1 - Vector Algebra and Geometry

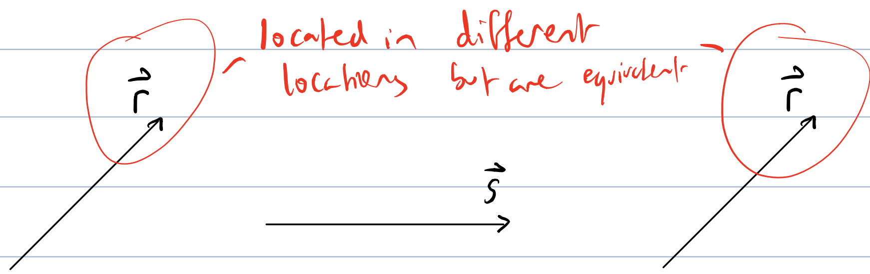
Def: A vector is a directed segment (arrow) that is characterized by its length (magnitude, norm) and its direction

Note: Vectors are equal if and only if their length and direction are identical.

Remark: Vectors are "free": they are not set in specific locations of a graph.

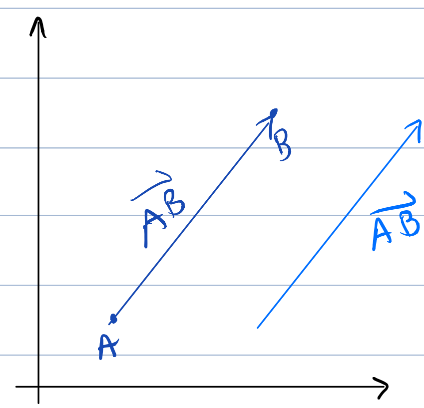
Notation: lower case letter (usually) with the "$\rightarrow$" symbol

ex.



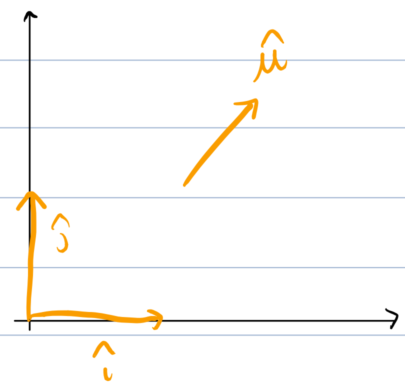
Exceptions:

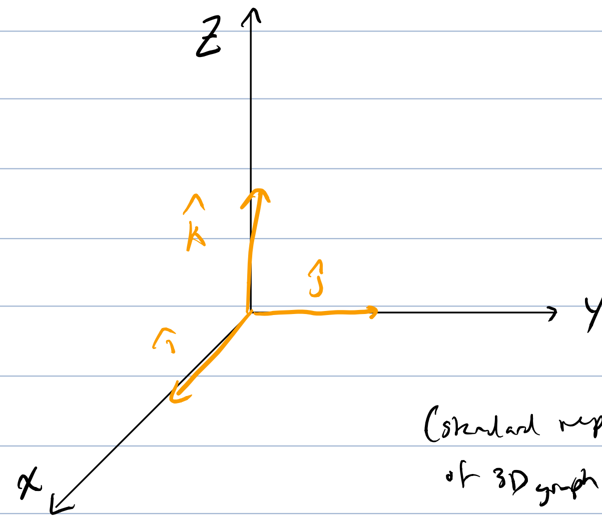
① $\overrightarrow{AB}$ to label a vector whose endpoints A and B are used to define it (pts A & B are fixed, but $\overrightarrow{AB}$ can be moved)



② We use a "^" symbol to clarify that a vector is a unit vector

in $\mathbb{R}^2$ in $\mathbb{R}^3$





(standard representation of 3D graph in this order)

For practical purposes, vectors are usually represented in component form.

ex. In $\mathbb{R}^2$:

$\vec{u} = \langle u_1, u_2 \rangle$ represents a vector whose displacement is u_1 in the x-direction and u_2 in the y-direction.

(i.e. $\vec{u} = u_1\hat{i} + u_2\hat{j}$)

In $\mathbb{R}^3$:

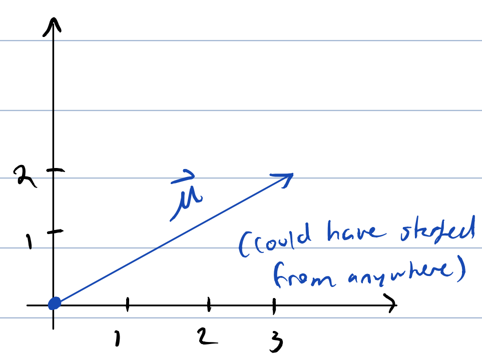
$\vec{u} = \langle u_1, u_2, u_3 \rangle$ represents a vector whose displace is u_1 in the x-direction, u_2 in the y-direction and u_3 in the z-direction. (i.e. $\vec{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$)

Vector component forms can be extended to $\mathbb{R}^n$

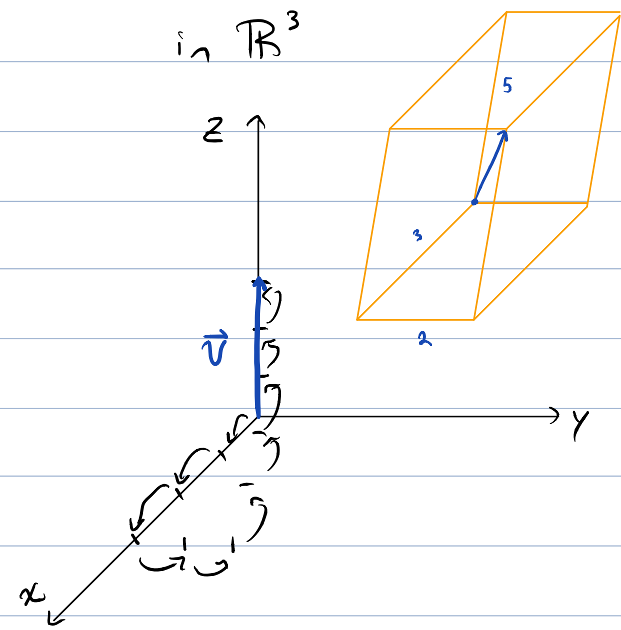
$\vec{u} = \langle u_1, u_2, u_3, \ldots, u_n \rangle$

ex. Draw vectors $\vec{u} = \langle 3, 2 \rangle$ and $\vec{v} = \langle 3, 2, 5 \rangle$

in $\mathbb{R}^2$



in $\mathbb{R}^3$



Operations on Vectors

① Vector Addition

Geometry

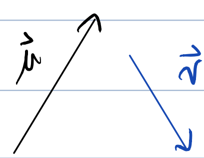
$\vec{u} + \vec{v}$ is obtained by placing $\vec{v}$'s tail where $\vec{u}$'s tip occurs

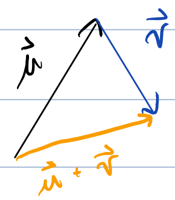
"tip-to-tail"

Algebra

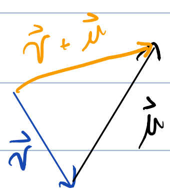
$$\vec{u} + \vec{v} = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle$$
$$= \langle u_1 + v_1, u_2 + v_2 \rangle$$

ex. Find $\vec{u} + \vec{v}$



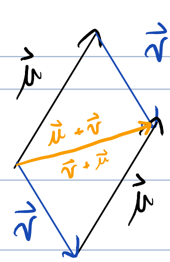


À vous Find $\vec{v} + \vec{u}$



Note: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

OR

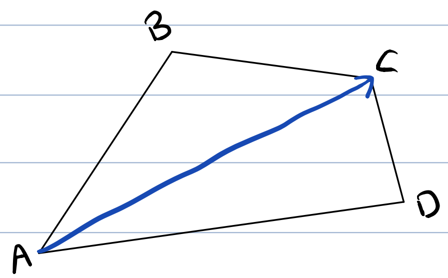


ex If $u = \langle 1, 3, 2, 7 \rangle$ and $\vec{v} = \langle 2, 5, 0, -1 \rangle$

then

$$\vec{u} + \vec{v} = \langle 1, 3, 2, 7 \rangle + \langle 2, 5, 0, -1 \rangle$$
$$= \langle 3, 8, 2, 6 \rangle$$

À vous Express $\vec{AC}$ as 2 different combinations of vectors.



