

MM2 Lecture 3: Ordinary Differential Equations II

First order ODEs (revision)

In this lecture we will concern ourselves with equations of the form:

$$\frac{dy}{dx} = F(x, y).$$

Analytical solutions to this equation are obtainable only for fairly simple right hand side functions $F(x, y)$ and a number of techniques are available. Some of these will have been covered already in Mathematical Methods I so aspects of this lecture are revision.

Separation of variables

For example, suppose we have the equation:

$$\frac{dy}{dx} = -\frac{x}{y},$$

with initial condition:

$$y(0) = 4.$$

That is to say, at $x = 0$, $y = 4$. We can transform this expression by multiplying by y and integrating with respect to dx :

$$\begin{aligned} y \frac{dy}{dx} &= -x \\ \int y \frac{dy}{dx} dx &= - \int x dx \\ \int y dy &= - \int x dx \\ y^2 &= -x^2 + c \end{aligned}$$

Substituting the initial condition into this general solution, we find $c = 16$ so the particular solution to this ODE is:

$$x^2 + y^2 = 16$$

In other words, a circle of radius 4. It is good practice to check the answer by taking it's derivative with respect to x :

$$\begin{aligned} \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= \frac{d}{dx}(16) \\ 2x + 2y \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

The method of variation of parameters

Suppose we have the general linear ODE of form:

$$\frac{dy}{dx} + f(x)y = r(x)$$

There are two important cases of this equation. If $r(x) \equiv 0$ then the equation is termed **homogeneous**. Otherwise, the system is **non-homogeneous** or **inhomogeneous**.

Provided the equation is linear (as above), in the non-homogeneous case, we can express the solution as the sum of the **general solution** of the homogeneous problem, denoted by y_h , and a **particular solution** to the full non-homogeneous problem, denoted by y_p :

$$y = y_h + y_p$$

In the rest of the text we will refer to y_p as the **particular integral** to avoid confusion with the terminology **particular solution** that we already used to indicate the solution of a differential equation where the arbitrary constants have been specified. The reason for decomposing the solution into homogeneous and non-homogeneous parts, is that the homogeneous part is easier to calculate, and can be used to calculate the non-homogeneous part. The technique known as *variation of parameters*, which is due to Lagrange, is a technique where the homogeneous solution is used in such a way to find the solution to the full non-homogeneous equation.

Therefore, as a first step we solve the homogeneous case of equation

$$\begin{aligned}\frac{dy}{dx} &= -f(x)y \\ \int \frac{1}{y} dy &= - \int f(x) dx \\ \ln(y) &= - \int f(x) dx + c \\ y &= ce^{-\int f(x) dx}\end{aligned}$$

So we can write:

$$y_h = ce^{-\int f(x) dx}$$

To find the non-homogeneous solution we try generalising the homogeneous solution, which can be written in the following form:

$$y_h = cv(x)$$

With:

$$v = e^{-\int f(x) dx}$$

One possible generalisation is to allow the parameter c to be a function. Essentially, we are **varying the parameter of the homogeneous solution**:

$$y_p = u(x)v(x)$$

where $u(x)$ is a yet to be determined function and $v(x)$ is the solution to the homogeneous equation, without the constant of integration. In this context, the function v is known as an **integrating factor**.

NOTE: In general, for any function $f(x)$, we can decompose this as $f(x) = g(x)h(x)$. Nothing stops us from doing this if we like! Example:

$$f(x) = x = x^2 \frac{1}{x} = g(x)h(x), \quad \text{with} \quad g(x) = x^2, h(x) = \frac{1}{x}.$$

In fact, an infinite number of choices are available to this. However, when using this method for solve ODEs we'll need to make specific choices!

