DEFINITE INTEGRAL AS THE LIMIT OF A SUM

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Question 01. Evaluate $\int_a^b \sin x \, dx$ by first principal / definition.

Solution: We know,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} f(a+rh), \qquad nh = b-a$$

Here $f(x) = \sin x$ and $f(a+rh) = \sin(a+rh)$.

Now

$$\int_a^b \sin x \, dx = \lim_{h\to 0} h \sum_{r=1}^{n} \sin(a+rh)$$

$$= \lim_{h\to 0} h[\sin(a+h) + \sin(a+2h) + \sin(a+3h) + \cdots \cdots + \sin(a+nh)]$$

$$\sin(a+h) + \sin(a+2h) + \sin(a+3h) + \cdots \cdots + \sin(a+nh)$$

$$= \frac{\sin \frac{Number\ of\ terms\ \times Difference\ between\ any\ two\ consecutive\ angles}{2}}{\sin\frac{a}{2}} \times \sin\frac{First\ angle + Last\ angle}{2}$$

$$= \frac{\sin\frac{nh}{2}}{\sin\frac{h}{2}} \times \sin\frac{a+h+a+nh}{2}$$

$$= \lim_{h\to 0} h \left[\frac{\sin\frac{nh}{2}}{\sin\frac{h}{2}} \times \sin\frac{a+h+a+nh}{2}\right]$$

$$= \lim_{h\to 0} \frac{h}{2} \times 2 \left[\sin\frac{nh}{2} \times \frac{1}{\sin\frac{h}{2}} \times \sin\frac{a+h+a+nh}{2}\right]$$

$$= \lim_{h\to 0} \frac{h}{2} \times 2 \times \sin\frac{nh}{2} \times \frac{1}{\sin\frac{h}{2}} \times \sin\frac{a+h+a+nh}{2}$$

$$= \lim_{h\to 0} \frac{h}{2} \times \frac{1}{\sin\frac{h}{2}} \times 2 \times \sin\frac{nh}{2} \times \sin\frac{a+h+a+nh}{2}$$

$$= \lim_{h\to 0} \frac{\frac{h}{2}}{\sin\frac{h}{2}} \times 2 \times \sin\frac{nh}{2} \times \sin\frac{a+h+a+nh}{2}$$

$$= 2 \times \lim_{h\to 0} \frac{\frac{h}{2}}{\sin\frac{h}{2}} \times \sin\frac{nh}{2} \times \sin\frac{2a+h+nh}{2}$$

$$= 2 \times \lim_{h\to 0} \frac{\frac{h}{2}}{\sin\frac{h}{2}} \times \sin\frac{(b-a)}{2} \times \sin\frac{2a+h+(b-a)}{2}$$

$$= 2 \times \sin\frac{(b-a)}{2} \times \lim_{h\to 0} \frac{\frac{h}{2}}{\sin\frac{h}{2}} \times \sin\frac{2a+h+(b-a)}{2}$$

$$= 2 \times \sin\frac{(b-a)}{2} \times 1 \times \sin\frac{2a+0+(b-a)}{2}$$

$$= 2 \times \sin\frac{b-a}{2} \times \sin\frac{b+a}{2}$$

$$= \cos a - \cos b$$

Question 02. Evaluate $\int_a^b \cos x \, dx$ by first principal / definition.

Solution: try yourself.

Note: $\cos(a+h) + \cos(a+2h) + \cos(a+3h) + \cdots \cdots + \cos(a+nh)$

$$= \frac{\sin\frac{Number\ of\ terms\ \times Difference\ between\ any\ two\ consecutive\ angles}{2}}{\sin\frac{a}{2}} \times \cos\frac{First\ angle + Last\ angle}{2}$$

$$= \frac{\sin\frac{nh}{2}}{\sin\frac{h}{2}} \times \cos\frac{a+h+a+nh}{2}$$

Question 03. Evaluate $\int_{0}^{\pi/2} \cos x \, dx$ by first principal / definition.

Solution: class work.

Question 04. Evaluate $\int_{0}^{\pi/2} \sin x \, dx$ by first principal / definition.

Solution: try yourself.

Question 05. Evaluate $\int_{a}^{b} x^2 \, dx$ by first principal / definition.

Solution: We know,

$$\int_{a}^{b} f(x)\, dx = \lim_{h \to 0} h \sum_{r=1}^{n} f(a+rh), \qquad nh = b-a$$

Here $f(x) = x^2$ and $f(a+rh) = (a+rh)^2$.

