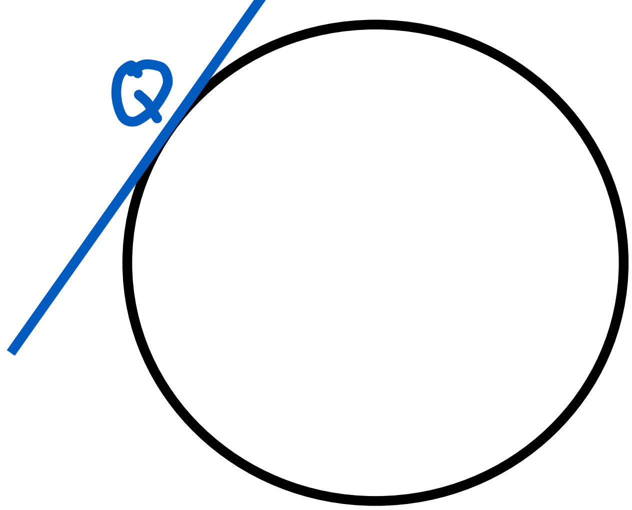
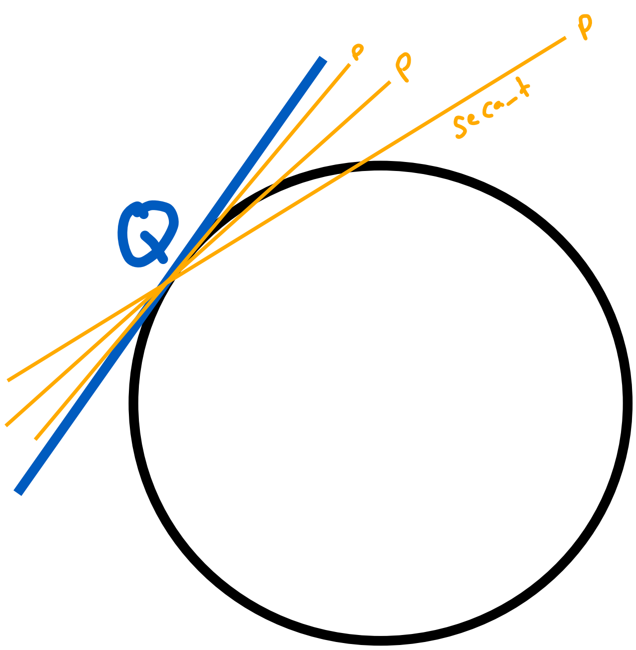
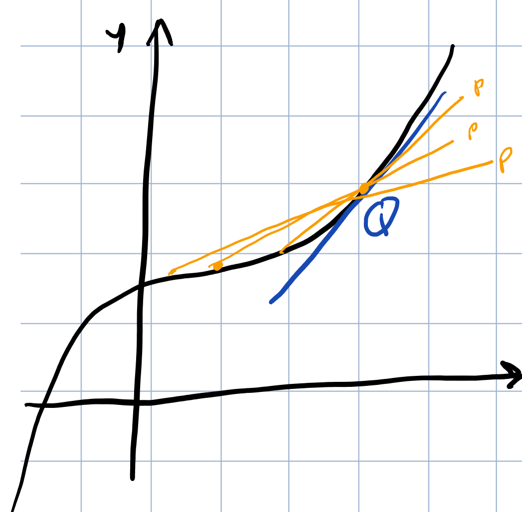
DERIVATIVES AND TANGENT LINES



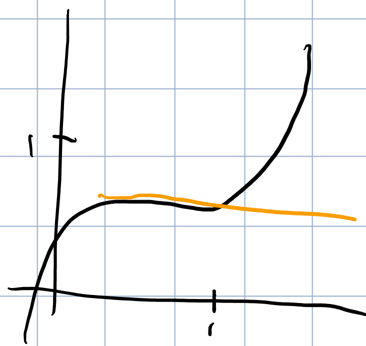




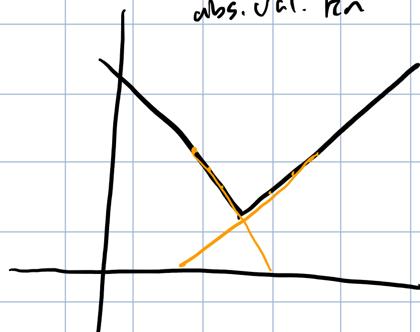
Chose pts P on the graph of y = f(x) approaching pt Q (from both sides, none touching pt Q).

If the "secant" lines PQ approach a unique "limiting" position, then the line that sits at this limiting position is defined to be the tangent line.

September 16, 2016

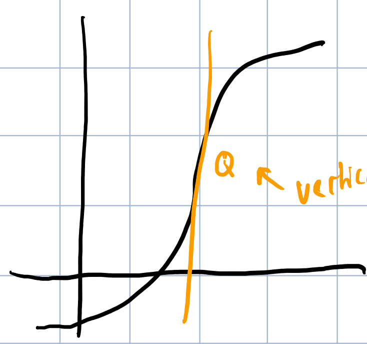


abs. val. fcn



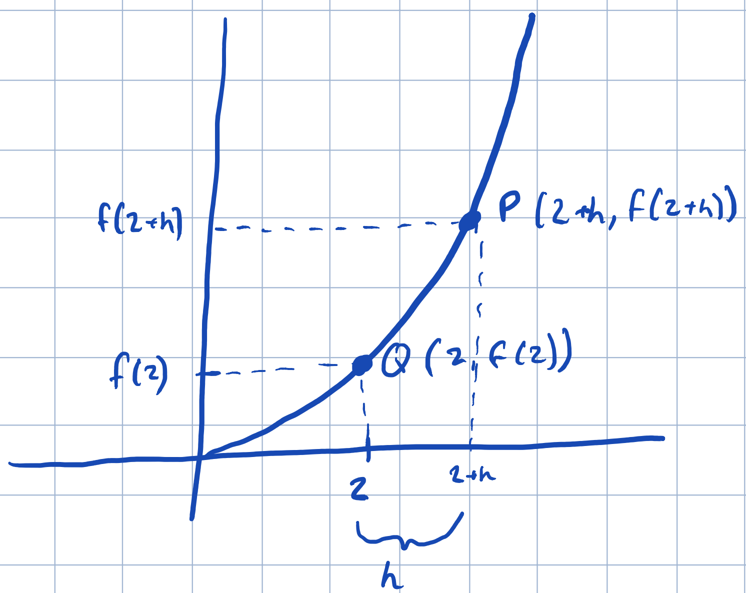
No unique "limiting position"

so no tangent line at pt Q



← vertical tangent line

Ex. Consider $y = f(x) = x^2$



<u>Slope of secant line $\overline{PQ}$:</u>

$$m_{sec} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(2+h) - f(2)}{(2+h) - 2} = \frac{f(2+h) - f(2)}{h}$$

To force pt P along the graph towards pt Q, let h –> 0

Then the positions of secant lines $\overline{PQ}$ will approach the position of the tangent line.

