Section 4 – Areas Volumes and Arc Lengths

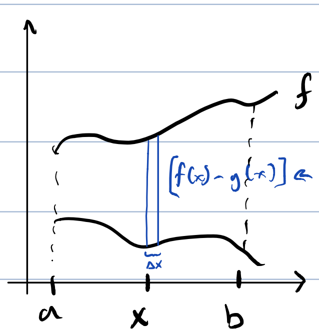
Area between curves

Let $y = f(x)$ and $y = g(x)$ represent two continuous functions over $[a,b]$

Assuming that $f \geq g$ over $[a,b]$

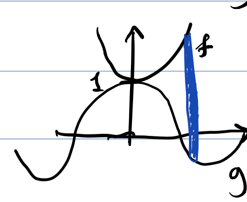
Then the area contained between them is :

$$\text{Area} = \int_a^b \overbrace{\underbrace{[\,f(x) - g(x)\,]}_{\text{height of the seperation between two curves}}\,dx}^{\text{for 1 rectangle}}$$



ex. Let $f(x) = x^2 + 1$ and $g(x) = \cos x$

Find the area contained between f and g within the interval $[0, \pi]$



← doesn't matter that g is negative

$$A = \int_0^{\pi} \left[(x^2+1) - \cos x \right] dx$$

$$= \left[\frac{x^3}{3} + x - \sin x \right]_0^{\pi}$$

$$= \frac{\pi^3}{3} + \pi$$

À vous !

Consider $f(x) = x^3 - x$ and $g(x) = x^2 - x$

Find the area of the region enclosed between f and g

pts of intersection: $x^3 - x = x^2 - x$

$$x^3 = x^2$$

$$x^3 - x^2 = 0$$

$$x(x^2 - x) = 0$$

$$x(x(x-1)) = 0$$

$$x = 0$$

$$x = 1$$

$[0,1] \rightarrow$ test $x = \frac{1}{2}$

$f(\frac{1}{2}) = -0.375$

$g(\frac{1}{2}) = -0.25$

So $g \geq f$ over $[0,1]$

$$A = \int_0^1 [g(x) - f(x)]\,dx$$

$$= \int_0^1 \left[(x^2 - x) - (x^3 - x)\right] dx$$

$$= \int_0^1 (x^2 - x^3)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{4}$$

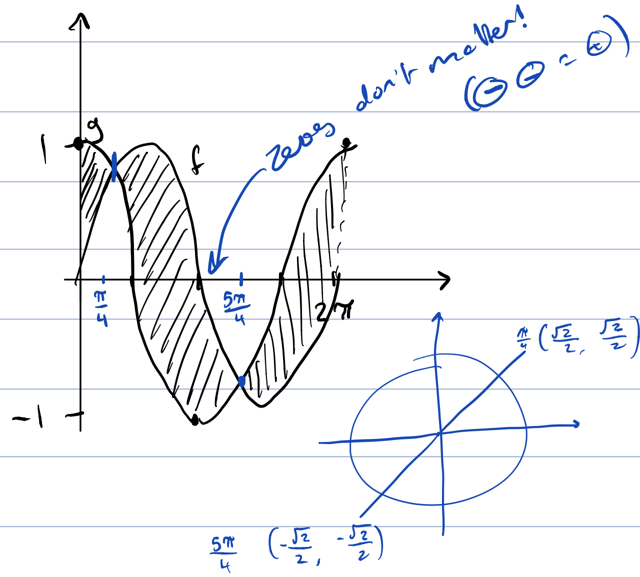
$$= \frac{1}{12}$$

Area

March 20, 2017

$$A = \int_a^b [f(x) - g(x)]\,dx$$ given that $f(x) \geq g(x)$ over $[a, b]$

Ex, Find the area that is found between $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[0, 2\pi]$



$$A = \int_0^{\frac{\pi}{4}} [\cos x - \sin x]\,dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} [\sin x - \cos x]\,dx + \int_{\frac{5\pi}{4}}^{2\pi} [\cos x - \sin x]\,dx$$

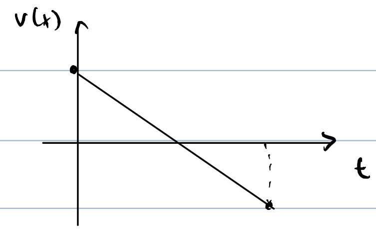
⋮

Special Case: DISPLACEMENT VS DISTANCE, INTEGRATIVE ACTIVITY

displacement = signed area of the velocity function

$$\text{displacement} = \int_{t_1}^{t_2} v(t)\,dt$$

distance = area between the velocity function and the t-axis



S.A. = disp

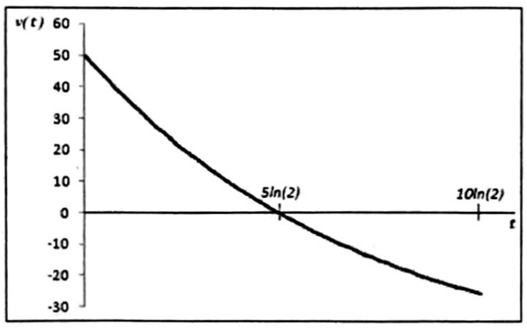
Area = distance

Area...
201 – NYB
March 20

Problem 39
A pilot ejects from his F-14 after it has been severely damaged by the attack of a trig sub (submarine, not substitution). The pilot's vertical velocity (in m/s) at time t is given by

$$\boxed{v(t) = -50 + 100e^{-0.2t}}$$

The graph of $v(t)$ is shown.



