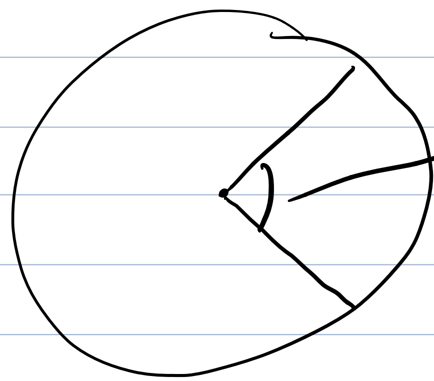
π radians $= 180°$

$\text{rad} = \frac{180°}{\pi} \approx 57.3°$



Note: Convention

$\theta = \frac{\pi}{2}$ ← understood it's in rad

$\beta = 30$ ← means 30 rad, not $30°$

Conversions

degrees	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$360°$
radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	2π

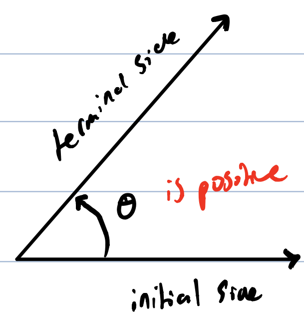
$120° = 2 \times (60°) = 2\left(\frac{\pi}{3}\right) = \frac{2\pi}{3}$

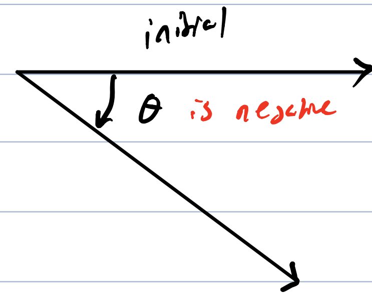
$-135° = -3 \times 45° = -\frac{3\pi}{4}$

$\frac{5\pi}{6} = 5\left(\frac{\pi}{6}\right) = 5(30°) = 150°$

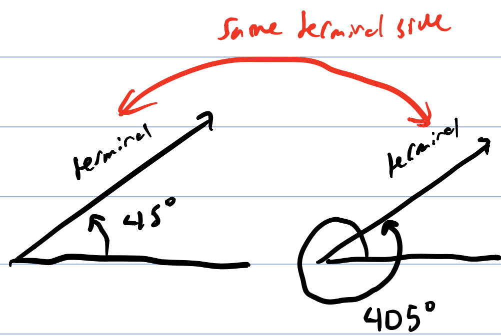
$= 5\left(\frac{\cancel{180°}^{30°}}{\cancel{6}}\right) = 150°$

Positive vs. Negative angles





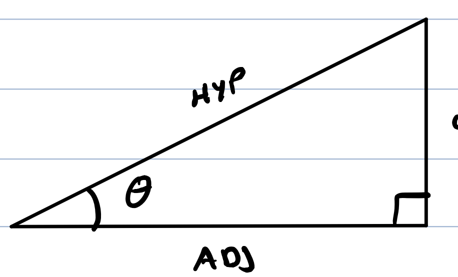
Coterminal angles



45° and 405 are coterminal ∡s, but are not equivalent

$\sin(45°) = \sin(405°)$ Trig fcns of coterminal ∡s are always equal

Trig fcns of ∡s ($0° < \theta < 90°$, $0 < \theta < \frac{\pi}{2}$)



→ Pythagorous $ADJ^2 + OPP^2 = HYP^2$

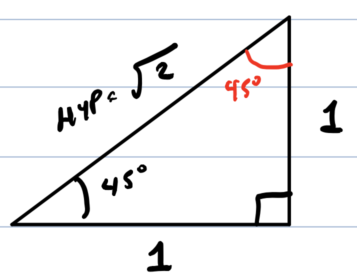
$\sin\theta = \frac{OPP}{HYP}$ $\csc\theta = \frac{HYP}{OPP}$ – ~~$\csc = \frac{HYP}{OPP}$~~ – need $\csc\theta$

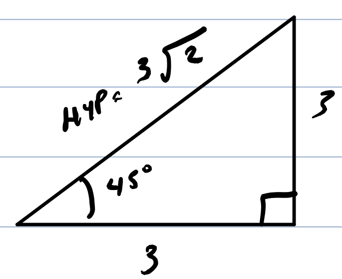
$\cos\theta = \frac{ADJ}{HYP}$ $\sec\theta = \frac{HYP}{ADJ}$

$\tan\theta = \frac{OPP}{ADJ}$ $\cot\theta = \frac{ADJ}{OPP}$

reciprocals

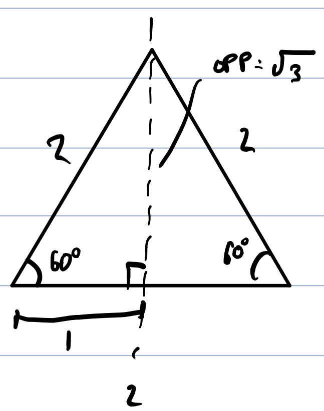
Trig f^{cns} of Special Δ's (30°, 45°, 60°)





$\sin 45° = \frac{O}{H} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\sin 45° = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$

$\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

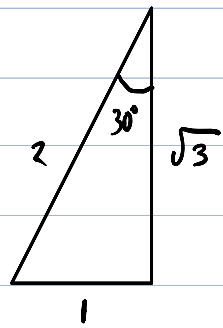


$\sin 60° = \frac{O}{H} = \frac{\sqrt{3}}{2}$

$\cot 60° = \frac{A}{H} = \frac{1}{\sqrt{3}}$

$\tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$

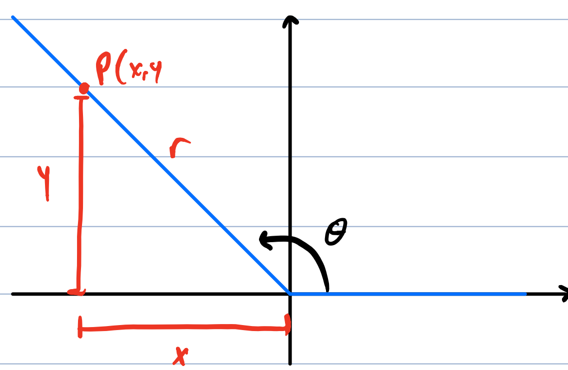
$OPP = \sqrt{1^2 + 2^2} = \sqrt{3}$



$$\sin 30^\circ = \frac{O}{H} = \frac{1}{2}$$
$$\cos 30^\circ = \frac{A}{H} = \frac{\sqrt{3}}{2}$$

