# Factoring Techniques:

→ Greatest common factor

ex: $a^2b^3c^4 - ab^2c^5 = ab^2c^4(ab - c)$

→ Grouping (mise en evidence double)

ex: $x^3 - 2x^2 - x + 2 =$
$= x^2(x-2) - (x-2)$
$= (x-2)(x^2-1)$

→ Difference of squares

$a^2 - b^2 = (a-b)(a+b)$

→ Difference / Sum of cubes

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$a^3 + b^3 = (a+b)(a^2-ab+b^2)$

→ Perfect square

$(a+b)^2 = a^2 + 2ab + b^2$ / $(a-b)^2 = a^2 - 2ab + b^2$

- completing the square $\pm(1/2 \text{ coef})^2$

ex: $x^2 - \underbrace{2}x + 2 =$ $(1/2\,(2))^2 = 1$
$= \underbrace{x^2 - 2x + 1} - 1 + 2$
$= (x-1)^2 + 1$

→ Trinomial factoring

$ax^2 + bx + c = a(x - x_1)(x - x_2)$

- ① Find m, n so that:
  $m \cdot n = a \cdot c$
  $m + n = b$

  $6x^2 + 5x - 4$
  $= 6x^2 + 8x - 3x - 4$
  $= 2x(3x+4) - (3x+4)$
  $= (3x+4)(2x-1)$

  $\underline{8} \cdot \underline{-3} = -24$
  $\underline{8} + \underline{-3} = 5$

- ② $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- ③ Sum (of $x_1, x_2$) $= -b/a$

  Product (of $x_1, x_2$) $= c/a$

# Lowest Common Denominator

- Find factors of our denom
- Adjust to the <u>lowest</u> common denom

ex: $\dfrac{x+1}{x^2+5x+6} - \dfrac{2x+3}{x+2} =$

Restrictions:
$x \neq -2$
$x \neq -3$

$= \dfrac{x+1}{(x+2)(x+3)} - \dfrac{(2x+3)}{(x+2)} \cdot \dfrac{(x+3)}{(x+3}$

$= \dfrac{x+1 - (2x^2+6x+3x+9)}{(x+2)(x+3)}$

$= \dfrac{-2x^2-8x-8}{(x+2)(x+3)}$

$= \dfrac{-2(x+2)\cancel{(x+2)}}{(x+3)\cancel{(x+2)}} = \dfrac{-2(x+2)}{(x+3)}$

ex: $\dfrac{x+y}{x^{-1}+y^{-1}} =$

Restrictions:
$x+y \neq 0$
$x \neq -y$

$= \dfrac{x+y}{\frac{1}{x}+\frac{1}{y}}$

$= \dfrac{x+y}{\frac{y+x}{xy}}$

$= \cancel{(x+y)} \cdot \dfrac{(xy)}{\cancel{(x+y)}} = xy$

# Laws of Exponents

$a^{-n} = \frac{1}{a^n}$

$a^m \cdot a^n = a^{(m+n)}$

$\frac{a^m}{a^n} = a^{(m-n)}$

$(a^m)^n = a^{(m \cdot n)}$

$(ab)^n = a^n \cdot b^n$

$a^0 = 1 \quad , \; a \neq 0$

$a^{1/n} = \sqrt[n]{a}$ if a is defined

$a^{m/n} = \left.\begin{matrix} = (a^{1/n})^m = (\sqrt[n]{a})^m \\ = (a^m)^{1/n} = \sqrt[n]{a^m} \end{matrix}\right\}$ if $m/n$ is in lowest terms

$\sqrt[n]{a} \quad \begin{cases} \text{if } n \text{ is even}, & a \geq 0 \\ \text{if } n \text{ is odd}, & a \in \mathbb{R} \end{cases}$ ex. $\sqrt[3]{-8} = -2$

$\sqrt[n]{a^n} = \begin{cases} |a|, & \text{if } n \text{ is even} \\ a, & \text{if } n \text{ is odd} \end{cases}$ ex: $\sqrt{(-5)^2} = 5$

Note: $\sqrt{4} = +2$

Note: $x^2 = 4$
$x = \pm\sqrt{4}$
$x = \pm 2$

ex: $\sqrt{4xy^{-2}} =$

$= \sqrt{\frac{4x}{y^2}}$

$= \frac{\sqrt{4x}}{\sqrt{y^2}}$ * $|y|$ ∵ $\sqrt[n]{a^n} = |a|$, if $n$ is even

$= \frac{2\sqrt{x}}{|y|}$

# Rationalizing

- moving a root (from denom. to numer. or numer. to denom.)
- multiplying by a special form of 1

ex: $\frac{2x}{3-\sqrt{y}} \cdot \frac{3+\sqrt{y}}{3+\sqrt{y}} = \frac{2x(3+\sqrt{y})}{9-y}$

ex: $\frac{7^{2/5}}{8} \cdot \frac{7^{3/5}}{7^{3/5}} = \frac{7}{8 \cdot 7^{3/5}}$

ex: $\frac{\sqrt[3]{b}-2}{3} =$

$= \frac{(\sqrt[3]{b}-2)}{3} \cdot \frac{(b^{2/3}+2b^{1/3}+4)}{(b^{2/3}+2b^{1/3}+4)}$

$= \frac{b-8}{3b^{2/3}+6b^{1/3}+12}$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

$a=\sqrt[3]{b}$

$b=2$

# Solving Equations

① Linear

ex: $3x+7=5x$

$-2x=-7$

$\boxed{\underline{\text{Sol.}} \quad x = 7/2}$

ex: $\frac{1}{2x-1} = \frac{3}{x}$

$x = 3(2x-1)$

$x = 6x-3$

$\boxed{\underline{\text{Sol.}} \quad x = 3/5}$

Restrictions:

$2x-1 \neq 0$

$x \neq 1/2$

$x \neq 0$

## ② Quadratic equation

$\Delta = b^2 - 4ac$

→ if $\Delta > 0$ ∴ can factorise ∴ solutions
→ if $\Delta < 0$ ∴ can't factorise ∴ no solution

- $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

ex: $3x^2 - 5x + 1 = 0$

$x = \dfrac{5 \pm \sqrt{13}}{6}$

Sol. $x \in \left\{ \dfrac{5-\sqrt{13}}{6}, \dfrac{5+\sqrt{13}}{6} \right\}$

- Factorise (method → Sum $(x_1, x_2) = -b/a$, Product $(x_1, x_2) = c/a$)

- Easily isolate $x$

ex: $x^2 - 5x + 6 = 0$

$(x-2)(x-3) = 0$

$x-2=0 \mid x-3=0$

$x=2 \mid x=3$

$S=5$, $P=6$ > $x_1 = 2$, $x_2 = 3$

Sol. $x \in \{2, 3\}$

ex: $x^2 = 9$

$x = \pm\sqrt{9}$

Sol. $x = \pm 3$

## ③ Cubic equation

ex: $6x^3 + 3x^2 - 9x = 0$

$3x(2x^2 + x - 3) = 0$

$(3x)(2(x+3/2)(x-1)) = 0$

$3x(2x+3)(x-1) = 0$

$x=0 \mid x=-3/2 \mid x=1$

$S = -1/2$, $P = -3/2$ > $x_1 = 1$, $x_2 = -3/2$

Sol. $x \in \{-3/2, 0, 1\}$

ex: $8x^3 - 27 = 0$

$(2x)^3 - 3^3 = 0$

$(2x-3)(4x^2+6x+3) = 0$

$x = 3/2 \mid$ No zero ∴ can't factorise

Sol $x \in \{3/2\}$

→ Distributing rule: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

⑤ Note: when there is a square root in the expression, and there are $x$'s inside and outside the square root:
→ Verify solutions in initial expression

# Inequality

property: if $a > b$, and $c < 0$ (neg)
$a(c) < b(c)$

## Quadratic inequality

ex: $x^2 - x > 2$
$x^2 - x - 2 > 0$
$(x-2)(x+1) > 0$

$S = 1$, $P = -2$ $\rangle$ $x_1 = 2$, $x_2 = -1$

| | $-\infty$ | $-1$ | | $2$ | $+\infty$ |
|---|---|---|---|---|---|
| $x-2$ | $-$ | | $-$ | $0$ | $+$ |
| $x+1$ | $-$ | $0$ | $+$ | | $+$ |
| | $+$ | $0$ | $-$ | $0$ | $+$ |

$x \in (-\infty, -1) \cup (2, +\infty)$

ex: $\dfrac{x^2 - 5x - 6}{2x - 1} \geq 0$

$S = 5$, $P = -6$ $\rangle$ $x_1 = 6$, $x_2 = -1$

$\dfrac{(x-6)(x+1)}{2x-1} \geq 0$

Restrictions
$2x - 1 \neq 0$
$x \neq 1/2$

| | $-\infty$ | $-1$ | | $6$ | $+\infty$ |
|---|---|---|---|---|---|
| $x-6$ | $-$ | | $-$ | $0$ | $+$ |
| $x+1$ | $-$ | $0$ | $+$ | | $+$ |
| | $+$ | $0$ | $-$ | $0$ | $+$ |

| | $-\infty$ | $-1$ | | $1/2$ | | $6$ | $+\infty$ |
|---|---|---|---|---|---|---|---|
| $x-6$ | $-$ | | $-$ | | $-$ | $0$ | $+$ |
| $x+1$ | $-$ | $0$ | $+$ | | $+$ | | $+$ |
| $2x-1$ | $-$ | | $-$ | $0$ | $+$ | | $+$ |
| | $-$ | | $+$ | | $-$ | | $+$ |

$x \in [-1, 1/2) \cup [6, +\infty)$

* $1/2$ not included because it's a restriction

ex: $x^3 - 5x^2 - 6x < 0$
$x(x^2 - 5x - 6) < 0$
$x(x-6)(x+1) < 0$

$S = 5$, $P = -6$ $\rangle$ $x_1 = 6$, $x_2 = -1$

| | $-\infty$ | $-1$ | | $0$ | | $6$ | $+\infty$ |
|---|---|---|---|---|---|---|---|
| $x$ | $-$ | | $-$ | $0$ | $+$ | | $+$ |
| $x-6$ | $-$ | | $-$ | | $-$ | $0$ | $+$ |
| $x+1$ | $-$ | $0$ | $+$ | | $+$ | | $+$ |
| | $-$ | | $+$ | | $-$ | | $+$ |

$x \in (-\infty, -1) \cup (0, 6)$

# Overview of Functions

A function is a correspondance between 2 sets:

- Domain $(x) \rightarrow$ Dom $(f)$
- ~~Range (y) → Range (f)~~
- Range $(y) \rightarrow$ Range $(f)$

For every value in Dom $(f)$, there exists only one value in Range$(f)$

$$\left(\underset{\substack{\uparrow \\ \text{For every}}}{\forall}\, x \in \text{Dom}(f),\ \underset{\substack{\uparrow \\ \text{There exists} \\ \text{only one}}}{\exists!}\, y \in \text{Range}(f)\right)$$

Finding Domain:

- Linear Function

  $f(x) = 3x + 2$

  Dom $(f)$: $x \in \mathbb{R}$ or $(-\infty, +\infty)$
  Range $(f)$. $y \in \mathbb{R}$

- Quadratic

  $f(x) = x^2 - 2x + 2$

  complete the square $\quad (\pm(\frac{1}{2}\text{ coef})^2)$

  $$\underbrace{x^2 - 2x \boxed{+1 - 1}}\, + 2$$

  $f(x) = (x-1)^2 + 1$

  vertex: $(x, y) = (h, k) \rightarrow (1, 1)$
  opening: ∪

  Dom $(f)$: $x \in \mathbb{R}$
  Range $(f)$: $y \in [1, +\infty)$

- Rational

  $f(x) = \dfrac{3}{x^2 - 4}$

  Restrictions.
  $x^2 - 4 \neq 0$
  $x^2 \neq 4$
  $x \neq \pm 2$

  Dom $(f)$: $x \in \mathbb{R} \setminus \{-2, 2\}$
  or
  Dom $(f)$: $x \in (-\infty, -2) \cup (-2, 2) \cup (2, +\infty)$

- Algebreic

  $y = \sqrt{x-3} + \dfrac{1}{x-4}$

  Restrictions:
  $x - 3 \geqslant 0 \qquad x - 4 \neq 0$
  $x \geqslant 3 \qquad x \neq 4$

  Dom $(f)$: $x \in [3, 4) \cup (4, +\infty)$

- Algebreic function
  ex: $f(x) = \sqrt{\dfrac{x-1}{2x+4}}$

Restrictions :
$2x+4 \neq 0$
$x \neq -2$
and
$\dfrac{x-1}{2x+4} \geq 0$

$-\infty \quad -2 \quad 1 \quad +\infty$

| | | | |
|---|---|---|---|
| $x-1$ | $-$ | $-$ | 0 $+$ |
| $2x+4$ | $-$ 0 | $+$ | $+$ |
| | $+$ | $-$ | $+$ |

$Dom(f): x \in (-\infty, 2) \cup [1, +\infty)$

# Graphs (Basic Functions)

Power Functions. $y = x^k$

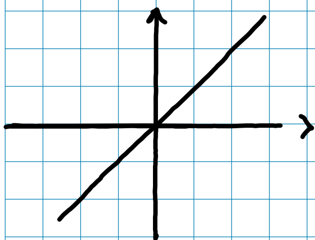
→ Even function : $k$ is even
$f(x) = f(-x)$
(symetric with y axis)

→ Odd function : $k$ is odd
$f(-x) = -f(x)$
(symetric with origin)

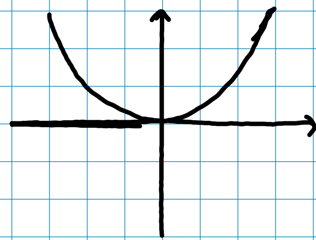
Most functions are neither odd or even (shift)

→ One to one function.
- For every x there's 1 y (verticle)
- For every y there's 1 x (horizontal)

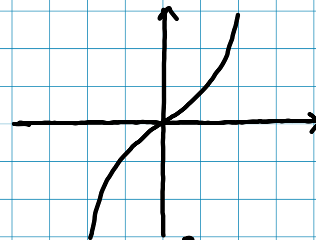
↳ The inverse will also be a function



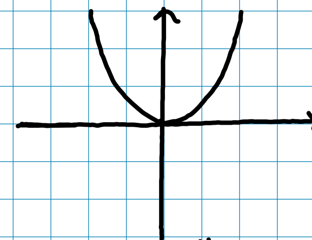
$y=x$
Dom: $\mathbb{R}$
Range: $\mathbb{R}$



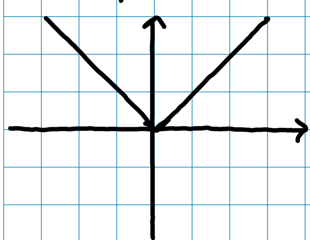
$y=x^2$
Dom: $\mathbb{R}$
Range: $[0,+\infty)$



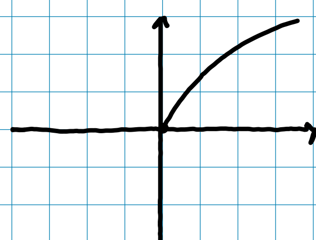
$y=x^3$
Dom: $\mathbb{R}$
Range: $\mathbb{R}$



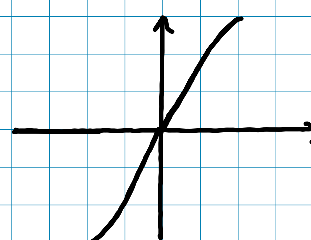
$y=x^4$
Dom: $\mathbb{R}$
Range: $[0,+\infty)$



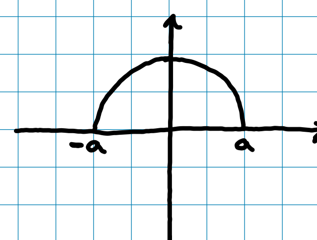
$y=|x| = \begin{cases} x, & x\geq 0 \\ -x, & x<0 \end{cases}$



$y=\sqrt{x}$
Dom: $[0,+\infty)$
Range: $[0,+\infty)$



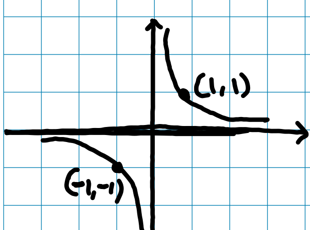
$y=x^{1/3}$
Dom: $\mathbb{R}$
Range: $\mathbb{R}$



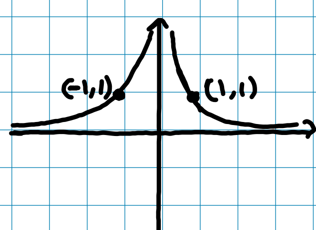

$y=\sqrt{a^2-x^2}$
Dom: $[-a,a]$

↓

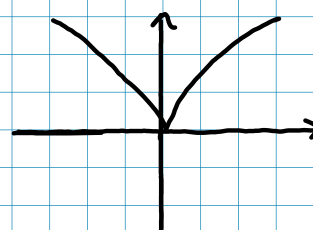
Half circle
$x^2+y^2=a^2$
$y=\pm\sqrt{a^2-x^2}$
Take only positive value for semi-circle




$y=1/x$
Dom: $\mathbb{R}\setminus\{0\}$

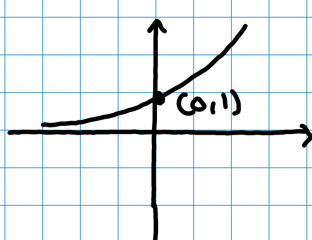



$y=1/x^2$
Dom $\mathbb{R}\setminus\{0\}$



$y=x^{2/3}$
Dom: $\mathbb{R}$
Range: $[0,+\infty)$

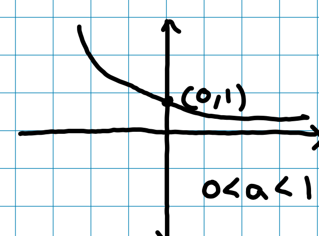
## Exponential fcn ($y=a^x$)




$a>1$

| x | y |
|---|---|
| 0 | 1 |
| 1 | 2 |
| 2 | 4 |
| -1 | 1/2 |
| -2 | 1/4 |

$\xrightarrow{\text{flip y axis}}$



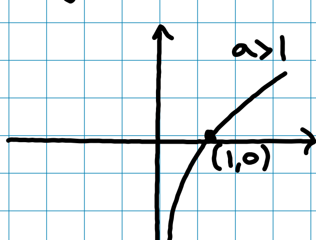

$0<a<1$

| x | y |
|---|---|
| 0 | 1 |
| 1 | 1/2 |
| 2 | 1/4 |
| -1 | 2 |
| -2 | 4 |

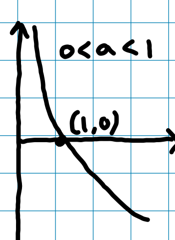
↓ inverse

a (base) is always positive
→ exp: $a^x > 0$ (restriction)
→ log: $x > (0)$ (restriction)

## Logarithmic fcn ($y=\log_a x$) ⇒ ($a^y = x$)

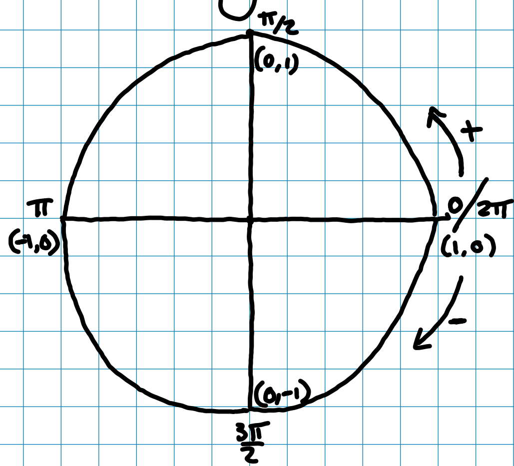



$y=a^x$
inverse
$x=a^y$
$\log_a x = \log_a a^y$
$\log_a x = y \quad (1)$
$\therefore f^{-1}(x) = \log_a x$




