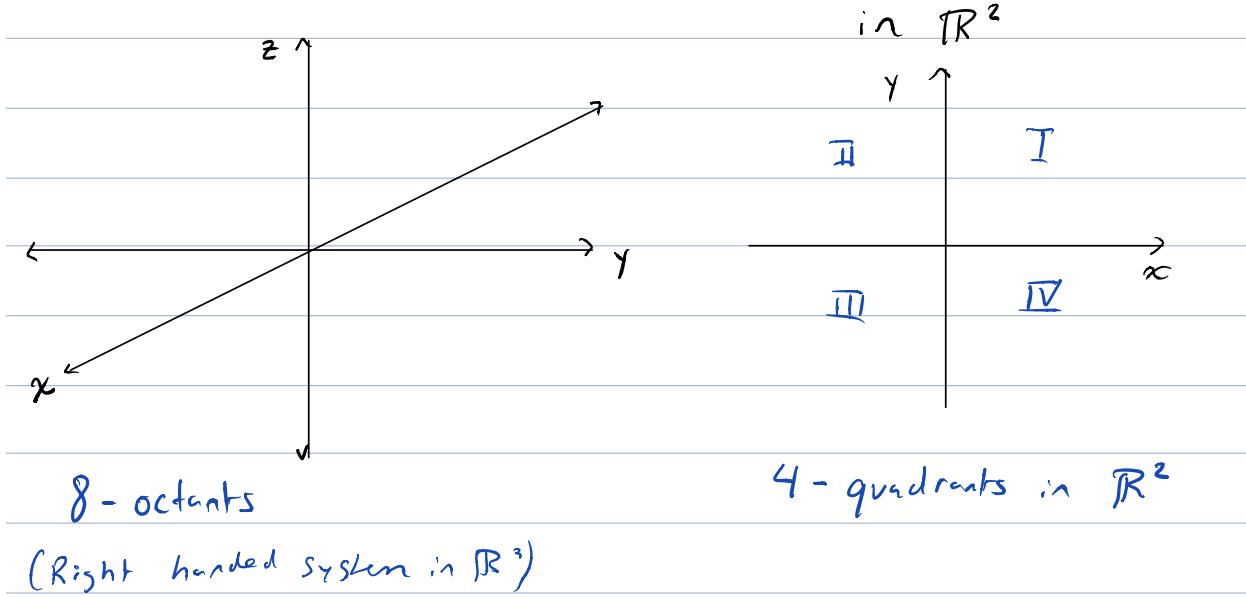
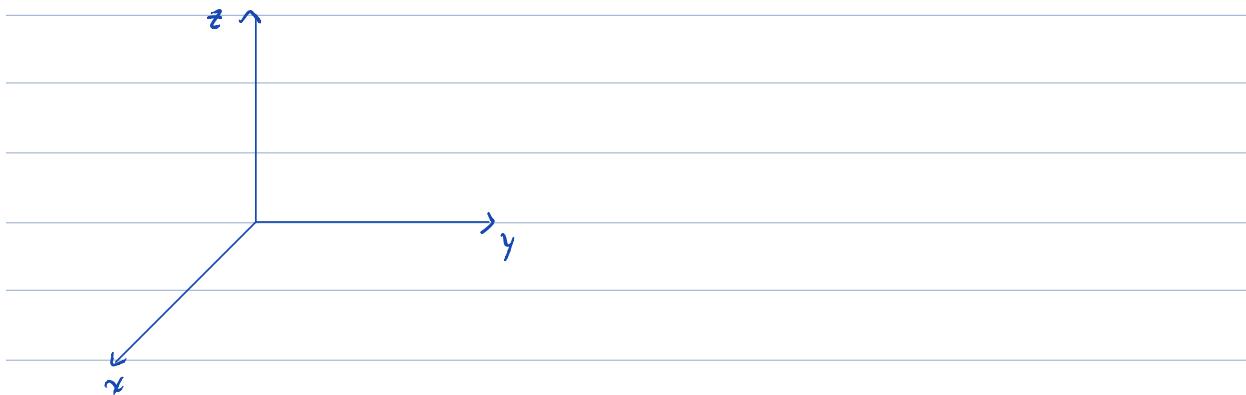


March 14

Types of Graphs in \mathbb{R}^3



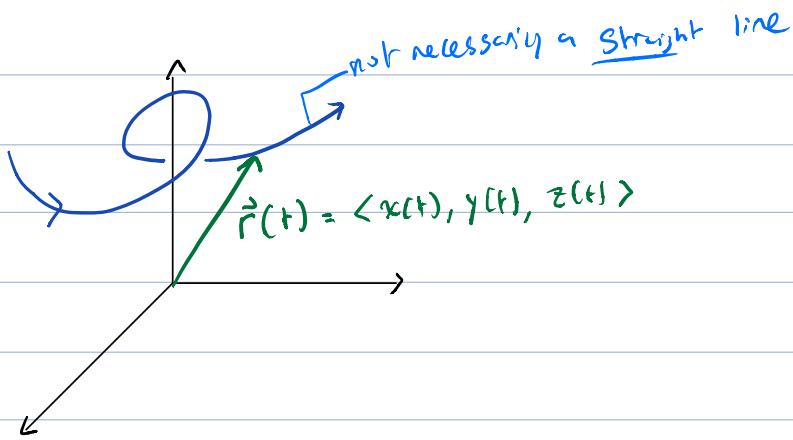
1st octant:



You can have 1D graphs in $\mathbb{R}^3 \rightarrow$ line

2D graphs in $\mathbb{R}^3 \rightarrow$ plane

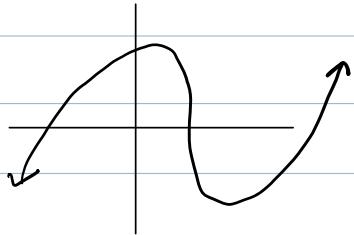
1D graphs: Vector functions or space curves



2D graphs:

$$\text{In } \mathbb{R}^2: y \stackrel{\text{def}}{\leftarrow} \begin{matrix} \text{up} \\ \text{ind} \end{matrix} f(x)$$

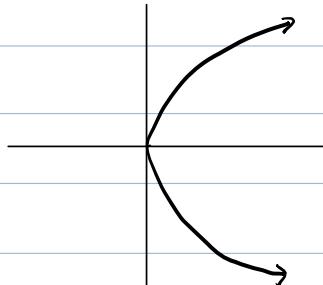
Functions



Vertical line test:

Any vertical intersects the graph at at most 1 point

Non-Functions



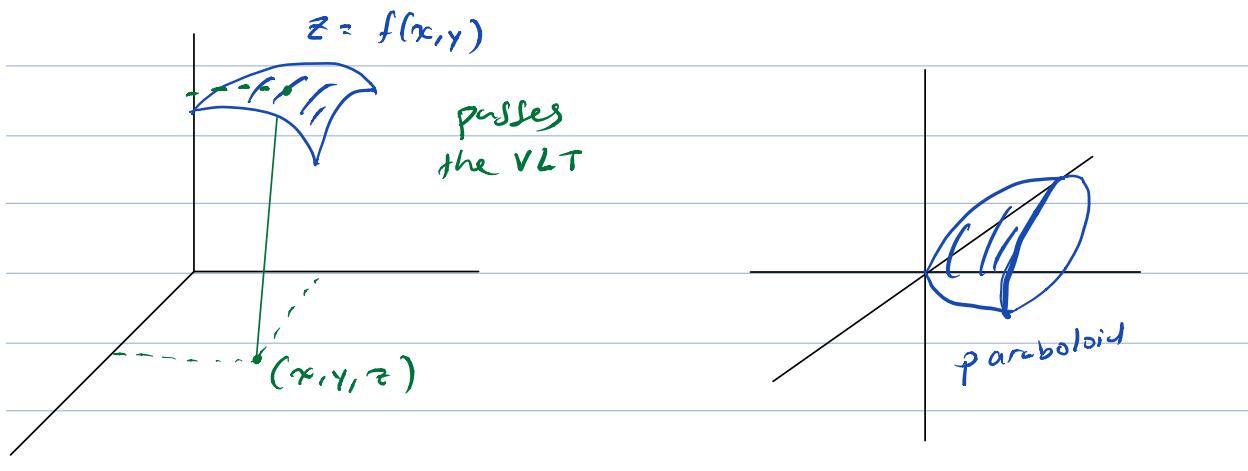
\mathbb{R}^3 :

Functions

$$z = f(x, y) \stackrel{\text{def}}{\leftarrow} \begin{matrix} \text{2 ind} \end{matrix}$$

Non-Functions

Cylinders quadratic surfaces
 may still pass the VLT that fail the VLT



Cylinders

A cylinder is the name that is given to a surface that is parallel to one of the axes.

In particular, it may or may not look like a "cylinder" from geometry.

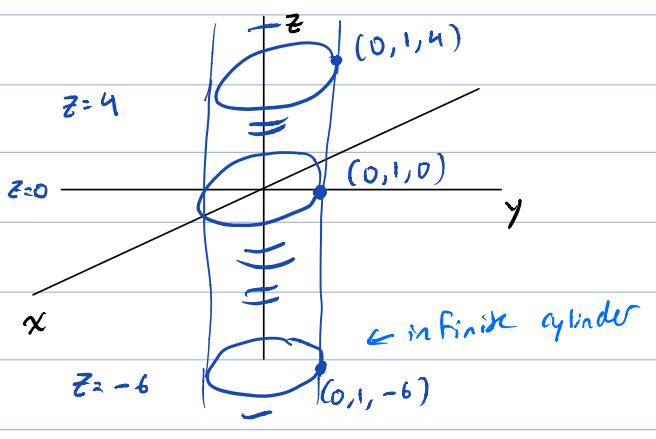
Algebraically: Cylinders occur when one of the coordinate variables (x, y, z) is missing from the equation of the surface.