WARNING: Given $\sin 12^\circ$ ← Degrees
$\sin 12$ ← RAD

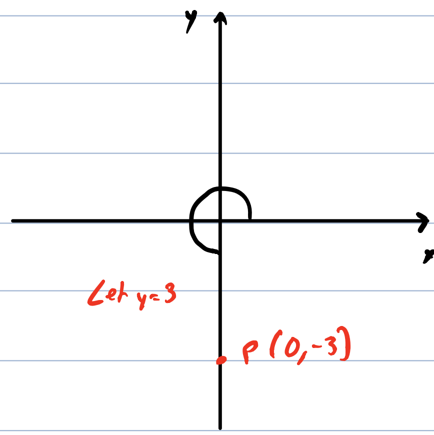
Trig f^{ens} of General $\angle^s$



$$\boxed{x^2 + y^2 = r^2}$$

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

cos 270°

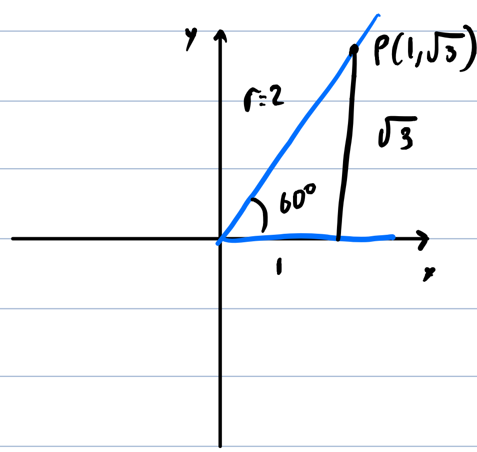


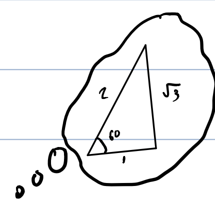
$\cos 270° = \frac{x}{y} = \frac{0}{3} = 0$

$\sin 270° = \frac{y}{r} = \frac{-3}{3} = -1$

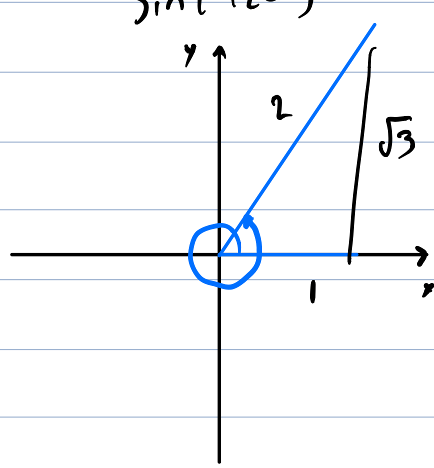
$\tan 270° = \frac{y}{x} = \frac{-3}{0} =$ UND

Sin 60°





sin(420°) → for trig, same ans. as 60° in trig



Proof of $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{\cancel{r}}}{\frac{x}{\cancel{r}}} = \frac{y}{r} = \tan\theta$$

Trig Identities You Need: - Nobody will ask you to prove it

1) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ 2) $\cot\theta = \frac{\cos\theta}{\sin\theta}$

3) $\sec\theta = \frac{1}{\cos\theta}$ 4) $\csc = \frac{1}{\sin\theta}$

5) $\sin^2\theta + \cos^2\theta = 1$

Proof of 5) - Not needed

$$\sin^2\theta + \cos^2\theta = (\sin\theta)^2 + (\cos\theta)^2$$

$$= \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2$$

$$= \frac{y^2 + x^2}{r^2}$$

$$= \frac{r^2}{r^2}$$

$$= 1$$

N.B.

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

6) $1 + \tan^2\theta = \sec^2\theta$

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

7) $1 + \cot^2\theta = \csc^2\theta$

If this is a '−' also a '−' minus

8) $\sin(A+B) = \sin A \cdot \cos B + \sin B \cdot \cos A$

ex. $\sin(75^\circ) = \sin(30^\circ + 45^\circ)$

$= \sin 30^\circ \cdot \cos 45^\circ + \sin 45^\circ \cdot \cos 30^\circ$

$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}$

$= \frac{\sqrt{2} + \sqrt{6}}{4}$

'−' '+'

9) $\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$

N.B.

$\sin(A+B) = \sin A \cos B + \sin B \cos A$

Suppose $\theta = A = B$

$\sin(\theta + \theta) = \sin\theta\cos\theta + \sin\theta\cos\theta$

10) $\sin(2\theta) = 2\sin\theta\cos\theta$

9) $\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$

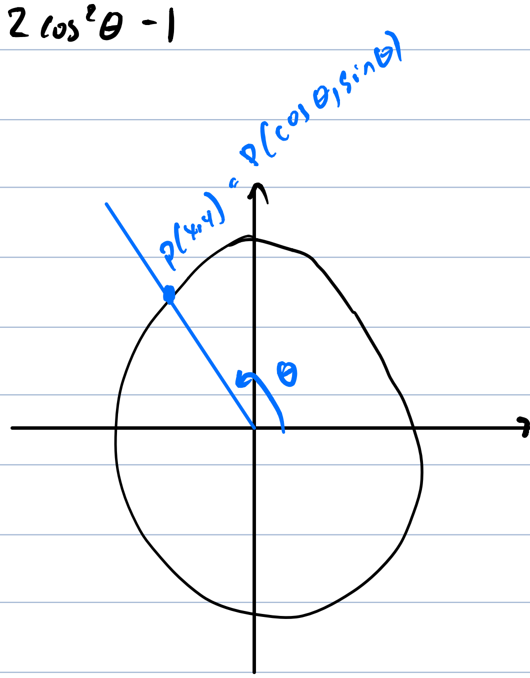
Let $\theta = A = B$

$\cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$

11 a) $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ $\quad\quad \sin^2\theta + \cos^2\theta = 1$