$$\vec{AC} = \vec{AB} + \vec{BC}$$

or

$$\vec{AD} = \vec{AD} + \vec{DC}$$

② Scalar Multiplication

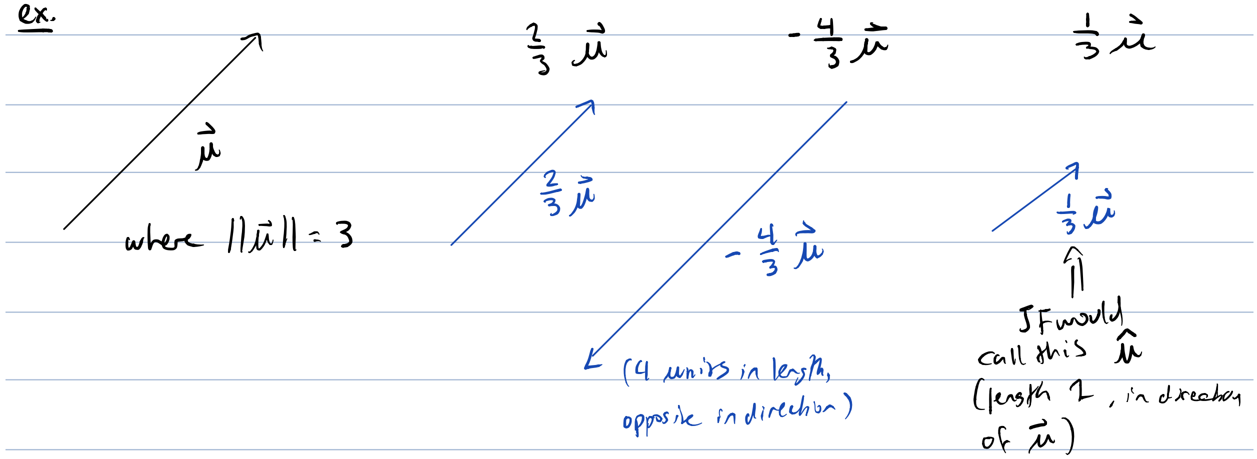
Geometry

$K\hat{u}$ is $|K|$ times the length of $\vec{u}$ and in the same (or opposite) direction as $\vec{u}$ if $K > 0$ (or $K < 0$)

Algebra

$$K \langle u_1, u_2, \ldots, u_n \rangle = \langle K u_1, K u_2, \ldots, K u_n \rangle$$

ex.



Def: Vectors $\vec{u}$ and $\vec{v}$ are parallel as long as they are scaler multiples of one another.

i.e. $\vec{u} = k\vec{v}$ or $\vec{v} = m\vec{u}$

ex. Let $\vec{a} = \langle 4, 7, k \rangle$ and $\vec{v} = \langle 3, c, 1 \rangle$.

Find values for k and c, so that $\vec{u}$ and $\vec{v}$ are parallel

$\vec{a} \parallel \vec{v}$ if $\vec{a} = m\vec{v}$

$\langle 4, 7, k \rangle = m \langle 3, c, 1 \rangle$

$\langle 4, 7, k \rangle = \langle 3m, mc, m \rangle$

This implies that:

$4 = 3m$ $\qquad$ $7 = mc$ $\qquad$ $k = m$

$m = \frac{4}{3}$ $\qquad$ $c = \frac{7}{m} = \frac{7}{4/3} = \frac{21}{4}$ $\qquad$ $k = \frac{4}{3}$

③ Vector Subtraction

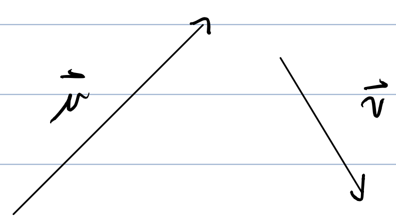
Geometry

$\vec{u} - \vec{v}$ is obtained from $\vec{u} + (-\vec{v})$

Algebra

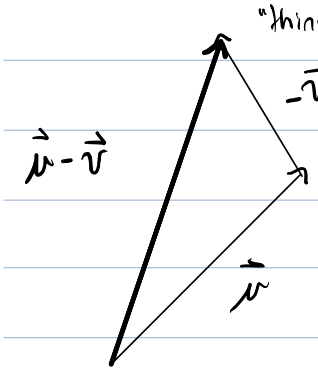
$$\vec{u} - \vec{v} = \langle u_1, u_2 \rangle - \langle v_1, v_2 \rangle$$
$$= \langle u_1 - v_1, u_2 - v_2 \rangle$$

ex.



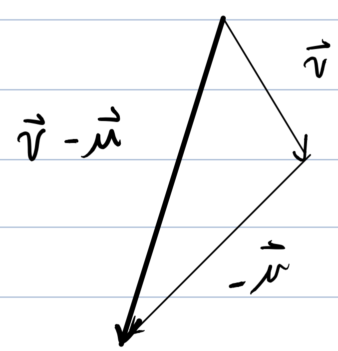
a) Find $\vec{u} - \vec{v}$

"think $\vec{u} + (-\vec{v})$"



b) Find $\vec{v} - \vec{u}$

"think $\vec{v} + (-\vec{u})$"



So $\vec{v} - \vec{u} = -(\vec{u} - \vec{v})$

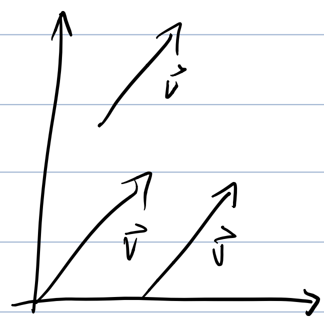
$= \vec{v} - \vec{u}$

Aug. 23

Review From Last Class

In general: $P(x_1, y_1)$, $Q(x_2, y_2)$

The vector $\overrightarrow{PQ}$ is given by $\langle x_2 - x_1, y_2 - y_1 \rangle$



All of these are the same vector $\vec{v}$, i.e. a vector is determined only by its direction & length.

Remember: Two vectors $\vec{a}$ and $\vec{b}$ if they are scalar multiples of each other, i.e. there exists a scalar $k \in \mathbb{R}$ (real numbers), such that $\vec{a} = k\vec{b}$ $(k \neq 0)$

ex. $\vec{a} = \langle 6, \frac{3}{2}, -2 \rangle$ and $\vec{b} = \langle -12, -3, 4 \rangle$

so $\vec{b} = -2\vec{a}$ $\left(\vec{a} = -\frac{1}{2}\vec{b}\right)$

Linear Combination of Vectors

A linear combination of vectors is an expression of the type $k_1\vec{v}_1 + k_2\vec{v}_2 + \ldots + k_r\vec{v}_r$

where $k_i \in \mathbb{R}$ for all i and $\vec{v}_i$ is a vector.

(k can $= 0$, so don't have to include all vectors)

ex. $3\vec{a} + 2\vec{b} - 7\vec{c} + \vec{d}$ is a linear combination of vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ with coefficients $3, 2, -7, 1$

ex. Let $\vec{v} = \langle 1, 3, 6 \rangle$, $\vec{a} = \langle 1, 0, 0 \rangle$, $b = \langle 0, 3, 3 \rangle$ and $\vec{c} = \langle 0, 0, -1 \rangle$, then write $\vec{v}$ as a linear combination of $\vec{a}, \vec{b}, \vec{c}$

$\vec{v} = 1\vec{a} + 1\vec{b} - 3\vec{c}$

$= \langle 1, 3, 6 \rangle$

Properties of vectors (addition, subtraction, multiplication by scaler)

(1) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ commutative property

(2) $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ associative property

(3) There exists a vector $\vec{0} = \langle 0, 0, \ldots, 0 \rangle$ called a zero vector such that $\vec{v} + \vec{0} = \vec{v}$

(identity element)

(4) $\vec{a} + (-\vec{a}) = \vec{0}$

inverse of $\vec{a}$

⑤ $0\vec{b} = \vec{0}$ (Note not 0, rather a vector of 0s, hence $\vec{0}$)

⑥ Let $k, \ell \in \mathbb{R}$, then $k(\ell\vec{v}) = (k\ell)\vec{v}$

ex $2(3\vec{v}) = (2\cdot3)\vec{v} = 6\vec{v}$

⑦ $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$ distributive property

⑧ $(k+\ell)\vec{a} = k\vec{a} + \ell\vec{a}$ (Note: don't write $(k+\ell)\cdot\vec{a}$ since that is the dot product)

Proof of ⑥

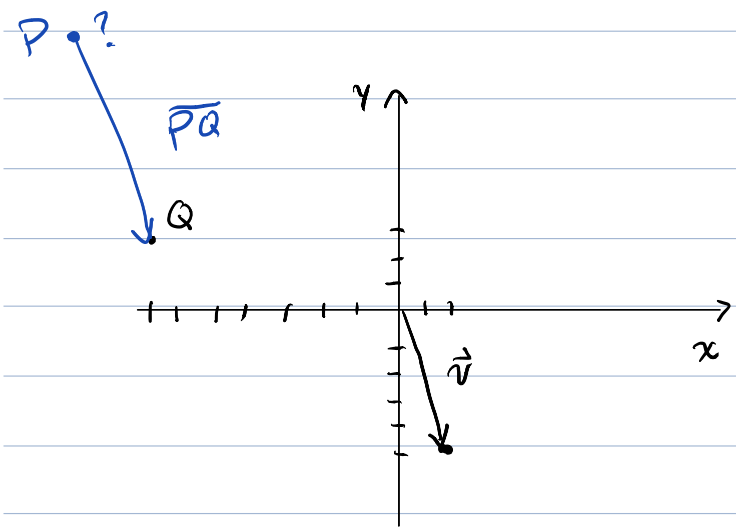
Let $k, \ell \in \mathbb{R}$ and let $\vec{v}$ be a vector in $\mathbb{R}^n$

we want to show that $k(\ell\vec{v}) = (k\ell)\vec{v}$.

