2 - Integration Methods

objective: To find antiderivatives of "non-basic" functions.

→ We know that $\int \cos dx = \sin x$

$$\int \cos(5x)\,dx = ?$$

$$\int \cos(\ln x)\,dx = ?$$

$$\int \cos(e^x)\,dx = ?$$

$$\int \cos(x^2)\,dx = ?$$

Basic Algebraic Approaches

ex. $\int (4x+3)^2\,dx$

→ expand!

$$= \int (16x^2 + 24x + 9)\,dx$$

$$= 16\int x^2 + 24\int x + 9\int 1 \; dx$$

$= 16\frac{x^3}{3} + 24\frac{x^2}{2} + 9x + k$

ex. $\int \frac{3x^2 + 2x + 5}{x}\, dx$

→ distribute the divider

$= \int \left(3x + 2 + \frac{5}{x}\right) dx$

$= \frac{3}{2}x^2 + 2x + 5\ln|x| + A$

N.B. not 'x'

(needs to be on same dom as $\frac{1}{x}$)

ex. $\int \sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) dx$

→ Trig Identities

$= \int \frac{1}{2}\sin x\, dx$

$= \frac{1}{2}\int \sin x\, dx$

$= -\frac{1}{2}\cos x + $ 웃

ex. $\int \tan^2 x \, dx$

$= \int (\sec^2 x - 1) \, dx$

$= \tan x - x + \underline{\Pi}$

ex. $\int (\sin x + \cos x)^2 \, dx$

$= \int (\sin^2 x + 2 \sin x \cos x + \cos^2 x) \, dx$

$= \int (2 \sin x \cos x + 1) \, dx$

$= \int (\sin(2x) + 1) \, dx$

?

SUBSTITUTION

ex. $\int \cos x \, dx = \sin x + C$ ← basic antiderivative

ex. $\int 3x^2 \cos(x^3) \, dx$

Let $u = x^3$

$\frac{du}{dx} = u' \rightarrow du = u' dx$

$du = 3x^2 \, dx$

$$\int 3x^2 \cos(x^3)\,dx = \int \cos u \, du$$

$$= \sin u + C$$

$$= \sin(x^3) + C$$

$$\int (2x+5)^{12}\,dx$$

Let $u = 2x + 5$

$du = u'\,dx$

$du = 2\,dx$

$$\int (2x+5)^{12}\,dx = \frac{1}{2}\int (2x+5)^{12}\,2\,dx$$

$$= \frac{1}{2}\int u^{12}\,du$$

$$= \frac{1}{2}\,\frac{u^{13}}{13} + C$$

$$= \frac{u^{13}}{26} + C$$

$$= \frac{(2x+5)^{13}}{26} + C$$

ex. $\int \frac{e^x}{16 + e^{2x}} \, dx$ $\qquad$ $\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right)$

$= \int \frac{1}{16 + (e^x)^2} e^x \, dx$

Let $u = e^x$

$du = e^x \, dx$

$$\int \frac{e^x}{16 + e^{2x}} \, dx = \int \frac{1}{16 + u^2} \, du$$

$$= \frac{1}{4} \arctan\left(\frac{u}{4}\right) + C$$

$$= \frac{1}{4} \arctan\left(\frac{e^x}{4}\right) + C$$

Survival Guide: Choices of 'u'

- → inner function
- → is there a $u \leftrightarrow u'$ combination
- → when all else fails, let u = denominator, or switch methods

ex. $\int \frac{24x^2 + 64x - 16}{4x^3 + 16x^2 - 8x + 14} \, dx$

Let $u = 4x^3 + 16x^2 - 8x + 14$

$du = (12x^2 + 32x - 8) \, dx$

$$\int \frac{24x^2 + 64x - 16}{4x^3 + 16x^2 - 8x + 14}\,dx = 2\int \frac{1}{u}\,du$$

$$= 2\ln|u| + C$$

$$= 2\ln\left|4x^3 + 16x^2 - 8x + 14\right| + C$$

<u>ex.</u> $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

Let $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{u}}\,dx$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx$$