a) Find the pilot's displacement over the time interval $[0, 10\ln(2)]$.

distance $= \int_0^{5\ln 2} \left[\left(-50 + 100e^{-0.2t}\right) - 0\right] dt + \int_{5\ln 2}^{10\ln 2} \left[0 - \left(-50 + 100e^{-0.2t}\right)\right] dt$

$100e^{-0.2t}$

$= \left[-50t - 500e^{-0.2t}\right]_0^{5\ln 2} + \left[50t + 500e^{-0.2t}\right]_{5\ln 2}^{10\ln 2}$

b) Find the distance the pilot has travelled over the time interval $[0, 10\ln(2)]$.

displacement $= \int_0^{10\ln 2} v(t)\, dt$

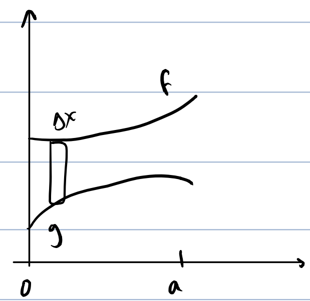
$= \int_0^{10\ln 2} \left[-50 + 100e^{-0.2t}\right] dt$

$= \left[-50t - 500e^{-0.2t}\right]_0^{10\ln 2}$

Horizontal rectangle representation

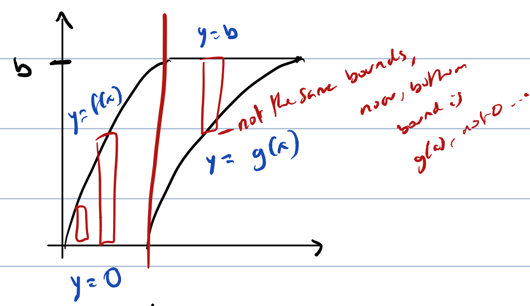
It may be convenient to express (convert) functions of x into functions of y in order to evaluate areas.

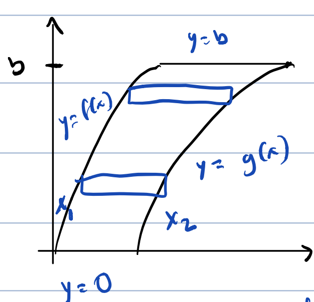
→ Classic x-interval



$$A = \int_0^a [f(x) - g(x)]\, dx$$

→





Horizontal rectangle representation

width of rectangle $= \Delta y \to dy$

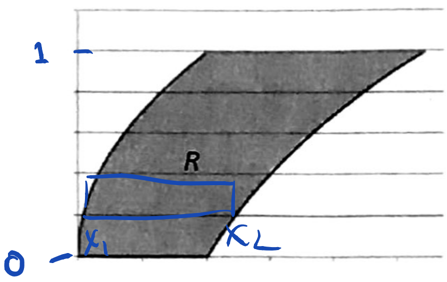
length of rectangle $= x_2 - x_1$

$= g^{-1}(y) - f^{-1}(y)$

$$\text{Area} = \int_0^b \overbrace{[g^{-1}(y) - f^{-1}(y)]\, dy}^{\text{area of one rectangle}}$$

Problem 41

Consider the region R (shown below) and its boundary functions:



$f(x) = \sqrt{x}$
$g(x) = \ln(x)$
$h(x) = 1$
$y = 0$

a) Set up the x-integral(s) that would be required to compute the area of region R.
b) Set up the y-integral(s) that would be required to compute the area of region R.
c) Compute the area of region R using either result of b) or c).

$f: \quad y = \sqrt{x} \quad \rightarrow \quad x_1 = y^2$

$g: \quad y = \ln x \quad \rightarrow \quad x_2 = e^y$

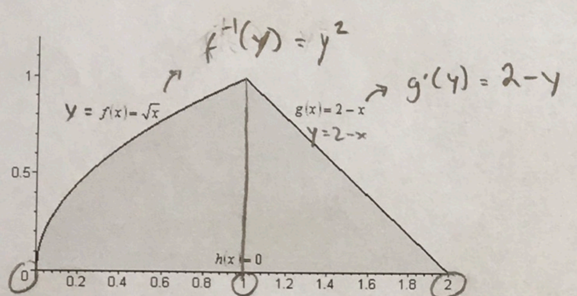
$$\text{Area} = \int_0^1 (\overbrace{e^y}^{x_2} - \overbrace{y^2}^{x_1})\, dy$$

$$= \left[e^y - \frac{y^3}{3}\right]_0^1$$

$$= \left(e - \frac{1}{3}\right) - 1$$

Problem 44

Consider the region that is bounded by the functions $f(x) = \sqrt{x}$, $g(x) = 2 - x$, and the x-axis (which we can think of as $h(x) = 0$).