At this stage we do not know if this technique will work for a given ODE, we can only try and see. Substituting into the original non-homogeneous ODE, we have:

$$\begin{aligned} \frac{d}{dx}(uv) + f(x)(uv) &= r(x) \\ v \frac{du}{dx} + u \frac{dv}{dx} + f(x)uv &= r(x) \\ v \frac{du}{dx} + \left(\frac{dv}{dx} + f(x)v \right) u &= r(x) \end{aligned}$$

This equation can be separated into two. Notice that the expression in the brackets is the homogeneous equation to which v is a solution:

$$\frac{dv}{dx} + f(x)v = 0$$

This leaves:

$$v \frac{du}{dx} = r(x)$$

Which we can solve by separation of variables:

$$\begin{aligned} \int du &= \int \frac{r(x)}{v} dx \\ u &= \int \frac{r(x)}{v} dx \end{aligned}$$

Note that we have left out the constant of integration. The reason is that the constant of integration is used to ensure that the solutions satisfies boundary conditions, and is accounted for later (since we still have one integral to evaluate which will provide a constant of integration):

$$\begin{aligned} y &= y_p + y_h \\ &= uv + y_h \\ &= v \int \frac{r(x)}{v} dx + cv \end{aligned}$$

We therefore have:

$$y = uv = v \int \frac{r(x)}{v} dx + cv$$

We can show that ignoring the constant of integration when solving y_p does not affect the general character of the solution. Lets try including all constants of integration:

$$\begin{aligned} \int du &= \int \frac{r(x)}{v} dx \\ u &= \int \frac{r(x)}{v} dx + c_1 \end{aligned}$$

Then,

$$y_p = uv = v \int \frac{r(x)}{v} dx + c_1 v$$

So the full solution is:

$$\begin{aligned} y &= y_p + y_h \\ &= uv + y_h \\ &= v \int \frac{r(x)}{v} dx + c_1 v + cv \\ &= v \int \frac{r(x)}{v} dx + c_2 v \end{aligned}$$

Thus, we end up with a solution of the same form as was obtained above.

To review

- We decomposed the solution as the sum of homogeneous and non-homogeneous parts $y(x) = y_h(x) + y_p(x)$.
- The decomposition is based on the fact that the problem is **linear**: The sum of two solutions to the ODE is itself a solution.
- The next step was to solve the homogeneous equation, using separation of variables.
- The next step was to generalise the homogeneous solution towards finding the non-homogeneous solution. In the method of variation of parameters, we do so by turning the constant parameters in the homogeneous solution into an unknown function: $y_p = uv$ where $y_h = cv$ (c is a constant) is the solution to the homogeneous problem.
- The solution of the homogeneous problem acts as an integrating factor, simplifying the non-homogeneous problem.

For completeness, we mention the distinction between linear and non-linear differential equations. In non-linear differential equations the sum of two solutions is not necessarily a solution. Non-linear equations are much harder to solve, and in this course we focus exclusively on linear Ordinary Differential Equations.

At this stage it is worth noting that we have found **one** solution to the ODE. Mathematically, it is necessary to consider whether this solution is the only one, however the study of uniqueness is beyond the scope of this course.

We refer to $y(x) = y_h(x) + y_p(x)$, where the unknown constant of integration is included, as the **general solution**.

The constant of integration can be determined by providing the solution of the ODE at a single point, or the derivative of the solution. It is common to formally state problems involving differential equations (ordinary or partial) in the following way:

$$\frac{dy}{dx} + f(x)y = r(x), \quad y(x_0) = a$$

or

$$y'' + f(x)y = r(x), \quad y'(x_0) = b$$

where where a and b are given constants. When modelling physical systems, such conditions on the solution or its derivative are usually a statement of the known state at the boundaries of the system, or at the 'start time'. Thus, such statements are known as **boundary conditions** or **initial conditions**, and can be used to determine the constants of integration in the general solution. Once the constants of integration have been determined, the solution is known as the **particular solution**. However, note that the the boundary or initial conditions are assigned to the full solution. Therefore, in non-homogeneous problems, the general solution must first be found, and following this boundary conditions can be applied to find the unknown constant of integration.

Example: variation of parameters

Solve:

$$\frac{dy}{dx} + \frac{3}{x}y = x^2$$

subject to the initial condition $y(1) = 1/6$.