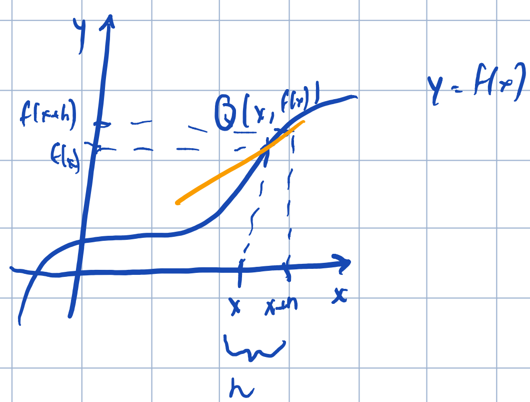
So then the slopes M_{sec} will approach M_{tan} of the tangent line

<u>i.e.</u> $\lim_{h \to 0} m_{sec} = m_{tan}$

<u>i.e.</u> $m_{tan} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$

$$= \lim_{h \to 0} \frac{(2+h)^2 - 4}{h} = \lim_{h \to 0} \frac{4 + 4h + h^2 - 4}{h} = \lim_{h \to 0} (4+h) \stackrel{\circ}{=} 4$$

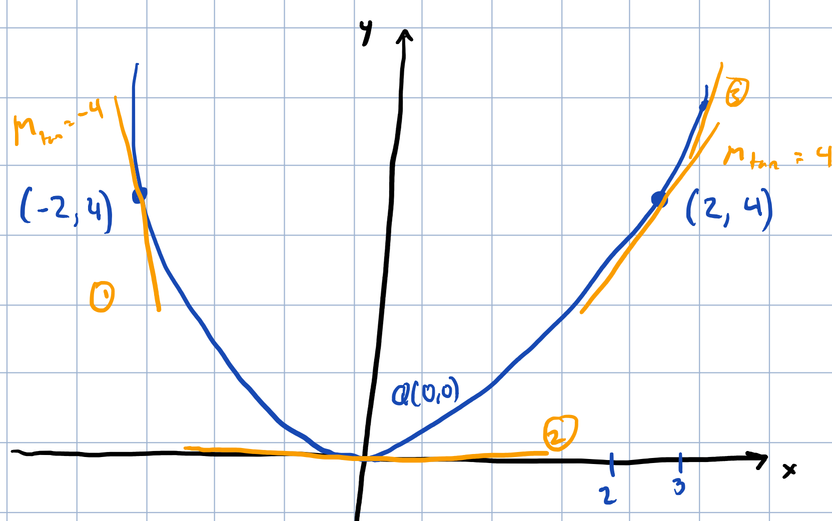
For any f^n $y = f(x)$ & at any pt $Q(x, f(x))$ we can apply the same logic



$$M_{sec} = \frac{f(x+h) - f(x)}{x+h-x} = \frac{f(x+h) - f(x)}{h}$$

$$M_{tan} = \lim_{h \to 0} \frac{f(x+h) + f(x)}{h}$$

ex. $f(x) = x^2$



1

$$m_{tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h}$$

$$= \lim_{h \to 0} \frac{(-2+h)^2 - 4}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{4} - 4h + h^2 - \cancel{4}}{h}$$

$$= \lim_{h \to 0} (-4 + h)$$

$$= -4$$

2

$$m_{tan} = \frac{f(x+h) - f(x)}{h}$$

$$m_{tan} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$m_{tan} = \lim_{h \to 0} \frac{h^2 - 0}{h}$$

$$m_{tan} = \lim_{h \to 0} h$$

$$m_{tan} = 0$$

3

$$m_{tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(3+h)^2 - (3)^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{9} + 6h + h^2 - \cancel{9}}{h}$$

$$= \lim_{h \to 0} (6 + h)$$

$$= \lim_{h \to 0}$$

$$= 6$$

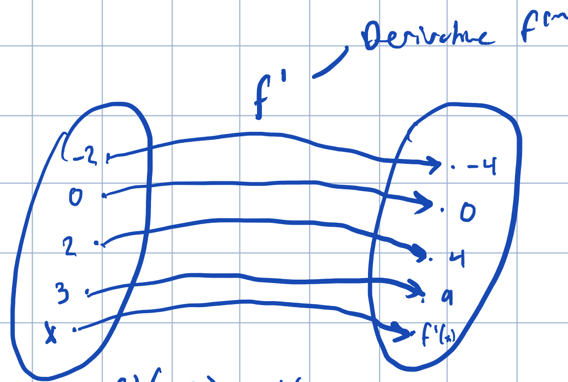
$y = mx + b$

$9 = 6(3) + b$

$9 = 18 + b$

$b = -9$

eqn tangent line: $y = 6x - 9$



$f'(-2) = -4$

$f'(0) = 0$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} (2x+h)$$

$\boxed{f'(x) = 2x}$ ← derivative f^n

Def^n If $y = f(x)$ is any f^n the the derivative (f') is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Mean

ex. If $f(x) = \sqrt{x}$, find $f'(x)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})}{h} \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$f'(x) \overset{ch}{=} \frac{1}{2\sqrt{x}}$$

ex. If $f(x) = x^3$, find the eqn of the tang line at $x = 2$

$$f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)(x+h)(x+h) - x^3}{h}$$

$$= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2)(x+h) - x^3}{h}$$

$$= \lim_{h\to 0} \frac{x^3 + 2x^2h + h^2x + x^2h + 2xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 \cancel{- x^3}}{h}$$

$$= \lim_{h\to 0} (3x^2 + 3xh + h^2)$$

$= 3x^2$ ← Derivative

So $f'(2) = 3(2)^2 = 12$ ← slope of tan line at $x = 2$

Ans $y = 12 - 16$

PASCAL'S TRIANGLE

$(a+b)^n$

$$1$$

$$1 \quad 1$$

$$\underbrace{1 \quad 2}_{1+2=3} \quad 1$$

$$1 \quad 3 \quad \underbrace{3 \quad 1}_{3+1=4}$$

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

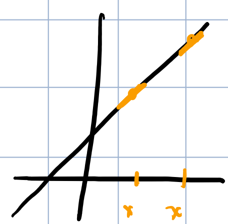
ex. $(a+b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$

$a^5b^0 \quad a^4b^1$

$a \quad 5 \to 4 \to 3$

$b \quad 0 \to 1 \to 2$

ex. $f(x) = 3x+2$ Find $f'(x)$



Geometrically, we expect $f'(x) = 3$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x) = 3x+2$

$f(x+h) = 3(x+h)+2$

$$= \lim_{h\to 0} \frac{3(x+h)+2 - (3x+2)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{3x}+3h \cancel{+2} - \cancel{3x} \cancel{-2}}{h}$$

$$= \lim_{h\to 0} 3$$

$f'(x) = 3$ as expected

ex. $f(x) = 7$

$f'(x)$ should be $= 0$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{7-7}{h} = \lim_{h\to 0} \frac{7}{h} = \lim_{h\to 0} 0 = 0$$

