lecture1-nb

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1 Maths Methods 2

2 Lecture 1 - Introduction and Complex Numbers I

Contact details:

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2.1 Lecture Schedule

Every Monday 14:00-17:00 from 19th of January until the end of term.

2.2 Lecture Structure

The course consists of 8 core lectures.

In the absence of any lecture needing to be re-arranged, the final lecture will be a revision lecture where we focus on problems brought to me by you and/or past exam questions.

Roughly speaking, the **first one and a half to two hours** of the lecture will be dedicated towards the theory and worked examples, the **final hour** will be run as a problem class. In this final hour, you'll work through various problems by yourselves and myself and GTAs will be present to answer any questions you have. As we approach the end of term, we'll spend more of this final hour going through past exam questions on the board.

2.3 Teaching assistants

- Zoe Leibowitz (zoe.leibowitz21@imperial.ac.uk)
- Alexandra Neagu (alexandra.neagu20@imperial.ac.uk)
- Oliver Coombes (oliver.coombes22@imperial.ac.uk)

2.4 Office hour

Tuesday's 12:00-13:00 (can be changed if another time is more convenient). Please come to my office with any questions about the course you may have!

2.5 Course material

- Course material will also be available via ESESIS and Lambda Feedback.
- The 'main' lecture notes are in the file `MathsMethods2_LectureNotes.pdf`.
- All material appearing in the main lecture notes is considered examinable unless otherwise stated.
- Each lecture will follow a Jupyter notebook which will also be posted on ESESIS prior to each lecture. The notebooks are also available on GitHub here.

2.6 Lecture notebooks

As noted above, each lecture will be accompanied by a Jupyter notebook (such as this one). These notebooks will largely mirror the corresponding lecture main course notes - but may contain a few extra details/examples or illustrative plots (via python code).

You can view and run the notebooks online using Google Colab. Once on the Colab webpage, click `File`, `Open notebook`, click the `GitHub` tab, then enter `rhodrin/MM2-Notebooks`. A list of the notebooks available should then appear.

To view notebooks locally you can install Anaconda 3 and then use the notebook viewer though the API.

If you have any issues running the notebooks please ask myself of a GTA for some assistance.

2.7 Examination

- 100% of the mark will come from a 60 minute exam in May.

2.8 Problem sheets

There are no assessed problem sets.

Each lecture will be accompanied by a problem sheet. Worked solutions for these sheets will be uploaded to ESESIS shortly after each lecture.

Important Additional problem sheets along with their solutions are available on Lambda Feedback. These are the *old* coursework sheets - it's critical you do each of these in order to properly prepare for the exam.

2.9 Module AI-assistant (Beta)

The module also has a dedicated AI-assistant available via Chat ESE and Lambda Feedback. See ESESIS for further details. Feel free to ask if you have any questions about using this assistant. Also, feel free to report any issues you find with the bot to me!

2.10 Some comments on the course in general

- Many of the interesting real world Geophysics applications (Remote sensing, Seismology, Plate-Tectonics, Mantle Dynamics etc.) can be Modeled using Partial Differential Equations (which you'll learn about in later methods courses).
- However solving partial differential equations (e.g. the Wave-Equation) relies on many of the techniques you'll learn in this course: 2nd order ODEs, Fourier series.

- Further, in the later years you'll solve more complex PDEs which will rely on, e.g., Fourier Transforms and Numerical Analysis.
- Using Fourier Transforms relies on using complex number Theory. Numerical Methods (and loads of other things!) utilize the Taylor Series to form the methods. (I'll highlight other examples as we go).
- That is, the course is to give you the foundation to be able to study these interesting topics later on.
- The material you learn will be very useful for the applications you're interested in at a later date! And (although it may not be immediately evident) you'll learn about the Taylor Series in this course which is among the most useful and important scientific theories you'll ever learn (it's the basis of many of the geophysical models you'll make use of later and is the basis of many numerical methods).

2.11 Feedback

Your main mechanisms to get individual feedback during this course are during the **problem classes** and **office hours**. Please take advantage of these opportunities to get assistance with any topic you find difficult or to check your understanding of topics.

Please also be proactive in asking questions during the lectures. Whilst time is somewhat limited I'll always be happy to clear up an explanation where possible!

3 Complex Numbers I

During the BSc/MSci in Geophysics, you're most likely to re-encounter complex numbers in the context of Fourier Transforms which are used for solving PDEs and in signal processing.

3.1 Complex numbers and their arithmetic

3.1.1 The imaginary unit

You will all be familiar with real numbers, both positive and negative and with operations involving them.