Note:
$\log_a a = 1$
$\log_e x = \ln x$
$\log_c x = y$
↳ $c^y = x$

# Trig Fcn




$P(\pi) =$ even $(1,0)$
odd $(-1,0)$

$P(\frac{\pi}{4}) = (\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2})$

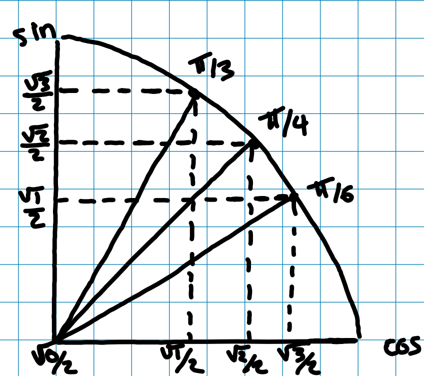
$P(\frac{\pi}{2}) = (0, \pm 1)$

$P(\frac{\pi}{6}) = (\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2})$

$P(\frac{\pi}{3}) = (\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$

$(\cos\theta, \sin\theta)$

| | |
|---|---|
| − + | + + |
| − − | + − |



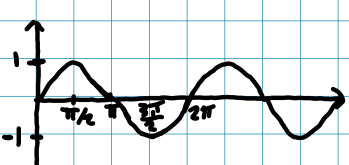

## Identities

$\tan\theta = \frac{\sin}{\cos}$

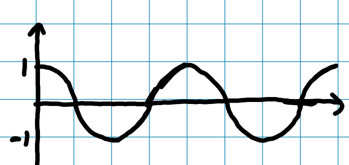
$\cot\theta = \frac{\cos}{\sin}$

$\csc\theta = \frac{1}{\sin}$

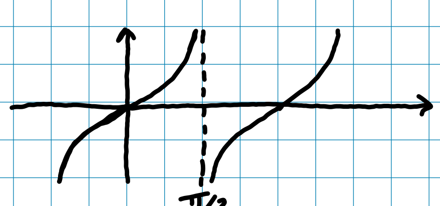
$\sec\theta = \frac{1}{\cos}$




$y = \sin x$ $\quad -1 \leq \sin x \leq 1$




$y = \cos x$ $\quad -1 \leq \cos x \leq 1$




$y = \tan x$

Note: $\sin^2 x + \cos^2 x = 1$

$\sin 2A = 2 \cdot \sin A \cdot \cos A$

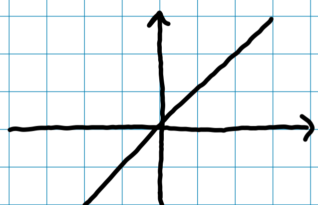
$\cos 2A = \cos^2 A - \sin^2 A$

# Graphing: Reflections, Horizontal/Vertical Shifts

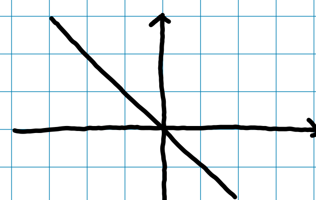
## Reflections

- Multiply $x$ by $-1$

↳ Reflection in $y$ axis



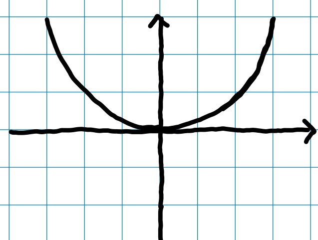
$y = 2x$



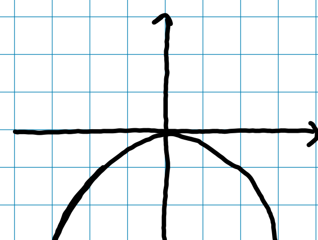
$y = 2(-x) \Rightarrow y = -2x$

- Multiply $y$ by $-1$

↳ Reflection in $x$ axis



$y = x^2$



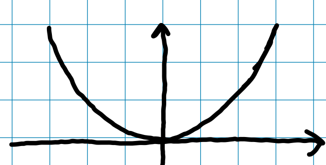
$(-y) = x^2 \Rightarrow y = -x^2$

## Horizontal Shift

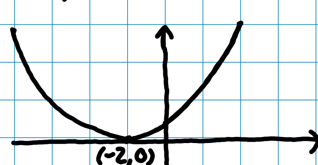
- Substract "h" to $x$ $\Rightarrow (x-h)$

↳ if $h$ is pos. shift right

↳ if $h$ is neg. shif left



$y = x^2$



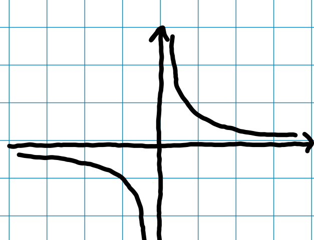

$y = (x+2)^2$ $h = -2 \therefore$ shift left 2

## Vertical Shift

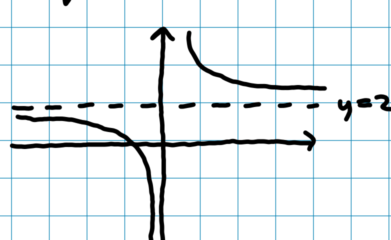
- Add "k" to $y$

↳ if $k$ is pos. shift up

↳ if $k$ is neg. shift down



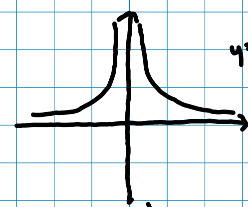
$y = \frac{1}{x}$



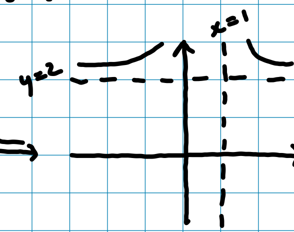

$y = \frac{1}{x} + 2$ $k = 2 \therefore$ shif up 2 ex: graph function of...

$y = 2 - \frac{1}{(x-1)^2}$

$y = \frac{-1}{(x-1)^2} + 2$

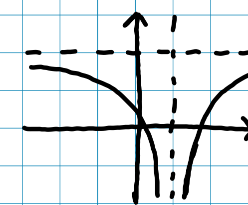


$y = \frac{1}{x^2}$




$y = \frac{1}{(x-1)^2} + 2$

$h = 1$
$k = 2$



$y = -\frac{1}{(x-1)^2} + 2$

mult y by $-1$ ∴ Reflect

# Finding the Domain

- Find restrictions
- Exclude them from the domain

ex: $f(x) = \ln(25 - x^2)$

Restrictions:

$25 - x^2 > 0$

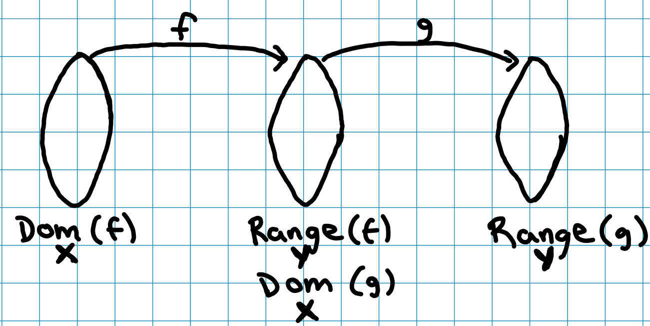
$-(x^2 - 25) > 0$ $S = 0$

$-(x-5)(x+5) > 0$ $P = -25$

| | | -5 | | 5 | |
|---|---|---|---|---|---|
| $-1$ | $-$ | | $-$ | | $-$ |
| $(x-5)$ | $-$ | | $-$ | 0 | $+$ |
| $(x+5)$ | $-$ | 0 | $+$ | | $+$ |
| | $-$ | | $+$ | | $-$ |

∴ Dom $(f)$: $x \in (-5, 5)$

# Composition of Functions




Given 2 functions $f(x)$ and $g(x)$

→ If Dom (g) $X$ = Range (f) $Y$ ∴ $(g \circ f)(x)$
$g(f(x))$

* Verify that the range of f is contained in the domain of g

ex: $f(x) = \sin x$ , $g(x) = x^2$

- $(f \circ g)(x) = f(g(x)) = f(x^2) = \sin(x^2)$ ∴ $(f \circ g)(x) = \sin x^2$
- $(g \circ f)(x) = g(f(x)) = g(\sin x) = (\sin x)^2$ ∴ $(g \circ f)(x) = \sin^2 x$

Decomposition:

→ Whatever is closest to x is what we do first

ex: $h(x) = (f \circ g)(x)$
$h(x) = \frac{1}{(x-2)^2}$

- $g(x) = x-2$ , $f(x) = \frac{1}{x^2}$
  check: $f(g(x)) = f(x-2) = \frac{1}{(x-2)^2}$
- $g(x) = \frac{1}{x-2}$ , $f(x) = x^2$
  check: $f(g(x)) = f(\frac{1}{x-2}) = (\frac{1}{x-2})^2 = \frac{1}{(x-2)^2}$

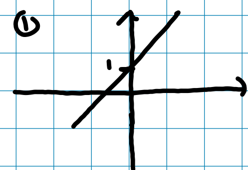
ex: $h(x) = (f \circ g \circ k)(x)$
$h(x) = \sin^2(3x+2)$

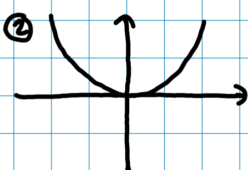
- $k(x) = 3x+2$ , $g(x) = \sin x$ , $f(x) = x^2$

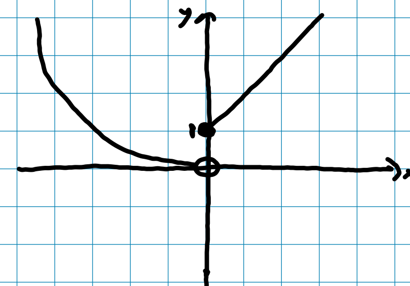
* $(f \circ f^{-1})(x) = x$ / $(f^{-1} \circ f)(x) = x$

# Piecewise-defined functions

ex: $f(x) = \begin{cases} x+1 & \text{if } x \geq 0 \quad ① \\ x^2 & \text{if } x < 0 \quad ② \end{cases}$

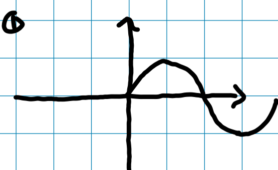
①


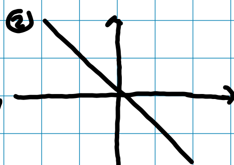
②


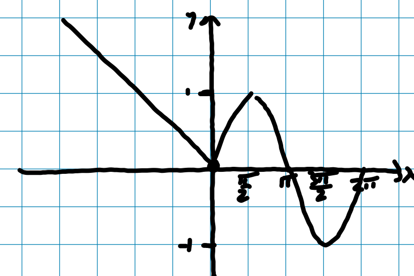


Dom (f) $x \in \mathbb{R}$

ex: $g(x) = \begin{cases} \sin x & \text{if } x \geq 0 \quad ① \\ -x & \text{if } x < 0 \quad ② \end{cases}$

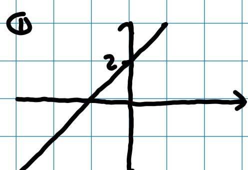
①


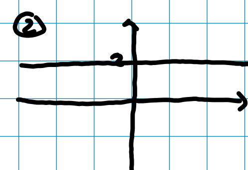
②


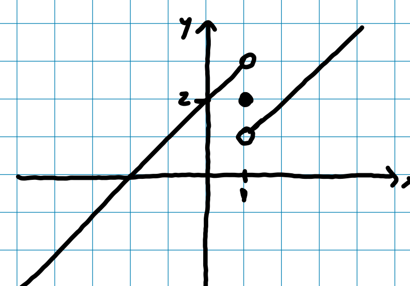


Dom (g) $x \in \mathbb{R}$

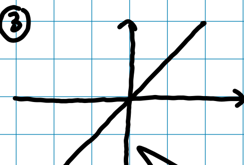
ex: $f(x) = \begin{cases} x+2 & \text{if } x < 1 \quad ① \\ 2 & \text{if } x = 1 \quad ② \\ x & \text{if } x > 1 \quad ③ \end{cases}$

①


②




Dom (f) : $x \in \mathbb{R}$

③


~~* Absolute value is peicewise~~

~~$|x| = \begin{cases} x & \text{if } x \geq 1 \\ -x & \text{if } x \leq 1 \end{cases}$ ← negative of negative is positive~~

~~$|x-4| = \begin{cases} x-4 & \text{if } x \geq 4 \\ -x+4 & \text{if } x \leq 4 \end{cases}$~~

* Absolute value = peicewise

$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ ← negative of negative is positive

$|x-4| = \begin{cases} x-4 & \text{if } x \geq 4 \\ -x+4 & \text{if } x < 4 \end{cases}$

# Limits

**Definition:** Let $y = f(x)$ be a function

- If the $y$ values approach $L \in \mathbb{R}$ as the $x$ values approach $a \in \mathbb{R}$, we say the limit of $f(x)$ as $x$ approaches $a$ is $L$

$$\lim_{x \to a} f(x) = L$$

**Definition (one-sided limits):** Let $y = f(x)$ be a function

- If the $y$ value approaches $L \in \mathbb{R}$ as $x$ approaches $a \in \mathbb{R}$ from left $(x < a)$, we say that the limit from the left as $x$ approaches $a$ is $L_L$

$$\lim_{x \to a^-} f(x) = L_L \qquad \text{(left hand limit)}$$

- If the $y$ value approaches $L \in \mathbb{R}$ as $x$ approaches $a \in \mathbb{R}$ from right $(x > a)$, we say that the limit from the right as $x$ approaches $a$ is $L_R$

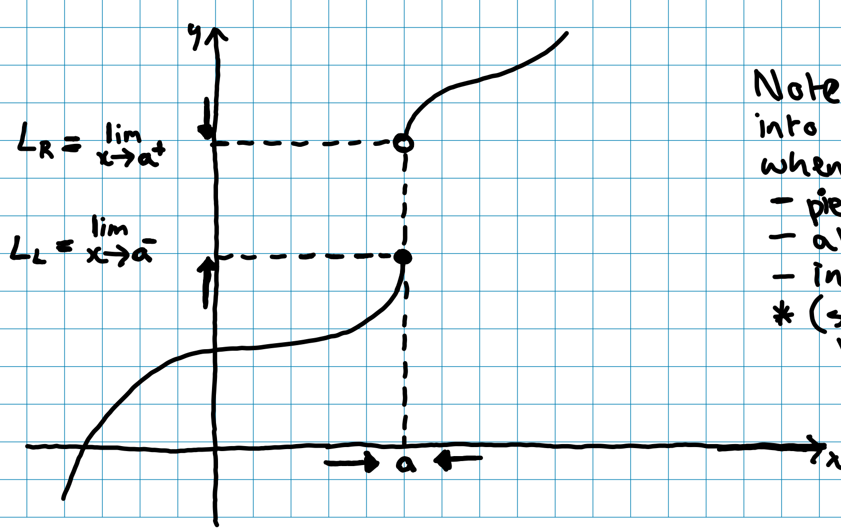
$$\lim_{x \to a^+} f(x) = L_R \qquad \text{(right hand limit)}$$

Note: → if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

then $\lim_{x \to a} f(x) = L$

→ if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+}$

then $\lim_{x \to a} f(x)$ DNE




Note: we split into one-sided limits when:

- piecewise
- absolute value
- infinite limit
- * (suspect different limits)

# General Idea of Limits

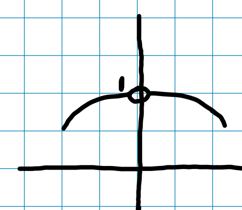
Some functions do not exist at a certain point (domain does not exist)

ex: $f(x) = \frac{\sin x}{x}$  $\quad$ Dom (f) : $x \in \mathbb{R} \setminus \{0\}$

If we want to know what the graph looks like at zero, we cannot use substitution $\therefore$ $f(0)$ is undefined

Consider a table of y values as x gets closer to 0

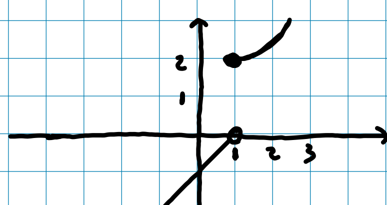
| X | -0.1 | -0.01 | -0.001 | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $y = \frac{\sin x}{x}$ | 0.998 | 0.9999 | 0.99999 | X | 0.99999 | 0.9999 | 0.998 |




"Y value approaches 1, when x approaches 0"

$\lim_{x \to 0} \frac{\sin x}{x} = 1$ $\therefore$ the limit of $\frac{\sin x}{x}$ as x approaches 0 is 1

ex: $y = \begin{cases} x^2+1 & \text{if } x \geq 1 \\ x-1 & \text{if } x < 1 \end{cases}$

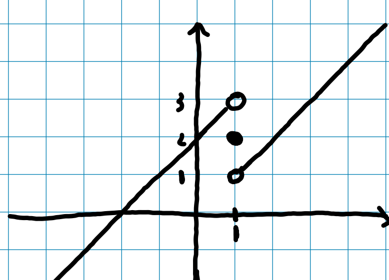



- $f(1) = 2$ (exists / defined)
- $\lim_{x \to 1^-} f(x) = 0$
- $\lim_{x \to 1^+} f(x) = 2$

$\therefore$ $\lim_{x \to 1} f(x)$ DNE $\left(\lim_{x \to 1^-} \neq \lim_{x \to 1^+}\right)$

* Even if the function exists at 1

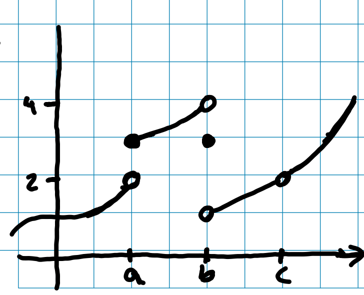
ex: $y = \begin{cases} x+2 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ x & \text{if } x > 1 \end{cases}$




- $f(1) = 2$
- $\lim_{x \to 1^-} f(x) = 3$
- $\lim_{x \to 1^+} f(x) = 1$

$\left.\begin{matrix} \lim_{x \to 1^-} f(x) = 3 \\ \lim_{x \to 1^+} f(x) = 1 \end{matrix}\right\} \therefore \lim_{x \to 1} f(x)$ DNE

ex:




$\left.\begin{matrix} \lim_{x \to a^-} f(x) = 2 \\ \lim_{x \to a^+} f(x) = 3 \end{matrix}\right\rangle \lim_{x \to a} f(x)$ DNE

$\left.\begin{matrix} \lim_{x \to b^-} f(x) = 4 \\ \lim_{x \to b^+} f(x) = 1 \end{matrix}\right\rangle \lim_{x \to b} f(x)$ DNE

$\left.\begin{matrix} \lim_{x \to c^-} f(x) = 2 \\ \lim_{x \to c^+} f(x) = 2 \end{matrix}\right\rangle \lim_{x \to c} f(x) = 2$

# Properties of limits

① $\lim_{x\to a} K = K$

② $\lim_{x\to a} x = a$

③ $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

④ $\lim_{x\to a} K \cdot f(x) = K \lim_{x\to a} f(x)$

⑤ $\lim_{x\to a} f(x) \cdot g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

⑥ $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

* Both limits exist
* $\lim_{x\to a} g(x) \neq 0$

⑦ $\lim_{x\to a} x^n = a^n$

⑧ $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$

⑨ $\lim_{x\to a} [f(x)]^2 = [\lim_{x\to a} f(x)]^2$

⑩ $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$

Note: For most regular functions we can use direct substitution to evaluate the $\lim_{x\to a} f(x)$
* if $a \in \text{Dom } f$
  - polynomials
  - rationals
  - roots
  - exponential / log
  - trig functions
  - algebreic function

# When can we not use direct sub?

→ $a \notin \text{Dom } f$

① Limits of type $\frac{0}{0}$

→ simplify the limit so that we eliminate the denominator
→ sub
* We can do this because even though the value of x we are approaching doesn't exist in the function we are looking for values approaching that number

ex: $\lim_{x\to 1} \frac{x^2-1}{x-1} = \frac{0}{0}$

$= \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)}$

$= \lim_{x\to 1} (x+1)$

$= 2$

explanation:
- 1 is not part of the domain
- But we are only using values approaching 1
- $f(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ \text{DNE} & \text{if } x = 1 \end{cases}$

ex: $\lim_{x\to 1} \frac{\frac{1}{x+4} - \frac{1}{5}}{x-1} = \frac{0}{0}$

$= \lim_{x\to 1} \frac{5-(x+4)}{5(x+4)} \cdot \frac{1}{(x-1)}$

$= \lim_{x\to 1} \frac{-x+1}{5(x+4)(x-1)}$

$= \lim_{x\to 1} \frac{-\cancel{(x-1)}}{5(x+4)\cancel{(x-1)}}$

$= \lim_{x\to 1} \frac{-1}{5(1+4)}$

$= -1/25$

explanation:
- I am trying to factor out $(x-1)$ so that the denom is not 0

ex: $\lim_{x\to 1} \frac{x-1-|x-1|}{1-|x|}$

$|x-1| = \begin{cases} x-1 & \text{if } x \geq 1 \\ -x+1 & \text{if } x < 1 \end{cases}$

$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

↳ because we are looking at $\lim_{x\to 1}$, $x$ is always pos $(x>0)$

• $\lim_{x\to 1^-} \frac{x-1-(-x+1)}{1-(x)}$ $\frac{0}{0}$

$= \lim_{x\to 1^-} \frac{-2(1-x)}{(1-x)}$

$= 2$

• $\lim_{x\to 1^+} \frac{x-1-(x-1)}{1-x}$

$= \lim_{x\to 1^+} \frac{(x-1)(1-1)}{-(x-1)}$

$= \lim_{x\to 1^+} \frac{1-1}{-1}$

$= 0$

explanation:
- The initial function and the factored form of the function are the same except at point $x=a$, where the initial function doesn't exist

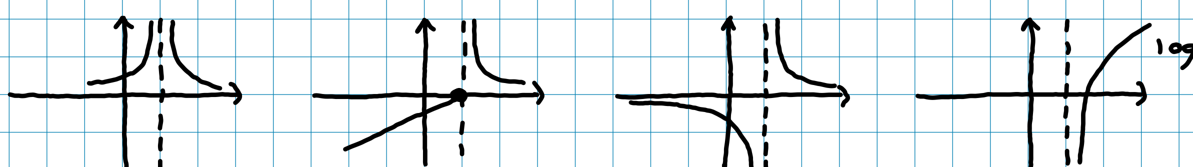
## ② Limits of type $\frac{k}{0}$ $(k \neq 0)$