September 14, 2016

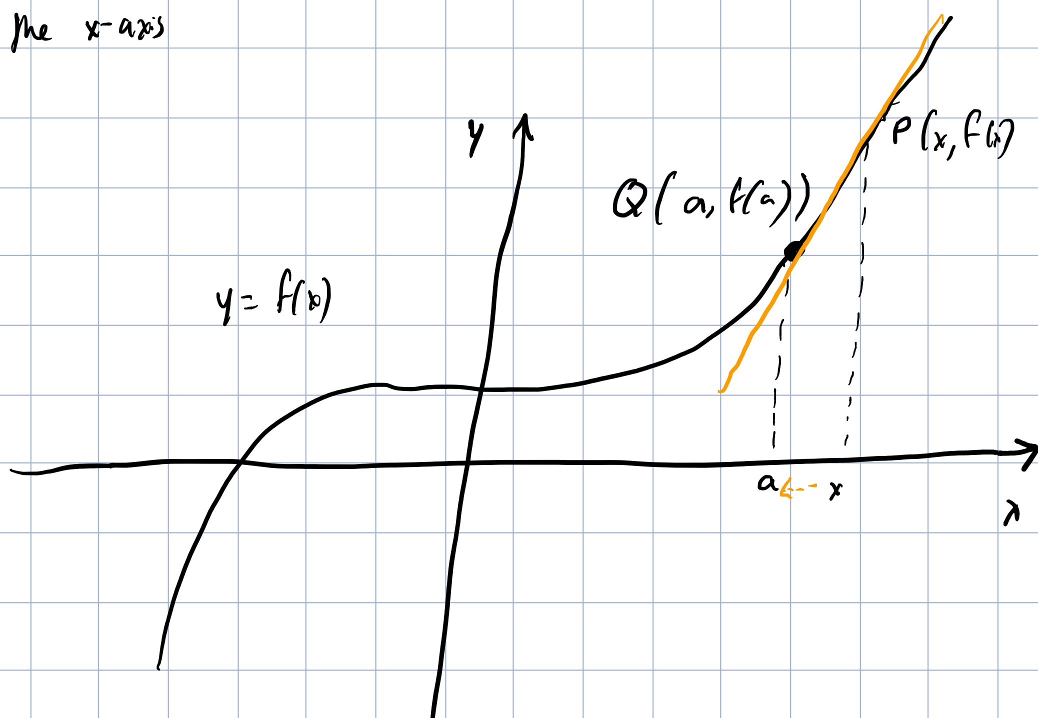
Constant fcn rule

If $y = f(x) = c$ (c is a constant), then $f'(x) = 0$

ex. $g(x) = \sqrt{3} - 2.7 \Rightarrow g'(x) = 0$

Alternative def^n of the derivative

Consider a fcn $y = f(x)$ and chose any fixed number 'a' on the x-axis



Slope m_{sec} through $\overline{PQ}$ is:

$$m_{sec} = \frac{f(x) - f(a)}{x - a}$$

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$ Let $x \to a$

Alt def^n

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

ex. Consider $f(x) = x^3$

Choose any fixed x in Dom f

$$f'(x) \overset{\text{original def}^n}{=} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2$$

$$= 3x^2$$

Chose any fixed 'a' in Dom f

$$f'(a) \overset{\text{alt def}^n}{=} \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{x^3 - a^3}{x - a}$$

$$= \lim_{x \to a} \frac{\cancel{(x-a)}(x^2 + ax + a^2)}{\cancel{x-a}}$$

$$= \lim_{x \to a} x^2 + ax + a^2$$

$$= a^2 + a^2 + a^2$$

$$= 3a^2$$

So $f'(x) = 3x^2$ for any x in Dom f

OBSERVING A PATTERN

SPECIAL FACTORING

$$x^3 - a^3 = (x-a)(x^2 + ax + a^2)$$

$$x^4 - a^4 \quad (x-a)(x^3 + x^2a + xa^2 + a^3)$$

$$x^5 - a^5 \quad (x-a)(x^4 + x^3a + x^2a^2 + xa^3 + a^4)$$

$$\vdots$$

$$x^n - a^n \quad (x-a)\underbrace{(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots a^{n-1})}_{n \text{ terms}}$$

If $f(x) = x^4$, find $f'(x)$ (alt def^n)

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{x^4 - a^4}{x - a}$$

$$= \lim_{x \to a} (x^3 + x^2a + xa^2 + a^3)$$

$= \lim_{x \to a} (a^3 + a \quad + a \quad + a)$

$= 4a^3$

So $f'(x) = 4x^3$

$f(x) = x^2 \implies f'(x) = 2x$

$f(x) = \sqrt{x} \implies f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-\frac{1}{2}}$

$f(x) = x^3 \implies f'(x) = 3x^2$

$f(x) = x^4 \implies f'(x) = 4x^3$

Based on pattern

$x^7 \implies f'(x) = 7x^6$

Power Rule

If $f(x) = x^r$ then $f'(x) = r x^{r-1}$ (fixed number "r")

ex. $g(x) = x^{14} \implies g'(x) = 14x^{13}$

ex. $h(t) = t^5 \implies g'(t) = 5t^4$

ex. $g(x) = \sqrt[3]{x} = x^{\frac{1}{3}} \implies g'(x) = \frac{1}{3} x^{-\frac{2}{3}}$

ex. $f(t) = \frac{1}{\sqrt[5]{t^2}} = \frac{1}{(t^2)^{\frac{1}{5}}} \implies f'(t) = -\frac{2}{5} t^{-\frac{2}{5}-1} = -\frac{2}{5} t^{-\frac{7}{5}}$

ex. $f(x) = \frac{1}{x^{10}} = x^{-10} \Rightarrow f'(x) = -10\, x^{-11}$ ← not -9

ex. $g(x) = \pi^4 \Rightarrow g'(x) = 0$ constant

ex. $f(x) = x^1 \Rightarrow f'(x) = 1 \cdot x^{1-1} = x^0 = 1$

Proof of Power Rule

If $f(x) = x^r$ then $f'(x) = r x^{r-1}$ (fixed number "r")

Proof for pos. integers only $(n = 1, 2, 3, \ldots)$

Proof

Assume n is some pos. integer and assume $f(x) = x^n$

$$f'(a) \overset{\text{alt. def}^n}{=} \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{x^n - a^n}{x - a}$$

$$= \lim_{x \to a} \frac{\cancel{(x-a)} \overbrace{(x^{n-1} - x^{n-2} a + x^{n-3} a^2 + \ldots + x a^{n-2} + a^{n-1})}^{n \text{ terms}}}{\cancel{x-a}}$$

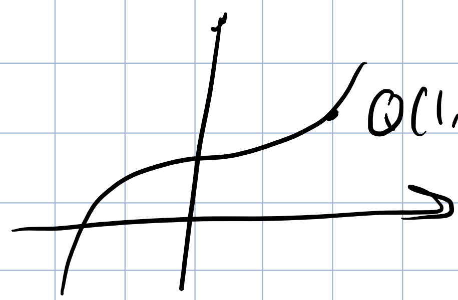
$$\overset{\text{ch}}{=} a^{n-1} + a^{n-2} \cdot a + a^{n-3} a^2 + \ldots a \cdot a^{n-2} + a^{n-1}$$

$$= a^{n-1} + a^{n-1} + a^{n-1} \ldots + a^{n-1}$$

$f'(a) = n a^{n-1}$

So $f'(x) = n x^{n-1}$

ex. $y = g(x) = x^5$ Find the eqn of tangent line at $x = 1$



$Q(1,1)$ when $x=1$, $y = 1^5 = 1$

$g'(x) = 5x^4$

slope of tangent line at $x=1$ is $g'(1) = 5(1)^4 = 5$

So eqn of tang. line

$y = mx + b$

$\downarrow$

$y = 5x + b$

$\downarrow$ $x=1, y=1$

$1 = 5(1) + b$

$b = -4$

eqn tang. line at $x=1$: $y = 5x - 4$

PRIME NOTATION / LEIBNITZ NOTATION

Prime Notation	Leibnitz Notation
$y = f(x) = x^4$	$\underbrace{\frac{d}{dx}}_{\text{depends on } x}^{\text{derivative}} y = 4x^3 \Rightarrow \frac{dy}{dx}$
$y' = 4x^3$	$\frac{d}{dx}(f(x)) = 4x^3$
$f'(x) = 4x^3$	$\frac{d}{dx}(x^4) = 4x^3$
$(x^4)' = 4x^3$	