Let us use x to designate a real number and investigate solutions to the equation:

$$x^2 = 4$$

Then we can immediately see that:

$$2^2 = 4 \qquad \text{and} \qquad (-2)^2 = 4$$

An equivalent way of saying that 2 and -2 are solutions to $x^2 = 4$ is to write:

$$\sqrt{4} = \pm 2$$

Similarly, we know that:

$$\sqrt{1} = \pm 1$$

Now, what if the equation we are given is instead:

$$x^2 = -1$$

There is no real number which gives a negative number when multiplied by itself. If we square a positive number we always get a positive number and the result of squaring a negative number is also always a positive number.

Suppose this number did somehow exist, though. For the moment, we might write it as $\sqrt{-1}$. Lets consider the square root of other negative numbers. Using the usual rules for operations under a square root sign we know that:

$$\sqrt{-4} = \sqrt{(4) \times (-1)} = \sqrt{4} \times \sqrt{-1} = 2\sqrt{-1}$$

More generally $\sqrt{x}$ for negative real x can be written as $a\sqrt{-1}$ for real a. That means that if we have a number $\sqrt{-1}$, we don't need more new numbers to express the square roots of other negative real numbers.

We can add (and subtract) multiples of $\sqrt{-1}$ in the normal way:

$$\sqrt{-4} + \sqrt{-9} = 2\sqrt{-1} + 3\sqrt{-1} = 5\sqrt{-1}$$

3.2 Imaginary numbers

So, we have a new set of numbers of the form $y\sqrt{-1}$. We can count and perform arithmetic with these numbers. But we **cannot** represent them in terms of real numbers. They are new numbers outside of the real number system.

In contrast to real numbers, these new numbers are referred to as *imaginary numbers*. It is usual to use the symbol i to stand for $\sqrt{-1}$. In the engineering literature, it is not unusual to see j instead of i.

You might argue that these are just abstract mathematical quantities, not relevant to the *real* world. You will see throughout this course and in other courses that this is not the case and that operations involving imaginary numbers can produce many physically meaningful results.

We can use the definition of i to work out some of the basic behaviour of this new number:

$$i^2 = (\sqrt{-1})^2 = -1$$
$$i^3 = i \times i^2 = -i$$
$$i^4 = i^2 \times i^2 = 1 \text{ and } \frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{i^2} = -i$$
$$\frac{1}{i^2} = -1$$
$$\frac{1}{i^3} = i$$
$$\frac{1}{i^4} = 1$$

3.3 Complex Numbers

We now create a new set of numbers encompassing both real and imaginary numbers. We name this new set **Complex numbers**. Complex numbers, for which we will often use the symbol z, have a real part and an imaginary part:

$$z = (x, y)$$

It is usual in practice to write complex numbers as:

$$z = x + iy$$

but do not be mislead by this notation: iy is not a real number so there is no way we can actually evaluate this 'addition'. The '+' just indicates that iy is positive. The above notation does show clearly how the complex numbers include all the real numbers and all the imaginary numbers since if $z = x + iy$ then:

$$y = 0 \Rightarrow z = x, \quad z \text{ is real}$$
$$x = 0 \Rightarrow z = iy, \quad z \text{ is imaginary}$$

There is also notation for extracting the real and imaginary parts of a complex number. If $z = x+iy$ again then:

$$x = \mathrm{Re}(z)$$
$$y = \mathrm{Im}(z)$$

3.4 Complex Analysis: Some more exotic examples of Geophysics/Physics applications

Lets look at a few research papers:

3.4.1 Well bore drilling

Exert: "This study presents a systematic analysis of stress changes near the horizontal well sections drilled in shale and clay formations using a new approach."

3.4.2 Waterflooding of resevroirs

Exert: "We apply an improved potential flow model based on conformal mapping to study the sweep pattern of waterfloods in bounded reservoirs."

3.4.3 Geophysical flows

Exert: "Recent observations obtained by NASA's Cassini spacecraft (Dyudinaet al.2009) have revealed the existence of an intense polar storm at Saturn's south pole. The storm is cyclonic and as noted in Polvani and Dritschel (1993) the vorticity structure of Saturn is such that the non-rotating system considered here may be, to first order, very relevant in studying the global dynamics of Saturn. Smith (1982) points to evidence suggesting that on Saturn there exists a so-called zonal"ribbon" with a strong vorticity gradient near the equator. The global circulation is anticyclonic to the north of the ribbon and cyclonic to its south resulting in the mean winds at the equator being generally to the east. There is therefore the possibility that there exists a strong cyclonic storm surrounded by a cyclonic sea i.e. the polar storm has the same-signed vorticity as its surrounding fluid. As discussed previously in this study, for a cyclonic southern cap, the south pole would be a stable equilibrium for a cyclonic storm, such as the one observed at Saturn's south pole. This result therefore offers a possible dynamic explanation explaining the presence and stability of Saturn's polar storm."