b) $\cos(2\theta) = 1 - 2\sin^2\theta$

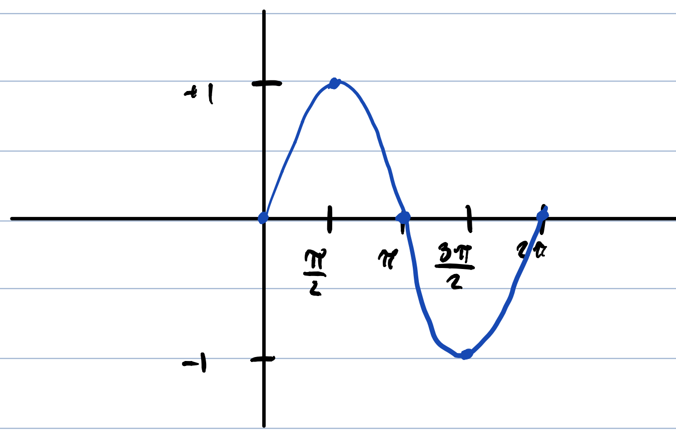
c) $\cos(2\theta) = 2\cos^2\theta - 1$



Graph of $y = \sin x$

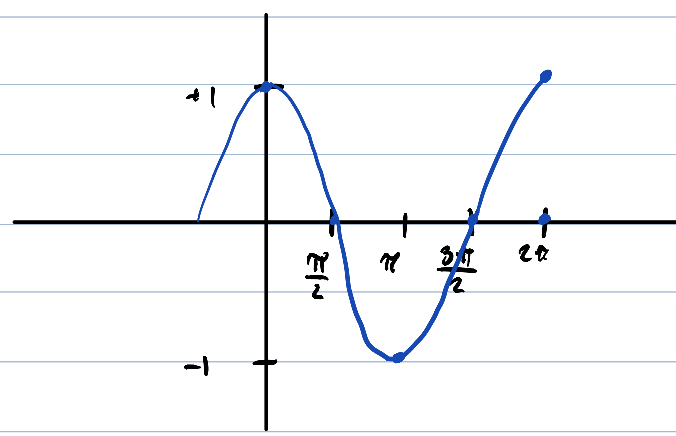
-know how to draw graph of sin, cos, tan

x	y
0	$\sin(0) = 0$
$\pi/2$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0

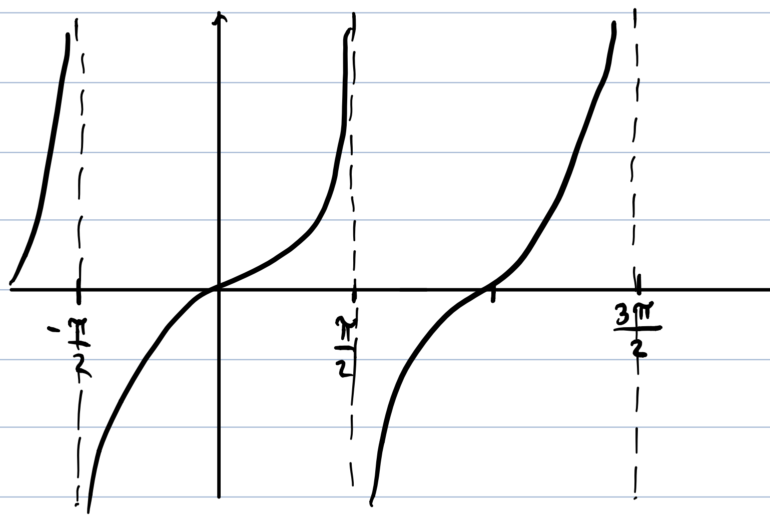


Graph of $y = \cos x$

x	y
0	$\cos(0) = 1$
$\pi/2$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1



$y = \tan x$



DOM $= \{ x \mid x \neq \frac{\pi}{2} + k\pi \quad k = \pm 0, \pm 1, \pm 2, \ldots \}$

Range $= \mathbb{R}$

$\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty$

$\lim_{x \to \frac{\pi}{2}^+} \tan x = -\infty$

Because y = sin(x) and y = cos(x) are both continuous everywhere, we can "cheat" to evaluate limits for sin and cos ex. $\lim\limits_{x \to \frac{\pi}{3}} \frac{x \sin x}{\cos x + 2} \overset{ch}{=} \frac{\frac{\pi}{3} \sin \frac{\pi}{3}}{\cos\left(\frac{\pi}{3}\right) + 2}$ use calculator $=$

ex. $\lim\limits_{x \to \frac{\pi}{2}} \frac{x \sin x}{\cos(x) + 2} \overset{ch}{=} \frac{\frac{\pi}{2} \sin \frac{\pi}{2}}{\cos\left(\frac{\pi}{2}\right) + 2} \overset{\text{use unit circle}}{=} \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4}$

ex. $\lim\limits_{x \to 0^+} x^2 \sin\left(\frac{1}{x}\right) \overset{ch}{=} 0^+ \sin\left(\frac{1}{0^+}\right)$ ← Indeterminate Form

Use Squeeze Theorem

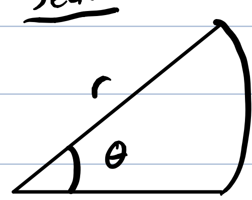
ex. $\lim\limits_{\theta \to 0} \frac{\sin \theta}{\theta} = \frac{0}{0}$ ← INDET

THIS IS AN IMPORTANT LIMIT! PROVE THAT IT'S = 1

← can be asked on test

Proof:

Sector

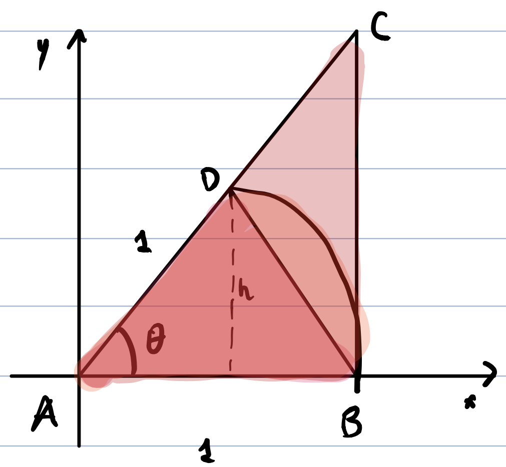


Area $A = \frac{\theta}{2\cancel{\pi}} \left(\cancel{\pi} r^2\right) = \frac{\theta}{2} r^2$