We construct an integrating factor:

$$\begin{aligned} v &= e^{-\int \frac{3}{x} dx} \\ &= e^{-3 \ln x} \\ &= \frac{1}{x^3} \end{aligned}$$

Note that the integrating factor does not contain a constant of integration as this is absorbed into the unknown function u . We can then use the general form of the solution:

$$\begin{aligned} y &= v \int \frac{r(x)}{v} dx + cv \\ &= \frac{1}{x^3} \int \frac{x^2}{1/x^3} dx + \frac{c}{x^3} \\ &= \frac{1}{x^3} \frac{x^6}{6} + \frac{c}{x^3} \\ &= \frac{x^3}{6} + \frac{c}{x^3} \end{aligned}$$

Once again we may check this general solution by differentiation:

$$\frac{dy}{dx} = \frac{x^2}{2} - \frac{3c}{x^4} \quad \text{and} \quad \frac{3}{x}y = \frac{x^2}{2} + \frac{3c}{x^4} \quad \text{and hence} \quad \frac{dy}{dx} + \frac{3}{x}y = x^2$$

as required. We now apply the initial conditions $y(1) = 1/6$ (or $x = 1, y = 1/6$):

$$\begin{aligned} \frac{1}{6} &= \frac{1}{6} + c \\ c &= 0 \end{aligned}$$

The particular solution is therefore:

$$y = \frac{x^3}{6}$$

Example: variation of parameters

Solve:

$$\frac{dx}{dt} + tx = t \quad x(0) = 2$$

The integrating factor is:

$$\begin{aligned} v &= e^{-\int t dt} \\ &= e^{-\frac{t^2}{2}} \end{aligned}$$

So that:

$$\begin{aligned} x &= e^{-\frac{t^2}{2}} \int t e^{\frac{t^2}{2}} dt + c e^{-\frac{t^2}{2}} \\ &= e^{-\frac{t^2}{2}} e^{\frac{t^2}{2}} + c e^{-\frac{t^2}{2}} \\ &= 1 + c e^{-\frac{t^2}{2}} \end{aligned}$$

The integral was evaluated by the change of variables $u = t^2/2$. Hence $du = t dt$ which we use with the well-known identity $\int e^u du = e^u$.

Applying the initial conditions, we find $c = 1$ so:

$$x = 1 + e^{-\frac{t^2}{2}}$$

Second order ODEs

We will now move on to higher order ODEs. This is a large and difficult subject so we will initially restrict ourselves to the general linear second order ODEs:

$$\frac{d^2 y}{dx^2} + f(x) \frac{dy}{dx} + g(x)y = r(x)$$

This gives us a basis for exploring theoretical considerations and is of practical significance as second order ODEs arise in many of the most important applications.

Solution form of linear homogeneous ODEs

As with first order ODEs, the distinction between homogeneous and non-homogeneous is important and we will often build up the solution out of homogeneous and non-homogeneous parts. Linear homogeneous ODEs have some special properties which make this approach attractive.

- A second order linear ODE has two independent solutions y_1 and y_2 .
- If y_1 is a solution to an ODE then $c y_1$ is also a solution, for any constant c .
- If y_1 and y_2 are both solutions then so is $c_1 y_1 + c_2 y_2$ for any constant c_1, c_2

Since any function which is linearly dependent on y_1 and y_2 is expressible as a weighted sum of these two functions, and there are only two linearly independent solutions to a second order ODE, the **general solution** to a linear homogeneous second order ODE is:

$$y = c_1 y_1 + c_2 y_2$$

Where y_1 and y_2 are linearly independent solutions to the ODE. This result may be extended into more dimensions but it is important to remember that it is only applicable to linear homogeneous ODEs.

