II - Series

Sequence $\{a_n\} = a_1, a_2, a_3, \ldots$

Series $\sum_{n\geq 1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$

→ the sum of the terms of a sequence

Question: How to add ∞-many terms?

Strategy: partial sums ① $S_n = a_1 + a_2 + \ldots + a_n$ (finite)

② let $n \to \infty$ $\lim_{n\to\infty} S_n$

$\{S_n\}$ a sequence

Feb. 9

Sequence $\{a_n\} = a_1, a_2, a_3, \ldots$ ← full sum $= S_\infty$

Series $\sum_{n\geq 1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$

Series = adding up elements of a sequence

associated sequences: $\{a_n\}$ ← terms that are being added

$\{S_n\} \leftarrow$ sequence of partial sums

$$
\begin{aligned}
S_n &= n^{th} \text{ partial sum} \\
&= a_1 + a_2 + \dots + a_n \\
&= \sum_{i=1}^{n} a_i
\end{aligned}
$$

- add up n terms
- let $n \to \infty$

Define: $\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = S_\infty$

Major Problem: To take $\displaystyle\lim_{n\to\infty} S_n$, we need the formula for

S_n (not given; only a_n's are given).

Finding a formula for S_n is sometimes difficult,

and often impossible.

Big Picture: Given $\displaystyle\sum_{n=1}^{\infty} a_n$,

is a formula for S_n findable?

NO / YES

Pass/Fail Tests

But, value of sum remains unknown

ex. integral test — p-series
comparison — direct / limit
ratio
root
n^{th} term test (test for divergence)

all required (+) terms

others: alternating series tests
absolute convergence

$\lim_{n\to\infty} S_n$ Done.

value can be found

ex. telescoping, geometric

Def'n:

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = S_\infty$$

If the $\lim_{n\to\infty} S_n$ exists, we say $\sum_{n=1}^{\infty} a_n$ converges to the limit value.

otherwise, $\sum_{n=1}^{\infty} a_n$ diverges.

ex. ① $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

a) Find a_3, a_4, S_1, S_2, S_3, S_4

$$a_3 = \frac{1}{3^2+3} = \frac{1}{12}$$

$$a_4 = \frac{1}{16+4} = \frac{1}{20}$$

$S_1 = a_1 = \frac{1}{2}$

$S_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{6} = \frac{1}{2}$

$S_3 = a_1 + a_2 + a_3$

$= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20}$

$= \frac{3}{4}$

$S_4 = a_1 + a_2 + a_3 + a_4$

$= \frac{4}{5}$

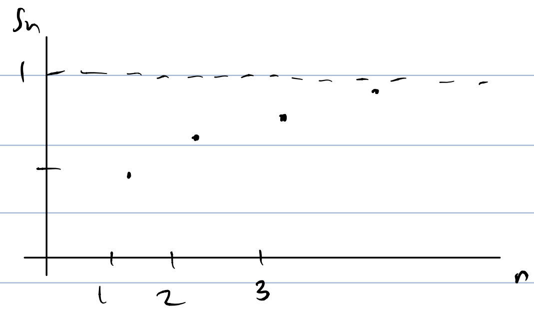
$\{S_n\} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

guess? $S_n = \frac{n}{n+1}$

b) Does $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converge? If so, find its value.

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

If our guess is correct, $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges to 1



d) Prove that this is right.

Notice: $\frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$ partial fraction

$$1 = A(n+1) + B(n)$$

Magic Values Method

$n=0: \quad 1 = A$

$n=-1: \quad 1 = -B$

$B = -1$

System of Eq Method

$1 = (A+B)n + A$

$n^1: \begin{cases} 0 = A + B \\ 1 = A \end{cases}$

$n^0:$

So, $\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$

$\therefore S_n = a_1 + a_2 + a_3 + \dots + a_n$

$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cancel{\dots} + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$

$= 1 - \frac{1}{n+1} = \frac{n}{n+1}$ ✓

$\therefore$ Our guess was right.

<u>ex.</u> For you

① Consider $\sum_{n=1}^{\infty} a_n$ where a_n is unknown.

You are told however that $S_n = \frac{n-1}{2n+3}$

a) Does $\sum_{n=1}^{\infty} a_n$ converge? If so, find its value.

$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{n-1}{2n+3} = \lim_{n \to \infty} \frac{\cancel{n}(1-\frac{1}{n})}{\cancel{n}(2+\frac{3}{n})} = \frac{1}{2}$ ✓

b) $a_n = S_n - S_{n-1}$

$= \frac{n-1}{2n+3} - \frac{(n-1)-1}{2(n-1)+3}$

$$= \frac{n-1}{2n+3} - \frac{n-2}{2n+2}$$

② Examine the following for convergence. If they converge, find their values.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3} = S_\infty$

$$\frac{1}{n^2+4n+3} = \frac{1}{(n+1)(n+3)} = \frac{A}{n+1} + \frac{B}{n+3}$$

$$1 = A(n+3) + B(n+1)$$

$n=-3$: $1 = -2B$

$B = -\frac{1}{2}$

$n=-1$: $1 = 2A$

$A = \frac{1}{2}$

$$S_\infty = \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+3}\right) \cdot \frac{1}{2}$$

$$S_n = \frac{1}{2} \cdot \left[\left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \cancel{\ldots} + \left(\cancel{\frac{1}{n}} - \frac{1}{n+2}\right) + \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+3}\right)\right]$$

$$= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} + \frac{1}{n+3}\right)$$

$$S_\infty = \lim_{n\to\infty} S_n = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3}\right)$$

$$= \frac{5}{12} \qquad \frac{3}{6} + \frac{2}{6} \quad \frac{5}{6} \cdot \frac{1}{2} = \frac{5}{12}$$

b) $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$

$$\ln\left(1 + \frac{1}{n}\right)$$

$$= \ln\left(\frac{n+1}{n}\right)$$

$$= \ln\left((n+1) \cdot \frac{1}{n}\right)$$

$$= \ln(n+1) + \ln\left(\frac{1}{n}\right)$$

$$= \ln(n+1) - \ln(n)$$

$$S_n = \left(\cancel{\ln(2)} - \ln(1)\right) + \left(\cancel{\ln(3)} - \cancel{\ln(2)}\right) + \left(\cancel{\ln(4)} - \cancel{\ln(3)}\right)$$

$$+ \cdots + \left(\ln(n+1) - \cancel{\ln(n)}\right)$$

$$S_n = \ln(n+1) - \ln(1)$$

$\lim_{n\to\infty} S_n = \ln(\infty) - 0 = \infty$ ∴ The series does not converge.

Geometric Series

geometric sequence: $\{ar^{n-1}\} = a, ar, ar^2, ar^3, \ldots$

geometric series: $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots$

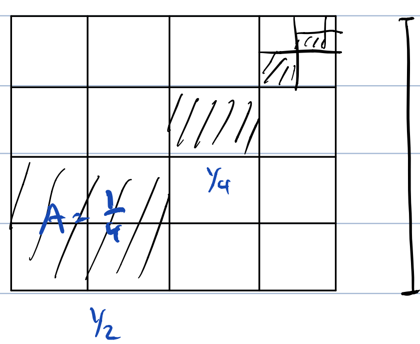
$a = 1^{st}$ term

$r =$ ratio of subsequent terms

$$\frac{a_{n+1}}{a_n} = r$$

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \dfrac{a}{1-r} & |r| < 1 \\ \text{diverges} & |r| \geq 1 \end{cases}$$

ex.



1 unit

or see for any size square that a third of the squares are covered.

$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots$

geometric

$a = \frac{1}{4}$

$r = \frac{1}{4}$

$$\frac{1/4}{1 - \frac{1}{4}} = \frac{\frac{1}{\cancel{4}}}{3/\cancel{4}} = \frac{1}{3}$$

Feb. 12

2. Parastu receives \$25 each year on her birthday from when she is 5 years old to when she is 20 years old. Each year she deposits the money and leaves it in the bank at 2% interest compounded yearly. How much money does she have on her 20^{th} birthday?

$n=0$	$n=1$	$n=2$	$n=3$	
5 yrs	6 yrs	7 yrs	8 yrs	time
25	25	25	25	
	$+\ 25(1.02)$	$+\ 25(1.02)$	$+\ 25(1.02)$	
		$+\ 25(1.02)^2$	$+\ 25(1.02)^2$	
			$+\ 25(1.02)^3$	

Finite geometric series.

$$\sum_{i=0}^{15} 25(1.02)^n$$

$$S_{16} = 25 + 25(1.02) + \ldots + 25(1.02)^{15}$$

$$S_{16} = \frac{A(1-R^N)}{1-R} = \frac{25(1-1.02^{16})}{1-1.02}$$

Geometric series 4 Power Series

↳ ex. Taylor series are power series

A power series is a series of the form

$$\sum_{n=0}^{\infty} C_n x^n \quad \text{or} \quad \sum_{n=0}^{\infty} C_n (x-a)^n$$

↑ center at 0 ↑ centered at a

where C_i's are coefficients and x is a variable.

So, the power series is really a function

$$f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$$

The domain of $f(x)$ is the set of x values for which the series converges.

