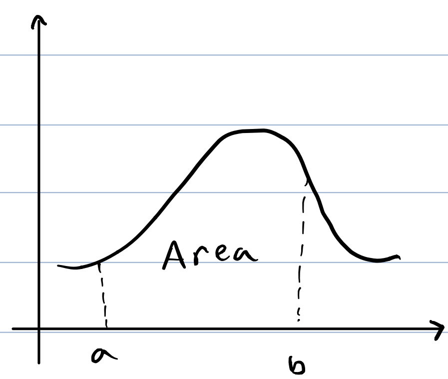
1 - Riemann Sums

Jan. 18

Objective: Find the area under a curve (and above the x-axis)

How do we find area?

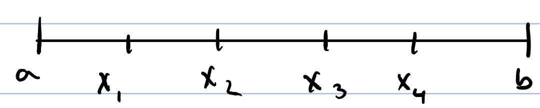
Suppose f is a cont's f^{cn} over $[a, b]$...

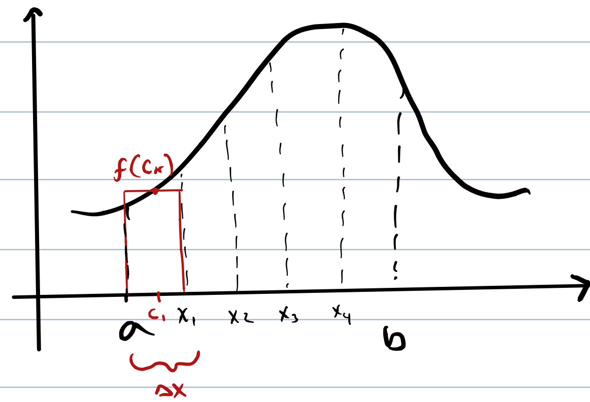


How would we evaluate this area?

Riemann approximation method

→ Divide $[a,b]$ into smaller intervals





→ Determine the width of each subinterval

for convenience: create equal subdivisions Δx

$$\Delta x = \frac{b-a}{n}$$

Δx will become the base of each rectangle

→ chose a point, c_k, within each subinterval, and use $f(c_k)$ as the rectangle's height

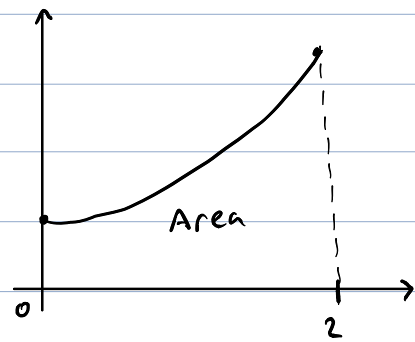
→ Area $\approx f(c_1)\,\Delta x + f(c_2)\,\Delta x + \ldots f(c_n)\,\Delta x$

$$\text{Area} \approx \sum f(c_k)\,\Delta x$$

↑ sigma (sum of)

ex. consider $f(x) = x^2 + 1$ over $[0, 2]$

Use 4 rectangles to estimate the area between $f(x)$ and the x-axis within $[0, 2]$.



① $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$

② Select pts from each subinterval

$[0, 0.5] \rightarrow c_1 = 0$ (choosing left bound)

$[0.5, 1] \rightarrow c_2 = 0.5$ (")

$[1, 1.5] \rightarrow c_3 = 1$

$[1.5, 2] \rightarrow c_4 = 1.5$

$f(c_1) = f(0) = 1$

$f(c_2) = f(0.5) = 1.25$

$f(c_3) = f(1) = 2$

$f(c_4) = f(1.5) = 3.25$

③ $\text{Area} \approx \sum f(c_k)\,\Delta x$

$= 1(0.5) + 1.25(0.5) + 2(0.5) + 3.25(0.5)$

Area ≈ 3.75

(underestimate of actual area)

Using right-bound : Area ≈ 5.75

(everything gets shifted over)

If the number of subdivisions is large, the pt you pick doesn't matter

Sigma Notation

Jan. 19

= a short-hand notation to describe a sum (finite or infinite)

upper bound of summation

$$\sum_{k=1}^{n} a_k$$

Summation symbol

lower bound of summation

terms of the sum

ex.

$$\sum_{k=1}^{5} (3k+2) = \underbrace{5 + 8 + 11 + 14 + 17}_{\text{expanded form}}$$

ex.

$$\sum_{k=1}^{5} (3a+2) = (3a+2) + (3a+2) + \underset{5 \text{ times}}{\cdots}$$

ex. Write the following series of terms in sigma form:

$$2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$$

$$\sum_{k=0}^{5} 2\left(\frac{1}{2}\right)^{k} \quad \text{or} \quad \sum_{k=0}^{5} \frac{1}{16}(2)^{k}$$