Solution form of linear non-homogeneous ODEs

Following the same approach as 1st order ODEs, we decompose the solution of non-homogeneous ODEs into a sum of the **homogeneous solution** y_h and the **non-homogeneous particular integral** y_p :

$$y = y_h + y_p$$

where particular integral, y_p , represents one particular solution of the non-homogeneous equation. Thus the solution can be written as

$$y = c_1 y_1 + c_2 y_2 + y_p$$

Where y_1 and y_2 are solutions to the **homogeneous** part of the equation with two arbitrary constants, c_1 and c_2 , while the particular solution of the non-homogeneous problem contains no constants of integration.

Mirroring the approach of first order ODEs, we can show that the decomposition works because of linearity:

$$\begin{aligned} \frac{d^2}{dx^2}(y_h + y_p) + f(x)\frac{d}{dx}(y_h + y_p) + g(x)(y_h + y_p) &= r(x) \\ \frac{d^2 y_h}{dx^2} + f(x)\frac{dy_h}{dx} + g(x)y_h + \frac{d^2 y_p}{dx^2} + f(x)\frac{dy_p}{dx} + g(x)y_p &= r(x). \end{aligned}$$

Since y_h is a solution to the homogeneous equation, the first line is equal to 0. Similarly, since y_p is a solution to the non-homogeneous equation, the second line represents an equality in its own right.

Homogeneous linear second order ODEs with constant coefficients

We will now examine a particular case where $f(x)$ and $g(x)$ are real constants and the problem is homogeneous, so $r(x) \equiv 0$. The ODE takes the following form:

$$\frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0,$$

where $a, b \in \mathbb{R}$.

Recall the first order homogeneous linear equation:

$$\frac{dy}{dx} + ky = 0,$$

As we showed towards the end of Lecture 2, general solutions take the form:

$$y = ce^{-kx}.$$

Inspired by this, we look for solutions of the form $e^{\lambda x}$. Substituting $y = e^{\lambda x}$ into equation the ODE gives:

$$\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0$$

Dividing by the common factor, this leads to the so-called **characteristic equation** or **characteristic polynomial**:

$$\lambda^2 + b\lambda + c = 0.$$

Having studied complex numbers, we know that this equation will have solutions, although they may be complex. Broadly, when equation the characteristic equation has complex roots the system described by the second order linear homogeneous 2nd order ODE exhibits oscillatory behaviour, otherwise it follows a decaying behaviour towards static equilibrium.

Recall that a quadratic $\alpha x^2 + \beta x + \gamma = 0$ has roots $x_{1,2} = \frac{-\beta \pm \sqrt{\Delta}}{2\alpha}$, $\alpha \neq 0$, where Δ is termed the discriminant, $\Delta = \beta^2 - 4\alpha\gamma$. Therefore, we can classify the solutions of the characteristic equation into three cases:

- Case I: Two distinct real roots for $\Delta > 0$.
- Case II: A pair of complex conjugate roots for $\Delta < 0$.
- Case III: A double real root for $\Delta = 0$.

In the following we explore each of the three cases.

Case I: Two distinct real roots λ_1 and λ_2

This means that $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are solutions of the Ordinary Differential Equation. Since linear combinations of solutions are also solutions, this gives us the general solution form:

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

for any A and B .

Example

Solve:

$$\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} - 3y = 0$$

This yields characteristic equation:

$$\begin{aligned}\lambda^2 + 2\lambda - 3 &= 0 \\ (\lambda + 3)(\lambda - 1) &= 0\end{aligned}$$

The roots of the characteristic equation are therefore:

$$\lambda_1 = -3 \quad \lambda_2 = 1$$

So the general form is:

$$y = Ae^{-3x} + Be^x$$

For a second order ODE, the solution has **two** arbitrary conditions. Therefore, to obtain a particular solution we need to specify either the value of the function at two different points:

$$y(x_1) = y_1 \quad y(x_2) = y_2$$

or, more often, the value of both y and its derivative at a single point:

$$y(x_0) = y_0 \quad \left. \frac{dy}{dx} \right|_{x=x_0} = f_0$$

The first condition is known as a **boundary condition** while the second is termed an **initial condition**. These names arise from the physical systems which ODEs model. If the independent variable is time then the **initial condition** specifies the whole state of the system at the start of time. Similarly if the independent variable is space then the **boundary condition** can be used to specify the value of the solution at the edges of the simulation domain.