Similarly, we can build series out of other functions.

$$f(x) = \sum_{n=0}^{\infty} c_n (g(x))^n$$

Ex. $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$ expanded or series form

geometric series

$a = 1$

$r = x$

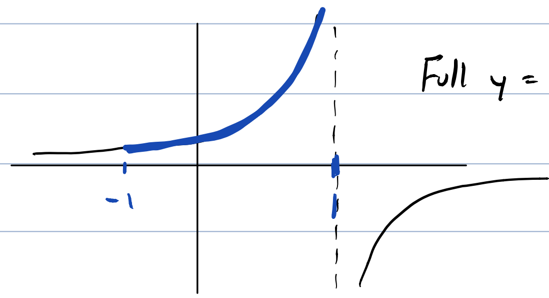
domain → interval of convergence

$|r| = |x| < 1 \Rightarrow -1 < x < 1$

or $x \in (-1, 1)$

Find a closed-form formula

$f(x) = \frac{a}{1-r} = \frac{1}{1-x}$ when it converges



Full $y = \frac{1}{1-x}$

ex. $f(x) = \dfrac{1}{2-x}$; express as a series

note: assymptote

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$f(x) = \frac{1}{2-x} = \frac{1}{1-(x-1)} \Rightarrow \begin{array}{l} a=1 \\ r = x-1 \end{array}$$

different spelling of same fcn

$$f(x) = \sum_{n=0}^{\infty} (x-1)^n \qquad \text{valid for } |x-1| < 1$$

$$-1 < x-1 < 1$$

$$0 < x < 2$$

$$x \in (0, 2)$$

3. Use a geometric series to express each of the following functions as a power series at the given centre. Give its interval of convergence.

a) $f(x) = \dfrac{1}{x+3}$, centered at 0 (in powers of x)

$$f(x) = \frac{1}{x+3} = \frac{1/3}{1+\frac{x}{3}} = \frac{1/3}{1-\left(-\frac{x}{3}\right)}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{3}\left(-\frac{x}{3}\right)^n \qquad \text{valid for } \left|-\frac{x}{3}\right| < 1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^{n+1}}$$

$$-1 < \frac{x}{3} < 1$$

$$-3 < x < 3$$

$$x \in (-3, 3)$$

b) $g(x) = \dfrac{2x^2}{1+8x^3}$, centered at 0

$$g(x) = \frac{2x^2}{1+8x^3}$$

$$g(x) = \frac{2x^2}{1-(-8x^3)}$$

$$g(x) = \sum_{n=0}^{\infty} 2x^2(-8x^3)^n$$

valid for $|-8x^3| < 1$

$$-1 < 8x^3 < 1$$

$$-\frac{1}{8} < x^3 < \frac{1}{8}$$

$$-\frac{1}{2} < x < \frac{1}{2}$$

$$x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$$

c) $h(x) = \dfrac{1}{x^2 - 6x + 10}$, centered at 3 (in powers of $(x-3)$) $+1-1$

$$h(x) = \frac{1}{1-(-x^2+6x-9)}$$

$$= \sum_{n=0}^{\infty} (-x^2+6x-9)$$

valid for $|-x^2+6x-9| < 1$

$$|-(x-3)^2| < 1$$

$$(x-3)^2 < 1$$

$$-1 < x-3 < 1$$

$2 < x < 4$

$x \in (2,4)$

d) $j(x) = \dfrac{1}{2x+3}$, centered at 1 (in powers of $(x-1)$). HINT: Let $t = x - 1$, so $x = t + 1$

Let $t = x-1 \Rightarrow x = t+1$

$$j(t) = \frac{1}{2(t+1)+3}$$

$$= \frac{1}{2t+5}$$

$$= \frac{1/5}{1-(-\frac{t}{2})}$$

$$j(x) = \frac{1/5}{1-(-\frac{x-1}{2})}$$

$$= \sum_{n=0}^{\infty} \frac{1}{5}\left(-\frac{x-1}{2}\right)^n \quad \text{valid for } \left|-\frac{x-1}{2}\right| < 1$$

$-1 < \frac{x-1}{2} < 1$

$-2 < x-1 < 2$

$-1 < x < 3$

$x \in (-1,3)$

Countrapositive

A (over "it's raining"), B (over "there are clouds in the sky")

→ If it's raining, then there are clouds in the sky

I) If there are clouds, then it's raining

II) If it's not raining, then there are no clouds

→ III) If there are no clouds, then it's not raining

Statement: If A, then B

contrapositive: if not B, then not A   Switched order and made them negative

A statement and its contrapositive are equivalent

Recall: n^{th} term test

$$\text{If } \lim_{n\to\infty} a_n \neq 0, \text{ then } \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

Contrapositive: If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

proof

Feb. 14

n^{th}-term Test (Test for Divergence)

$$\sum_{n=1}^{\infty} a_n \Rightarrow \sum a_n$$

infinity is implied

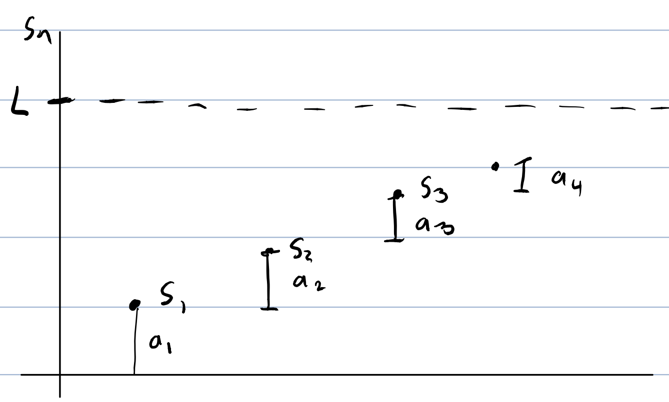
necessary only when value is important

If $\sum a_n$ is a series and if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

Proof by contraposition: we will prove the contrapositive

If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

Pictorally:



<u>Proof:</u> If $\sum a_n$ converges,

then $\lim_{n\to\infty} S_n$ exists

$\therefore$ let $\lim_{n\to\infty} S_n = L \in \mathbb{R}$ $(L \neq \pm\infty)$

Examine $S_n = S_{n-1} + a_n$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (S_{n-1} + a_n)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} + \lim_{n\to\infty} a_n$$

$$L = L + \lim_{n\to\infty} a_n$$

$$\therefore \lim_{n\to\infty} a_n = 0$$

<u>Remarks:</u> If $\lim_{n\to\infty} a_n = 0$, we can conclude <u>nothing</u>.

n^{th}-term test fails.

<u>ex.</u>

from Cal II

$$\left\{ \begin{array}{l} \sum \frac{1}{n} = \text{Harmonic Series} = \infty \\ \sum \frac{1}{n^2} = \frac{\pi^2}{6} \text{ conv.} \end{array} \right\} \text{p-series}$$

n^{th}-term test

$\lim_{n\to\infty} \frac{1}{n} = 0$ but div

$\lim_{n\to\infty} \frac{1}{n^2} = 0$ but conv

ex. Apply n^{th} term test to

a) $\sum \frac{\sqrt{n}}{\ln n}$

$\lim_{n\to\infty} \frac{\sqrt{n}}{\ln n} = \infty$ (growth) $\neq 0$ $\therefore$ $\sum \frac{\sqrt{n}}{\ln}$ div

b) $\sum \frac{\ln n}{\sqrt{n}}$

$\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = 0$ (growth) $\therefore$ we can conclude nothing

c) $\sum \left(1 - \frac{2}{n}\right)^n$

$\lim_{n\to\infty} \left(1 - \frac{2}{n}\right)^n = e^{-2} \neq 0$ $\therefore$ $\sum \left(1 - \frac{2}{n}\right)^n$ div

$\because$ $\left(1 + \frac{-2}{n}\right) = e^{-2}$

Integral Test

If $\sum a_n$ is a series and if $a_n = f(n)$ where

$f(x)$ is a continuous, decreasing, positive function,

then $\sum a_n$ & $\int_{c \in \mathbb{R}}^{\infty} f(x)\,dx$ either converge together or

↳ not necessarily to the same value

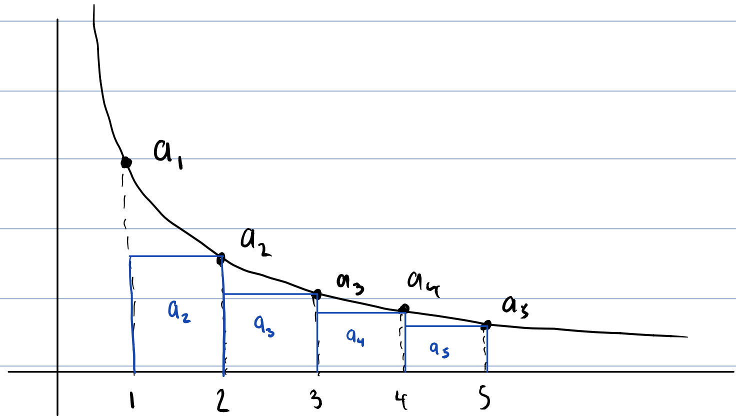
div together

$$\sum a_n \neq \int_c^{\infty} f(x)\,dx$$

Proof: Convergent Case

For ease of notation, let $c=1$.

