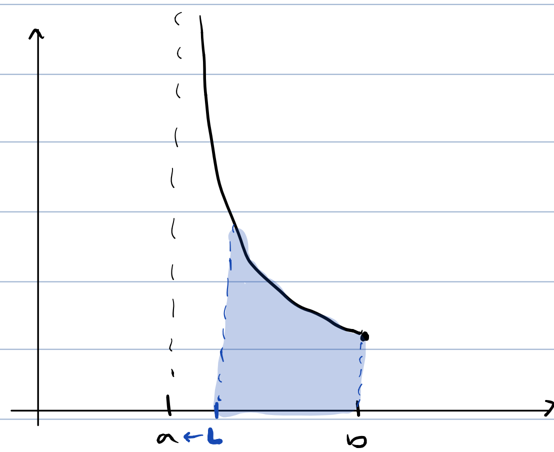
Section 5 - Improper Integrals

An integral is considered improper when its integral $(f(x))$ is
→ discontinuous at the bounds of integration
→ discontinuous between the bounds of integration
→ evaluated over infinite bounds

Case 1: Discontinuity at the left bound

Suppose $f(x)$ is discontinuous at $x = a$

Then $\int_a^b f(x)\,dx = \lim_{L \to a^+} \int_L^b f(x)\,dx$




ex. Evaluate the improper integral

$$\int_0^1 \frac{1}{2\sqrt{x}}\,dx$$

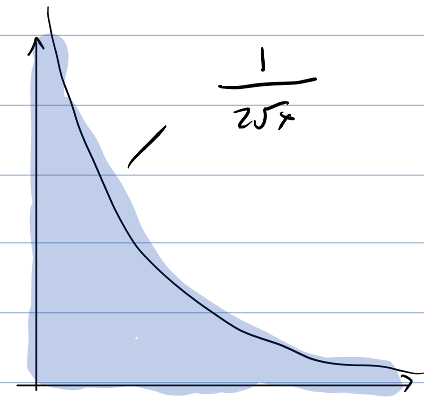
Note $\frac{1}{2\sqrt{x}}$ is discontinuous (undefined) at $x=0$

$$= \lim_{L \to 0^+} \int_L^1 \frac{1}{2} x^{-1/2} \, dx$$

$$= \lim_{L \to 0^+} \left[ \cancel{\frac{1}{2}} \cdot \frac{x^{1/2}}{\cancel{1/2}} \right]_L^1$$

$$= \lim_{L \to 0^+} \left[ \sqrt{1} - \sqrt{L} \right]$$

$$= 1 - 0 = 1$$

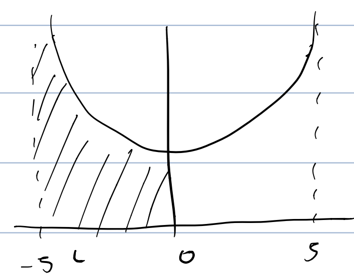



This improper interval converges to 1

<u>ex.</u> À vous

$$\int_{-5}^{0} \frac{1}{\sqrt{25 - x^2}} \, dx$$

$25 - x^2 = 0$ if $x = 5$, $x = -5$




$$= \lim_{L \to -5^+} \int_L^0 \frac{1}{\sqrt{25 - x^2}} \, dx$$

$$= \lim_{L \to -5^+} \left[ \arcsin \frac{x}{5} \right]_L^0$$

$$= \lim_{L \to -5^+} \left[ \arcsin 0 - \arcsin \frac{L}{5} \right]$$

$$= \lim_{L \to -5^+} \left[ -\arcsin \frac{L}{5} \right]$$

$$= -\arcsin \frac{-5}{5}$$

$$= -\left(-\frac{\pi}{2}\right)$$

$$= \frac{\pi}{2}$$

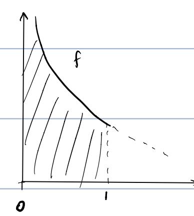
ex. À vous

N.B. $= (\sqrt{x})^3$ dom $x \geq 0$, and in denominator so dom $= ]0, +\infty[$

$$f(x) = \frac{1}{x^{3/2}} \qquad \text{dom}(f) = ]0, +\infty[$$

a) Find the Area between $x=0$, $x=1$

b) Find the volume in the 1st quadrant between $x=0$, $x=1$ when rotated around the $y$-axis




