SUCCESSIVE REDUCTION

Part two

ABU ABDILLAH MOHAMMAD NASIM
Lecturer, Department of CSE, RMU.
abuabdillahmath@outlook.com

Question: Find the reduction formula for $\int \sin^m x \cos nx\, dx$.

Solution:

$$\text{Let } I_{m,n} = \int \sin^m x \cos nx\, dx$$

$$= \sin^m x \int \cos nx\, dx - \int \frac{d}{dx}\sin^m x \int \cos nx\, dx\, dx$$

$$= \sin^m x \frac{\sin nx}{n} - \int m \sin^{m-1} x \cos x \frac{\sin nx}{n}\, dx$$

$$= \sin^m x \frac{\sin nx}{n} - \frac{m}{n}\int \sin^{m-1} x \cos x \sin nx\ dx$$

$$\sin(n-1)x = \sin nx \cos x - \cos nx \sin x$$

$$= \sin^m x \frac{\sin nx}{n} - \frac{m}{n}\int \sin^{m-1} x\, [\sin(n-1)x + \cos nx \sin x]\, dx$$

$$= \sin^m x \frac{\sin nx}{n} - \frac{m}{n}\left[\int \sin^{m-1} x\ \sin(n-1)x\ dx + \int \sin^{m-1} x \cos nx \sin x\ dx\right]$$

$$= \sin^m x \frac{\sin nx}{n} - \frac{m}{n}\left[\int \sin^{m-1} x\ \sin(n-1)x\ dx + \int \sin^{m} x \cos nx\ dx\right]$$

$$= \frac{1}{n}\sin^m x \sin nx - \frac{m}{n}\int \sin^{m-1} x\ \sin(n-1)x\ dx - \frac{m}{n}\int \sin^{m} x \cos nx\ dx$$

$$= \frac{1}{n}\sin^m x \sin nx - \frac{m}{n}\int \sin^{m-1} x\ \sin(n-1)x\ dx - \frac{m}{n} I_{m,n}$$

$$\frac{m+n}{n} I_{m,n} = \frac{1}{n}\sin^m x \sin nx - \frac{m}{n}\int \sin^{m-1} x\ \sin(n-1)x\ dx$$

$$I_{m,n} = \frac{1}{m+n}\sin^m x \sin nx - \frac{m}{m+n}\int \sin^{m-1} x\ \sin(n-1)x\ dx$$

$$= \frac{1}{m+n}\sin^m x \sin nx - \frac{m}{m+n} J_{m-1,n-1} \cdots \cdots \cdots (i)$$

Again

Let $J_{m-1,n-1} \quad = \int \sin^{m-1} x \ \sin(n-1)x \ dx$

$$= \sin^{m-1} x \int \sin(n-1)x \ dx - \int \frac{d}{dx}\sin^{m-1} x \int \sin(n-1)x \ dx \ dx$$

$$= \sin^{m-1} x \frac{-\cos(n-1)x}{n-1} - \int (m-1)\sin^{m-2} x \cos x \ \frac{-\cos(n-1)x}{n-1} \ dx$$

$$= \frac{-\sin^{m-1} x \ \cos(n-1)x}{n-1} + \frac{(m-1)}{n-1}\int \sin^{m-2} x \cos x \ \cos(n-1)x \ dx$$

$$\cos(n-2)x = \cos(n-1)x\cos x + \sin(n-1)x \sin x$$

$$= \frac{-\sin^{m-1} x \ \cos(n-1)x}{n-1} + \frac{(m-1)}{n-1}\int \sin^{m-2} x \left[\cos(n-2)x - \sin(n-1)x \sin x\right] dx$$

$$= \frac{-\sin^{m-1} x \ \cos(n-1)x}{n-1} + \frac{(m-1)}{n-1}\left[\int \sin^{m-2} x \cos(n-2)x \, dx - \int \sin^{m-1} x \sin(n-1)x \ dx\right]$$

$$= \frac{-\sin^{m-1} x \ \cos(n-1)x}{n-1} + \frac{(m-1)}{n-1}\left[\int \sin^{m-2} x \cos(n-2)x \, dx - J_{m-1,n-1}\right]$$

$$= \frac{-\sin^{m-1} x \ \cos(n-1)x}{n-1} + \frac{(m-1)}{n-1}\int \sin^{m-2} x \cos(n-2)x \, dx - \frac{(m-1)}{n-1} J_{m-1,n-1}$$

$$\frac{(m+n-2)}{n-1}J_{m-1,n-1} = \frac{-\sin^{m-1}x\ \cos(n-1)x}{n-1} + \frac{(m-1)}{n-1}\int \sin^{m-2}x\cos(n-2)x\,dx$$

$$J_{m-1,n-1} = \frac{-\sin^{m-1}x\ \cos(n-1)x}{(m+n-2)} + \frac{(m-1)}{(m+n-2)}\int \sin^{m-2}x\cos(n-2)x\,dx \quad \cdots\cdots\cdots (ii)$$

Now from (i)

$$I_{m,n} = \frac{1}{m+n}\sin^m x\sin nx - \frac{m}{m+n}\left[\frac{-\sin^{m-1}x\ \cos(n-1)x}{(m+n-2)} + \frac{(m-1)}{(m+n-2)}\int \sin^{m-2}x\cos(n-2)x\,dx\right]$$

$$I_{m,n} = \int \sin^m x\cos nx\,dx = \frac{1}{m+n}\sin^m x\sin nx - \frac{m}{m+n}\left[\frac{-\sin^{m-1}x\ \cos(n-1)x}{(m+n-2)} + \frac{(m-1)}{(m+n-2)}I_{m-2,n-2}\right]$$

$$I_{m,n} = \int \sin^m x\cos nx\,dx = \frac{1}{m+n}\sin^m x\sin nx + \frac{m}{(m+n)(m+n-2)}\sin^{m-1}x\ \cos(n-1)x - \frac{m(m-1)}{(m+n)(m+n-2)}I_{m-2,n-2}$$