

## Lecture 7 - Series and Convergence

objective: To identify whether or not a series,  $\sum_{k=1}^{\infty} a_k$ , converges to a fixed sum, or diverges to  $\infty$  (where  $a_k \geq 0$ )

ex. Does the series  $\sum_{k=0}^{\infty} 2 \left(\frac{1}{2}\right)^k$  converge or not

$$\sum_{k=0}^{\infty} 2 \left(\frac{1}{2}\right)^k = 2 + 1 + \frac{1}{4} + \frac{1}{8} + \dots$$

To be discussed...

ex. Does the series  $\sum_{k=1}^{\infty} \frac{1}{k}$  converge or not

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

To be discussed...

### Def

A series is convergent if its partial sums are convergent as

$$n \rightarrow \infty$$

i.e.  $\sum_{k=1}^{\infty} a_k$  converges if  $\underbrace{\sum_{k=1}^n a_k}_{S_n}$  converges as  $n \rightarrow \infty$

ex.

a) Find the  $n^{\text{th}}$  partial sum of  $\sum_{k=0}^n \left(\frac{1}{2}\right)^k$

b) Evaluate  $\lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^k$

$$S_0 = \sum_{k=0}^0 \left(\frac{1}{2}\right)^k = 1$$

$$S_1 = \sum_{k=0}^1 \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} = \frac{3}{2}$$

$$S_2 = \sum_{k=0}^2 \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

$$S_3 = \sum_{k=0}^3 \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$$

Based on our observations:

$$S_n = \frac{2 \cdot 2^n - 1}{2^n}$$

i.e.  $\sum_{k=0}^n \left(\frac{1}{2}\right)^k = \frac{2 \cdot 2^n - 1}{2^n}$

b)  $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$  converges if  $\lim_{n \rightarrow \infty} \sum_{k=0}^n \left(\frac{1}{2}\right)^k$  converges

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{2 \cdot 2^n - 1}{2^n}$$

$$= \lim_{n \rightarrow \infty} 2 - \frac{1}{2^n}$$

$$= 2$$

Since  $S_n \rightarrow 2$  as  $n \rightarrow \infty$

$$\text{then } \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2$$

$\therefore$  convergent series

ex. Consider the series  $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$

Does this series converge or not?

$$S_1 = \sum_{k=1}^1 \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{2}$$

$$S_2 = \sum_{k=1}^2 \left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3}$$

$$S_3 = \sum_{k=1}^3 \left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= 1 - \frac{1}{4}$$

$$S_n = 1 - \frac{1}{n+1}$$

as  $n \rightarrow \infty$ ,  $S_n \rightarrow 1$

$$\text{Conclusion: } \sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1$$

ex. À vous

Does  $\sum_{k=1}^{\infty} \ln\left(\frac{k}{k+1}\right)$  converge or diverge?

$$S_1 = \sum_{k=1}^1 \ln\left(\frac{k}{k+1}\right) = \ln\left(\frac{1}{2}\right)$$

$$S_2 = \sum_{k=1}^2 \ln\left(\frac{k}{k+1}\right) = \ln\left(\frac{1}{2}\right) + \ln\left(\frac{2}{3}\right) = \ln\left(\frac{2}{6}\right) = \ln\left(\frac{1}{3}\right)$$

$$S_3 = \sum_{k=1}^3 \ln\left(\frac{k}{k+1}\right) = \ln\left(\frac{1}{2}\right) + \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3}{4}\right) = \ln\left(\frac{1}{4}\right)$$

$$\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4}$$

$$S_n = \ln\left(\frac{1}{n+1}\right)$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \ln\left(\frac{1}{n+1}\right) \quad \left( = \ln\left(0^+\right) \right)$$

$$\lim_{n \rightarrow \infty} [\ln 1 - \ln(n+1)] = -\infty \quad \text{Diverges}$$

ex. Evaluate, if it exists, the sum of the following  
telescoping (collapsing) series.  
↑  
terms cancel

$$\sum_{k=2}^{\infty} \frac{2}{k^2 - 1}$$

$$= \sum_{k=2}^{\infty} \frac{2}{(k-1)(k+1)} \rightsquigarrow \frac{2}{(k-1)(k+1)} = \frac{A}{k-1} + \frac{B}{k+1}$$

$$2 = A(k+1) + B(k-1)$$

$$k=1 : 2 = -2B$$

$$B = -1$$

$$k=1 : 2 = 2A$$

$$A = 1$$

$$= \sum_{k=2}^{\infty} \left[ \frac{1}{k-1} - \frac{1}{k+1} \right]$$

$$S_2 = \frac{1}{1} - \frac{1}{3} = 1 - \frac{1}{3}$$

$$S_3 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right)$$

2 terms before (so notes arise that 1 &  $\frac{1}{2}$  won't cancel)