a) $$A = \int_0^1 x^{-3/2}\,dx$$

$$= \lim_{L \to 0^+} \int_L^1 x^{-3/2}\,dx$$

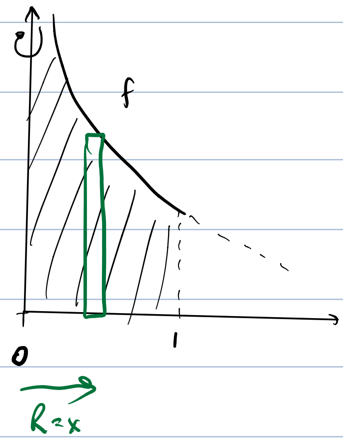
$$= \lim_{L \to 0^+} \left[\frac{x^{-1/2}}{-1/2}\right]_L^1$$

$$= \lim_{L \to 0^+} \left[\frac{-2}{\sqrt{x}}\right]_L^1$$

$$= \lim_{L \to 0^+} \left[-2 + \frac{1}{\sqrt{L}}\right]$$

$= \infty$ (divergent)

b)




$dV = 2\pi R h w$

$= 2\pi x \cdot x^{-3/2}\, dx$

$= 2\pi x^{-1/2}\, dx$

$V = \int_0^1 2\pi x^{-1/2}\, dx$

$= 2\pi \int_0^1 \frac{1}{\sqrt{x}}\, dx$

$= 2\pi \lim_{L \to 0^+} \int_L^1 x^{-1/2}\, dx$

$= 2\pi \lim_{L \to 0^+} \left[ \frac{x^{1/2}}{1/2} \right]_L^1$

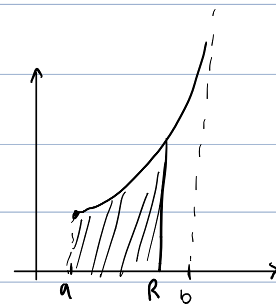
$= 2\pi \lim_{L \to 0^+} [2 - 2L]$

$= 4\pi$

## Case 2

If $f(x)$ is discontinuous at $x=b$,

then $\displaystyle\int_a^b f(x)\,dx = \lim_{R \to b^-} \int_a^R f(x)\,dx$

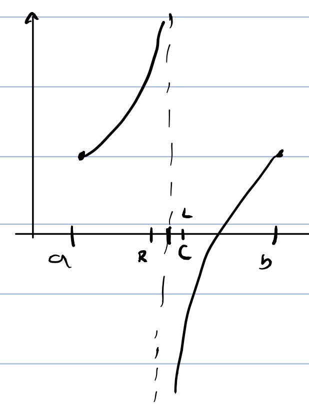



## Case 3

If $f(x)$ is discontinuous at $x=c$ where $a<c<b$

then $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$$= \lim_{R \to c^-} \int_a^R f(x)\,dx + \lim_{L \to c^+} \int_L^b f(x)\,dx$$

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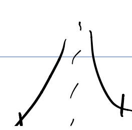
Improper Integral → Case 1, 2, 3

ex. $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$ $\qquad \frac{1}{x^2}$ is undefined at $x=0$

$$= \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_0^1 \frac{1}{x^2}\,dx$$



$$= \lim_{R\to 0^-} \int_{-1}^{R} \frac{1}{x^2}\,dx + \lim_{L\to 0^+} \int_{L}^{1} \frac{1}{x^2}\,dx$$

-1 0 1

$$= \lim_{R\to 0^-} \left[-\frac{1}{x}\right]_{-1}^{R} + \lim_{L\to 0^+} \left[-\frac{1}{x}\right]_{L}^{1}$$

$$= \lim_{R\to 0^-} \left[-\frac{1}{R} + \frac{1}{-1}\right] + \lim_{L\to 0^+} \left[-\frac{1}{1} + \frac{1}{L}\right]$$

$$= \underbrace{+\infty - 1}_{\text{diverging}} \quad \underbrace{-1 + \infty}$$

## Case 4 Infinite Bounds

Definite integrals with an infinite bound are considered improper:

$$\int_{a}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{a}^{R} f(x)\,dx$$

don't need to write $R \to \infty^-$

OR

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{L\to -\infty} \int_{L}^{a} f(x)\,dx$$