a) Set up the x-integral(s) that would be required to compute the area of region R.
b) Set up the y-integral(s) that would be required to compute the area of region R.
c) Compute the area of region R using either result of b) or c).

a) $A_x = \int_0^1 [\sqrt{x} - 0]\, dx + \int_1^2 [(2-x) - 0]\, dx$

b) $A_y = \int_0^1 [g^{-1}(y) - f^{-1}(y)]\, dy$

$$= \int_0^1 [(2-y) - y^2]\, dy$$

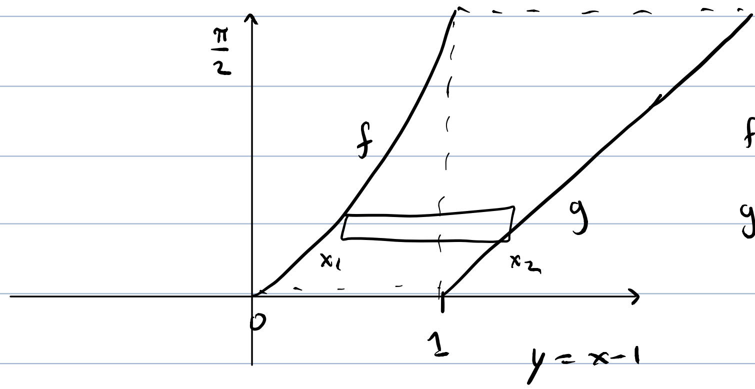
c)

$$= \left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$= 2 - \frac{1}{2} - \frac{1}{3}$$

$$= \frac{7}{6}$$

À vous Set up using both x- and y-integrals the expressions required to compute the area of the following region.



$f: y = \arcsin x$

$g: y = x - 1$

$y = x - 1$

$\frac{\pi}{2} = x - 1$

$x = \frac{5\pi}{2}$

x-integral

$$A_x = \int_0^1 \left[f(x) - 0 \right] dx + \int_1^{\frac{5\pi}{2}} \left[\frac{\pi}{2} - g(x) \right] dx$$

$$= \int_0^1 \arcsin x \, dx + \int_1^{\frac{5\pi}{2}} \left[\frac{\pi}{2} - (x-1) \right] dx$$

y-integral

$f: y = \arcsin x \rightarrow x = \sin y$

$g: y = x - 1 \rightarrow x = y + 1$

$$A_y = \int_0^{\frac{\pi}{2}} \left[g^{-1}(y) - f^{-1}(y) \right] dy$$

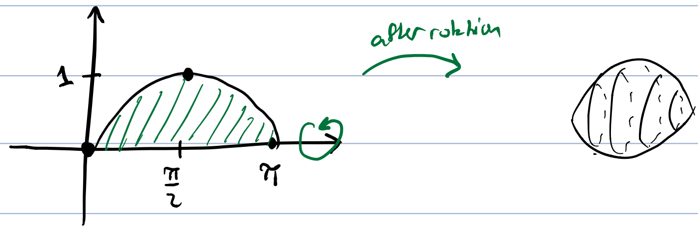
$$= \int_0^{\frac{\pi}{2}} \left[(y+1) - \sin y \right] dy$$

Volumes of revolution

March 22, 2017

→ evaluate volumes that are obtained by rotating regions (enclosed between curves) around an axis of symmetry

ex. Suppose $f(x) = \sin x$ over $[0, \pi]$ is rotated around the x-axis. What volume would be obtained?

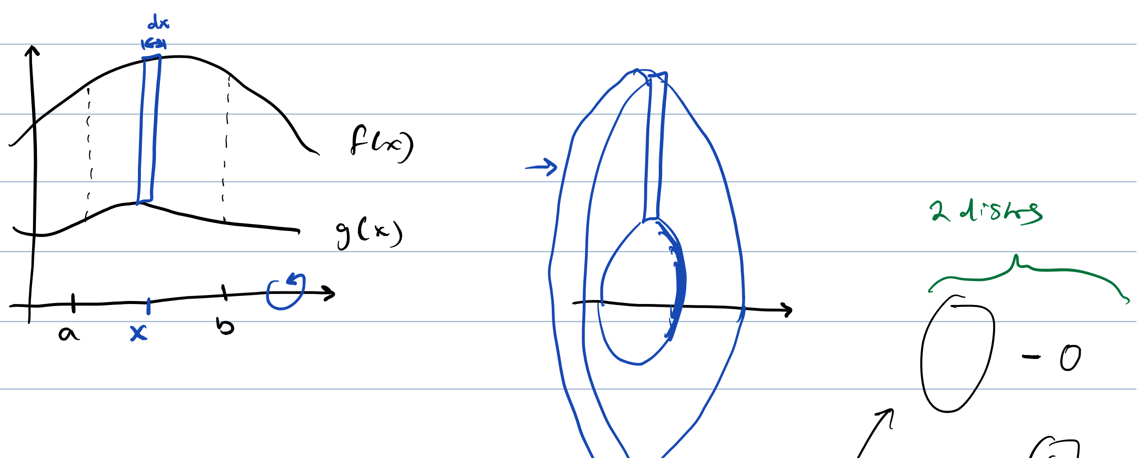


Method 1: Disk/Washer Method

→ Subdivide the area to be rotated into rectangles that are perpendicular to the axis of rotation.

