Lecture 5 - Power series

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- Introduction to power series
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Power series, of some form or another, are ubiquitous throughout mathematics and engineering. They are among the most important mathematical tools you will ever learn and use. Their application and manipulation form the basis of numerical analysis. For example, manipulations of the Taylor provide the foundation of the **Finite difference** method (which you'll see in later courses) which is used very widely throughout academia and industry (seismic imaging, weather modelling, ocean dynamics, stresses on structures and in many, many, **MANY** more areas).

Further, they play an important role in solving differential equations. At the end of the previous lecture we began to consider ODEs of the form:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}y}{\mathrm{d}x} + c(x)y = 0.$$

We examined a particular case where $b(x)$ and $c(x)$ were of a form that allowed us to, via substitution, transform the equation into an ODE with constant coefficients (which we know how to solve). In general, things aren't quite so easy.

A more general approach is to expand the function y as a power series and solve for the coefficients of this series. These types of solution are known as 'series solutions' of ODEs. Symbolic computation packages such as `SymPy`, `Mathematica` etc. make use of these ideas.

Power series expansions

A power series expansion of a function is a representation of that function as an infinite polynomial:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

frequently written in the following formal notation:

$$f(x)=\sum_{k=0}^{\infty} a_k x^k \tag{1}$$

The a_n are constants (and hence do not vary with x).

A power series representation of f is only valid for values of x for which the series converges. It is possible to test for the convergence of a series using the ratio test, otherwise known as d'Alembert's test (after the 18th century French mathematician Jean de Rond d'Alembert who originally published it). The test establishes the following sufficient test for convergence:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right|<1$$

or equivalently:

$$|x|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$$

Hence the convergence of the power series depends not only on the coefficients a_n, it is also linear in the magnitude of x. This enables us to define a new parameter R, the radius of convergence. The convergence criteria now becomes:

$$|x|<R \quad\iff\quad -R<x<R$$

where:

$$R\equiv\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

The radius of convergence of a power series is preserved under addition, subtraction, multiplication and differentiation. That is to say, if two power series with the same radius of convergence are added, subtracted or multiplied then the resulting power series has the same radius of convergence. Similarly, the derivative of a power series is a power series with the same radius of convergence.

Taylor and Maclaurin Series

Consider a power series expressed in a slightly different way:

$$f(x)=b_0+b_1(x-x_0)+b_2(x-x_0)^2+\ldots+b_n(x-x_0)^n+\ldots$$

Here x_0 is some fixed value. Note that:

$$f(x_0)=b_0$$

Differentiating the series we have:

$$f'(x) = b_1 + 2b_2(x - x_0) + 3b_3(x - x_0)^2 \ldots + nb_n(x - x_0)^{n-1} + \ldots$$

so that:

$$f'(x_0) = b_1$$

If we differentiate again, we have:

$$f''(x) = 2b_2 + 2 \times 3b_3(x - x_0) \ldots + (n-1) \times nb_n(x - x_0)^{n-2} + \ldots$$

yielding:

$$f''(x_0) = 2b_2$$

similarly:

$$\begin{aligned} f'''(x_0) &= 3 \times 2b_3 \\ &= 3!\, b_3 \end{aligned}$$

In general:

$$f^{(n)}(x_0) = n!\, b_n$$

This gives us a new way of representing a function:

$$\begin{aligned} f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\ = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \end{aligned}$$

provided, of course, that the series converges.

This means we have an algorithm for expressing $f(x)$ as a power series subject to the radius of convergence and assuming that the derivatives $f^{(n)}(x)$ are continuous within that radius of convergence.

This form of power series is known as a Taylor series. It is sometimes written in an alternative form replacing x by $x + x_0$:

$$f(x + x_0) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} x^k$$

A special case of the Taylor series is achieved by setting $x_0 = 0$:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

This form of Taylor series is referred to as a MacLaurin series.

Example: e^x

Expand $f(x) = e^x$ around $x = 0$.

e^x is a special function in that:

$$f(x) = f'(x) = f''(x) = f^{(n)}(x) = e^x$$

so at $x = 0$ all the derivatives are 1. This produces the series:

$$\begin{aligned} e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \end{aligned}$$

The radius of convergence for this series is:

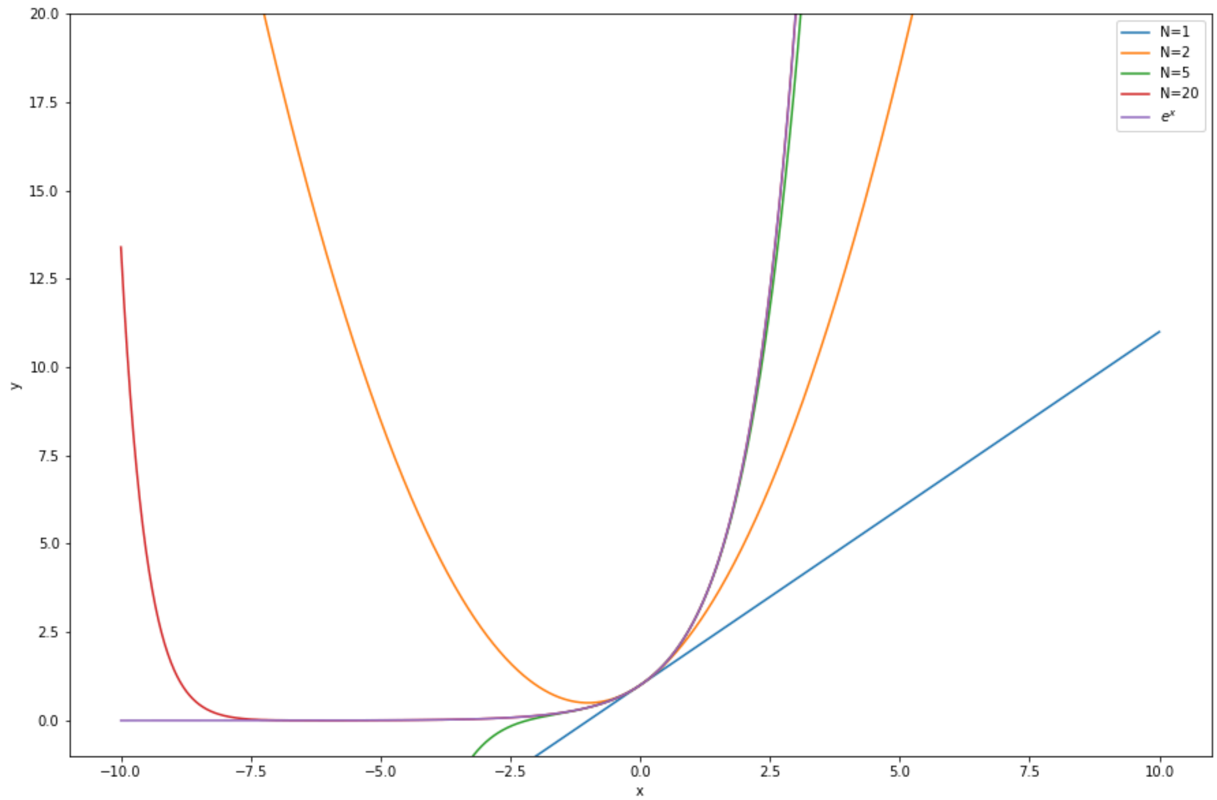
$$\begin{aligned} R &= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \\ &= \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \right| \\ &= \lim_{n\to\infty} n+1 \\ &= \infty \end{aligned}$$

That is to say, the expansion converges for any x.

Lets plot various orders of the expansion to `NumPy`'s implementation: (**Food for thought:** How do you think `NumPy` computes e^x?)

In [1]:
```
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt

N = 1000
x = np.linspace(-10,10,N)

def eval_e(x, N):
    e = 0
    for i in range(N+1):
        e+=x**i/np.math.factorial(i)
    return e
```

```
y1 = eval_e(x, 1)
y2 = eval_e(x, 2)
y3 = eval_e(x, 5)
y4 = eval_e(x, 20)
y5 = np.exp(x)

plt.rcParams['figure.figsize'] = [15, 10]
plt.plot(x, y1, label = "N=1")
plt.plot(x, y2, label = "N=2")
plt.plot(x, y3, label = "N=5")
plt.plot(x, y4, label = "N=20")
plt.plot(x, y5, label = "$e^x$")
plt.xlabel("x")
plt.ylabel("y")
plt.ylim((-1, 20))
plt.legend()
plt.show()
```



Example: $\cos x$

Expand $f(x) = \cos x$ around $x = 0$.