3.4.4 Black holes

Exert: "Applications of the Jimbo–Miwa–Ueno tau functions, closer in spirit to the topic of this paper, include the connection problem for the Heun differential equation, which was used to study scattering of scalar fields in **black hole** backgrounds [18,19] as well as the quantization of the Rabi model in quantum optics [20]."

All the above studies have relied heavily on complex analysis! There are many other applications. See some of Prof. Robert Zimmerman's rock mechanics papers, e.g. this one.

Along with applications such as those shown above, complex numbers often make the mathematics neater, and once familiar with them, simpler as we'll see later when covering, e.g. Fourier series.

You don't necessarily need to be and expert! Understanding the basics and potential applications is often enough - you then just need to find a friendly mathematician!

3.5 Arithmetic with complex numbers

3.5.1 Equality

The two complex numbers are only equal to each other if the real parts and the imaginary parts of two complex numbers are equal to each other.

$$x + iy = u + iv \Leftrightarrow x = u,\ y = v$$

3.5.2 Addition and subtraction

Addition and subtraction operate by adding like terms:

$$
\begin{aligned}
(x + iy) + (u + iv) &= (x + u) + i(y + v) \\
(x + iy) - (u + iv) &= (x - u) + i(y - v)
\end{aligned}
$$

3.5.3 Multiplication

Multiplication is achieved by expanding the brackets:

$$
\begin{aligned}
(x + iy)(u + iv) &= xu + i^2 yv + iyu + ixv \\
&= (xu - yv) + iyu + ixv
\end{aligned}
$$

Notice that yv is part of the real part of the product but both y and v belong to the imaginary part of the two operands.

There is a very special case of multiplication:

$$
\begin{aligned}
(x + iy)(x - iy) &= x^2 - i^2 y^2 + ixy - ixy \\
&= x^2 - i^2 y^2 \\
&= x^2 + y^2
\end{aligned}
$$

In this case, the multiplication of two complex numbers results in a real number. This can be very useful.

If $z = x + iy$ then the number $x - iy$ is called the **complex conjugate** of z and is written $\bar{z}$ or z^*. The complex conjugate is formed by changing the sign of the imaginary part of a complex number. As well as the addition property above, there are important additive properties of the complex conjugate:

$$
\begin{aligned}
z + \bar{z} &= (x + iy) + (x - iy) = 2x \\
z - \bar{z} &= (x + iy) - (x - iy) = 2iy
\end{aligned}
$$

3.5.4 Division

Division is achieved by using the complex conjugate to make the denominator real:

$$
\begin{aligned}
\frac{u + iv}{x + iy} &= \frac{u + iv}{x + iy}\frac{x - iy}{x - iy} \\
&= \frac{ux - vy + i(vx - uy)}{x^2 + y^2} \\
&= \frac{ux - vy}{x^2 + y^2} + i\frac{vx - uy}{x^2 + y^2}
\end{aligned}
$$

3.5.5 A complex arithmetic example

$$
\begin{aligned}
\left(\frac{1+7i}{1+2i} - (5-i)\right)^2 &= \left(\frac{1+7i}{1+2i}\frac{1-2i}{1-2i} - (5-i)\right)^2 \\
&= \left(\frac{1-14i^2+(7-2)i}{1+4} - (5-i)\right)^2 \\
&= \left(\frac{15+5i}{5} - (5-i)\right)^2 \\
&= (3+i-5+i)^2 \\
&= (-2+2i)^2 \\
&= 4-8i+4i^2 \\
&= 4-4-8i \\
&= -8i
\end{aligned}
$$

3.6 Argand diagrams and the polar representation of complex numbers

As a complex number represents a pair of numbers, one can also visualise it as a point on the plane. Representing complex numbers in this geometric fashion is of great practical importance.