Examine $\sum_{n=1}^{\infty} a_n$ & $\int_1^{\infty} f(x)\,dx$



We will show that $\{S_n\}$ is

- incr
- bounded above (it's already bounded below (by 1^{st} term) since it's increasing)

(i) incr: Show that $S_n > S_{n-1}$

$$S_n - S_{n-1} > 0$$

$$a_n > 0$$

$$f(n) > 0 \quad \text{(positive)} \checkmark$$

(ii) bounded above:

Let $\int_1^\infty f(x)\,dx = A$

missing a_1

by the picture clearly $a_2 + a_3 + \dots + a_n < A$

$$a_1 + a_2 + a_3 + \dots + a_n < a_1 + A$$

$$S_n < a_1 + A \quad \text{for any } n$$

$\therefore S_n$ is bounded above

$\therefore$ By BMCT $\lim_{n\to\infty} S_n$ exists

Since $\sum a_n = \lim_{n\to\infty} S_n$, the sum conv.

ex. Apply the int. test to:

(1) $\sum \frac{1}{n \ln n}$

note: can't start at $n=1$

Let $f(x) = \frac{1}{x \ln x}$ on $x \geq 2$

- continuous: quotient of product of continuous functions $\therefore$ continuous

- $x \geq 2$

$\ln x \geq \ln 2 \geq 0 \quad \therefore \quad \frac{1}{x \ln x} > 0$ for $x \geq 2$

- x, $\ln x$ $\uparrow$

$x \cdot \ln x$ $\uparrow$

$\frac{1}{x \cdot \ln x}$ $\downarrow$

or show $f'(x) < 0$

Examine $\int_2^\infty \frac{1}{x \ln x} \, dx = \lim_{t \to \infty} \int_2^t \frac{1}{x \ln x} \, dx$

Let $u = \ln x$

$du = \frac{1}{x} dx$

$$= \int_{\ln 2}^{\ln t} \frac{1}{u} du = \lim_{t \to \infty} \Big[\ln |u| \Big]_{\ln 2}^{\ln t}$$

$$= \lim_{t \to \infty} \Big(\ln |\underbrace{\ln t}_{\infty}| - \ln(\ln 2) \Big)$$

$$= \infty \quad \therefore \sum \frac{1}{n \ln n} \text{ div}$$

② Apply the int. test

$$\sum n e^{-n}$$

let $f(x) = x e^{-x} dx$ for $x \geq 1$

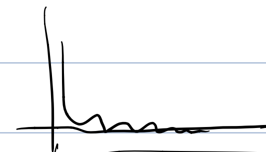
- continuous: product of two cont's fcns

- $x \geq 1$

$x e^{-x} \geq 1 e^{-x} \geq 0$

• $f(x) = \dfrac{x}{e^x}$

Note:



$$\boxed{\lim_{x \to \infty} f = 0 \not\Rightarrow \text{dec}}$$

$$f'(x) = \frac{e^x - x e^x}{e^{2x}}$$

$$= \frac{1}{e^x} - \frac{x}{e^x}$$

$$= \frac{1-x}{e^x} < 0 \text{ for } x > 1$$

Examine $I = \displaystyle\int x e^{-x}\, dx$

Let $f = x \qquad g = -e^{-x}$

$\quad\; f' = 1 \qquad g' = e^{-x}$

$$I = -x e^{-x} - \int_1^{\infty} -e^{-x}\, dx$$

$$= -x e^{-x} - e^{-x} + C$$

$$\therefore \int_1^{\infty} x e^{-x}\, dx = \lim_{t \to \infty} \Big[-x e^{-x} - e^{-x} \Big]_1^t$$

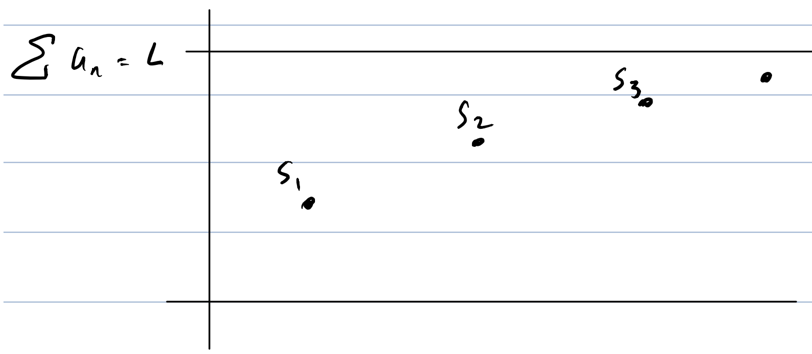
$$= \lim_{t \to \infty} \left[-\frac{t}{e^t} - \frac{1}{e^t} + \frac{2}{e} \right]$$

$$= 0 - 0 + \frac{2}{e}$$

$$= \frac{2}{e}$$

$$\therefore \sum n e^{-n} \text{ conv. (to an unknown value)}$$

Approximating Sums



Take one of the partial sums as an approx.

ex. $$\sum_{n=1}^{\infty} n e^{-n} \approx S_3 = e^{-1} + 2e^{-2} + 3e^{-3}$$

But how good an approx is this? What's the max error?

Remainder R (error)

$$\sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + a_3}_{S_n} + \underbrace{a_n + a_{n+1} + a_{n+2} + \dots}_{R}$$

for any n, R is the sum of the ommitted terms

R is another series.

Feb. 16

Integral test consequence: $\sum_1 \frac{1}{n^p} \longleftrightarrow \int_1^\infty \frac{1}{x^p}\,dx \begin{cases} \text{conv}: p>1 \\ \text{div}: p \leq 1 \end{cases}$

Estimating Series Values & Error

Conv. $\sum_1 a_n = \lim_{n\to\infty} S_n = S$

$$= \underbrace{a_1 + a_2 + a_3 + \dots + a_n}_{S_n} + \underbrace{a_{n+1} + a_{n+2} + \dots}_{\text{omitted}: R}$$

[remainder or error

Strategy:

pick a n-value

Use S_n as an estimate for S

Our goal will be to give a bound on R.

$R = a_{n+1} + a_{n+2} + \dots$

or $S = S_n + R$

$R = S - S_n$ ← for a series with positive terms

$a_n > 0$,

then $S - S_n > 0$

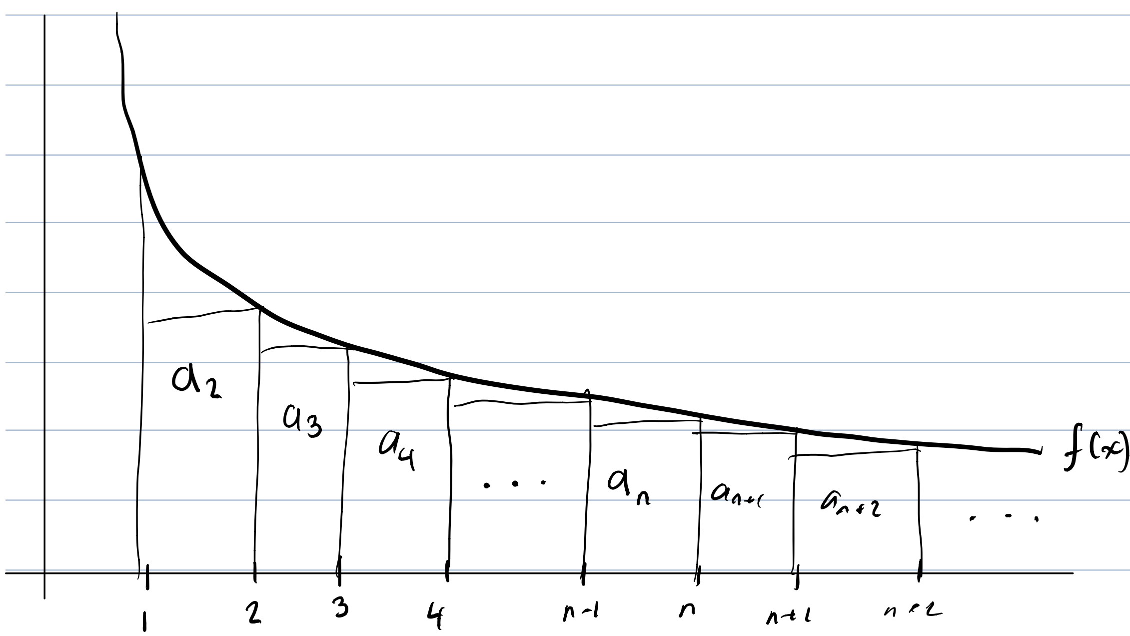
In general, the magnitude of the error is

$$R = |S - S_n|$$

Error in context of the integral test

Estimate the error by using S_n as an estimate for

$S = \sum a_n$ $\qquad f(n) = a_n$



$R = a_{n+1} + a_{n+2} + \ldots$

$$\boxed{R < \int_n^\infty f(x)\,dx}$$

Last day $\sum_{n=1}^{\infty} n e^{-n}$ conv by int-test

a) Estimate the value of this series by taking $n=4$ terms

b) Give a bound for the error in your estimate

a) $S = \sum_{n=1}^{\infty} n e^{-n} \approx S_4 = a_1 + a_2 + a_3 + a_4$

$= e^{-1} + 2e^{-2} + 3e^{-3} + 4e^{-4}$

≈ 0.8612

b) $R < \int_n^{\infty} f(x)\,dx$

$< \int_4^{\infty} x e^{-x}\,dx = \lim_{t\to\infty} \left[-x e^{-x} - e^{-x}\right]_4^t$

$= 5e^{-4}$

≈ 0.092

c) How many terms do we need for the error to be less than 10^{-3}.