The graph is parallel to the missing variable's axis

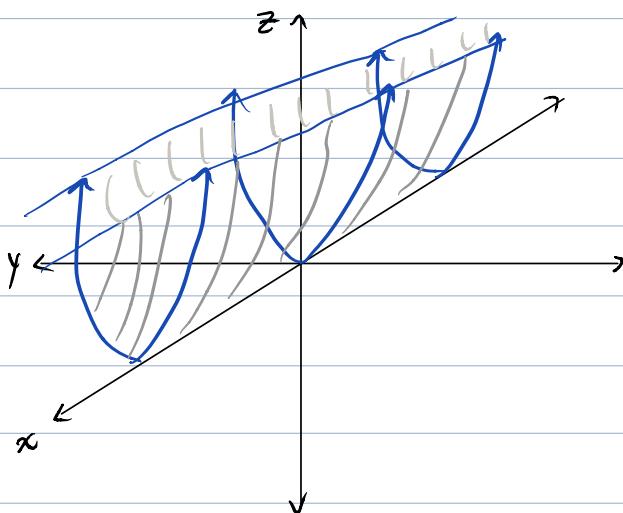
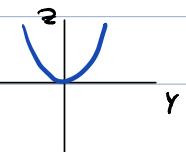
$$\text{ex. } x^2 + y^2 = 1 \text{ in } \mathbb{R}^3$$

no z

parallel to z -axis



ex. $z = y^2$ in \mathbb{R}^3
 ↪ parabolic cylinder

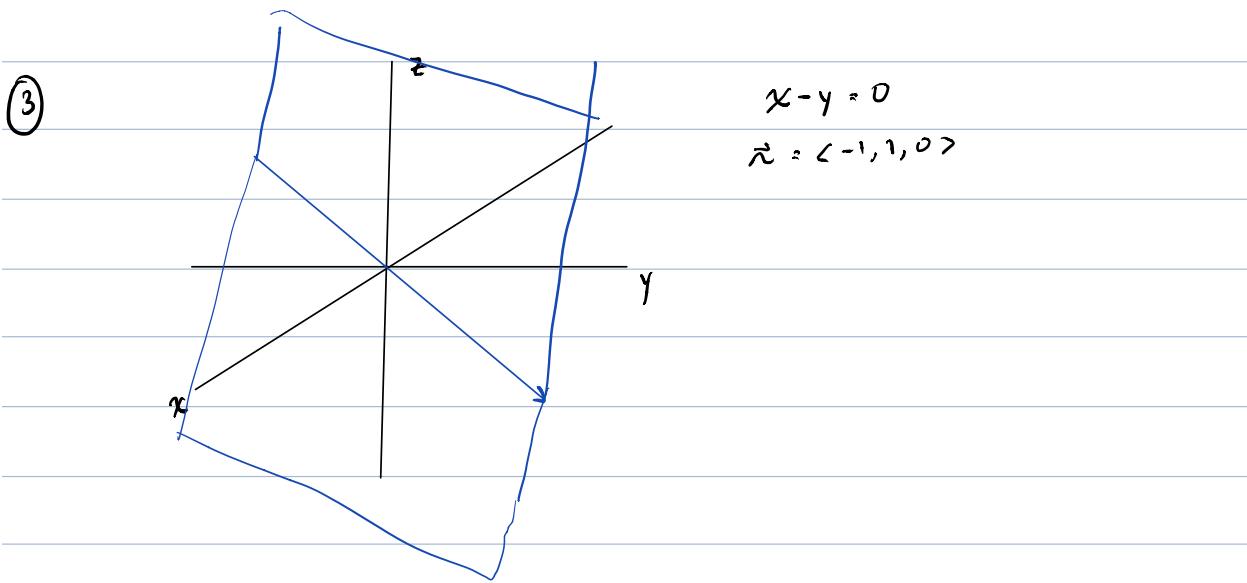
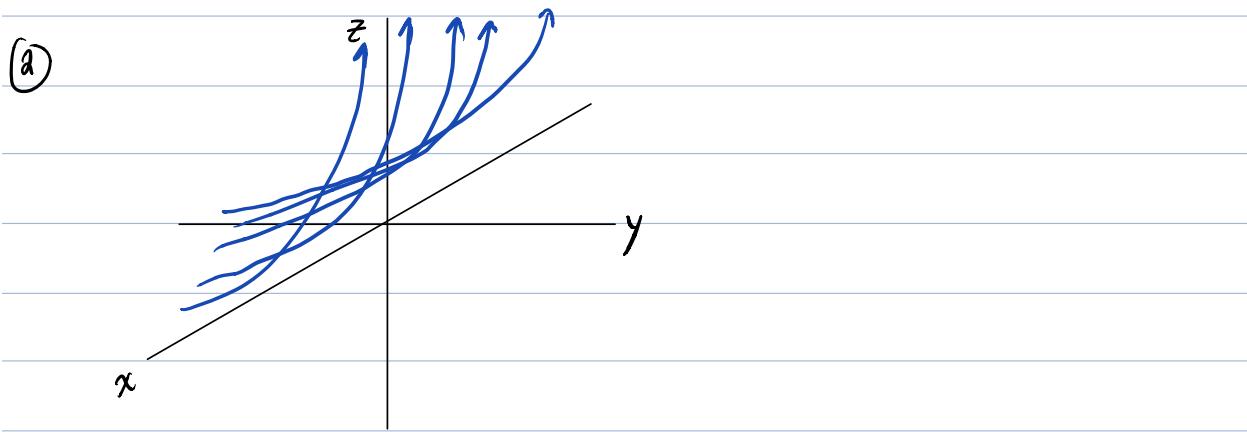
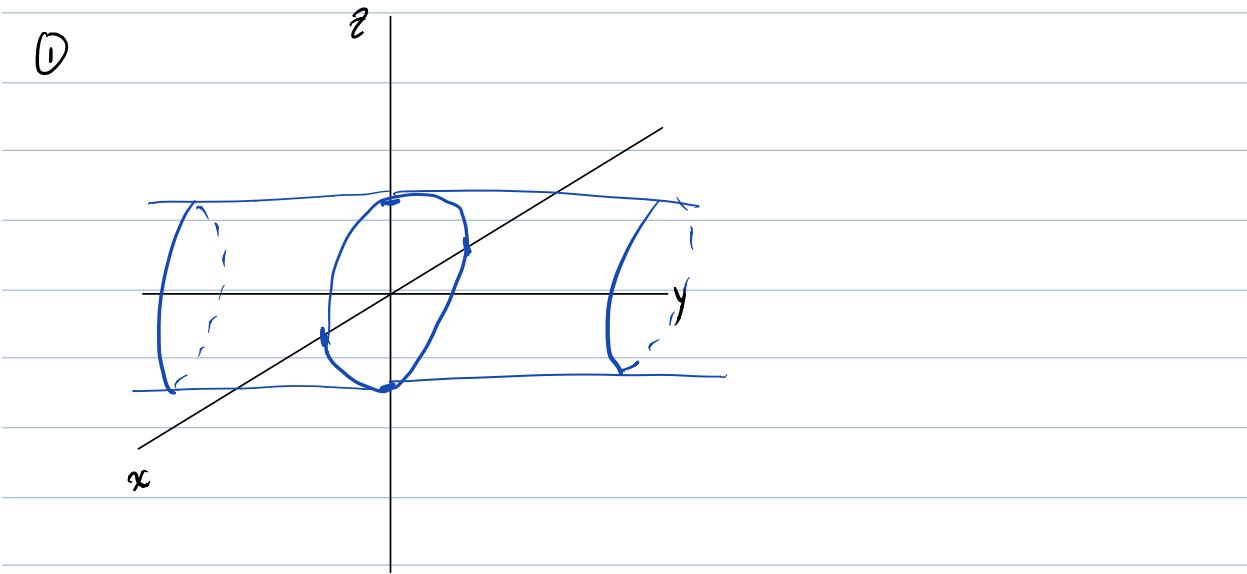


ex. For you: Sketch

$$\textcircled{1} \quad x^2 + z^2 = 1$$

$$\textcircled{2} \quad z = e^y$$

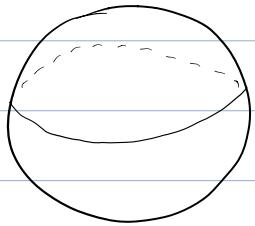
$$\textcircled{3} \quad x - y = 0$$



Quadratic Surfaces

3D analogy to conic sections

Sphere

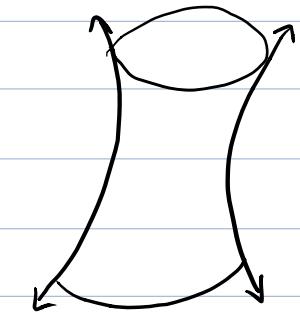


Ellipsoid



Hyperboloids

1 sheet

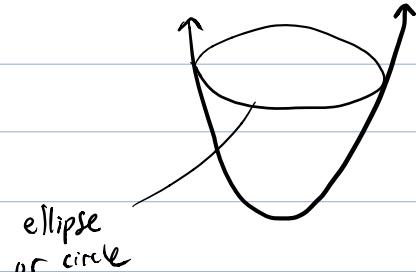


2 sheets

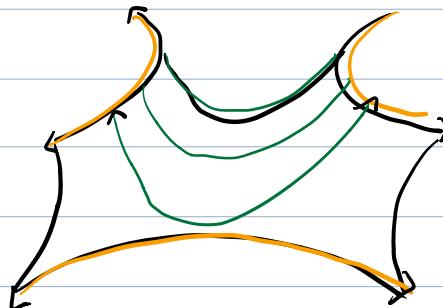


Paraboloids

Elliptic paraboloid ("paraboloid")



Hyperbolic paraboloid



Equations of Quadratic Surfaces

The most general form equation is

Rotational Terms

$$Ax^2 + By^2 + Cz^2 + Dx + Ey + Fz + Gxy + Hyz + Ixz + J = 0$$

Sphere: centred at $(0,0,0)$

$$\text{ex. } r=0 \quad x^2 + y^2 + z^2 = 0$$

radius $r > 0$

$$(0,0,0)$$

$$x^2 + y^2 + z^2 = r^2$$

single point

Ellipsoid: ex. $4x^2 + 4y^2 + z^2 = 12$

\Rightarrow Investigate by cross section analysis

Examine algebraically cross-sections in the planes:

$x = k$ \rightarrow for ellipsoids: ellipses

$y = k$ \rightarrow for arbitrary k

$z = k \rightarrow$ for ellipsoids: circle

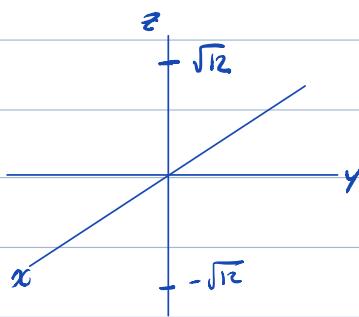
Let $z = k$ for $k \in \mathbb{R}$ $\left(z = k \text{ is a plane parallel to } xy\text{-plane} \right)$

$$4x^2 + 4y^2 = 12 - k^2$$

$$\underbrace{x^2 + y^2}_{\geq 0} = \frac{1}{4} \underbrace{(12 - k^2)}_{r^2} \quad \text{circles}$$

$$12 - k^2 \geq 0$$

$$-\sqrt{12} \leq k \leq \sqrt{12} \rightarrow \text{length is } 2\sqrt{12}$$



March 16

Review Cross Sections

sphere



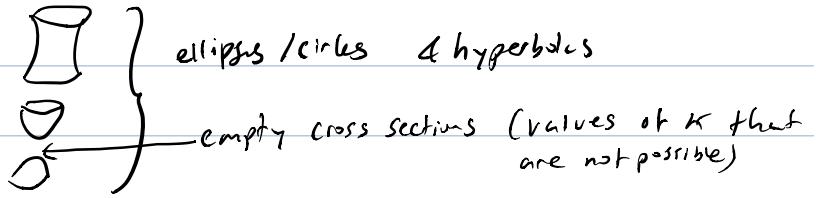
circles

ellipsoid



ellipses & circles

hyperboloid - 1 sheet
- 2 sheet

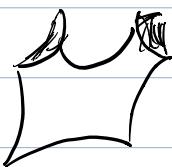


elliptic paraboloid



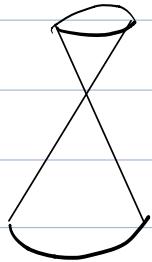
parabolas & ellipses / circles

hyperbolic paraboloid



hyperbolae & parabolae

Cone



circles / ellipses & hyperbolae
two intersecting lines

ex. Back to previous ex.