→ Determine the volume generated by that rectangle

Idea: Consider the area between $f(x)$ and $g(x)$ within interval $[a, b]$ rotated around the x-axis.



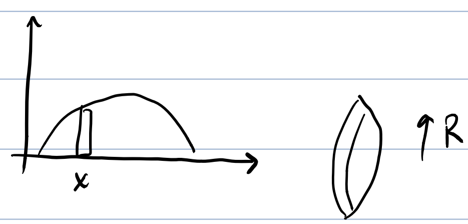
volume of disk $= \pi r^2 h$

$$dV = \pi R^2 w - \pi r^2 w$$
$$= \pi [f(x)]^2 dx - \pi [g(x)]^2 dx$$
$$= \pi \left([f(x)]^2 - [g(x)]^2 \right) dx$$

$$V = \int_{\text{domain}} dV$$

$$= \pi \int_a^b \left([f(x)]^2 - [g(x)]^2 \right) dx$$

ex. Let $f(x) = \sin x$ over $[0, \pi]$. What volume is obtained if $f(x)$ is rotated around the x-axis?



$$dV = \pi (f(x))^2 dx$$
$$= \pi \sin^2 x \; dx$$

$$V = \int_0^\pi \pi \sin^2 x \, dx$$

$$= \pi \int_0^\pi \frac{1}{2} (1 - \cos 2x) \, dx$$

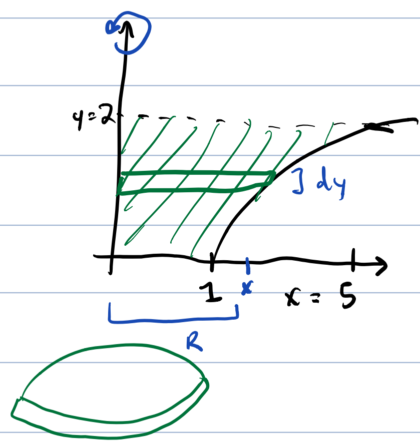
$$= \frac{\pi}{2} \left[\int_0^\pi 1 \, dx - \frac{1}{2} \int_0^\pi \cos 2x \; \overbrace{2 \, dx}^{dx} \right.$$

$$= \frac{\pi}{2}\left[x \Big|_0^{\pi} - \frac{1}{2}\sin 2x \Big|_0^{\pi} \right]$$

$$= \frac{\pi}{2}\left[\pi - \frac{1}{2}(0) \right] = \frac{\pi^2}{2}$$

ex. Let $f(x) = \sqrt{x-1}$

Find the volume generated by $f(x)$ when it is rotated around the y-axis (for values of y between 0 and 2)



$$2 = \sqrt{x-1}$$

$$x = 5$$

$$dV = \pi R^2 w$$

$$= \pi \left[f^{-1}(y) \right]^2 dy \quad \text{or} \quad = \pi x^2 dy$$

$$= \pi (y^2+1)^2 dy$$

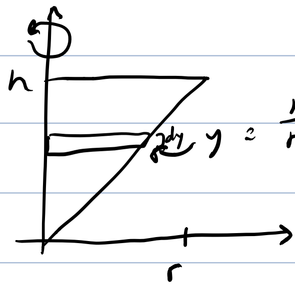
don't need to convert bounds b.c. it's with respect to y

$$V = \int_0^2 \pi (y^2+1)^2 dy$$

Note: when using a horizontal rectangle representation:

$$V = \int_c^d \pi \left[x_2^{\,2} - x_1^{\,2} \right] dy$$

ex.



$y = \frac{h}{r} x \rightarrow x = \frac{r}{h} y$

$dV = \pi \left(\frac{r}{h} \cdot y\right)^2 dy$

$V = \int_0^h \pi \left(\frac{r}{h} \cdot y\right)^2 dy$

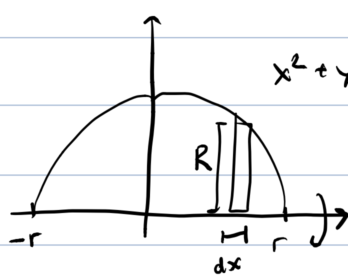
$= \pi \cdot \frac{r^2}{h^2} \int_0^h y^2 \, dy$

$= \pi \frac{r^2}{h^2} \left[\frac{y^3}{3} \right]_0^h$

$= \pi \frac{r^2}{h^2} \cdot \frac{h^3}{3}$

$= \frac{\pi r^2 h}{3}$

ex.