$A = \frac{\theta}{2} r^2$

Theorem: $\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$

Proof:



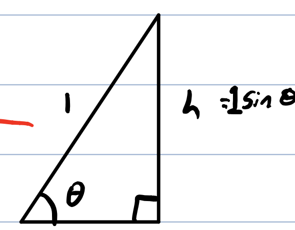
Area

$A_1 = \triangle ABD$

$A_2 = \overset{\frown}{ABD}$ sector

$A_3 = \triangle ABC$

$A_1 = \frac{1}{2}\, \text{base} \times \text{height} = \frac{1}{2}(1)\, h = \frac{1}{2} \sin\theta$



$A_2 = \frac{\theta}{2} r^2 = \frac{\theta}{2}(1)^2 = \frac{1}{2}\theta$

$\tan\theta = \frac{OPP}{ADJ}$

$A_3 = \frac{1}{2}\, \text{base} \times \text{height} = \frac{1}{2}(1)\, \overline{BC} = \frac{1}{2}\tan\theta$

$= \frac{1}{2} \frac{\sin\theta}{\cos\theta}$

$ADJ \tan\theta = OPP$

$\tan\theta = \overline{BC}$

N.B.

$A_1 \le A_2 \le A_3$

$\frac{1}{2}\sin\theta \le \frac{1}{2}\theta \le \frac{1}{2}\frac{\sin\theta}{\cos\theta}$

and

$$\frac{1}{2}\sin\theta \le \frac{1}{2}\theta \quad\Big|\quad \frac{1}{2}\theta \le \frac{1}{2}\frac{\sin\theta}{\cos\theta}$$

$$\frac{\sin\theta}{\theta} \le 1 \quad\Big|\quad \cos\theta \le \frac{\sin\theta}{\theta}$$

So $\cos\theta \le \dfrac{\sin\theta}{\theta} \le 1$

But...

$$\lim_{\theta\to 0^+} \cos\theta = \cos(0) = 1$$

&

$$\lim_{\theta\to 0^+} 1 = 1$$

By squeeze th^m:

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1$$

Also... $\displaystyle\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1$ (don't need to prove it)

① $\displaystyle\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

② $\displaystyle\lim_{\theta\to 0} \frac{\cos(\theta)-1}{\theta} = 0$

Proof: $\lim_{\theta \to 0} \frac{\cos(\theta)-1}{\theta} = \lim_{\theta \to 0} \frac{(\cos(\theta)-1)(\cos(\theta)+1)}{\theta(\cos\theta+1)}$

$$= \lim_{\theta \to 0} \frac{\cos^2\theta - 1}{\theta(1+\cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{-\sin^2\theta}{\theta(1+\cos\theta)}$$ $\sin^2\theta + \cos^2\theta = 1$

$$= \lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta} \cdot \frac{-\sin\theta}{1+\cos\theta} \right)$$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta \to 0} \frac{-\sin\theta}{1+\cos\theta}$$

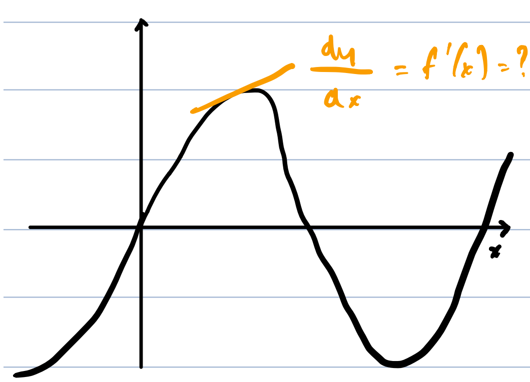
ch

$$= 1 \cdot \frac{\sin(0)}{1+\cos(0)}$$

$$= 1 \cdot \frac{0}{2}$$

$$= 0$$

Now, consider $y = f(x) = \sin(x)$



Important proof

1. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\sin(A+B) = \sin A \cos B + \sin B \cos A$

$= \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$

$= \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$

$= \lim_{h \to 0} \frac{\sin x \cos h - \sin x + \sin h \cos x}{h}$

$= \lim_{h \to 0} \frac{\sin x (\cos h - 1)}{h} + \frac{\sin h \cos x}{h}$

constant

$= \lim_{h \to 0} \left[\sin x \cdot \frac{\cos h - 1}{h} \right] + \lim_{h \to 0} \left[\frac{\sin h}{h} \cdot \cos x \right]$

$= \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$

$= \sin x \cdot 0 + \cos x (1)$

$\boxed{f'(x) = \cos x}$

If $y = \sin x$, then $\frac{dy}{dx} = \cos x$

i.e. $(\sin x)' = \cos x$

ex. If $f(x) = \sin x$, what is the slope of tang. line at

a) $\frac{\pi}{2}$

$f'\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

refer to unit circle

b) π

$f'(\pi) = \cos \pi = -1$

c) 0

$f'(0) = \cos 0 = 1$

ex. $y = x \sin x$

$\frac{dy}{dx} = (x)' \sin x + x (\sin x)'$

$= 1 \sin x + x \cos x$

$= \sin x + x \cos x$

ex. $y = \sin^2 x$

$\frac{dy}{dx} = (\sin x)^2 \leftarrow (\cos)^r$

chain rule $\frac{dy}{dx} = 2(\sin x)^1 \cdot (\sin x)'$

$= 2 \sin x \cdot \cos x$ ✓

$= \sin(2x)$ ← can simplify to this, but not necessary

Now, consider $y = \cos x$

$\frac{dy}{dx} = -\sin x$

i.e. $(\cos x)' = -\sin x$

Proof Let $f(x) = \cos x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$$

$$= \lim_{h \to 0} \frac{\cos x \cdot \cos h - \sin x \cdot \sin h - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x \cdot \cos h - \cos x}{h} - \frac{\sin x \cdot \sin h}{h}$$

$$= \cos x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} - \frac{\sin h}{h} \cdot \sin x$$

October 14, 2016

③ $y = \tan x \rightarrow \frac{dy}{dx} = \sec^2 x$

Proof: $y = \tan x = \frac{\sin x}{\cos x}$

$$\frac{dy}{dx} = \frac{(\sin x)' \cdot \cos x - \sin x \cdot (\cos x)'}{(\cos x)^2}$$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