RULES OF DERIVATIVES

① Constant Function Rule

$y = f(x) = c$	$\frac{dy}{dx} = 0$	ex. $(\sqrt{2})' = 0$
$y' = 0$	$\frac{d}{dx}(f(x)) = 0$	or $\frac{d}{dx}(\sqrt{2}) = 0 \leftarrow f(x) = \sqrt{2}$
$f'(x) = 0$	$\frac{d}{dx}(c) = 0$	or $\frac{d}{dx}(\sqrt{2}) = 0 \leftarrow f(t) = \sqrt{2}$
$(c)' = 0$		

② Power Rule

$y = f(x) = x^r$	$\frac{dy}{dx} = r x^{r-1}$	ex. $\frac{d}{dx}(x^5) = 5x^4$
$y' = r x^{r-1}$	$\frac{d}{dx}(f(x)) = r x^{r-1}$	$\frac{d}{dt}(t^5) = 5t^4$
$f'(x) = r x^{r-1}$ (r is a constant)		$\frac{d}{dx}(t^5) = 0 \leftarrow$ Constant

③ Constant Multiple Rule

$$\boxed{\text{If } y = k\,f(x) \text{, then } y' = k\,f'(x)}$$

i.e. $[k \circ f(x)]' = k[f(x)]' = k\,f'(x)$

ex. $y = 6x^4 \rightarrow \frac{dy}{dx} = 6(x^4)'$ ③

$6(4x^3)$ ②

$24x^3$

ex. $y = \sqrt{5x}$, find y'

$y = \sqrt{5}\sqrt{x}$

$= \sqrt{5}\,x^{\frac{1}{2}}$

$= \sqrt{5}\left(x^{\frac{1}{2}}\right)'$

$= \sqrt{5}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$

$= \frac{\sqrt{5}}{2}x^{-\frac{1}{2}}$

$= \frac{\sqrt{5}}{2\sqrt{x}}$

ex. What is the slope of the tangent line at $x=9$?

$y = \sqrt{5x}$

$y' = \frac{\sqrt{5}}{2\sqrt{9}} = \frac{\sqrt{5}}{6}$

Note: ~~$y = \sqrt{5x} = (5x)^{\frac{1}{2}}$~~

~~$y' = \frac{1}{2}(5x)^{-\frac{1}{2}}$~~

ex. $f(x) = \frac{5}{x^3}$, $f'(x) = ?$

$f(x) = 5x^{-3}$

$f'(x) = 5(x^{-3})'$

$f'(x) = 5(-3x^{-4})$

$f'(x) = -15x^{-4}$

Proof of Constant Multiple Rule

If $y = k f(x)$

then

$y' = k f'(x)$

Proof: Let $y = Q(x) = k \cdot f(x)$

$$y' = Q'(x) = \lim_{h \to 0} \frac{Q(x+h) - Q(x)}{h}$$

$$Q'(x) = \lim_{h \to 0} \frac{k f(x+h) - k f(x)}{h}$$

$$Q'(x) = \lim_{h \to 0} \left[k \cdot \frac{f(x+h) - f(x)}{h} \right]$$

$$= \lim_{h \to 0} k \cdot \lim \frac{f(x+h) - f(x)}{h}$$

$$= k \cdot f'(x)$$

ex. $y = 5x$, $y' = ?$

$$y' = (5x)' = 5(x)' = 5(1) = 5$$

4/5 Sum Rule & Difference Rule

$$\boxed{\text{If } y = f(x) \pm g(x) \text{ then } y' = f'(x) \pm g'(x)}$$

ex. $y = x^7 + 4$, Find y'

$$y' = (x^7)' + (4)'$$
$$y' = 7x^6 + 0$$
$$= 7x^6$$

ex. $y = t^3 + 6t^2 + 3$, Find $\frac{dy}{dt}$

$$\frac{dy}{dt} = (t^3)' + (6t^2)' + (3)'$$
$$= 3t^2 + 6(t^2)' + 0$$
$$= 3t^2 + 12t$$

Note: Sum rule extends:

$$(f+g+h+j)' = f' + g' + h' + i'$$

Proof of Sum Rule

Let $y = Q(x) = f(x) + g(x)$

So $y' = Q'(x) = \lim\limits_{h \to 0} \dfrac{Q(x+h) - Q(x)}{h}$

$$= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(x) + g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$y' = Q'(x) = f'(x) + g'(x)$

ex, $y = \frac{x^3 - x^4}{3x}$, $y' = ?$

$y = \frac{x^3}{3x} - \frac{x^4}{3x}$

$= \frac{1}{3}x^2 - \frac{1}{3}x^3$

$y' = \frac{2}{3}x - x^2$

$y = \sqrt{3x^5} - 2x^4 + \frac{4}{x^3}$

Find $\frac{dx}{dy}$

$y = (3x^5)^{\frac{1}{2}} - 2x^4 + \frac{4}{x^3}$

$\frac{dx}{dy}\left((3x^5)^{\frac{1}{2}}\right) = \frac{dx}{dy}\left(\sqrt{3} \cdot x^{\frac{5}{2}}\right) = \sqrt{3} \cdot \frac{5}{2} x^{\frac{3}{2}}$

$\frac{dx}{dy}\left(\frac{4}{x^3}\right) = \frac{dx}{dy}\left(4x^{-3}\right) = 4 \cdot -3x^{-4}$

$\frac{dy}{dx} = \sqrt{3} \cdot \frac{5}{2} x^{\frac{3}{2}} - 8x^3 - 12x^{-4}$

Quick Rules so far:

1. $(c)' = 0$
2. $(x^r)' = r\,x^{r-1}$
3. $(cf)' = c \cdot f'$
4. $(f+g)' = f' + g'$
5. $(f-g)' = f' - g'$

6. $(f \cdot g)' = f' \cdot g'$ is FALSE!!

ex. $(x^3 \cdot x^2)' = \cancel{(x^3)' \cdot (x^2)' = (3x^2)(2x) = 6x^3}$

$(x^3 \cdot x^2)' \downarrow (x^5)' \downarrow 5x^4$

6. Product Rule

If $y = f(x) \cdot g(x)$ then $y' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

i.e.

$$(f \cdot g)' = f' g + f g'$$

ex. Back to:

$$(x^3 \cdot x^2)' = (x^3)'(x^2) + (x^3)(x^2)'$$

$= (3x^2)(x^2) + (x^3)(2x)$

$= 3x^4 + 2x^4$

$= 5x^4$

ex.