$$\text{Ellipsoid: } 4x^2 + 4y^2 + z^2 = 12$$

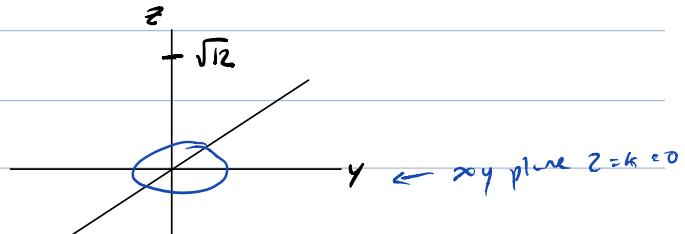
$$\text{Let } z=k \quad 4x^2 + 4y^2 + k^2 = 12$$

$$x^2 + y^2 = \underbrace{3 - \frac{1}{2}k^2}_{r^2} \quad \text{circles}$$

$$r^2 \geq 0$$

$$\therefore k^2 \leq 12$$

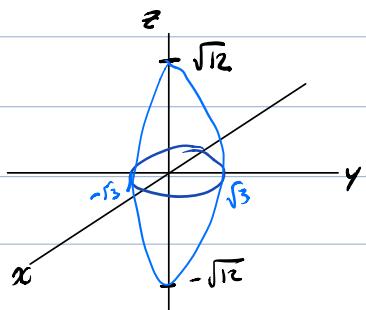
$$-\sqrt{12} > k > \sqrt{12}$$



parallel to xz plane

Let $y = k$: $4x^2 + z^2 = 12 - 4k^2$ ellipses
 $x^2 + \frac{z^2}{4} = 3 - k^2 \geq 0$

$-\sqrt{3} \leq k \leq \sqrt{3}$



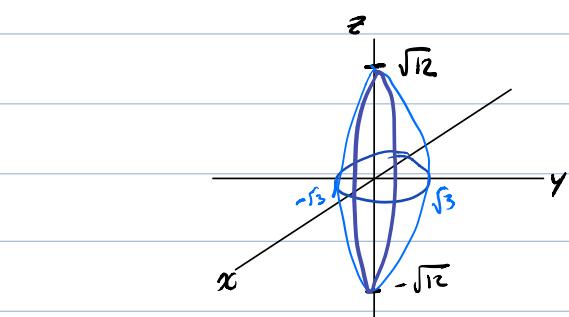
parallel to yz plane

Let $x = k$: $y^2 + \frac{z^2}{4} = 3 - k^2$ ellipses
 $-\sqrt{3} \leq k \leq \sqrt{3}$

$x^2 + y^2 = r^2$ circle

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipse

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$ hyperbola



Ex. For you

Perform a cross-section analysis in $x=k, y=k, z=k$ on the following
 Name the surface & sketch it.

$$\textcircled{1} \quad 4x^2 - 2y^2 - z^2 = 4$$

$$\text{Let } x = k \quad 4k^2 - 4 = 2y^2 + z^2 \geq 0$$

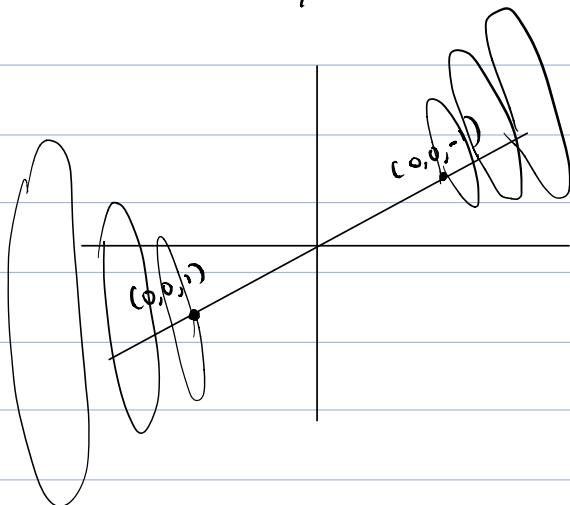
$$20 \quad 20$$

$k^2 \geq 1 \quad -1 \leq k \leq 1$ gap \rightarrow hyperboloid of 2 sheets
 \rightarrow looking for ellipse &
 hyperbolas in cross-sections

$$\text{at } k=1 : \quad 2y^2 + z^2 = 0$$

$$y=0, z=0$$

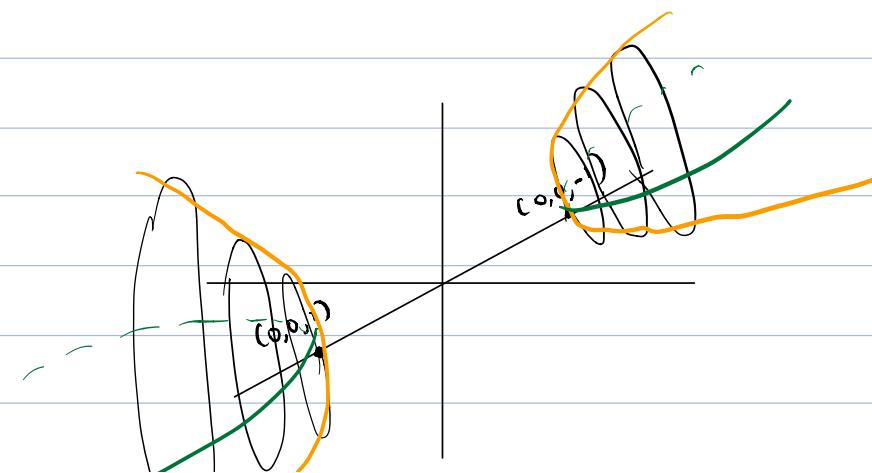
$$\text{at } k=-1 : \quad y=0, z=0$$

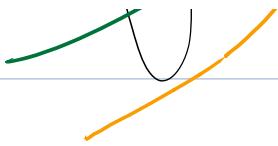


$$\text{Let } y = k: \quad 4x^2 - z^2 = 4 + 2k^2 \quad \text{hyperbola}$$

no restrictions

$$z \in \mathbb{R}$$





Let $z=k$... $z \in \mathbb{R}$ no restriction hyperbola

Tip: Find ellipses first

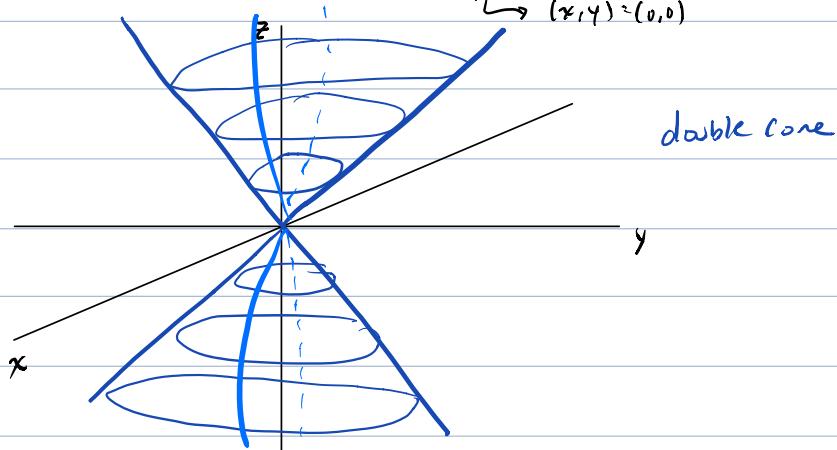
$$\textcircled{2} \quad 2x^2 + y^2 - z^2 = 0$$

Let $x=k$: $y^2 - z^2 = -2k^2$ hyperbola for $k \in \mathbb{R}$
 ↳ when $k=0$ $y=\pm z$ straight line

Let $y=k$: $2x^2 - z^2 = -k^2$ hyperbola for $k \in \mathbb{R}$
 ↳ $(0,0,0)$

Let $z=k$: $2x^2 + y^2 = k^2$ ellipse for $k \in \mathbb{R}$

↳ when $k \neq 0$: $2x^2 + y^2 = k^2$ ↳ $(x,y) = (0,0)$



Vector Functions

↙ for ↗

A vector function is a function with one real input and whose output is a vector.

$$\vec{r}(t) = \langle x(t), y(t) \rangle \in \mathbb{R}^2$$

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle \in \mathbb{R}^3$$

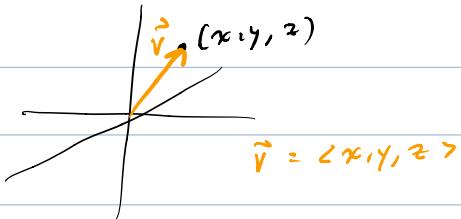
ex. You know some from linear

$$\text{line: } \langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle v_1, v_2, v_3 \rangle$$

$$\vec{r}(t) = \underbrace{\langle x_0 + tv_1, y_0 + tv_2, z_0 + tv_3 \rangle}_{\langle x(t), y(t), z(t) \rangle}$$

$$= x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

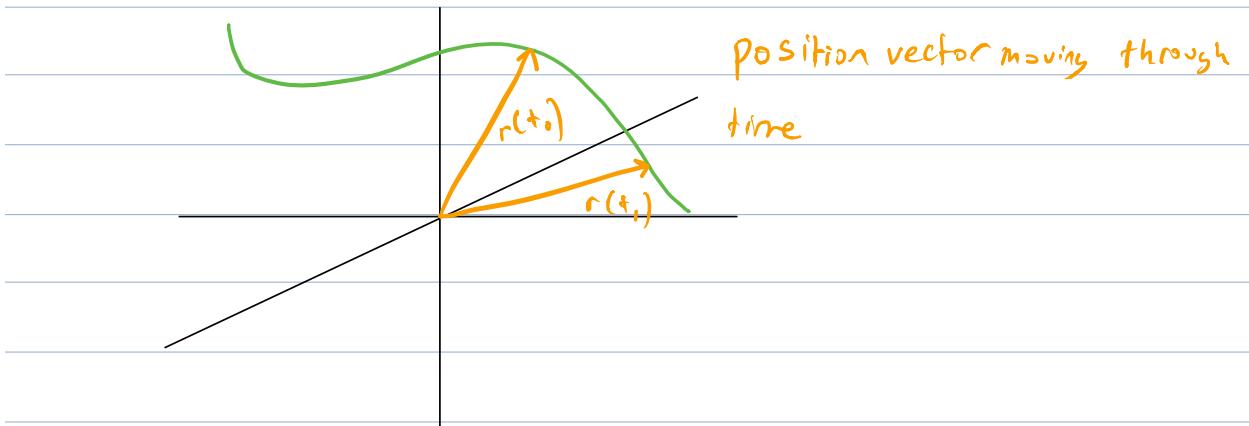
$\vec{r}(t)$ is giving the position vector at t that changes (through time)



March 19

Vector Functions (in $\mathbb{R}^3 \rightarrow$ space curves)

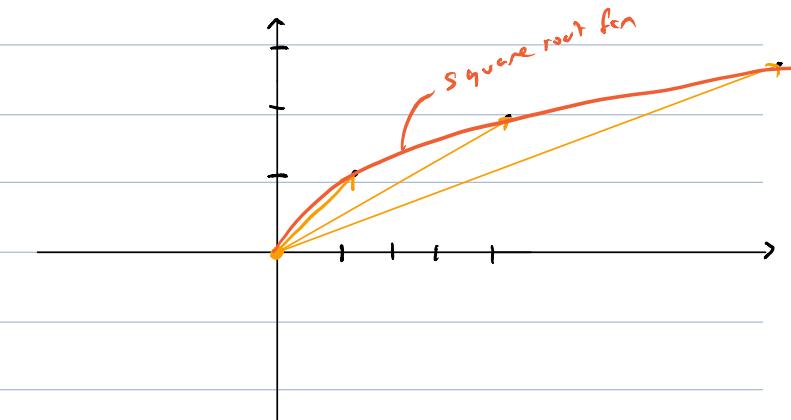
$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle \quad t \rightarrow \text{time}$$



ex. (in \mathbb{R}^2)

$$\textcircled{1} \quad \vec{r}(t) = \langle t, \sqrt{t} \rangle \quad t \in [0, \infty)$$

t	\vec{r}
0	$\langle 0, 0 \rangle = \vec{0}$
1	$\langle 1, 1 \rangle$
4	$\langle 4, 2 \rangle$
9	$\langle 9, 3 \rangle$