$$
\begin{aligned}
k(\ell\vec{v}) &= k(\ell \langle v_1, v_2, \ldots, v_n \rangle) \\
&= k \langle \ell v_1, \ell v_2, \ldots, \ell v_n \rangle \\
&= \langle k(\ell v_1), k(\ell v_2), \ldots, k(\ell v_n) \rangle \\
&= \langle (k\ell)v_1, (k\ell)v_2, \ldots, (k\ell)v_n \rangle \quad \text{assosiative property of real numbers} \\
&= (k\ell) \langle v_1, v_2, \ldots, v_n \rangle \\
&= (k\ell)\vec{v}
\end{aligned}
$$

Exercises

ex. Find $P(x_1, y_1)$ such that the vector $\overrightarrow{PQ}$, where Q is $(-7, 3)$, is parallel (and same direction) to the vector $\vec{v} = \langle 2, -5 \rangle$ and is twice as long.



We don't need to solve this graphically.

$$\overrightarrow{PQ} = 2\vec{v}$$

end-point - initial pt

$$\langle -7-x, 3-y \rangle = 2\langle 2, -5 \rangle$$

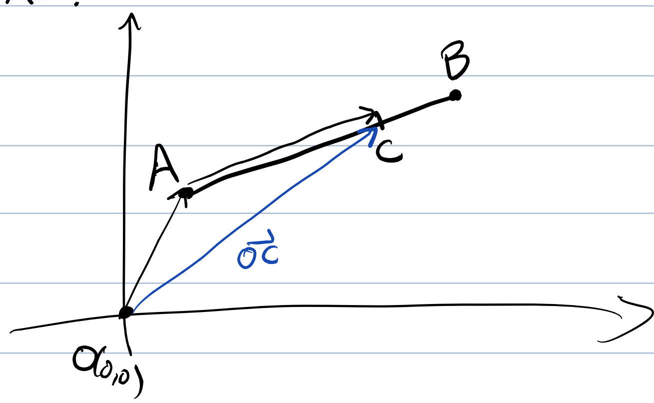
$$\langle -7-x, 3-y \rangle = \langle 4, -10 \rangle$$

$$-7-x = 4 \qquad 3-y = -10$$

$$x = -11 \qquad y = 13$$

$$\therefore P(-11, 13)$$

ex. Given the points $A(2,1)$ and $B(6,-5)$, find the point C that is $3/4$ of the way along the segment $\overline{AB}$.



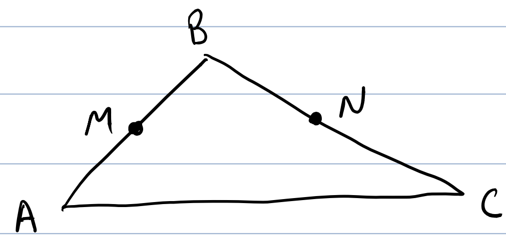
To find C, I can look for the vector $\vec{OC}$

$$\begin{aligned}
\text{So} \quad \vec{OC} &= \vec{OA} + \vec{AC} \\
&= \vec{OA} + \tfrac{3}{4}\vec{AB} \\
&= \langle 2, 1 \rangle + \tfrac{3}{4}\langle 6-2, -5-1 \rangle \\
&= \langle 2, 1 \rangle + \tfrac{3}{4}\langle 4, -6 \rangle \\
&= \langle 2, 1 \rangle + \langle 3, -\tfrac{9}{2} \rangle \\
&= \langle 5, -\tfrac{7}{2} \rangle
\end{aligned}$$

$\therefore C(5, -\tfrac{7}{2})$

*assignment due Wednesday — Aug. 25

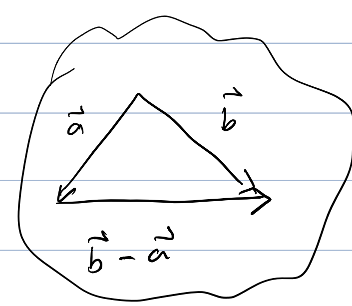
ex. Let M & N be the midpoints of two of the three sides of a triangle.



Show that the segment MN is parallel to the third side and half the length.

$$\overrightarrow{MN} = \frac{1}{2}\overrightarrow{AC}$$

$$\begin{aligned}\overrightarrow{MN} &= \overrightarrow{MB} + \overrightarrow{BN}\\ &= \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} \quad \text{b/c } M, N \text{ are midpoints}\\ &= \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{BC}\right)\\ &= \frac{1}{2}\overrightarrow{AC}\end{aligned}$$

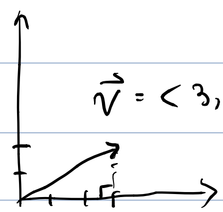


or
$$\begin{aligned}\overrightarrow{MN} &= \overrightarrow{BN} - \overrightarrow{BM}\\ &= \frac{1}{2}\overrightarrow{BC} - \frac{1}{2}\overrightarrow{BA}\\ &= \frac{1}{2}\left(\overrightarrow{BC} - \overrightarrow{BA}\right)\\ &= \frac{1}{2}\overrightarrow{AC}\end{aligned}$$

Norm of a Vector

The norm of a vector $\vec{v}$ is the length or magnitude and is denoted by $\|\vec{v}\|$

In $\mathbb{R}^2$



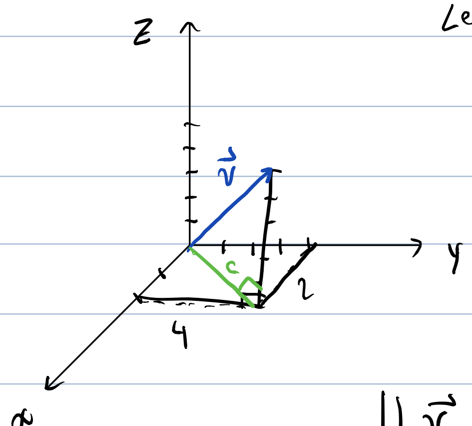
$$\|\vec{v}\| = \sqrt{3^2 + 2^2} = \sqrt{13}$$

In general: Given a vector $\vec{v} = \langle x_1, y_1 \rangle$,

then $\|\vec{v}\| = \sqrt{x_1^2 + y_1^2}$

In $\mathbb{R}^3$

Let $\vec{v} = \langle 2, 4, 5 \rangle$



$$\|\vec{v}\| = \sqrt{\underbrace{2^2 + 4^2}_{c^2} + 5^2}$$

In general, given the vector $\vec{v} = \langle x_1, y_1, z_1 \rangle$

$$\|\vec{v}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

In $\mathbb{R}^n$, let $\vec{v} = \langle V_1, V_2, \ldots, V_n \rangle$

then $\|\vec{v}\| = \sqrt{V_1^2 + V_2^2 + \ldots + V_n^2}$

Properties:

1) $\|\vec{v}\| \geq 0$

2) $\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0}$

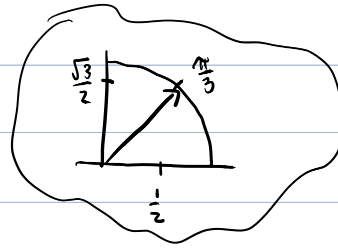
3) Let $k \in \mathbb{R}$
$\|k\vec{v}\| = |k| \, \|\vec{v}\|$

Unit Vectors

A vector $\vec{v}$ is called a unit vector if $\|\vec{v}\| = 1$

ex. $\vec{a} = \langle 1, 0 \rangle$ is a unit vector in $\mathbb{R}^2$

ex. $\vec{b} = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$



ex. $\vec{c} = \langle \frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} \rangle$

Standard Unit Vectors

In $\mathbb{R}^2$, the vectors $\hat{i} = \langle 1, 0 \rangle$ and $\hat{j} = \langle 0, 1 \rangle$ are the standard unit vectors.

In $\mathbb{R}^3$, the vectors $\hat{i} = \langle 1, 0, 0 \rangle$, $\hat{j} = \langle 0, 1, 0 \rangle$, $\hat{k} = \langle 0, 0, 1 \rangle$

Note: Any vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ can be written as a linear combination of the standard unit vectors.

ex. Let $\vec{a} = \langle 7, 2, -\frac{3}{2} \rangle$, then

$$\vec{a} = 7\hat{i} + 2\hat{j} - \frac{3}{2}\hat{k}$$

ex. Given vector $\vec{v} = \langle 2, 3, 1 \rangle$, find a unit vector in the same direction as $\vec{v}$.