$y = (x^3+1)(x^4-5) + 7x^4$ Find $\frac{dy}{dx}$

$y' = [(x^3+1)(x^4-5)]' + [7x^4]'$ Sum rule

$y' = (x^3+1)'(x^4-5) + (x^3+1)(x^4-5)' + 7[x^4]'$ Constant mult. rule

$y' = (3x^2)(x^4-5) + (4x^3) + 7(4x^3)$

$y' = (3x^2)(x^4-5) + (4x^3) + (28x^3)$

Proof of Product Rule

STANDARD EXAM QUESTION

If $y = f(x)\cdot g(x)$ then $y' = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

Proof!

$$y = Q(x) = f(x)\cdot g(x)$$

$$y' = Q'(x) \stackrel{\text{defn}}{=} \lim_{h\to 0} \frac{Q(x+h) - Q(x)}{h}$$

$$y' = Q'(x) = \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x+h) + f(x)\cdot g(x+h) - f(x)\cdot g(x)}{h}$$

Algebra trick

split in half

$$= \lim_{h \to 0} \left[\frac{\big(f(x+h) - f(x)\big)\, g(x+h)}{h} + \frac{f(x)\big(g(x+h) - g(x)\big)}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \cdot g(x+h) \right] + \lim_{h \to 0} \left[f(x) \cdot \frac{g(x+h) - g(x)}{h} \right]$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h \to 0} g(x+h) + \lim_{h \to 0} f(x) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

So $y' = Q'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

7 Quotient Rule

$\cancel{\left(\frac{f}{g}\right)' = \frac{f'}{g'} \text{ FALSE}}$

If $y = \dfrac{f(x)}{g(x)}$, then $y = \dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$

order matters

Do not need to know the proof

ex. $$\left(\frac{x^5}{x^3}\right)' = \frac{(x^5)' \cdot x^3 - x^5 \cdot (x^3)'}{(x^3)^2} = \frac{5x^4 \cdot x^3 - x^5 \cdot 3x^2}{x^5}$$

$$= \frac{5x^7 - 3x^7}{x^6} = \frac{2x^7}{x^6} = 2x$$

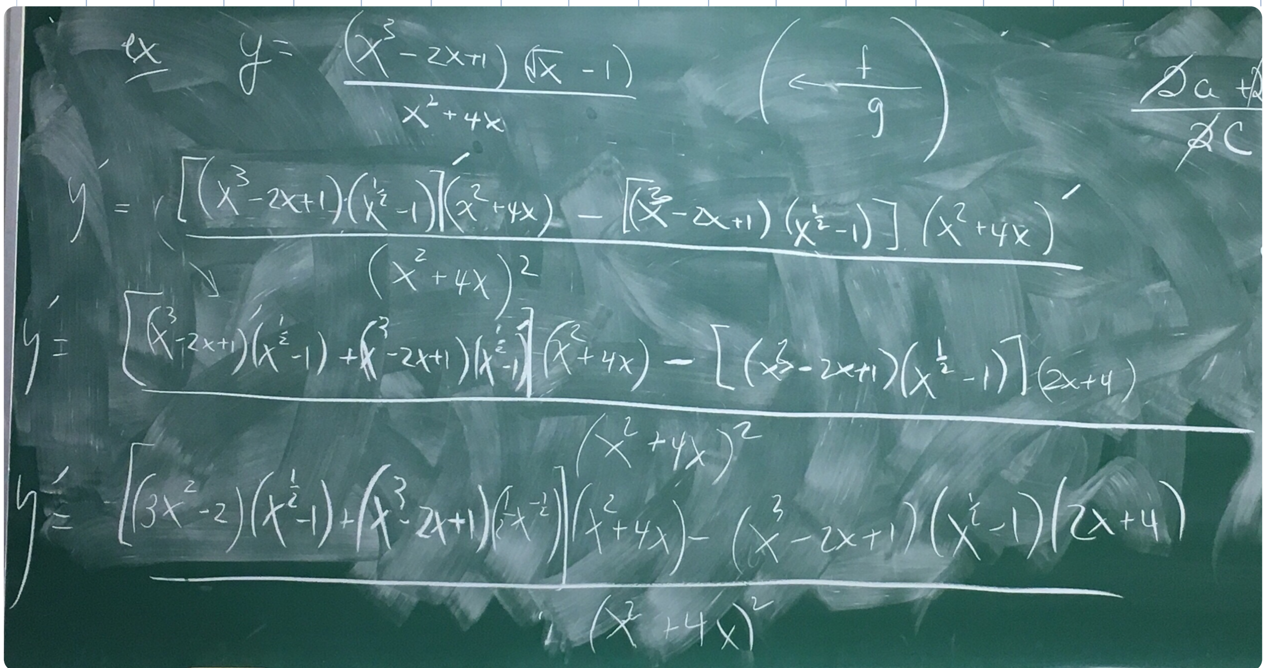
ex. $$y = \frac{(x^3 - 2x + 1)(\sqrt{x} - 1)}{x^2 + 4x} \Rightarrow \frac{(x^3 - 2x + 1)}{x^2 + 4x} \cdot (\sqrt{x} - 1)$$

can also use product rule 1st, but not recommended

$$y' = \frac{(x^3 - 2x + 1)'(x^{\frac{1}{2}} - 1) - (x^3 - 2x + 1)(x^{\frac{1}{2}} - 1)'}{(x^2 + 4x)^2}$$

$$y' = \frac{\left[(x^3 - 2x + 1)(x^{\frac{1}{2}} - 1) + (x^2 - 2x + 1)(x^2 - 1)\right](x^2 + 4) - \left[(x^3 - 2x + 1)(x^{\frac{1}{2}} - 1)\right](2x + 4)}{(x^2 + 4x)^2}$$

$$y' = \frac{\left[(3x^2 - 2)(x^{\frac{1}{2}} - 1) + (x^3 - 2x + 1)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)\right](x^2 + 4x - (x^3 - 2x + 1)(x^{\frac{1}{2}} - 1)(2x + 4)}{(x^2 + 4x)^2}$$



September 29, 2016

$$\left(\frac{f}{g}\right)' = \frac{f'\cdot g - f\cdot g'}{g^2}$$

ex.

$$y = \frac{(x-3)(x^2+4)}{2x-3}$$

$$\frac{dy}{dx} = \frac{\left[(x^3-1)(x^2+4)\right]'(2x-3) - \left[(x^3-1)(x^2+4)\right](2x-3)'}{(2x-3)^2}$$

$$\frac{dy}{dx} = \frac{\left[(x^3-1)(x^2+4)' + (x^3-1)'(x^2+4)\right] - \left[(x^3-1)(x^2+4)\right](2x-3)'}{(2x-3)^2}$$

$$= \frac{\left[(x^3-1)(2x) + (3x^2)(x^2+4)\right]\left[(2x-3) - \left[(x^3-1)(x^2+4)\right]2\right.}{(2x-3)^2}$$

Differenciate = "find the derivative"

Higher Order Derivatives

ex. $y = f(x) = x^5$

Prime notation

1st deriv. $f'(x) = 5x^4$

2nd $f''(x) = 20x^3$

3rd $f'''(x) = 60x^2$

4th deriv = $f^{(4)}(x) = 120x$

5th deriv. $f^{(5)}(x) = 120$

$f^{(6)}(x) = 0$

$f^{(7)}(x) = 0$

$\vdots$

ex. $y = f(x) = x^5$

Leibnitz Notation

$$\frac{dy}{dx} = 5x^4$$

2nd deriv.

$$\Rightarrow \frac{d^2y}{dx^2} = 20x^3$$

$$(f'(x))' = f''(x)$$

2 'd'

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

2 'dx'

Differentiability

ex. Consider $f(x) = |x|$ Find $f'(0)$, if possible

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

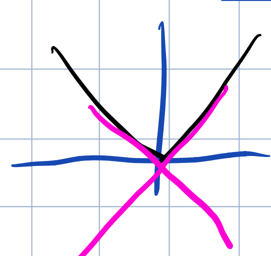
$$f'(0) = \lim_{h\to 0} \frac{|h| - 0}{h}$$

$$|h| = \begin{cases} -h & \text{if } h<0 \\ h & \text{if } h\geq 0 \end{cases}$$

$$= \lim_{h\to 0} \frac{|h|}{h}$$

$$\left.\begin{aligned} \lim_{h\to 0^-} \frac{|h|}{h} &= \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} (-1) = -1 \\ \lim_{h\to 0^+} \frac{|h|}{h} &= \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} (1) = 1 \end{aligned}\right\} f'(0) = \lim_{h\to 0} \frac{|h|}{h} \text{ DNE}$$

Concl: $f(x) = |x|$ is not differentiable at $x = 0$



[redacted]

In general, a f^{cn} that has a "sharp" pt at $x = 'a'$ will not be diff^{ble} at $x = a$.