$x^2 + y^2 = r^2 \rightarrow y^2 = r^2 - x^2$

$y = \sqrt{r^2 - x^2}$

$$dV = \pi\left(\sqrt{r^2-x^2}\right)^2 dx$$

$$= \pi\left(r^2-x^2\right) dx$$

$$V = \int_{-r}^{r} \pi\left(r^2-x^2\right) dx$$

$$= \pi \int_{-r}^{r} \left(r^2-x^2\right) dx$$

$$= \pi \cdot \left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r}$$

$$= \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right)\right]$$

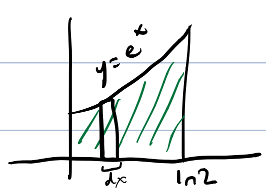
$$= \pi\left[2r^3 - \frac{2r^3}{3}\right]$$

$$= \frac{\pi}{3}\left(6r^3 - 2r^3\right)$$

$$= \frac{4}{3}\pi r^3$$

ex. Mr. Nasty

Consider the region:



Set up (but do not compute) the integral(s) needed to find the volume obtained when the region is rotated:

1) around the x-axis

2) around the y-axis

3) around $y=3$

4) around $y=-1$

① $R = y = e^x$

$$V = \int_0^{\ln 2} \pi (e^x)^2 \, dx$$

$$= \int_0^{\ln 2} \pi e^{2x} \, dx$$

Let $u = 2x$

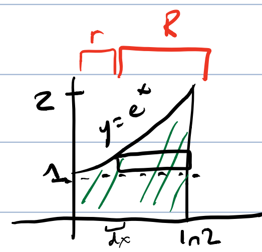
$$= \frac{1}{2} \int_0^{\ln 4} \pi e^u \, du$$

$du = 2dx$

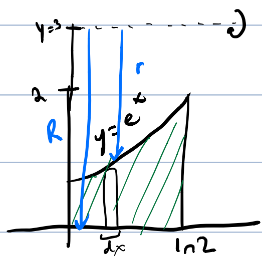
⋮

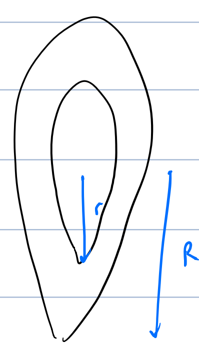
②

$$V = \int_0^1 \pi (\ln 2)^2 \, dy + \int_1^2 \left[\pi (\ln 2)^2 - (\ln y)^2\right] dy$$



③





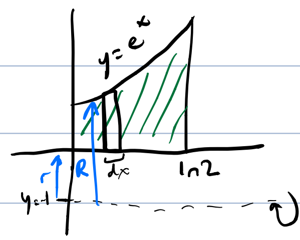
March 23, 2017

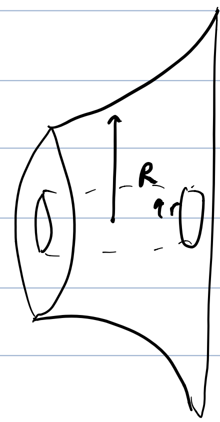
$R = 3$ $\quad r = 3 - e^x$

$$dV = \pi R^2 w - \pi r^2 w$$
$$= \pi (R^2 - r^2) w$$
$$= \pi \left(9 - (3 - e^x)^2\right) dx$$

$$V = \int_0^{\ln 2} \pi \left[9 - (3 - e^x)^2\right] dx$$

(4)





$$R = e^x - (-1) = e^x + 1$$
$$r = 0 - (-1) = 1$$

$$dV = \pi (R^2 - r^2) w$$
$$= \pi \left((e^x + 1)^2 - 1\right) dx$$

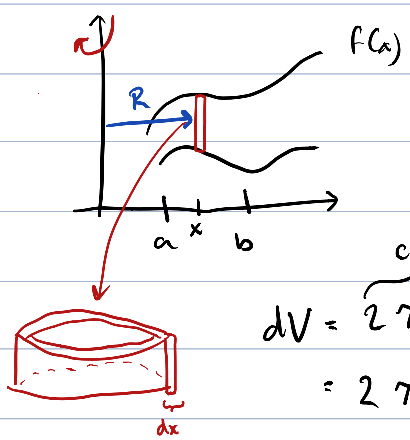
$$V = \int_0^{\ln 2} \pi \left[(e^x + 1)^2 - 1\right] dx$$

<u>Shell Method</u>

This method is used when rectangles are drawn <u>parallel</u> to the axis of rotation.

Visualization!

Suppose the area between f and g within $[a, b]$ is rotated along the y-axis.



$$dV = \overbrace{2\pi R}^{\text{circumference}}\, h\, w$$

$$= 2\pi x\left[f(x) - g(x)\right] dx$$

$$\boxed{V = \int_a^b 2\pi x\left[f(x) - g(x)\right] dx}$$

or

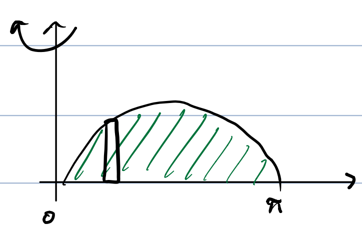
$$\int_c^d 2\pi y\left[y_2 - y_1\right] dy$$

y_2 or $f^{-1}(y)$, y_1 or $g^{-1}(y)$

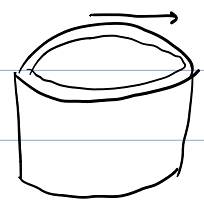
(if rotation is around x-axis)

ex. Let $f(x) = \sin x$ over $[0, \pi]$

What volume is generated if the region below $f(x)$ and above the x-axis is rotated around the y-axis?



