The power series method is the most useful to solve DE^s that have non constant coefficients
We want real coef^s and variables in this section.

Idea: Given, say a linear ODE:

$$P_n(x)y^{(n)} + P_{n-1}y^{(n-1)} + \dots + P_1 y' + P_0 y - r(x) = 0$$

1. Expand coef^s in power series
2. Try to express a soln of the DE expressed as a power series: $y(x) = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \dots$
3. "Plug in" $y(x) = \sum_{m=0}^{\infty} a_m x^m$, $y'(x) = \sum_{m=0}^{\infty} a_m m x^m$ etc. into the DE and solve for the $a_0, a_1, a_2, \dots$
   But because there are infinitely many a_m, we try to find a pattern (i.e. a "recurrence relation to express these)

Note: If necessary, could work with a change of center like $y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m$

E.g. Solve $y' = 2xy$ using power series
Let $y(x) = \sum_{m=0}^{\infty} a_m x^m$, $y'(x) = \sum_{m=0}^{\infty} a_m \cdot m x^{m-1}$

$$\sum_{m=0}^{\infty} a_m m x^{m-1} - 2x \sum_{m=0}^{\infty} a_m x^m = 0 \quad \text{must be true } \forall x$$

$$\sum_{m=0}^{\infty} a_m m x^{m-1} - \sum_{m=0}^{\infty} 2 a_m x^{m+1} = 0$$

Now we must get same power of x in both series, so we shift the index.

$$\sum_{m=0}^{\infty} m a_m x^{m-1} - \sum_{m=2}^{\infty} 2 a_{m-2} x^{m-1} = 0$$

$$0 \cdot a_0 + a_1 x^0 + \sum_{m=2}^{\infty} m a_m x^{m-1} - \sum_{m=2}^{\infty} 2 a_{m-2} x^{m-1} = 0$$

$$0 a_0 + a_1 x^0 + \sum_{m=2} x^{m-1} (m a_m - 2 a_{m-2}) = 0 x^0 + 0 x + 0 x^2 + \dots$$

a_0 is "free" to be anything

Comparing powers of x^0: $a_1 = 0$, $\therefore \boxed{a_1 = 0}$

Because we need $\sum_{m=0}^{\infty} x^{m-1}(m a_m - 2a_{m-2}) = 0 \quad \forall x$

$\Rightarrow m a_m - 2 a_{m-2} = 0$

$\boxed{a_m = -\frac{2}{m} a_{m-2}}$ recurrent relation

a_0 is free:

$a_2 = -\frac{2}{2} a_0 = -a_0$

$a_3 = -\frac{2}{3} \overset{0}{a_1} = 0$

$a_4 = -\frac{2}{4} a_2 = \frac{1}{2} a_0$

$a_5 = -\frac{2}{5} a_3 = 0$

(Follow that odd a_i should all be zero)

$a_6 = -\frac{2}{6} \cdot a_4 = -\frac{1}{3} \cdot \frac{-1}{2} a_0 = \frac{1}{3 \cdot 2 \cdot 1} a_0$

$a_8 = -\frac{2}{48} a_6 = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{a_0}{4!}$

$a_m = \frac{a_0}{(\frac{m}{2})!}$; $a_m = 0$

For even m ; For odd m

$y(x) = a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6$

$y(x) = a_0 + \frac{a_0}{1!} x^2 + \frac{a_0}{2!} x^4 + \frac{a_0}{3!} x^6 + \dots$

The above infinite series can be expressed as:

$$y(x) = a_0 \underbrace{\sum_{m=0} \frac{x^{2m}}{m!}}_{e^{x^2}} \Rightarrow \boxed{y(x) = a_0 e^{x^2}}$$

Just like C, here a_0 is free parameter.