$$= 2\int e^{u}\,du$$

$$= 2e^{u} + C$$

$$= 2e^{\sqrt{x}} + C$$

<u>ex.</u> $\displaystyle\int \frac{\ln(\ln t)}{t\ln t}\,dt$

Let $u = \ln t$ for $t \geq 0$

$du = \frac{1}{t}\,dt$

$$\int \frac{\ln(\ln t)}{t \ln t}\, dt = \int \frac{\ln(\ln t)}{\ln t} \cdot \frac{1}{t}\, dt$$

$$= \int \frac{\ln u}{u}\, du$$

Let $z = \ln u$

$dz = \frac{1}{u}\, du$

$$\int \frac{\ln u}{u}\, du = \int z\, dz$$

$$= \frac{z^2}{2} + C$$

$$= \frac{\ln^2(u)}{2} + C$$

or... $u = \ln(\ln t)$

$du = \frac{1}{\ln t} \cdot \frac{1}{t}\, dt$

$$= \frac{\ln^2(\ln t)}{2} + C$$

ex. $\int (2y+1)\sqrt{y-3}\, dy$

Let $u = y-3$

$du = dy$

$\int (2(u+3)+1)\sqrt{u}\, du$

$= \int (2u+7)\sqrt{u}\, du$

$= \int (2u^{3/2} + 7u^{1/2})\, du$

$= 2\int u^{3/2}\, du + 7\int u^{1/2}\, du$

$$= 2\frac{u^{5/2}}{5/2} + 7\frac{u^{3/2}}{3/2} + C$$

$$= \frac{4}{5}u^{5/2} + \frac{14}{3}u^{3/2} + C$$

$$= \frac{4}{5}(y-3)^{5/2} + \frac{14}{3}(y-3)^{3/2} + C$$

February 8, 2016

Problem 27

Use a substitution to solve each of the following (definite or indefinite) integrals

1) $\int \frac{2x+1}{4x^2+4x+3}dx$

2) $\int \frac{\cos(\sqrt{x})\sqrt{1+\sin(\sqrt{x})}}{\sqrt{x}}dx$

3) $\int \frac{xdx}{\sqrt{16-x^4}}$

4) $\int \frac{x^5}{\sqrt{x^3+4}}\,dx$

1) $\int \frac{2x+1}{4x^2+4x+3}\,dx$

Let $u = 4x^2 + 4x + 3$

$du = (8x + 4)\,dx$

$$\int \frac{2x+1}{4x^2+4x+3}\,dx = \frac{1}{4}\int \frac{1}{4x^2+4x+3}\,4(2x+1)\,dx$$

$$= \frac{1}{4}\int \frac{1}{u}\,du$$

$$= \frac{1}{4}\ln|u| + C$$

$$= \frac{1}{4}\ln|4x^2+4x+3| + C$$

2) $\int \frac{\cos(\sqrt{x})\sqrt{1+\sin(\sqrt{x})}}{\sqrt{x}}\,dx$

Let $u = 1 + \sin(\sqrt{x})$

$du = \left(\cos(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}\right) dx$

$$\int \frac{\cos(\sqrt{x})\sqrt{1+\sin(\sqrt{x})}}{\sqrt{x}}\,dx = 2\int \sqrt{1+\sin(\sqrt{x})}\,\cos(\sqrt{x})\,\frac{1}{2\sqrt{x}}\,dx$$

$$= 2\int u^{1/2}\,du$$

$$= 2 \cdot \frac{u^{3/2}}{3/2} + C$$

$$= \frac{4}{3}u^{3/2} + C$$

$$= \frac{4}{3}\left(1+\sin\sqrt{x}\right)^{3/2} + C$$

*3) $\int \frac{x}{\sqrt{16 - x^4}}\,dx \quad = \frac{1}{2}\int \frac{1}{\sqrt{16-(x^2)^2}}\,2x\,dx$

Let $u = x^2$ $\quad = \frac{1}{2}\int \frac{1}{\sqrt{16-u^2}}\,du$

$du = 2x\,dx$

$$= \frac{1}{2}\arcsin\left(\frac{x^2}{4}\right) + C$$

*4) $\int \frac{x^5}{\sqrt{x^3+4}}\,dx \quad = \frac{1}{3}\int \frac{x^3}{\sqrt{x^3+4}}\,3x^2\,dx$

Let $u = x^3 + 4$ $\quad = \frac{1}{3}\int \frac{u-4}{\sqrt{u}}\,du$ where $x^3 = u - 4$

