Section 7 - Seperable differential equations

def A differential equation is an equation that relates a dependant variable and its derivatives to an independant variable.

ex. $y'(x) + 3y(x) = e^x$

A differential equation is <u>seperable</u> if the independent and dependant variables can be seperated on either side of the equality.

ex. $y' = -\frac{x}{y}$ with $y(3) = 4$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Find the curve that respects $y' = -\frac{x}{y}$ and passes through $(3,4)$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$y\,dy = -x\,dx$$

$$\int y\,dy = \int -x\,dx$$

$$\frac{y^3}{2} = -\frac{x^3}{2} + C$$

$y^2 = -x^2 + K$ K can be identified by ensuring $x=3, y=4$ fits this eqn

$16 = -9 + k$

$k = 25$

$y^2 = -x^2 + 25$

$x^2 + y^2 = 25$

$\downarrow$ differentiate

Check: $2x + 2y\,y' = 25$

$y' = -\frac{x}{y}$

ex À vous

Solve $x \frac{dy}{dx} + 2\ln x = 0$ such that $y(1) = 4$

$\frac{dy}{dx} = \frac{-2\ln x}{x}$

$\int dy = -2\int \frac{\ln x}{x}\,dx$

$u = \ln x$

$y = \frac{-\cancel{2}(\ln x)^2}{\cancel{2}}\,dx$

$y = -(\ln x)^2 + C$

$4 = 0 + C$

$C = 4$

$y = -(\ln x)^2 + 4$

ex. Newton's Law of Cooling

$$\frac{dB}{dt} = k(B - M)$$

B = body temp
M = temp medium
k = constant (individual)
t = hours

Suppose the chef's body is found in the freezer at 7:00 AM. At 7:30 forensics officers are on the screen and take a 1st temperature reading (0°C). At 8:00 AM, a 2nd reading is taken (-6°C). Freezer Temp = -15°C (constant)

diff eq: $\frac{dB}{dt} = k(B+15)$

$$dB = k(B+15)\,dt$$

$$\frac{1}{B+15}\,dB = k\,dt$$

$$\int \frac{1}{B+15}\,dB = \int k\,dt$$

$\ln|B+15| = kt + C$ $\qquad$ $B \geq -15°C$ for all t

$\ln(B+15) = kt + C$

$\ln 15 = 0 + C$ $\qquad$ at 7:30, $B = 0$ $(t=0)$

$\Rightarrow C = \ln 15$

$\ln 9 = k(0.5) + \ln 15$ $\qquad$ at 8:00, $B = -6°C$

$k = 2\ln\frac{9}{15}$ $\qquad$ $t = 0.5$

$k = 2 \ln \frac{3}{5}$

$k = \ln \frac{9}{25}$

$\ln(B+15) = \ln\left(\frac{9}{25}\right) t + \ln(15)$

Find t when $B = 35$

$\ln 50 = \ln\left(\frac{9}{25}\right) t + \ln 15$

$$\frac{\ln \frac{50}{15}}{\ln\left(\frac{9}{25}\right)} = t$$

$t \approx -1.2$ hours or -72 minutes (roughly 6:18 AM)

Anyone say 'seperable'?

Problem 53

An electric circuit is set up with an 8 V battery, a 4 ohm resistor and a 2 Henry solenoid.

The differential equation that describes how the current flowing through the circuit will vary in time is:

$$2\frac{dI}{\mathrm{dt}} = 4I + 8$$

A. The process of separating the variables has been started below. Find the complete separated form of the differential equation.

$$2\frac{dI}{\mathrm{dt}} + 4I = 8$$
$$2\frac{dI}{\mathrm{dt}} = -4I + 8$$
$$\frac{dI}{\mathrm{dt}} = -2I + 4$$
$$\frac{dI}{\mathrm{dt}} = -2\left(I - 2\right)$$
$$\frac{1}{I-2}dI = -2\,\mathrm{dt}$$
$$\int \frac{1}{I-2}dI = -2\int \mathrm{dt}$$
$$\ln\left|I - 2\right| = -2t + c$$