④ $y = \cot x \rightarrow \frac{dy}{dx} = -\csc^2 x$

* memory tricks: $y = \cot x \rightarrow \frac{dy}{dx} = -\csc^2 x$ (same for rules 5 & 6)

⑤ $y = \sec x \rightarrow \frac{dy}{dx} = \sec x \tan x$

Proof $y = \sec x = \frac{1}{\cos x}$

$$\frac{dy}{dx} = \frac{(1)'\cos x - 1(\cos x)'}{(\cos x)^2} = \frac{0(\cos x) - (-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}$$

$$= \tan x \cdot \sec x$$

* Need to know Proofs

⑥ $y = \csc x \rightarrow \frac{dy}{dx} = -\csc x \cot x$

ex. $y = \frac{\tan^4 x}{x^2+1}$

$$\frac{dy}{dx} = \frac{(\tan^4)'(x^2+1) - (\tan^4 x)(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{4\tan^3 x\,(\tan x)'(x^2+1) - (\tan^4 x)(2x)}{(x^2+1)^2}$$

IN S chain rule

$$\left[(\tan x)^4\right]'$$

$$= 4\tan^3 x \cdot (\tan x)'$$

$$= \frac{4\tan^3 x\,(\sec^2 x)(x^2+1) - 2x\,(\tan^4 x)}{(x^2+1)^2}$$

ex. $y = \tan(x^4)$, $\frac{dy}{dx} = ?$

This is a composition of 2 f^cns: $y = f(x) = \tan(x)$

$y = g(x) = x^4$

<u>Chain rule</u>: If $y = f(g(x)) = f(u)$ where $u = g(x)$

then: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

For $y = \tan(x^4)$

Let $u = x^4 \rightarrow \frac{du}{dx} = 4x^3$

$y = \tan(u) \rightarrow \frac{dy}{du} = \sec^2(u)$

so $\frac{dy}{du} = \sec^2(u) \cdot 4x^3 = \sec^2(x^4) \cdot 4x^3$

<u>Summary</u>

$y = \tan x \rightarrow \frac{dy}{du} = \sec^2(x^4) \cdot (x^4)'$

<u>Chain Rule Structure:</u> $y = \tan(ins) \rightarrow \frac{dy}{du} = \sec^2(ins) \cdot (ins)'$

ex. $y = \tan(\sin x)$

$$\frac{dy}{dx} = \sec^2(\sin x) \cdot (\sin x)'$$

$$= \sec^2(\sin x) \cdot (\cos x)$$

ex. $y = \sin(5x)$

$$\frac{dy}{dx} = \cos(5x) \cdot (5x)'$$

$$= 5\cos(5x)$$

Deriv Formulas [illegible]	Chain Rule Version
① $y = x^r \rightarrow \frac{dy}{dx} = r x^{r-1}$	$y = (\text{ins})^r \rightarrow \frac{dy}{dx} = r(\text{ins})^{r-1} \cdot (\text{ins})'$
② $y = \sin x \rightarrow \frac{dy}{dx} = \cos x$	$y = \sin(\text{ins}) \rightarrow \frac{dy}{dx} = \cos(\text{ins}) \cdot (\text{ins})'$
③ $y = \cos x \rightarrow \frac{dy}{dx} = -\sin x$	$y = \cos(\text{ins}) \rightarrow$ " $=$ $-\sin(\text{ins}) \cdot (\text{ins})'$
④ " $= \tan x \rightarrow$ " $= \sec^2 x$	$= \tan(\text{ins}) \rightarrow$ " $=$ $\sec^2(\text{ins}) \cdot (\text{ins})'$
⑤ " $= \cot x \rightarrow$ " $= -\csc^2 x$	$\vdots$

⑥ " $= \sec x \to$ " $= \sec x \tan x$ | " $= \sec(\text{ins}) \to$ " $= \sec(\quad)\cdot\tan(\quad)\cdot(\quad)'$

⑦ " $= \csc x \to$ " $= -\csc x \cot x$ | " $= \csc(\quad) \to$ " $= -\csc(\quad)\cot(\quad)\cdot(\quad)'$

ex. $y = 5\sec(x^3 - 2x)$

$$\frac{dy}{dx} = 5\left[\sec(x^3-2x)\right]'$$

$$= 5\left[\sec(x^3-2x)\tan(x^3-2x)\cdot(x^3-2x)'\right]$$

$$= 5\sec(x^3-2x)\tan(x^3-2x)(3x^2-2)$$

ex. $y = \sin(x)$

$$\left.\begin{aligned} y' &= \cos(x)\cdot(x)' \\ &= \cos(x) \end{aligned}\right\}$$ stupid way to do it, but works

ex. $y = \sin(2)$ constant

$$\frac{dy}{dx} = 0$$

ex. $y = \sin^5(t^4)$

$$\frac{dy}{dx} = 0$$

$\frac{dy}{dt} = ?$

$$y = (\sin(t^4))^5$$
$$= 5(\sin(t^4))^4 (\sin(t^4))'$$
$$= 5\sin^4(t^4) \cdot \cos t^4 \cdot (t^4)'$$
$$= 5\sin^4(t^4)\cos t^4 \cdot 4t^3$$
$$= 20t^3 \sin^4 t^4 \cos t^4$$ ← Should simplify to this step

ex. $\sin(xy) = x^3 + y^4$, $\frac{dy}{dx} = ?$

must use implicit diff^n

$$[\sin(xy)]' = [x^3 + y^4]$$

$$\cos(xy) \cdot (xy)' = (x^3)' + (y^4)'$$

$$\cos(xy) \cdot [x'y + xy'] = 3x^2 + (4y^3 \cdot y')$$

$$\cos(xy) \cdot [y + xy'] = 3x^2 + 4y^3 y'$$

$$y\cos(xy) + xy' \cdot \cos(xy) = 3x^2 + 4y^3 y'$$
$$xy'\cos(xy) - 4y^3 y' = 3x^2 - y\cos(xy)$$
$$y'(x\cos(xy) - 4y^3) = 3x^2 - y\cos(xy)$$
$$y' = \frac{3x^2 - y\cos(xy)}{x\cos(xy) - 4y^3}$$