Summation Formulas

① $$\sum_{k=1}^{n} (1) = 1 + 1 + 1 + \dots + 1 = n$$

② $$\sum_{k=1}^{n} k = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

③ $$\sum_{k=1}^{n} k^2 = 1 + 4 + 9 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

④ $$\sum_{k=1}^{n} k^3 = 1 + 8 + 27 + 64 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

ex. $$\sum_{k=1}^{32} (1) = 32$$

ex. $\displaystyle\sum_{k=1}^{8} k = \frac{8(9)}{2} = 36$

ex. $\displaystyle\sum_{k=1}^{5} k^3 = \frac{5^2(6)^2}{4} = \frac{(25)(36)}{4} = 225$

ex. $\displaystyle\sum_{k=5}^{14} k^2 = \sum_{k=1}^{14} k^2 - \sum_{k=1}^{4} k^2$

$$= \frac{(14)(15)(29)}{6} - \frac{(4)(5)(9)}{6}$$

$$= 985$$

Summation Properties

① Constant multiple rule

$$\sum_{k=1}^{n} c a_k = c \sum_{k=1}^{n} a_k$$

② Addition / subtraction rule

$$\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$$

ex.

$$\sum_{m=1}^{200} (2m^2 - 10m)$$

$$= \sum_{m=1}^{200} 2m^2 - \sum_{m=1}^{200} 10m$$

$$= 2\sum_{m=1}^{200} m^2 - 10\sum_{m=1}^{200} m$$

$$= 2\left[\frac{200(201)(401)}{6}\right] - 10\left[\frac{200(201)}{2}\right]$$

$$= 5172400$$

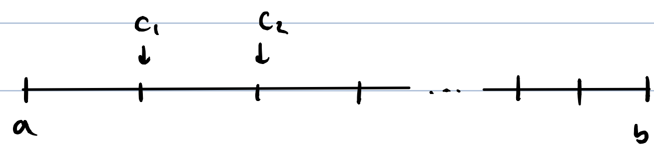
Sigma Notation and Riemann Sums

Recall that the area between a continuous function $f(x)$ and the x-axis within $[a,b]$ can be estimated by

$$\text{Area} \approx \sum_{k=1}^{n} f(c_k)\,\Delta x$$

Step 1 Find Δx ... $\Delta x = \frac{b-a}{n}$

Step 2 For convenience, chose all c_k's to be the right-hand pts of the subintervals



$c_1 = a + \Delta x$

$c_2 = a + 2\Delta x$

$C_3 = a + 3\Delta x$

$C_k = a + k\Delta x$

Step 3 Find an expression for $f(C_k)$

Step 4 Evaluate $\sum_{k=1}^{n} f(C_k)\,\Delta x$

using summation formulas and rules

ex. Apply Riemann Sums to $f(x) = x^2 + 1$ over $[0, 2]$ to estimate its area

① $\Delta x = \frac{2-0}{n} = \frac{2}{n}$

② $C_k = 0 + k\,\Delta x$

$= k \cdot \frac{2}{n}$

③ $f(x) = x^2 + 1$

So $f(C_k) = C_k^{\,2} + 1$

$= \left(k \cdot \frac{2}{n}\right)^2 + 1$

$f(C_k) = \frac{4k^2}{n^2} + 1$

④ $\text{Area}_{[0,2]} \approx \sum_{k=1}^{n} f(c_k)\,\Delta x$

$$\approx \sum_{k=1}^{n} \left(\frac{4k^2}{n^2} + 1 \right) \cdot \frac{2}{n}$$

$$\approx \sum_{k=1}^{n} \left(\frac{8}{n^3} k^2 + \frac{2}{n} \right)$$

$$\approx \sum_{k=1}^{n} \frac{8}{n^3} k^2 + \sum_{k=1}^{n} \frac{2}{n}$$

$$\approx \frac{8}{n^3} \sum_{k=1}^{n} k^2 + \frac{2}{n} \sum_{k=1}^{n} 1$$

n is not a variable

$$\approx \frac{8}{n^{\cancel{3}2}} \frac{\cancel{n}(n+1)(2n+1)}{6} + \frac{2}{\cancel{n}} \cdot \cancel{n}$$

$$\approx \frac{4}{3} \left(\frac{n+1}{2} \right) \left(\frac{2n+1}{n} \right) + 2$$

$$\approx \frac{4}{3} \left(1 + \frac{1}{n} \right) \left(2 + \frac{2}{n} \right) + 2$$

If $n = 4$... $A_{[0,2]} \approx \frac{4}{3} \left(1 + \frac{1}{4} \right) \left(2 + \frac{1}{4} \right) + 2$

$$\approx 5.75$$

Limit of a Riemann Sum

Jan. 23

Riemann's Principle :

According to Riemann, $\sum_{k=1}^{n} f(c_k)\Delta x$ converges towards one specific value regardless of the selected pts, c_k, as long as $\Delta x \to 0$ (or as long as $n \to \infty$)

(**N.B.** : Riemann's principle is true for all cont's fcns over an interval $[a, b]$)

The result of the limit is denoted by...