In this case we have:

$$
\begin{aligned}
f'(x) &= -\sin x & f''(x) &= -\cos x \\
f'''(x) &= \sin x & f''''(x) &= \cos x \\
f^{(5)}(x) &= -\sin x & f^{(6)}(x) &= -\cos x \\
f^{(7)}(x) &= \sin x & f^{(8)}(x) &= \cos x
\end{aligned}
$$

So at $x = 0$ all the odd derivatives are 0 while the even derivatives alternate between -1 and 1. This produces:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

once again, the radius of convergence is ∞. By a similar process:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Now, consider the Taylor series of $g(x) = e^{ix}$. We have

$$\begin{aligned} g'(x) &= ie^{ix} & g''(x) &= -e^{ix} \\ g'''(x) &= -ie^{ix} & g''''(x) &= e^{ix} \\ g^{(5)}(x) &= ie^{ix} & g^{(6)}(x) &= -e^{ix} \\ g^{(7)}(x) &= -ie^{ix} & g^{(8)}(x) &= e^{ix} \end{aligned}$$

and hence

$$e^{ix} \approx 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \dots$$

and notice that

$$\cos x + i \sin x \approx 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \dots.$$

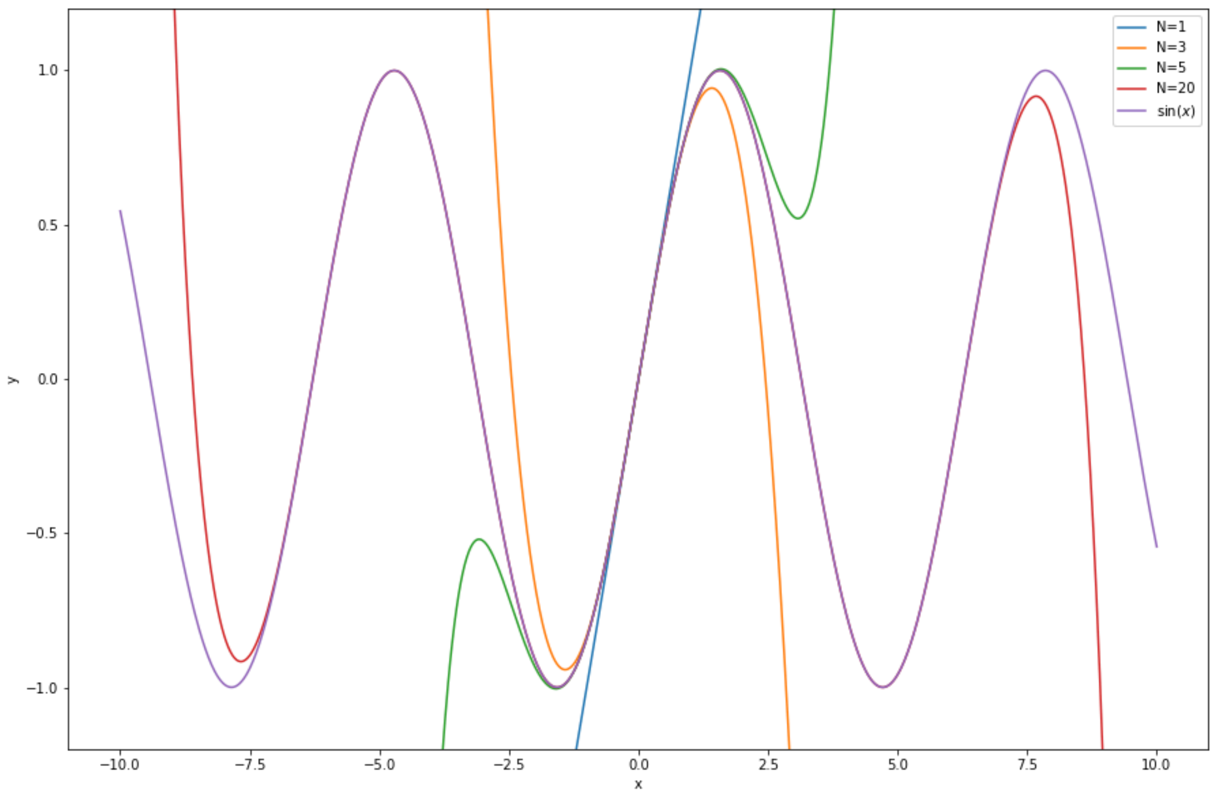
These are clearly equal and hence we have the important identity that we have used previously

$$e^{ix} = \cos x + i \sin x.$$

Lets again look at some plots for the $\sin$ series as we increase the number of terms present:

In [2]:
```
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt

N = 1000
x = np.linspace(-10,10,N)

def eval_s(x, N):
    s = 0
    for i in range(1,N+1,2):
        s+=-1j**(i+1)*x**i/np.math.factorial(i)
    return np.real(s)

y1 = eval_s(x, 1)
```

```
y2 = eval_s(x, 3)
y3 = eval_s(x, 5)
y4 = eval_s(x, 20)
y5 = np.sin(x)

plt.rcParams['figure.figsize'] = [15, 10]
plt.plot(x, y1, label = "N=1")
plt.plot(x, y2, label = "N=3")
plt.plot(x, y3, label = "N=5")
plt.plot(x, y4, label = "N=20")
plt.plot(x, y5, label = "$\sin(x)$")
plt.xlabel("x")
plt.ylabel("y")
plt.ylim((-1.2, 1.2))
plt.legend()
plt.show()
```



Example: $\ln x$

Since $\ln x$ is not defined at $x = 0$, there is no MacLaurin series for this function. Instead, produce a Taylor series for $\ln x$ around $x = 1$.