Now

$$\int_{a}^{b} x^2\, dx = \lim_{h \to 0} h \sum_{r=1}^{n} (a+rh)^2$$

$$= \lim_{h \to 0} h[(a+h)^2 + (a+2h)^2 + (a+3h)^2 + \cdots \cdots + (a+nh)^2]$$

$$= \lim_{h \to 0} h[n\, a^2 + 2ah(1+2+3+\cdots \cdots + n) + h^2(1^2+2^2+3^2+\cdots \cdots + n^2)]$$

Note: $1+2+3+\cdots \cdots + \mathrm{n} = \frac{\mathrm{n(n+1)}}{2}$

$1^2+2^2+3^2+\cdots \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

$$= \lim_{h \to 0} h\left[n\, a^2 + 2ah\frac{n(n+1)}{2} + h^2\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{h \to 0} \left[nh\, a^2 + a\, nh(nh+h) + \frac{nh(nh+h)(2nh+h)}{6}\right]$$

$$= \lim_{h \to 0} \left[(b-a)\, a^2 + a\,(b-a)\{(b-a)+h\} + \frac{1}{6} \times (b-a)\{(b-a)+h\}\{2(b-a)+h\} \right]$$

$$= \lim_{h \to 0} \left[(b-a)\, a^2 + a\,(b-a)\{(b-a)+h\} + \frac{1}{6} \times (b-a)\{(b-a)+h\}\{2(b-a)+h\} \right]$$

$$= (b-a)\, a^2 + a\,(b-a)\{(b-a)+0\} + \frac{1}{6} \times (b-a)\{(b-a)+0\}\{2(b-a)+0\}$$

$$= (b-a)\, a^2 + a\,(b-a)(b-a) + \frac{1}{6} \times (b-a)(b-a) \times 2(b-a)$$

$$= (b-a)\, a^2 + a\,(b-a)^2 + \frac{1}{3} \times (b-a)^3$$

$$= b\,a^2 - a^3 + a\,b^2 - 2\,b\,a^2 + a^3 + \frac{1}{3} \times (b^3 - 3\,b^2\,a + 3\,b\,a^2 - a^3)$$

$$= \frac{1}{3} \times (b^3 - a^3)$$

Question 06. Evaluate $\int_0^2 x^2\, dx$ by first principal / definition.

Solution: try yourself.

Question 07. Evaluate $\int_a^b e^x\,dx$ by first principal / definition.

Solution: We know,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} f(a+rh), \qquad nh = b-a$$

Here $f(x) = e^x$ and $f(a+rh) = e^{a+rh}$.

Now

$$\int_a^b e^x\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} e^{a+rh}$$

$$= \lim_{h\to 0} h \sum_{r=1}^{n} e^{a+rh}$$

$$= \lim_{h\to 0} h \sum_{r=1}^{n} e^a e^{rh}$$

$$= e^a \lim_{h\to 0} h \sum_{r=1}^{n} e^{rh}$$

$$= e^a \lim_{h\to 0} h[e^h + e^{2h} + e^{3h} + \cdots \cdots + e^{nh}]$$

$$= e^a \lim_{h\to 0} h\left[e^h + \left(e^h\right)^2 + \left(e^h\right)^3 + \cdots \cdots + \left(e^h\right)^n\right]$$

Note: $a + ar + ar^2 + ar^3 + \cdots \cdots + ar^{n-1} = \frac{a(r^n-1)}{r-1}$

$a + aa + aa^2 + aa^3 + \cdots \cdots + aa^{n-1} = \frac{a(a^n-1)}{a-1}$

$$= e^a \lim_{h\to 0} h\left[e^h + e^h e^h + e^h\left(e^h\right)^2 + \cdots \cdots + e^h\left(e^h\right)^{n-1}\right]$$

$$= e^a \lim_{h\to 0} h\left[\frac{e^h\left(\left(e^h\right)^n - 1\right)}{e^h - 1}\right]$$

$$= e^a \lim_{h\to 0} h\left[\frac{e^h\left(e^{nh} - 1\right)}{e^h - 1}\right]$$

$$= e^a \lim_{h\to 0} h\left[\frac{e^h\left(e^{b-a}-1\right)}{e^h-1}\right]$$

$$= e^a \times \left(e^{b-a}-1\right) \times \lim_{h\to 0} h\left[\frac{e^h}{e^h-1}\right]$$

$$= e^a \times \left(e^{b-a}-1\right) \times \lim_{h\to 0} \frac{he^h}{e^h-1}$$

$$= e^a \times \left(e^{b-a}-1\right) \times \lim_{h\to 0} \frac{he^h+e^h}{e^h} \qquad by\ L'Hospital\ rule.$$

$$= e^a \times \left(e^{b-a}-1\right) \times \frac{0e^0+e^0}{e^0}$$

$$= e^a \times \left(e^{b-a}-1\right) \times 1$$

$$= e^b - e^a$$

Question 08. Evaluate $\int_0^2 e^x\, dx$ by first principal / definition.