$R < \int_n^{\infty} f(x)\,dx < 10^{-3}$

$$\int_n^{\infty} x e^{-x}\,dx < 0.001$$

$\checkmark$ from maple

$(n+1)e^{-n} < 0.001 \rightsquigarrow$ by hand: guess & try

$n=9 \quad 0.0014$

$n=10 \quad 0.0005$

ex. Same thing for $\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv p-series $p=2<1$

a) $\sum_{n=1}^{\infty} \frac{1}{n^2} \approx S_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$

≈ 1.42

b) $R < \int_4^{\infty} \frac{1}{x^2}\,dx$

$= \int_4^{\infty} x^{-2}\,dx$

$= \lim_{t\to\infty} \left[-\frac{1}{x} \right]_4^t$

$= \lim_{t\to\infty} \left[\underbrace{-\frac{1}{t}}_{\to 0} + \frac{1}{4} \right]$

$= \frac{1}{4} = 0.25$

c) Error to be $\underline{10^{-6}}$:

$$R < \int_{n}^{\infty} \frac{1}{x^2}\,dx < 10^{-6}$$

$$\lim_{t\to\infty} \left[-\frac{1}{x}\right]_n^t < 10^{-6}$$

$$\lim_{t\to\infty} \left[-\frac{1}{t} + \frac{1}{n}\right] < 10^{-6}$$

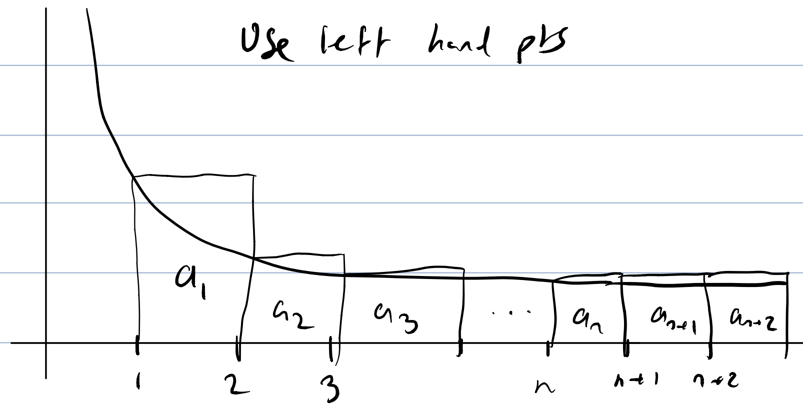
$$\frac{1}{n} < 10^{-6}$$

$$n > 10^{6}$$

<u>Creating a tighter interval for S in the special case of the Int test</u>

We have shown that $R < \int_{n}^{\infty} f(x)\,dx$

Now consider:



$$R = a_{n+1} + a_{n+2} + \dots$$

$$\boxed{R > \int_{n+1}^{\infty} f(x)\,dx}$$

Putting these together:

$$\int_{n+1}^{\infty} f(x)\,dx < R < \int_{n}^{\infty} f(x)\,dx$$

But, wait! $R = S - S_n$

$$\int_{n+1}^{\infty} f(x)\,dx < S - S_n < \int_{n}^{\infty} f(x)\,dx$$

$$S_n + \int_{n+1}^{\infty} f(x)\,dx < \underset{\sum a_n}{S} < S_n + \int_{n}^{\infty} f(x)\,dx$$

ex. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

Use S_5 to build an interval containing S

$$S_5 = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} \approx 1.1857$$

n+1

$$\int_6^{\infty} \frac{1}{x^3}\,dx = \lim_{t\to\infty} \left[-\frac{1}{2x^2} \right]_6^t = \frac{1}{2\cdot 6^2} = \frac{1}{72} \approx 0.0139$$

$$\int_5^{\infty} \frac{1}{x^3}\,dx = \lim_{t\to\infty} \left[-\frac{1}{2x^2} \right]_5^t = \frac{1}{50} \approx 0.02$$

$$1.857 + 0.0139 < \sum \frac{1}{n^3} < 1.857 + 0.02$$

$$1.1996 < \sum \frac{1}{n^3} < 1.2057$$

Direct Comparison Test

If $\sum a_n$ & $\sum b_n$ are series with $a_n, b_n > 0$

(i) If $0 < a_n < b_n$ & $\sum b_n$ converges,
then $\sum a_n$ conv as well

(ii) If $0 < a_n < b_n$ & $\sum a_n$ diverges,
then $\sum b_n$ div as well.

Proof: Case (i) $0 < a_n < b_n$, $\sum b_n$ conv

We will show that $\left\{ S_n = \sum_{i=1}^{n} a_n \right\}$ is incr & bounded above

Incr: Show $S_n > S_{n-1}$

$S_n - S_{n-1} > 0$

$a_n > 0$

Bounded above: Let $t_n = \sum_{i=1}^{n} b_n$ be the partial sums $\sum b_n$

$\sum b_n$ is convergent

$\therefore \sum b_n$ is bounded

which means there is a number M
such that $t_n \leq M$ for all n.

and since $a_n \leq b_n \leq M$

$\therefore S_n \leq t_n \leq M$

Since $\{S_n\}$ is incr & bdd above,

$\{S_n\}$ conv so $\sum a_n$ conv

February 19

Note on homework:

$$\frac{2x+3}{2x+2} = \frac{2x+3}{2x+3-1}$$

$\div 2x+3$

$$\frac{1}{1-\frac{1}{2x+3}}$$

$$\frac{-(2x+3)}{1-(2x+3)}$$

Direct Comparison Test (DCT)

If $\sum a_n$, $\sum b_n$ with $a_n, b_n > 0$ and if
$0 < a_n < b_n$ for all n.

(1) If $\sum b_n$ converges, then $\sum a_n$ converges as well

(2) If $\sum a_n$ diverges, then $\sum b_n$ diverges also

"known" series ex. p-series
geometric series
others: proven for other series

Limit Comparison Test (LCT)

If $\sum a_n$, $\sum b_n$ with $a_n, b_n > 0$

(1) If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ with $0 < L < \infty$, then

$\sum a_n$ & $\sum b_n$ either both converge or both diverge

(2) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then

$\sum a_n$ converges also

If $\sum b_n$ conv,

$\lim_{n\to\infty} b_n = 0$ ← 0 really fast

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{0} = 0$

(3) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ div, then $\sum a_n$ div

Proof of (1)

If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, therefore for $\boxed{\text{any } \varepsilon > 0}$,

there exists a N_ε such that for any $n \geq N_\varepsilon$,

$$\left| \frac{a_n}{b_n} - L \right| < \varepsilon$$

So chose $\varepsilon = \frac{1}{2} L$

Eventually $\left(\forall\, n \geq N_{\frac{1}{2}L}\right)$

$$\left| \frac{a_n}{b_n} - L \right| < \frac{1}{2} L$$

$$-\frac{1}{2} L < \frac{a_n}{b_n} - L < \frac{1}{2} L$$

$$\frac{1}{2} L < \frac{a_n}{b_n} < \frac{3}{2} L$$

$$\underbrace{\frac{1}{2} L b_n <}_{2} \underbrace{a_n < \frac{3}{2} L b_n}_{1} \quad \text{since } b_n > 0$$

① If $\sum b_n$ conv, then $\sum \frac{3}{2} L b_n$ is conv

(multiple of conv series)

$0 < a_n < \frac{3}{2} L b_n$

$\therefore \sum a_n$ conv as well by Direct Comparison Test (DCT)

② If $\sum b_n$ div, then $\sum \frac{1}{2} L b_n$ also div

$0 < \frac{1}{2} L b_n < a_n$

$\sum a_n$ div also by DCT

1. Decide if the Direct Comparison Test can be applied in each case. If it applies, use it to determine if the series converges or diverges.

a) $\sum \frac{1}{2^n+1}$

$0 < 2^n + 1 > 2^n$

$0 < \frac{1}{2^n+1} < \frac{1}{2^n}$

$\sum \frac{1}{2^n}$ is a conv geo series $r = \frac{1}{2} < 1$

$\therefore \sum \frac{1}{2^n+1}$ conv also by DCT

c) $\sum \frac{\ln n}{n}$

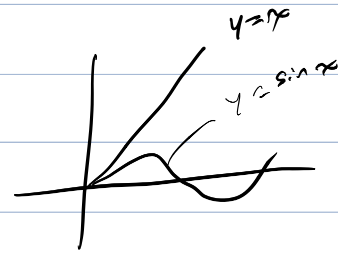
$0 < \frac{\ln n}{n} > \frac{1}{n}$

$\sum \frac{1}{n}$ is a p-series where $p = 1$ and thus diverges $\therefore$ By the DCT, $\sum \frac{\ln n}{n}$ div as well.

d) $\sum \frac{\sin(n)}{n^2}$

$\frac{\sin n}{n^2}$ is not always > 0 so the DCT does not apply

e) $\sum \sin\left(\frac{1}{n^2}\right)$

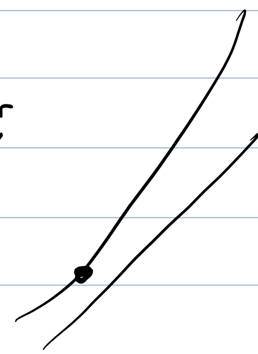


$0 < \sin\left(\frac{1}{n^2}\right) < \frac{1}{n^2}$

$\sum \frac{1}{n^2}$ is a p-series where $p=2>1$ and thus converges. $\therefore$ By the DCT, $\sum \sin\left(\frac{1}{n^2}\right)$ conv as well.

b) $\sum \frac{1}{2^n - 1}$

you can do LCT with $\sum 2^n$



$\frac{1}{2^n - 1} < \frac{1}{n^2}$ for $n \geq 100$

$2^n - 1$

$2^{100} - 1 > 100^2$?

$\sum \frac{1}{n^2}$ is a p-series where

and $\frac{d}{dn} 2^n = 2^n \ln 2$ 0 $2n$

$2^{100} \ln 2 > 2 \cdot 100$

$(2^n)'' = 2^n (\ln 2)^2 > 2$

$2^n = n^2$

$\ln 2^n = \ln n^2$

$n \ln 2 = 2 \ln n$

2. Apply the Limit Comparison Test to each of the following series to determine if possible whether the series converges or diverges.

a) $\sum \frac{1}{2^n - n}$

$\sum \left(\frac{1}{2}\right)^n$ is a converging series (geometric where $|r| < 1$)

$$\lim_{n\to\infty} \frac{2^n - n}{2^n} = \lim_{n\to\infty} \left(1 - \frac{n}{2^n}\right) = 1 = L$$

$\frac{n}{2^n} \to 0$

By the LCT, $2^n - n$ conv too

b) $\sum \frac{1}{n \ln n}$

will fail with LCT

c) $\sum \frac{(3n+1)\sqrt{n^3+3}}{n\sqrt[3]{n^6-3n+1}}$ $\sim \frac{\cancel{n} \cdot n^{3/2}}{\cancel{n} \cdot n^2} \sim \frac{1}{\sqrt{n}} = b_n$

Dominant terms!