A. Assuming initial conditions are I = 0 when t = 0, find the solution to the differential equation and express I as a function of t.

$$\ln\left|-2\right| = -2\left(0\right) + c$$
$$C = \ln 2$$
$$\ln\left|I - 2\right| = -2t + \ln 2$$
$$\left|I - 2\right| = e^{-2t+1n2}$$
$$I\left(t\right) = -2e^{-2t} + 2$$

A. Find the values of current 1 when t=2, t-5 and as t → oo and stretch a rough graph of I (t) based on your results.

$$t = 2$$
$$I = -2e^{-2(2)} + 2$$
$$-2 \times e^{-2 \times (2)} + 2 \simeq 1.963$$

$\overline{t=5}$

$$-2 \times e^{-2 \times (5)} + 2 \simeq 2.000$$

$\overline{t \to \infty}$

$$\lim_{t \to \infty} \left[-2e^{-2t} + 2\right] = 2$$

Curiosity: In the absense of a solenoid, the current in the circuit would have been I = V / R = 2 a s soon as the battery was hooked up, Discuss **briefly** the role the solenoid plays in this circuit.

Problem 54

The presence of a virus within the Mariano polis community has just been detected. To avoid the propagation of the disease outside the college, Mariano polis has been quarantined. Epidemiologists estimate that unless a vaccine is found soon, the infected proportion (denoted by p) will grow at a rate given by:

$$\frac{dp}{\mathrm{dt}} = 0.2p\left(1 - p\right)$$

where t is counted in hours.

A. Solve the differential equation and find the particular solution, p (t), that satisfies the initial condition

$$p\left(0\right) = 0.1$$

A. Evaluate the infected proportion after 6 hours and as t → oo

$\frac{dp}{\mathrm{dt}} = \frac{1}{5}p\left(1-p\right)$

$\frac{1}{p(1-p)}dp = \frac{1}{5}\mathrm{dt}$

$\int \frac{1}{p(1-p)}dp = \frac{1}{5}\int$

$\int \frac{1}{p}dp + \int \frac{1}{1-p}dp = \frac{1}{5}t$

$\ln\left|p\right| - \ln\left|1-p\right| = \frac{1}{5}t$

$\ln\frac{P}{1-p} = \frac{1}{5}t + c$

$0 \le p \le|$

$t = 0, p = 0.1$

$\ln\left|\frac{0.1}{0.9}\right| = C$

$\ln\left|\frac{0.1}{0.9}\right| \simeq -2.197$

$\boxed{C = \ln\frac{1}{9}}$

$\ln\frac{p}{1-p} - \ln\frac{1}{9} = \frac{1}{5}t$

$\ln\left|\frac{9p}{1-p}\right| = \frac{1}{5}t$

$\frac{9p}{1-p} = e^{\frac{1}{5}t}$

$9p = e^{\frac{1}{5}t}\left(1-p\right)$

$9p = e^{\frac{1}{5}t} - pe^{\frac{1}{5}t}$

$p\left(9 + e^{\frac{1}{5}}t\right) = e^{\frac{1}{5}t}$

$p\left(t\right) = \frac{e^{\frac{1}{5}t}}{9+e^{\frac{1}{5}t}}$

After 6 hours:

$p\left(6\right) =$

$\frac{e^{\frac{6}{5}}}{9+e^{\frac{6}{5}}} \simeq 0.269$

as t → oo

$\frac{1}{p(1-p)} = \frac{A}{p} + \frac{B}{1-p}$

$1 = A\left(1-p\right) + BP$

$p = 0 : A = 1$

$p = 1 : B = 1$

$$\lim_{t\to\infty} \frac{e^{\frac{1}{5}t}}{9+e^{\frac{1}{5}t}}$$

$$= \lim_{t\to\infty} \frac{e^{\frac{1}{5}t}}{e^{\frac{1}{5}t}\left(\frac{9}{e^{\frac{1}{5}t}}+1\right)} = 1$$