Example

Solve the previous ODE subject to the initial conditions:

$$y(0) = 0, \quad \frac{dy}{dx}(0) = 4$$

At the point $x = 0$ we have:

$$\begin{aligned}Ae^0 + Be^0 &= 0 \\ A + B &= 0 \\ A &= -B\end{aligned}$$

but also:

$$\begin{aligned}y &= Ae^{-3x} + Be^x \\ \frac{dy}{dx} &= -3Ae^{-3x} + Be^x \rightarrow \\ 4 &= -3A^0 + Be^0 \\ 4 &= -3A + B\end{aligned}$$

Solving these two linear equations we have:

$$A = -1 \quad B = 1$$

So that the particular solution we are seeking is:

$$y = e^x - e^{-3x}$$

Case II: complex conjugate pair of roots

In this case we can write the roots of the characteristic equation as:

$$\lambda_1 = p + iq \quad \lambda_2 = p - iq$$

So that the solutions are:

$$y_1 = e^{(p+iq)x} \quad y_2 = e^{(p-iq)x}$$

However in many situations, we are only interested in the real part of the solutions. We can obtain a real solution using Euler's formula:

$$\begin{aligned} y_1 &= e^{px} e^{iqx} \\ &= e^{px} (\cos qx + i \sin qx) \end{aligned}$$

and similarly:

$$\begin{aligned} y_2 &= e^{px} e^{-iqx} \\ &= e^{px} (\cos qx - i \sin qx) \end{aligned}$$

Now clearly both the real and imaginary parts of these results are solutions, however the real parts of y_1 and y_2 are identical, so we still only have one solution. Instead, we can combine the two complex solutions in a different way to produce real solutions. Looking back to lecture 3 we have:

$$\begin{aligned} \frac{1}{2}(y_1 + y_2) &= e^{px} \cos qx \\ \frac{1}{2i}(y_1 - y_2) &= e^{px} \sin qx \end{aligned}$$

These two solutions are real and linearly independent and therefore we can use them to write the general solution:

$$y = e^{px} (A \cos qx + B \sin qx)$$

We could also have obtained this result directly from the sum of the two complex solutions but we would have had to include the imaginary part in a new arbitrary constant which is fiddly and less neat.

Example

Find the solution to:

$$\frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 9y = 0$$

subject to the initial condition:

$$y(0) = 0 \quad \left. \frac{dy}{dx} \right|_{x=0} = \sqrt{5}$$

The characteristic equation is:

$$\lambda^2 + 4\lambda + 9 = 0$$

Using the quadratic formula, we have:

$$\begin{aligned} \lambda &= \frac{-4 \pm \sqrt{16 - 36}}{2} \\ &= -2 \pm i\sqrt{5} \end{aligned}$$

So this produces the general solution:

$$y = e^{-2x} (A \cos(\sqrt{5}x) + B \sin(\sqrt{5}x))$$

We can now consider the initial value problem:

$$\begin{aligned} y(0) &= e^{(-2 \times 0)} (A \cos(\sqrt{5} \times 0) + B \sin(\sqrt{5} \times 0)) \\ 0 &= 1(A \times 1 + B \times 0) \\ A &= 0 \end{aligned}$$

Using this result and the product rule we have:

$$\frac{dy}{dx} = -2y + e^{-2x} (\sqrt{5}B \cos(\sqrt{5}x))$$

Now applying the other initial condition:

$$\begin{aligned} \left. \frac{dy}{dx} \right|_{x=0} &= -2y(0) + e^{(-2 \times 0)} (\sqrt{5}B \cos(\sqrt{5} \times 0)) \\ \sqrt{5} &= 0 + 1 (\sqrt{5}B) \\ B &= 1 \end{aligned}$$

The particular solution is therefore:

$$y = e^{-2x} \sin(\sqrt{5}x)$$

Case III: real, double root $\lambda_1 = \lambda_2$

Looking back to the characteristic equation:

$$\lambda^2 + b\lambda + c = 0$$

and applying the quadratic formula, we have:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

A double real root occurs when $b^2 - 4c = 0$ with the result that $\lambda = -b/2$. Therefore, is our solution

$$y = Ae^{-bx/2} + Be^{-bx/2} = Ce^{-bx/2}?$$

While this is certainly one solution, shouldn't we be getting two? It's a 2nd order ODE after all! Are we potentially missing something here?

