum respectively:

$2x^3 - 21x^2 + 36x - 20$.

ii. Examine $f(x) = x^3 - 6x^2 + 24x - 12$ for maximum or minimum values.

iii. Examine whether $x^{1/x}$ possesses a maximum or a minimum and determine the same.

iv. Find the maximum and the minimum values of u where

$u = \frac{4}{x} + \frac{36}{y}$ and $x + y = 2$.

v. Show that of all rectangles of given area, the square has the smallest perimeter.

vi. Show that of all rectangles of given perimeter, the square has the largest area.

vii.

09 Partial differentiation

i. State and prove Euler's theorem on homogeneous function.

ii. If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

iii. Verify Euler's theorem for the function $u(x, y) = ax^2 + 2hxy + by^2$.

iv.

10 Tangent and Normal

i. Define tangent and normal.

ii. Define angle of intersection of two curves.

iii. Find the equation of tangent at the point (x, y) on the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

iv. Find the equation of tangent at the point θ on the curve $x = a\cos\theta\,, y = b\sin\theta$.

v. Find the equation of tangent at the point $(1, -1)$ on the curve

$$x^3 + xy^2 - 3x^2 + 4x + 5y + 2 = 0.$$

vi. Show that the normal at the point $\theta = \pi/4$ on the curve

$$x = 3\cos\theta - \cos^3\theta,$$
$$y = 3\sin\theta - \sin^3\theta$$

passes through the origin.

vii.

Co-ordinate Geometry

00. Co-ordinates

i. Find the poler co-ordinates of the point whose Cartesian co-ordinates are $(-3,2)$.
ii. Find the Cartesian co-ordinates of the point whose poler co-ordinates are $\left(4, \frac{5\pi}{4}\right)$.
iii. Find the distance between the points (x_1, y_1) and (x_2, y_2).
iv.

01. Transformation of Co-ordinates

i. Find the relation between the new co-ordinates and old co-ordinates when the origin is transferred to (α, β) without changing the direction of axes.
ii. Find the relation between the new co-ordinates and old co-ordinates when the origin is unchanged, and the axes are rotated through an angle θ.
iii. Transform the equation $2x^2 - 2xy + 9y^2 - x + y + 17 = 0$ to axes through the point $(-1,2)$ inclined at an angle $\frac{\pi}{4}$ to the original axes.
iv. Transform the equation $11x^2 + 3xy + 7y^2 + 19 = 0$ so as to remove the term xy.
v. Transform the equation $9x^2 + 24xy + 2y^2 - 6x + 20y + 41 = 0$ so as to remove the terms in x, y and xy.
vi.

02. The straight line

i. Derive the equation of straight line when the slope is known.
ii. Derive the equation of straight line passing through a fixed point (x_1, y_1) and having inclination α.
iii. Derive the equation of straight line passing through two points.
iv. Derive the equation of straight line in the interception form.
v. Derive the poler equation of a straight line.
vi. Find the distance of a point from a straight line.
vii. Find the angle between two straight lines.
viii. Find the co-ordinates of the point of intersection of two straight lines.
ix. Find the equation of the straight line that passes through the point $(-1,2)$ and parallel to the line $3x + 5y + 8 = 0$.
x. Find the equation of the line which passes through the point of intersection of the lines $7x - 6y + 6 = 0, 2x + 9y - 5 = 0$ and perpendicular to $x - 3y + 19 = 0$.
xi.

03. Pair of straight line

i. Prove that a homogeneous equation of the second degree always represent a pair of straight lines through the origin.

ii. Find the condition that the general equation of the second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ may represent a pair of straight lines.

iii. Prove that the equation $2x^2 + xy - y^2 - x - 7y - 10 = 0$ represents a pair of straight lines.

iv. Find for what value of λ the equation $12x^2 + 36xy + \lambda y^2 + 6x + 6y + 3 = 0$ represents a pair of straight lines.

v.