OR

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

$$= \lim_{L\to -\infty} \int_{L}^{c} f(x)\,dx + \lim_{R\to +\infty} \int_{c}^{R} f(x)\,dx$$

ex. $$\int_1^{\infty} \frac{1}{1+x^2}\, dx$$

$$= \lim_{R \to \infty} \int_1^{R} \frac{1}{1+x^2}\, dx$$

$$= \lim_{R \to \infty} \Big[ \arctan x \Big]_1^{L}$$

$$= \lim_{R \to \infty} \left[ \arctan L - \arctan 1 \right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

ex. $$\int_1^{\infty} \frac{1}{x^p}\, dx$$

$\boxed{\text{for } p = 1}$

$$= \lim_{R \to \infty} \int_1^{R} \frac{1}{x^1}\, dx$$

$$= \lim_{R \to \infty} \Big[ \ln|x| \Big]_1^{R}$$

$$= \lim_{R \to \infty} \left[ \ln|R| - 0 \right]$$

$= \infty$ Diverges

$\boxed{\text{for } p > 1}$

$$= \lim_{R\to\infty} \int_1^R \frac{1}{x^p}\, dx$$

$$= \lim_{R\to\infty} \left[ \frac{x^{-p+1}}{-p+1} \right]_1^R$$

$$= \lim_{R\to\infty} \left[ \frac{1}{(1-p)\, x^{p-1}} \right]_1^R$$ recall: $p > 1$

$$= \lim_{R\to\infty} \left[ \frac{1}{(1-p)\, R^{p-1}} - \frac{1}{1-p} \right]$$

$$= 0 - \frac{1}{1-p}$$

$$= \frac{1}{p-1}$$

$\boxed{\text{for } 0 < p < 1}$

$\vdots$

$$= \lim_{R\to\infty} \left[ \frac{x^{-p+1}}{-p+1} \right]_1^R$$ where $0 < p < 1$

$$= \lim_{R\to\infty} \left[ \frac{R^{-p+1}}{-p+1} - \frac{1}{-p+1} \right]$$

$= \infty$ diverges

## Conclusion

The "p-integral" $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx$

converges to $\dfrac{1}{p-1}$ if $p > 1$, but

diverges for all cases where $0 < p \leq 1$

ex. $\displaystyle\int_1^{\infty} \frac{1}{x^3}\,dx = \frac{1}{3-1} = \frac{1}{2}$

ex. $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$ diverges

ex.

a) Find an antiderivative for $f(x) = x^2 e^{-x}$

$F(x) = \displaystyle\int x^2 e^{-x}\,dx$

| | $f$ | $g'$ |
|---|---|---|
| $+$ | $x^2$ | $e^{-x}$ |
| $-$ | $2x$ | $-e^{-x}$ |
| $+$ | $2$ | $e^{-x}$ |
| $-$ | $0$ | $-e^{-x}$ |

$\displaystyle\int e^{-x}\,dx$ $\quad u = -x$, $du = -dx$

$= -\displaystyle\int e^{u}\,du$

$= -e^{-x}$

So

$$F(x) = -x^2 e^{x} - 2x\, e^{-x} - 2e^{-x} + K$$
$$= -e^{x}\left(x^2 + 2x + 2\right) + k$$

b) Solve $\displaystyle\int_0^{\infty} x^2 e^{-x}\,dx$

$$= \lim_{R\to\infty} \int_0^R x^2 e^{-x}\, dx$$

$$= \lim_{R\to\infty} \Big[ -e^{-x}(x^2 + 2x + 2) \Big]_0^R$$

$$= \lim_{R\to\infty} \Big[ -e^{-R}(R^2 + 2R + 2) + 1(2) \Big] = - \frac{\infty}{\infty} + 2$$

$$= -\lim_{R\to\infty} \left( \frac{R^2 + 2R + 2}{e^R} \right) + 2$$

$$\overset{L'H}{=} -\lim_{R\to\infty} \frac{2R + 2}{e^R} + 2$$

$$= -\lim_{R\to\infty} \frac{2}{e^R} + 2$$

$$= 0 + 2$$

$$= 2$$