$$S_4 = \left(1 - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{5}\right) = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}$$

$$S_5 = \left(1 - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6}\right)$$

$$= 1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}$$

$$S_n = 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$$

$$\text{as } n \rightarrow \infty, S_n \rightarrow 1 + \frac{1}{2}$$

So,

$$\sum_{k=2}^{\infty} \frac{2}{k^2-1} = \frac{3}{2}$$

April 12

### Geometric Series

A geometric series is a series whose form satisfies

$$\sum_{k=0}^{\infty} ar^k \quad \text{where } a \text{ is a constant (not } a_k, \text{ so nothing)} \\ r \text{ is the series' ratio}$$

ex. Show that the following are all geometric series.

$$4 + 2 + \frac{1}{2} + \frac{1}{4} + \dots = \sum_{k=0}^{\infty} 4 \left(\frac{1}{2}\right)^k$$

a)  $6 - 4 + \frac{8}{3} - \frac{16}{9}$

b)  $1 + \frac{5}{4} + \frac{25}{16} + \frac{125}{64} + \frac{625}{256} + \dots$

$$= \sum_{k=0}^{\infty} 1 \left(\frac{5}{4}\right)^k$$

## Partial sum of a geometric series

Consider the geometric series:

$$\sum_{k=0}^{\infty} ar^k$$

Let  $S_n$  represent the partial sum  $\sum_{k=0}^n ar^k$

$$S_n = a + ar + ar^2 + \dots + ar^{n-1} + ar^n$$

$$S_n = ar + ar^2 + ar^3 + \dots + ar^n + ar^{n+1}$$

$$S_n - S_n = a - ar^{n+1}$$

$$= a(r^{n+1} - 1)$$

$$n(r-1) = a(r^{n+1} - 1)$$

$$S_n = a \left[ \frac{r^{n+1} - 1}{r - 1} \right] \quad \text{or} \quad S_n = a \left[ \frac{1 - r^{n+1}}{1 - r} \right]$$

If  $r > 1$ ,  $\lim_{n \rightarrow \infty} S_n$  diverges  $\left( = \frac{\infty}{\text{finite}} \right)$

If  $0 < r < 1$ ,  $\lim_{n \rightarrow \infty} S_n = a \cdot \frac{1 - \frac{1}{\infty}}{1 - r} \approx a \cdot \frac{1}{1 - r}$

If  $r = 1$ ,  $\lim_{n \rightarrow \infty} S_n$  diverges unless  $a = 0$

More generally, (including negative numbers)

$$\sum_{k=0}^{\infty} ar^k \xrightarrow{r \neq 0} \frac{a}{1-r} \text{ if } |r| < 1$$

diverges if  $|r| \geq 1$

ex. Compute

$$\sum_{k=0}^{10} 4\left(\frac{2}{3}\right)^k = 4 \left[ \frac{1 - \left(\frac{2}{3}\right)^{11}}{1 - \frac{2}{3}} \right]$$

$$= 11.861268$$

$$\sum_{k=0}^{\infty} 4\left(\frac{2}{3}\right)^k = \frac{4}{1 - \frac{2}{3}} = \frac{4}{\frac{1}{3}} = 12$$

$$\underline{\text{ex.}} \quad \sum_{k=0}^{24} 500(1.005)^k = 500 \cdot \frac{1 - (1.005)^{24}}{1 - 1.005}$$

$$= 13270.56 \quad (\$700 \text{ more than } 500 \times 25 \text{ (without compound interest)})$$

$$\sum_{k=0}^{\infty} 500(1.005)^k \text{ diverges since } r = 1.005 > 1$$

$$\underline{\text{ex.}} \quad \sum_{k=4}^{\infty} 10\left(\frac{1}{2}\right)^k = \sum_{k=0}^{\infty} 10\left(\frac{1}{2}\right)^k - \sum_{k=0}^{3} 10\left(\frac{1}{2}\right)^k$$

$$\text{or} \quad \sum_{k=4}^{\infty} 10\left(\frac{1}{2}\right)^k = \frac{10}{16} + \frac{10}{16}\left(\frac{1}{2}\right) + \dots$$

$$= \sum_{j=0}^{\infty} \frac{1}{16} \left(\frac{1}{2}\right)^j$$

So far... in the world of convergence / divergence tests, we have:

① STD (stupid test for divergence)