$du = 3x^2\,dx$

$$= \frac{1}{3}\int \left(u^{1/2} - \frac{4}{\sqrt{u}}\right) du$$

$$= \frac{1}{3}\left[\int u^{1/2} - 4\int u^{-1/2}\right] du$$

$$= \frac{1}{3}\left[\frac{u^{3/2}}{3/2} - 4\,\frac{u^{1/2}}{1/2}\right]$$

$$= \frac{2}{9}u^{3/2} - \frac{8}{3}u^{1/2} + C$$

$$= \frac{2}{9}(x^3+4)^{3/2} - \frac{8}{3}(x^3+4)^{1/2} + C$$

Still to come...

- → Definite integrals involving a subs?
- → Initial condition problems

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u-sub involving definite integrals

$$\text{bounds of } x \left\{ \int_a^b f(x)\,dx \right. \longrightarrow \text{bounds of } u \left\{ \int_c^d g(u)\,du \right.$$

ex.

$$\int_1^e \frac{(1+\ln x)^3}{x}\,dx$$

✓ fcn cont's over specified interval

✓ fcn defined over specified interval

Let $u = 1 + \ln x$

$du = \frac{1}{x}\,dx$

Bounds

$x = e \longrightarrow u = 1 + \ln e$

$= 2$

$x = 1 \longrightarrow u = 1 + \ln 1$

$= 1$

$$\int_1^e \frac{(1+\ln x)^3}{x}\,dx = \int_1^e (1+\ln x)^3 \cdot \frac{1}{x}\,dx$$

$$= \int_1^2 u^3\,du$$

$$= \frac{u^4}{4}\Big|_1^2$$

No need to go back to x if bounds are in u

$$= 4 - \frac{1}{4}$$

$$= \frac{15}{4}$$

N.B. $\dfrac{(u^4+k)}{4}\Big|_1^2$

$$= \left(\frac{16+\cancel{k}}{4}\right) - \left(\frac{1+\cancel{k}}{4}\right)$$

So no need for k!

ex. $\displaystyle\int_{-2}^{2} \cos(t^2)\, t^3\, dt$

Let $f(t) = \cos(t^2)\, t^3\, dt$

$f(-t) = \cos(t^2)\,(-t^3)\,dt = -\cos(t^2)\, t^3\, dt$

$-f(t) = -\cos(t^2)\, t^3\, dt$

Recall even if $f(x) = f(-x)$

odd if $f(-x) = -f(x)$

So $f(-t) = -f(t)$

So $f(t)$ = odd

Also note $\cos(-x) = \cos(x)$

... would have been useful to prove $\cos(t^2)$ is even.

So $\displaystyle\int_{-2}^{2} \cos(t^2)\, t^3\, dt = 0$

or let $u = t^2$

$du = 2t\, dx$

Bounds

$t = 2 \rightarrow u = 4$

$t = -2 \rightarrow u = 4$

$$\int_{-2}^{2} \cos(t^2)\, t^3\, dt = \frac{1}{2}\int_{-2}^{2} \cos(t^2)\, t^2\, 2\,dt$$

$$= \frac{1}{2}\int_{4}^{4} \cos(u)\, u\, du$$

$$= 0$$

ex. $\int_{0}^{\frac{\pi}{6}} \sin x \cos x \, dx$

Let $u = \sin x$

$du = \cos x \, dx$

Bounds

$x = \frac{\pi}{6} \rightarrow u = \frac{1}{2}$

$x = 0 \rightarrow u = 0$

$$\int_{0}^{1/2} u \, du$$

$$= \left. \frac{u^2}{2} \right|_{0}^{1/2}$$

$$= \frac{\frac{1}{4}}{2}$$

$$= \frac{1}{8}$$

ex. $\int_{0}^{\pi/6} \tan x \, dx = \int_{0}^{\pi/6} \frac{\sin x}{\cos x} \, dx$