Let $\vec{u}$ be the desired vector, then

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$$

$$= \frac{1}{\sqrt{2^2+3^2+1^2}}\vec{v} = \frac{1}{\sqrt{14}}\langle 2, 3, 1\rangle$$

$$\text{or} = \left\langle \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right\rangle$$

ex. Given $\vec{a} = \langle 3, -2\rangle$, find a vector $\vec{b}$ with norm $= 3$ in the opposite direction of $\vec{a}$.

$$\vec{b} = -3 \cdot \frac{1}{\|\vec{a}\|}\vec{a}$$

$$= \frac{-3}{\sqrt{3^2+(-2)^2}}\langle 3, -2\rangle$$

$$= -\frac{3}{\sqrt{13}}\langle 3, -2\rangle$$

201-NYC **ICP 1**

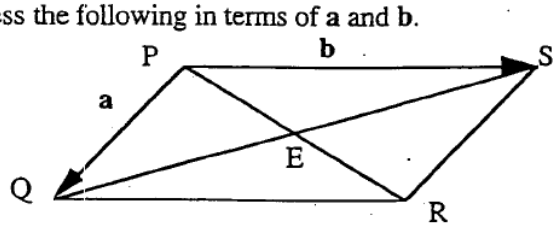
1.

Given the vectors **a** and **b**, sketch the following vectors:

a) $\mathbf{a} + \mathbf{b}$ b) $3(\mathbf{a} + \mathbf{b})$ c) $2\mathbf{a} + 3\mathbf{b}$ d) $\mathbf{a} - 2\mathbf{b}$

2.

If PQRS is the parallelogram below and $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{PS}$, express the following in terms of **a** and **b**.



a)	$\overrightarrow{QP}$	b)	$\overrightarrow{QR}$	c)	$\overrightarrow{SR}$	d)	$\overrightarrow{PR}$
e)	$\overrightarrow{SQ}$	f)	$\overrightarrow{PE}$	g)	$\overrightarrow{RE}$	h)	$\overrightarrow{SE}$
i)	$\overrightarrow{PS} + \overrightarrow{QR}$	j)	$\overrightarrow{PS} + \overrightarrow{RQ}$	k)	$\overrightarrow{PR} + \overrightarrow{SP}$		
l)	$\overrightarrow{PE} + \overrightarrow{EQ}$	m)	$\overrightarrow{PE} - \overrightarrow{EQ}$	n)	$\overrightarrow{PE} + \overrightarrow{RS}$		

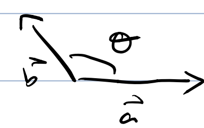
Dot Product (Scalar Product)

Aug. 29

There are two ways of defining the dot product. They yield the same result.

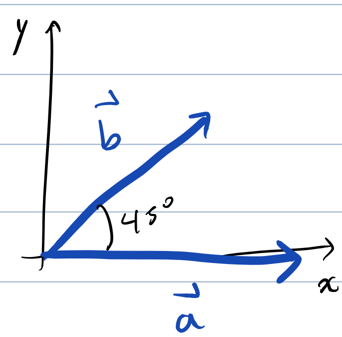
a) <u>Geometric definition</u>: Let $\vec{a}$ and $\vec{b}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. We define the dot product of $\vec{a}$ and $\vec{b}$ as ($\vec{a}, \vec{b} \neq \vec{0}$)

$$\vec{a} \cdot \vec{b} = \|a\| \, \|\vec{b}\| \cos\theta$$



How do you prove this?

<u>ex.</u> Let $\vec{a} = \langle 3, 0 \rangle$, $\vec{b} = \langle 2, 2 \rangle$



$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos\theta$$

$$= \sqrt{3^2 + 0^2} \sqrt{2^2 + 2^2} \cos 45°$$

$$= 3 \cdot \cancel{2}\sqrt{2} \cdot \frac{\sqrt{2}}{\cancel{2}}$$

$$= 6$$

b) <u>Algebraic definition</u>: Let $\vec{a}, \vec{b}$ be vectors in $\mathbb{R}^n$

$$\vec{a} = \langle a_1, a_2, \ldots, a_n \rangle, \quad \vec{b} = \langle b_1, b_2, \ldots, b_n \rangle$$

We define: $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$

<u>ex.</u> Let $\vec{a} = \langle 3, 0 \rangle$ and $\vec{b} = \langle 2, 2 \rangle$

$\vec{a} \cdot \vec{b} = 3 \cdot 2 + 0 \cdot 2 = 6$

ex. Let $\vec{a} = \langle -1, 0, 3 \rangle$, $\vec{b} = \langle -2, -3, 5 \rangle$

Find a) $\vec{a} \cdot \vec{b} = 2 + 15$
$= 17$

b) $\vec{b} \cdot \vec{a} = 17$

Note: Having both definitions allows us to find a formula for θ, the angle between two vectors.

$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$

$\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}$

$$\boxed{\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}\right)}$$

using algebraic components

ex $\vec{a} = \langle -1, 0, 3 \rangle$, $\vec{b} = \langle -2, -3, 5 \rangle$

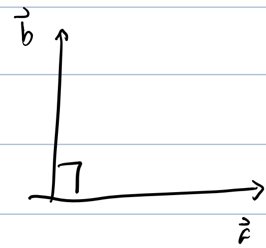
$\theta = \arccos\left(\dfrac{17}{\sqrt{10}\sqrt{38}}\right)$

$= \arccos\left(\dfrac{17}{\sqrt{380}}\right)$

$\approx 29.3^\circ$

Properties of dot product:

1) $\vec{a} \cdot \vec{b} = 0$ if and only if $\theta = 90^\circ$
$(\vec{a} \neq 0, \vec{b} \neq 0)$ (i.e. the vectors are orthogonal)



Proof: $\vec{a} \cdot \vec{b} = 0$

if and only if

$\|\vec{a}\| \, \|\vec{b}\| \cos\theta = 0$

if and only if

$\cos\theta = 0$ $\quad (\|\vec{a}\| \neq 0, \ \|\vec{b}\| \neq 0)$

if and only if

$\theta = 90^\circ$

2) $\vec{a} \cdot \vec{b} > 0$, then $0^\circ < \theta , 90^\circ$ $\therefore$ acute angle

3) $\vec{a} \cdot \vec{b} < 0$, then $90^\circ < \theta , 0^\circ$ $\therefore$ obtuse angle

A few more properties

a) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ commutative

* note: no associative property
since can't do $\vec{c} \cdot \vec{b} \cdot \vec{c}$
$\underbrace{\text{scalar} \cdot \text{vector}}_{\text{not a dot product}}$

b) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ distributive ex. $(\vec{a}+\vec{b})\cdot(\vec{c}+\vec{d})$ can foil

$=(\vec{a}+\vec{b})\cdot\vec{c}+(\vec{a}+\vec{b})\cdot\vec{d}$

$=\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}+\vec{a}\cdot\vec{d}+\vec{b}\cdot\vec{d}$

c) Let $k\in\mathbb{R}$, $k(\vec{a}\cdot\vec{b})=k\vec{a}\cdot\vec{b}=\vec{a}\cdot(k\vec{b})$

ex. $2(\vec{a}\cdot\vec{b})=2\vec{a}\cdot\vec{b}=\vec{a}\cdot2\vec{b}$

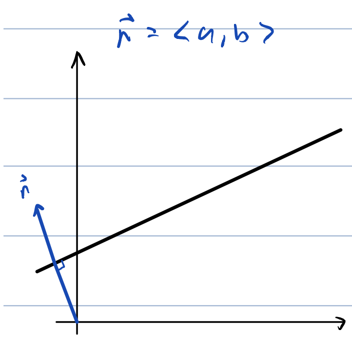
d) $\underline{\vec{a}\cdot\vec{a}=\|\vec{a}\|^2}$ important prove tetrahedral angle

b.c. $\vec{a}\cdot\vec{a}=\|\vec{a}\|\|\vec{a}\|\cos 0$

$=\|\vec{a}\|^2\cdot 1$

$=\|\vec{a}\|^2$

Normal Vector of a Line in $\mathbb{R}^2$ (using dot product)



General equation:

$ax+by+c=0 \rightsquigarrow$

$-by=ax+c$

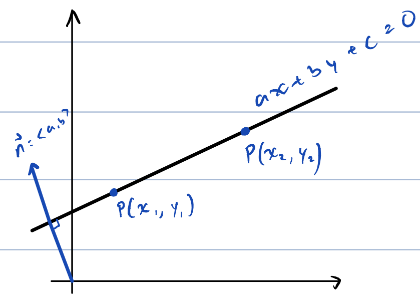
$y=-\frac{a}{b}x+\frac{c}{b}$

$y=mx+q$

$\perp\ -\frac{a}{b}=\frac{b}{a}$] normal line

The vector $\vec{n}=\langle a,b\rangle$ is a called a **normal vector** of the line and it is perpindiculer to the line.