② Integral Test

$$\int_1^{\infty} f(x) dx \quad \left\{ \begin{array}{l} \text{the behavior} \\ \text{is the same} \end{array} \right. \sum_{k=1}^{\infty} a_k$$

Ex Test the convergence of

$$\sum_{k=2}^{\infty} \frac{\ln k}{k^2}$$

The corresponding integral is  $\int_2^{\infty} \frac{\ln x}{x^2} dx$   
let  $u = \ln x$

$$\int_2^{\infty} \frac{u}{x^2} dx$$

$$du =$$

$$I = \int \ln x \cdot \frac{1}{x^2} dx$$

$$f = \ln x \quad g' = x^{-2}$$

$$f' = \frac{1}{x} \quad g = -\frac{1}{x}$$

$$I = -\frac{\ln x}{x} - \int \left( \frac{1}{x} \right) \left( -\frac{1}{x} \right) dx$$

$$\frac{-\ln x}{x} + \int \frac{1}{x^2} dx$$

$$\frac{-\ln x}{x} - \frac{1}{x} \quad (\text{Don't need } +C, \text{ need an antiderivative})$$

$$= -\frac{\ln x + 1}{x}$$

$$\frac{\ln x}{x^2} dx$$

$$\lim_{R \rightarrow \infty} \int_2^R \frac{\ln x}{x^2} dx$$

$$\lim_{R \rightarrow \infty} \left[ -\frac{\ln x + 1}{x} \right]_2^R$$

$$\lim_{R \rightarrow \infty} \left[ -\frac{\ln R + 1}{R} + \frac{\ln 2 + 1}{2} \right]$$

0  
 (from infinity)  
 R  
 0

$$= 0 + \frac{\ln 2}{2} + 1$$

Conclusion: Since  $\int_2^\infty \frac{\ln x}{x^2} dx$  converges (to  $\frac{\ln 2}{2} + 1$ ),

then  $\sum_{k=2}^\infty \frac{\ln k}{k^2}$  must also converge

ex. A vars

Test the convergence of all the following series.

$$a) \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

$$c) \sum_{k=1}^{\infty} \frac{1}{k}$$

$$b) \sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$d) \sum_{k=1}^{\infty} \frac{1}{k^3}$$

Reminder: the p-integral  $\int_1^{\infty} \frac{1}{x^p} dx$

diverges whenever  $0 < p \leq 1$ , but converges whenever  $p > 1$

a)  $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$  diverges (by Integral Test)

b)  $\sum_{k=1}^{\infty} \frac{1}{k^2}$  converges ( $p = 2$ )

c)  $\sum_{k=1}^{\infty} \frac{1}{k}$  diverges ( $p = 1$ )

d)  $\sum_{k=1}^{\infty} \frac{1}{k^3}$  converges ( $p = 3$ )

Remark

p-series is any series of form

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$
 and ① diverges if  $0 < p \leq 1$

② converges if  $p > 1$

## Direct Comparison Test

Let  $\sum_{k=1}^{\infty} a_k$  and  $\sum_{k=1}^{\infty} b_k$  represent two (positive term) series.

① If  $a_k \geq b_k$  for all  $k$  and  $\sum_{k=1}^{\infty} b_k$  diverges,  
then  $\sum_{k=1}^{\infty} a_k$  will diverge as well.

② If  $a_k \leq b_k$  for all  $k$  and  $\sum_{k=1}^{\infty} b_k$  converges,  
then  $\sum_{k=1}^{\infty} a_k$  will converge as well.

$$\underline{\text{ex.}} \quad \sum_{k=1}^{\infty} \frac{1}{k^2 + 7} \quad \text{Test its convergence/divergence}$$

$$\frac{1}{k^2 + 7} \leq \frac{1}{k^2} \text{ for all } k$$

$\sum_{k=1}^{\infty} \frac{1}{k^2}$  is a p-series for which  $p > 1$  and therefore converges.