Signed area

→ $S.A._{[a,b]}$

→ $\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x$

Sum

→ $\int_a^b f(x)\,dx$

ex. Use a limit process (as $n \to \infty$) to evaluate the exact the exact area* between $f(x) = 4x - x^2$ over the interval $[0, 10]$

① $\Delta x = \frac{b-a}{n} = \frac{10-0}{n} = \frac{10}{n}$

② $C_k = a + k\Delta x$

$= 0 + k \cdot \frac{10}{n} = \frac{10}{n}k$

③ $f(C_k) = 4\left(\frac{10}{n}k\right) - \left(\frac{10}{n}k\right)^2$

$= \frac{40}{n}k - \frac{100}{n^2}k^2$

④ Area $\approx \sum_{k=1}^{n} f(c_k)\,\Delta x$

$$\approx \sum_{k=1}^{n} \left(\frac{40}{n}k - \frac{100}{n^3}k^2\right) \cdot \left(\frac{10}{n}\right)$$

$$= \sum_{k=1}^{n} \left(\frac{400}{n^2}k - \frac{1000}{n^3}k^2\right)$$

$$= \frac{400}{n^2}\sum_{k=1}^{n} k - \frac{1000}{n^3}\sum_{k=1}^{n} k^2$$

$$= \frac{\overset{2}{\cancel{4}}00}{n^{\cancel{2}}} \cdot \frac{\cancel{n}(n+1)}{\cancel{2}} - \frac{1000}{n^{\cancel{3}2}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{6}$$

$$= 200\left(\frac{n+1}{n}\right) - \frac{500}{3}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

$$= 200\left(1 + \frac{1}{n}\right) - \frac{500}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

⑤ $SA_{[0,10]}$ or $\int_0^{10} f(x)\,dx$

$$= \lim_{n\to\infty}\left[200\left(1+\frac{1}{n}\right) - \frac{500}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

$$= 200(1) - \frac{500}{3}(1)(2)$$

$$= -\frac{400}{3}$$

N.B. $-\frac{400}{3}$ cannot truly represent area (which "should" be positive)

Riemann's process actually provides **"signed area"** (S.A.)

Signed area is the net result from:

AREA ABOVE X-AXIS − AREA BELOW X-AXIS

N.B. We could also write that

$$\overset{\text{sum}}{\int_0^{10}} \underbrace{(4x - x^2)}_{\text{height}}\,\underbrace{dx}_{\text{width}} = -\frac{400}{3}$$

ex. Evaluate

$$\int_0^{R} (4x - x^2)\,dx$$

R represents the right-hand bound of an arbitrary interval $[0, R]$

① $dx = \frac{R}{n}$

② $c_k = 0 + k \cdot \frac{R}{n}$

$= \frac{R}{n} k$

③ $f(c_k) = 4\left(\frac{R}{n}k\right) - \left(\frac{R}{n}k\right)^2$

$= \frac{4R}{n} k - \frac{R^2}{n^2} k^2$

④ $\text{Area} \approx \sum_{k=0}^{n} f(c_k)\, \Delta x$

$= \sum_{k=0}^{n} \left(\frac{4R}{n} k - \frac{R^2}{n^2} k^2\right)\left(\frac{R}{n}\right)$

$= \sum_{k=0}^{n} \left(\frac{4R^2}{n^2} k - \frac{R^3}{n^3} k^2\right)$

$= \frac{4R^2}{n^2} \sum_{k=0}^{n} k - \frac{R^3}{n^3} \sum_{k=0}^{n} k^2$

$= \frac{4R^2}{n^{\cancel{2}}} \cdot \frac{\cancel{n}(n+1)}{2} - \frac{R^3}{n^{\cancel{3}2}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{6}$

$= \frac{4R^2}{2} \cdot \left(1 + \frac{1}{n}\right) - \frac{R^3}{6} \cdot \left(1 + \frac{1}{n}\right) \cdot \left(2 + \frac{1}{n}\right)$

⑤ $\int_0^R (4x - x^2)\, dx$

$= \lim_{n \to \infty} \left[\frac{4R^2}{2} \cdot \left(1 + \frac{1}{n}\right) - \frac{R^3}{6} \cdot \left(1 + \frac{1}{n}\right) \cdot \left(2 + \frac{1}{n}\right)\right]$

$$= \frac{4R^2}{2} \cdot (1) - \frac{R^3}{\cancel{6}\,3}(1)\cancel{(2)}$$

$$= 2R^2 - \frac{1}{3}R^3$$

N.B.