In this case we have the following derivatives:

$$\begin{aligned} f'(x) &= \frac{1}{x} \\ f''(x) &= -\frac{1}{x^2} \\ f'''(x) &= \frac{2}{x^3} \\ f''''(x) &= -\frac{3 \times 2}{x^4} \\ f^{(n)}(x) &= (-1)^{n-1} \frac{(n-1)!}{x^n} \end{aligned}$$

So that at $x = 1$ we have:

$$\begin{aligned} f(1) &= 0 \\ f'(1) &= 1 \\ f''(1) &= -1 \\ f'''(1) &= 2 \\ f''''(1) &= -6 \\ &\vdots \end{aligned}$$

This leads us to:

$$\begin{aligned} \ln x &= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(k-1)!}{k!} (x-1)^k \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(x-1)^k}{k} \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \dots \end{aligned}$$

To calculate the radius of convergence, we make the substitution $t = x - 1$ to convert the series into the required form:

$$\ln(t+1) = t - \frac{t^2}{2} + \frac{t^3}{3} - \dots + (-1)^{n-1} \frac{t^n}{n}$$

This has radius of convergence:

$$\begin{aligned} R &= \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| \\ &= \lim_{n \to \infty} \left| \frac{n+1}{n} \right| \\ &= 1 \end{aligned}$$

So the series is shown to converge for $-1 < t < 1$ or $0 < x < 2$.

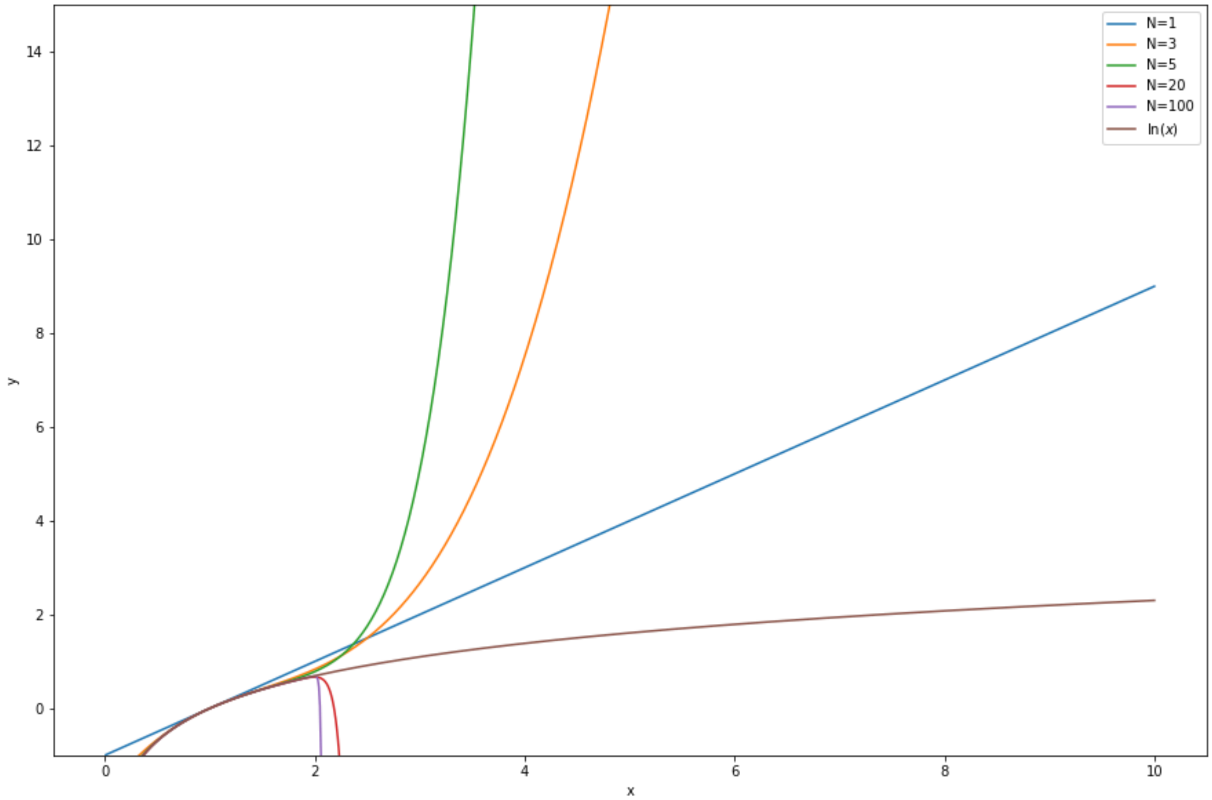
For example we can evaluate $\ln(1.1)$:

$$\begin{aligned}\ln(1.1) &= 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} + \dots \\ &\approx 0.095308\end{aligned}$$