→ infinite limits (limit dne ∵ not a real number)

→ vertical asymptotes: If $y = f(x)$ is a fcn, the line $x = a$ is a vertical asymptote of $f(x)$ if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$

$0^+$ = "approaching zero and positive"
$0^-$ = "approaching zero and negative"

If $k > 0$ ∴ $\frac{k}{0^+} = +\infty$ and $\frac{k}{0^-} = -\infty$




$\lim_{x \to a^-} f(x) = +\infty$
$\lim_{x \to a^+} f(x) = +\infty$

$\lim_{x \to a^-} f(x) = 0$
$\lim_{x \to a^+} f(x) = +\infty$

$\lim_{x \to a^-} f(x) = -\infty$
$\lim_{x \to a^+} f(x) = +\infty$

$\lim_{x \to a^+} f(x) = -\infty$
$\lim_{x \to a^-} f(x)$ undefined

### Evaluating limits algebreically:

Rule: let $n$ be a positive integer
- If $n$ is even, then $\lim_{x \to a} \frac{1}{(x-a)^n} = +\infty$
- If $n$ is odd, then split:
  $\lim_{x \to a^-} \frac{1}{(x-a)^n} = -\infty$
  $\lim_{x \to a^+} \frac{1}{(x-a)^n} = +\infty$

* One will be $+\infty$, other will be $-\infty$
* $\lim_{x \to a}$ DNE

### Find vertical asymptotes:
- Factor denom
- Find potential asymptotes (values that make denom = 0)
- Verify by findind the $\lim_{x \to a}$ for each potential asymptote

ex: $\lim_{x\to 0} \frac{1}{x^2} = \frac{K}{0}$ (even exponent)

$= \lim_{x\to 0} \frac{1}{0^+}$

$= +\infty$

ex: $\lim_{x\to 0} \frac{1}{x} = \frac{K}{0}$ (odd exponent)

- $\lim_{x\to 0^-} \frac{1}{x} = \frac{1}{0^-} = -\infty$
- $\lim_{x\to 0^+} \frac{1}{x} = \frac{1}{0^+} = +\infty$

$\Big\}$ $\lim_{x\to 0} f(x)$ DNE

ex: $\lim_{x\to 0} \frac{3x-6}{5x^6} = \frac{K}{0}$ (even exponent)

$= \lim_{x\to 0} \underbrace{\frac{3x-6}{5}}_{\downarrow\, -6/5} \cdot \underbrace{\frac{1}{x^6}}_{\downarrow\, +\infty}$

$= -\infty$

ex: $\lim_{x\to 2} \frac{5x^2+2x}{(x-2)^4} = \frac{K}{0}$ (even exponent)

$= \lim_{x\to 2} \underbrace{\frac{5x^2+2x}{(1)}}_{\downarrow\, 24} \cdot \underbrace{\frac{1}{(x-2)^4}}_{\downarrow\, +\infty}$

$= +\infty$

ex: $\lim_{x\to -2} \frac{3+x}{x^2-x-6} = \frac{K}{0}$

$= \lim_{x\to -2} \frac{3+x}{(x+2)(x-3)}$

$= \lim_{x\to -2} \frac{(3+x)}{(x-3)} \cdot \frac{1}{(x+2)}$ (odd exponent)

- $\lim_{x\to -2^-} \underbrace{\frac{3+x}{x-3}}_{\downarrow\, -1/5} \cdot \underbrace{\frac{1}{x+2}}_{\downarrow\, -\infty} = +\infty$
- $\lim_{x\to -2^+} \underbrace{\frac{3+x}{x-3}}_{\downarrow\, -1/5} \cdot \underbrace{\frac{1}{x+2}}_{\downarrow\, +\infty} = -\infty$

$\Big\}$ By default, if one is $+\infty$, the other will be $-\infty$

ex: Find vertical asymptotes of $y = \frac{3x^2-5x}{x^4-x^2}$

→ Factor denom: $y = \frac{3x^2-5x}{x^2(x^2-1)} = \frac{3x^2-5x}{x^2(x-1)(x+1)}$

→ Potential asymptotes: $x=0$, $x=1$, $x=-1$

$\lim_{x\to 0} \frac{3x^2-5x}{x^2(x^2-1)}$ $\frac{0}{0}$ ∴ not asympt

$= \lim_{x\to 0} \frac{\cancel{x}(3x-5)}{x^{\cancel{2}}(x^2-1)}$

$= \lim_{x\to 0} \frac{3x-5}{(x^2-1)} \cdot \frac{1}{x}$ $\frac{K}{0}$ (odd exp)

- $\lim_{x\to 0^-} \underbrace{\frac{3x-5}{x^2-1}}_{\downarrow\, 5} \cdot \underbrace{\frac{1}{x}}_{\downarrow\, 1/0^-}$

$= -\infty$

- $\lim_{x\to 0^+} f(x) = +\infty$

$\lim_{x\to 1} \frac{3x^2-5x}{x^2(x-1)(x+1)} = \frac{K}{0}$

$= \lim_{x\to 1} \frac{3x^2-5x}{x^2(x+1)} \cdot \frac{1}{(x-1)}$ (odd exp)

- $\lim_{x\to 1^-} \underbrace{\frac{3x^2-5x}{x^2(x+1)}}_{\downarrow\, -1} \cdot \underbrace{\frac{1}{x-1}}_{\downarrow\, 1/0^-}$

$= +\infty$

- $\lim_{x\to 1^+} f(x) = -\infty$

$\lim_{x\to -1} \frac{3x^2-5x}{x^2(x-1)(x+1)} = \frac{K}{0}$

$= \lim_{x\to -1} \frac{3x^2-5x}{x^2(x-1)} \cdot \frac{1}{(x+1)}$ (odd exp)

- $\lim_{x\to -1^-} \underbrace{\frac{3x^2-5x}{x^2(x-1)}}_{\downarrow\, -4} \cdot \underbrace{\frac{1}{(x+1)}}_{\downarrow\, 1/0^-}$

$= +\infty$

- $\lim_{x\to -1^+} f(x) = -\infty$

* A $\frac{0}{0}$ type limit can become a $\frac{k}{0}$ type limit as we simplify it

ex: $\lim\limits_{x\to 5} \dfrac{|x-5|}{x^2-10x+25}$ $\qquad |x-5| = \begin{cases} x-5 & \text{if } x>5 \\ -x+5 & \text{if } x<5 \end{cases}$

$= \lim\limits_{x\to 5} \dfrac{|x-5|}{(x-5)^2}$

- $\lim\limits_{x\to 5^-} \dfrac{-x+5}{(x-5)^2} = \frac{0}{0}$

$= \lim\limits_{x\to 5^-} \dfrac{-\cancel{(x-5)}}{(x-5)\cancel{(x-5)}}$

$= \lim\limits_{x\to 5^-} \dfrac{-1}{x-5}$ $\qquad \frac{k}{0}$

$= \lim\limits_{x\to 5^-} \underbrace{-1}_{\downarrow\,-1} \cdot \underbrace{\dfrac{1}{(x-5)}}_{\downarrow\,1/0^-}$

$= +\infty$

- $\lim\limits_{x\to 5^+} \dfrac{\cancel{(x-5)}}{\cancel{(x-5)}(x-5)} = \frac{0}{0}$

$= \lim\limits_{x\to 5^+} \underbrace{\dfrac{1}{(x-5)}}_{\downarrow\,1/0^+}$ $\qquad \frac{k}{0}$

$= +\infty$

ex: Find asymptotes of $f(x) = \dfrac{2x-2}{x^2-5x+4}$

→ Factor denom: $f(x) = \dfrac{2(x-1)}{(x-4)(x-1)}$

→ potential asymptotes: $x=4$, $x=1$

$\lim\limits_{x\to 4} \dfrac{2(x-1)}{(x-4)(x-1)}$ $\qquad \frac{k}{0}$

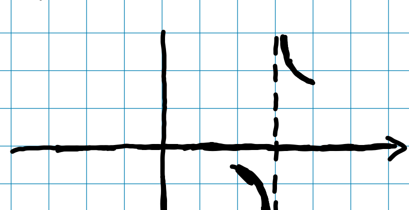
- $\lim\limits_{x\to 4^-} \dfrac{2\cancel{(x-1)}}{(x-4)\cancel{(x-1)}}$

$= \lim\limits_{x\to 4^-} \underbrace{2}_{\downarrow\,2} \times \underbrace{\dfrac{1}{(x-4)}}_{\downarrow\,1/0^-} = -\infty$

- $\lim\limits_{x\to 4^+} f(x) = +\infty$

$\therefore \lim\limits_{x\to 4} f(x)$ DNE

$\therefore x=4$ is vertical asym.


$x=4$

$\lim\limits_{x\to 1} \dfrac{2(x-1)}{(x-4)(x-1)}$ $\qquad \frac{0}{0}$

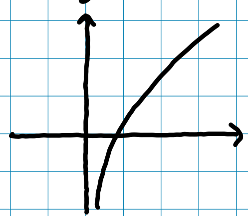
$= \lim\limits_{x\to 1} \dfrac{2}{(x-4)}$

$= \dfrac{2}{1-4}$

$= -2/3$

* There exists other types of infinite limits (asymptotes) which are not necessarily rationals

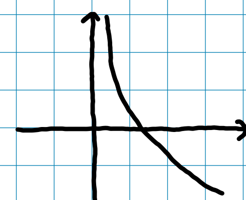
## Log Function



$\log_b x$

$b > 1$

$\lim_{x \to 0^+} f(x) = -\infty$



$\log_b x$

$0 < b < 1$

$\lim_{x \to 0^+} f(x) = +\infty$

- $\lim_{x \to 0^-} f(x)$ is undefined
  ↳ $x$ always has to be at least greater than 0

If $b > 1$ : $\lim_{x \to a^+} \log_b(\underbrace{x-a}_{\downarrow\, 0^+}) = -\infty$

If $0 < b < 1$ : $\lim_{x \to a^+} \log_b(\underbrace{x-a}_{\downarrow\, 0^+}) = +\infty$

ex: $\lim_{x \to 0^+} \ln \underbrace{x}_{\downarrow\, 0^+} = -\infty$

Note: $\ln x = \log_e x$
$e \approx 2.718$

ex: $\lim_{x \to 3^+} \log_2(x-3) = \log_2(3^+ - 3)$
$= \log_2(0^+)$
$= -\infty$

ex: $\lim_{x \to 2} \ln(x^2 - 4x + 4)$:
$= \lim_{x \to 2} \ln(x-2)^2$

- $\lim_{x \to 2^-} \ln(x-2)^2 = \ln(2^- - 2)^2$
  $= \ln(0^-)^2$
  $= \ln(0^+)$
  $= -\infty$

- $\lim_{x \to 2^+} \ln(x-2)^2 = \ln(2^+ - 2)^2$
  $= \ln(0^+)^2$
  $= \ln(0^+)$
  $= -\infty$

Restrictions:
$(x-2)^2 > 0$

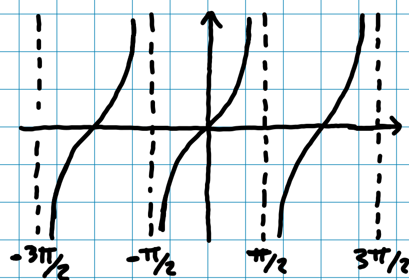
| | 2 | |
|---|---|---|
| $x-2$ | $-$ 0 $+$ | |
| $x-2$ | $-$ 0 $+$ | |
| | $+$ 0 $+$ | |

$x \in (-\infty, 2) \cup (2, +\infty)$
$x \neq 2$

$\therefore$ If $x \neq 2$, it's always greater than 0

# Tan function

$f(x) = \tan x$

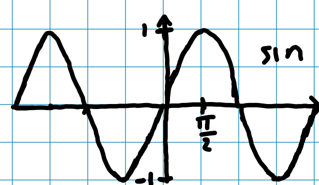


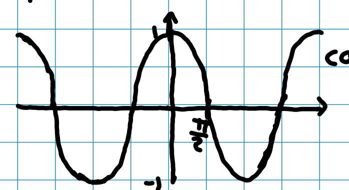

$y = \tan x$

- $\lim_{x \to \frac{\pi}{2}^-} f(x) = +\infty$
- $\lim_{x \to \frac{\pi}{2}^+} f(x) = -\infty$

The tan function is actually a type $\frac{k}{0}$

$\tan x = \frac{\sin x}{\cos x}$

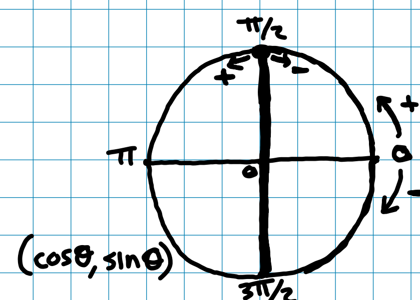



$\lim_{x \to \pi/2^-} \sin x = 1$
$\lim_{x \to \pi/2^+} \sin x = 1$

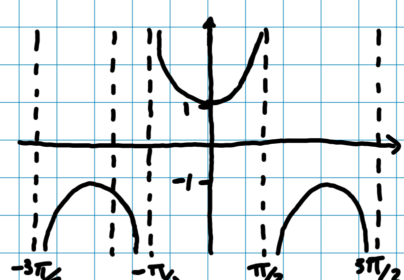
$\lim_{x \to \pi/2^-} \cos x = 0^+$
$\lim_{x \to \pi/2^+} \cos x = 0^-$

- $\lim_{x \to \pi/2^-} \tan x = \lim_{x \to \pi/2^-} \frac{\sin x}{\cos x} = \frac{1}{0^+} = +\infty$
- $\lim_{x \to \pi/2^+} \tan x = \lim_{x \to \pi/2^+} \frac{\sin x}{\cos x} = \frac{1}{0^-} = -\infty$



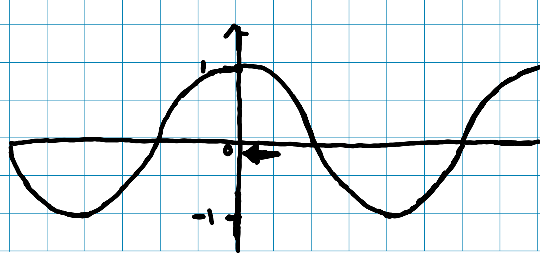

# Sec Function

$f(x) = \sec x$




- $\lim_{x \to \pi/2^-} \sec x = \lim_{x \to \pi/2^-} \frac{1}{\cos x} = \frac{1}{0^+} = +\infty$
- $\lim_{x \to \pi/2^+} \sec x = \lim_{x \to \pi/2^+} \frac{1}{\cos x} = \frac{1}{0^-} = -\infty$

ex: $\lim_{x \to 0^+} \frac{-2}{1 - \cos x} = \frac{-2}{1 - (1^-)} = \frac{-2}{0^+} = -\infty$




$\lim_{x \to 0^+} \cos x = 1^-$

# Continuity

**Definition:** The function $y = f(x)$ is continuous at $x = a$
if $\underbrace{\lim_{x \to a} f(x)}_{\text{exists}} = \underbrace{f(a)}_{\text{exists}}$

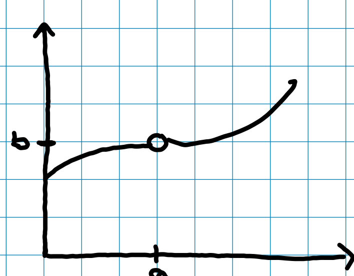
If not, the function is discontinuous at $x = a$

* Graphically, you can draw the function that is continuous without lifting your pencil

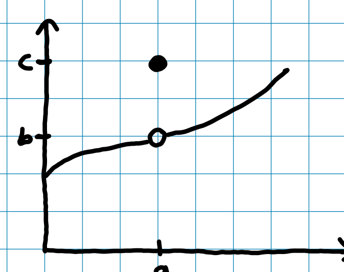
* To verify if $f(x)$ is continuous at $x = a$
  - → does $f(a)$ exist
  - → does $\lim_{x \to a} f(x)$ exist (as finite number)
    - $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$
  - → does $f(a) = \lim_{x \to a} f(x)$

## Types of discontinuity

① Removable discontinuity ($\lim_{x \to a} f(x)$ exists: $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$)



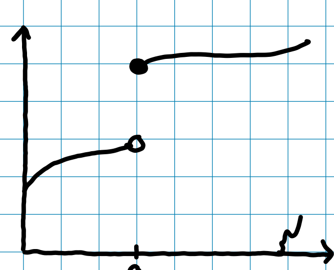

$f(a)$ dne
$\lim_{x \to a} f(x) = b$




$f(a) = c$
$\lim_{x \to a} f(x) = b$

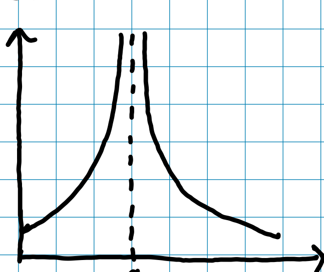
$f(a) \neq \lim_{x \to a} f(x)$

② Step discontinuity

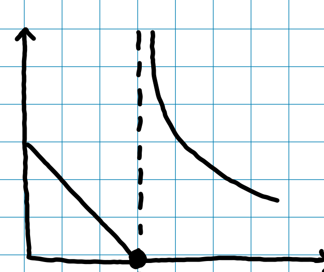



$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

③ Infinite discontinuity




$f(a)$ DNE




$f(a) = 0$
$\lim_{x \to a^+} f(x) = +\infty$

Most functions (we learnt) are continuous on their domain.
Even functions that have vertical asymptotes since that value of $x$ is not part of the domain.

* Piece-wise functions

ex: $f(x) = \begin{cases} x^2+1 & \text{if } x \leq 0 \\ 2-x & \text{if } 0<x<2 \\ 1 & \text{if } x=2 \\ (x-2)^2 & \text{if } x>2 \end{cases}$ Dom f : $x \in \mathbb{R}$

* potential values where f is discontinuous:

- at $x=0$ $f(0) = (0)^2+1 = 1$

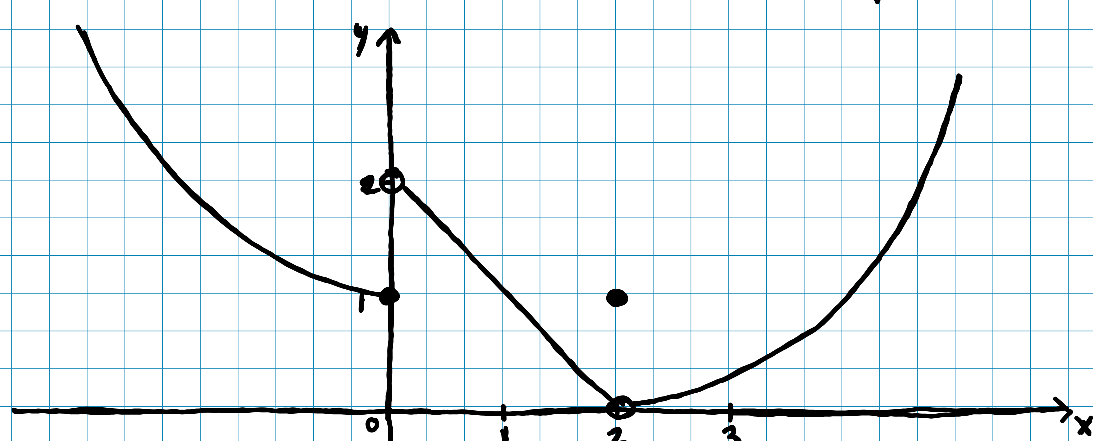
  $\lim_{x\to 0^-} x^2+1 = 1$
  $\lim_{x\to 0^+} 2-x = 2$
  $\lim_{x\to 0} f(x)$ DNE

  $\therefore$ f discontinuous at $x=0$

- at $x=2$ $f(2) = 1$

  $\lim_{x\to 2^-} 2-x = 0$
  $\lim_{x\to 2^+} (x-2)^2 = 0$
  $\lim_{x\to 2} f(x) = 0$

  $\therefore$ f discontinuous at $x=2$ $\left(\lim_{x\to 2} f(x) \neq f(2)\right)$
  $\therefore$ removable discontinuity ($\lim_{x\to 2} f(x)$ exists)

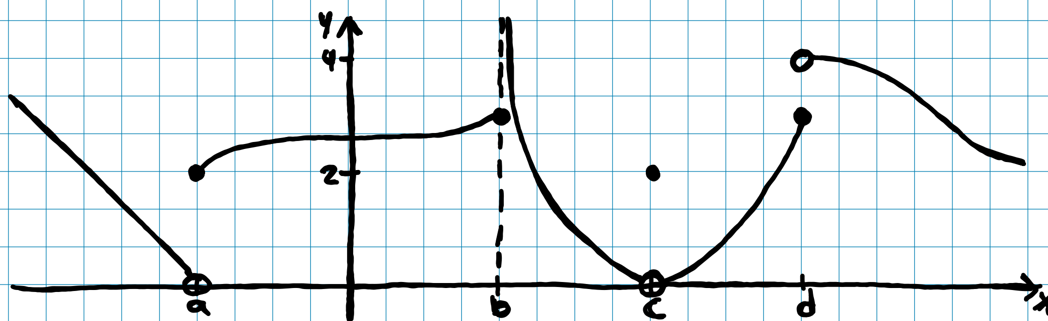



* Finding potential values where f is discontinuous
→ Restrictions
→ Seperation between piece-wise