By the comparison test,  $\sum_{k=1}^{\infty} \frac{1}{k^2 + k}$  must also converge

Ex. Test the following series for convergence

$$\sum_{j=3}^{\infty} \frac{(1 + 2\cos^2 j)}{5^j}$$

$$\frac{1+2\cos^2 j}{5^j} \geq \sum_{\text{geometric}} \frac{1}{5^j} = \left(\frac{1}{5}\right)^j$$

but... that doesn't tell us anything  
since  $\left(\frac{1}{5}\right)^j$  converges

So let's overestimate:

$$\frac{1+2\cos^2 j}{5^j} \geq \frac{3}{5^j} = 3\left(\frac{1}{5}\right)^j$$

$\sum_{j=3}^{\infty} 3\left(\frac{1}{5}\right)^j$  is a geometric series that converges since  $-1 \leq r = \frac{1}{5} \leq 1$

By the comparison test,  $\sum_{j=3}^{\infty} \frac{1+2\cos^2 j}{5^j}$  converges

ex.  $\sum_{k=2}^{\infty} \frac{\ln k}{k^2}$

$$\frac{\ln k}{k^2} \leq \frac{k}{k^2} = \frac{1}{k}$$

doesn't give us anything

$$\ln k \ll \ln k^p \rightarrow \ln k \ll k^{1/2}$$

$$\rightarrow \frac{\ln k}{k^2} \ll \frac{k^{1/2}}{k^2} = \frac{1}{k^{3/2}}$$

$\sum_{k=2}^{\infty} \frac{1}{k^{3/2}}$  is a p series where  $p = 3/2$  and thus ( $p > 1$ )

converges

$\sum_{k=2}^{\infty} \frac{\ln k}{k^2}$  is smaller than  $\sum_{k=2}^{\infty} \frac{1}{k^{3/2}}$  and therefore will also converge

ex. A vars

Using tests for convergence seen so far, test the following series

a)  $\sum_{n=1}^{\infty} \cos(\gamma_n)$

$$\lim_{n \rightarrow \infty} \cos(\gamma_n) = \cos 0 = 1 \neq 0$$

The series diverges in virtue of the sharpid test for divergence

b)  $\sum_{k=1}^{\infty} \frac{2 + \cos(\gamma_k)}{k}$

useless comparison

$$\frac{1}{k} \leq \frac{2 + \cos \gamma_k}{k} \geq \left( \frac{3}{k} \right) \text{ diverges}$$

useful

$\sum_{k=1}^{\infty} \frac{1}{k}$  is a diverging p-series ( $p=1$ ), therefore  $\sum \frac{2 + \cos \gamma_k}{k}$  must also diverge

c)  $\sum_{q=1}^{\infty} \left( \frac{5 + \sin(q)}{3} \right)^q$

diverges (geometric)

$$\left( \frac{4}{3} \right)^q \leq \left( \frac{5 + \sin(q)}{3} \right)^q \geq \left( \frac{6}{3} \right)^q$$

$\sum_{q=1}^{\infty} \left(\frac{4}{3}\right)^q$  is a divergent geometric series where  $r = \frac{4}{3} > 1$

$\int_0^{\infty} \sum_{q=1}^{\infty} \left(\frac{5 + \sin(q)}{3}\right)^q$  will diverge by the DCT

April 24, 2017

Done so far...

STD  $\sum (a_k) \rightarrow$  if they don't go to 0, series diverges

Integral Test  $\int_1^{\infty} f(x) dx \sim \sum_{k=1}^{\infty} a_k$

$$\frac{3k^2}{5k^3} \times \frac{(1 + \frac{1}{3k})}{(1 + \frac{2}{5k} + \frac{1}{5k^3})}$$

Direct Comparison

$$1 + \frac{1}{3k} \underset{k \rightarrow \infty}{\circlearrowleft} 1 + \frac{2}{5k} + \frac{1}{5k^3}$$

$$\frac{1}{3k} \underset{k \rightarrow \infty}{\circlearrowleft} \frac{2k^2 + 1}{5k^3}$$

$$5k^3 \underset{k \rightarrow \infty}{\circlearrowleft} 3k(2k^2 + 1)$$

$$5k^3 \underset{k \rightarrow \infty}{\circlearrowleft} 6k^3 + 3k$$

$$0 < k^3 < 3k$$

Limit Comparison Test

ex.  $\sum_{k=1}^{\infty} \frac{3k^2 + k}{5k^3 + 2k - 1}$

so bottom > top  
so multiplying by term < 1

$$\text{so } \frac{3k^2}{5k^3} > \frac{3k^2 + k}{5k^3 + 2k - 1}$$

Let  $\sum_{k=1}^{\infty} a_k$  and  $\sum_{k=1}^{\infty} b_k$  represent two positive term series

such that  $\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = L$  (where  $L$  is finite and non-zero)

Then, the series have the same (diverging or converging) behaviour  
 (Hint: rationals + radicals)

$$\text{Ex, } \sum_{k=1}^{\infty} \frac{3k^2 + k}{5k^3 + 2k - 1}$$

$$\frac{3k^2 + k}{5k^3 + 2k - 1} \sim \frac{3k^2}{5k^3} \sim \frac{k^2}{k^3} \sim \frac{1}{k}$$