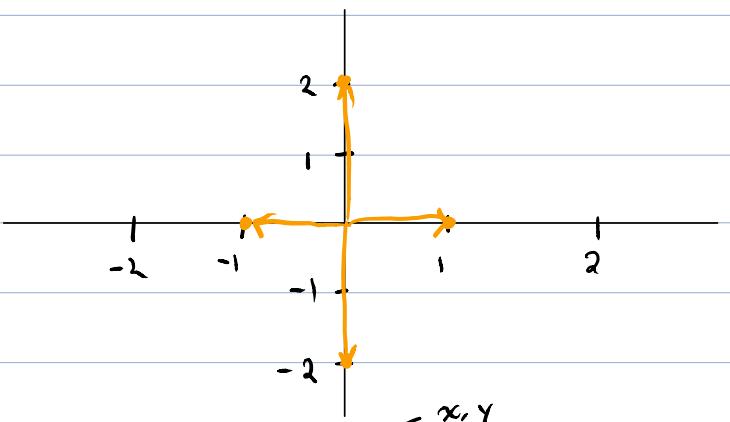


In general in \mathbb{R}^2 , $y = f(x)$ can be parameterized as
 $\vec{r}(t) = \langle t, f(t) \rangle$ ↳ t is the parameter

(2)

a) Make a table of values and sketch $\vec{r}(t) = \langle \cos t, 2\sin t \rangle$

t	$r(t)$
0	$\langle 1, 0 \rangle$
$\frac{\pi}{2}$	$\langle 0, 2 \rangle$
π	$\langle -1, 0 \rangle$
$\frac{3\pi}{2}$	$\langle 0, -2 \rangle$

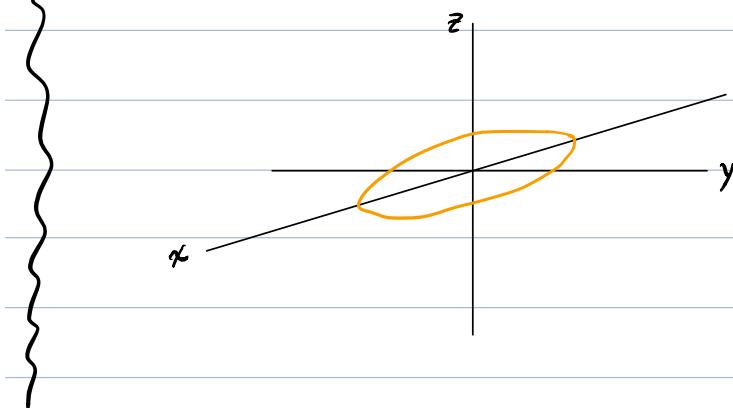


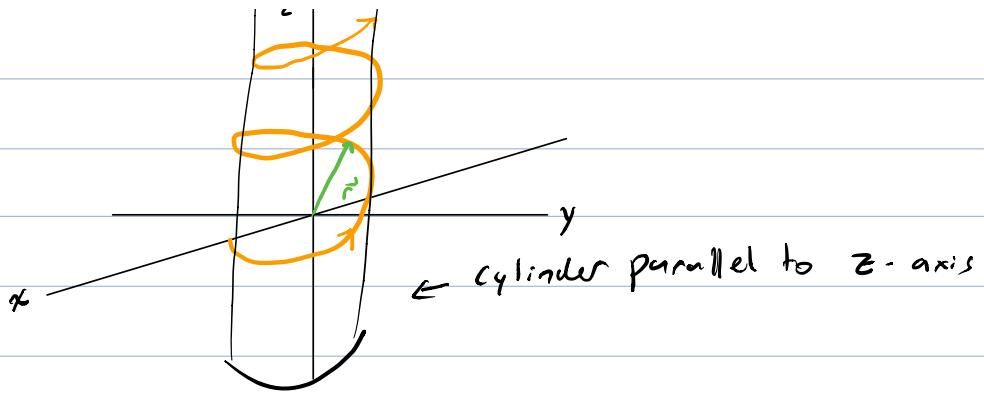
b) Identify the curve & give its cartesian equation

An ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$

(3) sketch the space curve $\vec{r}(t) = \langle \cos t, 2\sin t, t \rangle$

{ If $z=0$ hypothetically : $\langle \cos t, 2\sin t, 0 \rangle$





- ⑤ Show that $\vec{r}(t) = \langle \sin t \cos t, \sin^2 t, \cos t \rangle$ lies on the unit sphere.
 (show that $x(t)$, $y(t)$, $z(t)$ satisfy the equations of the surface)

- ⑥ Find the intersection of

$$\vec{r} = \langle t, t+1, t^2 - 2t + 1 \rangle$$

and the paraboloid $z = x^2 + y^2$

- (5) Unit Sphere: $x^2 + y^2 + z^2 = 1$

$$x(t) = \sin t \cos t \quad y(t) = \sin^2 t \quad z(t) = \cos t$$

$$(\sin t \cos t)^2 + (\sin^2 t)^2 + \cos^2 t \stackrel{?}{=} 1$$

$$\sin^2 t \cos^2 t + \sin^4 t + \cos^2 t \stackrel{?}{=} 1$$

$$\sin^2 t (1 - \sin^2 t) + \sin^4 t + (1 - \sin^2 t) \stackrel{?}{=} 1$$

$$\cancel{\sin^2 t - \sin^4 t} + \cancel{\sin^4 t} + 1 - \cancel{\sin^2 t} \stackrel{?}{=} 1$$

$$1 \stackrel{?}{=} 1$$

1 |
✓

$$⑥ (t^2 - 2t + 1) = t^2 + (t-1)^2$$

$$t^2 - 2t + 1 = t^2 + t^2 + 2t + 1$$

$$0 = t^2 + 4t$$

$$0 = t(t+4)$$

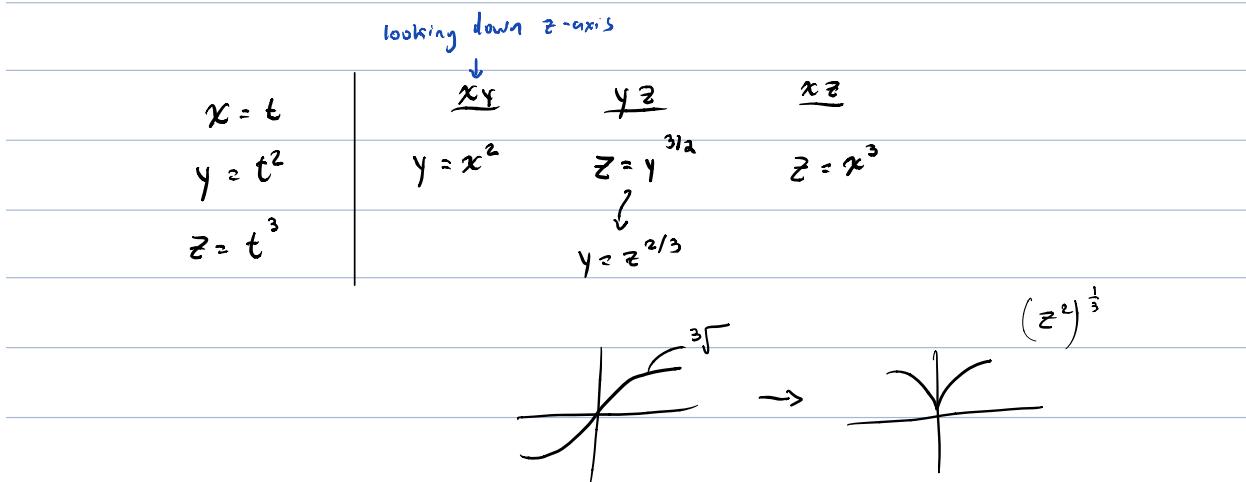
$$t=0, t=-4$$

$$\begin{cases} \vec{r}(0) = \langle 0, 1, 1 \rangle \\ r(-4) = \langle -4, -3, 25 \rangle \end{cases}$$

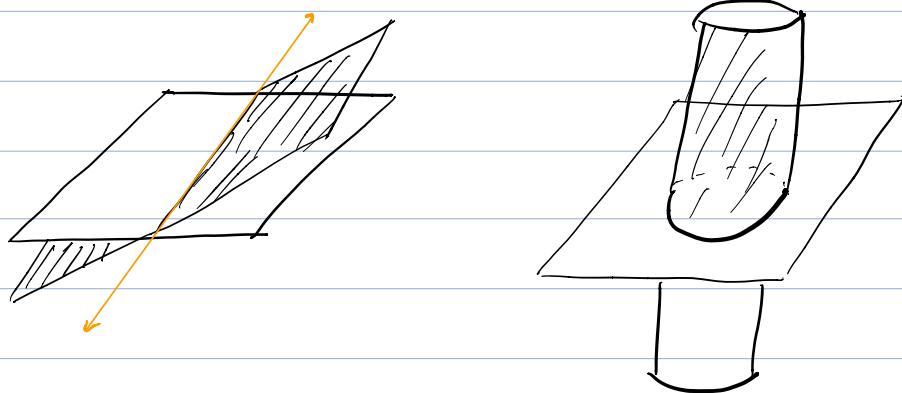
A note on projections:

The twisted cubic: $\vec{r}(t) = \langle t, t^2, t^3 \rangle$

The technique of projections



Intersection of surfaces



Two surfaces intersect in a curve

ex. Find a vector function that represents the curve of intersection of

$$\begin{cases} x^2 + y^2 = 1 & \leftarrow \text{circular cylinder parallel to } z \\ y + z = 2 & \leftarrow \text{plane parallel to } x \\ \hookrightarrow z = 2 - y \end{cases}$$

In \mathbb{R}^2 , we know that the circle can be parameterized as:

$$\vec{r}(t) = \langle \cos t, \sin t \rangle$$

$$\vec{r}_2(t) = \langle \cos t, \sin t, 2 - \sin t \rangle$$

Calculus of Vector Functions

Limits: $\lim_{t \rightarrow a} \vec{r}(t) \equiv \left\langle \lim_{t \rightarrow a} x(t), \lim_{t \rightarrow a} y(t), \lim_{t \rightarrow a} z(t) \right\rangle$

if these limits exist.

