SUCCESSIVE REDUCTION

Part one

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Question01: Find the reduction formula for $\int \sec^n x \, dx$.

Solution:

Let $I_n = \int \sec^n x \, dx$

$\Rightarrow I_n = \int \sec^{n-2} x \sec^2 x \, dx$

$\Rightarrow I_n = \sec^{n-2} x \int \sec^2 x \, dx - \int \frac{d}{dx} \sec^{n-2} x \int \sec^2 x \, dx \, dx$

$\Rightarrow I_n = \sec^{n-2} x \tan x - \int (n-2) \sec^{n-3} x \sec x \tan x \tan x \, dx$

$\Rightarrow I_n = \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x \, dx$

$\Rightarrow I_n = \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx$

$\Rightarrow I_n = \sec^{n-2} x \tan x - (n-2) \int (\sec^n x - \sec^{n-2} x) \, dx$

$\Rightarrow I_n = \sec^{n-2} x \tan x - (n-2)[\int \sec^n x \, dx - \int \sec^{n-2} x \, dx]$

$\Rightarrow I_n = \sec^{n-2} x \tan x - (n-2)[I_n - I_{n-2}]$

$\Rightarrow (1 + n - 2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

$\Rightarrow (n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

$\Rightarrow I_n = \frac{1}{(n-1)} \sec^{n-2} x \tan x + \frac{(n-2)}{(n-1)} I_{n-2} \quad \cdots \cdots \cdots \cdots (i)$

This is the required reduction formula.

Here $I_0 = \int \sec^0 x \, dx = \int 1 dx = x + c$

From (i) $I_2 = \frac{1}{(2-1)} \sec^{2-2} x \tan x + \frac{(2-2)}{(2-1)} I_{2-2}$

$\Rightarrow I_2 = \frac{1}{1} \sec^0 x \tan x + \frac{0}{1} I_0$

$\Rightarrow I_2 = \tan x + c$

Justification of I_2

$I_2 = \int \sec^2 x \, dx = \tan x + c$

Again $I_4 = \frac{1}{(4-1)}\sec^{4-2} x \tan x + \frac{(4-2)}{(4-1)} I_{4-2}$

$\Rightarrow I_4 = \frac{1}{3}\sec^2 x \tan x + \frac{2}{3} I_2$

$\Rightarrow I_4 = \frac{1}{3}\sec^2 x \tan x + \frac{2}{3}\tan x + c$

Again $I_6 = \frac{1}{(6-1)}\sec^{6-2} x \tan x + \frac{(6-2)}{(6-1)} I_{6-2}$

$\Rightarrow I_6 = \frac{1}{5}\sec^4 x \tan x + \frac{4}{5} I_4$

$\Rightarrow I_6 = \frac{1}{5}\sec^4 x \tan x + \frac{4}{5}\left(\frac{1}{3}\sec^2 x \tan x + \frac{2}{3}\tan x\right) + c$

$\Rightarrow I_6 = \left(\frac{1}{5}\sec^4 x + \frac{4}{5}\frac{1}{3}\sec^2 x + \frac{4}{5}\frac{2}{3}\right)\tan x + c$

$\Rightarrow I_6 = \left(\frac{1}{5}\sec^4 x + \frac{4}{5\times 3}\sec^2 x + \frac{4\times 2}{5\times 3}\right)\tan x + c$

And so on … …

Also $I_1 = \int \sec^1 x \, dx = \int \sec x \, dx = \ln|\sec x + \tan x| + c$

From (i) $I_3 = \frac{1}{(3-1)}\sec^{3-2} x \tan x + \frac{(3-2)}{(3-1)} I_{3-2}$

$\Rightarrow I_3 = \frac{1}{2}\sec^1 x \tan x + \frac{1}{2} I_1 = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + c$

Again $I_5 = \frac{1}{(5-1)}\sec^{5-2} x \tan x + \frac{(5-2)}{(5-1)} I_{5-2}$

$\Rightarrow I_5 = \frac{1}{4}\sec^3 x \tan x + \frac{3}{4} I_3$

$\Rightarrow I_5 = \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\left(\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x|\right) + c$

$$\Rightarrow I_5 = \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\ln|\sec x + \tan x| + c$$

And so on … …

Question02: Find the reduction formula for $\int \tan^n x\, dx$.