Let's compare, if we are allowed to,

$$\sum_{k=1}^{\infty} \frac{3k^2 + k}{5k^3 + 2k - 1} \quad \text{with} \quad \sum_{k=1}^{\infty} \frac{1}{k}$$

Allowed?

$$\lim_{k \rightarrow \infty} \frac{\frac{3k^2 + k}{5k^3 + 2k - 1}}{\frac{1}{k}}$$

$$= \lim_{k \rightarrow \infty} \frac{3k^3 + k^2}{5k^3 + 2k - 1}$$

$$= \lim_{k \rightarrow \infty} \frac{k^2 \left( 3 + \frac{1}{k} \right)}{k^3 \left( 5 + \frac{2}{k^2} + \frac{1}{k^3} \right)}$$

$$= \frac{3}{5}$$

$$\sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges (p-series, } p \leq 1)$$

By the limit comparison test,

$$\sum_{k=1}^{\infty} \frac{3k^2 + k}{5k^3 + 2k^2 + 1} \text{ will also diverge}$$

ex. À vous

$$\sum_{k=1}^{\infty} \frac{3k-1}{\sqrt{k^5 - 2k^2 + 17k}}$$

$$\frac{3k-1}{\sqrt{k^5 - 2k^2 + 17k}} \sim \frac{k}{\sqrt{k^5}} = \frac{k}{k^{5/2}} \sim \frac{1}{k^{3/2}}$$

Allowed?

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\frac{3k-1}{\sqrt{k^5 - 2k^2 + 17k}}}{\frac{1}{k^{3/2}}} &= \lim_{k \rightarrow \infty} \frac{3k-1}{\sqrt{k^5 - 2k^2 + 17k}} \cdot k^{3/2} \\ &= \lim_{k \rightarrow \infty} \frac{3k^{5/2} - k^{3/2}}{\sqrt{k^5 - 2k^2 + 17k}} \\ &\approx \lim_{k \rightarrow \infty} \frac{k^{5/2} \left( 3 - \frac{1}{k} \right)}{k^{5/2} \sqrt{1 - \frac{2}{k^3} + \frac{17}{k^4}}} = 3 \end{aligned}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \text{ is a convergent p-series } (p > 1)$$

By LCT,  $\sum_{k=1}^{\infty} \frac{3k-1}{\sqrt{k^5 - 2k^2 + 17k}}$  converges

$$\text{Ex. } \sum_{k=2}^{\infty} \frac{k \sin(\frac{1}{k})}{k^2 + 1}$$

Back to Calc 2  $\lim_{k \rightarrow \infty} \frac{\sin \frac{1}{k}}{\frac{1}{k}} = 1$

use  $\sin(\frac{1}{k}) \sim \frac{1}{k}$

$$\frac{k \sin(\frac{1}{k})}{k^2 + 1} \sim \frac{k \cdot \frac{1}{k}}{k^2 + 1} \sim \frac{1}{k^2 + 1} \sim \frac{1}{k^2}$$

Validation:  $\lim_{k \rightarrow \infty} \frac{\frac{k \sin(\frac{1}{k})}{k^2 + 1}}{\frac{1}{k^2}}$

$$= \lim_{k \rightarrow \infty} \frac{k^3 \sin \frac{1}{k}}{k^2 + 1}$$

$$= \lim_{k \rightarrow \infty} \frac{k^2}{k^2 + 1} \cdot \frac{\sin \frac{1}{k}}{\frac{1}{k}}$$

$$= \lim_{k \rightarrow \infty} \frac{k^2}{k^2(1 + \frac{1}{k^2})} \cdot \lim_{k \rightarrow \infty} \frac{\sin \frac{1}{k}}{\frac{1}{k}}$$

$$= 1$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \text{ converges (p-series } p > 1)$$

So  $\sum_{k=1}^{\infty} \frac{k \sin(\frac{1}{k})}{k^2 + 1}$  must also converge

April 26, 2017

Today ... Ratio and Root Tests

### Ratio Test

Consider a series  $\sum_{k=1}^{\infty} a_k$

① If  $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| < 1$ , the series converges

② If  $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| > 1$ , the series diverges

③ If  $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = 1$ , the test is inconclusive

When do we use the ratio test?