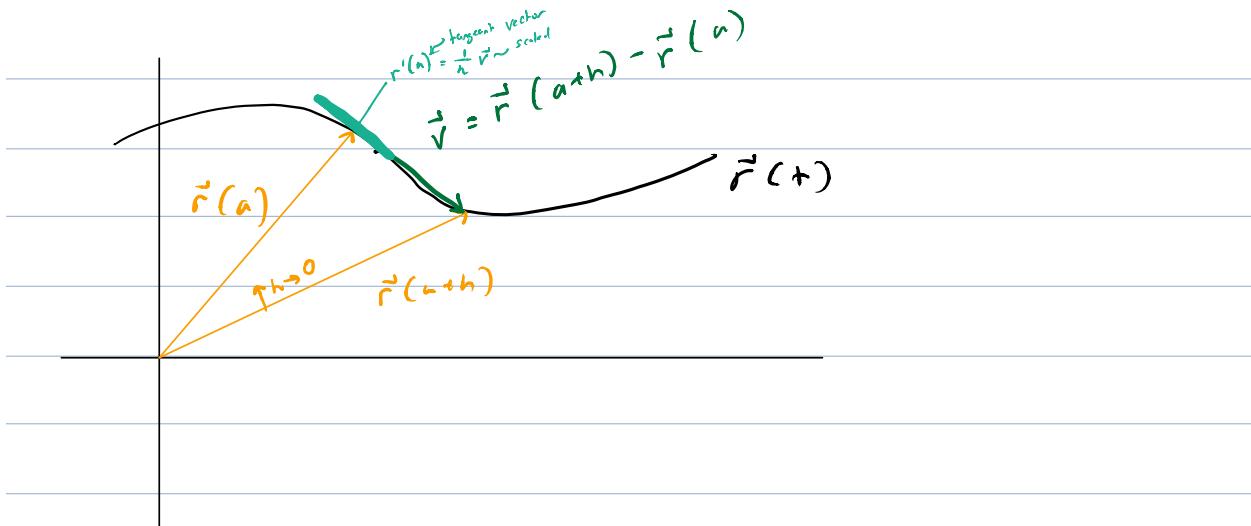
Continuity: A vector function $\vec{r}(t)$ is continuous at $t=a$ if each of its component functions is continuous at $t=a$.

The derivative of a vector function \rightarrow The tangent vector

Recall: From Cul I: $y = f(x)$ derivative at $x=a$

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

By analogy: $\vec{r}'(a) = \lim_{h \rightarrow 0} \frac{\vec{r}(a+h) - \vec{r}(a)}{h}$ ← vector subtraction



$\vec{r}'(a)$ is the tangent vector to $\vec{r}(t)$ at $t=a$

$$\vec{r}'(a) = \left\langle \lim_{h \rightarrow 0} \frac{x(a+h) - x(a)}{h}, \lim_{h \rightarrow 0} \frac{y(a+h) - y(a)}{h} \right\rangle,$$

$$\lim_{h \rightarrow 0} \frac{z(a+h) - z(a)}{h}$$

$= \langle x'(a), y'(a), z'(a) \rangle$ if these limits exist.

Unit tangent vector: $T(a) = \frac{\vec{r}'(a)}{\|\vec{r}'(a)\|}$ where $\|\vec{r}'(a)\| \neq 0$

ex. If $\vec{r}(t) = \langle t, t^2, t^3 \rangle$

Find $T(1)$

$$\vec{r}'(t) = \langle 1, 2t, 3t^2 \rangle$$

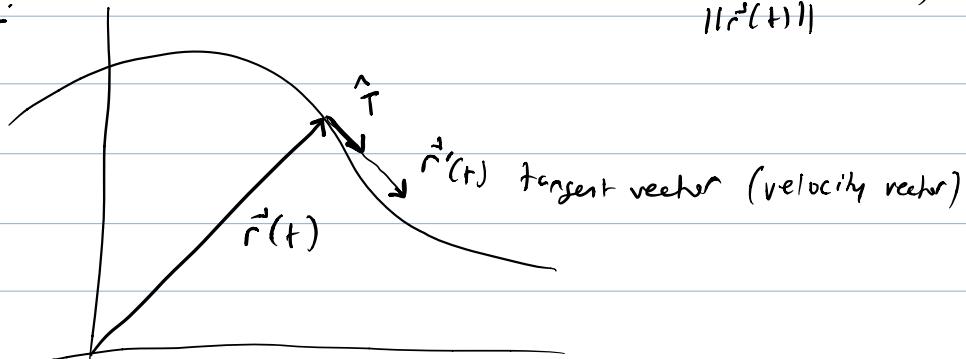
$$T(1) = \frac{\langle 1, 2, 3 \rangle}{\sqrt{14}} \rightarrow \text{unit tangent}$$

March 21

A5 due Monday

Recall:

$$\hat{r}(t) = \frac{1}{\|r'(t)\|} r'(t)$$



Note: $\|r'(t)\| \rightarrow \text{speed}$

ex. For you

a) Show that $P(-1, 2, -3)$ lies on the curve

$$\vec{r}(t) = t\hat{i} + (t^2+1)\hat{j} + (t^3+2t)\hat{k}$$

b) Find the angle between \hat{r} and \hat{r}' at P.

c) Find a vector equation for the tangent line to \hat{r} at P.

$$\begin{cases} t = -1 \\ t^2+1 = 2 \\ t^3+2t = -3 \end{cases} \text{ candidate, must check that other eqns are satisfied}$$

a) $\boxed{-1 = t}$

$$2 = t^2 + 1$$

$$-3 = t^3 + 2t$$

$$2 \stackrel{?}{=} (-1)^2 + 1 \quad t = -1$$

$$-3 \stackrel{?}{=} (-1)^3 + 2(-1) \quad t = -1$$

$$\boxed{= 2 \checkmark}$$

$$\boxed{= -1 - 2 = -3 \checkmark}$$

$$b) \vec{r}'(t) = \hat{i} + 2t\hat{j} + (3t^2 + 2)\hat{k}$$

$$\vec{r}(-1) = -\hat{i} + 2\hat{j} - 3\hat{k}$$

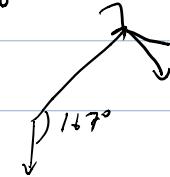
$$\vec{r}'(-1) = \hat{i} - 2\hat{j} + 5\hat{k}$$

$$\vec{r}(-1) \cdot \vec{r}'(-1) = -1 - 4 - 15$$

$$= -20 = \|\vec{r}(-1)\| \|\vec{r}'(-1)\| \cos \theta$$

$$\cos \theta = \frac{-20}{\sqrt{14} \sqrt{30}} = 167.4^\circ$$

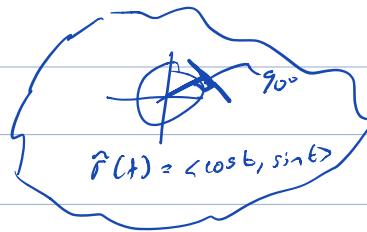
$$\phi = 180^\circ - 167.4 = 12.6^\circ$$



c)

$$\langle x, y, z \rangle = \langle -1, 2, -3 \rangle + t \langle 1, -2, 5 \rangle$$

Note: Nut = 90°

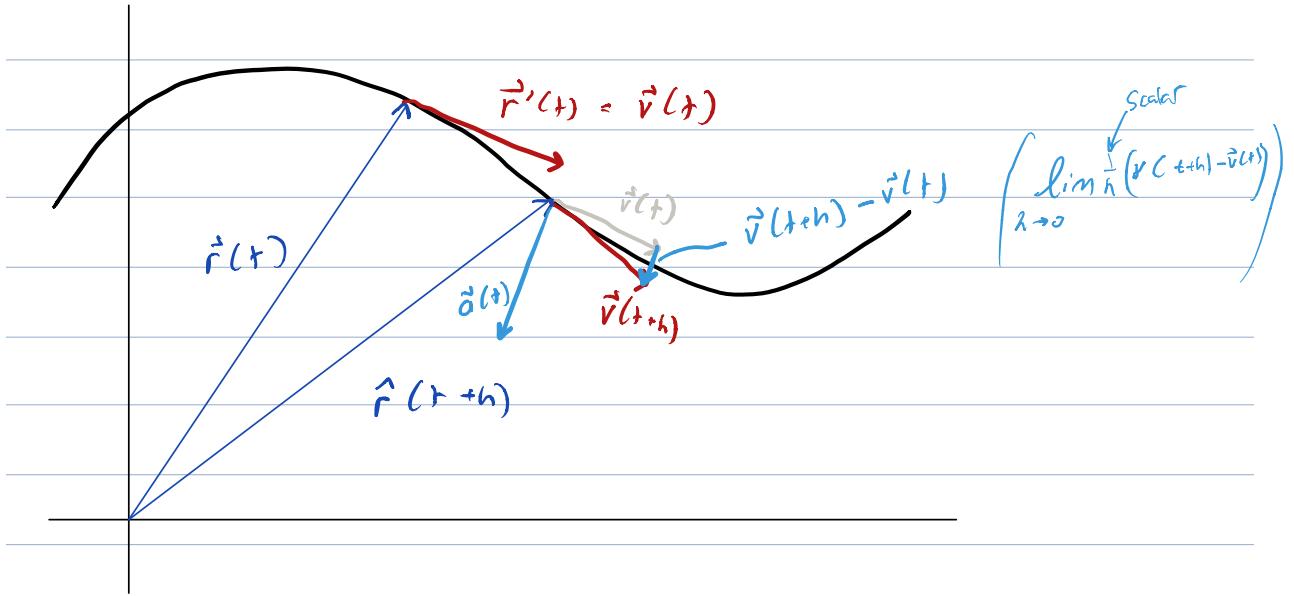


Motion in 3D-Space

If the position of a particle (travelling through space) at time t is given by $\vec{r}(t)$.

Then, the velocity of the particle is given by $\vec{v}(t) = \vec{r}'(t)$

and the acceleration $\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$



acceleration points in the direction of curvature

ex. ① A moving particle starts at position $\vec{r}_0 = \hat{i}$, with $\vec{v}_0 = \hat{i} - \hat{j} + \hat{k}$ and acceleration at time t given by $\vec{a}(t) = 4t\hat{i} + 6t\hat{j} + \hat{k}$.

Find the velocity and the position at any time t .

$$\left\{ \begin{array}{l} \vec{a}(t) = \langle a_1(t), a_2(t), a_3(t) \rangle \\ \int \vec{a}(t) dt = \langle \int a_1(t) dt, \int a_2(t) dt, \int a_3(t) dt \rangle \\ = \langle A_1(t) + C_1, A_2(t) + C_2, A_3(t) + C_3 \rangle \\ = \langle A_1(t), A_2(t), A_3(t) \rangle + \langle C_1, C_2, C_3 \rangle \\ = \vec{A}(t) + \vec{C} \end{array} \right.$$

$$\vec{v}(t) = \int \vec{a}(t) dt$$

$$= \int \langle 4t, 6t, 1 \rangle dt$$

$$= \langle 2t^2, 3t^2, t \rangle + \vec{C}$$

$$\vec{v}(0) = \langle 1, -1, 1 \rangle = 2(0)^2, 3(0)^2, 0 \rangle + \vec{C}$$

$$\vec{C} = \langle 1, -1, 1 \rangle$$

$$\therefore \vec{v}(t) = \langle 2t^2, 3t^2, t \rangle + \langle 1, -1, 1 \rangle = \langle 2t^2+1, 3t^2-1, t+1 \rangle$$

$$\vec{x}(t) = \int \langle 2t^2+1, 3t^2-1, t+1 \rangle dt$$

$$= \langle \frac{2}{3}t^3 + t, t^3 - t, \frac{1}{2}t^2 + t \rangle + \vec{D}$$

$$\vec{x}(0) = \langle 1, 0, 0 \rangle = \langle 0, 0, 0 \rangle + \vec{D}$$

$$\begin{aligned}\vec{x}(t) &= \langle \frac{2}{3}t^3 + t, t^3 - t, \frac{1}{2}t^2 + t \rangle + \langle 1, 0, 0 \rangle \\ &= \langle \frac{2}{3}t^3 + t + 1, t^3 - t, \frac{1}{2}t^2 + t \rangle\end{aligned}$$

(2) A particle moves with acceleration $\vec{a}(t) = \langle t, e^t, e^{-t} \rangle$
and $\vec{v}_0 = \vec{k}$.