$p = \frac{1}{2} < 1$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{(3n+1)\sqrt{n^3+3}}{n\sqrt[3]{n^6-3n+1}} \cdot n^{1/2}$$

$$= \frac{\cancel{n}\left(3 + \frac{1}{n}\right) n^{3/2} \sqrt{1 + \frac{3}{n^3}}}{\cancel{n}\; n^2 \sqrt{1 - \frac{3}{\ldots} + \frac{1}{\ldots}}} \cdot n^{1/2}$$

$$= \frac{\cancel{n^3}}{\cancel{n^2}} \cdot \frac{\left(3 + \frac{1}{n}\right)\sqrt{1 + \frac{3}{n^3}}}{\sqrt{1 - \frac{3}{n^3} + \frac{1}{n^6}}}$$

$$= 3$$

$$0 < L = 3 < \infty$$

$\sum a_n$ behaves like $\sum \frac{1}{n^{1/2}}$

$\therefore$ By LCT, <u>div.</u>

<u>DCT & Error Estimation</u>

$$S_n = \sum_{i=1}^{n} a_i \qquad t_n = \sum_{i=1}^{n} b_i$$

$$R = |S - S_n| \qquad T = |t - t_n|$$

$$0 < a_n < b_n$$ known

$$\therefore \quad 0 < R < T$$

error using S_n as approx

error using the nth partial sum of b_n

Use S_n with error T (which is easier to calculate then R)

3. For each of a) $\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$ and b) $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}$

   i) Use the DCT to show that the sum converges.

   ii) Use s_5 to estimate the value of the sum.

   iii) Estimate the error involved in your answer to (ii).

a) $\sum \frac{1}{2^n+1}$ compared to $\sum \frac{1}{2^n}$ conv. geo series

$$S_5 = \frac{1}{2^1+1} + \frac{1}{2^2+1} + \cdots + \frac{1}{2^5+1}$$

$$\approx 0.73357$$

$$R = a_6 + a_7 + a_8 + \cdots < b_6 + b_7 + b_8 + \cdots$$

$$< \sum_{n=6}^{\infty} \frac{1}{2^n} = \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \cdots$$

$$= \frac{a}{1-r} = \frac{1/2^6}{1-\frac{1}{2}}$$

$$= \frac{1}{2^5} = \frac{1}{32}$$

b) $\sum \frac{\sin^2 n}{n^3}$

i) $0 \le \frac{\sin^2 n}{n^3} < \frac{1}{n^3}$

$\sum \frac{1}{n^3}$ is a p-series where $p = 3 > 1$ and thus converges.

$\therefore$ By the DCT, $\sum \frac{\sin^2 n}{n^3}$ also converges

ii) $S_5 = \sum_{n=1}^{5} \frac{\sin^2 n}{n^3}$

≈ 0.82847

iii)

$$R = a_6 + a_7 + a_8 + \dots < b_6 + b_7 + b_8 + \dots < \int_5^\infty \frac{1}{x^3}\,dx$$

$$= \lim_{t \to \infty} \left[-\frac{1}{2} x^{-2} \right]_5^t$$

$$= \frac{1}{50}$$

Feb. 21

Alternating Series Test

If $\{a_n\}$ of (1) positive terms $(a_n > 0)$ with

(2) $\{a_n\}$ is decreasing

(3) $\lim_{n\to\infty} a_n = 0$

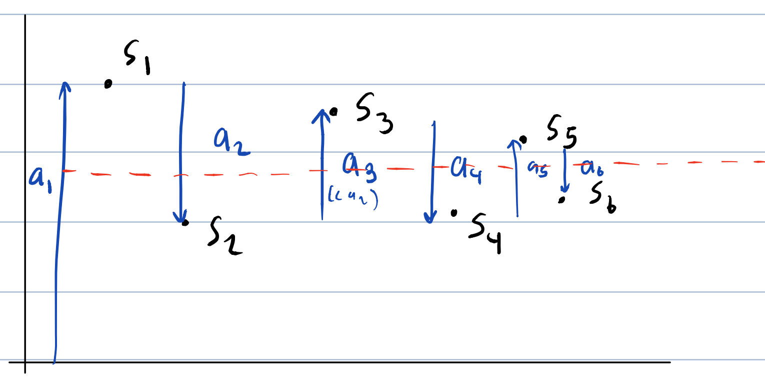
and if $b_n = (-1)^{n+1} a_n$ (or $b_n = (-1)^n a_n$)

then $\sum b_n$ converges.

<u>Proof:</u> $S_n = b_1 + b_2 + \ldots + b_n = a_1 - a_2 + a_3 + \ldots \pm a_n$

$\sum b_n = \lim_{n\to\infty} S_n$ ← we need to show that $\{S_n\}$ is conv

<u>Examine graph of S_n</u>



<u>Examine:</u> $\{S_{2n}\}$ the subsequence of even terms

$$S_{2n} = \underbrace{\underbrace{\underbrace{\underbrace{a_1 - a_2}_{S_2} + a_3 - a_4}_{S_4} + a_5 - a_6}_{S_6} + \dots + a_{2n-3} - a_{2n-2}}_{S_{2n-2} \,=\, S_{2(n-1)}} + a_{2n-1} - a_{2n}$$

Consecutive even terms

$$\Rightarrow S_{2n} = S_{2n-2} + a_{2n-1} - a_{2n}$$

$$\Rightarrow S_{2n} - S_{2n-2} = a_{2n-1} - a_{2n}$$

$$> 0 \quad \text{since } \{a_n\} \downarrow$$

$\therefore \{S_{2n}\}$ is incr

$$S_{2n} = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots + a_{2n-3} - a_{2n-2} + a_{2n-1} - a_{2n}$$

$$= a_1 - \underbrace{(a_2 - a_3)}_{+} - \underbrace{(a_4 - a_5)}_{+} + \dots + - \underbrace{a_{2n}}_{+}$$

$\therefore \{S_{2n}\}$ is bounded above by a_1

$\therefore \{S_{2n}\} \uparrow$, bounded

$\therefore \{S_{2n}\}$ converges by the BMCT

Examine $\{S_{2n+1}\}$ the odd terms and let $\lim_{n\to\infty} S_{2n} = S$

notice:

$$S_{2n+1} = S_n + a_{2n+1}$$

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to 0} a_{2n+1}$$

$$= S + 0$$

$$= S$$

Since $\lim_{n\to\infty} S_{2n} = S$

& $\lim_{n\to\infty} S_{2n+1} = S$

$\therefore \lim_{n\to\infty} S_n = S$ also

$\therefore \sum b_n = S$

conv

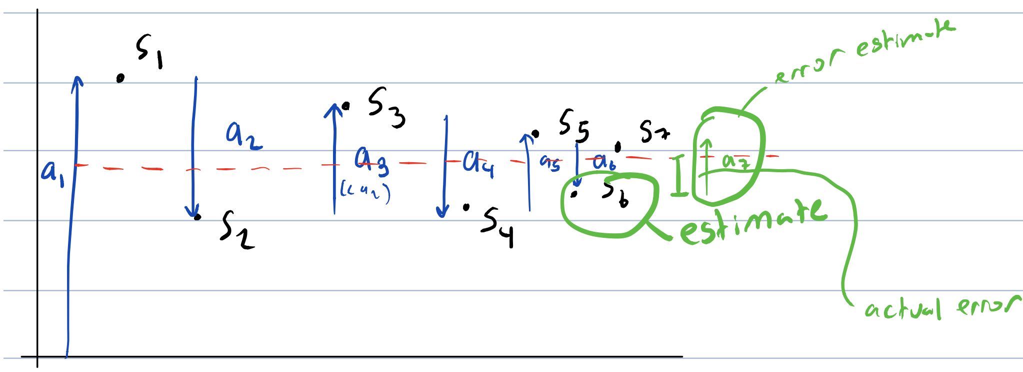
Error estimates

Using S_n as an estimate for $S = \sum b_n$,

error

$$R_n = |S - S_n| < a_{n+1} = |b_{n+1}|$$

"first neglected term"



Note: Error Estimates:

- integral test $R_n < \int_n^\infty f(x)\,dx$

- DCT $R_n \begin{matrix} < \text{geo} \\ < \text{int} \end{matrix}$

  ↑ direct comparison test

- AST $R_n < a_{n+1}$

  ↑ alternating series test

ex. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ vs $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

↳ from before: Int test: div

apply AST: $a_n = |b_n| = \frac{1}{n \ln n}$

① $a_n > 0$ $\quad n > 2;\ n, \ln n > 0$

② $\{a_n\} \downarrow$ $\quad n, \ln n \uparrow \therefore n \ln n \uparrow \therefore \frac{1}{n \ln n} \downarrow$

$n \ln n$

③ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n \ln n} = 0$

$\therefore$ By AST $\sum \frac{(-1)^n}{n \ln n}$ converges

Vocabulary: $\sum \frac{(-1)^n}{n \ln n}$ is conditionally convergent
(original is div)

ex. estimate $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ using 4 terms and give a bound on the error

$S_4 = \frac{1}{2 \ln 2} - \frac{1}{3 \ln 3} + \frac{1}{4 \ln 4} - \frac{1}{5 \ln 5}$

with error $< a_5 = \frac{1}{6 \ln 6}$ ← recall starting at $n=2$

ex. Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

a) Use the AST to show that the series converges

b) Use S_6 to approximate the series & give a bound on the error

c) How many terms are needed to guarantee an approximation correct to 7 decimals?

a) $a_n = |b_n| = \frac{1}{n^2}$

① $a_n > 0$ for $n > 1$, $n^2 > 0 \therefore \frac{1}{n^2} > 0$

② $\{a_n\} \downarrow$ $n^2 \uparrow$ for $n > 1 \therefore \frac{1}{n^2} \downarrow$

③ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$

$\therefore$ By ACT $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ Conv

b) $S_6 = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots \pm \frac{1}{36}$

≈ 0.810833

$R_n < |S_7| = \frac{1}{49}$

7th decimal place

c) $R_n < \left|\frac{1}{(n+1)^2}\right| < 0.0000000|5$ (round down will be 10^{-7})

$5 \times 10^{-8} > \frac{1}{(n+1)^2}$

$(n+1)^2 > \frac{1}{5} \times 10^8$

$n+1 > \sqrt{\frac{1}{5} \times 10^8}$

$n > \sqrt{2 \times 10^7} - 1$

Absolute Convergence

$\oplus$ or $\ominus$

Def^n: • A series $\sum a_n$ is called absolutely convergent if

$\sum |a_n|$ converges.