→ factorials are involved ( $K!$ )

→ geometries are involved ( $r^k$ )

→ combinations of the above with  $k^k$  or  $k^p$

ex Test the following series for convergence / divergence

a)  $\sum_{k=1}^{\infty} \frac{1}{k!}$        $a_k = \frac{1}{k!}$   
 $a_{k+1} = \frac{1}{(k+1)!}$

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}}$$

$$= \lim_{k \rightarrow \infty} \frac{k!}{(k+1)!}$$

$$= \lim_{k \rightarrow \infty} \frac{1 \cdot 2 \cdot 3 \cdots (k+1) \cdot k}{1 \cdot 2 \cdot 3 \cdots (k-1)k(k+1)}$$

$$= \lim_{k \rightarrow \infty} \frac{1}{k+1}$$

$$= 0 < 1$$

Conclusion  $\sum_{k=1}^{\infty} \frac{1}{k!}$  converges

$$\text{b)} \sum_{k=1}^{\infty} \frac{k^2}{2^k}$$

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \frac{\frac{(k+1)^2}{2^{k+1}}}{\frac{k^2}{2^k}}$$

$$= \lim_{k \rightarrow \infty} \frac{(k+1)^2}{2^{k+1}} \cdot \frac{2^k}{k^2}$$

$$\approx \lim_{k \rightarrow \infty} \frac{(k+1)^2}{2k^2}$$

$$= \lim_{k \rightarrow \infty} \frac{1}{2} \left( \frac{k+1}{k} \right)^2$$

$$\approx \lim_{k \rightarrow \infty} \frac{1}{2} \left( 1 + \frac{1}{k} \right)^2$$

$$= \frac{1}{2} < 1$$

Conclusion:  $\sum_{k=1}^{\infty} \frac{k^2}{2^k}$  converges

c)  $\sum_{j=2}^{\infty} \frac{(-1)^j 2^j}{(2j)!}$

$(2j+1)! = (2j+2)!$  not even necessary since abs.

val

just keep care  
of sign

$$\lim_{n \rightarrow \infty} \left| \frac{a_{kn}}{a_k} \right| = \lim_{j \rightarrow \infty} \left| \frac{(-1)^{j+1} 2^{j+1}}{(2j+2)!} \cdot \frac{(2j)!}{(-1)^j 2^j} \right|$$

$$= \lim_{j \rightarrow \infty} \left| 2(-1) \cdot \frac{1 \cdot 2 \cdot 3 \cdots 2(j-1) \cdot 2j}{1 \cdot 2 \cdot 3 \cdots 2(j+1)(2j+1)(2j+2)} \right|$$

$$= \lim_{j \rightarrow \infty} \frac{2}{(2j+1)(2j+2)} = 0 < 1$$

Therefore, the series

$\sum_{j=2}^{\infty} \frac{(-1)^j 2^j}{(2j)!}$  converges

ex. Mr. Nasty

Part 1 : Show that  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$

$$\text{Let } L = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \quad (1^\infty \text{ form is indeterminate})$$

$$\ln L = \lim_{n \rightarrow \infty} \ln \left(1 + \frac{1}{n}\right)^n \quad (\ln can be added inside)$$

$$= \lim_{n \rightarrow \infty} n \ln \left(1 + \frac{1}{n}\right) \quad (\infty \cdot 0)$$

$$= \lim_{n \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{n}\right)}{\frac{1}{n}}$$

$$\stackrel{\text{L'H}}{=} \lim_{n \rightarrow \infty} \frac{\frac{1}{1+\frac{1}{n}} \cdot \cancel{\left(-\frac{1}{n^2}\right)}}{-\cancel{\left(\frac{1}{n^2}\right)}}$$

$$= 1$$

$$\text{So } \ln L = 1$$

$$\rightarrow L = e^1$$

$$= e$$

Part 2 : Test the following series :

$$\sum_{k=1}^{\infty} \frac{n!}{n^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)}{(n+1)(n+1)^n}$$

$$= \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^n$$

$$= \lim_{n \rightarrow \infty} \left( \frac{1}{1 + \frac{1}{n}} \right)^n \quad \left( 1^{\text{any power}} = 1 \right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n}$$

$$= \frac{1}{e} < 1$$

Conclusion:  $\sum_{k=1}^{\infty} \frac{n!}{n^n}$  converges

### Root Test

$$\sum \sqrt[n]{a_n r^n} \text{ geometric}$$

Consider a series  $\sum_{n=1}^{\infty} a_n$

① If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$ , the series converges

② If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$ , the series diverges

③ If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$ , the test is inconclusive

Use the root test if all terms of the series have "n" as a common exponent

ex.  $\sum_{w=1}^{\infty} \left( \frac{2w+1}{5w+3} \right)^w$

$$\lim_{n \rightarrow \infty} \sqrt[w]{\left( \frac{2w+1}{5w+3} \right)^w}$$

$$= \lim_{w \rightarrow \infty} \frac{2w+1}{5w+3}$$

$$= \lim_{w \rightarrow \infty} \frac{w(2 + \frac{1}{w})}{w(5 + \frac{3}{w})}$$

$$= \frac{2}{5} < 1$$

By the root test, the series converges.