Proof:



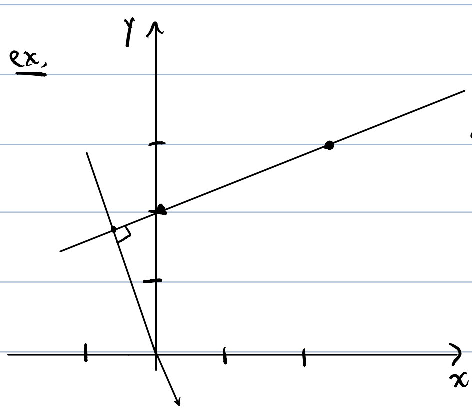
Take two random pts on line $P_1(x_1,y_1)$, $P_2(x_2,y_2)$.

Let's show that $\vec{n}\cdot\overrightarrow{P_1P_2}=0$ (i.e. orthogonal vectors)

$$\begin{aligned}\vec{n}\cdot\overrightarrow{P_1P_2} &= \langle a,b\rangle\cdot\langle x_2-x_1,\ y_2-y_1\rangle\\ &= a(x_2-x_1)+b(y_2-y_1)\\ &= ax_2-ax_1+by_2-by_1\\ &= ax_2+by_2-(ax_1+by_1)\\ &= -c-(-c)\\ &= 0\ \checkmark\end{aligned}$$

b.c. P_1 and P_2 are on the line so they satisfy the equation $ax+by+c=0$

ex.



$\frac{1}{2}x-y+3=0$

So $\vec{n}=\langle \frac{1}{2},-1\rangle$

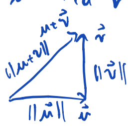
3. Use the properties of the dot product to show that $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\vec{u}\cdot\vec{v}$

$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}\|)$

$= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$

$= \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\vec{u}\cdot\vec{v}$

Note: If $\vec{u}$ and $\vec{v}$ are orthogonal



$\|u+v\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 + \cancel{2\vec{u}\cdot\vec{v}}$

0 b.c. $\theta = 90°$

4. Under what conditions on a and b are the vectors $\vec{v} = [1,2,3]$ and $\vec{w} = [a,b,-4]$

a) orthogonal?

iff $\vec{v}\cdot\vec{\omega} = 0$ ∴ vectors are orthogonal

$\langle 1,2,3\rangle \cdot \langle a,b,-4\rangle = 0$

$\boxed{a + 2b - 12 = 0}$

b) parallel?

iff $\vec{v} = k\vec{\omega}$ ∴ vectors are parallel

$\langle 1,2,3\rangle = k\langle a,b,-4\rangle$

$\langle 1,2,3\rangle = \langle ka, kb, -4k\rangle$

$3 = -4k$ $\qquad$ $1 = ka$ $\qquad$ $2 = kb$

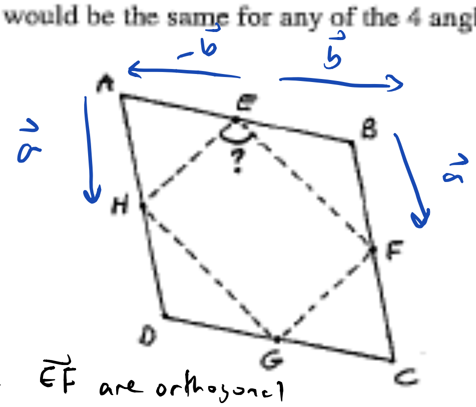
$\boxed{k = -\frac{3}{4}}$ $\qquad$ $1 = -\frac{3}{4}(a)$ $\qquad$ $2 = -\frac{3}{4}b$

$\boxed{a = -\frac{4}{3}}$ $\qquad$ $b = -\frac{4}{3}\cdot 2$ $\qquad$ $\boxed{b = -\frac{8}{3}}$

5.

Let $ABCD$ be an arbitrary rhombus (parallelogram with 4 sides of equal length), and consider midpoints E, F, G and H of its sides, as shown on the diagram below. Using vector methods, show that the inside quadrilateral $EFGH$ must actually be a rectangle, that is, the angles at E, F, G and H are right.

Note: since the procedure would be the same for any of the 4 angles, you need only show that $\angle HEF$ is right.



Show that $\vec{EH}$ and $\vec{EF}$ are orthogonal

i.e. $\vec{EH}\cdot\vec{EF} = 0$

$\vec{EH}\cdot\vec{EF}$

$= (\vec{EA} + \vec{AH})\cdot(\vec{EB} + \vec{BF})$

$= (-\vec{b} + \vec{a})\cdot(\vec{b} + \vec{a})$

$= -\vec{b}\cdot\vec{b} - \cancel{\vec{b}\cdot\vec{a}} + \cancel{\vec{a}\cdot\vec{b}} + \vec{a}\cdot\vec{a}$

$= -\|b\|^2 + \|\vec{a}\|^2$

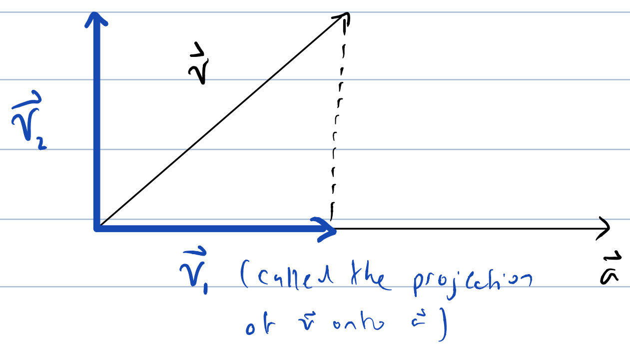
But $\|\vec{a}\| = \|\vec{b}\|$

$= 0$

Aug. 30

Orthogonal Projections

Consider $\vec{v}$, $\vec{a}$ two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$

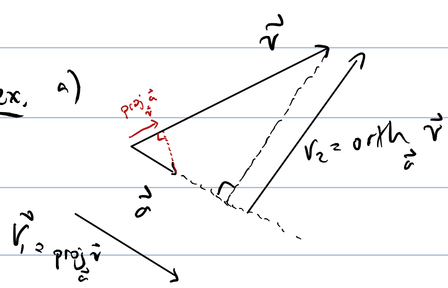


We can split vector $\vec{v}$ into two vectors $\vec{v}_1$ and $\vec{v}_2$, where $\vec{v}_1$ is parallel to $\vec{a}$, and $\vec{v}_2$ is orthogonal to $\vec{a}$, such that $\vec{v} = \vec{v}_1 + \vec{v}_2$.

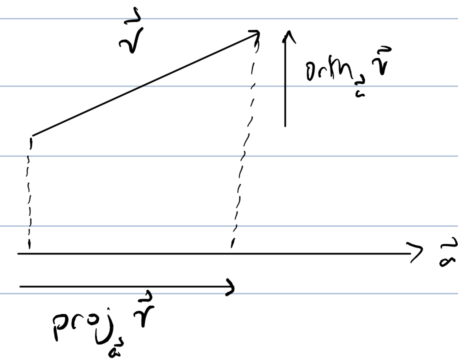
We call the vector $\vec{v}_1$, parallel to $\vec{a}$, the projection of $\vec{v}$ onto $\vec{a}$ denoted by $\text{proj}_{\vec{a}}\, \vec{v}$

We call the vector $\vec{v}_2$, orthogonal to $\vec{a}$, the component of $\vec{v}$ orthogonal to $\vec{a}$, denoted by $\text{orth}_{\vec{a}}\, \vec{v}$

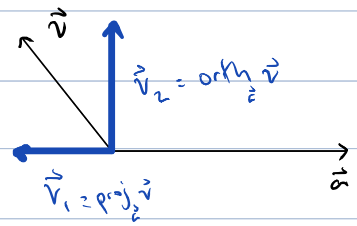
ex. a)



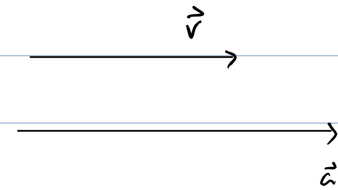
b)



c)



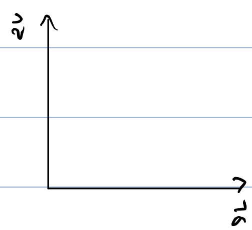
d)



$\vec{v}_1 = \vec{v} = proj_{\vec{a}}\,\vec{v}$

$\vec{v}_2 = orth_{\vec{a}}\,\vec{v} = \vec{0}$

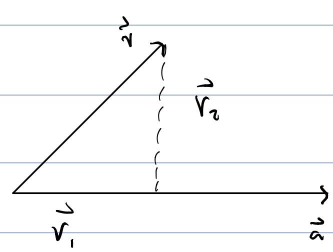
e)