Solution: Class work.

Question 09. Find the value of $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots\cdots+\frac{1}{2n}\right]$

Solution: Here,

$$\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots\cdots+\frac{1}{2n}\right]$$

$$=\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots\cdots+\frac{1}{n+n}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\frac{r}{n}}$$

$h=\frac{1}{n}$

$$=\lim_{h\to 0}h\sum_{r=1}^{n}\frac{1}{1+rh}$$

$$\int_a^b f(x)\,dx=\lim_{h\to 0}h\sum_{r=1}^{n}f(a+rh),\qquad nh=b-a$$

$$\int_0^1\frac{1}{1+x}dx=\lim_{h\to 0}h\sum_{r=1}^{n}\frac{1}{1+(0+rh)},\quad nh=1-0$$

$$=\lim_{h\to 0}h\sum_{r=1}^{n}\frac{1}{1+(0+rh)}$$

$$=\int_0^1\frac{1}{1+x}dx$$

$$=\int_0^1\frac{1}{1+x}dx$$

$$=\left[\ln(1+x)\right]_0^1$$

$$=\ln(1+1)-\ln(1+0)$$

$$=\ln 2$$

Question 10. Find the value of $\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots\cdots+\frac{1}{2n}\right]$

Solution: Here,

$$\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots\cdots+\frac{1}{2n}\right]$$

$$\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots\cdots+\frac{n}{n^2+n^2}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{n^2+r^2}$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{n^2\left(1+\frac{r^2}{n^2}\right)}$$

$$\int_a^b f(x)\,dx=\lim_{h\to 0}h\sum_{r=1}^{n}f(a+rh),\qquad nh=b-a$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\left(\frac{r}{n}\right)^2}\qquad h=\frac{1}{n}$$

$$\int_0^1\frac{1}{1+x^2}dx=\lim_{h\to 0}h\sum_{r=1}^{n}\frac{1}{1+(0+rh)^2},$$

$$nh=1-0$$

$$=\lim_{h\to 0}h\sum_{r=1}^{n}\frac{1}{1+(rh)^2}$$

$$=\lim_{h\to 0}h\sum_{r=1}^{n}\frac{1}{1+(0+rh)^2}$$

$$=\int_0^1\frac{1}{1+x^2}dx$$

$$=[\arctan x]\begin{matrix}1\\0\end{matrix}$$

$$=\arctan 1-\arctan 0$$

$$=\frac{\pi}{4}-0$$

$$=\frac{\pi}{4}$$

Question 11. Find the value of $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{\sqrt{n^2-1}}+\frac{1}{\sqrt{n^2-2^2}}+\cdots\cdots+\frac{1}{\sqrt{n^2-(n-1)^2}}\right]$

Solution: Here,

$$\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{\sqrt{n^2-1}}+\frac{1}{\sqrt{n^2-2^2}}+\cdots+\frac{1}{\sqrt{n^2-(n-1)^2}}\right]$$

$$=\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2-0^2}}+\frac{1}{\sqrt{n^2-1^2}}+\frac{1}{\sqrt{n^2-2^2}}+\cdots\right.$$
$$\left.+\frac{1}{\sqrt{n^2-(n-1)^2}}\right]$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{\sqrt{n^2-r^2}}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n-1}\frac{1}{\sqrt{1-\frac{r^2}{n^2}}}$$

h

$=\frac{1}{n}$

$$=\lim_{h\to 0}h\sum_{r=1}^{n-1}\frac{1}{\sqrt{1-(rh)^2}}$$

$$\int_a^b f(x)\,dx=\lim_{h\to 0}h\sum_{r=0}^{n-1}f(a+rh),\qquad nh$$

$$=b-a$$

$$=\lim_{h\to 0}h\sum_{r=0}^{n-1}\frac{1}{\sqrt{1-(0+rh)^2}}$$

$$\int_0^1\frac{1}{\sqrt{1-x^2}}dx=\lim_{h\to 0}h\sum_{r=0}^{n-1}\frac{1}{\sqrt{1-(0+rh)^2}},$$

$$=\int_0^1\frac{1}{\sqrt{1-x^2}}dx$$

$$nh=1-0$$

$$=[\arcsin x]_0^1$$

$$=\arcsin 1-\arcsin 0$$

$$=\frac{\pi}{2}$$