Solution:

Let $I_n = \int \tan^n x\, dx$

$\Rightarrow I_n = \int \tan^{n-2} x \tan^2 x\, dx$

$\Rightarrow I_n = \int \tan^{n-2} x\, (\sec^2 x - 1) dx$

$\Rightarrow I_n = \int \tan^{n-2} x \sec^2 x - \tan^{n-2} x\, dx$

$\Rightarrow I_n = \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} x\, dx$

$\Rightarrow I_n = \int \tan^{n-2} x \sec^2 x\, dx - I_{n-2}$

$\Rightarrow I_n = \int \tan^{n-2} x \sec^2 x\, dx - I_{n-2}$ Let $\tan x = z$

$\Rightarrow I_n = \int z^{n-2} dz - I_{n-2}$ $\Rightarrow \sec^2 x\, dx = dz$

$\Rightarrow I_n = \frac{1}{n-1} z^{n-1} - I_{n-2}$

$\Rightarrow I_n = \frac{1}{n-1} \tan^{n-1} x - I_{n-2}$

$\Rightarrow I_n = \frac{1}{n-1} \tan^{n-1} x - I_{n-2} \cdots \cdots \cdots \cdots (i)$

This the required reduction formula.

Here $\Rightarrow I_0 = \int \tan^0 x\, dx = \int 1\, dx = x + c$

From (i) $I_2 = \frac{1}{2-1} \tan^{2-1} x - I_{2-2}$

$\Rightarrow I_2 = \tan x - I_0$

$\Rightarrow I_2 = \tan x - x + c$

Again $I_4 = \frac{1}{4-1} \tan^{4-1} x - I_{4-2}$

$\Rightarrow I_4 = \frac{1}{3} \tan^3 x - I_2$

$\Rightarrow I_4 = \frac{1}{3}\tan^3 x - (\tan x - x) + c$

$\Rightarrow I_4 = \frac{1}{3}\tan^3 x - \tan x + x + c$

Again $I_6 = \frac{1}{6-1}\tan^{6-1} x - I_{6-2}$

$\Rightarrow I_6 = \frac{1}{5}\tan^5 x - I_4$

$\Rightarrow I_6 = \frac{1}{5}\tan^5 x - \left(\frac{1}{3}\tan^3 x - \tan x + x\right) + c$

$\Rightarrow I_6 = \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + c$

And so on … …

Also $I_1 = \int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx$

$\Rightarrow I_1 = -\ln|\cos x| + c$

$\Rightarrow I_1 = \ln|\sec x| + c$

Again $I_3 = \frac{1}{3-1}\tan^{3-1} x - I_{3-2}$

$\Rightarrow I_3 = \frac{1}{2}\tan^2 x - I_1$

$\Rightarrow I_3 = \frac{1}{2}\tan^2 x - \ln|\sec x| + c$

And $I_5 = \frac{1}{5-1}\tan^{5-1} x - I_{5-2}$

$\Rightarrow I_5 = \frac{1}{4}\tan^4 x - I_3$

$\Rightarrow I_5 = \frac{1}{4}\tan^4 x - \left(\frac{1}{2}\tan^2 x - \ln|\sec x|\right) + c$

$\Rightarrow I_5 = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x| + c$

And so on … …

Question03: Find the reduction formula for $\int \sin^n x\, dx$.

Try yourself.

Answer: $I_n = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}$

Question04: Find the reduction formula for $\int \cos^n x\, dx$.

Try yourself.

Answer: $I_n = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}$

Question05: Find the reduction formula for $\int \cos^m x \cos nx\, dx$.

Solution:

Let $I_{m,n} = \int \cos^m x \cos nx\, dx$

$\Rightarrow I_{m,n} = \cos^m x \int \cos nx\, dx - \int \frac{d}{dx}\cos^m x \int \cos nx\, dx\, dx$

$\Rightarrow I_{m,n} = \cos^m x \frac{\sin nx}{n} - \int m \cos^{m-1} x\,(-\sin x)\frac{\sin nx}{n}\, dx$

$\Rightarrow I_{m,n} = \cos^m x \frac{\sin nx}{n} + \frac{1}{n}\int m \cos^{m-1} x \sin x \sin nx\ \, dx$

$\cos(n-1)x = \cos nx \cos x + \sin nx \sin x$

$\Rightarrow I_{m,n} = \cos^m x \frac{\sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x\,[\cos(n-1)x - \cos nx \cos x]\ \, dx$

$\Rightarrow I_{m,n} = \cos^m x \frac{\sin nx}{n} + \frac{m}{n}[\int \cos^{m-1} x \cos(n-1)x\, dx -$
$\int \cos^{m-1} x \cos nx \cos x\, dx]$

$\Rightarrow I_{m,n} = \cos^m x \frac{\sin nx}{n} + \frac{m}{n}[\int \cos^{m-1} x \cos(n-1)x\, dx - \int \cos^m x \cos nx\, dx]$