$\vec{v}_1 = proj_{\vec{a}}\,\vec{v} = \vec{0}$

$\vec{v}_2 = \vec{v} = orth_{\vec{a}}\,\vec{v}$

Formula for $proj_{\vec{a}}\,\vec{v}$



$$\boxed{\vec{v}_1 = proj_{\vec{a}}\,\vec{v} = \left(\frac{\vec{v}\cdot\vec{a}}{\|a\|^2}\right)\vec{a}}$$

$\vec{v}_2 = orth_{\vec{a}}\,\vec{v} = \vec{v} - proj_{\vec{a}}\,\vec{v}$

Proof

We know that $\text{proj}_{\vec{a}} \vec{v} = k\vec{a}$ where $k \in \mathbb{R}$

Show that $k = \dfrac{\vec{v} \cdot \vec{a}}{\|\vec{a}\|^2}$

$\vec{v} = \vec{v}_1 + \vec{v}_2$

$\vec{v} = k\vec{a} + \vec{v}_2$ multiply by $\vec{a}$ using dot product

$\vec{v} \cdot \vec{a} = (k\vec{a} + \vec{v}_2)\, \vec{a}$

$\vec{v} \cdot \vec{a} = k\vec{a} \cdot \vec{a} + \vec{v}_2 \cdot \vec{a}$

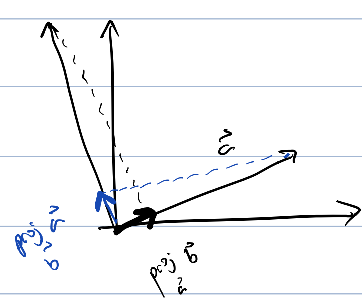
$\vec{v} \cdot \vec{a} = k\|\vec{a}\|^2 + 0$ b.c. $v_2 \perp \vec{a}$ ($=0$ in dot product)

$k = \dfrac{\vec{v} \cdot \vec{a}}{\|\vec{a}\|^2}$

ex, Let $\vec{a} = \langle 3,1 \rangle$, $\vec{b} = \langle -1,5 \rangle$

Find a) $\text{proj}_{\vec{a}} \vec{b}$

$= \left(\dfrac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2}\right) \vec{a}$



$$= \frac{2}{10} \langle 3, 1 \rangle$$

$$= \frac{1}{5} \langle 3, 1 \rangle$$

b) $\text{proj}_{\vec{b}} \vec{a}$

$$= \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2} \vec{a}$$

$$= \frac{2}{26} \langle -1, 5 \rangle$$

$$= \frac{1}{13} \langle -1, 5 \rangle$$

c) $\| \text{proj}_{\vec{a}} \vec{b} \| = \left\| \frac{1}{5} \langle 3, 1 \rangle \right\|$

$$= \frac{1}{5} \| \langle 3, 1 \rangle \|$$

$$= \frac{\sqrt{10}}{5}$$

Finding the norm of a projection

<u>Shortcut!</u>

$$\| \text{proj}_{\vec{a}} \vec{v} \| = \left\| \frac{\vec{v} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a} \right\|$$

$$= \left| \frac{\vec{v} \cdot \vec{a}}{\|\vec{a}\|^2} \right| \|\vec{a}\|$$

$$= \frac{|\vec{v} \cdot \vec{a}|}{\|\vec{a}\|^{\cancel{2}}} \cancel{\|\vec{a}\|}$$

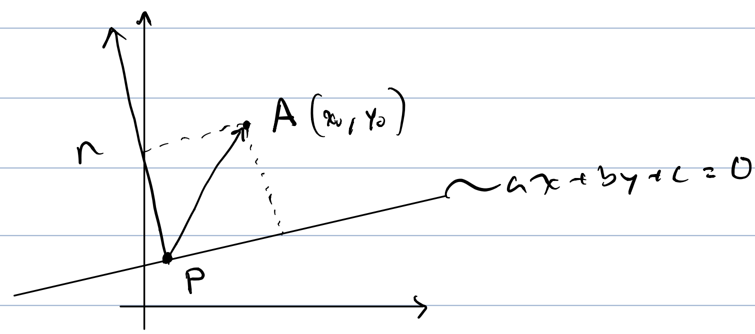
Reminder:

$$\|k \vec{v}\| = \underset{\text{abs. val}}{|k|} \|\vec{v}\|$$

$$\boxed{\|\text{proj}_{\vec{a}}\vec{v}\| = \frac{|\vec{v}\cdot\vec{a}|}{\|\vec{a}\|}}$$

ex, Consider the line in $\mathbb{R}^2$, $ax+by+c=0$ and the point $A(x_0,y_0)$ in the xy plane.

Use a projection to find the shortest distance from $\vec{a}$ to the line.



Let P be a pt on the line.

Then $d = \|\text{proj}_{\vec{n}}\vec{PA}\|$ where $\vec{n} = \langle a, b\rangle$ from line

$\therefore$ $d = \dfrac{|\vec{PA}|}{\|\vec{n}\|}$

We can turn this into an algebraic formula.

Let P be (x_1, y_1), then:

$d = |\langle x_0 - x_1, y_0 - y_1\rangle \cdot \langle a, b\rangle|$

$$\frac{\left| \quad \right|}{\sqrt{a^2+b^2}}$$

$$= \left| \frac{a(x_0 - x_1) + b(y_0 - y_1)}{\sqrt{a^2+b^2}} \right|$$

$$= \left| \frac{a x_0 + b y_0 + \overbrace{(-a x_1 - b y_1)}^{c \text{ since } (x_1, y_1) \text{ belongs to the line}}}{\sqrt{a^2+b^2}} \right|$$

So $$d = \frac{|a x_0 + b y_0 + c|}{\sqrt{a^2+b^2}}$$

ex. Find the distance from pt. $A(1,3)$ to the line

$$y = -\frac{1}{2}x - 1$$

$\downarrow$ General form

$$x + 2y + 2 = 0$$

$$n = \langle 1, 2 \rangle$$

Chose pt on the line

$P(0,-1)$

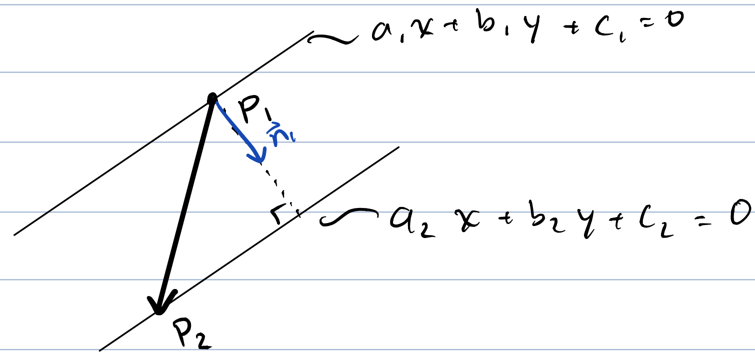
$\vec{PA} = \langle 1, 4 \rangle$

So

$$d = \left\| \text{proj}_{\vec{n}} \vec{PA} \right\| = \frac{|\vec{PA} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|\langle 1, 4 \rangle \cdot \langle 1, 2 \rangle|}{\sqrt{5}}$$

$$= \frac{9}{\sqrt{5}}$$

Distance between parallel lines in $\mathbb{R}^2$



$$\vec{n}_1 = \langle a_1, b_1 \rangle$$
$$\vec{n}_2 = \langle a_2, b_2 \rangle$$

$\therefore \vec{n}_1$ and $\vec{n}_2$ are parallel

Choose random pts P_1 and P_2 on lines ℓ_1 and ℓ_2 respectively.

