

NB

$$b^x = b^y \quad (b > 0, b \neq 1)$$

then $x = y$

Exponential Equations

ex. solve $2^x = 8$ ex. solve $2^{x^2-1} = 16$ ex. solve $2^{x^2-1} = 3$

$x = 3$ $2^{x^2-1} = 2^4$ Need logs

$x^2-1 = 4$
 $x^2 = 5$
 $x = \pm\sqrt{5}$

LOG REVIEW

$$2^3 = 8 \leftrightarrow \log_2 8 = 3$$

Defn:

$$b^y = x \leftrightarrow \log_b x = y$$

Exponential Form	Log Form
$10^3 = 1000$	$\log_{10} 1000 = 3$
$16^{\frac{1}{2}} = 4$	$\log_{16} 4 = \frac{1}{2}$
$(\frac{1}{16})^{\frac{1}{2}} = \frac{1}{4}$	$\log_{\frac{1}{16}} \frac{1}{4} = \frac{1}{2}$

Ex. Find $\log_2 32$

$$\text{Let } y = \log_2 32$$

$$2^y = 32$$

$$y = 5$$

Ex. Find $\log_2 (-16)$

$$\text{Let } y = \log_2 (-16)$$

$$2^y = -16$$

No sol'n for $y = \log_2 (-16) !!$

So $\log(-16)$ undefined

ex. $y = \log_2 0$

Let $y = \log_2 0$

$$2^y = 0$$

No sol'n

so $\log_2 0$ is vnd

Fact! $\log_b N$ is vnd if $N \leq 0$ Can't have a negative log

RULES OF LOGS

Rules: Assume $b > 0$, $b \neq 1$ and $M, N > 0$

① $\log_b b = 1$ ($b^1 = b$)

② $\log_b 1 = 0$ ($b^0 = 1$)

③ $\log_b(MN) = \log_b M + \log_b N$ (not necessarily)

ex. $\log_{10} 100$

Method A:

$$y = \log_{10} 100$$

$$10^y = 100$$

$$y = 2$$

Method B:

$$\log_{10}(10 \cdot 10)$$

$$= \log_{10} 10 + \log_{10} 10$$

$$= 1 + 1 = 2$$

$$\underline{\text{N.B.}} \quad \log_b(-3)(-4) \cancel{=} \underbrace{\log_b(-3)}_{b \text{ und}} + \underbrace{\log_b(-4)}_{b \text{ und}}$$

Proof of ⑤ - Not necessary

$$\begin{aligned} \text{Let } x &= \log_{10} M \quad \& y = \log_{10} N \\ &\downarrow \qquad \qquad \downarrow \\ b^x &= M \qquad b^y = N \\ b^x \cdot b^y &= MN \\ b^{x+y} &= MN \\ &\downarrow \\ \log(MN) &= x+y \\ &= \log_{10} M + \log_{10} N \end{aligned}$$

$$\textcircled{4} \quad \log_b \frac{M}{N} = \log_b M - \log_b N$$

$$\text{ex. } \log_{10} \frac{1}{10} = \log_{10} 1 - \log_{10} 10 = 0$$

Euler's constant "e"

$$e \approx 2.71\ldots \quad (\text{irrational } \mathbb{Q})$$

$$\log x = \log_{10} x$$

$$\ln x = \log_e x$$

$$\textcircled{5} \log_b N^r = r \cdot \log_b N$$

$$\text{ex. } \log_7 (7)^5$$

$$= 5 \log_7 7 = 5 \cdot 1 = 5$$

\textcircled{6} Change of base formula

$$\log_b N = \frac{\log_c N}{\log_c b} \quad \text{for any } c > 0, c \neq 1$$

$$\text{ex. } \log_3 2 = \frac{\log_{10} 2}{\log_{10} 3} \xrightarrow{\text{calculator}} 0.630924784$$

$$\text{i.e. } 3^{0.630924784} = 2$$

$$\textcircled{7} \log_b b^x = x$$

$$\text{ex. } \log_3 3^7 = 7$$

$$\textcircled{8} b^{\log_b x} = x \quad (\rightarrow \log_b x = \log_b x)$$

$$\text{ex. } 2^{\log_2 (x^3 - 4)} = x^3 - 4$$

Log Rules with e ($M, N > 0$)