rectangles parallel to the axis $\to$ Shell method



$dV = 2\pi R h w$

$= 2\pi x [\sin x - 0]\, dx$

$= 2\pi x \sin x\, dx$

$V = \int_0^{\pi} 2\pi x \sin x\, dx$

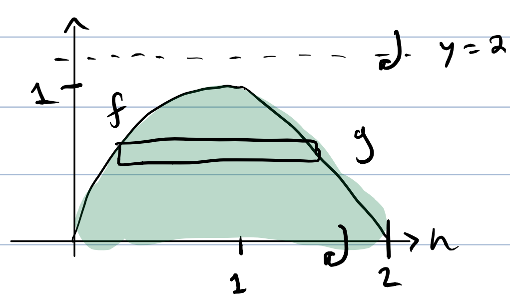
$= 2\pi \int_0^{\pi} x \sin x\, dx$

$= 2\pi \left[-x\cos x + \sin x \right]_0^{\pi}$

$\vdots$

	f	g'
+	x	$\sin x$
−	1	$-\cos x$
+	0	$-\sin x$

ex. Consider the following region:



$f: y = \sqrt[3]{x}$

$g: y = 2 - x$

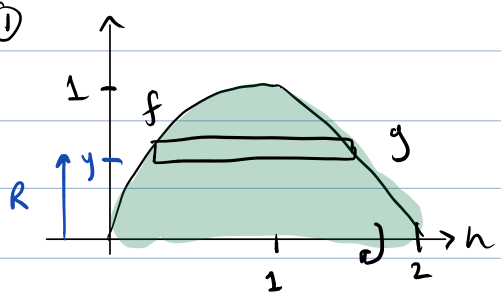
$h: y = 0$

Set up (using the Shell Method) the integrals required to find the volume obtained when the region is rotated:

① around the x-axis

② around $y = 2$

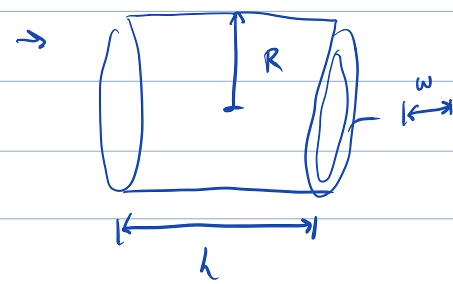
①



f: $y = \sqrt[3]{x} \rightarrow y^3 = x$

g: $y = 2 - x \rightarrow x = 2 - y$

→



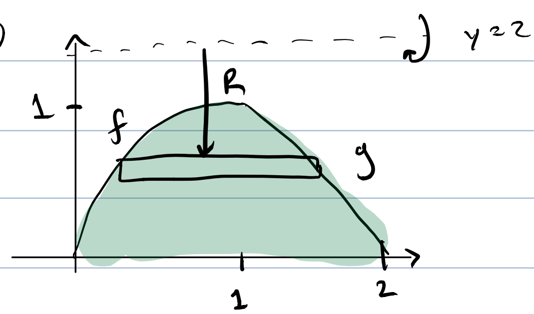
$dV = 2\pi R h w$

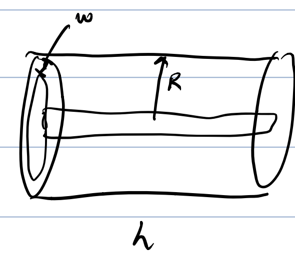
$= 2\pi y\left[g^{-1}(y) - f^{-1}(y)\right] dy$

$= 2\pi y\left[(2-y) - y^3\right] dy$

$V = \int_0^1 2\pi y(2-y-y^3)\, dy$

②





$R = 2 - y$

$h = x_2 - x_1 = 2 - y - y^3$

$w = dy$

$0 \leq y \leq 1$

$dV = 2\pi R h w$

$= 2\pi(2-y)(2-y-y^3)\, dy$

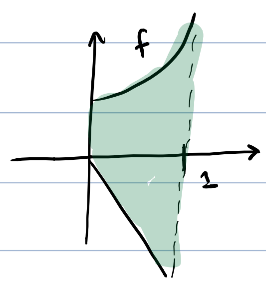
$$V = \int_0^1 2\pi (2-y)(2-y-y^3)\, dy$$

Advice

When evaluating areas or volumes of revolution

1. Chose your rectangle representation
2. Adapt the method to the rectangles

ex. Set up



f: $y = x^2 + 1$

g: $y = -x$

$x = 0$ and $x = 1$

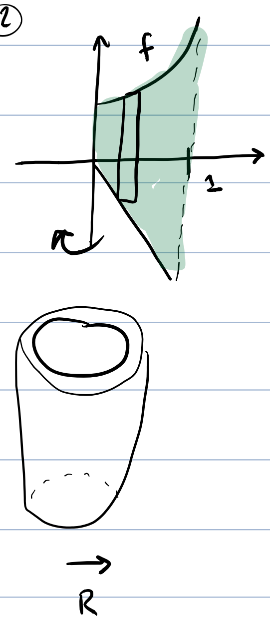
1. Area
2. Volume around y-axis
3. Volume around $x = 1$