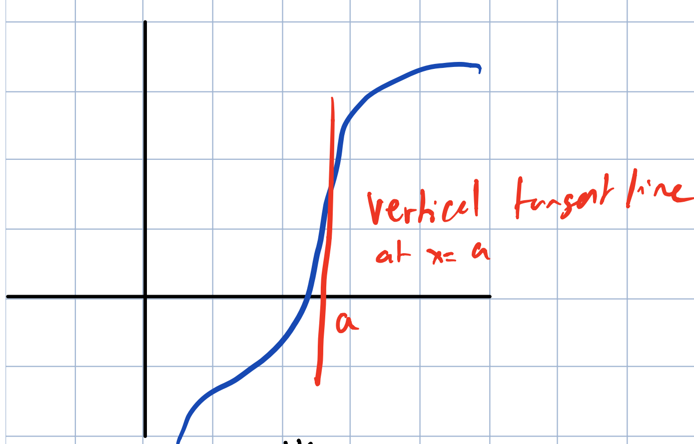
<u>Def^n</u>: A fc^n $y = f(x)$ is said to be diff^{ble} at $x = 'a'$ if:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exists}$$

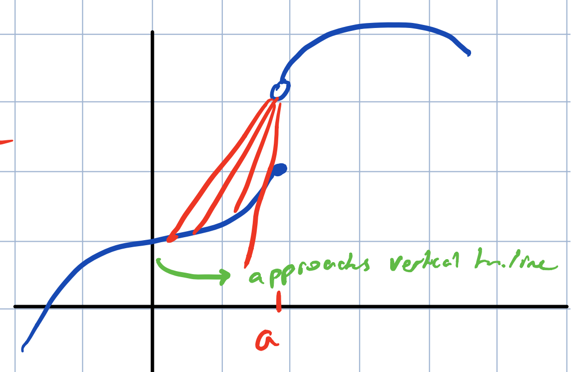
<u>i.e.</u>

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ exists}$$

Other Geometric Indicators of Non-Differentiability



$y = f(x)$ is not diff^{ble} at $x = a$



$y = f(x)$ is not diff^{ble} at $x = a$

ex. Determine whether $f(x) = \sqrt[3]{x+1}$ is diff^{ble} at $x = -1$

$$f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt[3]{-1+h+1} - \sqrt[3]{-1+1}}{h}$$

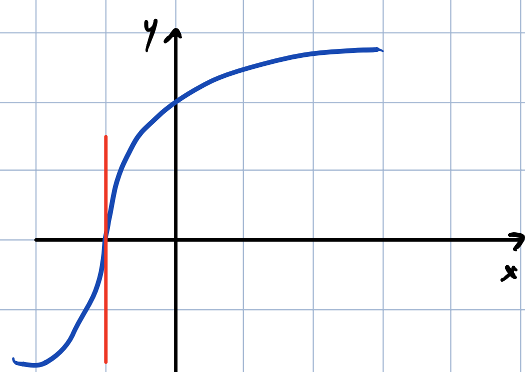
$$= \lim_{h \to 0} \frac{\sqrt[3]{h}}{h}$$

$$= \lim_{h \to 0} h^{-\frac{2}{3}}$$

$$= \lim_{h \to 0} \frac{1}{h^{2/3}} \overset{ch}{=} \frac{1}{0^+} = +\infty$$

(DNE)

Concl $f(x) = \sqrt[3]{x+1}$ is not diff^{ble} at $x = -1$



ex. $f(x) = x^3$ Is f diff^ble at $x=2$?

Method A

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0} \frac{(2+h)^3 - 2^3}{h}$$

$$= \lim_{h\to 0} \frac{2^3 + 3(2^2)h + 3(2)h^2 + h^3 - 8}{h}$$

$$= \lim_{h\to 0} \frac{12h + 6h^2 + h^3}{h}$$

$$= \lim_{h\to 0} 12 + 6h + h^2 \overset{\text{sh}}{=} 12$$

Concl $f(x)$ is diff^ble at $x=2$

Method B

By power rule, since $f(x) = x^3$, $f'(x) = 3x^2$

$f'(2) = 3(2)^2 = 12$

So $f(x)$ is diff^ble at $x=2$

ex. Is $f(x) = \sqrt[3]{x}$ diff^{ble} at $x=0$?

We can use method A or B.

Method B

$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

$f'(x) = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3x^{2/3}}$

$f'(0) = \frac{1}{0}$

Concl: f is not diff^{ble} at $x=0$

If a f^{cn} $y=f(x)$ is not cont^s at $x=a$ then $y=f(x)$ is not diff^{ble} ← TRUE

If a f^{cn} $y=f(x)$ is cont^s at $x=a$ then $y=f(x)$ is diff^{ble} ← FALSE

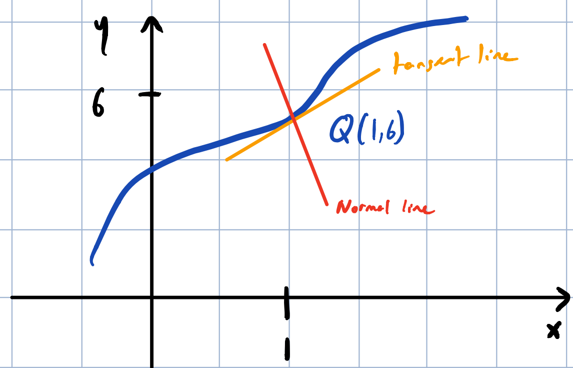
If a f^{cn} is diff^{ble} at $x=a$ then $y=f(x)$ is cont^s at $x=a$ ← TRUE

PROOF?

NORMAL LINES

ex. What is the eqn of the **NORMAL LINE** to the graph of $f(x) = 4x^4 + x + 1$ at $x=1$

normal = perpendicular



when $x=1$, $y = 4(1)^4 + (1) + 1 = 6$

$f'(x) = 16x^3 + 1$

$f'(1) = 17$ ← slope of tan line

so slope of NORMAL LINE is $-\dfrac{1}{17}$

$y = mx + b$

$y = -\dfrac{1}{17}x + b$

subst $x=1$, $y=6$

$6 = -\dfrac{1}{17}(1) + b$

$6 + \dfrac{1}{17} = b$

$b = \dfrac{103}{17}$

17

September 30, 2016

Proof: Assume $y = f(x)$ is diff^ble at $x = 'a'$

i.e. $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists

(Note this implies $f(a)$ is def^d)