So $d = \left\| \text{proj}_{\vec{n}_1} \overrightarrow{P_1P_2} \right\| = \dfrac{|\overrightarrow{P_1P_2} \cdot \vec{n}_1|}{\|\vec{n}_1\|}$

Cross Product (vector product)

<u>Note</u>: Only in $\mathbb{R}^3$

Let $\vec{a} = \langle x_1, y_1, z_1 \rangle$ and $\vec{b} = \langle x_2, y_2, z_2 \rangle$

We define the cross product of $\vec{a}$ and $\vec{b}$ as

$$\vec{a} \times \vec{b} = \langle y_1 z_2 - z_1 y_2,\ z_1 x_2 - x_1 z_2,\ x_1 y_2 - y_1 x_2 \rangle$$

$x_1 \ y_1 \ z_1$

$x_2 \ y_2 \ z_2$

Note:

$$\vec{a} \times \vec{b} = \left\langle \underbrace{\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}}_{\text{hide } x \text{ column}},\ -\underbrace{\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}}_{\text{hide } y \text{ column}},\ \underbrace{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}}_{\text{hide } z \text{ column}} \right\rangle$$

where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ ← the determinant

ex. Let $\vec{a} = \langle 1, 2, 3 \rangle$, $\vec{b} = \langle 0, -2, 5 \rangle$. Find $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$

$$\vec{a} \times \vec{b} = \left\langle \begin{vmatrix} 2 & 3 \\ -2 & 5 \end{vmatrix},\ -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix},\ \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} \right\rangle$$

$$= \langle 16, -5, -2 \rangle$$

$$\vec{b} \times \vec{a} = \left\langle \begin{vmatrix} -2 & 5 \\ 2 & 3 \end{vmatrix},\ -\begin{vmatrix} 0 & 5 \\ 1 & 3 \end{vmatrix},\ \begin{vmatrix} 0 & -2 \\ 1 & 2 \end{vmatrix} \right\rangle$$

$$= \langle -16, 5, 2 \rangle$$

Properties

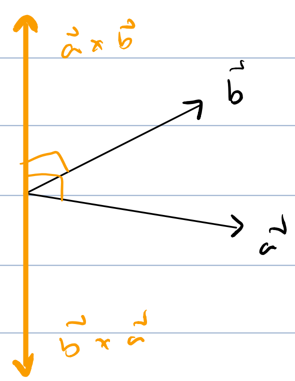
1) $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$ anti - commutative

2) i) $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$

ii) $(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$

$\vec{a} \times b$ is a vector that is orthogonal to $\vec{a}$ and $\vec{b}$

Property 2 tells us that the vector $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$



Proof of 2(i)

$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \times \vec{b} = \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle$$

$$\cdot \langle a_1, a_2, a_3 \rangle$$

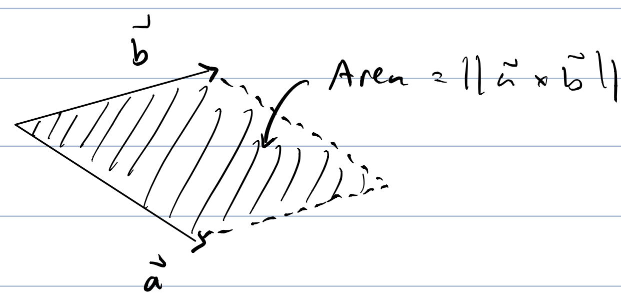
$$= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1)$$

$$+ a_3(a_1b_2 - a_2b_1)$$

$$= \cancel{a_1a_2b_3} - \cancel{a_1a_3b_2} - \cancel{a_1a_2b_3} + \cancel{a_2a_3b_1}$$
$$+ \cancel{a_1a_3b_2} - \cancel{a_2a_3b_1}$$

$$= 0$$

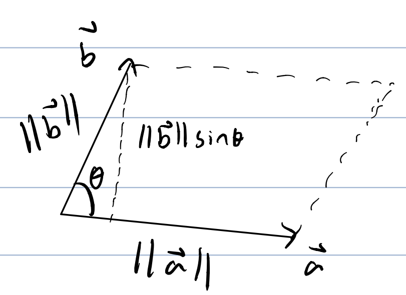
3) $\|\vec{a} \times \vec{b}\|$ = area of the parallelogram determined by $\vec{a}$ and $\vec{b}$



Half-Proof

We can show that $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\,\|\vec{b}\| \sin\theta$

But...



$A = b \times h$

$= \|\vec{a}\| \, \|\vec{b}\| \sin\theta$

Sept. 5

Properties (Cont'd)

(3) $\vec{a} \times \vec{a} = \vec{0}$ $\quad = \left\langle \begin{vmatrix} a_2 & a_3 \\ a_2 & a_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ a_1 & a_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix} \right\rangle$

$= \dots$

$\begin{cases} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{cases}$ $\quad = \langle 0, 0, 0 \rangle$

$= \vec{0}$

$$\boxed{\text{So, } \vec{a} \text{ and } \vec{b} \text{ are parallel iff } \vec{a} \times \vec{b} = \vec{0}}$$

Proof: $\vec{a}$ and $\vec{b}$ are parallel

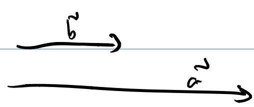
iff they are scaler multiples of each other

i.e. $\vec{a} = k\vec{b}$ for some $k \in \mathbb{R}$

iff $\vec{a} \times \vec{b} = k\vec{b} \times \vec{b} = k(\vec{b} \times \vec{b})$

Note: If $\vec{a}$ and $\vec{b}$ are parallel, then

$\|\vec{a} \times b\| = \|\vec{0}\| = 0$



$\therefore$ area of parallelogram $= 0$

④ $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

$(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

⑤ $k(\vec{a} \times \vec{b}) = k\vec{a} \times \vec{b} = \vec{a} \times k\vec{b}$

⑥ $\vec{a} \times \vec{0} = \vec{0}$

ICP 2

1) Find all vectors $\vec{w} = \langle a, b, c \rangle$ of norm $\|\vec{w}\| = 1$

(i.e. $\vec{w}$ is a unit vector)

that are orthogonal to

$\vec{u} = \langle 3, -1, 2 \rangle$, $\vec{v} = \langle 2, 3, 5 \rangle$

Method ① Using Dot Product

$\vec{w} \cdot \vec{u} = 0$ So $\begin{cases} 3a - b + 2c = 0 \\ 2a + 3b + 5c = 0 \end{cases}$

$\vec{w} \cdot \vec{v} = 0$ So

$\begin{cases} ① \; 3a - b + 2c = 0 \\ ② \; 2a + 3b + 5c = 0 \end{cases}$ $\rightarrow$ $b = 2c + 3a$

So, $2a + 3(2c + 3a) + 5c = 0$

$11a = -11c$

$\boxed{a = -c}$

Replace in 2

$b = 2c - 3c = -c$

$\boxed{b = -c}$

So $(-c)^2 + (-c)^2 + c^2 = 1$

$3c^2 = 1$

$c^2 = \frac{1}{3}$

$c = \pm \frac{1}{\sqrt{3}}$

So $\vec{w}_1 = \langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$

$w_2 = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \rangle$

Method 2 Using Cross product

$\vec{u} \times \vec{v} = \langle \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix}, - \begin{vmatrix} 3 & 2 \\ 2 & 5 \end{vmatrix}, \begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix} \rangle$

$= \langle -5 - 6, -(15 - 4), 9 + 2 \rangle$

$= \langle -11, -11, 11 \rangle$

$\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$

$= - \langle -11, -11, 11 \rangle$

$= \langle 11, 11, -11 \rangle$

$\|\vec{u} \times \vec{v}\| = \| \langle -11, -11, 11 \rangle \|$

$= \| 11 \langle -1, -1, 1 \rangle \|$

$= 11 \sqrt{(-1)^2 + (-1)^2 + (1)^2}$

$= 11\sqrt{3}$

$\vec{w}_1 = \frac{1}{11\sqrt{3}} \langle -11, -11, 11 \rangle$

$\boxed{\vec{w}_1 = \frac{1}{\sqrt{3}} \langle -1, -1, 1 \rangle}$

$w_2 = \frac{1}{11\sqrt{3}} \langle 11, 11, -11 \rangle$

$\boxed{w_2 = \frac{1}{\sqrt{3}} \langle 1, 1, -1 \rangle}$

2) If the angle between 2 unit vectors $\vec{a}$ and $\vec{b}$ is $120°$, then find the value of $\|\vec{a} + 2\vec{b}\|$

$\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

$\|\vec{a} + 2\vec{b}\|^2 = (\vec{a} + 2\vec{b}) \cdot (\vec{a} + 2\vec{b})$

$= \vec{a} \cdot \vec{a} + 4\vec{b} \cdot \vec{a} + 4\vec{b} \cdot \vec{b}$

$= \|a\|^2 + 4\left(-\frac{1}{2}\right) + 4\|b\|^2$

$= 1 - 2 + 4$

$= 3$

$\vec{a} \cdot \vec{b} = \|a\| \|b\| \cos\theta$

$= \cos(120°)$

$= -\frac{1}{2}$

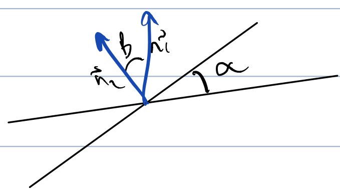
$\|\vec{a} + 2\vec{b}\| = \sqrt{3}$

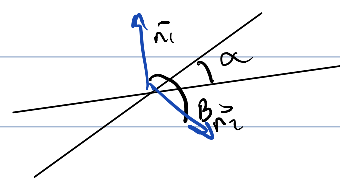
Sept. 6

For angle between two lines on assignment 2:

Let α be angle between lines.

β normal vectors.





