Exercise 1:

1. Compute the Maclaurin series of $1/(1+x)$ and $1/(1-x)$.
2. Hence, or otherwise, compute the series for $x/(1-x^2)$.
3. Repeat 1. and 2. but now finding the Taylor series for when $x_0 = 2$.

Solution 1

Let $g(x) = 1/(1+x)$. Differentiating this a few times we see:

$$\begin{aligned} g'(x) &= -1/(1+x)^2 & g''(x) &= 2/(1+x)^3 \\ g'''(x) &= -6/(1+x)^4 & g^{(n)}(x) &= (-1)^n n!/(1+x)^{n+1} \end{aligned}$$

Similarly, letting $h(x) = 1/(1-x)$ we get

$$\begin{aligned} h'(x) &= 1/(1-x)^2 & h''(x) &= 2/(1-x)^3 \\ h'''(x) &= 6/(1-x)^4 & h^{(n)}(x) &= n!/(1-x)^{n+1} \end{aligned}$$

When the expansion point $x_0 = 0$

$$\begin{aligned} g(0) &= 1 & h(0) &= 1 \\ g'(0) &= -1 & h'(0) &= 1 \\ g''(0) &= 2 & h''(0) &= 2 \\ g'''(0) &= -6 & h'''(0) &= 6 \\ g^{(n)}(0) &= (-1)^n n! & h^{(n)}(0) &= n! \end{aligned}$$

$$g(x) = 1 - x + x^2 - x^3 + x^4 + \ldots,$$
$$h(x) = 1 + x + x^2 + x^3 + x^4 + \ldots.$$

Solution 2

First, to make our lives easier lets split up our function as:

$$f(x) = \frac{x}{1-x^2} = \frac{x}{(1+x)(1-x)} = \frac{x}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right).$$

and hence

$$f(x) = \frac{x}{2}(g(x) + h(x)),$$

and plugging in the above two 'sub-expansions' from 1. we get:

$$f(x) = \frac{x}{2}\left(1 - x + x^2 - x^3 + x^4 + \ldots + 1 + x + x^2 + x^3 + x^4 + \ldots\right) = \frac{x}{2}\left(2\sum_{n=0}^{\infty} x^{2n}\right) = \sum_{n=0}^{\infty} x^{2n+1}$$

Solution 3

Plugging $x_0 = 2$ in to $g(x)$, $h(x)$ and their derivatives we get

$$\begin{aligned} g(2) &= 1/3 & h(2) &= -1 \\ g'(2) &= -1/9 & h'(2) &= 1 \\ g''(2) &= 2/27 & h''(2) &= -2 \\ g'''(2) &= -6/81 & h'''(2) &= 6 \\ g^{(n)}(2) &= (-1)^n n!/3^{n+1} & h^{(n)}(2) &= (-1)^{n+1} n! \end{aligned}$$

$$g(x) = \frac{1}{3} - \frac{1}{9}(x-2) + \frac{1}{27}(x-2)^2 - \frac{1}{81}(x-2)^3 + \ldots,$$
$$h(x) = -1 + (x-2) - (x-2)^2 + (x-2)^3 + \ldots.$$

Combining our series, the expansion for $f(x)$ about $x_0 = 2$ can be written as

$$f(x) = \frac{x}{2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}(x-2)^n + \sum_{n=0}^{\infty}(-1)^{n+1}(x-2)^n\right)$$

A note on the radii of convergence

Recall that the *Radius of convergence* is defined as

$$R \equiv \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

and based on the value of R we can expect our series expansion to converge between:

$$x_0 - R < x_0 < x_0 + R.$$

$x_0 = 0$:

For each of our expansions $g(x)$ and $h(x)$ $R = 1$ (since $|a_n| = 1$ for all n). Hence the radius of convergence for $f(x)$ will also be 1. We therefore expect out expansion to converge between -1 and 1.