# One-sided Continuity (weaker condition)

**Definition:** The function $y=f(x)$ is
- continuous from the left (left-continuous) if $\lim_{x\to a^-} f(x) = f(a)$
- continuous from the right (right-continuous) if $\lim_{x\to a^+} f(x) = f(a)$

**Note:** If $f$ is both continuous from the right and left at $x=a$ $\therefore f$ is continuous at $x=a$




* Closed circle on the side of the one-sided continuity

| f discontinuous at | b/c | One-sided continuity |
|---|---|---|
| $x=a$ | $\lim_{x\to a} f(x)$ DNE | Right - continuous |
| $x=b$ | $\lim_{x\to b} f(x)$ DNE | Left - continuous |
| $x=c$ | $\lim_{x\to c} f(x) \neq f(c)$ | $\emptyset$ none |
| $x=d$ | $\lim_{x\to d} f(x)$ DNE | Left - continuous |

# Horizontal Asymptote (Limits to infinity)

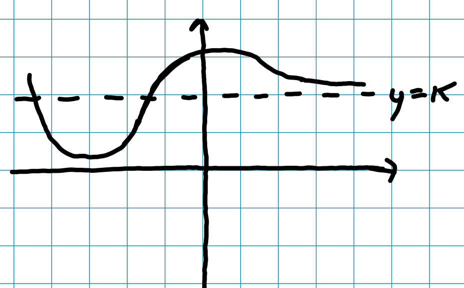
**Definition:** (limit at infinity)
Let $y = f(x)$ be a function
$\lim_{x \to \pm\infty} f(x) = K$ if the y value gets closer to K a x grows without bouds

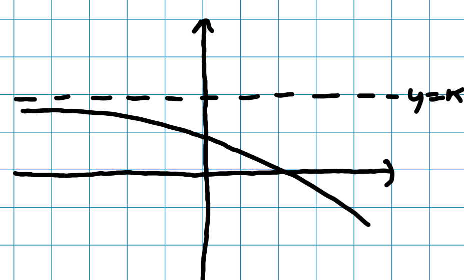
**Definition:** (horiz. asympt)
The line $y = K$ is a horizontal asymptote (h.a) of the function $y = f(x)$
if $\lim_{x \to -\infty} f(x) = K$ (asymptote on left)
or $\lim_{x \to +\infty} f(x) = K$ (asymptote on right)

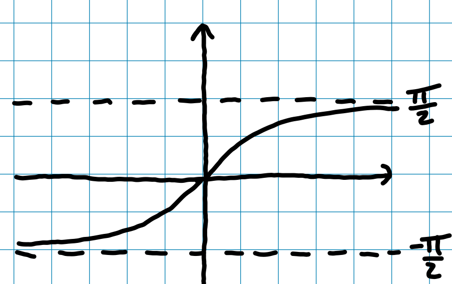
$K \in \mathbb{R}$ (finite number → not $\infty$)

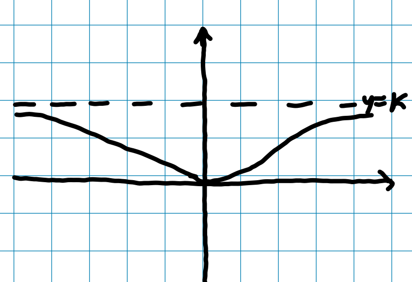
Note: it is false to believe that the function never touches vertical or horizontal asymptotes

ex:




$\lim_{x \to -\infty} f(x) = K = \lim_{x \to +\infty} f(x)$


$\lim_{x \to +\infty} f(x) = K$

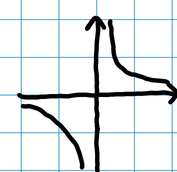
# Computing limits at infinity

Rule: Let $n \in \mathbb{R}$ $(n > 0)$

- If $n$ is even : $\lim_{x \to +\infty} \frac{1}{x^n} = 0^+$, $\lim_{x \to -\infty} \frac{1}{x^n} = 0^+$
- If $n$ is odd : $\lim_{x \to +\infty} \frac{1}{x^n} = 0^+$, $\lim_{x \to -\infty} \frac{1}{x^n} = 0^-$

ex: $\lim_{x \to +\infty} \frac{1}{x} = \frac{1}{+\infty} = 0^+$

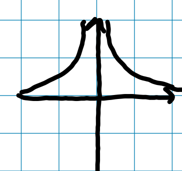
$\lim_{x \to -\infty} \frac{1}{x} = \frac{1}{-\infty} = 0^-$



$\therefore$ $y=0$ is h.a

ex. $\lim_{x \to +\infty} \frac{1}{x^2} = \frac{1}{+\infty} = 0^+$

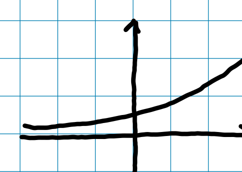
$\lim_{x \to -\infty} \frac{1}{x^2} = \frac{1}{(-\infty)^2} = \frac{1}{+\infty} = 0^+$



$\therefore$ $y=0$ is h.a.

ex $\lim_{x \to -\infty} e^x = e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{+\infty} = 0^+$

$\lim_{x \to +\infty} e^x = e^{+\infty} = +\infty$ (not finite num $\therefore$ not h a)



* Complicated rational functions

↳ factor out highest power of $x$ $\Rightarrow$ Type $\frac{\infty}{\infty}$

ex: $\lim_{x \to +\infty} \frac{3x^2 + x - 17}{5 - x^2}$

$= \lim_{x \to +\infty} \frac{\cancel{x^2}\left(3 + \frac{1}{x} - \frac{17}{x^2}\right)}{\cancel{x^2}\left(\frac{5}{x^2} - 1\right)} = \frac{3 + 0^+ - 0^+}{0^+ - 1} = -3$

Note: $\sqrt{x^2} = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

$\therefore$ if $\lim_{x \to -\infty} \sqrt{x^2} = \lim_{x \to -\infty} |x| = \lim_{x \to -\infty} (-x)$ $\because -\infty < 0$

$\therefore$ if $\lim_{x \to \infty} \sqrt{x^2} = \lim_{x \to \infty} |x| = \lim_{x \to \infty} (x)$ $\because \infty \geq 0$

## Algebra of infinity

$K > 0 \quad K \in \mathbb{R}$

$\infty + \infty = +\infty$
$-\infty - \infty = -\infty$

$\infty \cdot \infty = +\infty$
$-\infty \cdot -\infty = +\infty$
$-\infty \cdot \infty = -\infty$

$K(\infty) = +\infty$
$-K(\infty) = -\infty$

$\frac{K}{\infty} = 0^+$

$\frac{K}{-\infty} = 0^-$

$\frac{0}{\infty} = 0$

$\frac{\infty}{K} = \infty$

$\frac{-\infty}{K} = -\infty$

$\frac{\infty}{0^-} = -\infty$

$\frac{\infty}{0^+} = +\infty$

* We can evaluate all of these directly

## Indeterminate forms

* Cannot evaluate directly

$\rightarrow \frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \cdot \infty$

ex: $\lim_{x\to-\infty} \dfrac{2x^2-x+2}{5x^3+x+7}$ $\quad \dfrac{\infty}{\infty}$

$= \lim_{x\to-\infty} \dfrac{\cancel{x^2}\left(2-\frac{1}{x}+\frac{2}{x^2}\right)}{x^{\cancel{3}}\left(5+\frac{1}{x^2}+\frac{7}{x^3}\right)}$

$= \lim_{x\to-\infty} \dfrac{2}{5}\cdot \dfrac{1}{x} = 0$, $\quad \dfrac{1}{x} \to 0$

ex: $\lim_{x\to\infty} \dfrac{3x^3+5x+2}{x-1}$ $\quad \dfrac{\infty}{\infty}$

$= \lim_{x\to\infty} \dfrac{x^3\left(3+\frac{5}{x^2}+\frac{2}{x^3}\right)}{x\left(1+\frac{1}{x}\right)}$

$= \lim_{x\to\infty} x^2 \cdot 3 = \infty$, $\quad x^2 \to \infty$ $\quad \therefore$ no horizontal asymptote

ex: $\lim_{x\to\infty} \dfrac{\sqrt{x^2+2}}{(x+1)(x-2)}$ $\quad$ * expand denom

$= \lim_{x\to\infty} \dfrac{\sqrt{x^2+2}}{x^2-x-2}$

$= \lim_{x\to\infty} \dfrac{\sqrt{x^2\left(1+\cancel{\frac{2}{x^2}}\right)}}{x^2\left(1-\cancel{\frac{1}{x}}-\cancel{\frac{2}{x^2}}\right)}$

$= \lim_{x\to\infty} \dfrac{\sqrt{x^2}}{x^2}$ $\quad$ * $|x| = x$ when $x = +\infty$

$= \lim_{x\to\infty} \dfrac{|x|}{x^2} = \dfrac{\cancel{x}}{x^{\cancel{2}}} = \dfrac{1}{\infty} = 0$

ex: $\lim_{x\to-\infty} \dfrac{3x^2-x+5}{7x^3+2x-1}$ $\quad \dfrac{\infty}{\infty}$

$= \lim_{x\to-\infty} \dfrac{\cancel{x^2}\left(3-\cancel{\frac{1}{x}}-\cancel{\frac{5}{x^2}}\right)}{x^{\cancel{3}}\left(7+\cancel{\frac{2}{x^2}}-\cancel{\frac{1}{x^3}}\right)}$

$= \lim_{x\to-\infty} \dfrac{3}{7}\cdot\dfrac{1}{x} = 0$, $\quad \dfrac{1}{x} \to 0$

Find all asymptotes

ex: $y = \frac{\sqrt{x^2+2x}}{x-5}$

Vertical asymptotes:

potential values: $x = 5$

Restrictions: $x \neq 5$

$x^2 + 2x \geq 0$

$\therefore x \in (-\infty, -2] \cup [0, +\infty)$

• at $x = 5$ $\quad \lim_{x \to 5} \frac{\sqrt{x^2+2x}}{x-5}$ $\quad \frac{k}{0}$

• $\lim_{x \to 5^-} \underbrace{\sqrt{x^2+2x}}_{\downarrow \sqrt{35}} \cdot \underbrace{\frac{1}{x-5}}_{\downarrow -\infty} = -\infty$

• $\lim_{x \to 5^+} f(x) = +\infty$ $\quad \therefore x = 5$

Horizontal asymptotes:

$\lim_{x \to -\infty} \frac{\sqrt{x^2+2x}}{x-5}$ $\quad \frac{\infty}{\infty}$

$= \lim_{x \to -\infty} \frac{\sqrt{x^2(1+\cancel{\frac{2}{x}})}}{x(1-\cancel{\frac{5}{x}})}$

$= \lim_{x \to -\infty} \frac{|x|}{x}$ $\quad * \ |x| = -x$ when $x = -\infty$

$= \lim_{x \to -\infty} \frac{-x}{x} = -1$

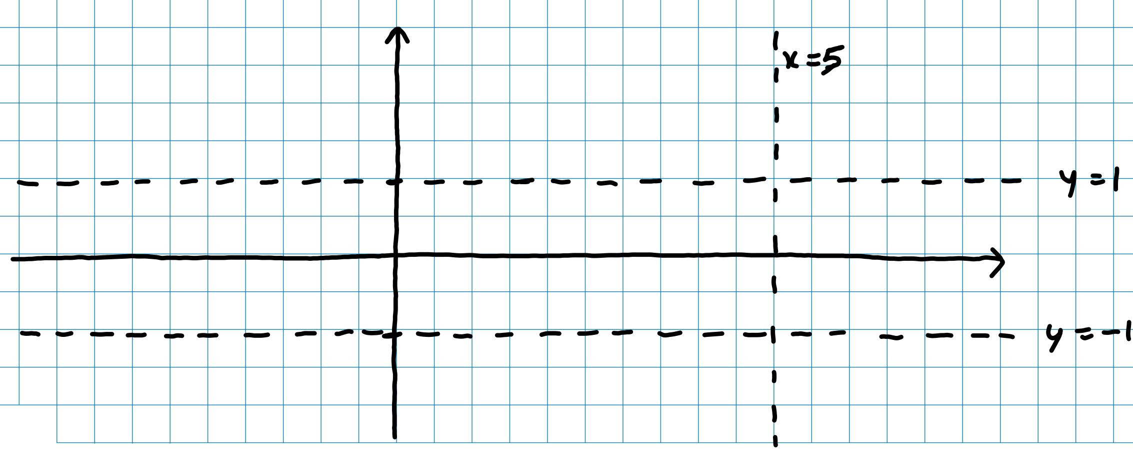
$\therefore y = -1$

$\lim_{x \to \infty} \frac{\sqrt{x^2+2x}}{x-5}$

$= \lim_{x \to \infty} \frac{|x|}{x}$

$= \lim_{x \to \infty} \frac{x}{x} = 1$

$\therefore y = +1$



Other functions with limits at infinity:

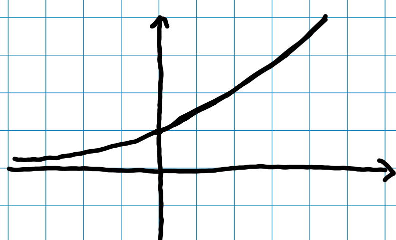
→ $e^x$

→ trig

* Theorem: If $f$ is continuous at $b$ and $\lim_{x \to a} g(x) = b$

$\therefore \lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(b)$

<u>$e^x$ (exponential)</u>



$\lim_{x \to -\infty} e^x = e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0^+$

$\lim_{x \to \infty} e^x = e^{\infty} = \infty \therefore$ no asym

ex: $\lim_{x \to -\infty} \frac{3}{1+2e^x} = \frac{3}{1+0} = 3$

ex: $\lim_{x \to \infty} \frac{e^{1/x}+2}{e^{-x}-3} =$

$= \lim_{x \to \infty} \frac{e^{1/x}+2}{0-3}$

$= \lim_{x \to \infty} -\frac{1}{3}(e^{1/x}+2)$ * Theorem

$= -1/3 \left(e^{\lim_{x \to \infty} 1/x} + 2\right)$

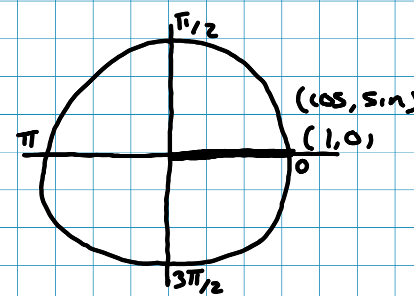
$= -1/3 \left(e^{1/\infty} + 2\right) = -1/3 \left(e^0 + 2\right) = -1/3 (3) = -1$

ex $\lim_{x \to \infty} \frac{e^x}{e^{2x}+1}$

$= \lim_{x \to \infty} \frac{e^x}{e^{2x}(1+\frac{1}{e^x})}$

$= \lim_{x \to \infty} \frac{1}{e^x(1+\frac{1}{\infty})} = \frac{1}{e^{\infty}} = 0$

## Trig Function




ex: $\lim_{x\to\infty} \sin \frac{1}{x}$

$= \lim_{x\to\infty} \sin \frac{1}{\infty}$

$= \sin 0 \quad = 0$

ex: $\lim_{x\to\infty} \tan\left(\frac{3}{x^2+2}\right)$

$= \lim_{x\to\infty} \tan\left(\frac{3}{\infty}\right)$

$= \tan(0)$

$= \frac{\sin 0}{\cos 0} \quad = \frac{0}{1} = 0$

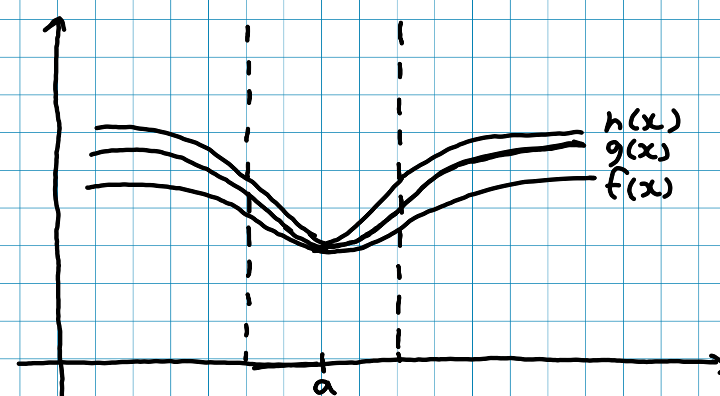
## Squeeze Principal

If $f(x) \leq g(x) \leq h(x)$
for x-values near a

$\to \lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$

$\therefore \lim_{x\to a} g(x) = L$

$\because$ g(x) "squeezed" in between




| Reminder : | $-1 \leq \sin x \leq 1$ |
|---|---|
| | $-1 \leq \cos x \leq 1$ |

ex: $\lim_{x\to 0} x^2 \sin \frac{1}{x}$ $\Rightarrow$ undefined b/c $\sin \frac{1}{0} = \sin(\infty)$

$\therefore$ squeeze $\Rightarrow$ $-1 \leq \sin 1/x \leq 1$

$-1(x^2) \leq x^2 \sin 1/x \leq 1(x^2)$

$-x^2 \leq x^2 \sin 1/x \leq x^2$

(f(x)): $\bullet \lim_{x\to 0} -x^2 = 0$

(h(x)): $\bullet \lim_{x\to 0} x^2 = 0$

$\Big\rangle \lim_{x\to 0} x^2 \sin \frac{1}{x} = 0$

# Type $\infty - \infty$ (indeterminate form)

- Rational or normal function : factor highest power of $x$
- Square root : multiply by conjugate (rationalyse)

ex: $\lim_{x\to\infty} 5x^3 - 3x^2$ $\quad = \lim_{x\to\infty} x^3(5 - \frac{3}{x})$

$\infty - \infty$

$= \infty^3 \cdot 5$

$= \infty$

ex: $\lim_{x\to\infty} \sqrt{x^2+1} - x$ $\quad = \lim_{x\to\infty} \sqrt{x^2+1} - x \cdot \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x}$

$\infty - \infty$

RATIONALISE $\quad = \lim_{x\to\infty} \frac{(x^2+1) - x^2}{\sqrt{x^2+1} + x}$

$= \lim_{x\to\infty} \frac{1}{\infty + \infty}$

$= \frac{1}{\infty} = 0$

* Always verify if you have $\infty + \infty$ or something else you can compute

ex: $\lim_{x\to-\infty} x - \sqrt{x^2+1}$

$-\infty - (\infty)$

* This can be computed : $-\infty - \infty = -\infty$

$= \lim_{x\to-\infty} -\infty - \sqrt{(-\infty)^2 + 1}$

$= -\infty - \sqrt{\infty + 1}$

$= -\infty - (\infty)$

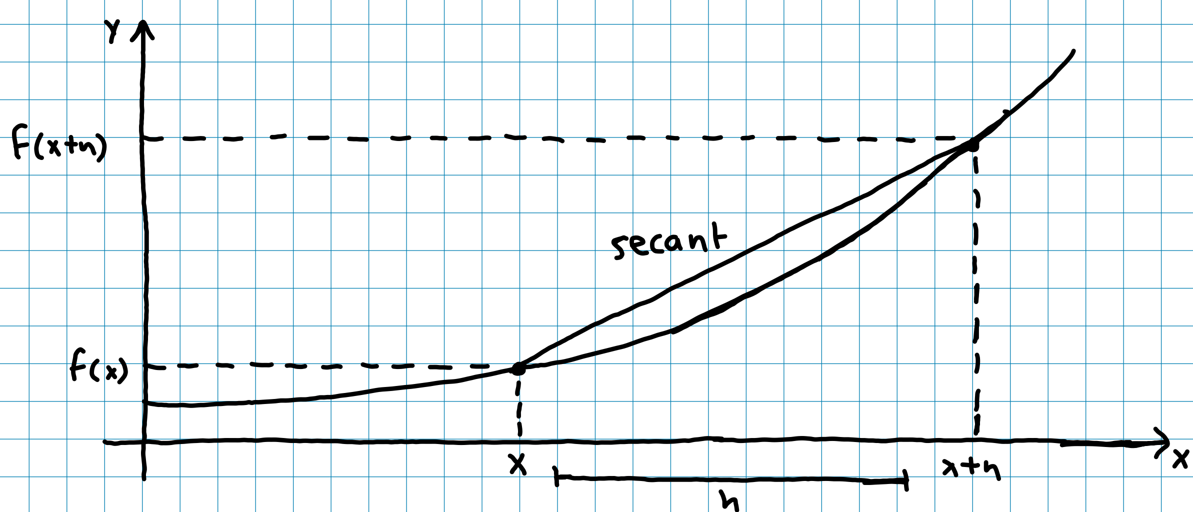
$= -\infty$

# Derivatives

Let $f$ be a continuous function

The derivative of $f$ at point $(x, f(x))$ is the slope of the tangent to the graph.