In fact the first 6 decimal places of the full series are 0.095310. The error in this case is the result of summing insufficiently many terms for the precision required.

And again, lets plot the Taylor series of $\ln(x)$ expanded around $x = 1$ for various orders:

In [3]:
```
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt

N = 1000
x = np.linspace(0.01,10,N)

def eval_ln(x, N):
    s = 0
    for i in range(1,N+1):
        s+=(-1)**(i+1)*(x-1)**i/i
    return s

y1 = eval_ln(x, 1)
y2 = eval_ln(x, 3)
y3 = eval_ln(x, 5)
y4 = eval_ln(x, 20)
y5 = eval_ln(x, 100)
y6 = np.log(x)

plt.rcParams['figure.figsize'] = [15, 10]
plt.plot(x, y1, label = "N=1")
plt.plot(x, y2, label = "N=3")
plt.plot(x, y3, label = "N=5")
plt.plot(x, y4, label = "N=20")
plt.plot(x, y5, label = "N=100")
plt.plot(x, y6, label = "$\ln(x)$")
plt.xlabel("x")
plt.ylabel("y")
plt.ylim((-1., 15))
plt.legend()
plt.show()
```



Evaluating series limits with L'Hôpital's rule

Suppose we have two continuous and differentiable functions which both vanish at a. That is:

$$\lim_{x \to a} f(x) = 0$$
$$\lim_{x \to a} g(x) = 0$$

What is:

$$\lim_{x \to a} \frac{f(x)}{g(x)}?$$

A classic example of this is

$$\lim_{x \to 0} \frac{\sin(x)}{x},$$

which we'll look at below. But first, let's expand f and g as Taylor series around a :

$$
\begin{aligned}
\lim_{x \to a} \frac{f(x)}{g(x)} &= \lim_{x \to a} \frac{f(a) + (x-a)f'(a) + (x-a)^2 f''(a)/2 + \dots}{g(a) + (x-a)g'(a) + (x-a)^2 g''(a)/2 + \dots} \\
&= \lim_{x \to a} \frac{0 + f'(a) + (x-a)f''(a)/2 + \dots}{0 + g'(a) + (x-a)g''(a)/2 + \dots} \\
&= \lim_{x \to a} \frac{f'(x)}{g'(x)}
\end{aligned}
$$

The result is known as **L'Hôpital's rule**, although historically also attributed to Johann Bernoulli:

$$
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{2}
$$

If the $f'(a)$ and $g'(a)$ are also both zero, then the same process continues and the second or subsequent derivatives can be used:

$$
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)} = \lim_{x \to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}
$$

More broadly, L'Hôpital's rule rule can be used when the limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ is of the $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ indeterminate forms. A general proof was derived by Taylor: https://doi.org/10.2307%2F2307183.

Example: $\sin(x)/x$

Evaluate:

$$
\lim_{x \to 0} \frac{\sin x}{x}
$$

Using L'Hospital's rule. We have:

$$
\begin{aligned}
\lim_{x \to 0} \frac{\sin x}{x} &= \lim_{x \to 0} \frac{\cos x}{1} \\
&= 1
\end{aligned}
$$

In longhand, by expanding $\sin x$ as a Taylor series about 0 we have:

$$
\begin{aligned}
\lim_{x \to 0} \frac{\sin x}{x} &= \lim_{x \to 0} \frac{x - x^3/3! + x^5/5! - \dots}{x} \\
&= \lim_{x \to 0} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots \\
&= 1
\end{aligned}
$$

Example: $\ln(x)/(x^2-1)$

Evaluate:

$$\lim_{x \to 1} \frac{\ln x}{x^2 - 1}$$

Using L'Hospital's rule we have:

$$\begin{aligned} \lim_{x \to 1} \frac{\ln x}{x^2 - 1} &= \lim_{x \to 1} \frac{1}{x} \times \frac{1}{2x} \\ &= \lim_{x \to 1} \frac{1}{2x^2} \\ &= \frac{1}{2} \end{aligned}$$