Find its speed at $t=1$.

$$\vec{v}(t) = \int \langle t, e^t, e^{-t} \rangle dt$$

$$\vec{v}(t) = \left\langle \frac{1}{2}t^2, e^t, -e^{-t} \right\rangle + \vec{c}$$

$$\vec{v}(0) = \langle 0, 0, 1 \rangle = \langle 0, 1, -1 \rangle + \vec{c}$$

$$\vec{c} = \langle 0, -1, -2 \rangle$$

$$\therefore \vec{v}(t) = \left\langle \frac{1}{2}t^2, e^t, -e^{-t} \right\rangle + \langle 0, -1, 2 \rangle$$

$$\vec{v}(1) = \left\langle \frac{1}{2}, e - 1, -\frac{1}{e} + 2 \right\rangle$$

$$\|\vec{v}(1)\| = 2.42 \text{ time } v/\text{distn}$$

Derivative Rules

$$\textcircled{1} \quad \frac{d}{dt} \vec{c} = \vec{0}$$

$$\textcircled{2} \quad \frac{d}{dt} (\vec{u}(t) \pm \vec{v}(t)) = \vec{u}'(t) \pm \vec{v}'(t)$$

$$\textcircled{3} \quad i) \quad \frac{d}{dt} (c \vec{u}(t))$$

$$ii) \quad \frac{d}{dt} \left(\underset{\substack{\downarrow \\ \text{Scalar fn}}}{\alpha(t)} \vec{u}(t) \right)$$

$$\text{ex. } \langle t^2, t^4 - t^2, 1 \rangle = t^2 \langle 1, t^2 - 1, \frac{1}{t^2} \rangle$$

$$\textcircled{4} \quad \frac{d}{dt} (\vec{u}(t) \cdot \vec{v}(t)) = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \vec{v}'(t)$$

$$\textcircled{5} \quad \frac{d}{dt} (\vec{u}(t) \times \vec{v}(t)) = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$$

$$\textcircled{6} \text{ Chain rule } \frac{d}{dt} (\vec{u}(\alpha(t))) = \vec{u}'(\alpha(t)) \alpha'(t)$$

March 23

Derivative Rule

$$[\vec{u}(t) \cdot \vec{v}(t)]' = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$$

Proof: Let $\vec{u}(t) = \langle u_1(t), u_2(t), u_3(t) \rangle$
 (in \mathbb{R}^3) $\vec{v}(t) = \langle v_1(t), v_2(t), v_3(t) \rangle$

$$\begin{aligned} [\vec{u}(t) \cdot \vec{v}(t)]' &= (u_1(t)v_1(t) + u_2(t)v_2(t) + u_3(t)v_3(t))' \\ &= (u_1(t)v_1(t))' + (u_2(t)v_2(t))' + (u_3(t)v_3(t))' \\ &= u_1'(t)v_1(t) + u_1(t)v_1'(t) + u_2'(t)v_2(t) + u_2(t)v_2'(t) \\ &\quad u_3'(t)v_3(t) + u_3(t)v_3'(t) \end{aligned}$$

$$\begin{aligned} &= u_1'(t)v_1(t) + u_2'(t)v_2(t) + u_3'(t)v_3(t) \\ &\quad + u_1(t)v_1'(t) + u_2(t)v_2'(t) + u_3(t)v_3'(t) \end{aligned}$$

$$= \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$$

Result #1

If $\vec{r}(t)$ is a curve with constant magnitude of \vec{r} , then
 $\vec{r}(t)$ and $\vec{r}'(t)$ are perpendicular.

Show $\vec{r} \cdot \vec{r}' = 0$

Proof: $\|\vec{r}(t)\| = k$

$$\|\vec{r}(t)\|^2 = k^2$$
$$(\vec{r}(t) \cdot \vec{r}(t))' = (k^2)'$$

$$\vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t) = 0$$

$$d\vec{r}'(t) \cdot \vec{r}(t) = 0$$

$$\vec{r}'(t) \cdot \vec{r}(t) = 0$$

$$\therefore \vec{r}(t) \perp \vec{r}'(t)$$

Result #2

$$\text{If } \vec{r}(t) \neq \vec{0}, \text{ then } \|\vec{r}(t)\|' \stackrel{\text{?}}{=} \frac{1}{\|\vec{r}(t)\|} \vec{r}(t) \cdot \vec{r}'(t)$$

Let $\|\vec{r}(t)\|$

$$(\|\vec{r}(t)\|^2)' = (\vec{r}(t) \cdot \vec{r}(t))' = (k^2)'$$

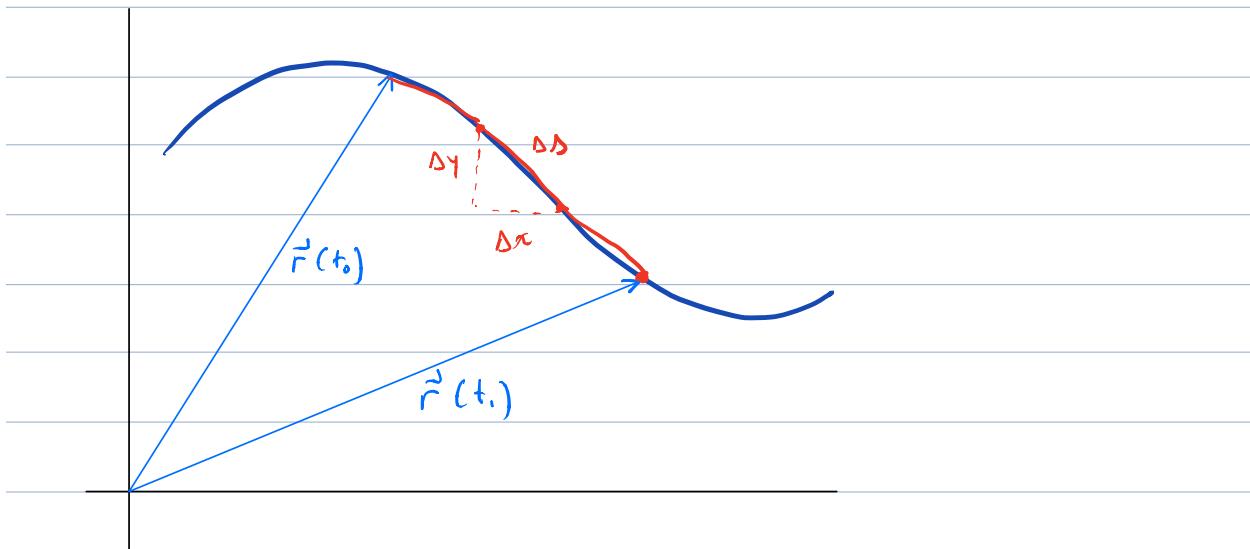
$$= \vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t)$$

$$\cancel{\|\vec{r}(t)\|} (\|\vec{r}(t)\|)' = \cancel{\vec{r}'(t) \cdot \vec{r}(t)}$$

$$\|\vec{r}(t)\|' = \frac{1}{\|\vec{r}(t)\|} \vec{r}'(t) \cdot \vec{r}(t)$$

Arc Length

Idea: Approximate a curve by line segments



$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2} \cdot \frac{\Delta t}{\Delta t}$$

$$= \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \Delta t$$

Let n be ∞ segments

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

as $n \rightarrow \infty$

$$\Delta t \rightarrow dt \quad \Delta y \rightarrow dy$$

$$= \sqrt{x'(t)^2 + y'(t)^2} dt$$

$$\Delta x \rightarrow dx \quad \Delta s \rightarrow ds$$

$$= \|r'(t)\| dt$$

$r(t) = \langle x(t), y(t) \rangle$
 $r'(t) = \langle x'(t), y'(t) \rangle$
 $\|r'(t)\| = \sqrt{x'(t)^2 + y'(t)^2}$

$$S = \int_{t_0}^{t_1} \|\vec{r}'(t)\| dt$$

Alternate way of thinking: motion

$\vec{r}(t)$ - position

$\|\vec{r}'(t)\|$ - speed

$\int \|\vec{r}'(t)\| dt$ - distance along path

ex. ① Let $\vec{r}(t) = \langle \cos t, \sin t, t \rangle \rightarrow \text{helix}$

Find the length of the helix from $t=0$ to $t=\pi$

(A particle is travelling along the helix. Find the total distance it travels from $t=0$ to $t=\pi$)

$$\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$\|\vec{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$$

$$S = \int_0^{\pi} \sqrt{2} dt = \sqrt{2} t \Big|_0^{\pi} = \sqrt{2} \pi$$

$$\textcircled{2} \quad \vec{r}(t) = \frac{1}{2} t^2 \hat{i} + \frac{2\sqrt{2}}{3} t^{3/2} \hat{j} + t \hat{k}$$

Find the length on $[1, 4]$

$$\vec{r}'(t) = \langle t, \sqrt{2} \sqrt{t}, 1 \rangle$$

$$\|\vec{r}'(t)\| = \sqrt{t^2 + 2t + 1}$$

$$= \sqrt{(t+1)^2}$$

$$= t+1$$

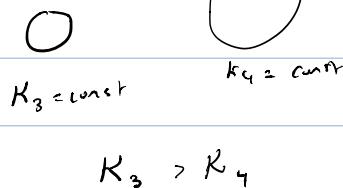
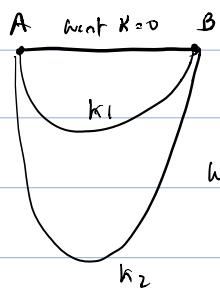
$$S = \int_1^4 (t+1) dt$$

$$= \left(\frac{t^2}{2} + t \right) \Big|_1^4 = 8 + 4 - \frac{1}{2} - 1 = \frac{21}{2}$$

Curvature

→ how tightly a curve changes direction over a unit distance

K - Krüppen



$$K_3 > K_4$$

The Arclength Function

$$S(t) = \int_0^t \|\vec{r}(u)\| du$$

dummy variable

A function which measures the arclength from $t=0$ to any input value along a given curve $\vec{r}(t)$

ex. a) Find $S(t)$ for $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

$$\vec{r}(u) = \langle \cos u, \sin u, u \rangle$$

$$\vec{r}'(u) = \langle -\sin u, \cos u, 1 \rangle$$

$$\|\vec{r}'(u)\| = \sqrt{2}$$

$$S(t) = \int_0^t \sqrt{2} du$$

$$= \sqrt{2} u \Big|_0^t$$

$$\lambda = \sqrt{2} t$$

b) Reparameterize the helix in terms of arclength

Change from t to λ

$$\lambda = \sqrt{2} t \Rightarrow t = \frac{1}{\sqrt{2}} \lambda = t(\lambda)$$

$$\vec{r}_2(\lambda) = \vec{r}(t(\lambda))$$

$$= \langle \cos\left(\frac{1}{\sqrt{2}} \lambda\right), \sin\left(\frac{1}{\sqrt{2}} \lambda\right), \frac{1}{\sqrt{2}} \lambda \rangle$$

 the helix with distance instead of time as a parameter

ex. Let $\vec{r} = \langle t, 1+2t, 3-2t \rangle$. Reparameterize \vec{r} in terms of arclength

$$\vec{r}' = \langle 1, 2, -2 \rangle$$

$$\|\vec{r}'\| = \sqrt{9} = 3$$

$$s(t) = \int_0^t 3 du$$

$$\Delta = 3t$$

$$\therefore t = \frac{\Delta}{3}$$

$$r(\Delta) = \left\langle \frac{\Delta}{3}, 1 + \frac{2}{3}\Delta, 3 - \frac{2}{3}\Delta \right\rangle$$

March 26

Recall: $s(t) = \int_0^t \|\vec{r}'(u)\| du \rightarrow$ arc length func

$$\frac{ds}{dt} = \|\vec{r}'(u)\| \quad (\text{FTC})$$

Curvature

Def'n: The curvature K of a vector curve is defined as

$$K = \left\| \frac{d\hat{T}}{ds} \right\| \text{ ~as scalar}$$

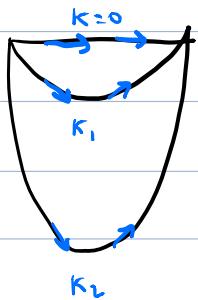
Unpacking

\hat{T} - unit tangent vector

\hookrightarrow magnitude = constant

$\hat{T}(s)$ parametrized in terms $\frac{d\hat{T}}{ds}$ - the instantaneous change in \hat{T} with respect to s .