$$\int_3^4 (4x - x^2)\,dx = \int_0^4 (4x - x^2)\,dx - \int_0^3 (4x - x^2)\,dx$$

Jan. 25, 2017

Fundamental Theorem of Calculus

Let $f(x)$ represent a continuous function over $[a, b]$ and let $F(x)$ represent an antiderivative of $f(x)$

i.e. $F'(x) = f(x)$

Then $\int_a^b f(x)\,dx = F(x)\Big|_a^b$

$$= F(a) - F(b)$$

ex. $\int_1^2 \underbrace{\frac{1}{x}}_{f(x)}\,dx = \ln x \Big|_1^2 = \ln 2 - \cancel{\ln 1}^{\,0} = \ln 2$

$f(x)$ ← which must be continuous! (in this case, it is)

ex. $\int_0^3 (10 - 9.8x)\,dx = \left(10x - 4.9x^2\right)\Big|_0^3$

$$= 30 - 4.9(9) - 0$$

$= -14.1$

ex. $\displaystyle\int_0^{\ln 7} e^a \, da = e^a \Big|_0^{\ln 7}$

$= e^{\ln 7} - e^0$

$= 7 - 1$

$= 6$

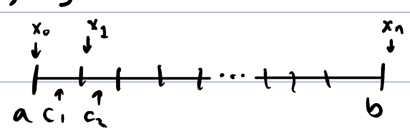
Proof of the FTC

Reminder: Mean Value Theorem

If $g(x)$ is conts over $[a,b]$ and diff^{ble} within (a,b),
then there exists at least one value $x = c$ within (a,b)
such that $g'(c) = \dfrac{g(b) - g(a)}{b-a}$ or $g(b) - g(a) = g'(c)(b-a)$

Proof of FTC:

Suppose $[a,b]$ is divided into n subintervals



and let $X_0, X_1, x_2, \ldots, X_n$ be the extremities of each subinterval

$$F(b) - F(a) = F(x_n) - F(x_0)$$
$$= \underline{F(x_n)} - F(x_{n-1}) + F(x_{n-1}) - \ldots - F(x_1) + \boxed{F(x_1) - \underline{F(x_0)}}$$

Applying MVT $= F'(c_n)(x_n - x_{n-1}) + F'(c_{n-1})(x_n - x_{n-1}) + \dots$
$+ F'(c_2)(x_2 - x_1) + F'(c_1)(x_1 - x_0)$

But... F needs to be cont! But... $F' = f$ so since F is diff'ble, then F is cont!

(Note: $c_1, c_2, c_3, \dots, c_n$ are selected within each subinterval)

Now, $F' = f$, and $x_k - x_{k-1} = \Delta x$

$$\text{so } F(b) - F(a) = f(c_n)\Delta x + f(c_{n-1})\Delta x + \dots + f(c_1)$$

$$= \sum_{k=1}^{n} f(c_k)\Delta x$$

The number of subdivisions is arbitrary and so, as $n \to \infty$

$$F(b) - F(a) = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x$$

$$= \int_a^b f(x)\,dx$$ "  "

<u>ON TEST</u>: Might write proof with blanks, or give justifications to proof parts (<u>eg</u>, use MVT, assume $\Delta x \to 0$, ...)

<u>Do you remember your derivatives / anti-derivatives</u>

① $(x)' = 1$ $\qquad \int 1\,dx = x$ or $\int 1\,dq = q$

② $(x^n)' = nx^{n-1}$ $\qquad$ $\int x^n \, dx = \frac{x^{n+1}}{n+1}$ (for $n \neq 1$)

③ $(\ln x)' = \frac{1}{x}$ $\qquad$ $\int \frac{1}{x} \, dx = \ln|x|$

④ $(\sin x)' = \cos x$ $\qquad$ $\int \cos x \, dx = \sin x$

⑤ $(\cos x)' = -\sin x$ $\qquad$ $\int \sin x \, dx = -\cos x$

⑥ $(\tan x)' = \sec^2 x$ $\qquad$ $\int \sec^2 x \, dx = \tan x$

⑦ $(\cot x)' = -\csc^2 x$ $\qquad$ $\int \csc^2 x = -\cot x$

⑧ $(\sec x)' = \sec x \tan x$ $\qquad$ $\int \sec x \tan x = \sec x$

⑨ $(\csc x)' = -\csc x \tan x$ $\qquad$ $\int \csc x \cot x \, dx = -\csc x$

⑩ $(e^x)' = e^x$ $\qquad$ $\int e^x \, dx = e^x$

⑪ $(a^x)' = a^x \ln a$ $\qquad$ $\int a^x \, dx = \frac{a^x}{\ln a}$

⑫ $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ $\qquad$ $\int \frac{1}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right)$