Slope $m_{TAN}$ in $y = m_{tan}x + b$




consider the slope of the secant

$$m_{sec} = \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{x+h - x} = \frac{f(x+h) - f(x)}{h}$$

When the value of $h$ approaches $0$, we get the slope of the tangent

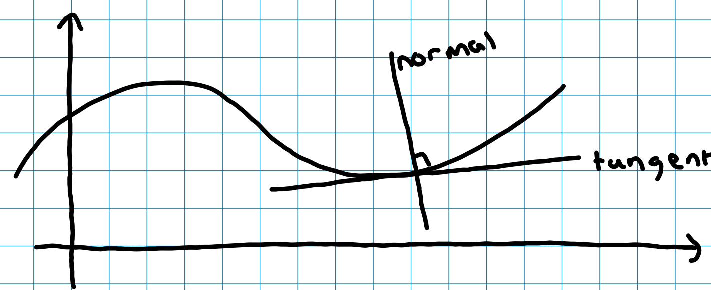
$$m_{TAN} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivative Function

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Derivative at $x = a$ : $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

# Tangent and normal line




$$m_{tan} \cdot m_{nor} = -1$$

$$m_{nor} = \frac{-1}{f'(x)}$$

ex: Find equation of tangent line of $y = x^2 + 1$ at $x = 1$

$$m_{tan} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[(x+h)^2 + 1] - [x^2 + 1]}{h}$$

sub $x = 1$

$$= \lim_{h \to 0} \frac{(1+h)^2 + 1 - 1^2 - 1}{h}$$

$$= \lim_{h \to 0} \frac{1 + 2h + h^2 + 1 - 1 - 1}{h}$$

$$= \lim_{h \to 0} \frac{2h + h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2+h)}{h}$$

$$= \lim_{h \to 0} 2 + h = 2$$

Finding the equation:

$$y = mx + b$$
$$y = 2x + b$$

at $x = 1$, $y = (1)^2 + 1 = 2$

$$2 = 2(1) + b$$
$$b = 0$$

$\therefore$ tan line: $y = 2x$

ex: Find equation of the normal line to $y = \sqrt{x+2}$ at $x = -1$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{(-1+h) + 2} - \sqrt{-1+2}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+h} - \sqrt{1}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+h} - \sqrt{1}}{h} \cdot \frac{\sqrt{1+h} + \sqrt{1}}{\sqrt{1+h} + \sqrt{1}}$$

$$= \lim_{h \to 0} \frac{(1+h) - 1}{h(\sqrt{1+h} + \sqrt{1})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{1+h} + \sqrt{1}}$$

$$f'(1) = 1/2$$

$$m_{nor} = \frac{-1}{f'(1)} = -2$$

Equation:

$$y = mx + b$$
$$y = -2x + b$$

at $x = -1$, $y = \sqrt{-1+2} = 1$

sub $(x, f(x))$

$$1 = -2(-1) + b$$
$$b = -1$$

$\therefore$ nor line: $y = -2x - 1$

Finding the derivative function

→ Can use it to find the slope of the tangent at different values of x

→ can sub $x=a$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

ex Find the derivative function of $y=x^2+1$

$$f'(x) = \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[(x+h)^2+1]-[x^2+1]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2+\cancel{1}-\cancel{x^2}-\cancel{1}}{h}$$

$$= \lim_{h\to 0} \frac{2xh+h^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \lim_{h\to 0} 2x+(0) = \boxed{2x = f'(x)}$$

ex: find derivative function of $f(x)=\sqrt{x+2}$
use it to find the tangent line at $x=1$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{(x+h)+2}-\sqrt{x+2}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h+2}-\sqrt{x+2}}{h} \cdot \frac{\sqrt{x+h+2}+\sqrt{x+2}}{\sqrt{x+h+2}+\sqrt{x+2}}$$

$$= \lim_{h\to 0} \frac{(\cancel{x}+h+\cancel{2})-(\cancel{x}+\cancel{2})}{h(\sqrt{x+h+2}+\sqrt{x+2})}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h+2}+\sqrt{x+2})}$$

$$= \lim_{h\to 0} \frac{1}{2\sqrt{x+2}}$$

$$f'(x) = \frac{1}{2\sqrt{x+2}} = m_{TAN}$$

At $x=1$

$$f'(1) = \frac{1}{2\sqrt{3}} = m_{tan}$$

sub $(x, f(x))$
$(1, \sqrt{3})$

$$y = mx+b$$

$$\sqrt{3} = \frac{1}{2\sqrt{3}}(1)+b$$

$$b = \frac{5}{2\sqrt{3}}$$

$$\therefore\ y = \frac{1}{2\sqrt{3}}(x) + \frac{5}{2\sqrt{3}}$$

## Notation and terminology

→ If $y = f(x)$ is a function, the derivative of the function can be denoted by:

- $f'(x)$
- $y'$
- $\left(\frac{d}{dx}\right) f(x)$ or $\frac{dy}{dx}$ or $\left(\frac{d}{dx}\right) y$ (Leibniz notation)

ex: if $y = f(x) = x^2 + 1$

- $f'(x) = 2x$ or $\frac{d}{dx} f(x) = 2x$
- $y' = 2x$ or $\frac{dy}{dx} = 2x$
- $(x^2+1)' = 2x$ or $\frac{d}{dx}(x^2+1) = 2x$

* Note: Finding the derivative = to differentiate

$\therefore$ the function is differentiable

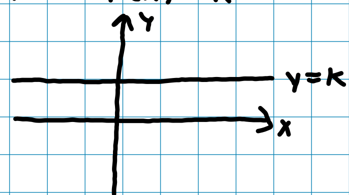
# Differentiation Techniques

## 1. Constant Rule

$$\boxed{f(x) = K \quad \rightarrow \quad f'(x) = 0}$$

* The derivative of a number (constant with no x) is always 0

Proof: $f(x) = K$




Slope of the tangent is always zero

## 2. Power Rule

$$\boxed{f(x) = a \cdot x^n \quad \rightarrow \quad f'(x) = a \cdot n\, x^{n-1}}$$

ex: $\frac{d}{dx}(x^5) = 5 \cdot x^{5-1} = 5x^4$

ex: $\left(\frac{1}{\sqrt{x}}\right)' = (x^{-1/2})' = -1/2 \cdot x^{-1/2-1} = -\frac{1}{2\,x^{3/2}}$

ex: $f(x) = \frac{1}{x^{3/2}} = x^{-3/2} \rightarrow f'(x) = -\frac{3}{2} \cdot x^{-3/2-1} = -\frac{3}{2\,x^{5/2}}$

ex: $f(x) = 5x \quad \rightarrow f'(x) = 5 \cdot 1 \cdot x^{1-1} = 5x^0 = 5$

## 3. Constant - multiple Rule

$$\boxed{f(x) = K \cdot g(x) \quad \rightarrow \quad f'(x) = K \cdot g'(x)}$$

ex: $(7x^3)' = 7(x^3)' = 7 \cdot 3x^2 = 21x^2$

* Always take out constant first (anything that doesn't have an x)

4. <u>Sum / Difference Rule</u>

$$\boxed{f(x) = g(x) \pm h(x) \longrightarrow f'(x) = g'(x) \pm h'(x)}$$

or $\frac{d}{dx}\left(h(x) \pm g(x)\right) = \frac{d}{dx} g(x) \pm \frac{d}{dx} h(x)$

* Proof : $f(x) = g(x) + h(x)$

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left(g(x+h) + h(x+h)\right) - \left(g(x) + h(x)\right)}{h}$$

$$= \lim_{h \to 0} \frac{g(x+h) - g(x) + h(x+h) - h(x)}{h}$$

$$= \lim_{h \to 0} \left( \underbrace{\frac{g(x+h) - g(x)}{h}}_{g'(x)} + \underbrace{\frac{h(x+h) - h(x)}{h}}_{h'(x)} \right)$$

$$= g'(x) + h'(x)$$

ex: $\left(3x^2 - \frac{1}{x} + 5\right)' = (3x^2)' - (x^{-1})' + (5)'$

$$= (3 \cdot 2x^1) - (-x^{-2}) + (0)$$

$$= 6x + \frac{1}{x^2}$$

## 5. Product Rule

$$\boxed{f(x) = g(x)\cdot h(x) \longrightarrow f'(x) = g'(x)\cdot h(x) + h'(x)\cdot g(x)}$$

or $\frac{d}{dx}\left(f(x)\cdot g(x)\right) = \frac{d}{dx}f(x)\cdot g(x) + \frac{d}{dx}g(x)\cdot f(x)$

* Note: $(f\cdot g\cdot h)' = f'\cdot g\cdot h + g'\cdot f\cdot h + h'\cdot f\cdot g$

→ Product rule for the product of 2 fcn, not the product of a coefficient and variable
ex: $5.x^2$ → no use of product rule

* Proof: $f(x) = g(x)\cdot h(x)$

$$f'(x) = \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{g(x+h)\cdot h(x+h) - g(x)\cdot h(x)}{h}$$

$$= \lim_{h\to 0} \frac{g(x+h)h(x+h) \overbrace{- g(x+h)h(x) + g(x+h)h(x)}^{+0} - g(x)h(x)}{h}$$

$$= \lim_{h\to 0}\left(\underbrace{g(x+h)}_{g(x)}\cdot \underbrace{\frac{h(x+h)-h(x)}{h}}_{h'(x)} + \underbrace{h(x)}_{h(x)}\cdot \underbrace{\frac{g(x+h)-g(x)}{h}}_{g'(x)}\right)$$

$$f'(x) = g'(x)\cdot h(x) + h'(x)\cdot g(x)$$

ex: $(x^2\cdot e^x)' = (x^2)'\cdot e^x + (e^x)'\cdot x^2$
$= 2x^{2-1}\cdot e^x + e^x\cdot x^2$
$= e^x(2x + x^2)$
$= x\cdot e^x(2+x)$

6. <u>Quotient Rule</u>

$$\boxed{f(x) = \frac{g(x)}{h(x)} \longrightarrow f'(x) = \frac{g'(x) \cdot h(x) - h'(x) \cdot g(x)}{(h(x))^2}}$$

$$\text{or} \quad \left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

$$* \text{ Note} : \left(\frac{1}{g(x)}\right)' = \frac{-g'(x)}{(g(x))^2}$$

$$\text{ex}: \left(\frac{3x^2+5x}{2x+1}\right)' = \frac{(3x^2+5x)' \cdot (2x+1) - (2x+1)' \cdot (3x^2+5x)}{(2x+1)^2}$$

$$= \frac{(6x+5) \cdot (2x+1) - (2) \cdot (3x^2+5x)}{(2x+1)^2}$$

$$\text{ex}: y = \frac{1}{x^2+2x} \longrightarrow y' = \frac{(1)'(x^2+2x) - (x^2+2x)' \cdot (1)}{(x^2+2x)^2}$$

$$= \frac{0(x^2+2x) - (2x+2)}{(x^2+2x)^2}$$

$$= \frac{-(2x+2)}{(x^2+2x)^2}$$

$$\text{ex}: \left(\frac{2}{xe^x}\right)' = 2\left(\frac{1}{xe^x}\right)' = 2 \cdot \frac{(1)' \cdot (xe^x) - (xe^x)' \cdot (1)}{(xe^x)^2}$$

$$= 2 \cdot \frac{-(xe^x)'}{(xe^x)^2}$$

7. <u>Derivative of $e^x$</u>

$$\boxed{f(x) = e^x \longrightarrow f'(x) = e^x}$$

$$\frac{d}{dx} e^x = e^x$$

8. <u>Derivative of $\ln x$</u>

$$\boxed{f(x) = \ln x \longrightarrow f'(x) = \frac{1}{x}}$$

* Reminder log rules:

$\log_a b = \frac{\log_c b}{\log_c a}$

$c^{(\log_c m)} = m$

$\log m \cdot n = \log m + \log n$

$\log \frac{m}{n} = \log m - \log n$

$\log_c m^n = n \log_c m$

ex: $(\log_2 x)' = \left(\frac{\ln x}{\ln 2}\right)' = \frac{1}{\ln 2} \cdot (\ln x)' = \frac{1}{x \ln 2}$

Finding tan line given a slope

→ parralel to ... ⇒ same slope
→ perpendicular to ... ⇒ slope × slope = −1
→ horizontal ... ⇒ slope = 0

$f(x) = e^x + 1$, find tan line parralel to $y = 5x + 1$

$f'(x) = 5 = m_{tan}$

$f'(x) = (e^x)' + (1)'$
$= e^x + 0$

$e^x = 5$
$\log_e 5 = x$

equation:
$y = 5x + b$
sub $(\ln 5, f(\ln 5))$

$6 = 5(\ln 5) + b$
$b = 6 - 5\ln 5$

$\therefore y = 5x + 6 - 5\ln 5$

## 9. Derivative of Trigonometric Functions

$$\boxed{\begin{aligned}
(\sin x)' &= \cos x \\
(\cos x)' &= -\sin x \\
\\
(\tan x)' &= \sec^2 x \\
(\cot x)' &= -\csc^2 x \\
\\
(\sec x)' &= \sec x \cdot \tan x \\
(\csc x)' &= -\csc x \cdot \cot x
\end{aligned}}$$

Reminder:

Special limits:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$$

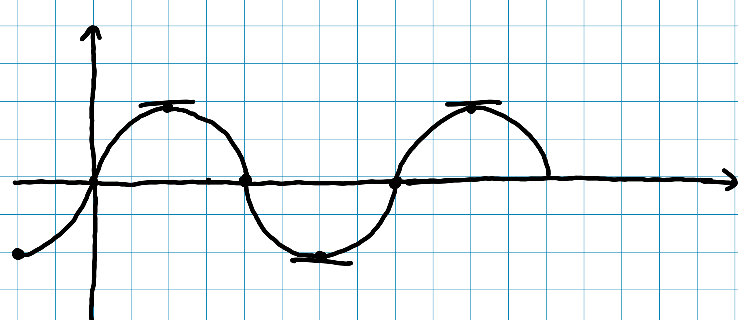
Pythagorus:

$$\sin^2 x + \cos^2 x = 1$$

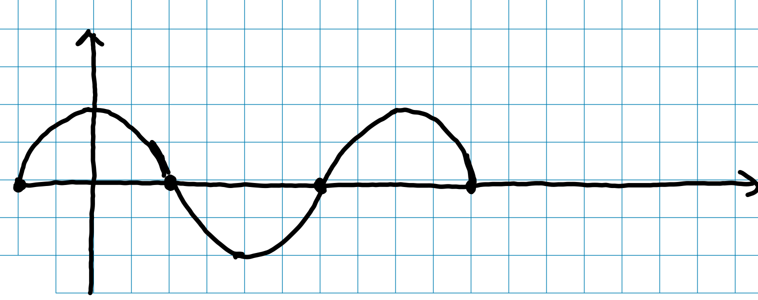
### Proof of $(\sin x)' = \cos x$

$$\boxed{\begin{aligned}
\sin(a+b) &= \sin a \cos b + \sin b \cos a \\
\cos(a+b) &= \cos a \cos b - \sin a \sin b
\end{aligned}}$$

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h \to 0} \frac{\sin x \cdot \cos h + \sin h \cdot \cos x - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x (\cos h - 1)}{h} + \frac{\sin h \cdot \cos x}{h} \\
&= \lim_{h \to 0} \underbrace{\sin x}_{} \cdot \underbrace{\frac{\cos h - 1}{h}}_{} + \underbrace{\cos x}_{} \cdot \underbrace{\frac{\sin h}{h}}_{} \\
&= \sin x \cdot 0 + \cos x \cdot 1 \\
&= \cos x
\end{aligned}$$



$y = \sin x$



$y' = \cos x$

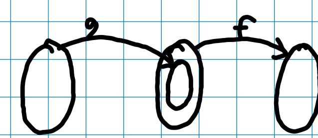
## 10. Chain Rule

$$\boxed{(f \circ g)'(x) = f'(g(x)) \cdot g'(x)}$$

Composition of fcn.

$(f \circ g)(x) = f(g(x))$

outside     inside




* can name inside fcn a $u$

$$\frac{d}{dx}(f \circ g) = \frac{d}{du} f(u) \cdot \frac{d}{dx} u$$
$$= f'(u) \cdot u'$$

ex: $y = (x^2+2x)^7$

Let $u = x^2+2x$, $f(u) = u^7$

$$\frac{dy}{dx} = \frac{d}{du} f(u) \cdot \frac{du}{dx}$$
$$= (u^7)' \cdot (x^2+2x)'$$
$$= 7u^6 \cdot (2x+2)$$

sub $u \rightarrow$
$$= 7(x^2+2x)^7 \cdot (2x+2)$$

ex: $f(x) = \sqrt{3x^2+5}$

Let $u = 3x^2+5$, $f(u) = \sqrt{u}$

$$\frac{d}{dx} f(x) = \frac{d}{du} f(u) \cdot \frac{d}{dx} u$$
$$= (u^{1/2})' \cdot (3x^2+5)'$$
$$= (\tfrac{1}{2} u^{-1/2}) \cdot (6x+0)$$

sub $u \rightarrow$
$$= \frac{1}{2\sqrt{3x^2+5}} \cdot 6x$$

ex. $y = \sin^2 x^2$

Let $f(u) = u^2$, $u = \sin v$, $v = x^2$

$$\frac{dy}{dx} = \frac{d}{du} \cdot f(u) \cdot \frac{d}{dv} u \cdot \frac{d}{dx} v$$
$$= (u^2)' \cdot (\sin v)' \cdot (x^2)'$$
$$= 2u \cdot \cos v \cdot 2x$$

sub $u$ and $v \rightarrow$
$$= 2 \sin x^2 \cdot \cos x^2 \cdot 2x$$
$$= 4x \cdot \sin x^2 \cdot \cos x^2$$

ex: $y = \sqrt{\dfrac{1}{x^2+5x+3}}$

$y = (x^2+5x+3)^{-1/2}$

Let $u = x^2+5x+3$, $f(u) = u^{-1/2}$

$y' = \frac{d}{du} f(u) \cdot \frac{d}{dx} u$

$= (u^{-1/2})' \cdot (x^2+5x+3)'$

$= (-1/2\ u^{-3/2})\ (2x+5+0)$

$= \dfrac{2x+5}{-2\ u^{3/2}}$

sub $u \to$

$= -\dfrac{2x+5}{2(x^2+5x+3)^{3/2}}$

ex: $f(x) = [\ln(\sec^2 x^3)]^3$

Let $f(u) = u^3$, $u = \ln v$, $v = w^2$, $w = \sec z$, $z = x^3$

$\frac{d}{dx} f(u) = \frac{d}{du} f(u) \cdot \frac{d}{dv} u \cdot \frac{d}{dw} v \cdot \frac{d}{dz} w \cdot \frac{d}{dx} z$

$= (u^3)'\ (\ln v)'\ (w^2)'\ (\sec z)'\ (x^3)'$

$= 3u^2 \cdot \frac{1}{v} \cdot 2w \cdot \sec z \tan z \cdot 3x^2$

sub $\to$

$= 3(\ln \sec^2 x^3)^2 \cdot \dfrac{1}{\sec^2 x^3} \cdot 2 \sec x^3 \cdot \sec x^3 \cdot \tan x^3 \cdot 3x^2$

$= 18(\ln \sec^2 x^3)^2 \cdot \dfrac{\sec^2 x^3 \cdot \tan x^3 \cdot \cancel{3}x^2}{\sec^2 x^3}$

$= 18(\ln \sec^2 x^3)^2 \cdot \tan x^3 \cdot x^2$

ex: $f(x) = \sqrt{\sin 3x + e^x \cot x}$

$= (\sin 3x + e^x \cot x)^{1/2}$

Let $u = \sin 3x + e^x \cot x$, $f(u) = u^{1/2}$

$f'(u) = (u^{1/2})' \cdot (\sin 3x + e^x \cot x)'$

$= (1/2\ u^{-1/2}) \cdot ((\sin 3x)' + (e^x)' \cot x + (\cot x)' \cdot e^x)$

$= \dfrac{1}{\sqrt{\sin 3x + e^x \cot x}} \cdot (3\cos 3x + e^x \cot x - \csc^2 x \cdot e^x)$

$= \dfrac{3\cos(3x) + e^x \cot x - e^x \csc^2 x}{\sqrt{\sin 3x + e^x \cot x}}$

$f(v) = \sin v$
$v = 3x$

$(\sin 3x)' =$
$= (\sin v)' \cdot (3x)'$
$= (\cos v) \cdot 3$
$= 3 \cos(3x)$

* Chain rule within chain rule

Patterns using chain rule

Let $u = g(x)$

① if $y = f(ax+b) \to y' = a \cdot f'(ax+b)$ * Linear $ax+b$

ex: $y = \sin(3x+5) \to y' = 3\cos(3x+5)$
$y = e^{7x-2} \to y' = 7 \cdot e^{7x-2}$

② $\frac{d}{dx} u^K = \frac{d}{du} u^K \cdot \frac{du}{dx} = K u^{K-1} \cdot \frac{du}{dx}$

③ $\frac{d}{dx} e^u = \frac{d}{du} e^u \cdot \frac{du}{dx} = e^u \cdot \frac{du}{dx}$

④ $\frac{d}{dx}(\ln u) = \frac{u'}{u}$

⑤ $\frac{d}{dx}(\sin u) = \cos u \cdot \frac{du}{dx}$

$\frac{d}{dx}(\tan u) = \sec^2 u \cdot \frac{du}{dx}$

$\frac{d}{dx}(\sec u) = \sec u \cdot \tan u \cdot \frac{du}{dx}$

Note:

$\boxed{\frac{d}{dx}(a^x) = a^x \cdot \ln a}$ $0 < a < 1$, $a > 1$

Proof. $(a^x)' =$
$= (e^{\ln a^x})'$

Let $u = \ln a^x$, $f(u) = e^u$

$= \frac{d}{du} e^u \cdot \frac{du}{dx}$
$= e^{\ln a^x} \cdot (\ln a^x)'$
$= e^{\ln a^x} \cdot (x \cdot \ln a)'$
$= a^x \cdot (\ln a)$

Reminder:

$e^{\ln a} = a$

$\ln e^a = a$

$e^{\ln 1} = 1$

$\ln e = 1$

$a^x = e^{\ln a^x}$

ex: $(2^{\sin^2 x})'$ Let $f(u) = 2^u$, $u = v^2$, $v = \sin x$

$= \frac{d}{du} 2^u \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$

$= 2^u \cdot \ln 2 \cdot (2v)(\cos x)$

$= 2^{\sin^2 x}(\ln 2)(2\sin x)(\cos x)$

* use $(a^x)' = a^x \cdot \ln a$

# Differentiable or not

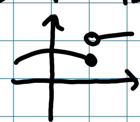
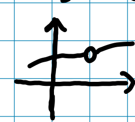
**Definition:** A function $f$ is differentiable at $x=a$ if $f'(a)$ exists.