An **Argand diagram** represents a complex number as a point on the Cartesian plane. The x axis shows the real part of the number while the imaginary part is shown on the y axis:

This allows for a different representation of z: instead of using Cartesian coordinates, we can specify z in terms of the polar coordinates r and θ:

To do this, we must express $z = x + iy$ in terms of (r, θ). Using Pythagoras' theorem we have:

$$
\begin{aligned}
r^2 &= x^2 + y^2 \\
r &= \sqrt{x^2 + y^2}
\end{aligned}
$$

The quantity r is termed the **modulus** of z, written $|z|$. Looking back to the complex conjugate $\bar{z} = x - iy$, it is clear that an equivalent definition of the modulus of z is:

$$|z| = \sqrt{\bar{z}z}$$

To find θ we use a little trigonometry:

$$
\begin{aligned}
\tan\theta &= y/x \\
\theta &= \tan^{-1}(y/x)
\end{aligned}
$$

θ is termed the **argument** of z and is written $\arg z$. Note that the *argument* is always represented in radians and is measured anticlockwise from the x axis. This definition represents a slight ambiguity as (r, θ) and $(r, \theta \pm 2\pi)$ represent the same point on the Argand diagram and hence the same complex number. To avoid this ambiguity, it is conventional to always specify $-\pi < \theta < \pi$. This value of θ, known as the **principle value** defines a unique relationship between z and (r, θ).

Of course it is also sometimes necessary to convert back from polar to Cartesian coordinates which can be achieved with a little more trigonometry:

$$x = r\cos\theta, \quad y = r\sin\theta \;\text{ and hence: }\; z = r(\cos\theta + i\sin\theta)$$

3.6.1 Polar coordinates example

Express $z = -2 + 2i$ in polar form.

$$r = \sqrt{x^2 + y^2} = \sqrt{4+4} = 2\sqrt{2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(-\frac{2}{2}\right) = \tan^{-1}(-1) = -\frac{\pi}{4} \quad (-45^\circ) \quad \text{or} \quad \frac{3\pi}{4} \quad (135^\circ)$$

There are two answers for θ because $\tan\theta$ is periodic with period π. It is therefore always necessary to check which quadrant the point lies in:

So we can see that $-2 + 2i$ is written as $(2\sqrt{2}, 3\pi/4)$ in polar coordinates. As a check, we can convert back to Cartesian coordinates:

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta) \\ &= 2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right) \\ &= 2\sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) \\ &= -2 + 2i \end{aligned}$$

3.6.2 Multiplication in polar coordinates

Let:

$$\begin{aligned} z_1 &= r_1(\cos\theta_1 + i\sin\theta_1) \\ z_2 &= r_2(\cos\theta_2 + i\sin\theta_2) \end{aligned}$$

So that:

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2 \left((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right) \\ &= r_1 r_2 \left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right) \end{aligned}$$

In other words, to multiply two complex numbers, we multiply the moduli and add the arguments. Note, we used **compound angle formulae** to consolidate the products of trigonometric functions into trigonometric functions of the angle sums.

3.6.3 Division in polar form

$$
\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2} \\
&= \frac{r_1}{r_2}\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)}{\cos^2\theta_2 + \sin^2\theta_2} \\
&= \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right)
\end{aligned}
$$

So to divide, we divide the moduli and subtract the arguments. Note that this makes division the inverse of multiplication. As before, we used **compound angle formulae** to consolidate the products of trigonometric functions into trigonometric functions of the angle differences.

3.7 The exponential representation of complex numbers

We may write the trigonometric functions $\sin\theta$ and $\cos\theta$ as power series (Maclaurin series):

$$
\begin{aligned}
\cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + ... \\
\sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + ...
\end{aligned}
$$

Using the relations $(i\theta)^2 = -\theta^2$, $(i\theta)^4 = \theta^4$, $(i\theta)^6 = -\theta^6$, ... we can rewrite these formula with only positive coefficients:

$$
\begin{aligned}
\cos\theta &= 1 + \frac{i^2\theta^2}{2!} + \frac{i^4\theta^4}{4!} + ... \\
\sin\theta &= \theta + \frac{i^2\theta^3}{3!} + \frac{i^4\theta^5}{5!} + ...
\end{aligned}
$$

This enables us to write the sum:

$$
\begin{aligned}
\cos\theta + i\sin\theta &= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + ... + \frac{i^n\theta^n}{n!} + ... \\
&= e^{i\theta}
\end{aligned}
$$

This very important relationship is known as **Euler's formula** after the 18th century Swiss mathematician Leonard Euler who published it in 1748. This establishes a third way of writing complex numbers:

$$
z = re^{i\theta}
$$

where r and θ have the same meaning as in the polar notation. Using the usual rules for multiplying exponents:

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

Once again we see that the moduli are multiplied while the arguments are added.

We can use $e^{i\theta}$ to demonstrate some more relationships:

$$\begin{aligned}
e^{-i\theta} &= \frac{1}{\cos\theta + i\sin\theta} \\
&= \frac{1}{\cos\theta + i\sin\theta}\frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta} \\
&= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} \\
&= \cos\theta - i\sin\theta
\end{aligned}$$

Observe that this last expression is the complex conjugate of $e^{i\theta}$.