Since the magnitude of \hat{T} is constant, any change in \hat{T} comes from a change in direction.



K_2 has a bigger change in the vector components than K_1

Unfortunately, calculating $\left\| \frac{d\hat{T}}{ds} \right\|$ directly is painful.

Given $\vec{r}(t)$, steps:

1. Reparameterize $s(t) = \int_0^t \|\vec{r}'(u)\| du \Rightarrow \vec{r}(s)$ often very difficult
2. Find $\hat{T}(s) = \frac{1}{\|\vec{r}'(s)\|} \vec{r}'(s)$
3. $\frac{d\hat{T}}{ds}$
4. $\left\| \frac{d\hat{T}}{ds} \right\|$

To avoid this heartache, we observe:

$$K = \left\| \frac{d\hat{T}}{ds} \right\| = \left\| \frac{d\hat{T}}{dt} \cdot \frac{dt}{ds} \right\| \quad \text{chain rule}$$

$$= \left\| \frac{\frac{d\hat{T}}{dt}}{\frac{ds}{dt}} \right\| = \frac{\left\| \hat{T}'(t) \right\|}{\left\| \vec{r}'(t) \right\|} \quad \text{related to the 2nd derivative}$$

ex. Find the curvature K of a straight line ($t \neq 0$)

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle = \langle x_0 + tv_1, y_0 + tv_2, z_0 + tv_3 \rangle$$

$$K = \frac{\|\hat{T}'(t)\|}{\|\vec{r}'(t)\|}$$

$$\vec{r}'(t) = \langle v_1, v_2, v_3 \rangle$$

$$\hat{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{1}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \langle v_1, v_2, v_3 \rangle$$

$$\hat{T}'(t) = \left(\frac{1}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \langle v_1, v_2, v_3 \rangle \right)'$$

$$= \langle 0, 0, 0 \rangle$$

$$K = \frac{\| \langle 0, 0, 0 \rangle \|}{\sqrt{v_x^2 + v_y^2 + v_z^2}} = \frac{0}{\sqrt{\dots}} = 0 \quad \checkmark$$

Ex. Show that the curvature of a circle with radius 'a' is constant.

$$\text{TR}^2: \vec{r}(t) = \langle a \cos t, a \sin t \rangle$$

$$\vec{r}'(t) = \langle -a \sin t, a \cos t \rangle$$

$$\hat{T}(t) = \frac{1}{a \sqrt{\sin^2 t + \cos^2 t}} \langle -a \sin t, a \cos t \rangle$$

$$= \langle -\sin t, \cos t \rangle$$

$$\hat{T}'(t) = \langle -\cos t, -\sin t \rangle$$

$$K = \frac{\sqrt{\cos^2 t + \sin^2 t}}{a \sqrt{\sin^2 t + \cos^2 t}} = \frac{1}{a} = C \quad \checkmark$$

where C is constant

Def'n: For any curve $\vec{r}(t)$, we define the radius of curvature

$$r_K = \frac{1}{K}$$

Also, we can show that:

$$K = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

ex. a) Apply this formula to find $K(t)$ for

$$\vec{r}(t) = \langle t, t^2, t^3 \rangle$$

$$\vec{r}'(t) = \langle 1, 2t, 3t^2 \rangle$$

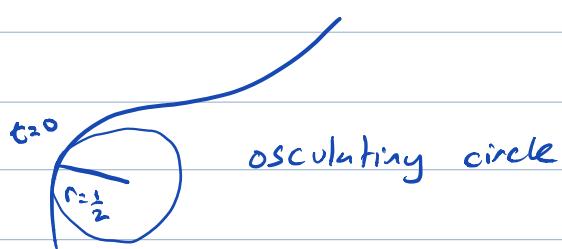
$$\vec{r}''(t) = \langle 0, 2, 6t \rangle$$

$$\vec{r}'(t) \times \vec{r}''(t) = \langle 6t^2, -6t, 2 \rangle = 2 \langle 3t^2, -3t, 1 \rangle$$

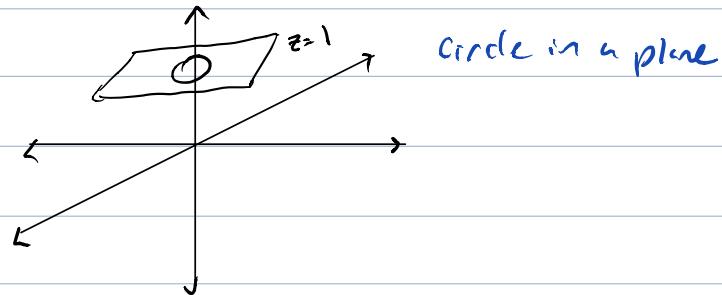
$$K(t) = \frac{2 \sqrt{9t^4 + 4t^2 + 1}}{(9t^4 + 4t^2 + 1)^{3/2}}$$

b) Find the radius of curvature at $t=0$

$$r_H = \frac{1}{K(0)} = \frac{1}{2}$$



ex. a) Find $K(0)$ for $\vec{r}(t) = \langle 3 \cos 2t, 3 \sin 2t, 1 \rangle$



$$\vec{r}'(t) = \langle -6 \sin 2t, 6 \cos 2t, 0 \rangle$$

$$\vec{r}''(t) = \langle -12 \cos 2t, -12 \sin 2t, 0 \rangle$$

don't need the whole thing for single pt

$$\begin{aligned}\vec{r}'(0) \times \vec{r}''(0) &= \langle 0, 6, 0 \rangle \times \langle -12, 0, 0 \rangle \\ &= \langle 0, 0, 72 \rangle\end{aligned}$$

$$K(0) = \frac{72}{6^3} = \frac{1}{3}$$

b) Find $K(t)$ for any t for $\vec{r}(t) = e^t \hat{i} + \sqrt{2} t \hat{j} + e^{-t} \hat{k}$

$$\vec{r}(t) = \langle e^t, \sqrt{2} t, e^{-t} \rangle$$

$$\vec{r}'(t) = \langle e^t, \sqrt{2}, -e^{-t} \rangle$$

$$\vec{r}''(t) = \langle e^t, 0, e^{-t} \rangle$$

$$\vec{r}'(t) \times \vec{r}''(t) = \langle \sqrt{2} e^{-t}, -2, -\sqrt{2} e^t \rangle$$

$$\|\vec{r}'(t) \times \vec{r}''(t)\| = \sqrt{2e^{-2t} + 4 + 2e^{2t}} = \sqrt{2} \sqrt{e^{2t} + 2 + e^{-2t}}$$

$$= \sqrt{2} (e^t + e^{-t})$$

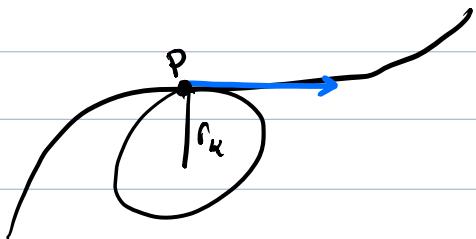
$$k(t) = \frac{\sqrt{2}(e^t + e^{-t})}{(e^t + e^{-t})^3} = \frac{\sqrt{2}}{(e^t + e^{-t})^2}$$

Note: In \mathbb{R}^2 , you will show that

$$k = \frac{|y''|}{(1+(y')^2)^{3/2}}$$

Osculating Circle

Def'n: The osculating circle for a curve at a point P is the circle passing through P which shares the tangent vector (has a parallel tangent vector) to the curve at P and which has radius $r_k = \frac{1}{k}$, where k is the curvature of the curve at P .



Ex. Find the osculating circle to $y = \sin x$ at $x = \frac{\pi}{2}$.

Sketch.

$$y' = \cos x \rightarrow y'\left(\frac{\pi}{2}\right) = 0$$

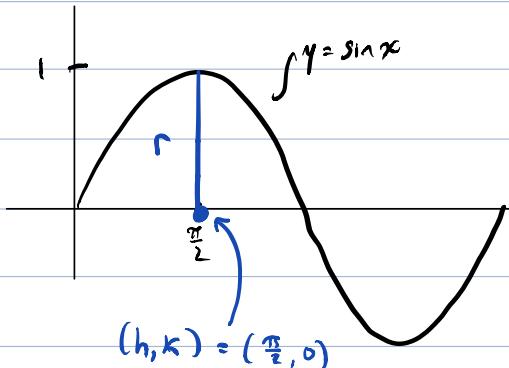
$$y'' = -\sin x \rightarrow y''\left(\frac{\pi}{2}\right) = -1$$

$$K = \frac{|y''|}{(1 + (y')^2)^{3/2}} = \frac{|-1|}{1} = 1$$

$$r_K = \frac{1}{K} = 1$$

$$(x-h)^2 + (y-k)^2 = r^2$$

$$\therefore (x-\frac{\pi}{2})^2 + y^2 = 1$$



ex. Find the eqn of the osculating circle at each pt. Sketch.

a) $y = 3x - x^3$ at local max

b) $y = x^2$ at $x=1$

(a) $y' = 3 - 3x^2$

$y' = 0 \Rightarrow 3 - 3x^2 = 0$

$x = 1, x = -1$



$$K = \frac{|y''|}{(1 + (y')^2)^{3/2}} = \frac{6x}{(1 - (3 - 3x^2)^2)^{3/2}}$$

$$\text{Recall: } K > 0 \quad K = \left\| \frac{d\hat{T}}{ds} \right\| = \frac{\|\hat{T}'(t)\|}{\|\hat{r}'(t)\|} = \frac{\|\hat{r}'(t) \times \hat{r}''(t)\|}{\|\hat{r}'(t)\|^3} \quad \text{March 28}$$

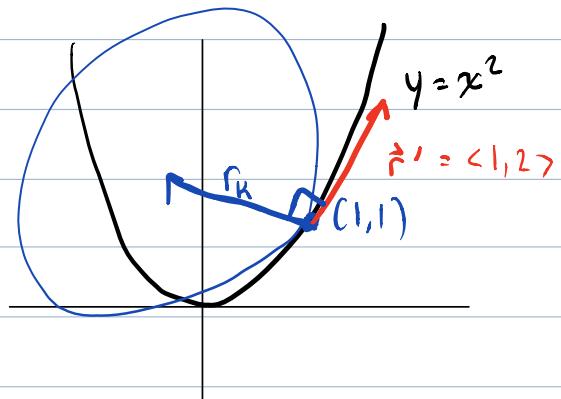
def'n

$$\text{In } \mathbb{R}^2: \quad y = f(x) \quad K = \frac{|y''|}{(1 + (y')^2)^{3/2}}$$