Let $u = \cos x$

$du = -\sin x$

Bounds

$x = \frac{\pi}{6} \rightarrow u = \frac{\sqrt{3}}{2}$

$x = 0 \rightarrow u = 1$

$$= -\int_0^{\pi/6} \frac{1}{\cos x}(-\sin x)\,dx$$

$$= -\int_1^{\sqrt{3}/2} \frac{1}{u}\,du$$

$$= -\ln|u| \Big|_1^{\sqrt{3}/2}$$

$$= -\ln\frac{\sqrt{3}}{2} = \ln\frac{2}{\sqrt{3}}$$ properties of ln

$$\ln\left(\frac{\sqrt{3}}{2}\right)^{-1} = -\ln\frac{\sqrt{3}}{2}$$

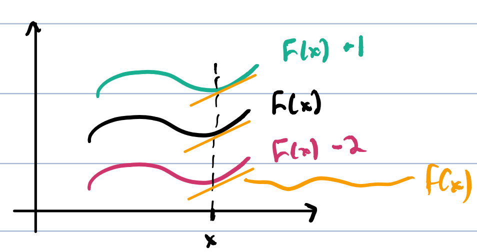
February 7, 2016

Antiderivative families

Let $f(x)$ represent a continuous function over $[a,b]$ and assume that $F(x)$ is an antiderivative of $f(x)$

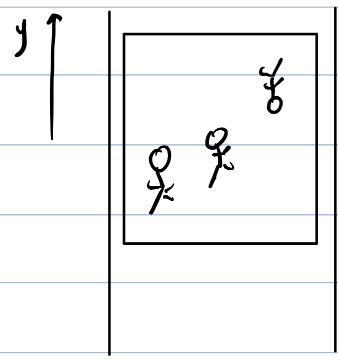
Then $F(x)+k$ is also an antiderivative of $f(x)$

$$F(x) \xrightarrow{\text{derivative}} F'(x) = f(x)$$
$$F(x)+k \xrightarrow{\text{derivative}} F'(x) = f(x)$$



Initial Condition Problems

ex. The terrible trio are in an elevator whose cables (yes all of them) happened to rupture



$a = -9.8$

Find $y(t)$ if the elevator was 150 m above the ground and heading up at a speed of 5 m/s when the cables broke $(t=0)$

$a(t) = -9.8$

$v'(t) = -9.8$

$v(t) = \int v'(t)\,dt$

$= \int -9.8\,dt$

$v(t) = -9.8t + C$

If $t = 0$, $v = 5$

$5 = -9.8(0) + C$

$C = 5$

so $v(t) = -9.8t + 5 = y'(t)$

$$y(t) = \int y'(t)\,dt$$

$$= \int (-9.8t + 5)\,dt$$

$$= \frac{9.8t^2}{2} + 5t + k$$

$$y(t) = -4.9t^2 + 5t + k$$

$y = 150$ when $t = 0$

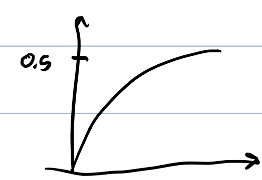
$$150 = -4.9(0)^2 + 5(0) + k \Rightarrow k = 150$$

$$y(t) = -4.9t^2 + 5t + 150$$

ex. The rate of taxation in Sweden is represented by the function:

$$T'(x) = 0.5 - 0.5e^{-0.2x}$$

where x is the annual income in thousands of euros



A person who earns $x = 100^{(th)}$ pays $35^{(th)}$ of euros in tax. Find $T(x)$.

$$T'(x) = 0.5 - 0.5e^{-0.2x}$$

$$T(x) = \int T'(x)\,dx$$

$= \int \left(0.5 - 0.5e^{-0.2x}\right) dx$

$= 0.5x - 0.5 \int e^{-0.2x} dx$

Let $u = -0.2x$

$du = -0.2\,dx$

$= 0.5x - 0.5\left(\frac{1}{-0.2}\right) \int e^{-0.2x} (-0.2\,dx)$

$= 0.5x + 2.5 \int e^{u} du$

$= 0.5x + 2.5e^{u} + C$

$T(x) = 0.5x + 2.5e^{-0.2x} + C$

when $x = 100$, $T = 35$

$35 = 0.5(100) + 2.5e^{-0.2(100)} + C$

$C = -15$

So $T(x) = 0.5x + e^{-0.2x} - 15$

Alternative (algebraic) strategies:

If substitution fails, the following strategies can be considered.