→ If not, we say the function is not differentiable at $x=a$

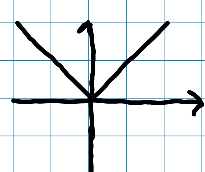
* If $y=f(x)$ is differentiable at $x=a$, $\therefore$ it is continuous at $x=a$

## How can $f'(a)$ fail to exist?
( $\therefore$ not differentiable)

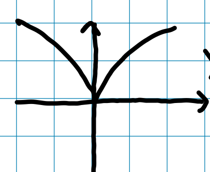
① $f$ is discontinuous at $a$

② $f$ has a corner or cusp at $x=a$

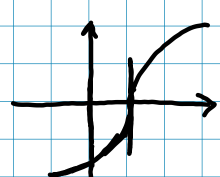


$y=|x|$

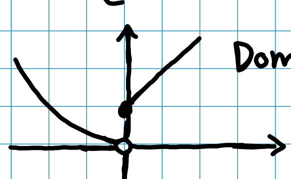


$y=x^{2/3}$

③ $f$ has vertical tangent line at $x=a$ (no slope)



ex: $f(x)=\begin{cases} x^2 & \text{if } x<0 \\ x+1 & \text{if } x \geq 0 \end{cases}$



Dom $f$: $\mathbb{R}$

$f$ discontinuous $x=0$

$f'(0)$ DNE

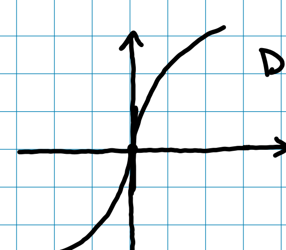
$\therefore$ not differentiable

$f'(x)=\begin{cases} 2x & \text{if } x<0 \\ 1 & \text{if } x>0 \end{cases}$

note we remove "or equal" $\because$ $f'(0)$ DNE

Dom $f'$: $\mathbb{R}\setminus\{0\}$

ex: $f(x)=x^{1/3}$



Dom $f$: $\mathbb{R}$

$f'(0)$ DNE

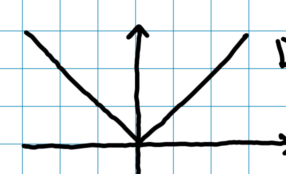
$\because$ vertical tangent line

$\therefore$ not differentiable

$f'(x)=\dfrac{1}{3x^{2/3}}$

Dom $f'$: $\mathbb{R}\setminus\{0\}$

ex: $f(x)=|x|$



Dom $f$: $\mathbb{R}$

$f$ ~~does~~ has corner at $x=0$

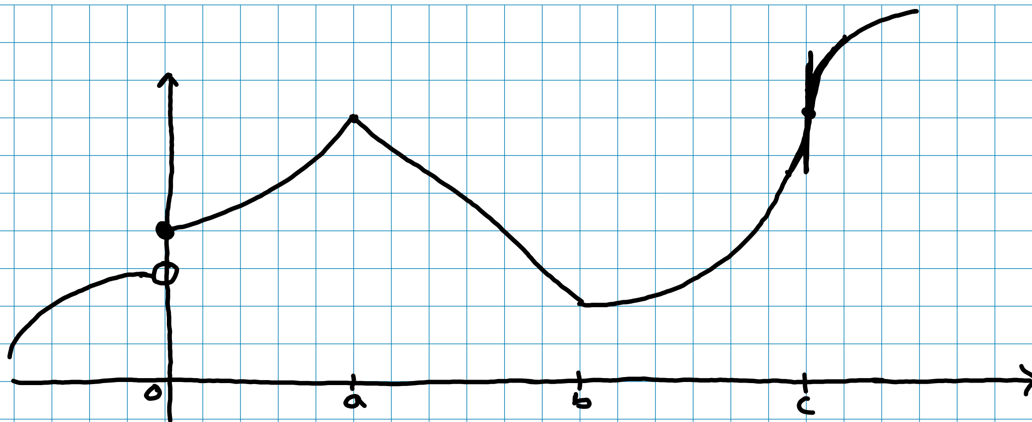
$\therefore$ $f'(0)$ DNE

$\therefore$ not differentiable

$f'(x)=\begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$

Removed "or equal"

Dom $f'$: $\mathbb{R}\setminus\{0\}$

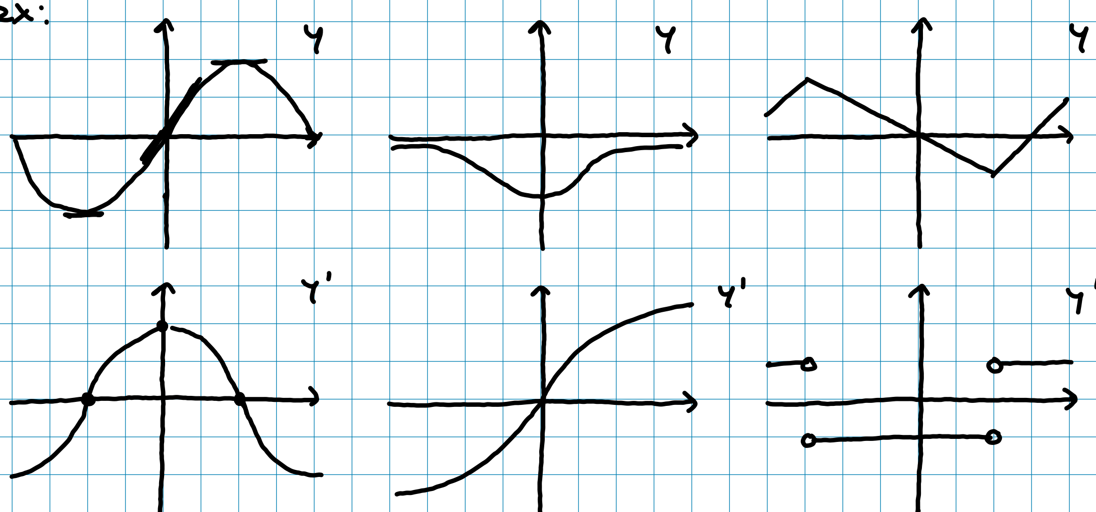



f not differentiable at:

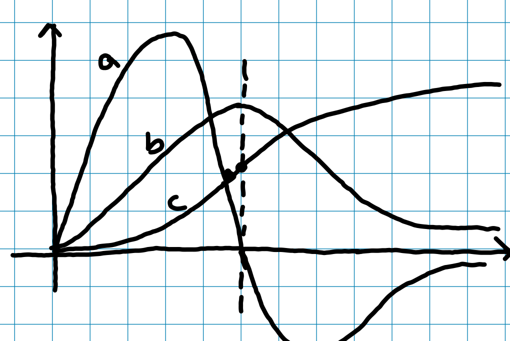
$x = 0$ b/c discontinuous
$x = a$ b/c corner
$x = b$ b/c corner
$x = c$ b/c tan line is vertical

**Note:** if $f(x) \nearrow$ at $x=a$, $f'(a) > 0$ (pos)
if $f(x) \searrow$ at $x=a$, $f'(a) < 0$ (neg.)
if $f(x)$ has <u>horizontal</u> tan line at $x=a$, $f'(a) = 0$

ex:




ex:




c is f

b is f' : b ⊕ when c $\nearrow$

a is f'' : a ⊕ when b $\nearrow$
a ⊖ when b $\searrow$
a = 0 when b —

**Note:** if f is differentiable and f' is also differentiable
$\therefore$ we define the second derivative $= f''(x) = \frac{d^2}{dx^2} f(x)$

Third $= f'''$, fourth $= f^4$, $n^{th}$ derivative $= f^{(n)}(x)$

# Implicit Differentiation

- Looking for $y'$
- The function isn't presented as $y = \dots$
- $y'$ depends on the value of $y$

ex: $x^3 - 6xy + y^3 = 0$

In this case we have to do implicit differentiation
1. Differentiate both sides
2. Isolate $y'$

We also have to find the derivative of a function of $y$ ($f(y)$):

$$\frac{d}{dx} f(y) = \frac{d}{dy} f(y) \cdot \frac{dy}{dx}$$

* Let $f(y)$ = "term with $y$ we want to differentiate"

Implicit differentiation and regular differentiation yield the same answer

$$\boxed{\frac{d}{dx} \log_a x = \frac{1}{x \cdot \ln a}}$$

proof 1: $(\log_a x)' =$

$= \left(\frac{\ln x}{\ln a}\right)'$

$= \frac{1}{\ln a} \cdot (\ln x)'$

$= \frac{1}{\ln a} \cdot \frac{1}{x}$

$y' = \frac{1}{x \ln a}$

proof 2: $y = \log_a x \Leftrightarrow a^y = x$

implicit diff

$(a^y)' = (x)'$

$a^y \cdot \ln a \cdot y' = 1$

$y' = \frac{1}{a^y \cdot \ln a}$

$y' = \frac{1}{x \cdot \ln a}$

ex: $x^2 + y^2 = 9$

$\frac{d}{dx} x^2 + \frac{d}{dx} y^2 = (9)'$

$2x + 2y \cdot y' = 0$

$y' = \frac{-2x}{2y} = \frac{-x}{y}$

Let $f(y) = y^2$

$\frac{d}{dx} f(y) = \frac{d}{dy} y^2 \cdot \frac{dy}{dx}$

$= 2y \cdot y'$

ex $x^3 + y^3 - 6xy = 0$

$(x^3)' + (y^3)' - (6xy)' = (0)'$

$3x^2 + 3y^2 \cdot y' - 6((x)'y + (y)' \cdot x) = 0$

$y'$ stays $y'$: Let $f(y) = y$

$\frac{d}{dx} f(y) = \frac{dy}{dy} \cdot \frac{dy}{dx}$

$= 1 \cdot y'$

isolate $y'$

$3y^2 \cdot y' - 6y'x = -3x^2 + 6y$

$y'(3y^2 - 6x) = -3x^2 + 6y$

$y' = \frac{6y - 3x^2}{3y^2 - 6x}$

Find $y'$ and $y''$ using implicit diff.

$$y' = \frac{d}{dx}$$
$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

1. Find $\frac{dy}{dx} = y'$
2. Find $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$
3. sub $y'$ from part 1

ex: $3x^2 + 4y^3 = 5$

Find $y'$

$$\frac{d}{dx}3x^2 + \frac{d}{dx}4y^3 = \frac{d}{dx}5$$
$$6x + 12y^2 \cdot y' = 0$$
$$y' = \frac{-6x}{12y^2}$$
$$y' = \frac{-x}{2y^2}$$

Find $y'' = \frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$
$$= \frac{d}{dx}\left(\frac{-x}{2y^2}\right)$$
$$= -\tfrac{1}{2} \cdot \frac{d}{dx}(x \cdot y^{-2})$$
$$= -\tfrac{1}{2} \cdot \left((x)' \cdot y^{-2} + (y^{-2})' \cdot x\right)$$
$$= -\tfrac{1}{2}\left(y^{-2} - 2y^{-3} \cdot x \cdot y'\right)$$
$$= -\tfrac{1}{2}\left(\frac{1}{y^2} - \frac{2x}{y^3} \cdot \frac{-x}{2y^2}\right) \quad \text{sub } y'$$
$$= -\tfrac{1}{2}\left(\frac{1}{y^2} + \frac{x^2}{y^5}\right)$$

Let $f(y) = y^{-2}$

$$\frac{d}{dx}f(y) = \frac{d}{dy}y^{-2} \cdot \frac{dy}{dx}$$
$$= -2y^{-3} \cdot y'$$

# Logarithmic Differentiation

→ Find $y'$ of $(f(x))^{(g(x))}$

* Take "ln" on both sides, then implicit diff

ex: $y = x^x$, Find $y'$

$$\ln(y) = \ln(x^x)$$
$$\ln y = x \ln x$$
$$\frac{d}{dx} \ln y = \frac{d}{dx}(x \ln x)$$
$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln x + \frac{1}{x} \cdot x$$
$$\frac{dy}{dx} = y(\ln x + 1)$$

sub $y = x^x$

$$\boxed{\frac{dy}{dx} = x^x(\ln x + 1)}$$

ex: $y = (2x+1)^{\sin x}$, find $y'$

$$\ln y = (\sin x) \cdot \ln(2x+1)$$
$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\sin x \cdot \ln(2x+1))$$
$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}\sin x \cdot \ln(2x+1) + \frac{d}{dx}\ln(2x+1) \cdot \sin x$$
$$\frac{dy}{dx} = y\left(\cos x \cdot \ln(2x+1) + \frac{2\sin x}{2x+1}\right)$$

Let $u = 2x+1$, $f(u) = \ln u$

$$\frac{d}{du} f(u) \cdot \frac{d}{dx} u =$$
$$= \frac{1}{u} \cdot 2$$

sub $y = (2x+1)^{\sin x}$

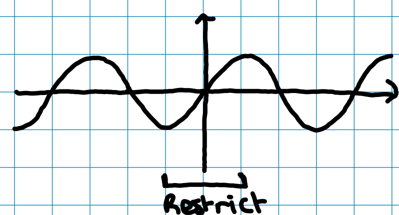
$$\boxed{\frac{dy}{dx} = (2x+1)^{\sin x}\left(\cos x \cdot \ln(2x+1) + \frac{2\sin x}{2x+1}\right)}$$

# Inverse Trig. Function

* For a function to be invertible, it must be a "one-to-one" function (passes vertical line test AND horizontal line test).

$\sin x$ and $\cos x$ ARE NOT one-to-one
$\therefore$ we set restrictions

① $y = \sin x$




Dom f : $[-\frac{\pi}{2}, \frac{\pi}{2}]$
(restricted)

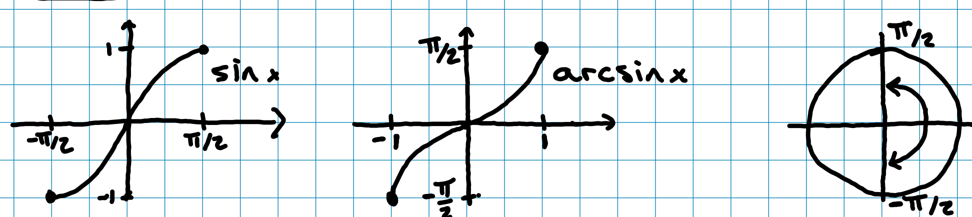
Range f : $[-1, 1]$

$y = \sin x \Rightarrow x = \sin^{-1} y$

$-1 \leq y \leq 1$ and $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

Inverse sin function:

$\boxed{y = \arcsin x}$




* $\underset{\text{angle}}{\sin \theta} = \underset{\text{value}}{x} \quad \not\Rightarrow \quad \arcsin \underset{\text{value}}{x} = \underset{\text{angle}}{\theta}$

Note: $\sin(\arcsin x) = x$ (always)
$\arcsin(\sin \theta) = \theta$ (if $-\pi/2 \leq \theta \leq \pi/2$
if not, find corresponding angle with same sin value)

ex: $\frac{\sqrt{2}}{2} = \sin \frac{\pi}{4} \Rightarrow \frac{\pi}{4} = \arcsin\left(\frac{\sqrt{2}}{2}\right)$

ex: $\arcsin \frac{\sqrt{3}}{2} = \frac{\pi}{3}$ b/c $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

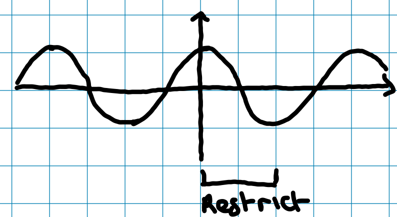
ex: $\arcsin 0 = 0$

ex: $\arcsin 1/2 = \pi/6$

ex: $\arcsin 1 = \pi/2$

ex: $\arcsin(-1) = -\pi/2$ b/c $\sin(-\pi/2) = -1$

② $y = \cos x$




Dom f : $[0, \pi]$
(restricted)

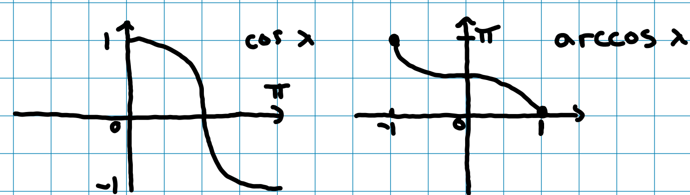
Range f : $[-1, 1]$

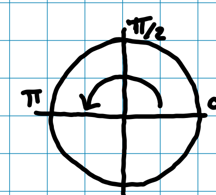
$y = \cos x \Rightarrow x = \cos^{-1} x$

$-1 \le y \le 1$ and $0 \le x \le \pi$

Inverse cosine function :

$\boxed{y = \arccos x}$




* $\cos\theta = x \not\Rightarrow \arccos x = \theta$

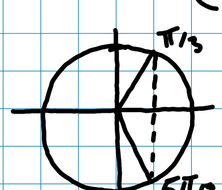
Note: $\cos(\arccos x) = x$
$\arccos(\cos\theta) = \theta$ (if $0 \le \theta \le \pi$
if not, find corresponding angle with same cos )

ex: $\arccos 0 = \pi/2$ b/c $\cos \pi/2 = 0$

ex: $\arccos 1/2 = \pi/3$ b/c $\cos \pi/3 = 1/2$

ex: $\arccos(\cos \pi/2) = \pi/2$ ($\theta = \pi/2$ between 0 and $\pi$)

ex: $\arccos(\cos 5\pi/3)$ ($\theta = 5\pi/3$ not between 0 and $\pi$)



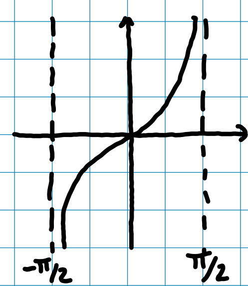

$\cos \frac{5\pi}{3} = \cos \frac{\pi}{3}$

$\arccos(\cos \frac{\pi}{3}) = \pi/3$

or

$\arccos(\cos \frac{5\pi}{3}) = \arccos(1/2) = \pi/3$

③ $y = \tan x$

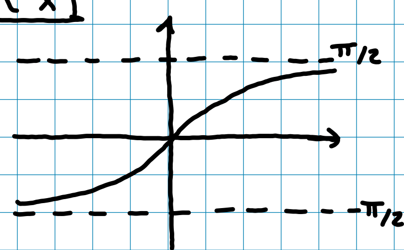



Dom $f$ : $(-\pi/2, \pi/2)$
Range $f$ : $\mathbb{R}$

Note: $\left.\begin{array}{l} \lim_{x \to -\pi/2^+} \tan x = -\infty \\ \lim_{x \to \pi/2^-} \tan x = \infty \end{array}\right\}$ v.a

Inverse trig function:

$\boxed{y = \arctan x}$




Dom (f) : $\mathbb{R}$
Range f : $(-\pi/2, \pi/2)$

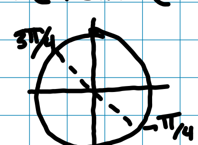
Note:
$\left.\begin{array}{l} \lim_{x \to -\infty} \arctan x = -\frac{\pi}{2} \\ \lim_{x \to +\infty} \arctan x = \frac{\pi}{2} \end{array}\right\}$ h.a

* $\tan \theta = x \quad \not\Rightarrow \quad \arctan x = \theta$

Note: $\tan(\arctan x) = x$ (always)
$\arctan(\tan \theta) = \theta$ (if $-\pi/2 < \theta < \pi/2$
if not find angle corresponding with same tan value)

ex: $\tan(\arctan 10) = 10$

ex: $\arctan\left(\tan \frac{3\pi}{4}\right) =$

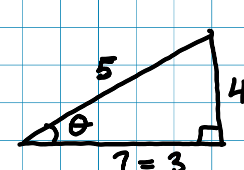



$\therefore \arctan\left(\tan \frac{3\pi}{4}\right) = \arctan\left(\tan -\frac{\pi}{4}\right) = -\pi/4$

corresponding angle

⊛ ex: $\tan \underbrace{(\arcsin 4/5)}_{\theta}$

$\theta = \arcsin 4/5$
$\sin \theta = \frac{4}{5} = \frac{\text{opp}}{\text{hyp}}$