$$\textcircled{1} \ln e = 1$$

$$\textcircled{2} \ln 1 = 0$$

$$\textcircled{3} \ln(M \cdot N) = \ln M + \ln N$$

$$\textcircled{4} \ln\left(\frac{M}{N}\right) = \ln M - \ln N$$

$$\textcircled{5} \quad \ln N^r = r \ln N$$

$$\textcircled{6} \quad \log_b N = \frac{\ln N}{\ln b}$$

$$\textcircled{7} \quad \ln e^x = x$$

$$\textcircled{8} \quad e^{\ln x} = x$$

ex. $2^{x^2-1} = 3$

$$\ln(2^{x^2-1}) = \ln 3$$

$$(x^2-1) \ln 2 = \ln 3$$

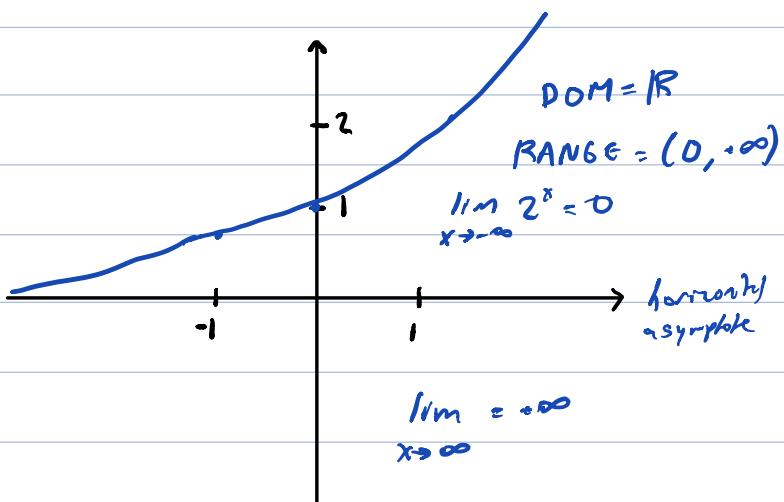
$$x^2-1 = \frac{\ln 3}{\ln 2}$$

$$x^2 = \frac{\ln 3}{\ln 2} + 1$$

$$x = \pm \sqrt{\frac{\ln 3}{\ln 2} + 1}$$

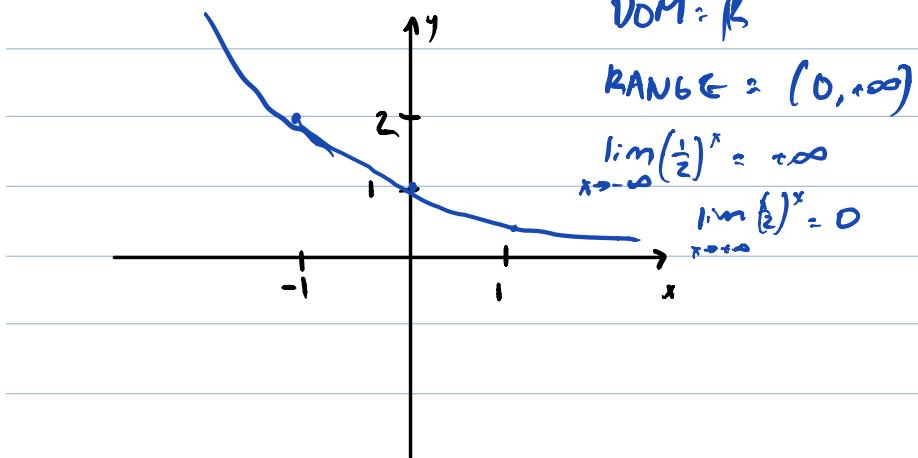
Graphs of Basic Exponential Functions

x	$y = 2^x$
-1	$\frac{1}{2}$
0	1
1	2

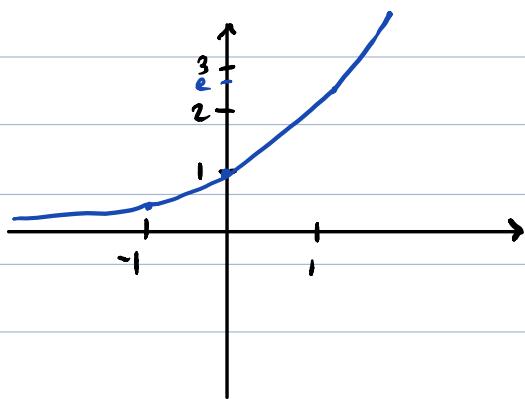


$$y = b^x \quad (b > 1)$$

ex. $y = \left(\frac{1}{2}\right)^x$ ($0 < b < 1$)