② $\lim_{x \to a} f(x) = f(a)$

① $\lim_{x \to a} [f(x) - f(a)] = 0$

① $\lim_{x \to a} [f(x) - f(a)]$

$= \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \cdot (x - a) \right]$

$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} (x - a)$

$= f'(a) \cdot 0$

$= 0$

$\lim_{x \to a} f(x) = \lim_{x \to a} [(f(x) - f(a)) + f(a)]$

constant

$= \lim_{x \to a} [f(x) - f(a)] + \lim_{x \to a} f(a)$

$= 0 + f(a)$

$= f(a)$

i.e. $\lim_{x \to a} f(x) = f(a)$

So f is cont^s at $x = 'a'$

CHAIN RULE

Consider $y = (x^2+1)^3$

Let $g(x) = x^2+1$

& $f(x) = x^3$

$$(f \circ g)(x) = f(g(x)) = [g(x)]^3 = (x^2+1)^3 = y$$

So $y = (x^2+1)^3$ is a composition of two "simpler" f^{ens}

$f(x) = x^3$ & $g(x) = x^2+1$

Chain rule let's us find the deriv. of a composition of two f^{ens}

⑧ Chain Rule

If $y = f(g(x))$ then $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$

Back to $y = (x^2+1)^3 = f(g(x))$

where $g(x) = x^2+1 \rightarrow 2x$

& $f(x) = x^3 \rightarrow 3x^2 \rightarrow f'(g(x)) = 3(g(x))^2$

By chain rule

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$
$$= 3(g(x))^2 \cdot g'(x)$$
$$= 3(x^2+1)^2 \cdot 2x$$

i.e. $\frac{dy}{dx} = 6x(x^2+1)^2$

Summary

Started with $y = (x^2+1)^3$

$$\frac{dy}{dx} = 3(x^2+1)^2 \cdot (x^2+1)'$$

It appears that the "structure" of result looks like:

$$\boxed{y\,(\text{inside})^r \rightarrow \frac{dy}{dx} = r(\text{inside})^{r-1}(\text{inside})'}$$

ex, $y = \sqrt{3x-1}$ $\quad \frac{dy}{dx} = \frac{1}{2}(3x-1)^{-\frac{1}{2}} \cdot (3x-1)'$

ex. $f(t) = \left(\frac{t-1}{t+1}\right)^6$, $f'(t) = ?$

$$f'(t) = 6\left(\frac{t-1}{t+1}\right)^5 \left(\frac{t-1}{t+1}\right)'$$ Chain rule

$$= 6\left(\frac{t-1}{t+1}\right)^5 \cdot \frac{(t-1)'(t+1) - (t-1)(t+1)'}{(t+1)^2}$$

$$= 6\left(\frac{t-1}{t+1}\right)^5 \cdot \frac{(t+1) - (t-1)}{(t+1)^2}$$

$$= 6\left(\frac{t-1}{t+1}\right)^5 \cdot \frac{2}{(t+1)^2}$$

$$= \left(\frac{t-1}{t+1}\right)^5 \cdot \frac{12}{(t+1)^2}$$

WARNING

ex. $y = (x^3-2x+1)^4$, $y' = ?$

$$y' = \cancel{(3x^2-2)}^3\,(3x^2-2)$$

$$y' = 4(x^3-2x+1)^3(x^3-2x+1)'$$

$$= 4(x^3-2x+1)^3(3x-2)$$

ex. $y = -5(3x^2-1)^7$; $\frac{dy}{dx} = ?$

$$\frac{dy}{dx} = -5\left[(3x^2-1)^7\right]'$$ Constant mult. rule

$$= -5\left[7(3x^2-1)^6(3x^2-1)'\right]$$

$$= -5\left[7(3x^2-1)^6(6x)\right]$$

$$= -210x(3x^2-1)^6$$

ex. $y = \frac{1}{\sqrt[3]{(x^2-1)^5}} = \frac{1}{\left[(x^2-1)^5\right]^{\frac{1}{3}}} = (x^2-1)^{-\frac{5}{3}}$

$$y' = -\frac{5}{3}(x^2-1)^{-\frac{5}{3}-1}(x^2-1)'$$

$$= -\frac{5}{3}(x^2-1)^{-\frac{8}{3}}(2x)$$

$$= -\frac{10}{3}x(x^2-1)^{-\frac{8}{3}}$$

<u>Chain Rule using only Leibnitz Notation</u>

$y = f(x) \Rightarrow \frac{dy}{dx} = f'(x)$

$y = f(t) \Rightarrow \frac{dy}{dt} = f'(t)$

$y = f(u) \Rightarrow \frac{dy}{du} = f'(u)$

Chain Rule - Notation

If $y = f(g(x))$ then $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$

Let $u = g(x) \Rightarrow \frac{du}{dx} = g'(x)$

& $f'(g(x)) = f'(u) = \frac{dy}{du}$

So chain rule says:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

ex. $y = (x^5+1)^4$

Let $u = x^5+1 \longrightarrow \frac{du}{dx} = 5x^4$

So $y = u^{14} \longrightarrow \frac{dy}{du} = 14u^{13}$

So $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$14u^{13}(5x^4)$

$= 14(x^5+1)^{13}(5x^4)$

ex. Find $\frac{dy}{dx}$ if $y = \sqrt{7-4x^3}$

using ① $y = (\text{inside})^n$

② $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

① $y = (7-4x^3)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}(7-4x^3)^{-\frac{1}{2}} \cdot (7-4x^3)'$$

$$\frac{dy}{dx} = \frac{1}{2}(7-4x^3)^{-\frac{1}{2}} \cdot (-12x^2)$$

$$\frac{dy}{dx} = -\frac{6x^2}{\sqrt{7-4x^3}}$$

② Let $u = 7-4x^3 \Rightarrow \frac{du}{dx} = -12x^2$

So $y = \sqrt{u} \Rightarrow \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$

So $\frac{dy}{dx} = \frac{du}{dx} \cdot \frac{dy}{du}$

$$= (-12x^2)\left(\frac{1}{2}u^{\frac{1}{2}}\right)$$

$$= (-12x^2)\,\frac{1}{2}(7-4x^3)^{-\frac{1}{2}}$$

$$= \frac{-6x^2}{\sqrt{7-4x^3}}$$

$y = t\sqrt{3t+4}$

$\frac{dy}{dx} = (t)'(3t+4)^{\frac{1}{2}} + t\left[(3t+4)^{\frac{1}{2}}\right]'$

$= (3t+4)^{\frac{1}{2}} + t\left[\frac{1}{2}(3t+4)^{-\frac{1}{2}}(3t+4)'\right]$

$\vdots$

October 6, 2016

$y = Q(x) = f(g(x))$

$Q'(x) = f'(g(x)) \cdot g'(x)$

If we let $u = g(x)$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Recall $y = Q(x) = (x^2+1)^3 = f(g(x))$