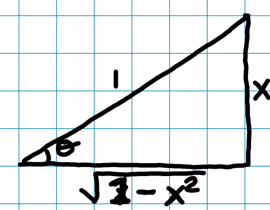


$\therefore \tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{3}$

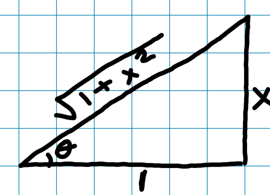
$\therefore \tan(\arcsin 4/5) = \frac{4}{3}$

# Derivatives of Inverse Trig functions

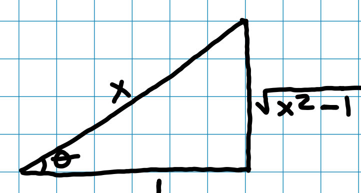
$$\left.\begin{aligned} \frac{d}{dx}(\arcsin x) &= \frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx}(\arccos x) &= -\frac{1}{\sqrt{1-x^2}} \end{aligned}\right\} \text{①}$$




$$\left.\begin{aligned} \frac{d}{dx}(\arctan x) &= \frac{1}{1+x^2} \\ \frac{d}{dx}(\text{arccot } x) &= -\frac{1}{1+x^2} \end{aligned}\right\} \text{②}$$




$$\left.\begin{aligned} \frac{d}{dx}(\text{arcsec } x) &= \frac{1}{x\sqrt{x^2-1}} \\ \frac{d}{dx}(\text{arccsc } x) &= -\frac{1}{x\sqrt{x^2-1}} \end{aligned}\right\} \text{③}$$




① proof: $y = \arcsin x \Rightarrow \sin y = x$ , if $-\pi/2 \leq y \leq \pi/2$

implicit differentiation

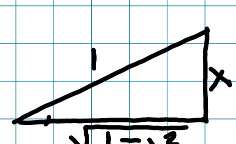
$(\sin y)' = (x)'$

$\cos y \cdot y' = 1$

$\boxed{y' = \frac{1}{\cos y}}$

* Want a Fcn of $x$ → $\sin y = x$

$\sin y = \frac{x}{1} = \frac{opp}{hyp}$




$\therefore \cos y = \frac{adj}{hyp} = \sqrt{1-x^2}$

$\sin^2 y + \cos^2 y = 1$

$\cos^2 y = 1 - \sin^2 y$

$\cos y = \sqrt{1-\sin^2 y}$

sub $\sin y = x$

$\cos y = \sqrt{1-x^2}$

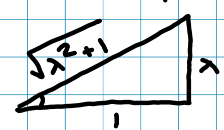
$$\therefore y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

② proof: $y = \arctan x \Rightarrow \tan y = x$ , if $-\pi/2 < y < \pi/2$

$$(\tan y)' = (x)'$$
$$\sec^2 y \cdot y' = 1$$
$$\boxed{y' = \frac{1}{\sec^2 y}}$$

* want a fcn of $x$, $\Rightarrow$ $\{\tan y = x\}$

$$\tan y = \frac{x}{1} = \frac{opp}{adj}$$




$$\sec y = \frac{1}{\cos y} = \frac{hyp}{adj} = \sqrt{x^2+1}$$

$$\therefore \sec^2 y = x^2+1$$

$$\frac{\sin^2 y}{\cos^2 y} + \frac{\cos^2 y}{\cos^2 y} = \frac{1}{\cos^2 y}$$
$$\tan^2 y + 1 = \sec^2 y$$
sub $\tan y = x$
$$\sec^2 y = x^2+1$$

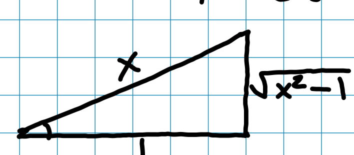
$$\therefore y' = \frac{1}{\sec^2 y} = \frac{1}{x^2+1}$$

③ proof: $y = \operatorname{arcsec} x \Rightarrow \sec y = x$

$$(\sec y)' = (x)'$$
$$\sec y \cdot \tan y \cdot y' = 1$$
$$\boxed{y' = \frac{1}{\sec y \cdot \tan y}}$$

* want a fcn of $x$ $\Rightarrow$ $\{\sec y = x\}$

$$\sec y = \frac{x}{1} = \frac{1}{\cos} = \frac{hyp}{adj}$$




$$\therefore \sec y = x$$
$$\therefore \tan y = \frac{\sqrt{x^2-1}}{1}$$

$$\therefore y' = \frac{1}{\sec y \cdot \tan y} = \frac{1}{x\sqrt{x^2-1}}$$

ex: $\frac{d}{dx}\left(\arctan(x^2)\right) =$ Let $u = x^2$, $f(u) = \arctan(u)$

$= \frac{d}{du}\arctan u \cdot \frac{d}{dx}x^2$

$= \frac{1}{u^2+1} \cdot 2x \quad = \frac{2x}{x^4+1}$

ex: $\frac{d}{dx}\left(\arctan\sqrt{\frac{1-x}{1+x}}\right) =$ Let $w = \frac{1-x}{1+x}$, $u = w^{1/2}$, $f(u) = \arctan u$

$= \frac{d}{du}\arctan(u) \cdot \frac{d}{dw}w^{1/2} \cdot \frac{d}{dx}\left(\frac{1-x}{1+x}\right)$

$= \frac{1}{u^2+1} \cdot \frac{1}{2\sqrt{w}} \cdot \frac{(1-x)'(1+x) - (1+x)'(1-x)}{(1+x)^2}$

$= \frac{1}{\frac{1-x}{1+x}+1} \cdot \frac{1}{2\sqrt{\frac{1-x}{1+x}}} \cdot \frac{-(1+x) - (1-x)}{(1+x)^2}$

$= \frac{1}{\frac{1-x+(1+x)}{1+x}} \cdot \frac{1}{\cancel{2}\sqrt{\frac{1-x}{1+x}}} \cdot \frac{-\cancel{2}}{(1+x)^2}$

$= \frac{\cancel{(1+x)}}{2} \cdot \sqrt{\frac{1+x}{1-x}} \cdot \frac{-1}{(1+x)^{\cancel{2}}} \quad = -\frac{1}{2}\sqrt{\frac{1+x}{1-x}} \cdot \frac{1}{1+x}$

ex: $\left(x\sin^{-1}x + \sqrt{1-x^2}\right)' =$

$= (x)' \cdot \sin^{-1}x + (\sin^{-1}x)' \cdot x + \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x)$

$= \sin^{-1}x + \frac{1x}{\sqrt{1-x^2}} + \frac{-1x}{\sqrt{1-x^2}}$

$= \sin^{-1}x$

# Theorems

Notation: $\exists$ = there exists

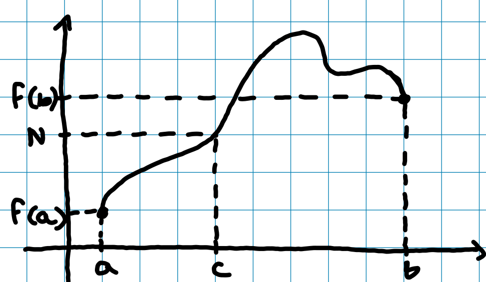
$\exists!$ = there exists exactly one

$\forall$ = for all ...

s.t = such that

## Intermediate Value theorem (IVT)

Suppose $y = f(x)$ is continuous on a closed interval $[a,b]$
Let $N$ be a number such that $f(a) < N < f(b)$
$(f(a) \neq f(b))$
There must be a number $c$ s.t $c \in (a,b)$ and $f(c) = N$




N between f(a) and f(b),
$\therefore$ There must be a value c

ex: use IVT to show that $4x^3 - 6x^2 + 3x - 2$ has at least one real root (zero).

$f(x) = 4x^3 - 6x^2 + 3x - 2$

$f(1) = 4 - 6 + 3 - 2 = -1 < 0$
$f(2) = 32 - 24 + 6 - 2 = 12 > 0$

$f(1) < 0 < f(2)$

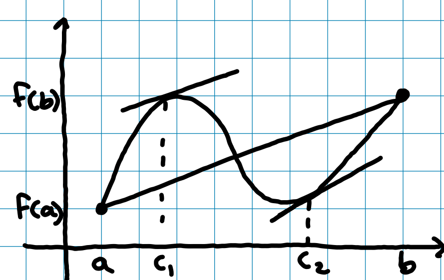
$\therefore$ there must be a number $c \in (1,2)$ s.t $f(c) = 0$

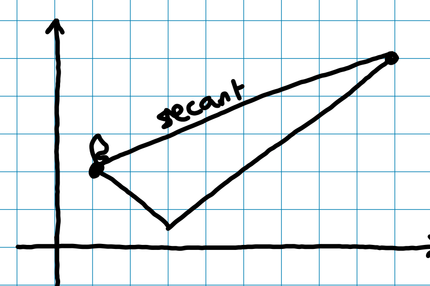
# Mean Value theorem (MVT)

Let $f$ be a function that satisfies the following :
→ $f$ is continuous on $[a,b]$
→ $f$ is differentiable $(a,b)$

Then there must be a number $c \in (a,b)$ s.t :

$$\underbrace{f'(c)}_{\text{slope tangent}} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{slope secant}}$$




MVT doesn't work/apply b/c the function is not differentiable everywhere

ex: $f(x) = x^3 - x$ on $[0,2]$
Find $c \in (0,2)$ guaranteed by MVT

i.e. Find $c$ s.t. $f'(c) = \frac{f(2)-f(0)}{2-0}$

$f'(x) = 3x^2 - 1$

$f'(c) = 3c^2 - 1 = \frac{f(2)-f(0)}{2-0}$

$3c^2 - 1 = 6/2$

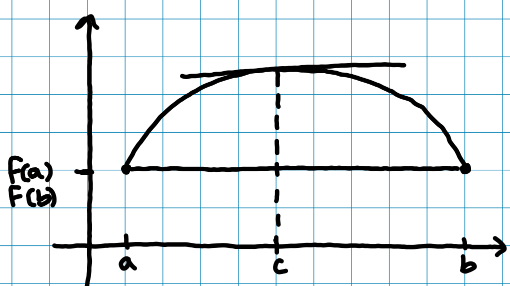
$\boxed{c = \pm\sqrt{4/3}}$ ommit $c = -\sqrt{4/3}$ b/c $\notin (0,2)$

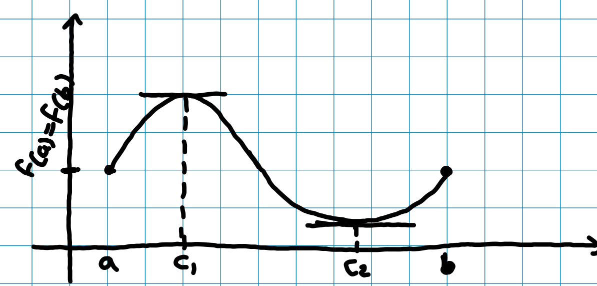
$\boxed{\therefore\ c = 2/\sqrt{3}}$

# Rolle's theorem (special MVT)

Let f be a fcn that satisfies the following:
- → f is continuous on [a, b]
- → f is differentiable on (a, b)
- → f(a) = f(b)

Then there exists $c \in (a, b)$ s.t. $f'(c) = 0$ (horizontal tangent line)




ex: Let $s(t)$ be position vs time ($s'(t)$ = velocity)
Suppose object is at the same position at times
$t=a$, $t=b$ $(s(a) = s(b))$
∴ there must be a moment $c \in (a,b)$ where $s'(c) = 0$

ex. Show that $x^3 + x - 1$ has exactly one real root.

1) Use IVT to prove it has at least one zero
$f(1) = 1 > 0$
$f(0) = -1 < 0$
∴ there must be a value $c \in (0,1)$ where $f(c) = 0$

2) Suppose there are 2 real roots $(c_1, c_2)$ where
$f(c_1) = f(c_2) = 0$
Rolle's theorem should apply on $(c_1, c_2)$
i.e. there should be $d \in (c_1, c_2)$ s.t. $f'(d) = 0$

$f'(x) = 3x^2 + 1$

Always greater than zero

Let $f'(x) = 0$
$3x^2 + 1 = 0$
$x^2 = -1/3$
impossible

∴ there cannot be $d \in (c_1, c_2)$ s.t. $f'(d) = 0$
∴ there are not 2 real roots

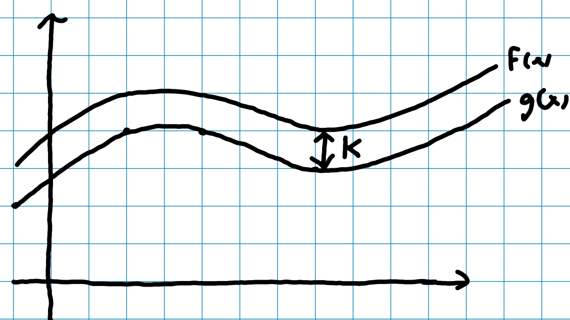
Note: $f'(x) > 0$ ∴ f always increasing

## Theorem

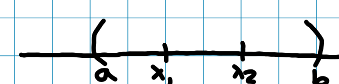
If $f'(x) = 0$ for all $x \in (a,b)$, then $f(x) = k$ on $(a,b)$
$\rightarrow$ ($k$ is a constant)

## Corollary

If $f'(x) = g'(x)$ for all $x \in (a,b)$, then $f - g$ = constant
$\Rightarrow$ $f(x) = g(x) + k$
$\rightarrow$ $f(x)$ and $g(x)$ are //




Proof: Let $x_1, x_2 \in (a,b)$, $x_1 < x_2$




$\because$ $f$ is differentiable on $(a,b)$, $\therefore$ it is differentiable (continuous) on $[x_1, x_2]$

$\therefore$ By MVT, $\exists c \in (x_1, x_2)$ s.t. $f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

But $f'(c) = 0$ $\therefore$ $f(x_2) - f(x_1) = 0$
$f(x_2) = f(x_1)$ $\therefore$ $f$ is constant

ex: Prove that $\arctan x + \operatorname{arccot} x = \pi/2$

1) show that $f(x) = \arctan x + \operatorname{arccot} x$ = constant
(i.e. show $f'(x) = 0$)

$$f'(x) = (\arctan x)' + (\operatorname{arccot} x)' \quad * \text{ for all } x$$
$$= \frac{1}{x^2+1} - \frac{1}{x^2+1}$$
$$= 0 \quad \therefore \; f(x) \text{ is constant}, \; f(x) = k$$

2) Sub any value to prove $k = \pi/2$

sub $x = 1$

$$f(1) = \underbrace{\arctan(1)}_{\theta_1} + \underbrace{\operatorname{arccot}(1)}_{\theta_2}$$

$\tan\theta_1 = 1$ $\qquad$ $\cot\theta_2 = 1$

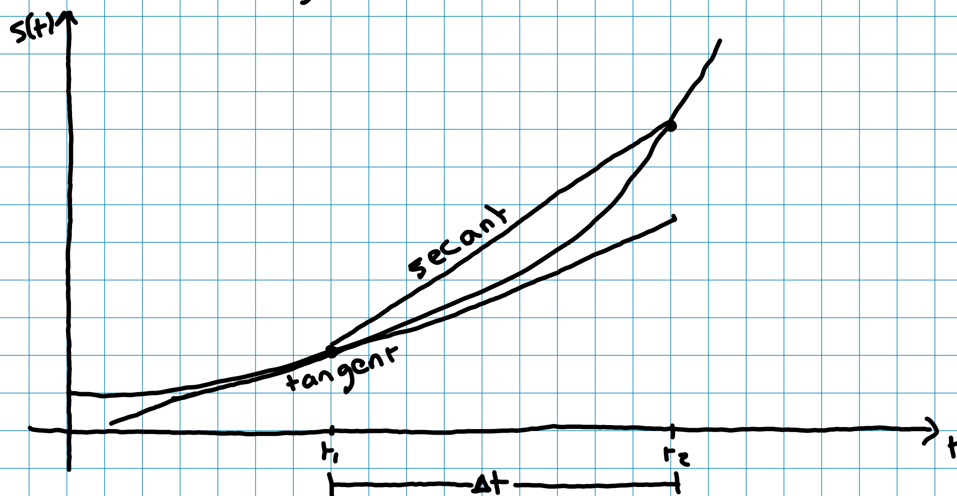
$\dfrac{\sin\theta}{\cos\theta} = 1$ $\qquad$ $\dfrac{\cos\theta}{\sin\theta} = 1$

$\theta_1 = \frac{\pi}{4}$ $\qquad$ $\theta_2 = \frac{\pi}{4}$

$\therefore$ $f(1) = \pi/2$
$\therefore$ $f(x) = \pi/2$

# Rate of Change

Let $s(t)$, with $t \geq 0$ be the position of an object moving in a straight line at time $t$.




Slope of secant:
Average rate of change between $t_1$ and $t_2$

$$\frac{S(t_2) - S(t_1)}{t_2 - t_1} = \frac{S(t_1 + \Delta t) - S(t_1)}{\Delta t} = \text{Average velocity}$$

Slope of tangent:
If $t_2 \to t_1$, we have an instantaneous rate of change at time $t$
i.e $\Delta t \to 0$

$$\lim_{\Delta t \to 0} \frac{S(t + \Delta t) - S(t)}{\Delta t} = S'(t) = \text{Instantaneous velocity} \quad v(t)$$

The rate at which velocity changes is acceleration
$a(t) = s''(t) = v'(t)$

ex. conside $s(t) = t^3 - 6t^2 + 9t$

a) Velocity at time $t$

$v(t) = s'(t) = 3t^2 - 12t + 9$

b) Velocity at time $t=2$

$v(2) = -3$

c) When is the object at rest

$v(t) = 0$

$3t^2 - 12t + 9 = 0$

$3(t-1)(t-3) = 0$

$t=1 \quad , \quad t=3$

* Note : object changes direction of motion a $v(t) = 0$

d) When is object moving forward / backward

| | | 1 | | 3 | |
|---|---|---|---|---|---|
| $t-1$ | $-$ | 0 | $+$ | | $+$ |
| $t-3$ | $-$ | | $-$ | 0 | $+$ |
| | $+$ | | $-$ | | $+$ |

$v(t) > 0$ forward $\therefore$ $t \in [0, 1) \cup (3, +\infty)$

↳ $t \geq 0$

$v(t) < 0$ backward $\therefore$ $t \in (1, 3)$

e) Distance travelled in first 5s

Separate into intervals

- moving forward on $[0,1) \rightarrow |s(1) - s(0)|$
- moving backward on $(1,3) \rightarrow |s(3) - s(1)|$
- moving forward on $(3,5] \rightarrow |s(5) - s(3)|$

* Remarks about differential equations

→ equations involving a function and possibly its first and/or second derivative $(y'(x)$ and $y''(x))$

ex: $y'' + 9y = 0$

consider $y = \sin 3x$

$(\sin 3x)'' + 9\sin 3x$

$= (3\cos 3x)' + 9\sin 3x$

$= -9\sin 3x + 9\sin 3x$

$= 0$

ex: $\frac{dy}{dx} = -2y$

$y' + 2y = 0$

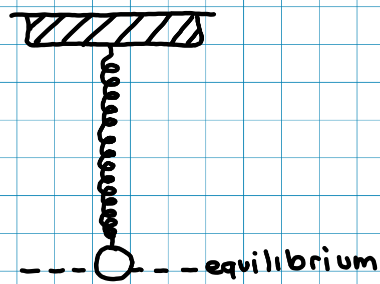
consider $y = e^{-2x}$

$(e^{-2x})' + 2e^{-2x} =$

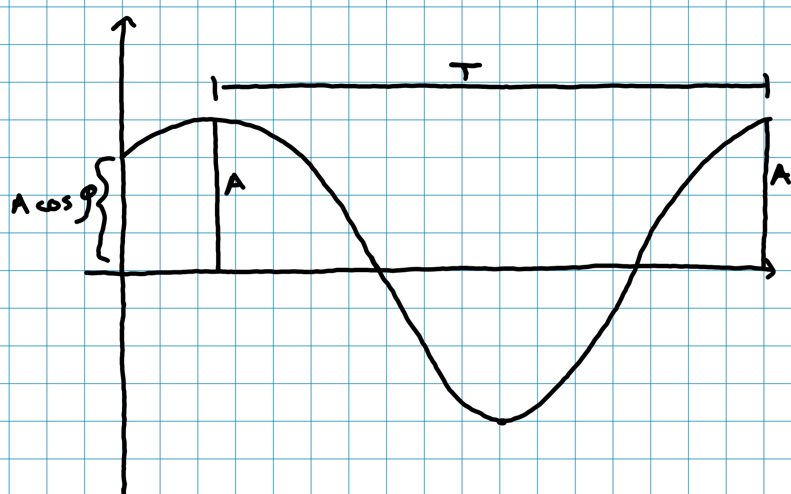
$= -2e^{-2x} + 2e^{-2x}$

$= 0$

# Simple Harmonic Motion (SHM)




- Push or pull passed equilibrium point
- If no friction or outside force, at release, object will continuously go up and down.