⑬ $(\arctan x)' = \frac{1}{1+x^2}$ $\qquad$ $\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right)$

⑭ $(\operatorname{arcsec} x)' = \frac{1}{|x|\sqrt{x^2-1}}$ $\qquad$ $\int \frac{1}{|x|\sqrt{x^2-a^2}} \, dx = \frac{1}{a} \operatorname{arcsec}\left(\frac{x}{a}\right)$

ex. $\int_0^2 \frac{1}{4+x^2} dx = \frac{1}{2} \arctan\left(\frac{x}{2}\right) \Big|_0^2$

$= \frac{1}{2}\arctan\left(\frac{2}{2}\right) - \frac{1}{2}\arctan\left(\frac{0}{2}\right)$

$= \frac{1}{2} \cdot \frac{\pi}{4}$

$= \frac{\pi}{8}$

FTC ... suite

Jan. 26

$$F(b) - F(a) = \int_a^b f(x)\, dx$$

where f is cont^s over $[a, b]$

and where F is an antiderivative of f (or $F' = f$)

ex. $\int_1^3 \frac{1}{x^3} dx = \int_1^3 x^{-3} dx$

$= \frac{x^{-2}}{-2} \Big|_1^3$

$= -\frac{1}{2x^2} \Big|_1^3$

$= \left(\frac{-1}{18}\right) - \left(\frac{-1}{2}\right)$

$= \frac{4}{9}$

ex. $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \, dx = \left(-\cos x\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 0 - 0$$

$$= 0$$

ex. $$\int_{0}^{4} \frac{1}{\sqrt{25 - x^2}} \, dx$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right)$$

$$= \left[\arcsin\left(\frac{x}{5}\right)\right]\Big|_{0}^{4}$$

$$= \arcsin\left(\frac{4}{5}\right) - \arcsin 0$$

$$= \arcsin\left(\frac{4}{5}\right) \approx 0.927$$

ex. $$\int_{-2}^{1} \frac{1}{x^2} \, dx$$

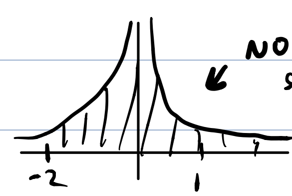
$$= \int_{-2}^{1} x^{-2} \, dx$$

$$= \left[\frac{x^{-1}}{-1}\right]\Big|_{-2}^{1}$$

$$= \left(-\frac{1}{1}\right) - \left(\cancel{-}\frac{1}{\cancel{-}2}\right)$$

$$= -1 - 0.5$$

$= -\frac{3}{2}$ ← DOESN'T MAKE SENSE
(can't be negative)



NOT CONTS within $[-2, 1]$
so FTC doesn't work

ex. Find R such that

$$\int_0^R \sqrt[3]{x}\, dx = 50$$

$$\int_0^R x^{\frac{1}{3}}\, dx = \left.\frac{x^{4/3}}{4/3}\right|_0^R$$

$$= \left.\frac{3}{4} x^{4/3}\right|_0^R$$

$$= \frac{3}{4}(R)^{4/3} - 0$$

So $\frac{3}{4} R^{4/3} = 50$

$$R^{4/3} = \frac{200}{3}$$

$$R = \left(\frac{200}{3}\right)^{3/4}$$

Properties of the definite integral

① If $f(x) \geq 0$ over $[a,b]$, then $\int_a^b f(x)\, dx \geq 0$ (assuming that $b > a$)

② If $f(x) \geq g(x)$ over $[a,b]$ then $\int_a^b f(x) \geq \int_a^b g(x)\, dx$

③ $\int_a^a f(x)\, dx = 0$

④ $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$

$$\underset{\text{FTC}}{\downarrow} \qquad \underset{\text{FTC}}{\downarrow}$$

$$F(a) - F(b) = -(F(b) - F(a))$$

⑤ $\int_a^b K f(x)\,dx = K \int_a^b f(x)\,dx$ (allowed in sigma notation)

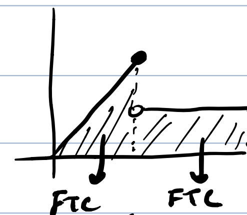
⑥ $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

(also allowed in sigma notation)

⑦ $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$ Additive interval property

WHY * as long as f is integrable over both intervals

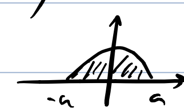
ex.