ex. $y = e^x$



Basic Graphs of Log Fns

x	$\log_2 x$
$\frac{1}{2}$	-1
1	0
2	1

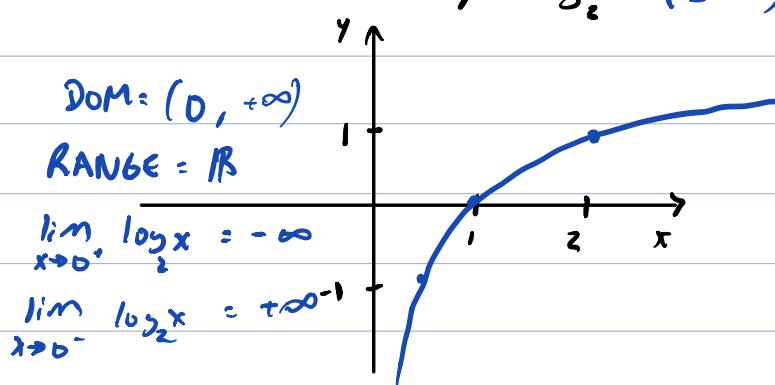
$\text{DOM} = (0, +\infty)$

$\text{RANGE} = \mathbb{R}$

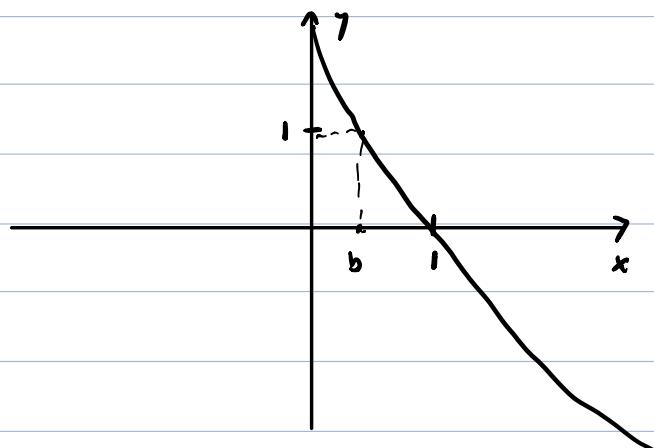
$\lim_{x \rightarrow 0^+} \log_2 x = -\infty$

$\lim_{x \rightarrow 0^-} \log_2 x = +\infty$

$y = \log_2 x$ ($b > 1$)

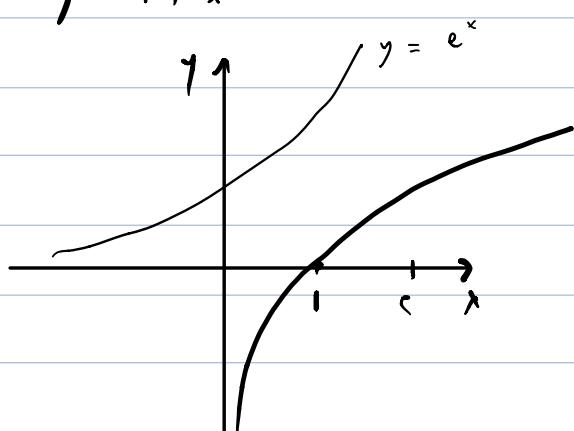


If $0 < b < 1$: $\log_b x$



$$y = \ln x$$

$$\begin{aligned}f(x) &= \ln x \\f'(x) &= e^x\end{aligned}$$



$$f^{-1}(f(x)) = e^{f(x)} = e^{\ln x} = x$$

Derivatives

$$f(x) = e^x$$

$$f'(x) = ?$$

N.B. Special limit: Don't have to prove

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h}$$

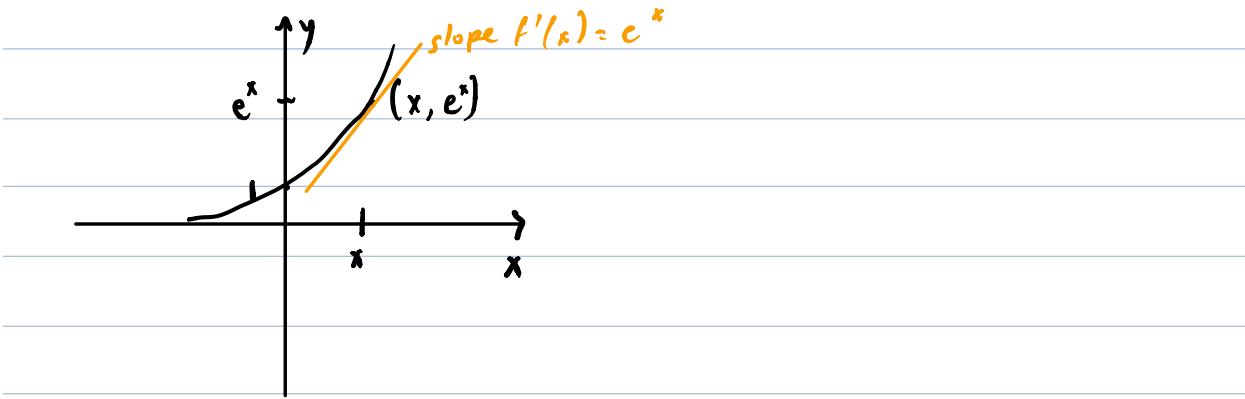
$$= \lim_{h \rightarrow 0} \frac{e^x \cdot e^h - e^x}{h}$$