E.g. Solve $9y''+y=0$ using the power series method.

$$y(x)=\sum_{m=0}^{\infty}a_m x^m,\quad y'(x)=\sum_{m=0}^{\infty}a_m\cdot m x^{m-1},\quad y''(x)=\sum_{m=0}^{\infty}a_m m(m-1)x^{m-2}$$

$$9\sum_{m=0}^{\infty}a_m m(m-1)x^{m-2}+\sum_{m=0}^{\infty}a_m x^m=0$$ ← Substitute power series into the DE

$$9\sum_{m=0}^{\infty}a_m\cdot m(m-1)x^{m-2}+\sum_{m=2}^{\infty}a_{m-2}x^{m-2}=0$$ ← Shift index to get same powers of x in both power series

$$0a_0+0a_1x+\sum_{m=2}^{\infty}9a_m m(m-1)x^{m-2}+\sum_{m=2}^{\infty}a_{m-2}x^{m-2}=0$$ ← Get index same on both power series

a_0 and a_1 are "free"; i.e. they can be anything

$$\sum_{m=2}^{\infty}x^{m-2}\underbrace{\left[9a_m m(m-1)+a_{m-2}\right]}_{\text{must be zero } \forall x}=0$$ ← Join into one power series then set the stuff multiplying x^{m-n} to zero and isolate for recurrence relations

$$9a_m m(m-1)+a_{m-2}=0$$

$$\boxed{a_m=\frac{-a_{m-2}}{9m(m-1)}}$$ recurrence relation

a_0 is free:

$$a_2=\frac{-a_0}{9\cdot 2\cdot 1}$$

$$a_4=\frac{-a_2}{9\cdot 4\cdot 3}=\frac{(-1)^2}{9\cdot 9\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$a_6=\frac{-a_4}{9\cdot 6\cdot 5}=\frac{(-1)^3}{9\cdot 9\cdot 9\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$a_m=\frac{(-1)^{m/2}a_0}{9^{m/2}\cdot m!}$$

For even m: $$a_m=\frac{(-1)^{m/2}}{3^m m!}a_0$$

a_1 is free:

$$a_3=-\frac{a_1}{9\cdot 3\cdot 2}$$

$$a_5=-\frac{a_3}{9\cdot 5\cdot 4}=\frac{(-1)^2a_0}{9\cdot 9\cdot 5\cdot 4\cdot 3\cdot 2}$$

$$a_7=-\frac{a_5}{9\cdot 7\cdot 6}=\frac{(-1)^3a_0}{9\cdot 9\cdot 9\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}$$

$$a_m=\frac{(-1)^{\frac{m-1}{2}}a_0}{9^{\frac{m-1}{2}}\cdot m!}$$

For odd m: $$a_m=\frac{(-1)^{\frac{m-1}{2}}a_0}{3^{m-1}m!}$$

Thus,

$$y(x)=a_0+a_1x+a_2x^2+a_3x^3+\dots$$

$$y(x)=(a_0+a_2x^2+a_4x^4+\dots)+(a_1x+a_3x^3+a_5x^5+\dots)$$

$$y(x)=a_0\left(1-\frac{1}{3^2\,2!}x^2+\frac{1}{3^4\,4!}x^4+\dots\right)+3a_1\left(\frac{x}{3}-\frac{1}{3^3\,3!}x^3+\frac{1}{3^5\,5!}x^5+\dots\right)$$

$$y(x)=a_0\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{3^{2n}(2n)!}+3a_1\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{3^{2n+1}(2n+1)!}$$

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n (x/3)^{2n}}{(2n)!} + 3a_1 \sum_{n=0}^{\infty} \frac{(-1)^n (x/3)^{2n+1}}{(2n+1)!}$$

$$y(x) = \underbrace{a_0 \cos\left(\tfrac{x}{3}\right) + \tilde{a}_1}_{\text{these are basically like the constants } A \text{ and } B.} \sin\left(\tfrac{x}{3}\right)$$

The differential equation:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

is called LeGendre's Differential equation

* Used in potential theory... x is usually r in spherical coordinates and n describes energy levels.

Any soln to LeGendre's DE is called a LeGendre function.

One soln we will obtain will be a polynomial $P_n(x)$, suitably normalized so that $P_n(x)$ satisfies $P_n(1) = 1$, is called a <u>LeGendre Polynomial</u>.