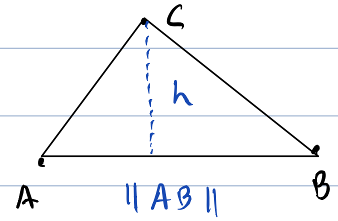
$\alpha = \beta$
if $0 \le \beta \le 90°$

$\alpha = 180° - \beta$
if $90° \le \beta \le 180°$

If β between $\vec{n}_1$ and $\vec{n}_2$ > 90, then $\alpha = 180 - \beta$.
If not, $\alpha = \beta$

ex. Find the area of the triangle in $\mathbb{R}^3$ determined by
$A(1,1,1)$, $B(2,-1,3)$, $C(4,1,0)$.

Method ①

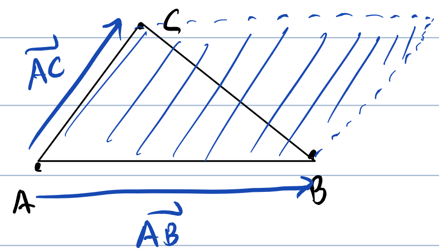


$$A = \frac{b\,h}{2}$$

$$b = \|\vec{AB}\|$$

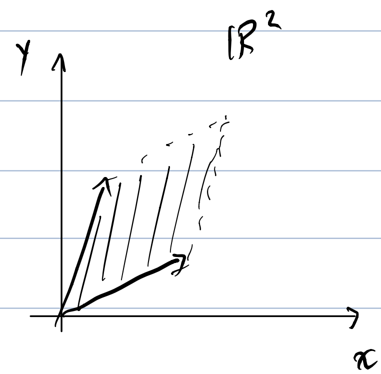
$$h = \|\text{orth}_{\vec{AB}}\,\vec{AC}\|$$

Method ②

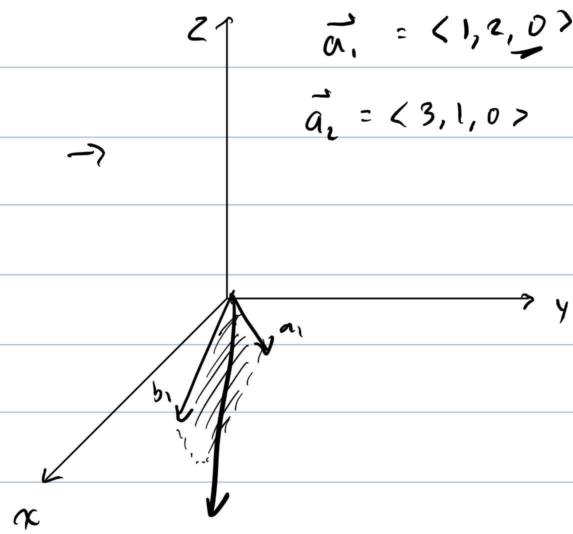


$$A = \frac{\|\vec{AB} \times \vec{AC}\|}{2}$$ ← for cross product in $\mathbb{R}^3$ (only)

ex. Find the area of the parallelogram in $\mathbb{R}^2$ determined by vectors $\vec{a} = \langle 1, 2 \rangle$, $\vec{b} = \langle 3, 1 \rangle$



→



$\vec{a}_1 = \langle 1, 2, 0 \rangle$

$\vec{a}_2 = \langle 3, 1, 0 \rangle$

$$A = \|\vec{a}_1 \times \vec{b}_1\|$$

3. a) Show that the points $A(2,-1,1)$, $B(1,2,1)$, $C(3,0,3)$ *and* $D(0,1,-1)$ are vertices of a parallelogram.

$\vec{AB} = \langle -1, 3, 0 \rangle$
$\vec{CD} = \langle -3, 1, -4 \rangle$] Nope.

Trial and error

$\vec{AC} = \langle 1, 1, 2 \rangle$

$\vec{BD} = \langle -1, -1, -2 \rangle \rightarrow \vec{DB} = \langle 1, 1, 2 \rangle$

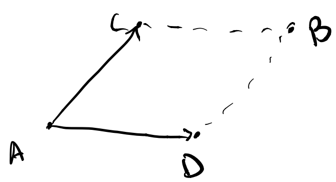
So $\vec{DB} = \vec{AC}$

So A, B, C & D are the vertices of a parallelogram

b) Find the area of the parallelogram.

$\vec{AB} = \langle -1 \quad 3 \quad 0 \rangle$
$\vec{AD} = \langle -2, 2, -2 \rangle$

$\text{Area} = \|\vec{AB} \times \vec{AD}\|$

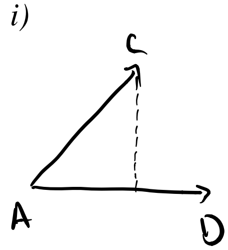


$= \|\langle \quad , \quad , \quad \rangle\|$

$=$

c) Find the height of the parallogram in two different ways.

i)



$h = \|\text{orth}_{\vec{AD}} \vec{AC}\|$

$\text{proj}_{\vec{AB}} \vec{AC} = \dfrac{\vec{AC} \cdot \vec{AD}}{\|\vec{AD}\|^2} \vec{AD}$

ii) $\text{Area} = b \times h$

$h = \dfrac{\text{Area}}{b}$

$= \dfrac{\|\vec{AD} \times \vec{AC}\|}{\|\vec{AD}\|}$

$\left(\dfrac{\vec{v} \cdot a}{\|a\|^2}\right) \vec{a}$ $\quad \vec{v}_2 = \quad \vec{v} - \text{proj}_{\vec{a}} \vec{v}$

Scalar Triple Product

Given three vectors $\vec{a}, \vec{b}, \vec{c}$ in $\mathbb{R}^3$, we define the scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$ as

$$\vec{a} \cdot (\vec{b} \times \vec{c})$$

ex. Let $\vec{a} = \langle 2, 1, -1 \rangle$

$\vec{b} = \langle 2, 2, 0 \rangle$

$\vec{c} = \langle -1, 3, 1 \rangle$

$\vec{a} \cdot (\vec{b} \times \vec{c})$

$= \vec{a} \cdot \langle 2, -2, 4 \rangle$

$= \langle 2, 1, -1 \rangle \cdot \langle 2, -2, 8 \rangle$

$= 4 - 2 - 8$

$= -6$

Shortcut

① Copy first two columns

$\vec{a} \cdot (\vec{b} \times \vec{c})$

$$\begin{array}{c|ccccc} a & 2 & 1 & -1 & 2 & 1 \\ b & 2 & 2 & 0 & 2 & 2 \\ c & -1 & 3 & 1 & -1 & 3 \end{array}$$

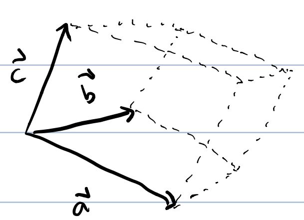
$$2(2)(1) + (1)(0)(-1) + (-1)(2)(3)$$
$$-\Big((-1)(2)(-1) + (2)(0)(3) + (1)(2)(1)\Big)$$

$$= \cancel{4} - 6 \; \cancel{-2} - 2$$

$$= -6$$

Volume of a Parallelepiped

Given three vectors in $\mathbb{R}^3$, we can determine a parallelepiped as follows:



Volume $= \left| \vec{a} \cdot (\vec{b} \times \vec{c}) \right|$ or $\left| \vec{b} \cdot (\vec{c} \times \vec{a}) \right|$ or...

doesn't matter b.c. absolute val

ex. Let $\vec{a} = \langle 1, 2, 1 \rangle$, $\vec{b} = \langle 3, 0, 2 \rangle$, $\vec{c} = \langle 0, 1, -1 \rangle$

Find the volume of the parallelepiped determined by $\vec{a}, \vec{b}, \vec{c}$.

$$V = \left| \vec{a} \cdot (\vec{b} \times \vec{c}) \right|$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 3 & 0 & 2 & 3 & 0 \\ 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

$$= \left| 0 + 0 + 3 - 0 - 2 + 6 \right|$$

$$= 7$$

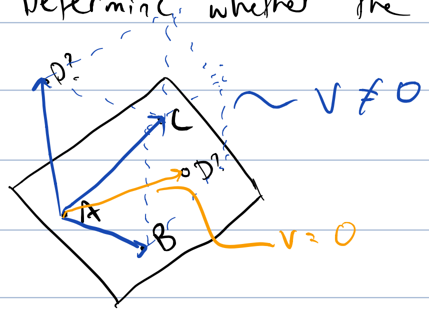
<u>Note</u>: We get the same answer if we change the order of vectors.

$$V = \left| \vec{b} \cdot (\vec{c} \times \vec{a}) \right|$$

$$= |7|$$

$$= 7$$

ex. Determine whether the 4 points are coplanar
(same plane)



* If D is not on the same plane as A, B, C, then $V \neq 0$
* If D <u>is</u> ———— " ————, then $V = 0$

$$\text{Vol} = \left| \vec{AB} \cdot (\vec{AC} \times \vec{AD}) \right| = |-10| \neq 0$$

$\therefore$ pts A, B, C, D are not coplanar