$$= \lim_{h \rightarrow 0} e^x \cdot \frac{e^h - 1}{h}$$

$$= e^x \lim_{h \rightarrow 0} \frac{e^h - 1}{h}$$

$$= e^x$$

↑
Proof of: If $y = e^x$, $\frac{dy}{dx} = e^x$



ex. $y = \frac{x e^x}{x^2 + 1}$

$$\begin{aligned} y' &= \frac{(x e^x)'(x^2 + 1) - (x e^x)(x^2 + 1)'}{(x^2 + 1)^2} \\ &= \frac{(1)e^x + x(e^x)'(x^2 + 1) - (x e^x)(2x)}{(x^2 + 1)^2} \\ &= \frac{(e^x + x e^x)(x^2 + 1) - 2x^2 e^x}{(x^2 + 1)^2} \end{aligned}$$

Chain Rule Version

$$y = e^{(\text{ins})} \rightarrow y' = e^{(\text{ins})} \cdot (\text{ins})'$$

$$\begin{aligned} \text{ex. } y &= e^{5x} \rightarrow y' = e^{5x}(5x)' \\ &= 5e^{5x} \end{aligned}$$

$$\text{ex. } y = (e^{5x})^3 = e^{15x} \dots$$

ex. $e^y = x$ Find $\frac{dy}{dx}$ Implicit diff.

$$(e^y)' = x'$$

$$e^y \cdot y' = 1 \rightarrow \text{Not! } e^{y \cdot y'}$$

$$y' = \frac{1}{e^y}$$

ex. $y = \ln x$

$$y = \log_e x$$

↓

$$e^y = x$$

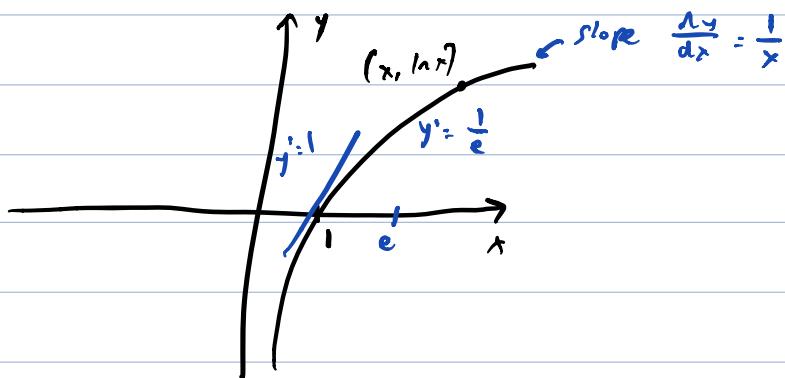
!

$$y' = \frac{1}{e^y}$$

$$y' = \frac{1}{x}$$

proof of $y = \ln x$

so prove that $y = \ln x$, $y' = \frac{1}{x}$



Chain rule version

$$y = \ln(\ln s) \rightarrow y' = \frac{1}{\ln s} \cdot (\ln s)'$$

ex. $y = \ln(x^3)$

(A) $y' = \frac{1}{x^3} \cdot 3x^2 = \frac{3}{x}$

(B) $y = 3 \ln x = 3 [\ln x]'$
 $= 3\left(\frac{1}{x}\right) = \frac{3}{x}$

ex. $y = 2^x$ Not e^x

$$\ln(y) = \ln(2^x)$$

$$\ln(y) = x \ln 2$$

$$[\ln(y)]' = [x \ln 2]'$$

$$\frac{1}{y} \cdot y' = \ln 2 [x]'$$

$$y' = y \ln 2$$

$$y' = 2^x \ln 2$$

$$y = b^x \rightarrow y' = b^x \ln b$$

$$y = e^x$$

$$y = \ln x$$

$$y = b^x$$

Chain Rule Version

$$y = b^{\sin x} \rightarrow y' = b^{\sin x} \ln b \cdot (\sin x)'$$

ex. $y = 7^{\sin x}$

$$\frac{dy}{dx} = 7^{\sin x} \ln 7 \cdot (\sin x)'$$

$$y' = 7^{\sin x} \ln 7 \cdot \cos x$$

$$y'' = \ln 7 \left[7^{\sin x} \cos x \right]' \quad \text{: product rule}$$

Given $y = \log_2 x$, $\frac{dy}{dx}$?