Next:

$$\begin{aligned}
e^{i\theta} + e^{-i\theta} &= \cos\theta + i\sin\theta + \cos\theta - i\sin\theta \\
&= 2\cos\theta
\end{aligned}$$

and hence:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

By a similar process:

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

In this way we have arrived at a new definition of the trigonometric functions in terms of imaginary powers of e. Here one begins to see how complex numbers make the analysis of the physical world more powerful.

There is one particular value of θ which is of special note. Let $\theta = \pi$:

$$\begin{aligned}
e^{i\pi} &= \cos\pi + i\sin\pi \\
&= -1
\end{aligned}$$

This identity, which is known as **Euler's Identity**, is usually written:

$$e^{i\pi} + 1 = 0$$

This one formula links 5 of the most important numbers in mathematics.

3.7.1 Examples

Evaluate:

$$(4 + 3i)e^{i\pi/3}$$

First convert $4 + 3i$ to exponent form:

$$r = \sqrt{16 + 9} = 5$$
$$\theta = \tan^{-1}\left(\frac{3}{4}\right) \approx 0.6435$$

Now multiply:

$$5e^{0.6435i}e^{i\pi/3} = 5e^{1.6907i}$$

Transform back into $x + iy$ form:

$$\tan\theta = -8.3 = \frac{y}{x}$$
$$x^2 + y^2 = 25$$

which has solutions:

$$x = \pm 0.598$$
$$y = \mp 4.964$$

Now we note that $\theta = 1.6907 \approx 97^\circ$ is in the second quadrant so:

$$z = -0.598 + 4.964i$$

Evaluate:

$$\ln(-1)$$

First we convert to exponential notation. From Euler's identity we know:

$$e^{i\pi} = -1$$

From which:

$$\ln(-1) = \ln\left(e^{i\pi}\right) = i\pi$$

Once again we can see that complex numbers can give meaning to expressions, like $\ln(-1)$, which are otherwise undefined.

Material below this point is either non-examinable or will be covered in a future lecture.

3.8 Further reading

Consider the complex number $z = -i$. This can of course be expressed as $z = re^{i\theta}$ where $r = |z|$ and θ is the argument of the complex number ($\theta = \tan^{-1}(\mathrm{Im}(z)/\mathrm{Re}(z))$). Hence, here $z = e^{i3\pi/2}$.

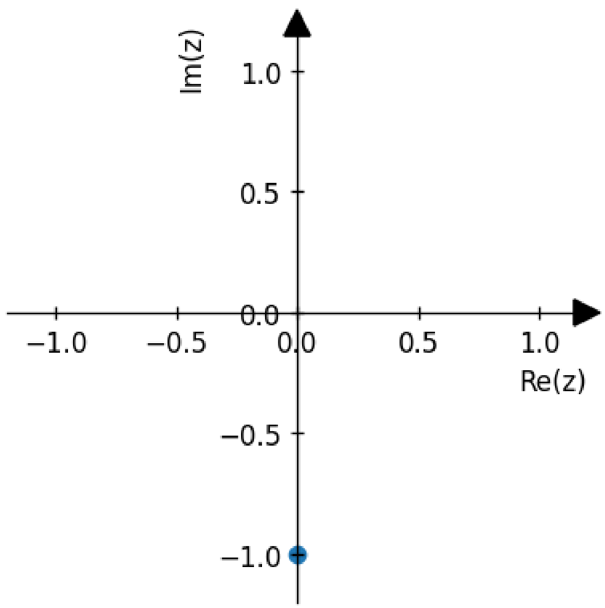
Lets plot this on an Argand diagram.

[1]:
```
# Define our complex number:
z = 0-1j
```

[2]:
```
# Define a plotter function:

%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt

def argand(z, xl=-1.2, xr=1.2, yl=-1.2, yr=1.2):

    rc = {"xtick.direction" : "inout", "ytick.direction" : "inout",
          "xtick.major.size" : 5, "ytick.major.size" : 5,}
    with plt.rc_context(rc):
        fig, ax = plt.subplots(figsize=(6,4), dpi= 120, facecolor='w',␣
 ↪edgecolor='k')
        ax.scatter(z.real,z.imag)

        ax.spines['left'].set_position('zero')
        ax.spines['right'].set_visible(False)
        ax.spines['bottom'].set_position('zero')
        ax.spines['top'].set_visible(False)
        ax.xaxis.set_ticks_position('bottom')
        ax.yaxis.set_ticks_position('left')
        ax.set_xlim([xl, xr])
        ax.set_ylim([yl, yr])
        ax.set_aspect('equal')
        ax.set_xlabel('Re(z)', loc='right')
        ax.set_ylabel('Im(z)', loc='top')
```