04. The circle

i. Define circle.

ii. Derive the equation of a circle whose centre is at (α, β) and radius r.

iii. Find the co-ordinates of the centre and the radius of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

iv. Derive the poler equation of circle.

v. Find the condition that the general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ may represent a circle.

vi. Find the equation of the circle passing through the points $(-3,2), (1,7)$ and $(5,-3)$.

vii. Find the equation of tangent at the point (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

viii. Find the condition that the line $y = mx + c$ should be a tangent to the circle $x^2 + y^2 = a^2$.

ix. Find the condition that the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 + 2g'x + 2f'y + c' = 0$ should cut orthogonally.

x. Find the condition that the two circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ may touch.

xi.

05. Conics in general

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$
$$\Delta = \begin{vmatrix} a & g & h \\ h & b & f \\ g & f & c \end{vmatrix}$$
$$C = ab - h^2$$
$$I = a + b$$

Case	Conditions on the invariants	Types of locus
Proper conic $\Delta \neq 0$	$C > 0; a = b, h = 0;\ I, \Delta$ opposite in sign.	Circle
	$C > 0;\ I, \Delta$ opposite in sign.	Ellipse
	$C < 0.$	Hyperbola
	$C = 0.$	Parabola
	$C > 0;\ I, \Delta$ same sign.	No real locus.
Degenerate conic $\Delta = 0$	A pair of straight lines.	

i. Reduce the equation $x^2 - 4xy + y^2 + 8x + 2y - 5 = 0$ to its standard form.
ii. Reduce the equation $x^2 - 6xy + 9y^2 - 2x - 3y + 1 = 0$ to its standard form.
iii.

06. The parabola

Elements of parabola	$y^2 = 4ax$	$x^2 = 4ay$
Vertex	$(0,0)$	$(0,0)$
Focus	$(a, 0)$	$(0, a)$
Equation of directrix	$x = -a$	$y = -a$
Equation of axis	$y = 0$	$x = 0$
Equation of latus rectum	$x = a$	$y = a$
Length of latus rectum	$\|4a\|$	$\|4a\|$
Equation of tangent at vertex	$x = 0$	$y = 0$

Find vertex, focus, length of latus rectum, equation of latus rectum, equation of axis, and equation of directrix of the following parabolas:

i. $y^2 = 4x + 4y - 8 = 0$
ii. $x^2 - 8x + 2y + 7 = 0$
iii.

07. The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Elements of ellipse	$a > b$	$a < b$
Centre	$(0,0)$	$(0,0)$
Length of major axis	$2a$	$2b$
Length of minor axis	$2b$	$2a$
Equation of major axis	$y = 0$	$x = 0$
Equation of minor axis	$x = 0$	$y = 0$
Eccentricity	$e = \sqrt{1 - \frac{b^2}{a^2}}$	$e = \sqrt{1 - \frac{a^2}{b^2}}$
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrix	$x = \pm\frac{a}{e}$	$y = \pm\frac{b}{e}$
Equation of axis	$y = 0$	$x = 0$
Equation of latus rectum	$x = \pm ae$	$y = \pm be$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Distance between the foci	$2ae$	$2be$

i. Find the eccentricity and length of latus rectum of the ellipse $4x^2 + 5y^2 - 16x + 10y + 1 = 0$.
ii. Find focus and equation of directrix of the ellipse $6x^2 + 4y^2 - 36x - 4y - 43 = 0$.
iii.

08. Hyperbola

Elements of hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$
Centre	$(0,0)$	$(0,0)$
Eccentricity	$e = \sqrt{1 - \frac{b^2}{a^2}}$	$e = \sqrt{1 - \frac{a^2}{b^2}}$
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrix	$x = \pm \frac{a}{e}$	$y = \pm \frac{b}{e}$
Equation of latus rectum	$x = \pm ae$	$y = \pm be$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Distance between the foci	$2ae$	$2be$

i. Find vertex, eccentricity, focus, equation of directrix, length of latus rectum of the hyperbola $x^2 - 3y^2 - 2x - 8 = 0$.

ii.

Preference:

i. Midterm questions
ii. Red marks questions