$$y = \log_2 x \stackrel{\text{change of base}}{=} \frac{\ln x}{\ln 2} = \frac{1}{\ln 2} \cdot \ln x$$

$$\begin{aligned} y' &= \frac{1}{\ln 2} [\ln x]' \\ &= \frac{1}{x \ln 2} \end{aligned}$$

$$\boxed{y = \log_b x \rightarrow y' = \frac{1}{x \ln b}}$$

$$\text{ex. } y = \log_7 x \Rightarrow \frac{dy}{dx} = \frac{1}{x \ln 7}$$

Chain Rule Version

$$y = \log_b (\text{ins}) \Rightarrow y' = \frac{1}{(\text{ins}) \ln b} \cdot (\text{ins})'$$

$$\textcircled{1} \quad y = e^x \Rightarrow y' = e^x$$

$$\textcircled{2} \quad y = b^x \Rightarrow y' = b^x \ln b$$

$$\textcircled{3} \quad y = \ln x \Rightarrow y' = \frac{1}{x}$$

$$\textcircled{4} \quad y = \log_b x \Rightarrow y' = \frac{1}{x \ln b}$$

$$\text{ex. } y = x^e \Rightarrow y' = e x^{e-1} \text{ Note (just power rule)}$$

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$$\text{ex. } e^{xy} + \ln y = x^3$$

$$[e^{xy} + \ln y]' = [x^3]',$$

$$[e^{xy}]' + [\ln y]' = [x^3]',$$

$$e^{xy}(xy)' + \frac{1}{y} y' = 3x^2$$

$$e^{xy}(x'y + xy') + \frac{y'}{y} = 3x^2$$

$$e^{xy}(y + xy') + \frac{y'}{y} = 3x^2$$

$$ye^{xy} + xy'e^{xy} + \frac{y'}{y} = 3x^2$$

$$y^2 e^{xy} + xy'e^{xy} + y' = 3x^2 y$$

$$y'(xye^{xy} + 1) = 3x^2 y - 3y^2 e^{xy}$$

$$y' = \frac{3x^2 y - 3y^2 e^{xy}}{xye^{xy} + 1}$$

$$\text{ex. } y = \frac{\sqrt[3]{x(x^3+1)^4}}{x-1}$$

Can use quotient rule then...

Alternative method:

$$\ln y = \ln \frac{\sqrt[3]{x(x^3+1)^4}}{x-1}$$

$$\begin{aligned} & \text{(can skip)} \\ & \text{to this step} \\ & = \ln \sqrt[3]{x} + \ln(x^3+1)^4 - \ln(x-1) \end{aligned}$$

$$\ln y = \frac{1}{3} \ln x + 4 \ln(x^3+1) - \ln(x-1)$$

$$[\ln y]' = \left[\frac{1}{3} \ln x + 4 \ln(x^3+1) - \ln(x-1) \right]'$$

$$\frac{1}{y} y' = \frac{1}{3} \cdot \frac{1}{x} + 4 \cdot \frac{1}{x^3} (x^3+1)' - \frac{1}{x-1} (x-1)'$$

$$y' = y \left[\frac{1}{3x} + \frac{12}{x^3} - \frac{1}{x-1} \right]$$

Need to present in terms of 'x'

$$y' = \frac{\sqrt[3]{x}(x^3+1)^4}{x-1} \left[\frac{1}{3x} + \frac{12}{x^3} - \frac{1}{x-1} \right]$$

Ex. $y = x^4 \rightarrow \frac{dy}{dx} = 4x^3$ ← must be a constant

Ex. $y = 4^x \rightarrow \frac{dy}{dx} = 4^x \ln 4$ ← must be a constant

Ex. $y = x^x \rightarrow \frac{dy}{dx} = \cancel{x^x} \cancel{x^{x-1}}$
 $= \cancel{x^x} \ln x$

Use Logarithmic Differentiation:

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

$$[\ln y]' = [x \ln x]'$$

$$\begin{aligned} \frac{1}{y} \cdot y' &= x' \ln x + x(\ln x)' \\ &= 1 \ln x + x \cdot \frac{1}{x} \end{aligned}$$

$$\frac{1}{y} \cdot y' = \ln x + 1$$

$$y' = y (\ln x + 1)$$

$$y' = x^x (\ln x + 1)$$

In general, if:

$$y = f(x)^{g(x)} \quad (f(x) \text{ & } g(x) \text{ are not constant fns})$$

USE LOGARITHMIC DIFFERENTIATION

ex. $y = 2^{\sin x}$

$$\begin{aligned} \frac{dy}{dx} &= 2^{\sin x} \cdot \ln 2 \cdot (\sin x)' \\ &= 2^{\sin x} \cdot \cos x \end{aligned} \quad \leftarrow \text{N.B. log diff would have worked, but is longer}$$

ex. $y = (x^3 + 1)^{\sin x}$

Must use log diff^n

$$\ln y = \ln [(x^3 + 1)^{\sin x}]$$

$$\ln y = \sin x \cdot \ln(x^3 + 1)$$

$$[\ln y]' = [\sin x \cdot \ln(x^3 + 1)]'$$

$$\frac{1}{y} \cdot y' = (\sin x)' (\ln(x^3 + 1)) + \sin x (\ln(x^3 + 1))'$$

$$= \cos x \cdot \ln(x^3 + 1) + \sin x \cdot \frac{1}{x^3 + 1} \cdot 3x^2$$

$$y' = (x^3 + 1)^{\sin x} \left(\cos x \cdot \ln(x^3 + 1) + \frac{3x^2}{x^3 + 1} \right)$$