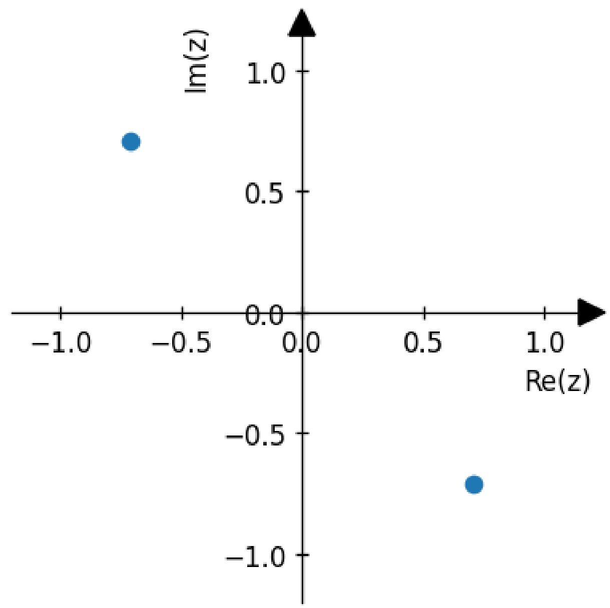
```
    # make arrows
    ax.plot((1), (0), ls="", marker=">", ms=10, color="k",
            transform=ax.get_yaxis_transform(), clip_on=False)
    ax.plot((0), (1), ls="", marker="^", ms=10, color="k",
             transform=ax.get_xaxis_transform(), clip_on=False)

    plt.show()
```

[3]:
```
# Plot:
argand(z)
```



Now, consider the operation $\sqrt{z} = \sqrt{r}e^{i\theta/2}$ which here would result in $z = e^{i3\pi/4}$. However, we can also write $z = -1i = e^{-i\pi/2}$ and hence $\sqrt{z} = e^{-i\pi/4}$. Despite taking the square root of the same number we've clearly ended up with two different solutions:

[4]:
```
argand(np.array([np.exp(-1j*np.pi/4), np.exp(3j*np.pi/4)]))
```



Notice that in fact be the solution can be written as $z = e^{i(3\pi/2+2n\pi)}$ for *any* (positive or negative) integer n. In general, this doesn't matter since owing to the 2π periodicity of the complex exponential different values of n will correspond to one of the two points on the Argand diagram above.

So when does this matter? Notice however what happens if we take log of our complex number:

$$\log(z) = \log(re^{i\theta}) = \log(r) + i\theta.$$

Whilst the real part $\log(r)$ is well determined it's clear that the complex part take a number of different values depending on how we choose θ. This is what's known as a multivalued function.

Something to think about: *What happens when we treat, e.g.,* $z^{\frac{1}{3}}$ *in a similar way?*

3.8.1 Which root should we choose?

If many applications we'll find ourselves taking logarithms of complex numbers as we did above, for example to model fluid flow in porous media. However, we need our solution to be unique for it to make sense physically. This is where the idea of 'branches' comes in!

By branch, we mean choosing θ to have continuous values over a range 2π. Common branch choices are $[-\pi, \pi)$ and $[0, 2\pi)$, but we can choose the branch as we like, e.g. $[-17.8\pi, -15.8\pi)$ is a potential choice. In reality, the problem we are solving will often dictate the choice we need to make i.e. only a certain choice will result in a physically permissible solution.

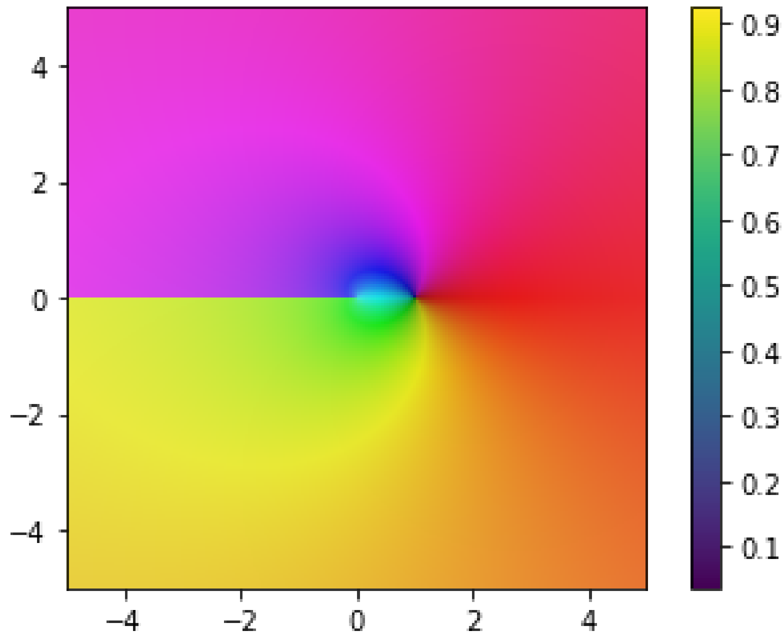
In the above case, if $z = -i$ and we set our branch to be $[-\pi, \pi)$, then the result of $\sqrt{z}$ is unique.