Inverse Functions - Trig Twist

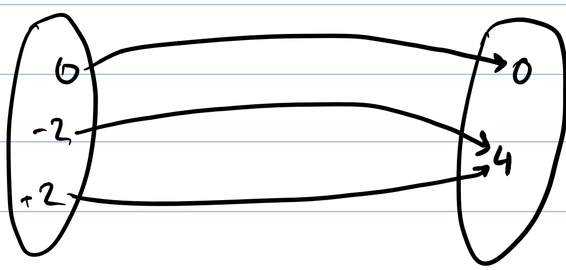
$f^{-1}(f(f(x))) = x$ (x in dom f)

p.205 #45 ↑ Not on Assignment

$f(f^{-1}(x)) = x$ (x in dom f^{-1})

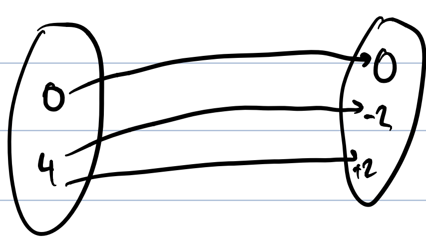
Consider $g(x) = x^2$

Dom: $\mathbb{R}$ Range: $[0, +\infty]$



many to one

If we "reverse" arrows



one to many

So $g(x) = x^2$ has no inverse f^n, but there is an inverse relationship

We can still talk about an inverse f^n related to $g(x) = x^2$ by restricting its domain:

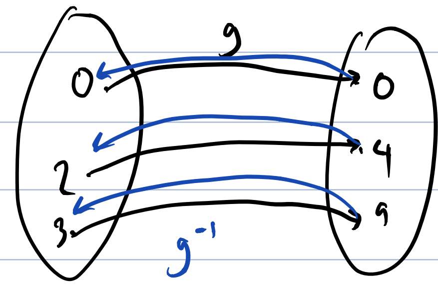
$g(x) = x^2 \quad (x \geq 0)$

changes

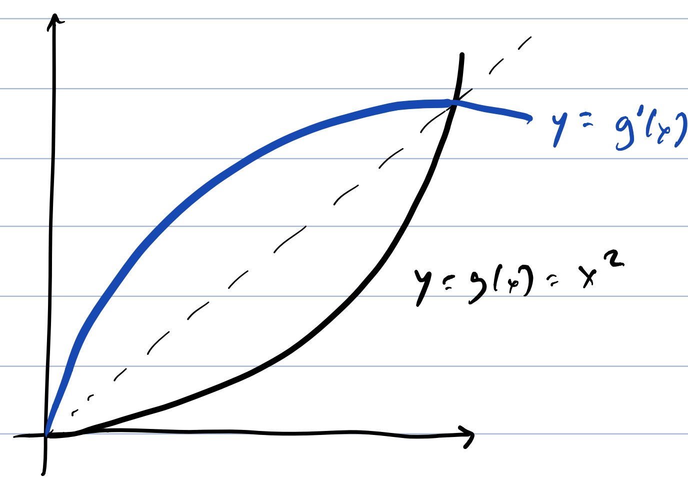
DOM $= [0, +\infty)$

Doesn't change

RANGE $= [0, +\infty)$



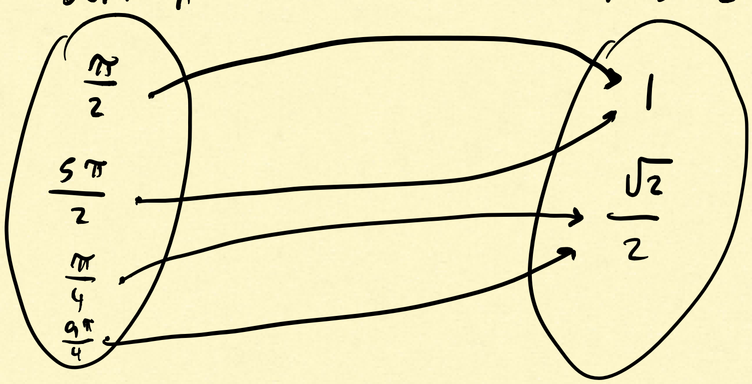
Graph:



INVERSE OF TRIG FCNS

ex. $y = f(x) = \sin x$

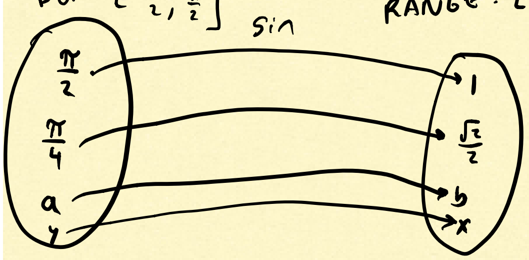
Dom $= \mathbb{R}$ Range $= [-1, 1]$



many to one, so inverse is not a f^{cn}

In order to tlk about an inverse f^{cn} for $f(x) = \sin x$, we restrict the domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

DOM $= [-\frac{\pi}{2}, \frac{\pi}{2}]$ RANGE $= [-1, 1]$



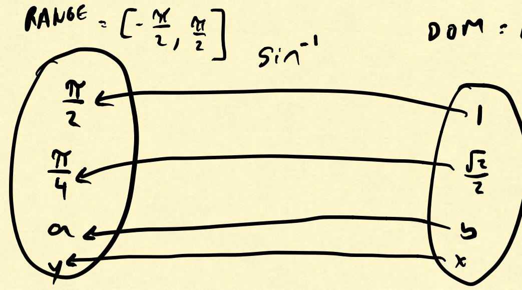
$\sin \frac{\pi}{2} = 1$

$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\sin a = b$

$\sin y = x$

RANGE $= [-\frac{\pi}{2}, \frac{\pi}{2}]$ DOM $= [-1, 1]$



$\sin^{-1}(1) = \frac{\pi}{2}$

$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

$\sin^{-1}(b) = a$

$\sin^{-1}(x) = y$

Defn

$$y = \sin^{-1}(x) \leftrightarrow \sin(y) = x$$

where $-1 \leq x \leq 1$ & $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