L'HOPITAL'S RULE (L'H)

INTRODUCTION

$$\text{ex: } \lim_{x \rightarrow 2} \frac{x^2 - 4}{x-2} \stackrel{\text{indef.}}{=} \frac{0}{0}$$

$$\lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{x-2} = 4$$

$$\begin{aligned} \text{ex: } & \lim_{x \rightarrow \infty} \frac{\sqrt[3]{x^3 - 2x + 1}}{2x^3 + 4} = \frac{\infty}{\infty} \quad \text{indef.} \\ &= \lim_{x \rightarrow \infty} \frac{x^{2/3} \left(1 - \frac{2}{x^2} + \frac{1}{x^3}\right)}{x^3 \left(2 + \frac{4}{x^3}\right)} \\ &\quad \begin{matrix} \nearrow 0 & \nearrow 0 \\ \downarrow 0 & \end{matrix} \\ &= \frac{1}{2} \end{aligned}$$

$$\text{ex. } \lim_{x \rightarrow 0} \frac{e^{3x} - 1}{x} = \frac{0}{0}$$

what else ??

Need another method called L'HOPITAL'S RULE

L'Hôpital's Rule:

If $\lim \frac{f(x)}{g(x)}$ leads to an indet. form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$

then $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$
(any form of lim)

ex. $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \frac{0}{0}$

L'H $\lim_{x \rightarrow 2} \frac{2x}{1}$

= 4

ex. $\lim_{x \rightarrow \infty} \frac{x^3 - 2x + 1}{2x^3 + 4} = \frac{\infty}{\infty}$

L'H $\lim_{x \rightarrow \infty} \frac{3x^2 - 2}{6x^2}$

L'H $\lim_{x \rightarrow \infty} \frac{6x}{12x} = \frac{1}{2}$

ex. $\lim_{x \rightarrow 0} \frac{e^{3x} - 1}{x} = \frac{0}{0}$

L'H $\lim_{x \rightarrow 0} \frac{3e^{3x}}{1}$

= $3e^0$

= 3

Ex. $\lim_{x \rightarrow 1} \frac{3x^2 - 1}{x^2 + 1}$ Need to check to see if result is $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$$\text{L'H} = \frac{6x}{2x} = 3$$

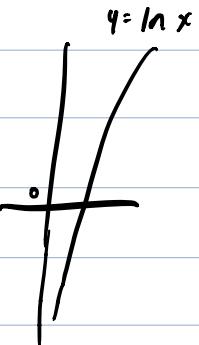
$\downarrow = \frac{2}{2} = 1$

CAUTION

Ex. $\lim_{x \rightarrow 0^+} x \ln x \xrightarrow{\text{rec.}} \infty \cdot (-\infty)$

$= (0) \cdot (-\infty)$

\uparrow
INDEF. FORM



Cannot apply L'H rule yet

$$\lim_{x \rightarrow 0^+} x \cdot \ln x$$

$$\approx \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} \approx \frac{-\infty}{+\infty}$$

$$\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-x^{-2}}$$

$$= \lim_{x \rightarrow 0^+} (-x)$$

$$\approx 0$$

N.B. For $0 \cdot (-\infty)$, ans will not always be 0.

Consider: $\lim_{x \rightarrow \infty} \left[\frac{4}{x} \cdot (-x) \right] = 0 \cdot (-\infty)$