(can include pt b.c. $\int_a^a f(x)\,dx = 0$)

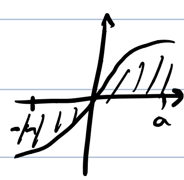
⑧ If $f(x)$ is an even function (symmetrical : $f(-x) = f(x)$)

then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$



⑨ If $f(x)$ is an odd function ($f(-x) = -f(x)$)

then $\int_{-a}^{a} f(x)\,dx = 0$



symmetry at $y=0$

ex. Consider the function $f(x) = \begin{cases} \cos x, & x < 0 \\ 2x-1, & x \geq 0 \end{cases}$

Evaluate $\int_{-\frac{\pi}{2}}^{3} f(x)\,dx$

$= \int_0^3 2x-1 + \int_{-\frac{\pi}{2}}^{0} \cos x$

$$= \left(x^2 - x\right)\Big|_0^3 + \sin x \Big|_{-\frac{\pi}{2}}^{0}$$

Could also split it up, factor out 2,... but not necessary

$$= (9-3) + (0-(-1))$$

$$= 6 + 1$$

$$= 7$$

ex. $$\int_1^2 \frac{(x+1)^2 - 3x}{x^2}\,dx$$

$$= \int_1^2 \frac{x^2+2x+1-3x}{x^2}\,dx$$

$$= \int_1^2 \frac{x^2-x+1}{x^2}\,dx$$

$$= \int_1^2 \left(1 - \frac{1}{x} + \frac{1}{x^2}\right)dx$$

$$= \left(x - \ln(|x|) + \frac{x^{-1}}{-1}\right)\Big|_1^2$$

$$= \frac{3}{2} - \ln 2$$

Jan. 30

ex. Riemann Sum $\to$ Integral

Find an exact value for the following Riemann sum:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{4+\frac{4k^2}{n^2}} \cdot \frac{2}{n}$$

$\Delta x = \frac{2}{n}$

$\frac{b-a}{\cancel{n}} = \frac{2}{\cancel{n}}$

$2 = b-a$ Chose $a=0$, $b=2$

$c_k = a + k\Delta x$

$= 0 + k\Delta x$

$= k\frac{2}{n}$

$= \frac{2}{n}k$

Notice that $\frac{4k^2}{n^2} = \left(\frac{2k^2}{n^2}\right) = c_k^2$

Therefore, $\frac{1}{4+\frac{4k^2}{n^2}} = \frac{1}{4+c_k^2} = f(c_k)$

Then $f(x) = \frac{1}{4+x^2}$

Oooo... oh! That means that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{4+\frac{4k^2}{n^2}} \cdot \frac{2}{n} = \int_0^2 \frac{1}{4+x^2}\,dx$$

$$= \frac{1}{2}\arctan\left(\frac{x}{2}\right)\Big|_0^2$$

$$= \frac{1}{2}\arctan(1) - \frac{1}{2}\arctan(0)$$

$$= \frac{1}{2}\cdot\frac{\pi}{4}$$

$$= \frac{\pi}{8}$$

À vous: Convert $\int_1^3 \ln(2+x)\,dx$ into a Riemann sum:

$b = 3$

$a = 1$

① $\Delta x = \frac{b-a}{n} = \frac{2}{n}$

② $c_k = 1 + k\Delta x$

$= 1 + \frac{2}{n}k$

③ $f(c_k) = \ln(2+c_k)$

$= \ln\left(2 + \frac{2}{n}k\right)$

④ $\text{Area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(2+1+\frac{2}{n}k\right)\frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(3+\frac{2}{n}k\right)\frac{2}{n}$$

(T) or $F = \int_3^5 \ln x \, dx$

FTC Part I

Let $f(x)$ represent a continuous function over $[a,b]$, then

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x) \qquad \text{where } a \le x \le b$$

Recall (from FTC part 2) that if f is continuous over $[a,b]$ and if F is an antiderivative of f (i.e. $F' = f$)

then $\int_a^b f(t)\,dt = F(b) - F(a)$

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] \overset{\text{FTC 2}}{=} \frac{d}{dx}\left[F(t)\Big|_a^x\right]$$

$$= \frac{d}{dx}\left[F(x) - F(a)\right]$$

$$= F'(x) - 0 \quad \text{a is a constant}$$

$$= f(x)$$

ex. $\dfrac{d}{dx}\left[\int_0^x \cos t \, dt\right]$

$= \dfrac{d}{dx}\left[\sin t \Big|_0^x\right]$

$= \dfrac{d}{dx}\left[\sin x - \overset{\text{const.}}{\sin 0}\right]$

$= \cos x$

ex. $\dfrac{d}{dx}\left[\int_1^x \sqrt{\underset{\text{cont's}}{\sin^2 t} + \dfrac{t^3}{5}} \, dt\right]$