N.B. $\sin^{-1} \neq \frac{1}{\sin}$

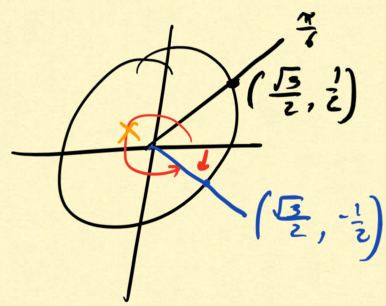
ex. Find $\sin^{-1}\left(-\frac{1}{2}\right)$

Let $y = \sin^{-1}\left(-\frac{1}{2}\right)$

$\sin y = -\frac{1}{2}$ $\left(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\right)$

From unit circle: $y = -\frac{\pi}{6}$

i.e. $\sin^{-1}\left(\frac{1}{2}\right) = -\frac{\pi}{6}$



ex. Find $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

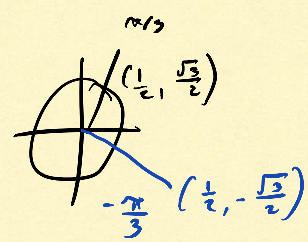
Let $y = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$\downarrow$ defn

$\sin y = -\frac{\sqrt{3}}{2}$ $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$y = -\frac{\pi}{3}$

i.e. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$



$f^{-1}(f(x)) = x \quad (x \text{ in dom } f)$

$f(f^{-1}(x)) = x \quad (x \text{ in dom } f^{-1})$

$\downarrow$

Fact ① $\sin^{-1}(\sin x) = x$ true for all x in $[-\frac{\pi}{2}, \frac{\pi}{2}]$

② $\sin(\sin^{-1} x) = x$ " $[-1, 1]$

N.B.

$\sin^{-1}(3) \leftarrow$ UND. not in DOM $= [-1, 1]$

ex. $\sin(\sin^{-1} 1/3) = 1/3$

$\sin(\sin^{-1} 3) =$ UND

ex. $\sin\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = -\frac{\sqrt{3}}{2}$

ex. $\sin^{-1}\left(\sin \frac{\pi}{7}\right) = \frac{\pi}{7}$

ex. $\sin^{-1}\left(\sin \frac{5\pi}{2}\right) \leftarrow$ Rule ① cannot be used, but still defined

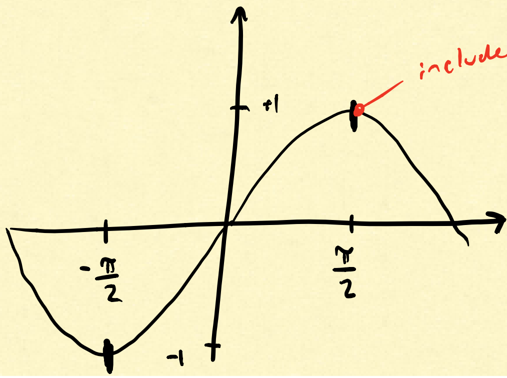
$\frac{5\pi}{2}$ & $\frac{\pi}{2}$ are coterminal since $\frac{5\pi}{2} = \frac{\pi}{2} + 2\pi$

So $\sin\left(\frac{5\pi}{2}\right) = \sin \frac{\pi}{2}$

So $\sin^{-1}\left(\sin \frac{5\pi}{2}\right) = \sin^{-1}\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

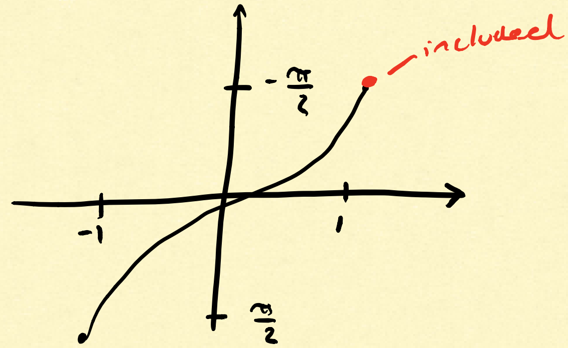
Graphs

$\sin x$



$\Rightarrow$

$\sin^{-1} x = \arcsin x$



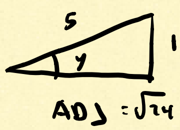
ex. $\cos\left(\sin^{-1} \frac{1}{5}\right)$

$\downarrow$

$y = \sin^{-1} \frac{1}{5} \qquad \left(-\frac{\pi}{2} \le y \le \frac{\pi}{2}\right)$

$\downarrow$ defn

$\sin y = \frac{1}{5}$



$ADJ^2 + 1^2 = 5^2$

$ADJ = \sqrt{24}$

So $\cos\left(\sin^{-1} \frac{1}{5}\right) = \cos(y) = \frac{A}{O} = \frac{\sqrt{24}}{5}$

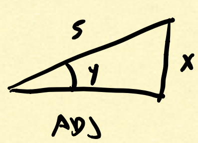
ex. Rewrite $\cos\left(\sin^{-1} \frac{x}{5}\right)$ in "algebraic form"

= no trig

Let $y = \sin^{-1} \frac{x}{5}$

$\downarrow$

$\sin y = \frac{x}{5}$



$ADJ^2 + x^2 = 5^2$

$ADJ = \sqrt{25 - x^2}$

Restrictions: $-1 \le \frac{x}{5} \le 1$

Don't need to include, it's implied

So $\cos\left(\sin^{-1} \frac{x}{5}\right) = \cos(y) = \frac{A}{H} = \frac{\sqrt{25-x^2}}{5}$

Question: If $y = \sin^{-1} x$, $\frac{dy}{dx} = ?$ ← PROOF (know it!)