$$= \lim_{x \rightarrow \infty} \frac{-4x}{x} = -4$$

N.B. Checking with 0's and ∞ 's Determinate vs. Indeterminate

Consider: $\lim_{x \rightarrow 0^+} x = \ln 0^+ = -\infty$

\uparrow
Determinate
Form

ex. $\lim_{x \rightarrow \infty} e^x = e^\infty = +\infty$

\uparrow
Determinate
form

$\infty + \infty = \infty$

ex. $\lim_{x \rightarrow \infty} (x + e^x) = \infty + e^\infty = \infty$

ex. $\infty - \infty \leftarrow \text{INDET}$

ex. $\lim_{x \rightarrow \infty} [(x+4) - x] = \infty - \infty$

\uparrow
INDET

$$= 4$$

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INDETERMINATE FORMS

$$\frac{0}{0} \text{ or } \frac{\pm\infty}{\pm\infty}$$

- L'HOPITAL'S

$$\infty - \infty$$

$$0 \cdot \infty$$

$$\infty^0, 0^\infty, 1^\infty$$

$$\frac{1}{0^-} = -\infty ; \frac{1}{0^+} = +\infty$$

$$\arctan(\infty) = \frac{\pi}{2}$$

$$\arctan(-\infty) = -\frac{\pi}{2}$$

$$\infty \cdot \infty = \infty$$

$$\sqrt{\infty} = \infty$$

$$\infty^3 = \infty$$

$$\frac{1}{\infty} = 0$$

$$\lim_{x \rightarrow 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$$

$$= \frac{1}{0^+} - \frac{1}{0^+}$$

$$= \infty - \infty \leftarrow \text{INDET}$$

DETERMINATE FORMS

$$\infty + \infty = \infty$$

$$\ln(0^+) = -\infty$$

$$\ln(\infty) = \infty$$

$$e^\infty = \infty$$

$$e^{-\infty} = 0$$

$$\frac{1}{0^-} = -\infty ; \frac{1}{0^+} = +\infty$$

$$\lim_{x \rightarrow 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$$

$$\underset{x \rightarrow 1^+}{\text{lim}} \frac{\frac{x-1 + \ln x}{(\ln x)(x-1)}}{=} \frac{0}{0}$$

$$\stackrel{\text{L'H}}{=} \underset{x \rightarrow 1^+}{\lim} \frac{1 - \frac{1}{x}}{\frac{1}{x}(x-1) + \ln x}$$

$$\underset{x \rightarrow 1^+}{\lim} \frac{\left(1 - \frac{1}{x}\right)x}{\left(\frac{1}{x}(x-1) + \ln x\right)x} \quad \text{like a fraction}$$

$$\underset{x \rightarrow 1^+}{\lim} \frac{x-1}{x-1 + x \ln x} = \frac{0}{0}$$

$$\stackrel{\text{L'H}}{=} \underset{x \rightarrow 1^+}{\lim} \frac{1}{1 + \ln x + x \cdot \frac{1}{x}}$$

$$\underset{x \rightarrow 1^+}{\lim} \frac{1}{2 + \ln x} = \frac{1}{2}$$

Don't see
nursafe

$$\text{ex. } \underset{x \rightarrow \infty}{\lim} \left[\frac{1}{x} + x \right] = \frac{1}{\infty} \cdot \infty = 0 \cdot \infty$$

$$\underset{x \rightarrow \infty}{\lim} 1 = 1$$

$$\underline{\text{ex.}} \lim_{x \rightarrow 0^+} x \ln x = 0^+ \ln(0^+) \\ = 0 \cdot -\infty$$

$$= \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} \stackrel{x \rightarrow 0^+}{\sim} \frac{\ln(0^+)}{\frac{1}{0^+}} = \frac{-\infty}{\infty}$$

$$\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \stackrel{x \rightarrow 0^+}{\sim} \lim_{x \rightarrow 0^+} (-x) = 0$$

$$\underline{\text{ex.}} \lim_{x \rightarrow \infty} (x^3 + e^x) = \infty^3 + e^\infty \\ = \infty + \infty \\ = \infty$$

$$\underline{\text{ex.}} \lim_{x \rightarrow \infty} x e^{-x} = \infty \cdot e^{-\infty} = \infty \cdot 0$$

$$= \lim_{x \rightarrow \infty} \frac{x}{e^x} = \frac{\infty}{\infty}$$

$$\stackrel{\text{L'H}}{=} \frac{1}{e^x} = \frac{1}{\infty} = 0$$

$$\underline{\text{ex.}} \lim_{x \rightarrow \infty} \frac{-x^4 - 2x^3 + 4x - 1}{5x^4 + 7x^2 - 7x + 2}$$

$$= \lim_{x \rightarrow \infty} \frac{-x^4 - 2x^3 + 4x - 1}{5x^4 + 7x^2 - 7x + 2}$$

$$\stackrel{\text{L'H}}{=} \frac{-x^4 - 2x^3 + 4x - 1}{5x^4 + 7x^2 - 7x + 2} = \frac{-\infty}{\infty}$$

For polynomials, look at the first term for $x \rightarrow \infty$

$$\text{L'H} \lim_{x \rightarrow \infty} \frac{-4x^3 - 6x + 4}{20x^3 + 14x - 7} = \frac{-\infty}{\infty}$$

$$\text{L'H} \lim_{x \rightarrow \infty} \frac{-12x^2 - 6}{60x^2 + 14} = \frac{-\infty}{\infty}$$

$$\text{L'H} \lim_{x \rightarrow \infty} \frac{-24x}{120x} = \frac{-24}{120} = -\frac{1}{5}$$

$$\lim_{x \rightarrow \infty} \frac{x^4 \left(-1 - \frac{6}{x} - \frac{4}{x^3} + \frac{1}{x^4} \right)}{x^4 \left(5 + \frac{7}{x^2} - \frac{7}{x^3} + \frac{2}{x^4} \right)} = -\frac{1}{5}$$