E.g. Consider LeGendres equation with $n=2$.

$$(1-x^2)y'' - 2xy' + 6y = 0$$

Solve using the power series method and find the LeGendre polynomial $P_2(x)$.

Put: $y(x) = \sum_{m=0}^{\infty} a_m x^m$, $y'(x) = \sum_{m=0}^{\infty} m a_m x^{m-1}$, $y''(x) = \sum_{m=0}^{\infty} m(m-1) a_m x^{m-2}$

$$\Rightarrow (1-x^2)\sum_{m=0}^{\infty} a_m m(m-1) x^{m-2} - 2x \sum_{m=0}^{\infty} m a x^{m-1} + 6 \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=0}^{\infty} m(m-1) a_m x^{m-2} - \sum_{m=0}^{\infty} m(m-1) a_m x^m - 2\sum_{m=0}^{\infty} a_m m x^m + 6\sum_{m=0}^{\infty} a_m x^m = 0$$

$$0 + 0 + \sum_{m=2}^{\infty} m(m-1)\, a_m\, x^{m-2} - \sum_{m=0}^{\infty} \left[m(m-1)\, a_m - 2 a_m m + 6 a_m \right] x^m = 0$$

want m

must shift

$$0 + 0 + \sum_{m=0}^{\infty} (m+2)(m+1)\, a_{m+2}\, x^m - \sum_{m=0}^{\infty} a_m x^m \left[m^2 - m - 2m + 6 \right] = 0$$

$$\sum_{m=0}^{\infty} x^m \overbrace{\left[(m+2)(m+1)\, a_{m+2} - a_m (m^2 + m - 6) \right]}^{\text{must be zero so that power series is zero } \forall x.} = 0$$

$$a_{m+2} (m+2)(m+1) - a_m (m+3)(m-2) = 0$$

$$\boxed{a_{m+2} = \frac{(m+3)(m-2)}{(m+2)(m+1)}\, a_m} \quad \text{Recurrence relation}$$

a_0 free.

$a_2 = \frac{3 \cdot -2}{2 \cdot 1} a_0$

$a_4 = \frac{5 \cdot 0}{4 \cdot 3} a_2 = 0$

$a_6 = \frac{7 \cdot 2}{6 \cdot 5}$

etc.

a_1 free

$a_3 = \frac{4 \cdot -1}{3 \cdot 2} a_1$

$a_5 = \frac{6 \cdot 1}{5 \cdot 4} a_3 = \frac{6 \cdot 1 \cdot 4 \cdot -1}{5 \cdot 4} a_1$

$a_7 = \frac{8 \cdot 3}{7 \cdot 6} a_5$

$a_9 = \frac{10 \cdot 5}{9 \cdot 8} a_7$

$$\therefore a_{13} = - \frac{(14 \cdot 12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2)\,(9 \cdot 7 \cdot 5 \cdot 3 \cdot 1)}{2 \cdot 13!}$$

Aside: Double Factorials:

$$n!! = \begin{cases} n(n-2)(n-4) \cdot \ldots \cdot 4 \cdot 2 & n \text{ even} \\ n(n-2)(n-4) \cdot \ldots \cdot 3 \cdot 1 & n \text{ odd} \end{cases}$$

$7!! = 7 \cdot 5 \cdot 3 \cdot 1$

$10!! = 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2$

Also, $0!! = 1$

$\square$

Using this notation, $a_m = - \frac{(m+1)!!\,(m-4)!!}{2 \cdot m!}$

For odd m

Soln of LeGendre's DE is

$$y(x) = a_0(\underbrace{1-3x}_{\text{LeGendre Polynomial?}}+0+0+\dots) + (\underbrace{a_1 x + a_3 x^3 + a_5 x^5 + \dots}_{\text{given by formula found}})$$

$$P_2(x) = \underset{\substack{\uparrow \\ \text{scaling}}}{c}(1-3x^2)$$

$$P_2(1) = c(1-3) = -2c$$

$$-2c = 1 \Rightarrow c = -\frac{1}{2}$$

$\therefore$ $P_2(x) = -\frac{1}{2}(1-3x^2)$ $\square$

End of LC2!