• A convergent that is not absolutely convergent is called

conditionally convergent.

ex. (Back to previous)

$\sum \frac{(-1)^{n+1}}{n^2}$ conv by AST

$\sum \left| \frac{(-1)^{n+1}}{n^2} \right|$ conv by p-series where $p = 2 > 1$

$\therefore \sum \frac{(-1)^{n+1}}{n^2}$ is absolutely conv.

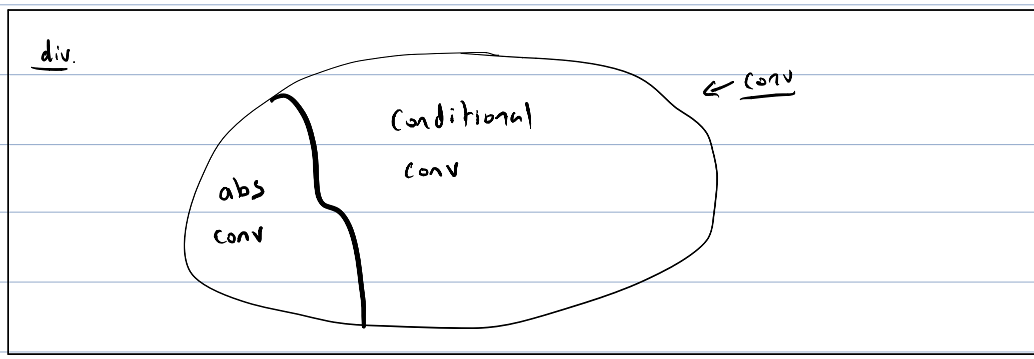
ex. If $\sum b_n$ is conv and $b_n > 0$

$\sum |b_n| = \sum b_n$ (all conv $\oplus$ve term series are abs conv)

Absolute Conv Implies Conv

Theorem: If $\sum |a_n|$ is conv, then $\sum a_n$ is also conv

All series



ex.

a) $\sum \frac{(-1)^{n+1}}{n^2}$ is conv by AST

alternate logic: Examine $\sum \left| \frac{(-1)^{n+1}}{n^2} \right| = \sum \frac{1}{n^2}$ conv p-series

$\therefore \sum \frac{(-1)^{n+1}}{n^2}$ is abs conv, and is therefore conv

b) $\sum \frac{\sin(n)}{n^2}$ ← not always positive ← not alternating so AST does not apply

Examine $\sum \left| \frac{\sin(n)}{n^2} \right|$

$0 < |\sin(n)| < 1$

$\therefore\ 0 \quad \left|\dfrac{\sin(n)}{n^2}\right| < \dfrac{1}{n^2}$

$\therefore\ \sum \left|\dfrac{\sin(n)}{n^2}\right|$ conv by DCT

$\therefore\ \sum \dfrac{\sin(n)}{n^2}$ conv also by abs conv

Feb. 23

<u>Theorem</u>: Abs conv $\Rightarrow$ conv

$\sum |a_n|$ conv $\Rightarrow \sum a_n$ conv

<u>Proof:</u> Notice

$$0 \leq a_n + |a_n| \leq 2|a_n|$$

Let $\sum |a_n|$ be conv, then $\sum 2|a_n|$ is also conv

By DCT, $\sum \left(a_n + |a_n|\right)$ is conv.

Notice that $\sum a_n = \underbrace{\sum (a_n + |a_n|)}_{\text{conv by DCT}} - \underbrace{\sum |a_n|}_{\text{conv by hypothesis}}$

$\therefore \sum a_n$, as the difference of 2 conv series is conv as well.

ex. Classify each as div, abs conv, or cond conv.

① $\sum \frac{(-1)^{n+1}}{n}$

② $\sum \frac{(-1)^{n+1}}{2^n+1}$

③ $\sum (-1)^n \left(\frac{n}{n+1}\right)^n$

1) $\sum \frac{(-1)^{n+1}}{n}$

$\sum |a_n| = \sum \frac{1}{n}$ div since it's a p-series where $p = 1 \leq 1$

AST:

① $\left|\frac{(-1)^{n+1}}{n}\right| = \frac{1}{n} > 0$

② $\{\frac{1}{n}\} \downarrow$ since $n \uparrow \therefore \frac{1}{n} \downarrow$

③ $\lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore$ By AST, $\sum \frac{(-1)^{n+1}}{n}$ conv

$\therefore \sum \frac{(-1)^{n+1}}{n}$ is cond conv

2) $\sum \left| \frac{(-1)^{n+1}}{2^n+1} \right| = \sum \frac{1}{2^n+1} \le \sum \frac{1}{2^n}$

$\sum \left(\frac{1}{2}\right)^n$ is a geo series that conv since $r = \frac{1}{2} < 1$ by the

$\therefore \sum \frac{1}{2^n+1}$ conv as well by the DCT

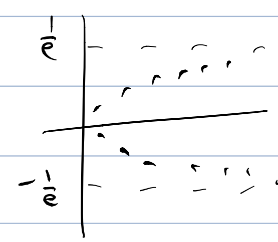
$\therefore \sum \frac{1}{2^n+1}$ is abs conv

3) $\sum |a_n| = \sum \left(\frac{n}{n+1}\right)^n$

$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{e}$

$\therefore \sum |a_n|$ div by n^{th} term test

$\lim_{n\to\infty} (-1)^n \left(\frac{n}{n+1}\right)^n$ DNE



How do we know? Even: $\lim_{n\to\infty} a_{2n} = \frac{1}{e}$

even & odds are going to different places

odd: $\lim_{n\to\infty} a_{2n+1} = -\frac{1}{e}$

$\therefore$ div by n^{th} term test

Rearrangements

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ conv.}$$

Let $s = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ $\quad s = \ln 2$

$$s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots$$

$$\frac{1}{2} s = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8}$$

$$\frac{3}{2} s = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots$$

adding in a different order

Same terms in a different order

~~Commutativity of addition~~

gives a different value

Riemann Rearrangement Theorem

If $\sum a_n$ is <u>conditionally</u> conv, then the terms of the series can be rearranged to give any value we like.

This is <u>not</u> true for <u>abs conv</u>

Ratio Test

idea: geo series $r = \frac{a_{n+1}}{a_n}$

$|r| < 1$

← A series that passes the ratio test is "almost" geometric

If $\sum a_n$ is a series of positive terms $(a_n > 0)$

if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and equals $L < 1$

then $\sum a_n$ conv. (if $L > 1$, series div; $L = 1$, test fails)

Root Test

If $\sum a_n$ is a series with positive terms $(a_n > 0)$

and if $\lim_{n\to\infty} \sqrt[n]{a_n}$ exists and equals $L < 1$,

then $\sum a_n$ conv. (if $L > 1$, series div; $L = 1$, test fails)

ex. Categorize each as div, abs conv, or cond conv

① $\sum \frac{(-1)^n n!}{n^n}$

② $\sum \left(\frac{n}{n+1}\right)^{n^2}$

③ $\sum n\left(-\frac{3}{2}\right)^n$

1) $\sum |a_n| = \sum \frac{n!}{n^n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= \lim_{n\to\infty} \cancel{(n+1)} \cdot \frac{n^n}{(n+1)^{n+1}}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \frac{1}{e} < 1$$

$\therefore \sum |a_n|$ conv by the ratio test

$\therefore \sum a_n$ is abs conv

2) $$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \frac{1}{e} < 1$$

$\therefore$ By the root test, $\sum a_n$ conv. Also $\sum |a_n| = \sum a_n$

$\therefore \sum a_n$ is abs conv

3) $\sum n\left(-\frac{3}{2}\right)^n = \sum (-1)^n n\left(\frac{3}{2}\right)^n$

Note: by default $\sum|a_n|$ is not geo. But can compare to geo series

$\sum |a_n| = \sum n\left(\frac{3}{2}\right)^n$

$\lim_{n\to\infty} n\left(\frac{3}{2}\right)^n = \infty$

By the n'th term test, $\sum |a_n|$ div

$\lim_{n\to\infty} n\left(-\frac{3}{2}\right)^n$ DNE since even terms: $\lim_{n\to\infty} a_n = -\infty$

odd terms : $\lim_{n\to\infty} a_n = \infty$

$\therefore \sum a_n$ div

Power Series

$\cancel{(\ln 2x)^n}$

$\sum_{n=0}^{\infty} C_n(x-a)^n$ C_n's constant

a constant

Theorem:

If $\sum C_n(x-a)^n$ is a power series,

there is a value $R \in [0, \infty)$ (radius of convergence), such that the power series converges for all x

such that $|x-a| < R$, diverges for all x

such that $|x-a| > R$ (if $|x-a| = R$, either conv or div is possible)

Feb. 26

<u>Power Series</u>

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

Why power series?