From last day

Find the eqn of the osculating circle to $y = x^2$ at $x=1$. ^{kissing}

Sketch.



$$y' = 2x \rightarrow y'(1) = 2$$

$$y'' = 2$$

$$K(1) = \frac{|2|}{(1 + 2^2)^{3/2}} = \frac{2}{5^{3/2}}$$

$$r_K = \frac{1}{k} = \frac{5^{3/2}}{2}$$

$\vec{r}(t) = \langle t, f(t) \rangle$
 $= \langle t, t^2 \rangle \rightarrow \text{at } t=1 \Rightarrow (1, 1)$

$r'(t) = \langle 1, 2t \rangle$
 $r'(1) = \langle 1, 2 \rangle$

Centre (h, k)

$$(x-h)^2 + (y-k)^2 = r^2$$

$\vec{n} \perp \vec{r}'$

$$\vec{n} = \langle 2, -1 \rangle \quad \text{checks } \vec{n}' \cdot \vec{n} = 0 \checkmark$$

But need left hand up

$$\vec{n} = \langle -2, 1 \rangle$$

$$\vec{n}_1 = \frac{r_K}{\|\vec{n}\|} \vec{n} \quad \text{scale so it's the right length}$$

$$= \frac{5^{3/2}/2}{5^{1/2}} \langle -2, 1 \rangle$$

$$= \frac{5}{2} \langle -2, 1 \rangle$$

$$= \langle -5, \frac{5}{2} \rangle$$

$$\langle h, k \rangle = \langle 1, 1 \rangle + \langle -5, \frac{5}{2} \rangle = \langle -4, \frac{7}{2} \rangle$$

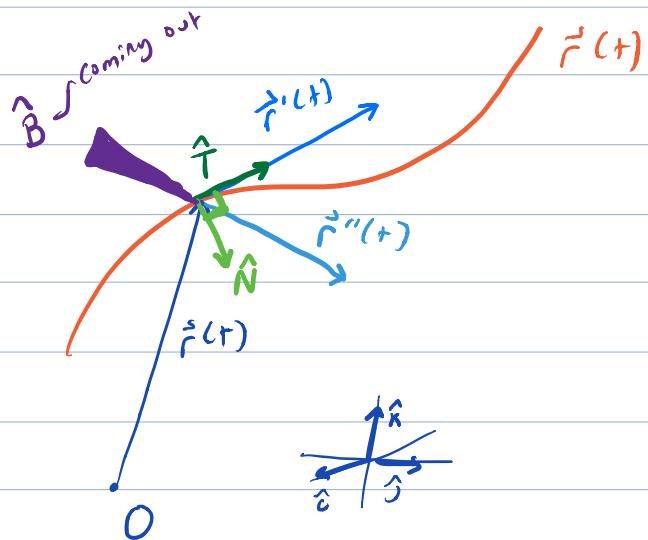
$$(x+4)^2 + (y - \frac{7}{2})^2 = \frac{125}{4}$$

TNB - Frame

\hat{T} - unit tangent

\hat{N} - unit normal

\hat{B} - unit binormal



$$\hat{T}(t) = \frac{1}{\|\vec{r}'(t)\|} \vec{r}'(t)$$

$$\hat{N}(t) = \frac{1}{\|\hat{T}'(t)\|} \hat{T}'(t)$$

$$\hat{B}(t) = T'(t) \times \hat{N}(t) \rightarrow \|\hat{B}\| = 1 \quad \text{recall} \quad \|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin \theta$$

$$\|\hat{B}(t)\| = \|\hat{T}(t) \times \hat{N}(t)\|$$

$$= \|\hat{T}(t)\| \|\hat{N}(t)\| \sin 90^\circ$$

$$= 1 \cdot 1 \cdot 1$$

3 related planes

osculating plane

contains \hat{T} , \hat{N}

normal plane

contains \hat{N} , \hat{B}

rectifying plane

contains \hat{B} , \hat{T}

$$\text{ex } \vec{r}(t) = \langle t, \frac{1}{2}t^2, t^2 \rangle$$

Find $\hat{T}(0)$, $\hat{N}(0)$, $\hat{B}(0)$ & the three planes

$$\vec{r}'(t) = \langle 1, t, 2t \rangle$$

$$\|\vec{r}'(t)\| = \sqrt{1 + 5t^2}$$

$$\hat{T}(t) = \frac{1}{\sqrt{1+5t^2}} \langle 1, t, 2t \rangle$$

$$\hat{T}'(t) = \left(\frac{1}{\sqrt{1+5t^2}} \right)' \langle 1, t, 2t \rangle + \frac{1}{\sqrt{1+5t^2}} (\langle 1, t, 2t \rangle)'$$

$$= -\frac{1}{2} (1+5t^2)^{-3/2} \cdot 10t \langle 1, t, 2t \rangle + (1+5t^2)^{-1/2} \langle 0, 1, 2 \rangle$$

No more derivatives \rightarrow plug in $t=0$

$$\hat{T}(0) = \frac{1}{\sqrt{1+0}} \langle 1, 0, 0 \rangle = \langle 1, 0, 0 \rangle$$

$$\hat{T}'(0) = \vec{0} + \langle 0, 1, 2 \rangle$$

$$\hat{N}(0) = \frac{1}{\sqrt{5}} \langle 0, 1, 2 \rangle$$

$$\begin{aligned}\hat{B}(0) &= \langle 1, 0, 0 \rangle \times \frac{1}{\sqrt{5}} \langle 0, 1, 2 \rangle \\ &= \frac{1}{\sqrt{5}} \langle 0, -2, 1 \rangle\end{aligned}$$

planes

From linear: $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

point-normal form: (x_0, y_0, z_0) point on plane

$\langle a, b, c \rangle = \vec{n}$ normal of plane

$$\text{point } \vec{r}(0) = \langle 0, 0, 0 \rangle \quad (x_0, y_0, z_0) = (0, 0, 0)$$

osculating	normal	rectifying
$\hat{n} \parallel \hat{B}$	$\hat{n} \parallel \hat{T}$	$\hat{n} \parallel \hat{N}$
$\vec{n} = \langle 0, -2, 1 \rangle$	$\vec{n} = \langle 1, 0, 0 \rangle$	$\vec{n} = \langle 0, 1, 2 \rangle$
$\therefore -2y + z = 0$	$\therefore x = 0$	$\therefore y + 2z = 0$

For each of the situations below:

- Find the vectors \hat{T} , \hat{N} and \hat{B} (the unit tangent, unit normal and unit binormal vectors) to \vec{r} at P .
- Find the general equations of the osculating, normal and rectifying planes to \vec{r} at P .
- Decompose the acceleration vector $\vec{a}(t) = \vec{r}''(t)$ into its tangential and normal components

1. $\vec{r}(t) = \langle t^2, 3, -1 + 2t \rangle$ at the point $P(1, 3, -3)$

a) $\vec{r}'(t) = \langle 2t, 0, 2 \rangle$

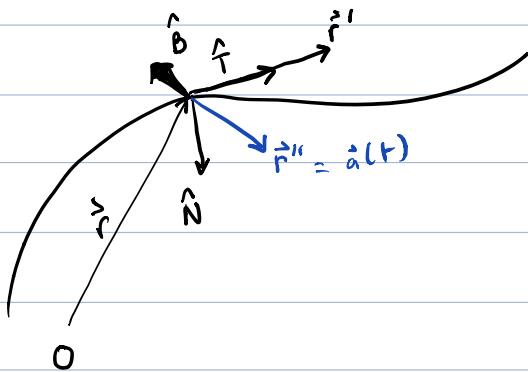
$$\hat{T}(t) = \frac{1}{\sqrt{4t^2+4}} \langle 2t, 0, 2 \rangle$$

$$\hat{T}'(t) = \left(\frac{1}{\sqrt{4t^2+4}} \right)' \langle 2t, 0, 2 \rangle + \frac{1}{\sqrt{4t^2+4}} \langle 2, 0, 0 \rangle$$

$$= (4t^2+4)^{-3/2} \cdot 8t \cdot \langle 2t, 0, 2 \rangle + (4t^2+4)^{-1/2} \langle 2, 0, 0 \rangle$$

$$\begin{cases} t^2 = 1 \\ 3 = 3 \\ -1 + 2t = -3 \end{cases} \rightsquigarrow t = -1$$

$$\hat{T}(-1) = \frac{1}{\sqrt{8}} \langle -2, 0, 2 \rangle$$



April 3

Goal: ① Decompose $\ddot{a}(t)$ into tangential component $a_T(t)$ and normal component $a_N(t)$

$$\text{Express } \ddot{a}(t) = a_T(t) \hat{T}(t) + a_N(t) \hat{N}(t)$$

can be done using projectors (but we're not going to...)

$$\textcircled{2} \text{ Prove } K = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

① Examine

$$\hat{T}(t) = \frac{1}{\|\vec{r}'(t)\|} \vec{r}'(t)$$

$$\left(\text{also } \hat{N}(t) = \frac{1}{\|\vec{T}'(t)\|} \vec{T}'(t) \right)$$

$$\Rightarrow r'(t) = \|\vec{r}'(t)\| \hat{T}(t)$$

$$\begin{aligned}\ddot{a}(t) &= r''(t) = \left(\|\vec{r}'(t)\| \hat{T}(t) \right)' \\ &= \|\vec{r}'(t)\|^2 \hat{T}(t) + \|\vec{r}'(t)\| \hat{T}'(t) \cdot \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} \cdot \frac{\|\vec{r}''(t)\|}{\|\vec{r}'(t)\|} \\ &\quad \text{product rule} \quad \hat{N} \quad K\end{aligned}$$

$$\text{Recall: } K = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

$$\text{Let } \gamma(t) = \text{speed} = \|\vec{r}'(t)\|$$

$$\ddot{a}(t) = \underbrace{\gamma'(t) \hat{T}(t)}_{a_T(t) = \gamma(t)} + \underbrace{\gamma^2 K \hat{N}(t)}_{a_N(t) = \gamma(t)^2 K}$$

② K formula

$$\vec{r}''(t) = \|\vec{r}'(t)\|^2 \hat{T}(t) + \|\vec{r}'(t)\|^2 K \hat{N}(t)$$

Examine $\vec{r}'(t) \times \vec{r}''(t) = \|\vec{r}'(t)\|^2 \left(\underbrace{\vec{r}'(t) \times \hat{T}(t)}_{\vec{r}'(t) \parallel \hat{T}(t)} \right) + \|\vec{r}'(t)\|^2 K \left(\vec{r}'(t) \times \hat{N}(t) \right)$

$$\|\vec{r}'(t) \times \vec{r}''(t)\| = \|\vec{r}'(t)\|^2 K \|\vec{r}'(t) \times \hat{N}(t)\|$$

$$= \|\vec{r}'(t)\|^2 K \|\vec{r}'(t)\| \|\hat{N}(t)\| \sin \theta \hookrightarrow 90^\circ$$

(unit vector)

$$K = \frac{\|\vec{r}''(t) \times \vec{r}'(t)\|}{\|\vec{r}'(t)\|^3}$$

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin \theta$$