ex. $\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt[3]{x}} = \frac{\infty}{\infty}$

ex. $\lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{3x^{2/3}}} = \lim_{x \rightarrow \infty} \frac{3x^{2/3}}{x} = \lim_{x \rightarrow \infty} \frac{3}{x^{1/3}} = \frac{3}{\infty} = 0$

ex. $\lim_{x \rightarrow \infty} (2^x)^{\frac{1}{x}} = (2^\infty)^{\frac{1}{\infty}} = \infty^0$

$$\lim_{x \rightarrow \infty} 2^x = \infty$$

Indef Forms of Type 0^0 , 1^∞ , 0^∞

ex. $\lim_{x \rightarrow \infty} \left(\frac{1}{2^x}\right)^{\frac{1}{x}} = \left(\frac{1}{2^\infty}\right)^{\frac{1}{\infty}} = \frac{1}{\infty} = 0^0$

$$= \lim_{x \rightarrow \infty} \frac{1^{\frac{1}{x}}}{(2^x)^{\frac{1}{x}}} = \frac{1}{2}$$

If we change '2' to '3' above we still get 0^0 , but answer is $\frac{1}{3}$. So 0^0 is INDEF.

To deal with forms 0^0 , 1^∞ , ∞^∞

we need to recall a few things:

① $\ln N^r = r \ln N$

② $e^{\ln N} = N$

③ $\lim_{x \rightarrow 0^+} \ln x = \ln(0^+) = -\infty$

④ $\lim_{x \rightarrow 0^+} e^{f(x)} = e^{\lim_{x \rightarrow 0^+} f(x)}$

ex. $\lim_{x \rightarrow 0^+} x^x = 0^0$ No algebra to come up with answer

$$\lim_{x \rightarrow 0^+} e^{\ln(x^x)} \quad \text{by ②}$$

$$= e^{\lim_{x \rightarrow 0^+} \ln(x^*)} \text{ by } ④$$

$$\text{So } \lim_{x \rightarrow 0^+} \ln(x^*) \stackrel{①}{=} \lim_{x \rightarrow 0^+} x \ln x \stackrel{③}{=} 0^+ \cdot \ln(0^+) = 0 \cdot \infty$$

$$= \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} = \frac{\ln 0^+}{\frac{1}{0^+}} = -\frac{\infty}{\infty}$$

$$\stackrel{L'H}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -x = 0$$

$$\text{So } e^{\lim_{x \rightarrow 0^+} \ln(x^*)} = e^0 = 1$$

$$\text{ex. } \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = 1^x = 1$$

$$\text{ex. } \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = 1^\infty$$

$$\lim_{x \rightarrow \infty} e^{\ln \left(1 + \frac{1}{x}\right)^x} = e^{\lim_{x \rightarrow \infty} \ln \left(1 + \frac{1}{x}\right)^x}$$

$$\lim_{x \rightarrow \infty} \ln \left(1 + \frac{1}{x}\right)^x$$

$$= \lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x}\right) = \infty \cdot 0$$

$$\lim_{x \rightarrow \infty} \frac{\ln(1 + \frac{1}{x})}{\frac{1}{x}} = \frac{0}{0}$$

$$\stackrel{\text{L'H}}{\lim_{x \rightarrow \infty} \frac{\frac{1}{1+\frac{1}{x}}}{-\frac{1}{x^2}}} = \lim_{x \rightarrow \infty} \frac{1}{1 + \frac{1}{x}} = 1$$

so $e^{\lim_{x \rightarrow \infty} \ln(1 + \frac{1}{x})^x} = e^1 = e$

$$\begin{aligned} y &= \frac{1}{x} \\ y' &= x^{-1} \\ &= -x^{-2} \\ &= -\frac{1}{x^2} \end{aligned}$$

ex. $\lim_{x \rightarrow \infty} x^{\frac{1}{x}} = \infty^0$

$$\lim_{x \rightarrow \infty} e^{\ln(x^{\frac{1}{x}})}$$

$$= e^{\lim_{x \rightarrow \infty} \ln(x^{\frac{1}{x}})} \stackrel{\text{see below}}{=} e^0 = 1$$

so $\lim_{x \rightarrow \infty} \ln x^{\frac{1}{x}}$

$$\lim_{x \rightarrow \infty} \frac{1}{x} \cdot \ln x = 0^+ \cdot \infty$$

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$$

$$\stackrel{\text{L'H}}{\lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{x}}} = \lim_{x \rightarrow \infty} 1 = 0 \quad \text{- Don't need to write } 0^+$$

$$\text{so } e^{\lim_{x \rightarrow \infty} \ln(x^{\frac{1}{x}})} = e^0 = 1$$

instead of this