$= \sqrt{\sin^2 x + \dfrac{x^3}{5}}$

À vous $\dfrac{d}{dx}\left[\int_x^3 \cos(t^2) \, dt\right]$

$= \dfrac{d}{dx}\left[-\int_3^x \cos(t^2) \, dt\right]$

$= -\dfrac{d}{dx}\left[\int_3^x \cos(t^2) \, dt\right]$

$= -\cos(x^2)$

ex. $\dfrac{d}{dx}\left[\int_{\ln x}^{x^2} \underbrace{\sqrt{4+t^3}}_{f} \, dt\right]$

$$= \frac{d}{dx}\left[F(t) \Big|_{\ln x}^{x^3} \right]$$

$$= \frac{d}{dx}\left[F(x^2) - F(\ln x) \right]$$

$$= F'(x^2) \cdot (x^2)' - F'(\ln x) \cdot (\ln x)'$$

$$= f(x^2) \cdot 2x - f(\ln x) \cdot \frac{1}{x}$$

$$= \sqrt{4 + x^6} \cdot 2x - \sqrt{4 + (\ln x)^3} \cdot \frac{1}{x}$$

"Our general FTC part I"

$$\frac{d}{dx}\left[\int_{u_1(x)}^{u_2(x)} f(t)\, dt \right]$$

$$= f(u_2(x)) \cdot u_2'(x) - f(u_1(x)) \cdot u_1'(x)$$

Application

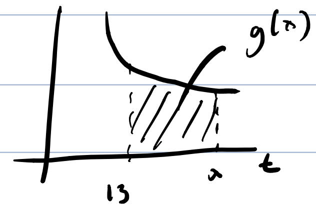
Let $g(x) = \int_{13}^{x} \frac{1}{1+t^3}\, dt$

a) Find $g(13)$

b) Find $g'(13)$

c) Find the eq of the line that is tangent to $g(x)$ at $x = 13$

$$g(x) = \int_{13}^{x} \frac{1}{1+t^3}\,dt$$



a) $g(13) = \int_{13}^{13} \frac{1}{1+t^3}\,dt$

$= 0$

b) $g'(x) = \frac{d}{dx}\left[\int_{13}^{x} \frac{1}{1+t^3}\,dt\right]$

$= \frac{1}{1+x^3}$

$g'(13) = \frac{1}{1+13^3} = \frac{1}{2198}$

c) Let y = eqn tangent line of $g(x)$ at $x=13$

$y = \frac{1}{2198}x + b$

$0 = \frac{1}{2199}(13) + b$

$b = -\frac{13}{2199}$

So $y = \frac{1}{2198}x - \frac{13}{2199}$

ex. Suppose

$$x^2 + \cos x = \int_{\pi}^{x} \frac{f(\bar{x})}{\sqrt{1+\bar{x}}}\, d\bar{x} + k$$

a) If this eqn is true for all x, what must k be?

Chose $x = \pi$

$\pi^2 - 1 = 0 + k \longrightarrow k = \pi^2 - 1$

b) Can you find $f(x)$ that ensures this eqn is true for all x.

$$\left(x^2 + \cos x\right)' = \left(\int_{\pi}^{x} \frac{f(\bar{x})}{\sqrt{1+\bar{x}^2}}\, dx + k\right)'$$

$$2x - \sin x = \frac{f(x)}{\sqrt{1+x^2}}$$

$$f(x) = \sqrt{1+x^2}\,(2x - \sin x)$$

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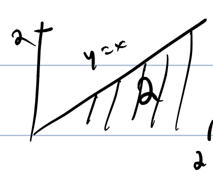
Def: Let $f(x)$ represent a function defined over $[a,b]$. Its average value is denoted by $\bar{f}_{[a,b]}$ and is obtained by

$$\bar{f}_{[a,b]} = \frac{1}{b-a} \int_a^b f(x)\, dx \qquad \frac{\text{Area}}{\text{width}}$$



$2/2 = 1$ ✓

Recall avg $= \frac{\Sigma x}{n}$

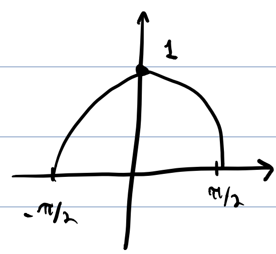
ex. Let $f(x) = \cos x$. Find the average value of f over $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\bar{f}_{[-\pi/2, \frac{\pi}{2}]} = \frac{1}{\frac{\pi}{2} - (-\frac{\pi}{2})} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\, dx$$

$$= \frac{1}{\pi} [\sin x] \Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{\pi}(1 - (-1)) = \frac{2}{\pi}$$

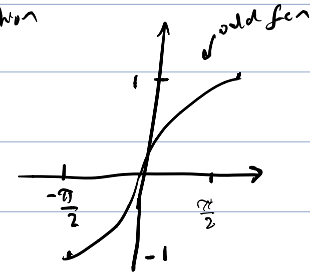
visualization:



ex. for $\sin x$...