- They're cool!
- good for approximations $\rightarrow$ calculators
- evaluating numerical series

$e^2 = e \cdot e$

$e^{-4} = \frac{1}{e^2 e^2}$

$e^{2.5} = e^2 e^{1/2} = e^2 \sqrt{e}$

$e^{\pi} = ?$

$= \sum_{n=0}^{\infty} \frac{\pi^n}{n!}$ ← conv

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

fill in "gaps" of transcendel functions

ex, $\int e^{x^2} dx$ FTC → guarantees answer

Answer is expressable as a series

Power series

$\sum_{n=0}^{\infty} C_n(x-a)^n$ ~ Function series:

ex, $\sum_{n=0}^{\infty} C_n(g(x))^n$

Interval of Convergence

R - radius of convergence $R \in [0, \infty)$

$|x-a| < R$ conv → $a-R < x < a+R$

$|x-a| > R$ div

$|x-a| = R$ anything possible conv or div when $R = 0$: "interval" $[a, a]$

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \; ; \; f(a) = c_0$$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 \ldots$$

Series always conv at $x = a$

ex. Find the interval of conv of $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n} (x-2)^n$.

Classify the pts of conv as absolute or conditional

Sol'n: By ratio test, series will conv when

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

$$= \lim_{n\to\infty} \left| \frac{\ln(n+1)}{n+1} (x-2)^{n+1} \cdot \frac{n}{\ln n \; (x-2)^n} \right| < 1$$

$$= \lim_{n\to\infty} \left| \frac{\ln(n+1)}{\ln(n)} \cdot \frac{n}{n+1} \cdot (x-2) \right| < 1$$

$= |x-2| \lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n)} \cdot \lim_{n\to\infty} \frac{n}{n+1} < 1$

$= |x-2| < 1$

$-1 < x-2 < 1$

$1 < x < 3$

Test end pts

$(1, 3)$ absolute convergence since ratio test looks at abs val

$R = 1$

centre $= a = 2$

at $x=1$

$\sum \frac{\ln n}{n}(1-2)^n$

$= \sum \frac{(-1)^n \ln n}{n} \rightarrow \sum a_n$

at $x=3$

$\sum \frac{\ln n}{n}(3-2)^n$

$= \sum \frac{\ln n}{n} \rightarrow \sum |a_n|$

Test this one first

Use DCT

$\ln n > 1$ (for $n > e$)

$\frac{\ln n}{n} > \frac{1}{n} > 0$

Since $\sum \frac{1}{n}$ is div p-series ($p=1$),

then $\sum \frac{\ln n}{n}$ div also by DCT

AST: ① $\ln n$, n, $\oplus$ $\therefore$ $\frac{\ln n}{n}$ $\oplus$

② $\frac{\ln n}{n} \downarrow$ since $f'(x) = \frac{\frac{1}{x} \cdot x - \ln x}{x^2}$

$$= \frac{1-\ln x}{x^2} < 0 \text{ for } x > e$$

③ $\lim_{n\to\infty} \frac{\ln n}{n} = 0$ by growth hierarchy

$\therefore \sum \frac{(-1)^n \ln n}{n}$ conv by AST

$\therefore \sum_{n=1}^{\infty} \frac{\ln n}{n}(x-2)^n$ conv on $[1,3)$

absolutely on $(1,3)$

Conditionally at $x=1$

For each of the following, find the center, radius and interval of convergence (including the endpoints). State clearly where the convegence is absolute, and where it is conditional.

Remember the general procedure:

- Examine $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ (or $\lim_{n\to\infty}\sqrt[n]{|a_n|}$) to determine if the series converges only at $x = a$ or if the series converges absolutely on some open interval.
- If the radius of convergence R is such that $0 < R < \infty$, i.e. the series converges absolutely not on $\mathbb{R}$ but on $(a-R, a+R)$, test separately for convergence at the endpoints: $x = a-R$ and $x = a + R$.

  Often, this will mean testing two related series $\sum c_n$ and $\sum b_n$, where $b_n > 0$, and $c_n = (-1)^n b_n$ is alternating. For efficiency:
  - test $\sum b_n$ first, since if it converges, $\sum c_n$ will also converge (because absolute convergence guarantees convergence).
  - test $\sum c_n$ (using the AST) only if $\sum b_n$ diverges.

1. $\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{\sqrt{n+1}}(5-2x)^n$

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{(-3)^{\cancel{n+2}}}{\sqrt{n+2}}(5-2x)^{\cancel{n}+1}\cdot\frac{\sqrt{n+1}}{\cancel{(5-2x)^n}\cdot\cancel{(-3)^{n+1}}}\right|$$

$$= 3|5-2x| \lim_{n\to\infty} \sqrt{\frac{n+1}{n+2}}$$

$$= 3|5-2x| < 1$$ ~ or $3|2x-5| < 1$

$$3\cdot 2\left|x-\tfrac{5}{2}\right| < 1$$

$$\Big|x - \underbrace{\tfrac{5}{6}}_{a}\Big| < \underbrace{\tfrac{1}{6}}_{R}$$

$$-1 < 15-6x < 1$$

$$1 > 6x-15 > -1$$

$$16 > 6x > 14$$

$$\frac{8}{3} > x > \frac{7}{3}$$

$\left(\frac{7}{3}, \frac{8}{3}\right)$ ~ abs conv

$R = \frac{1}{6}$

center $= \frac{15}{6} = \frac{5}{2}$

<u>at $x = \frac{7}{3}$:</u>

$$\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{\sqrt{n+1}} \left(5 - 2\cdot\frac{7}{3}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{\sqrt{n+1}} \left(\frac{1}{3}\right)^n$$

$$= 3\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$$

<u>at $x = \frac{8}{3}$:</u>

$$\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{\sqrt{n+1}} \left(5 - 2\cdot\frac{8}{3}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{\sqrt{n+1}} \left(-\frac{1}{3}\right)^n$$ ~ $(-3)^{-n}$

$$= \sum_{n=0}^{\infty} \frac{-3}{\sqrt{n+1}}$$

$$= -3\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$$

<u>LCT:</u> $\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}}$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = \frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\sqrt{1+\frac{1}{n}}} = 1$$

$\therefore \sum \frac{1}{\sqrt{n}}$ is a p-series where $p = \frac{1}{2} \leq 1$ and thus diverges.

$\therefore \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$ div also

AST:

① $\frac{1}{\sqrt{n+1}} > 0$

② $\sqrt{n} \uparrow \therefore \sqrt{n+1} \uparrow \therefore \frac{1}{\sqrt{n+1}} \downarrow$

③ $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$

$\therefore$ By AST, $3 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$ conv

$\therefore \sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{\sqrt{n+1}} (5-2x)^n$ conv on $\left[\frac{7}{3}, \frac{8}{3}\right)$

Conditionally at $x = \frac{7}{3}$

absolutely on $\left(\frac{7}{3}, \frac{8}{3}\right)$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(2x+3)^n}{5^n(n^2+1)}$

$$\lim_{n\to\infty} \left| \frac{(2x+3)^{\cancel{n+1}}}{5^{n+1}\left((n+1)^2+1\right)} \cdot \frac{\cancel{5^n}(n^2+1)}{\cancel{(2x+3)^n}} \right|$$

$= \frac{1}{5}|2x+3| \lim_{n\to\infty} \left| \frac{n^2+1}{(n+1)^2+1} \right|$

$= \frac{1}{5}|2x+3| \lim_{n\to\infty} \left| \frac{\cancel{n^2}(1+\frac{1}{n^2})}{\cancel{n^2}(1+\frac{1}{n})^2+\frac{1}{n^2}} \right|$

$= \frac{1}{5}|2x+3| < 1$

$\frac{1}{5} 2 |x+\frac{3}{2}| < 1$

$|x+\frac{3}{2}| < \frac{5}{2}$

$a = -\frac{3}{2}$, $R = \frac{5}{2}$

$(-4, 1)$

at $x=-4$:

$\sum_{n=0}^{\infty} \frac{(-\cancel{5})^n}{\cancel{5^n}(n^2+1)}$

$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$

at $x=1$:

$S_2 = \sum_{n=0}^{\infty} \frac{\cancel{5^n}}{\cancel{5^n}(n^2+1)}$

$= \sum_{n=0}^{\infty} \frac{1}{n^2+1}$

$0 < \frac{1}{n^2+1} < \frac{1}{n^2}$

$\sum \frac{1}{n^2}$ conv since it's a p-test where $p=2>1$

$\therefore \sum_{n=0}^{\infty} \frac{1}{n^2}$ conv also by the DCT

absolute convergence

Since $\sum \left|\frac{(-1)^n}{n^2+1}\right| = \sum \frac{1}{n^2+1}$, the series conv at $x=-4$ also.