To find another solution, we can look to variation of parameters again:

$$y_2 = u(x)e^{\lambda x}$$

We can now take the derivative of this expression:

$$\frac{dy_2}{dx} = \lambda u e^{\lambda x} + \frac{du}{dx} e^{\lambda x}$$

Since this is a second order ODE, we also need the second derivative:

$$\begin{aligned} \frac{d^2 y_2}{dx^2} &= \lambda^2 u e^{\lambda x} + \lambda \frac{du}{dx} e^{\lambda x} + \lambda \frac{du}{dx} e^{\lambda x} + \frac{d^2 u}{dx^2} e^{\lambda x} \\ &= \lambda^2 u e^{\lambda x} + 2\lambda \frac{du}{dx} e^{\lambda x} + \frac{d^2 u}{dx^2} e^{\lambda x}. \end{aligned}$$

Now we can substitute into the original ODE:

$$\lambda^2 u e^{\lambda x} + 2\lambda \frac{du}{dx} e^{\lambda x} + \frac{d^2 u}{dx^2} e^{\lambda x} + b(\lambda u e^{\lambda x} + \frac{du}{dx} e^{\lambda x}) + c u e^{\lambda x} = 0$$

Dividing by the exponential term and gathering like terms, we have:

$$\frac{d^2 u}{dx^2} + \underbrace{(2\lambda + b)}_{=0} \frac{du}{dx} + \underbrace{(\lambda^2 + b\lambda + c)}_{=0} u$$

The first bracketed expression is zero because $\lambda = -b/2$ while the second is the characteristic equation to which λ is a solution. This leaves us with the rather simple second order ODE:

$$\frac{d^2 u}{dx^2} = 0$$

Which has, leaving out the constants of integration for now, solution:

$$u(x) = x$$

The arbitrary constants we be re-introduced later when writing the general solution. Substituting back into our definition of y_2 we have the second solution to the ODE:

$$y_2 = x e^{\lambda x}$$

This yields the general solution to the ODE:

$$y = (A + Bx)e^{\lambda x}$$

Example

Solve:

$$\frac{1}{4} \frac{d^2 y}{dx^2} - \frac{dy}{dx} + y = 0$$

subject to the initial conditions:

$$y(0) = 3 \quad \left. \frac{dy}{dx} \right|_0 = 1$$

Multiplying by 4 to transform the equation into canonical form, we can arrive at the characteristic equation:

$$\lambda^2 - 4\lambda + 4 = 0$$

which we can factorise:

$$(\lambda - 2)^2 = 0$$

which has a double root at $\lambda = 2$ so that the general solution is:

$$y = (A + Bx)e^{2x}$$

Now we apply the initial conditions to this:

$$\begin{aligned} 3 &= (A + B \times 0)e^0 \\ A &= 3 \end{aligned}$$

and:

$$\begin{aligned} \frac{dy}{dx} &= 2(A + Bx)e^{2x} + Be^{2x} \\ 1 &= 2(3 + B \times 0)e^0 + Be^0 \\ B &= -5 \end{aligned}$$

So the solution to this initial value problem is:

$$y = (3 - 5x)e^{2x}$$

NOTE: How can we be sure we've found all solutions to the various categories of ODEs above? Why exactly do we need to use variation of parameters in one specific case but not the others? We'll explore this next week!

Note on higher order equations

The methods discussed here based on the characteristic equation can be generalised to higher order systems of equations but these are beyond the scope of this course.