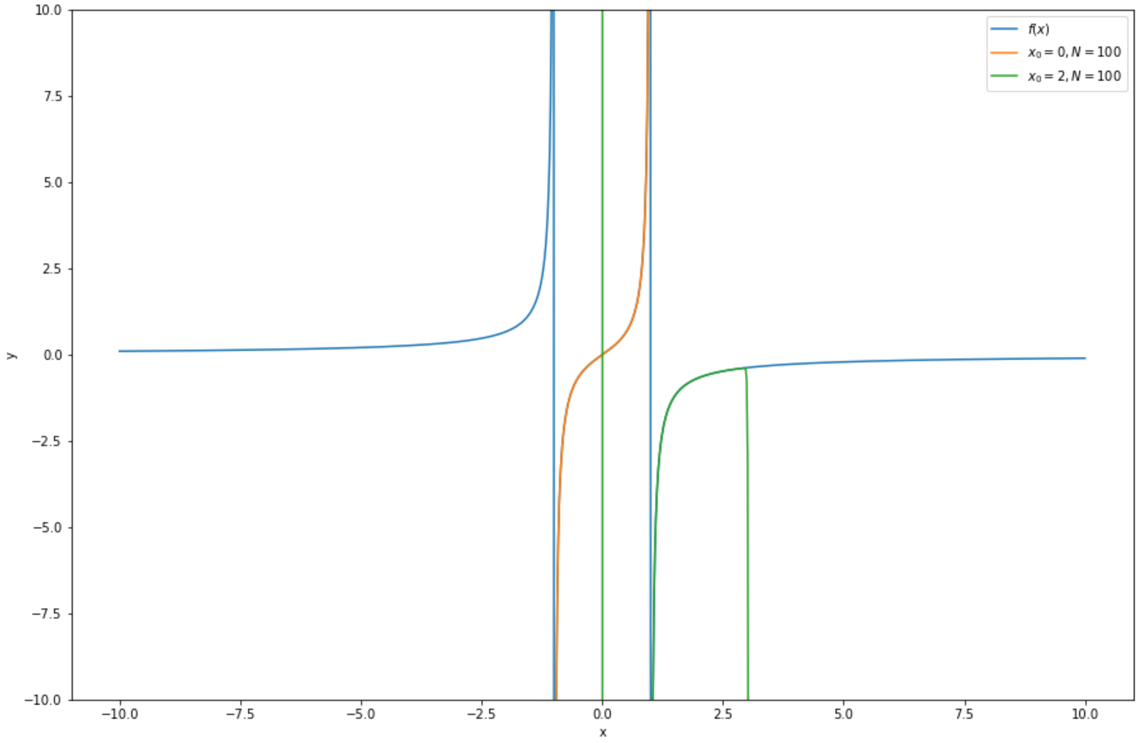
$x_0 = 2$:

I'll leave this case to you for the time being.

Let us however explore the expansion vs. the true function in the two cases above:

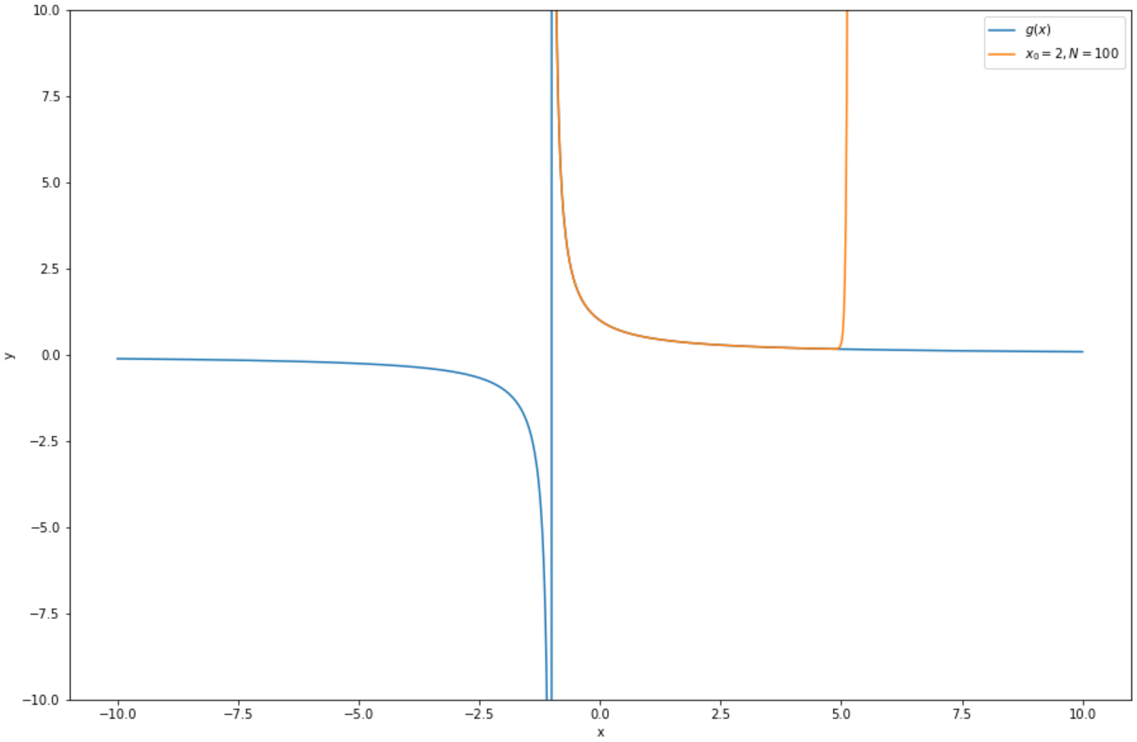
In [1]:

```
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt

N = 1000
x = np.linspace(-10,10,N)

def eval_f(x):
    return x/(1-x**2)

def eval_fe0(x, N):
    e = 0
    for i in range(N):
        e+=x**(2*i+1)
    return e

def eval_fe2(x, N):
    e = 0
    for i in range(N+1):
        e+=(-1)**i/3**(i+1)*(x-2)**i+(-1)**(i+1)*(x-2)**i
    e = e*x/2
    return e

y1 = eval_f(x)
y2 = eval_fe0(x, 100)
y3 = eval_fe2(x, 100)

plt.rcParams['figure.figsize'] = [15, 10]
plt.plot(x, y1, label = "$f(x)$")
plt.plot(x, y2, label = "$x_0=0, N=100$")
plt.plot(x, y3, label = "$x_0=2, N=100$")
plt.xlabel("x")
plt.ylabel("y")
plt.ylim((-10, 10))
plt.legend()
plt.show()
```



We see that our expansion about $x_0 = 0$ is performing well between roughly -1 and 1 and our expansion about $x_0 = 2$ between roughly 1 and 3. Note, that if we look at our expansions about $x_0 = 2$ individually we see the following:

In [2]:
```
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt

N = 1000
x = np.linspace(-10,10,N)

def eval_g(x):
    return 1/(1+x)

def eval_ge2(x, N):
    e = 0
    for i in range(N+1):
        e+=(-1)**i/3**(i+1)*(x-2)**i
    return e

y1 = eval_g(x)
y2 = eval_ge2(x, 100)

plt.rcParams['figure.figsize'] = [15, 10]
plt.plot(x, y1, label = "$g(x)$")
plt.plot(x, y2, label = "$x_0=2, N=100$")
plt.xlabel("x")
plt.ylabel("y")
plt.ylim((-10, 10))
plt.legend()
plt.show()
```



In [3]:
```python
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt

N = 1000
x = np.linspace(-10,10,N)

def eval_h(x):
    return 1/(1-x)

def eval_he2(x, N):
    e = 0
    for i in range(N+1):
        e+=(-1)**(i+1)*(x-2)**i
    return e

y1 = eval_h(x)
y2 = eval_he2(x, 100)

plt.rcParams['figure.figsize'] = [15, 10]
plt.plot(x, y1, label = "$h(x)$")
plt.plot(x, y2, label = "$x_0=2, N=100$")
plt.xlabel("x")
plt.ylabel("y")
plt.ylim((-10, 10))
plt.legend()
plt.show()
```