$$\bar{f}_{[-\pi/2, \pi/2]} = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\, dx$$

visualization



$$= \frac{1}{\pi} - \cos x \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 0$$

ex. Find $\bar{h}_{[0,2]}$ if $h(x) = \begin{cases} 1 - x ; & x \leq 1 \\ x^2 + 1 ; & x > 1 \end{cases}$

$$\bar{h}_{[0,2]} = \frac{1}{2} \int_0^2 h(x)\, dx$$

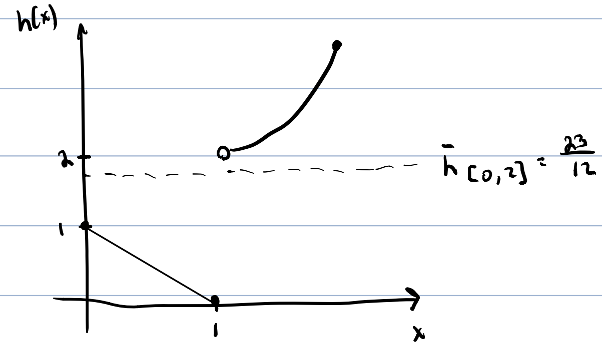
$$= \frac{1}{2}\left[\int_0^1 (1-x)\,dx + \int_1^2 (x^2+1)\, dx\right]$$

$$= \frac{1}{2}\left[\left(x - \frac{x^2}{2}\right)\Big|_0^1 + \left(\frac{x^3}{3} + x\right)\Big|_1^2\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} + \frac{8}{3} + 2 - \left(\frac{1}{3} + 1\right)\right]$$

$$= \frac{23}{12}$$

Visualization:



MEAN VALUE THM OF INTEGRAL CALCULUS

Let $f(x)$ represent a cont^s fcn over $[a,b]$, then there exists at least one value of c in $]a,b[$ such that

$$f(c) = \bar{f}_{[a,b]}$$

or

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Proof: Let $g(x) = \int_a^x f(t)\,dt$ where f is cont^s

① What is $g(a)$?

$$g(a) = \int_a^a f(t)\,dt = 0$$

② What is $g(b)$?

$$g(b) = \int_a^b f(t)\,dt$$

According to MVT [CAL I]:

$$\frac{g(b) - g(a)}{b-a} = g'(c)$$ for at least one value 'c' in $]a,b[$

③ What is $g'(c)$?

$$g'(x) = \frac{d}{dx}\, g(x)$$

$$g'(x) = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

so $g'(c) = f(c)$

Substitute all accumulated info ①, ②, ③

$$\underbrace{\frac{\int_a^b f(t)\,dt - 0}{b-a}}_{\bar{f}_{[a,b]}} = f(c)$$

Conclusion: $f(c) = \frac{1}{b-a} \int_a^b f(t)\,dt$

Application:

Let $f(x) = \frac{1}{x^2}$. Find $\bar{f}_{[1,7]}$. Find, if it exist, the value of c such that $f(c) = \bar{f}_{[1,7]}$

$$\bar{f}_{[1,7]} = \frac{1}{6}\int_1^7 f(x)\,dx$$

$$= \frac{1}{6}\int_1^7 \frac{1}{x^2}\,dx$$

$$= \frac{1}{6}\left(-\frac{1}{x}\right)\Big|_1^7$$

$$= \frac{1}{6}\left(-\frac{1}{7}+1\right)$$

$$= \frac{1}{7}$$

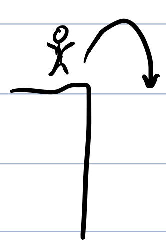
$f(c) = \frac{1}{c^2}$

$\frac{1}{7} = \frac{1}{c^2}$

$c^2 = 7$

$\boxed{c = \sqrt{7}}$ or $\cancel{c = -\sqrt{7}}$ Not in $]1,7[$

ex.



Let $v(t) = 10 - 9.8t$

a) Find $\bar{v}_{[0,2]}$

b) Find t_c such that $v(t_c) = \bar{v}_{[0,2]}$

a) $\bar{v}_{[0,2]} = \frac{1}{2} \int_0^2 v(t)\,dt$

$= \frac{1}{2} \left(10t - 4.9t^2 \right) \Big|_0^2$

$= \frac{1}{\cancel{2}} \left(\frac{\cancel{2}}{5} \right)$

$= \frac{1}{5}$

b) $v(t_c) = 10 - 9.8t_c$

$\frac{1}{5} = 10 - 9.8t_c$

$\frac{10 - \frac{1}{5}}{9.8} = t_c$

$\boxed{t_c = 1}$