April 27, 2017

### Reminder on Taylor Polynomials

Let  $f$  represent an infinitely differentiable function at  $x=a$ . Then, the taylor polynomial of degree  $n$  associated to  $f$  near  $x=a$  is:

$$T_n(x) = f(a) + f'(a) \cdot (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots$$

$$+ \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$T(x)$  is a Taylor series when  $\lim_{n \rightarrow \infty} T_n(x)$

ex. Consider  $f(x) = e^x$  near  $x=0$

<u>n</u>	<u><math>f^{(n)}(x)</math></u>	<u><math>f^{(n)}(0)</math></u>	<u><math>f^{(n)}(0)/n!</math></u>
0	$e^x$	1	1
1	$e^x$	1	1
2	$e^x$	1	$\frac{1}{2!} = \frac{1}{2}$
3	$e^x$	1	$\frac{1}{3!} = \frac{1}{6}$

So,  $T_n(x)$ , the  $n^{\text{th}}$  order Taylor Polynomial for  $f(x) = e^x$

centered at  $x=0$  is...

$$T_n(x) = 1 + 1x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \dots + \frac{1}{n!} x^n$$

The taylor series expansion for  $f(x) = e^x$  when  $x=0$  is

$$T(x) = 1 + 1x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \dots$$

$$T(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

What values of  $x$  guarantees this series will converge?

Ratio test:  $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{\frac{1}{(k+1)!} x^{k+1}}{\frac{1}{k!} x^k} \right|$

$$= \lim_{k \rightarrow \infty} \left| \frac{1}{(k+1)!} x^{k+1} \cdot \frac{k!}{x^k} \right|$$
$$= \lim_{k \rightarrow \infty} \left| \frac{x}{k+1} \right| \quad \left( \frac{x^{k+1}}{x^k} = \frac{x^k \cdot x}{x^k} \right)$$

$= 0 < 1$  This series converges for all  $x$

Application:

According to our work, we can predict that

$$\sum_{k=0}^{\infty} \frac{1}{k!} = e^1 \quad \text{or} \quad e^x = \underbrace{\sum_{k=0}^{\infty} \frac{x^k}{k!}}_{T(x)}$$

check: (for  $x=1$ )

$$\sum_{k=0}^{\infty} \frac{1}{k!} = \underbrace{1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \dots}_{2.7166}$$

$$e = 2.71828$$

ex. Let  $f(x) = \sin x$

Find  $T(x)$ , the Taylor series expansion for  $f(x)$  around  $x=0$ .

<u>n</u>	<u><math>f^{(n)}(x)</math></u>	<u><math>f^{(n)}(0)</math></u>	<u><math>f^{(n)}(0)/n!</math></u>
0	$\sin x$	0	0
1	$\cos x$	1	1
2	$-\sin x$	0	$0$
3	$-\cos x$	-1	$-\frac{1}{3!} = -\frac{1}{6} x^3$
4	$\sin x$	0	0
5	$\cos x$	1	$\frac{1}{5!} = \frac{1}{120} x^5$

$$T(x) = 1x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \dots$$

$$T(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

Ratio test:  $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)^{k+1} x^{2(k+1)+1}}{(-1)^k x^{2k+1} (2(k+1)+1)!} \right|$

$$= \lim_{k \rightarrow \infty} \left| \frac{(-1)^{k+1} x^{2k+3}}{(2k+3)!} \times \frac{(2k+1)!}{(-1)^k \cdot x^{2k+1}} \right|$$

$$(2k+1)! = 1 \cdot 2 \cdot 3 \cdots (2k+1)$$

$$\begin{aligned} & \approx \lim_{k \rightarrow \infty} \left| \frac{x^2}{(2k+2)(2k+3)} \right| \\ & (2k+3)! = 1 \cdot 2 \cdot 3 \cdots (2k+1)(2k+2) \end{aligned}$$

$x \rightsquigarrow \text{constant}$

$$= \lim_{k \rightarrow \infty} \frac{x^k}{(2k+2)(2k+3)} = 0 < 1 \text{ for all } x$$

$\hookrightarrow_{k \rightarrow \infty}$

True or false?