1. vertical rectangles

$$\int_0^1 (y_2 - y_1)\, dx$$

$$= \int_0^1 \left[(x^2+1) - (-x)\right] dx \quad = \int_0^1 (x^2 + x + 1)\, dx$$

②



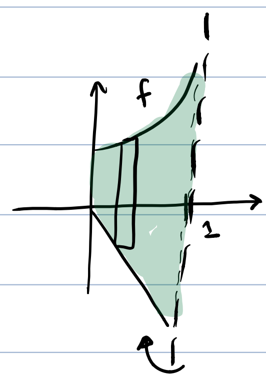
$$dV = 2\pi R h w$$

$$= 2\pi x \left((x^2+1) - (-x)\right) dx$$

$$= 2\pi x \left(x^2 + x + 1\right) dx$$

$$V = \int_0^1 2\pi x (x^2 + x + 1)\, dx$$

③



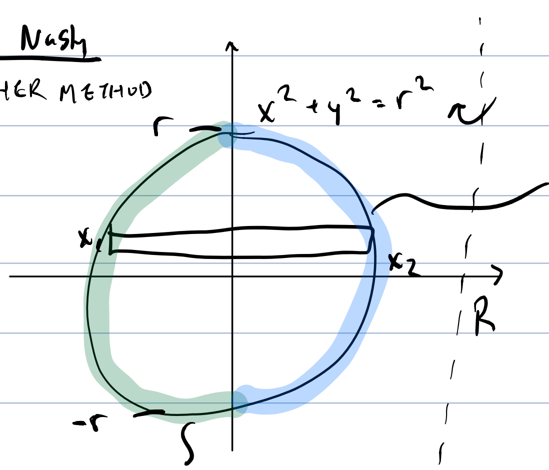
$$dV = 2\pi R h w$$

$$= 2\pi (1-x)(x^2 + x + 1)\, dx$$

$$V = \int_0^1 2\pi (1-x)(x^2 + x + 1)\, dx$$

Mr. Nash

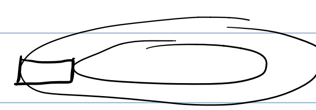
WASHER METHOD



$x^2 + y^2 = r^2$

$g: x_2 = +\sqrt{r^2 - y^2}$

$f: x_1 = -\sqrt{r^2 - y^2}$



$$R_{out} = R - x_1 = R - \left(-\sqrt{r^2 - y^2}\right)$$
$$= R + \sqrt{r^2 - y^2}$$

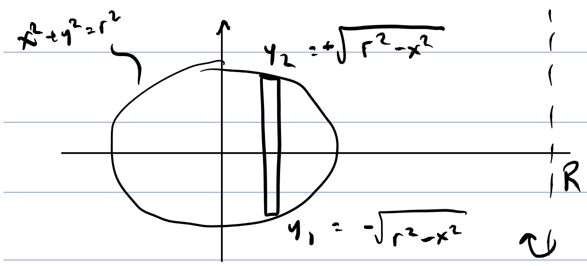
$$R_{in} = R - x_2 = R - \sqrt{r^2 - y^2}$$

$$dV = \pi R^2 w$$

$$= \pi \left[\left(R + \sqrt{r^2 - y^2}\right)^2 - \left(R - \sqrt{r^2 - y^2}\right)^2 \right] dy$$

$$V = \int_{y=-r}^{y=+r} \pi \left[\left(R + \sqrt{r^2 - y^2}\right)^2 - \left(R - \sqrt{r^2 - y^2}\right)^2 \right] dy$$

SHELL METHOD



$$dV = 2\pi(\text{radius})\, hw$$

$$= 2\pi\,(R - x)\left(2\sqrt{r^2 - x^2}\right) dx$$

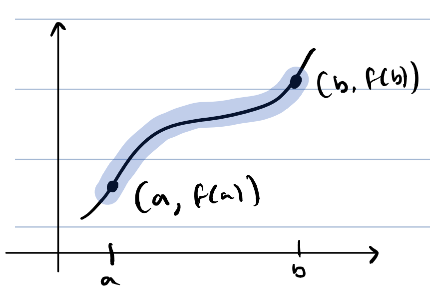
radius $= R - x$

$$h = \left(\sqrt{r^2 - x^2}\right) - \left(-\sqrt{r^2 - x^2}\right)$$

$$= 2\sqrt{r^2 - x^2}$$

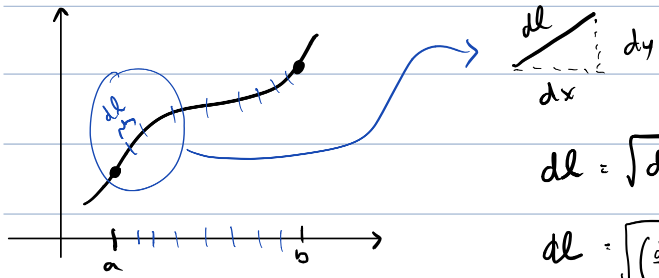
$$V = \int_{-r}^{r} \left[2\pi (R - x)\left(2\sqrt{r^2 - x^2}\right) dx \right.$$

Arc Length of a Curve



Arc length = distance required to travel from (a, f(a)) to (b, f(b)) while remaining on the curve.