$\therefore \displaystyle\sum_{n=0}^{\infty} \frac{(2x+3)^n}{5^n (n^2+1)}$ conv absolutely on $[-1, 4]$

Constructing Power Series

- Geometric Series Formula (when it applies)
- Taylor's Formula

→ new series from old ones

- composition
- multiplication / division
- integration / differentiation

Taylor's Formula

If $f(x)$ is an infinitely differentiable function at $x=a$,

then, the Taylor series generated by $f(x)$ at $x=a$ is:

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Notes:

- We aren't guaranteed that $f(x) = T(x)$
  ($f(x) \neq T(x)$ is rare, but does occur)

- Note that $T^{(n)}(a) = f^{(n)}(a)$
  by design (easy to check)
  (derivatives $\to$ shape of graph)

$$T(x) = \frac{\overbrace{f^{(0)}(a)}^{f(a)}}{\underbrace{0!}_{1}} + \frac{\overbrace{f^{(1)}(a)}^{f'(a)}}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \ldots$$

- Convergence of $T(x) \to$ conv of a power series
  (guaranteed to converge at $x=a$)

- Maclaurin series : Taylor series at $a=0$

ex. Use the formula to generate the Maclaurin series for
$f(x) = e^x$

$f^{(0)}(x) = e^x \qquad f^{(0)}(0) = e^0 = 1$

$f^{(1)}(x) = e^x \qquad f^{(1)}(0) = e^0 = 1$

$f^{(2)}(x) = e^x \qquad \vdots$

$f^{(3)}(x) = e^x$

$\vdots$

$f^{(n)}(x) = e^x \qquad f^{(n)}(0) = 1$

$$\sum \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{n=0}^{\infty} \frac{1 \cdot x^n}{n!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{on } (-\infty, \infty)$$ can be proven

- Convergence is quickest close to a

Feb. 28

<u>Taylor Series</u>

$$T(x) = \sum \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$T^{(n)}(a) = f^{(n)}(a)$$

But it needs to be proven $T(x) = f(x)$

<u>ex.</u> $g(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$

(maclaurin series generated by $g(x)$) $= 0 \neq g(x)$

matches at $a = 0$

<u>i.e.</u> radius of conv $R = 0$

ex. Find the Mauclaurin series generated by $\sin(x)$

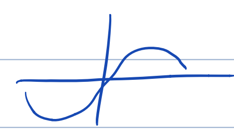
	$f^{(n)}(0)$
$f^{(0)} = \sin x$	0
$f^{(1)} = \cos x$	1
$f^{(2)} = -\sin x$	0
$f^{(3)} = -\cos x$	-1
$f^{(4)} = \sin x$	0

$$T(x) = 0 + \frac{1x^1}{1!} + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} \ldots$$

$$= x - \frac{x^3}{3!} + \frac{x^3}{5!} + \ldots$$

odd function : $-f(x) = f(-x)$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$



rotational symmetry

ex. Find the interval of convergence

we want $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$

$$\lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{\cancel{(-1)^n} x^{2n+1}} \right| < 1$$

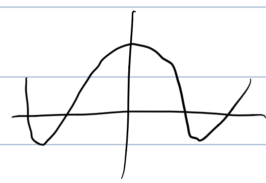
$$= x^2 \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} < 1$$

$$x^2 \cdot 0 < 1$$

$0 < 1$ ✓ for any value of x

$\therefore$ series converges for $x \in \mathbb{R}$

ex. Can you guess the formula for $\cos(x)$



even

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Result: $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ on $(-\infty, \infty)$

Using Known Power Series to Evaluate Numerical Series

ex. Evaluate (a) $\sum_{n=0}^{\infty} \frac{1}{n!}$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$

(c) $1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} + \dots$

(a) $\sum_{n=0}^{\infty} \frac{1}{n!} = e^1$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} = e^{-2}$

(c) $= 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n!} = \cos(1)$

ex. Rewrite $e^{i\theta}$ where $i = \sqrt{-1}$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\,\theta^3}{3!} + \frac{\theta^4}{4!} + \dots$$

exponential form of a complex number

$$= i \sin\theta + \cos\theta$$

ex. Find the Maclaurin Series for $\sin(x^2)$

$$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2(2n+1)}}{2n!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n!}$$

ex. $xe^{x} = x\sum_{n=0}^{\infty}\frac{x^{n}}{n!} = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}$

ex. Find the first 4 terms of the Maclaurin series for

(1) $e^{x}\sin(x)$

(2) $\tan(x)$

(1) $e^{x}\sin x = \sum_{n=0}^{\infty}\frac{x^{n}}{n!} \cdot \sum_{n=0}^{\infty}(-1)^{n}\cdot\frac{x^{2n+1}}{(2n+1)!}$

$$= \left(1 + \frac{x^{1}}{2!} + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \frac{x^{4}}{4!}\right)\left(x - \frac{x^{3}}{6} + \frac{x^{5}}{120} - \frac{x^{7}}{5040} + \dots\right)$$

$$= \underline{1}\,x + \underline{1}\,x^{2} + \underline{-\tfrac{1}{6}+\tfrac{1}{2}}\,x^{3} + \underline{-\tfrac{1}{3!}+\tfrac{1}{3!}}\,x^{4} + \dots$$

(2) $\tan x = \frac{\sin x}{\cos x}$

$$x + \frac{1}{3}x^{3}$$

$$1 - \frac{x^{2}}{2!} + \frac{x^{4}}{4!} - \frac{x^{6}}{6!} + \dots \;\Big)\; x - \frac{x^{3}}{3!} + \frac{x^{5}}{5!} - \frac{x^{7}}{7!} + \dots$$

$$-\left(x - \frac{1}{2}x^{3} + \frac{1}{24}x^{5} - \frac{1}{720}x^{7} + \dots\right)$$

$$\frac{1}{3}x^{3} + \dots$$

$$-\left(\frac{1}{3}x^{3} + \dots\right)$$

□ repeat as desired

March 2

Theorem: If $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges on $(a-R, a+R)$

Then, (1) $\frac{d}{dx} \sum c_n (x-a)^n = \sum \frac{d}{dx} c_n (x-a)^n$

$$= \sum n c_n (x-a)^{n-1}$$

(2) $\int \sum c_n (x-a)^n \, dx = \sum \int c_n (x-a)^n \, dx$

$$= \sum \frac{c_n (x-a)^{n+1}}{n+1} + C$$

Note: - when differentiating, conv may be lost at an end pt (but never gained)

- when integrating, conv may be gained at an end pt (but never lost)

Series for arctan x

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$R = (-1, 1)$

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt$$

$$= \Big[\arctan t\Big]_0^x$$

$$= \arctan x$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n$$

So, $$\arctan x = \int_0^x = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n}\,dt$$

$$= \sum_{n=0}^{\infty}\int_0^x (-1)^n t^{2n}\,dt$$

$$= \sum_{n=0}^{\infty}\left[(-1)^n \frac{t^{2n+1}}{2n+1}\right]_0^x$$

$$= \sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n+1}}{2n+1} - 0\right)$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$$

ex. Evaluate $4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} + \dots$

$$= 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}\right)$$

$$= 4\sum_{n=0}^{\infty} \frac{(-1)^n (1)^{2n+1}}{2n+1} = 4\arctan(1)$$

$$= \cancel{4} \cdot \frac{\pi}{\cancel{4}}$$

$$= \pi$$

ex. ① Find the series for $\ln(x+1)$ by integrating a geometric series

② Evaluate $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$

$$\frac{d}{dx}\ln(x+1) = \frac{1}{x+1} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$$

$$\ln(x+1) = \int_0^x \frac{1}{t+1}\,dt = \int_0^x \sum_{n=0}^{\infty} (-t)^n\,dt$$

$$= \sum_{n=0}^{\infty} \int_0^x (-t)^n\,dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

② $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(1)^n}{n}$

$= \ln(2)$

③ $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ on $(-1,1)$.

So (a) What function has series $\sum_{n=0}^{\infty} n x^{n-1}$?

(b) What function has series $\sum_{n=0}^{\infty} n x^n$?

(c) Evaluate $\sum_{n=0}^{\infty} \frac{n}{3^n}$

(a) $\sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} \frac{d}{dx} x^n = \frac{d}{dx} \sum_{n=0}^{\infty} x^n$

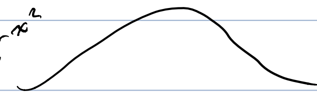
$= \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$

(b) $\frac{x}{(1-x)^2}$

(c) $\sum_{n=0}^{\infty} \frac{n}{3^n} = \sum_{n=0}^{\infty} n\left(\frac{1}{3}\right)^n$

$= \frac{1/3}{\left(\frac{2}{3}\right)^2} = \frac{1/3}{\frac{4}{9}} = \frac{9}{3 \cdot 4} = \frac{3}{4}$

ex. Approximate $\int_0^1 e^{-x^2}\,dx$

$y = e^{-x^2}$ 

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$\int_0^1 e^{-x^2}\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,dx$$

$$= \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n x^{2n}}{n!}\,dx$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1} \right]_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,n!}$$ conv since $R = \infty$

conv on $(-\infty, \infty)$

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

So integral conv

Binomial Series

Taylor Series of $(1+x)^k$ at $x=a=0$

if $k \in \mathbb{N}$, $(1+x)^k$ is a polynomial

$\Rightarrow$ finite series

But, if $k \in \mathbb{N}$

Apply: $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

	$f^{(n)}(x)$	$f^{(n)}(0)$
0:	$(1+x)^k$	1
1:	$k(1+x)^{k-1}$	k
2:	$k(k-1)(1+x)^{k-2}$	$k(k-1)$
3:	$k(k-1)(k-2)(1+x)^{k-3}$	$k(k-1)(k-2)$
$\vdots$		
n:	$k(k-1)\cdots(k-n+1)(1+x)^{k-n}$	$k(k-1)\cdots(k-n+1)$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!} + \ldots$$

$$= \sum_{n=0}^{\infty} \binom{k}{n} x^n \qquad + \boxed{\frac{k(k-1)\cdots(k-n+1)}{n!}} x^n + \ldots$$

on $(-1, 1)$

notation $\binom{k}{n}$ "k chose n"

End pts can change depending on k

ex. Use the binomial series to expand

$$f(x) = \frac{1}{\sqrt{1-x}} = \left(1 + (-x)\right)^{-1/2} \qquad k = -1/2$$

$$= 1 + \underbrace{\left(-\tfrac{1}{2}\right)(-x)}_{kx} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{\underbrace{2!}_{k(k-1)x^2}}(-x)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x)^3$$

$$+ \dots$$

$$= 1 + \frac{1}{2}x + \frac{1\cdot 3}{2^2\, 2!}x^2 + \frac{1\cdot 3\cdot 5}{2^3\, 3!}x^3$$

$$+ \dots + \frac{2\cdot 4\cdot 6\cdots 2n}{2\cdot 4\cdot 6\cdots 2n}\,\frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{2^n\, n!}x^n + \dots$$

$$\to \frac{(2n)!}{(2^n n!)^2}x^n$$

not all that important