Note: The notation $[a, b)$ means that the 'lower limit' is included and the 'upper limit' is excluded. Hence $(a, b]$, which is also perfectly valid, would mean the opposite.

Lets look at a plot of $\log(z)$ to understand more about what's going on.

[5]:
```
# Helper function:

from colorsys import hls_to_rgb

def colorize(z):
    n,m = z.shape
    c = np.zeros((n,m,3))
    c[np.isinf(z)] = (1.0, 1.0, 1.0)
    c[np.isnan(z)] = (0.5, 0.5, 0.5)

    idx = ~(np.isinf(z) + np.isnan(z))
    A = (np.angle(z[idx]) + np.pi) / (2*np.pi)
    A = (A + 0.5) % 1.0
    B = 1.0 - 1.0/(1.0+abs(z[idx])**0.3)
    c[idx] = [hls_to_rgb(a, b, 0.8) for a,b in zip(A,B)]
    return c
```

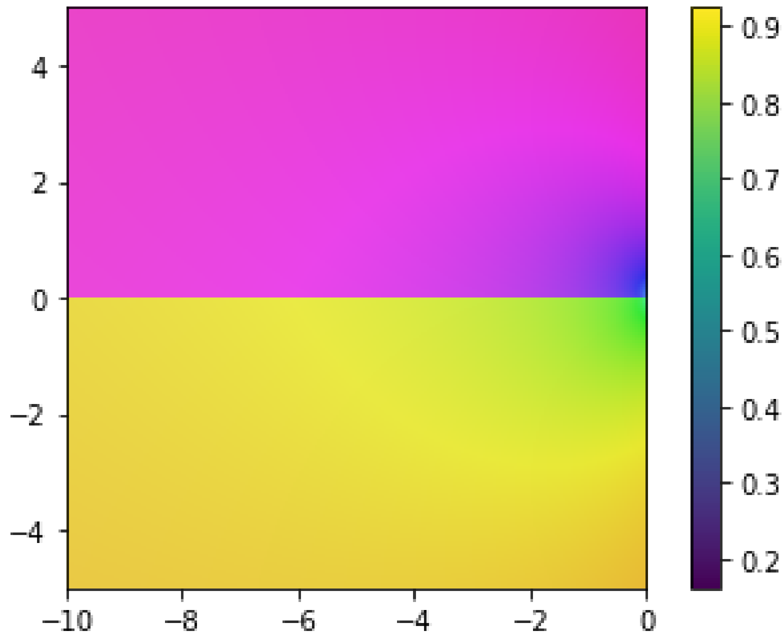
[6]:
```
# Make a grid and plot it:

N = 1000
A = np.zeros((N,N),dtype='complex')
axis_x = np.linspace(-5,5,N)
axis_y = np.linspace(-5,5,N)
X,Y = np.meshgrid(axis_x,axis_y)
Z = X + Y*1j

A = np.log(Z)

# Plot the array "A" using colorize
import pylab as plt
plt.imshow(colorize(A), interpolation='none',extent=(-5,5,-5,5))
plt.colorbar()
plt.show()
```



Notice the discontinuity along the negative real axis. Hence, by default Python has chosen our branch to be the $[-\pi, \pi)$ branch (or the $(-\pi, \pi]$ branch but it doesn't really for the point we're trying to illustrate).

Now, consider the following problem: our domain consists of the left half of the domain above (i.e. there's a solid wall along the Imaginary axis which extends into $x > 0$) and a 'jet' is located at $z = 0$ firing out e.g. air:

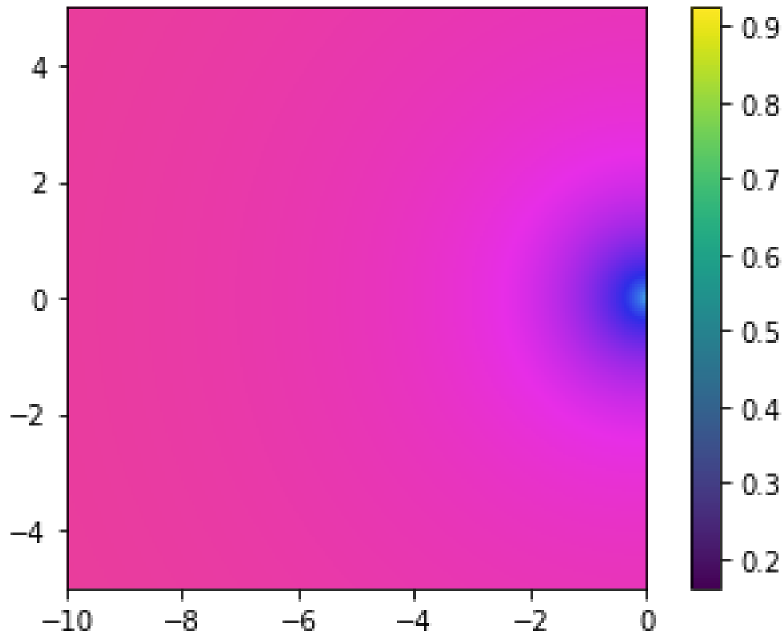
[7]:
```
N = 1000
A = np.zeros((N,N),dtype='complex')
axis_x = np.linspace(-10,0,N)
axis_y = np.linspace(-5,5,N)
X,Y = np.meshgrid(axis_x,axis_y)
Z = X + Y*1j

A = np.log(Z)

# Plot the array "A" using colorize
import pylab as plt
plt.imshow(colorize(A), interpolation='none',extent=(-10,0,-5,5))
plt.colorbar()
plt.show()
```



But what if some physical quantity* is related in $\mathrm{Im}(w(z))$? In our domain this quantity is discontinuous! But, if we choose our branch cut differently, e.g. to be the branch $[0, 2\pi)$ then we end up with the following:

(*In this particular case $\mathrm{Im}(w(z))$ will give the *streamlines* of the flow - *streamlines* are the trajectories that particles will follow).

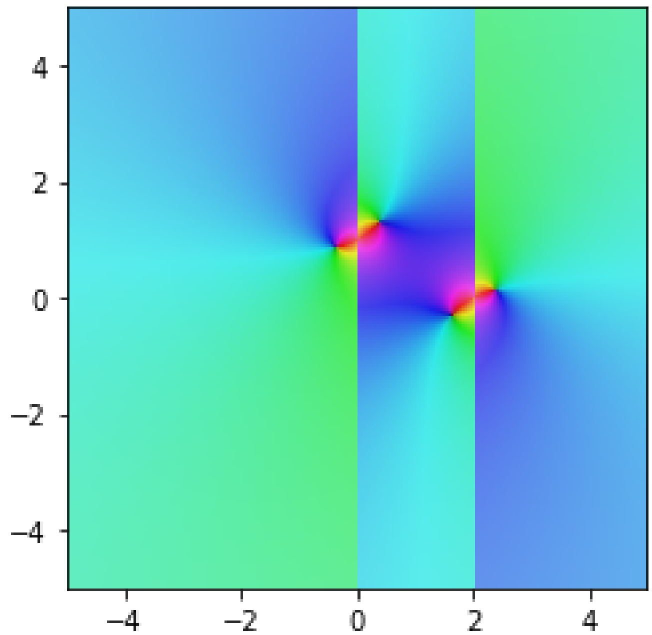
[8]:
```
A = np.log(np.abs(Z))+1j*np.arctan2(z.imag, z.real) # The arctan2 function␣
↪picks the [0,2\pi) by default

# Plot the array "A" using colorize
import pylab as plt
plt.imshow(colorize(A), interpolation='none',extent=(-10,0,-5,5))
plt.colorbar()
plt.show()
```



With this 'new' branch chosen our function is now continuous (no jump sudden jump in its real or imaginary component) within the physical domain.

Depending on what our complex potential is, things can get complicated:

[9]:
```
N = 1000
A = np.zeros((N,N),dtype='complex')
axis_x = np.linspace(-5,5,N)
axis_y = np.linspace(-5,5,N)
X,Y = np.meshgrid(axis_x,axis_y)
Z = X + Y*1j

A = np.log(1/(Z+1j)**2) + np.log(1/(Z-2)**2)

# Plot the array "A" using colorize
import pylab as plt
plt.imshow(colorize(A), interpolation='none',extent=(-5,5,-5,5))
plt.show()
```



Picking suitable branches can get messy but the general principles, outlined above, remain the same - it just becomes a ‘complicated’ book-keeping exercise.

Whilst branches of complex functions are not explicitly examinable in this course, as we’ll see later, appreciating their existence will help with some of the material we’ll see later!