How do we do this?



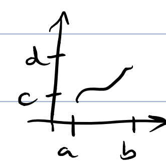
$$dl = \sqrt{dx^2 + dy^2}$$

$$dl = \sqrt{\left(\frac{dy^2}{dx^2} + 1\right) dx^2}$$

$$dl = \sqrt{[f'(x)]^2 + 1} \; dx$$

Therefore,

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} \; dx \qquad \text{or } L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \; dy$$



ex. Let $f(x) = \ln(\sec x)$ $\quad 0 \le x \le \frac{\pi}{4}$

Find the arc length generated by $f(x)$ over the given interval.

$$f'(x) = \left[\ln(\sec x)\right]' = \frac{1}{\cancel{\sec x}} \cdot \cancel{\sec x} \tan x$$

$$= \tan x$$

$$1 + [f'(x)]^2 = 1 + \tan^2 x$$

$$= \sec^2 x$$

$$\sqrt{1+[f'(x)]^2} = \sec x$$ (sign is taken care of in the given interval)

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1+[f'(x)]^2}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \sec x \, dx$$

$$= \Big[\ln|\sec x + \tan x| \Big]_0^{\frac{\pi}{4}}$$

$$= \ln(\sqrt{2}+1)$$

<u>ex</u> À vous

Let $f(x) = \frac{1}{3} x^{3/2} - x^{1/2}$ over $[1,4]$. Find L

$$f'(x) = \frac{1}{2} x^{1/2} - \frac{1}{2} x^{-1/2}$$

$$= \frac{1}{2}\sqrt{x} - \frac{1}{2\sqrt{x}}$$

$$1 + (f'(x))^2 = 1 + \left(\frac{1}{2}\sqrt{x} - \frac{1}{2\sqrt{x}}\right)^2$$

$$= 1 + \left[\frac{1}{4}x - \frac{1}{2} + \frac{1}{4x}\right]$$

$$= \frac{1}{2} + \frac{1}{4}x + \frac{1}{4x} = \left(\frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2}\right)^2$$

$$\sqrt{1+[f(x)]^2} = \sqrt{\frac{1}{2} + \frac{1}{4}x + \frac{1}{4x}}$$

$\downarrow$

OR

$$L = \int_1^4 \sqrt{\frac{1}{2} + \frac{1}{4}x + \frac{1}{4x}}\, dx$$

$$= \int_1^4 \sqrt{\frac{1}{4}\left(2 + x + \frac{1}{x}\right)}\, dx$$

$$= \sqrt{\frac{1}{4}} \int_1^4 \sqrt{\frac{x^2 + 2x + 1}{x}}\, dx$$

$$= \sqrt{\frac{1}{4}} \int_1^4 \sqrt{\frac{(x+1)^2}{x}}\, dx$$

$$= \sqrt{\frac{1}{4}} \int_1^4 (x+1)\, x^{-1/2}\, dx$$

$$= \sqrt{\frac{1}{4}} \int_1^4 \left[x^{1/2} + x^{-1/2}\right] dx$$

$$L = \int_1^4 \sqrt{\left(\frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2}\right)}\, dx$$

$$= \int_1^4 \left[\frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2}\right] dx$$

$$= \left[\frac{x^{3/2}}{\cancel{2} \cdot 3/\cancel{2}} + \frac{x^{1/2}}{\cancel{2} \cdot 1/\cancel{2}}\right]_1^4$$

$$= \left[\frac{1}{3}x^{3/2} + x^{1/2}\right]_1^4$$

...

ex. $y = \frac{2}{3}x^{3/2}$ over $[1,4]$

March 24, 2017

Arc Length of a curve

ex. Let $f(x) = \frac{1}{2}\ln(\sin x) + \frac{1}{2}\ln(\cos x)$

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx \quad \text{or} \quad L = \int_c^d \sqrt{1 + (g'(x))^2}\, dy \quad \text{where } g' = f^{-1}$$

Find the length of curve f within $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

$$f'(x) = \frac{1}{2} \cdot \frac{1}{\sin x} \cdot \cos x - \frac{1}{2} \cdot \frac{1}{\cos x} \cdot \sin x$$

$$= \frac{1}{2} \cot x - \frac{1}{2} \tan x$$

$$[f'(x)]^2 = \frac{1}{4} \cot^2 x - 2 \cdot \frac{1}{2} \cot x \cdot \frac{1}{2} \tan x + \frac{1}{4} \tan^2 x$$

$$= \frac{1}{4} \cot^2 x - \frac{1}{2} + \frac{1}{4} \tan^2 x$$

$$1 + (f'(x))^2 = \frac{1}{4} \cot^2 x - \frac{1}{2} + \frac{1}{4} \tan^2 x$$

$$= \left(\frac{1}{2} \cot x + \frac{1}{2} \tan x\right)^2$$

$$\sqrt{1 + (f'(x))^2} = \frac{1}{2} \cot x + \frac{1}{2} \tan x$$

$$L = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left[\frac{1}{2} \cot x + \frac{1}{2} \tan x\right] dx$$

$$= -\left[\frac{1}{2} \ln(\sin x) - \frac{1}{2} \ln(\cos x)\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$