If $y = \sin^{-1} x$, then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

i.e. $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$

$(\arcsin x)' =$ "

Soln: $y = \sin^{-1} x$

$\sin(y) = x$ ← def^n

use implicit diff^n

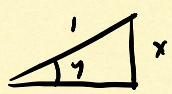
$[\sin(x)]' = [x]'$

↓

$\cos(y) \cdot y' = 1$

$y' = \frac{1}{\cos(y)}$

Back to $\sin(y) = x$

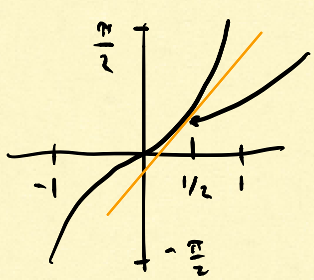


$ADJ = \sqrt{1-x^2}$ ← Pyth

↓

$\cos(y) = \frac{A}{H} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

So $y' = \frac{1}{\sqrt{1-x^2}}$



$\left(\frac{1}{2}, \sin^{-1}\frac{1}{2}\right) = \left(\frac{1}{2}, \frac{\pi}{6}\right)$

Slope $= \frac{1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-\left(\frac{1}{2}\right)^2}} = \frac{2}{\sqrt{3}}$

Chain Rule Version:

If $y = \sin^{-1}(\text{ins})$ then $\frac{dy}{dx} = \frac{1}{\sqrt{1-(\text{ins})^2}} (\text{ins})'$

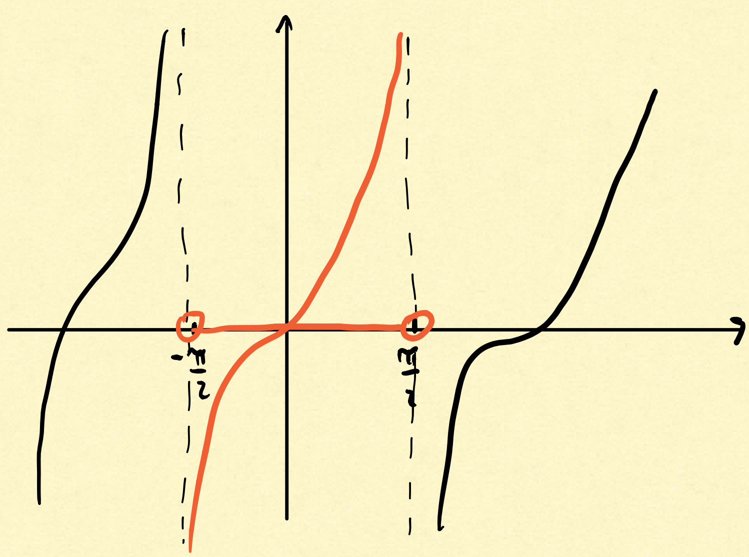
ex. $y = \sin^{-1}(3x)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(3x)^2}} (3x)'$$

$$= \frac{3}{\sqrt{1-9x^2}}$$

* Only responsible for $\sin^{-1}(x)$ & $\tan^{-1}(x)$ [not $\cos^{-1}(x)$]

$y = \tan^{-1} x = \text{Arctan } x$

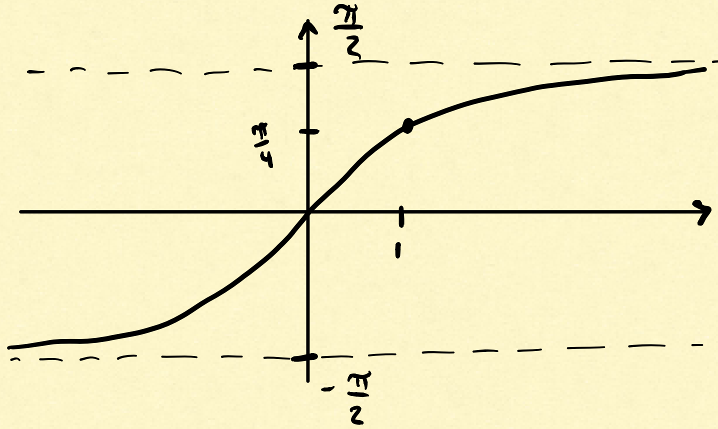


Working Def^n

$y = \tan^{-1} x \leftrightarrow \tan y = x$

$-\frac{\pi}{2} < y < \frac{\pi}{2}, \quad -\infty < x < +\infty$

Graph of Inverse $y = \tan^{-1}(x)$



$\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$

$\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$

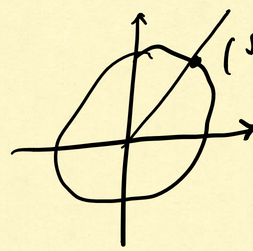
ex. $\tan^{-1}(1)$

Let $y = \tan^{-1}(1)$

$\tan y = 1 \qquad \left(-\frac{\pi}{2} < y < \frac{\pi}{2}\right)$

$y = \frac{\pi}{4}$

So $\tan^{-1}(1) = \frac{\pi}{4}$



$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ $\tan\theta = \frac{\sin\theta}{\cos\theta} = 1$

① $\tan^{-1}(\tan x) = x \qquad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$

② $\tan(\tan^{-1} x) = x \qquad (-\infty < x < +\infty)$

ex. $\tan(\tan^{-1} 3) = 3$

ex. $\tan^{-1}(\tan \frac{\pi}{4}) = \frac{\pi}{4}$

ex. $\tan^{-1}(\tan \frac{9\pi}{4}) = \underbrace{\cancel{\frac{9\pi}{4}}}_{\text{co terminal}} = \tan^{-1}(\tan \frac{\pi}{4}) = \frac{\pi}{4}$

so $\tan \frac{9\pi}{4} = \tan \frac{\pi}{4}$

Derivatives of $\tan^{-1}x$

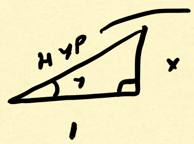
If $y = \tan^{-1}x$, then $\frac{dy}{dx} = \frac{1}{1+x^2}$

Proof

$$y = \tan^{-1}x$$

$$\tan(y) = x$$

$$\tan y = \frac{x}{1}$$



$$= \sqrt{1+x^2}$$

$$[\tan(y)]' = [x]'$$

$$\rightarrow \sec y = \frac{H}{A} = \sqrt{1+x^2}$$

$$\sec^2(y) \cdot y' = 1$$

$$y' = \frac{1}{\sec^2 y}$$

$$y' = \frac{1}{1+x^2}$$

Chain Rule Version:

$$y = \tan^{-1}(ins) \rightarrow \frac{dy}{dx} = \frac{1}{1+(ins)^2}\,(ins)'$$

ex. $q(t) = 3t \operatorname{Arctan} 5t$

$$q'(t) = [3t]' \operatorname{Arctan} 5t + 3t\,[\operatorname{Arctan} 5t]'$$

$$= 3 \operatorname{Arctan} 5t + 3t \cdot \frac{1}{1+(5t)^2} \cdot (5t)'$$

$$= 3 \operatorname{Arctan} 5t + \frac{5t}{1+25t^2}$$