where $g(x) = x^2+1 \rightarrow g'(x) = 2x$

$f(x) = x^3 \rightarrow f'(x) = 3x^2$

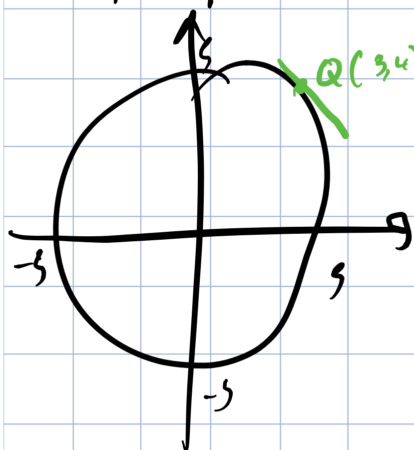
$Q'(x) = f'(g(x)) \cdot g'(x)$

$Q'(3) = f'(g(3)) \cdot g'(3)$

$= f'(10) \cdot g'(3)$

IMPLICIT FCNS

ex/ $x^2+y^2=25$ ← Not a f^{cn}

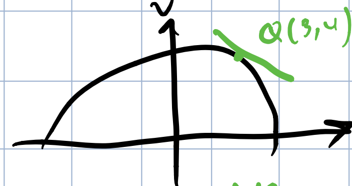


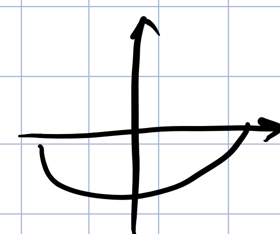
$x^2+y^2=25$

$y^2=25-x^2$

$y=\pm\sqrt{25-x^2}$

$y=+\sqrt{25-x^2}$ or $y=-\sqrt{25-x^2}$





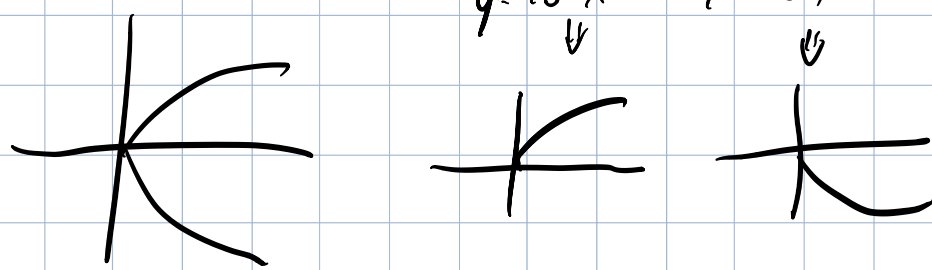
Now it's a f^{cn}

We say that the eqn $x^2+y^2=25$ implicitly defines 2 f^{cns}:

$y=+\sqrt{25-x^2}$ and $y=-\sqrt{25-x^2}$

ex. $x = y^2$ implicitly define two f^{ns}

$y = +\sqrt{x}$ $\quad$ $y = -\sqrt{x}$



ex. $x + y = 5$ implicitly defines $y = 5 - x$

ex. Back to $x^2 + y^2 = 25$

$$y = \sqrt{25 - x^2}$$

-1

$$y = (25 - x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(25 - x^2)^{-\frac{1}{2}}(25 - x^2)'$$

$$= \frac{1}{2} \times \frac{1}{\sqrt{25 - x^2}} (-2x)$$

$$= -\frac{x}{\sqrt{25 - x^2}} = -\frac{x}{y}$$

$$y = -\sqrt{25 - x^2}$$

$$y = -(25 - x^2)^{1/2}$$

$$\frac{dy}{dx} = -\frac{1}{2}(25 - x^2)^{-\frac{1}{2}}(25 - x^2)'$$

$$= -\frac{1}{2} \cdot \frac{1}{\sqrt{25 - x^2}} (-2x)$$

$$= -\frac{x}{-\sqrt{25 - x^2}}$$

$$-\frac{x}{y}$$

We have found 1 formula $\frac{dy}{dx} = -\frac{x}{y}$ which describes the slope of the tang. line at any $Q(x, y)$ on the circle

$$x^2 + y^2 = 25$$

IMPLICIT DIFFERENTIATION

$x^2 + y^2 = 25$

assume that it implicitly defines y as a f^n of x.

i.e. assume $y = f(x)$

$x^2 + (f(x))^2 = 25$

Take deriv. of both sides

$$\frac{d}{dx}\left[x^2 + (f(x))^2\right] = \frac{d}{dx}[25]$$

$$\frac{d}{dx}[x^2] + \frac{d}{dx}\left[(f(x))^2\right] = \frac{d}{dx}[25]$$

$$2x + 2[f(x)][f(x)]' = 0$$

$$2x + 2y \cdot y' = 0$$

$$y' = -\frac{x}{y}$$

Shorter Method...

$$\frac{d}{dx}\left[x^2+(y)^2\right]=\frac{d}{dx}[25]$$

$$\frac{d}{dx}[x^2]+\frac{d}{dx}[y^2]=\frac{d}{dx}[25]$$

chain

$$2x+2y\cdot y'=0$$

$\vdots$

Even Shorter...

$$\left[x^2+y^2\right]'=[25]'$$

$$(x^2)'+(y^2)'=(25)'$$

not 2y

$$2x+2yy'=0$$

ex. $5x^2 + 4y^3 = y$, find $\frac{dy}{dx}$

$$\left(5x^2 + 4y^3\right)' = y'$$

$$\left(5x^2\right)' + \left(4y^3\right)' = y'$$

$$10x + 4\left[3y^2(y')\right] = y'$$

$$10x + 12y^2 y' = y'$$

$$10x = y' - 12y^2 \cdot y'$$

$$10x = y'(1 - 12y^2)$$

$$y' = \frac{10x}{1-12y^2}$$

ex. $x^2 + 1 = y^3$

Find $\frac{dy}{dx}$ by finding the derivative

① Explicitly
② Implicitly

① $y = \left(x^2 + 1\right)^{\frac{1}{3}}$ ← explicit form

$$y' = \frac{1}{3}\left(x^2+1\right)^{-\frac{2}{3}}\left(x^2+1\right)'$$

$$= \frac{2x}{3\left(x^2+1\right)^{2/3}}$$

(2) $(x^2+1)' = (y^3)'$

$$2x = 3y^2 \cdot y'$$

$$y' = \frac{2x}{3y^2}$$

ex. Find the eqn of the tangent line to the graph of $x^3+y^3 = x+1$ at pt $Q(0,1)$

$$x^3+y^3 = x+1$$

$$(x^3+y^3)' = (x+1)'$$

$$3x^2+3y^2 \cdot y' = 1$$ ← Could have subst $x=0, y=1$ in here

$$y' = \frac{1-3x^2}{3y^2}$$

At pt $Q(0,1)$

$$y' = \frac{1-3(0)^2}{3(1)^2}$$

$$m = y' = \frac{1}{3}$$

Tangent line

$y = \frac{1}{3}x + b$

$1 = \frac{1}{3}(0) + b$

$b = 1$

$$y = \frac{1}{3}x + 1$$

ex. $3y^2 + x^2y^3 = 2-y$

$\frac{dy}{dx} = ?$

$$\left(3y^2 + x^2y^3\right)' = (2-y)'$$

$$6y \cdot y' + (x^2)'y^3 + x^2(y^3)' = 2' - y'$$

$$6y \cdot y' + 2xy^3 + x^2(3y^2y') = -y'$$

$$6yy' + 2xy^3 + 3x^2y^2y' = -y'$$

$$6yy' + 3x^2y^2y' + y' = -2xy^3$$

$$y'(6y + 3x^2y^2 + 1) = -2xy^3$$

$$y' = \frac{-2xy^3}{6y + 3x^2y^2 + 1}$$

$$y = \sqrt[3]{x^4 + x}$$

$$y = (x^4 + x)^{\frac{1}{3}}$$

$$y' = \frac{1}{3}(x^4 + x)^{-\frac{2}{3}}(4x^3 + 1)$$

OR

Implicit

$$y^3 = x^4 + x$$

$$(y^3)' = (x^4 + x)'$$

$$3y^2y' = 4x + 1$$

$$y' = \frac{4x+1}{3y^2}$$

$$y' = \frac{4x+1}{3\left(\sqrt[3]{x^4+x}\right)^2}$$