- → expansion
- → trig identities
- → split the numerator (in fraction)
- → completion of squares
- → long division

ex. $\int \frac{2x+5}{x^2+4}\,dx$ split the numerator

$= \int \frac{2x}{x^2+4}\,dx + \int \frac{5}{x^2+4}\,dx$

$= \int \frac{1}{x^2+4}\,2x\,dx + 5\int \frac{1}{x^2+4}\,dx$

Don't need to write usub

$= \ln(x^2+4) + 5\cdot\frac{1}{2}\arctan\frac{x}{2} + C$

abs. val can be removed since it is always positive

ex. $\int \frac{1}{x^2+8x+25}\,dx$

Completion of square! $\rightarrow$ $x^2+8x+\left(\frac{1}{2}8\right)^2-\left(\frac{1}{2}8\right)^2+25$

$= x^2+8x+16-16+25$

$= (x+4)^2+9$

$= \int \frac{1}{(x+4)^2+9}$

let $u = x+4$

$du = dx$

$a^2+2ab+b^2$

$= \int \frac{1}{u^2+3^2}\,du$

$= \frac{1}{3}\arctan\left(\frac{x+4}{3}\right) + C$

ex. $\int \frac{1}{\sqrt{20x-x^2}}\,dx$

$= \int \frac{1}{-(x^2-20x+100)+100}\,dx$ Completing the square

$= \int \frac{1}{100-(x-10)^2}\,dx$

$$= \int \frac{1}{100 - u^2}\, du \qquad \text{Let } u = x - 10$$

$$du = dx$$

$$= \arcsin\left(\frac{u}{10}\right)$$

$$= \arcsin\left(\frac{x-10}{10}\right) + C$$

February 13, 2016

À vous

$$\int \frac{2x + 1}{x^2 + 6x + 10}\, dx$$

$$= \int \frac{2x + 1}{x^2 + 6x + 9 - 9 + 10}\, dx$$

$$= \int \frac{2x + 1}{(x+3)^2 + 1}\, dx$$

Let $u = x + 3$

$$= \int \frac{2(u-3) + 1}{u^2 + 1}\, du \qquad du = dx$$

$$= \int \frac{2u - 5}{u^2 + 1}\, du$$

$$= \int \frac{2u}{u^2+1}\, du \quad - 5 \int \frac{1}{u^2+1}\, du$$

Let $w = u^2 + 1$

$dw = 2u\, du$

$$= \int \frac{1}{w}\, dw \quad - 5 \arctan(u)$$

$$= \ln\left|(x+3)^2 + 1\right| - 5 \arctan(x+3) + C$$

ex. What would you do?

higher or equal power on top

$$\int \frac{6x^3 + 5x^2 + 3x + 1}{x^2 + 6x + 10}\, dx$$

improper rational expression

$$\begin{array}{r} 6x - 31 \\ x^2 + 6x + 10 \overline{\big)\, 6x^3 + 5x^2 + 3x + 1} \\ -(6x^3 + 36x^2 + 60x) \\ \hline -31x^2 - 57x + 1 \\ -(-31x^2 - 186x - 310) \\ \hline 129x + 311 \end{array}$$

(+), not (×)

$$\text{So } \int \frac{6x^3 + 5x^2 + 3x + 1}{x^2 + 6x + 10}\, dx = \int \underbrace{6x - 31}_{\text{direct}} + \underbrace{\frac{129x + 311}{x^2 + 6x + 10}}_{\text{Completing the square + substitution + split numerator...}}$$

À vous

ex. $$\int \frac{1}{1 - \sin x}\, dx$$

$$= \int \frac{1}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x}\, dx$$

$$= \int \frac{1 + \sin x}{1 - \sin^2 x}\, dx$$

$$= \int \frac{1+\sin x}{\cos^2 x}\,dx$$

$$= \int \frac{1}{\cos^2 x} + \int \frac{\sin x}{\cos^2 x}\,dx$$

$$= \int \sec^2 x\,dx + \int \sec x \tan x\,dx$$

$$= \tan x + \sec x + C$$

ex. $\displaystyle\int \frac{1}{e^x + e^{-x}}\,dx$

$$= \int \frac{e^x \cdot 1}{e^x(e^x + e^{-x})}\,dx$$

$$= \frac{e^x}{(e^x)^2 + 1}$$

$\vdots$

ex. $\displaystyle\int \sec x\,dx$

$$= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{\sec^2 x + \tan x}{\sec x + \tan x}\,dx \quad u - u' \text{ combo}$$