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \cdot \underbrace{\frac{x^{2k+1}}{(2k+1)!}}_{T(x)} \text{ for all } x$$

ex. Let  $f(x) = \frac{1}{x}$

Find  $T(x)$ , The Taylor series expansion for  $f(x)$  around  $x=1$

<u>n</u>	<u><math>f^{(n)}(x)</math></u>	<u><math>f^{(n)}(1)</math></u>	<u><math>f^{(n)}(1)/n!</math></u>	
0	$\frac{1}{x}$	1	1	$(x-1)^0$
1	$-1/x^2$	-1	-1	$(x-1)^1$
2	$2/x^3$	2	1	$(x-1)^2$
3	$-6/x^4$	-6	-1	$(x-1)^3$
4	$24/x^5$	24	1	$(x-1)^4$

So,

$$T(x) = 1(x-1)^0 - 1(x-1)^1 + 1(x-1)^2 - 1(x-1)^3 + \dots$$

$$= \sum_{k=0}^{\infty} (-1)^k (x-1)^k = \sum_{k=0}^{\infty} [-1(x-1)]^k \text{ geometric}$$

This geometric series converges as long as

$$-1 < -1(x-1) < 1$$

$$1 > (x-1) > -1 \quad \rightarrow \quad 0 < x < 2$$

$$-1 < x-1 < 1$$

May 3, 2017

Reminder:  $T(x)$ , the Taylor series expansion around  $x=a$  associated to  $f(x)$  is obtained from:

$$T(x) = f(a) + \frac{f'(a)}{2!}(x-a) + \frac{f''(a)}{3!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

In sigma notation:  $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

ex.  $f(x) = \ln x$  centered at  $x=1$

<u>n</u>	<u><math>f^{(n)}(x)</math></u>	<u><math>f^{(n)}(1)</math></u>	<u><math>f^{(n)}(1)/n!</math></u>	
0	$\ln x$	0	0	$(x-1)^0$
1	$\frac{1}{x} = x^{-1}$	1	1	$(x-1)^1$
2	$-x^{-2}$	-1	$-\frac{1}{2}$	:
3	$2x^{-3}$	2	$\frac{2}{3!} = \frac{2}{6} = \frac{1}{3}$	
4	$-6x^{-4}$	-6	$-\frac{6}{4!} = -\frac{6}{24} = -\frac{1}{4}$	
5	$24x^{-5}$	24	$\frac{24}{5!} = \frac{24}{120} = \frac{1}{5}$	

$$T_n(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 + \dots$$

$$T_n(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^n}{n!}$$

Ratio Test

$$\lim_{n \rightarrow \infty} \left| \frac{(x-1)^{n+1}}{n+1} \cdot \frac{n}{(x-1)^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| (x-1) \cdot \frac{n}{n+1} \right|$$

$$= \lim_{n \rightarrow \infty} \left| (x-1) \cdot \frac{1}{1+\frac{1}{n}} \right|$$

$$= |x-1|$$

The geometric series converges as long as

$$|x-1| > 1$$

$$-1 > x-1 > 1$$

$$0 > x > 2$$

ex. Mr. Nasty

Around what values of  $x$  is the Taylor series of  $\sqrt{x}$  is a good approximation

Centred around  $x=1$

$n$	$f^{(n)}(x)$	$f^{(n)}(1)$	$f^{(n)}(1)/n!$
0	$x^{-\frac{1}{2}}$	1	1
1	$-\frac{1}{2}x^{-\frac{3}{2}}$	$-\frac{1}{2}$	$-\frac{1}{2}$
2	$-\frac{1}{2} \cdot \left(-\frac{3}{2}\right)x^{-\frac{5}{2}} = \frac{3}{4}x^{-\frac{5}{2}}$	$\frac{3}{4}$	$\frac{3}{8}$

$$\begin{array}{r}
 3 \quad -\frac{15}{8} x^{-3/2} \\
 4 \quad \frac{105}{16} x^{-5/2} \\
 \hline
 T_n(x) = 1 - \frac{1}{4}(x-1) + \frac{3}{8}(x-1)^2 - \frac{5}{16}(x-1)^3 + \frac{35}{128}(x-1)^4 \dots
 \end{array}$$

$$T(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(\dots)}{2^n} (x-1)^n$$

Ratio Test:

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \left| \frac{(2n+3)}{2^{n+1}} \cdot (x-1)^{n+1} \right| \frac{2}{(2n+1)(x-1)} \\
 &= \lim_{n \rightarrow \infty} \left| \frac{1}{2} \cdot \frac{2n+3}{2^{n+1}} \cdot (x-1) \right| \\
 &= \frac{1}{2} (x-1)
 \end{aligned}$$