$\Rightarrow I_{m,n} = \cos^m x \frac{\sin nx}{n} + \frac{m}{n}\left[I_{m-1,n-1} - I_{m,n}\right]$

$\Rightarrow I_{m,n} = \cos^m x \frac{\sin nx}{n} + \frac{m}{n} I_{m-1,n-1} - \frac{m}{n} I_{m,n}$

$\Rightarrow I_{m,n} + \frac{m}{n} I_{m,n} = \cos^m x \frac{\sin nx}{n} + \frac{m}{n} I_{m-1,n-1}$

$$\Rightarrow \frac{m+n}{n} I_{m,n} = \cos^m x \frac{\sin nx}{n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow (m+n) I_{m,n} = \cos^m x \sin nx + m I_{m-1,n-1}$$

$$\Rightarrow I_{m,n} = \frac{1}{(m+n)} \cos^m x \sin nx + \frac{m}{(m+n)} I_{m-1,n-1} \cdots \cdots \cdots \cdots (i)$$

This is the required formula.

Example:

Find $\int \cos^3 x \cos 2x \, dx$.

Solution:

Let $I_{3,2} = \int \cos^3 x \cos 2x \, dx$

By the reduction formula (i)

$$\Rightarrow I_{3,2} = \frac{1}{(3+2)} \cos^3 x \sin 2x + \frac{3}{(3+2)} I_{2,1}$$

$$\Rightarrow I_{3,2} = \frac{1}{5} \cos^3 x \sin 2x + \frac{3}{5} \left[\frac{1}{2+1} \cos^2 x \sin x + \frac{2}{2+1} I_{1,0} \right]$$

$$\Rightarrow I_{3,2} = \frac{1}{5} \cos^3 x \sin 2x + \frac{3}{5} \left[\frac{1}{3} \cos^2 x \sin x + \frac{2}{3} I_{1,0} \right]$$

$$\Rightarrow I_{3,2} = \frac{1}{5} \cos^3 x \sin 2x + \frac{1}{5} \cos^2 x \sin x + \frac{2}{5} I_{1,0} \quad \cdots \cdots \cdots \cdots (ii)$$

Here

$$I_{1,0} = \int \cos x \cos 0 \, dx$$

$$\Rightarrow I_{1,0} = \int \cos x \, dx = \sin x + c$$

Therefore (ii) becomes

$$I_{3,2} = \frac{1}{5} \cos^3 x \sin 2x + \frac{1}{5} \cos^2 x \sin x + \frac{2}{5} \sin x + c$$

That is

$$\int \cos^3 x \cos 2x \, dx = \frac{1}{5}\cos^3 x \sin 2x + \frac{1}{5}\cos^2 x \sin x + \frac{2}{5}\sin x + c$$

Follow-up Question: Find $\int \cos^5 x \cos 2x \, dx$.

Try yourself.

Question06: Find the reduction formula for $\int \sin^m x \sin nx \, dx$.

Try yourself.

Question07: Find the reduction formula for $\int \cos^m x \sin nx \, dx$.

Try yourself.

Answer: $I_{m,n} = \frac{-1}{(m+n)} \cos^m x \cos nx + \frac{m}{(m+n)} I_{m-1,n-1}$

Question08: Find the reduction formula for $\int \sin^m x \ \cos^n x \, dx$.

Try yourself.

Answer: $I_{m,n} = \frac{-1}{(m+n)} \sin^{m-1} x \cos^{n+1} x + \frac{m-1}{(m+n)} I_{m-2,n}$

Follow-up question: find $\int \sin^4 x \ \cos^3 x \, dx$.

Question09: Find the reduction formula for $\int x^n e^{ax}\, dx$.

Solution:

Let $I_n = \int x^n e^{ax}\, dx$

$\Rightarrow I_n = x^n \int e^{ax}\, dx - \int \frac{d}{dx} x^n \int e^{ax} dx\; dx$

$\Rightarrow I_n = x^n \frac{e^{ax}}{a} - \int n\, x^{n-1} \frac{e^{ax}}{a}\, dx$

$\Rightarrow I_n = x^n \frac{e^{ax}}{a} - \frac{n}{a} \int\, x^{n-1} e^{ax}\, dx$

$\Rightarrow I_n = \frac{1}{a} x^n e^{ax} - \frac{n}{a} I_{n-1}$

This is the required formula.

Follow-up Questions: Find the integrals:

i. $\int x^3 e^{5x}\, dx$
ii. $\int x^4 e^{3x}\, dx$