Equation describing motion:

$y(t) = A\cos(\omega t + \varphi)$

$A$ = amplitude

$\omega = 2\pi f = \frac{2\pi}{T}$

Note: $f = \frac{1}{T}$

$T$ = period $= \frac{2\pi}{\omega}$ [s]

$f$ = frequency $= \frac{\omega}{2\pi}$ [Hz]

$\varphi$ = phase (how much away from equilibrium we start)

$y(0) = A\cos\varphi$ at $t=0$

* Differential equation

$\boxed{y''(t) + \omega^2 y(t) = 0}$ or $y''(t) = -\omega^2 y(t)$

∴ If an objects movement satisfies this equation, we can conclude that the object moves in SHM

What determines $\omega$

Hookes Law: $F = -k \cdot y(t)$ $\quad$ k: spring constant

Newton's II: $F = m \; a(t)$

$$\therefore \quad m \cdot a(t) = -k \cdot y(t)$$
$$m \cdot y''(t) = -k \cdot y(t)$$
$$\left.\begin{array}{l} y''(t) = -\frac{k}{m} \cdot y(t) \\ \text{and} \quad y''(t) = -\omega^2 \cdot y(t) \end{array}\right\} \therefore \omega^2 = \frac{k}{m} = \frac{\text{spring constant}}{\text{mass}}$$

If $y(t)$ is position function (where $y(t)$ = distance from equilibrium) of a partical in SHM, the function can be written in 2 standard forms:

① $y(t) = A \cos(\omega t + \varphi)$

② $y(t) = a \cos(\omega t) + b \sin(\omega t)$

Going from ① to ②: $\boxed{\begin{array}{l} a = A \cos \varphi \\ b = -A \sin \varphi \end{array}}$

Going from ② to ①: $\boxed{\begin{array}{l} A = \sqrt{a^2 + b^2} \\ \left.\begin{array}{l} \cos \varphi = \frac{a}{A} \\ \sin \varphi = -\frac{b}{A} \end{array}\right\} \varphi \end{array}}$

ex: Rewrite

$$\underbrace{\sqrt{3}}_{a} \cos \underbrace{3}_{\omega}t \underbrace{-1}_{b} \sin 3t = y(t)$$

$$A = \sqrt{3+1} = 2$$
$$\left.\begin{array}{l} \cos \varphi = \frac{\sqrt{3}}{2} \\ \sin \varphi = \frac{1}{2} \end{array}\right\} \varphi = \pi/6$$

$$\therefore \quad y(t) = 2 \cos(3t + \pi/6)$$

ex: Rewrite $y(t) = \underbrace{5}_{b} \sin t \underbrace{-5}_{a} \cos t$ (always with cos)

$$\therefore y(t) = \sqrt{50} \cos\left(t - \frac{3\pi}{4}\right)$$

$$A = \sqrt{(-5)^2 + 5^2} = \sqrt{50}$$
$$\left.\begin{array}{l} \cos \varphi = \frac{-5}{\sqrt{50}} = -\frac{\sqrt{2}}{2} \\ \sin \varphi = \frac{-5}{\sqrt{50}} = -\frac{\sqrt{2}}{2} \end{array}\right\} \varphi = -\frac{3\pi}{4}$$

* Go back

$\omega = 1$

$$a = \sqrt{50} \cos\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} \cdot 5\sqrt{2} = -5$$
$$b = -\sqrt{50} \sin\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} \cdot -5\sqrt{2} = 5$$

$$\therefore y(t) = -5 \cos t + 5 \sin t$$

Note: $y(t)=0 \Rightarrow$ object at equilibrium
$y'(t)=v(t)=0 \Rightarrow$ object changes direction
$y''(t)=a(t)=0 \Rightarrow$ object at equilibrium (no force at e)

ex: $y(t)=\sqrt{3}\sin t + \cos t$
verify that $y(t)$ describes a SHM

$y'(t)=\sqrt{3}\cos t - \sin t$
$y''(t)=-\sqrt{3}\sin t - \cos t$
$= -(\sqrt{3}\sin t + \cos t)$ $\qquad \therefore\ y''(t) = -1\,y(t)\ \therefore$ SHM

ex: object oscillating according to a position fcn $x(t)$ that satisfies the differential equation : $x''(t)+4\pi^2 \cdot x(t)=0$

a) If $x(0)=1$ and $v(0)=2\pi\sqrt{3}$ m/s, find $x(t)$
b) Find position, velocity, acceleration at $t=1$
c) Find period, frequency, amplitude

$x(t) = a\cos wt + b\sin wt$
$w^2 = 4\pi^2 \Rightarrow w = 2\pi$
$x(t) = a\cos 2\pi t + b\sin 2\pi t$
$v(t)=x'(t) = -a \cdot 2\pi \cdot \sin 2\pi t + b \cdot 2\pi \cdot \cos 2\pi t$

$x(0) = a\underbrace{\cos(0)}_{1} + b\underbrace{\sin(0)}_{0} = 1$

$\boxed{a=1}$

$v(0) = -a\,2\pi\underbrace{\sin(0)}_{0} + b\,2\pi\underbrace{\cos(0)}_{1} = 2\pi\sqrt{3}$

$b\,2\pi = 2\pi\sqrt{3}$
$\boxed{b=\sqrt{3}}$

a) $\therefore\ x(t) = 1\cos 2\pi t + \sqrt{3}\sin 2\pi t$

b)
$x(1) = \cos 2\pi + \sqrt{3}\sin 2\pi = 1+0 = 1$
$v(1) = -2\pi\sin 2\pi + \sqrt{3}\cdot 2\pi \cdot \cos 2\pi = 2\sqrt{3}\pi$
$a(t) = x''(t) = -4\pi^2 x(t)$
$\therefore\ x''(1) = -4\pi^2 \cdot (1) \Rightarrow a(1) = -4\pi^2$

c) $A = \sqrt{1^2+3} = 2$

$T = \frac{2\pi}{w} = \frac{2\pi}{2\pi} = 1$

$f = \frac{1}{T} = 1$

ex: A partical is moving in line according to:
$s(t) = 2 - 4\cos^2 2t$

a) Find $v(t)$ and $a(t)$
b) Show that it satisfies SHM

$v(t) = s'(t) = 0 - 4 \cdot 2\cos 2t \cdot (-\sin 2t) \cdot 2$
$= 16 \cos 2t \cdot \sin 2t$

$a(t) = v'(t) = 16\left((\cos 2t)' \cdot \sin 2t + (\sin 2t)' \cdot \cos 2t\right)$
$= 16\left(-2\sin 2t \cdot \sin 2t + 2\cos 2t \cdot \cos 2t\right)$
$= 16\left(2\cos^2 2t - 2\sin^2 2t\right)$
$= 32\left(\cos^2 2t - \sin^2 2t\right)$

Show that $s''(t) + w^2 s(t) = 0$
$s''(t) = -w^2 s(t)$

$s''(t) = 32\left(\cos^2 2t \underbrace{- \sin^2 2t}_{\text{have to get rid of sin}}\right)$

$= 32\left(\cos^2 2t - (1 - \cos^2 2t)\right)$
$= 32\left(2\cos^2 2t - 1\right)$
$s''(t) = -16\underbrace{\left(-4\cos^2 2t + 2\right)}_{s(t)}$

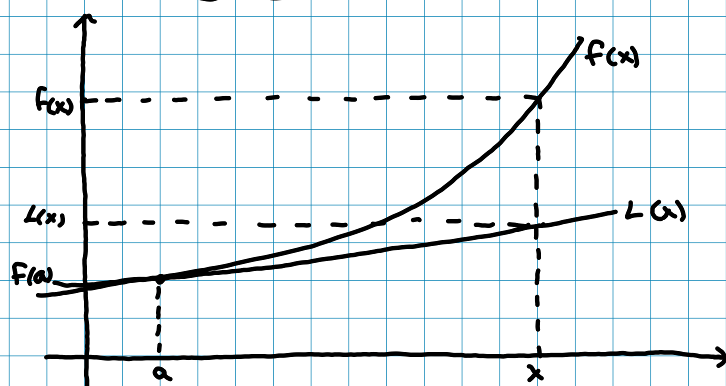
$\therefore s''(t) = -w^2 s(t) \Rightarrow$ SHM

# Approximations and Differentials

## Linear Approximation

Given a fcn $y = f(x)$ (differentiable at $x=a$)
Let $L(x)$ be the tangent line of the graph at a given point $(a, f(a))$

$\therefore \{L(x) \approx f(x)\}$, (close to $x=a$)




slope of tangent: $f'(a) = \dfrac{L(x) - f(a)}{x - a}$

Linearization of f(x) close to a $\Rightarrow$ $\therefore \boxed{L(x) = f(a) + f'(a) \cdot (x-a)} \approx f(x)$

ex: Find the linearization of $f(x) = \sqrt{x+3}$ centered at $a=1$

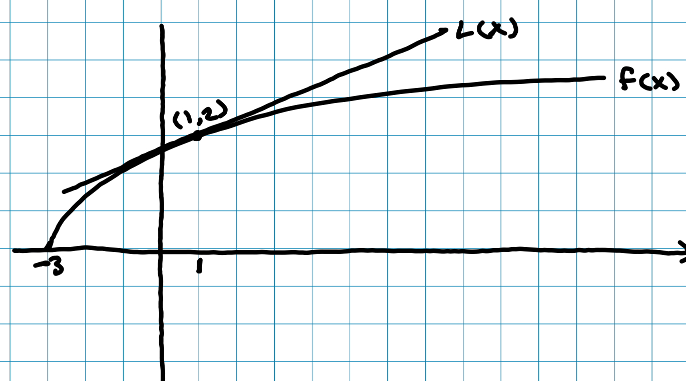
$L(x) = f(a) + f'(a)(x-a)$

$f'(x) = \dfrac{1}{2\sqrt{x+3}}$

- $f'(a) = 1/4$
- $f(a) = 2$

$\therefore L(x) = 2 + 1/4(x-1)$

$L(x) = \frac{x}{4} + \frac{7}{4}$




Appoximate $\sqrt{3.98}$

$\sqrt{3.98} = \sqrt{0.98+3} = f(0.98) \approx L(0.98)$

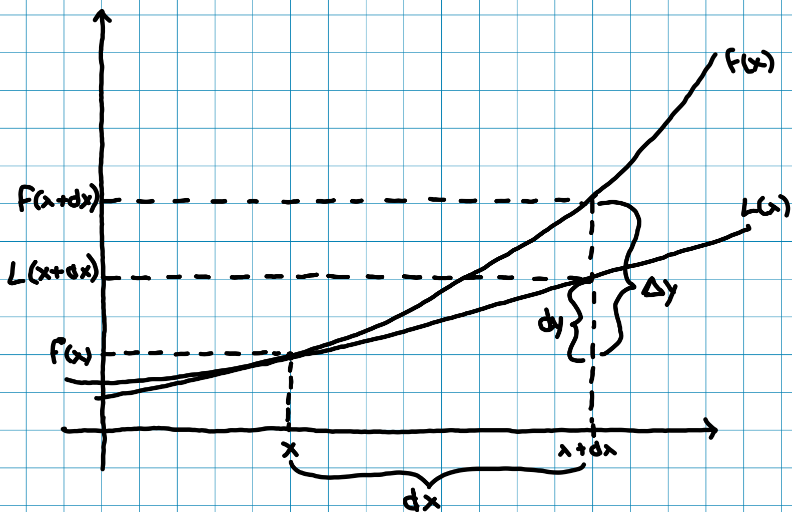
$= \dfrac{(0.98)}{4} + \dfrac{7}{4} = 1.995$

Approximate $\sqrt{4.05}$

$\sqrt{4.05} = f(1.05) \approx L(1.05)$

$= \dfrac{(1.05)}{4} + \dfrac{7}{4} = 2.0125$

## Alternate notation (Differentials)




Note:
$\Delta y = f(x+dx) - f(x)$
$dy = L(x+dx) - f(x)$

Note: $f'(x) = \frac{dy}{dx}$

$$\boxed{dy = f'(x) \cdot dx}$$

$dy$ = "differential"

$$\boxed{L(x+dx) \approx f(x+dx)}$$, close to $x$ (ie: small $dx$)

Slope of tangent: $\frac{L(x+dx) - f(x)}{dx} = \frac{dy}{dx} = f'(x)$

Linearization of $f(x)$ centered at $x$ $\Longrightarrow$ $\boxed{L(x+dx) = f(x) + \underbrace{f'(x) \cdot dx}_{dy}} \approx f(x+dx)$

ex: Find the linearization of $f(x) = (x+2)^5$ centered at $x=0$

$L(x+dx) = f(x) + f'(x) \cdot dx$

$L(0+dx) = f(0) + f'(0) \cdot dx$

$f(0) = 2^5$
$f'(x) = 5(x+2)^4$
$f'(0) = 5 \cdot (2)^4$

$\therefore L(0+dx) = 2^5 + 5(2)^4 \cdot dx$
$= 32 + 80\, dx$

Approximate $(2.001)^5$

$(2.001)^5 = (0.001+2)^5 = f(0.001) \approx L(0+0.001)$
$f(0+0.001) \approx 32 + 80(0.001) = 32\ 080$

ex: Find linearization of $\sqrt{x+3}$ centered at $x=1$

Approximate $\sqrt{4.05}$

$L(1+dx) = f(1) + f'(1) \cdot dx \approx f(1+dx)$

$\sqrt{4.05} = \sqrt{1.05+3} = f(1.05) = f(1+\underbrace{0.05}_{dx}) \approx f(1) + f'(1) \cdot (0.05) = 2.0125$

## Quadratic Approximation

$\{Q(x) \approx f(x)\}$ , near $x=a$ (better approximation than $L(x)$)

$$\boxed{Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}\cdot(x-a)^2} \approx f(x)$$

## Taylor Polynomial

$\{T_n(x) = f(x)\}$

* The $n^{th}$ degree approximation of $f(x)$ centered at $x=a$

$$\cancel{T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \ldots + \frac{f^n(a)}{n!}(x-a)^n}$$

$$\boxed{T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \ldots + \frac{f^n(a)}{n!}(x-a)^n}$$

Note: Factorials : $0! = 1$
$n! = n \times (n-1)!$

ex: Find $Q(x)$ for $y=\sqrt{x+3}$ near $x=1$ (or $a=1$)

$f(1) = 2$

$f'(x) = \frac{1}{2\sqrt{x+3}} \rightarrow f'(1) = 1/4$

$f''(x) = 1/2\left(-1/2(x+3)^{-3/2}\right)$
$= \frac{-1}{4(x+3)^{3/2}} \rightarrow f''(1) = -1/32$

$\therefore Q(x) = 2 + 1/4(x-1) + \frac{-1/32}{2}(x-1)^2$

$= 2 + \frac{x}{4} - \frac{1}{4} - \frac{1}{64}(x^2-2x+1)$

$Q(x) = \frac{-x^2}{64} + \frac{9x}{32} + \frac{111}{64}$

Approximate $\sqrt{3.1}$

$\sqrt{3.1} = \sqrt{0.1+3} = f(0.1) \approx Q(0.1)$

$= -\frac{(0.1)^2}{64} + \frac{9(0.1)}{32} + \frac{111}{64} = 1.76$

# L'Hopital's Rule

Suppose $f(x)$ and $g(x)$ are differentiable functions and $g'(x) \neq 0$ for values of $x$ near $a$

- If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$

$\therefore \boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$ type $\frac{0}{0}$, if the limit exists

- If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$

$\therefore \boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$ type $\frac{\infty}{\infty}$, if the limit exists

* L'Hopital's Rule only applies if indeterminate form is type $\frac{0}{0}$ or $\frac{\infty}{\infty}$

ex: $\underset{\frac{\infty}{\infty}}{\lim_{x\to\infty} \frac{\ln x}{x-1}} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \frac{1}{\infty} = 0$

ex: $\underset{\frac{\infty}{\infty}}{\lim_{x\to\infty} \frac{e^x}{x^2}} \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{2x} \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty$

ex: $\underset{\frac{0}{0}}{\lim_{x\to 1} \frac{\ln x}{x-1}} \overset{H}{=} \lim_{x\to 1} \frac{\frac{1}{x}}{1} = \lim_{x\to 1} \frac{1}{x} = 1$

ex: $\underset{\frac{\infty}{\infty}}{\lim_{x\to\infty} \frac{x^2-1}{2x^2+1}} \overset{H}{=} \underset{\frac{\infty}{\infty}}{\lim_{x\to\infty} \frac{2x}{4x}} = \lim_{x\to\infty} \frac{2}{4} = 1/2$

ex: $\underset{\frac{\infty}{\infty}}{\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}}} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3}x^{-2/3}} = \underset{\frac{\infty}{\infty}}{\lim_{x\to\infty} \frac{3x^{2/3}}{x}} \overset{H}{=} \lim_{x\to\infty} \frac{3 \cdot \frac{2}{3} x^{-1/3}}{1} = \lim_{x\to\infty} \frac{2}{x^{1/3}} = 0$

ex: $\underset{\frac{0}{0}}{\lim_{x\to 0} \frac{\tan x - x}{x^3}} \overset{H}{=} \underset{\frac{0}{0}}{\lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2}} \overset{H}{=} \underset{\frac{0}{0}}{\lim_{x\to 0} \frac{(2\sec x)(\sec x \tan x)}{6x}} = \underset{\frac{0}{0}}{\lim_{x\to 0} \frac{\sec^2 x \tan x}{3x}}$

$\overset{H}{=} \lim_{x\to 0} \frac{(\sec^2 x)' \tan x + (\tan x)' \sec^2 x}{(3x)'} = \lim_{x\to 0} \frac{2\sec^2 x \cdot \sec x \tan x \cdot \tan x + \sec^4 x}{3}$

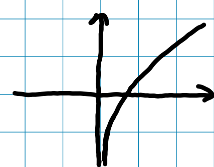
$= \lim_{x\to 0} \frac{2\sec^3 x \tan^2 x + \sec^4 x}{3} = \frac{0+1}{3} = 1/3$

## Other Indeterminate Forms

$\infty - \infty$ , $0 \cdot \infty$

* Strategy: Rewrite limit in form $\frac{0}{0}$ or $\frac{\infty}{\infty}$
→ Use L'Hopital's

ex: $\lim_{x\to 0^+} x \ln x$
$0 \cdot -\infty$



$= \lim_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}}$ $\quad \frac{\infty}{\infty}$

$\overset{H}{=} \lim_{x\to 0^+} \dfrac{\frac{1}{x}}{-1\,x^{-2}} = \lim_{x\to 0^+} -\dfrac{x^2}{x} \underset{\frac{0}{0}}{\overset{H}{=}} \lim_{x\to 0^+} \dfrac{-2x}{1} = 0$

ex: $\lim_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{e^x - 1}\right) = \lim_{x\to 0^+} \underset{\frac{0}{0}}{\dfrac{e^x - 1 - x}{x(e^x-1)}} \overset{H}{=} \lim_{x\to 0} \underset{\frac{0}{0}}{\dfrac{e^x - 1}{1(e^x-1) + e^x \cdot x}}$
$\infty - \infty$

$\overset{H}{=} \lim_{x\to 0^+} \dfrac{e^x}{e^x + 1e^x + xe^x} = \lim_{x\to 0^+} \dfrac{e^x}{e^x(1+1+x)} = \lim_{x\to 0^+} \dfrac{1}{2+x} = 1/2$

ex: $\lim_{x\to\infty} x \sin\frac{\pi}{x} \overset{H}{=} \lim_{x\to\infty} \dfrac{\sin \pi/x}{\frac{1}{x}} \; \frac{0}{0} \overset{H}{=} \lim_{x\to\infty} \dfrac{(\cos\frac{\pi}{x})\cdot(-\frac{\pi}{x^2})}{-1\,x^{-2}} = \lim_{x\to\infty} \cos\frac{\pi}{x} \cdot \frac{\pi}{x^2} \cdot x^2$
$\infty \cdot 0$

$= \lim_{x\to\infty} \pi \cdot \cos \pi/x = \lim_{x\to\infty} \cos\frac{\pi}{\infty} \cdot \pi = \cos(0)\cdot\pi = \pi \cdot 1 = \pi$

ex: $\lim_{x\to\infty} x^3 e^{-x^2} = \lim_{x\to\infty} \dfrac{x^3}{e^{x^2}} \underset{\frac{\infty}{\infty}}{\overset{H}{=}} \lim_{x\to\infty} \underset{\frac{\infty}{\infty}}{\dfrac{3x^2}{2x \cdot e^{x^2}}} = \lim_{x\to\infty} \dfrac{3x}{2e^{x^2}} \; \frac{\infty}{\infty}$
$\infty \cdot 0$

$\overset{H}{=} \lim_{x\to\infty} \dfrac{3}{2 \cdot 2x \cdot e^{x^2}} = \dfrac{3}{\infty} = 0$

# Indeterminate Powers

$\infty^0$ , $\cancel{\infty\cdot 0}$ , $0^0$ , $1^\infty$

* Strategy . use
1) $x = e^{\ln x}$
2) $\ln x^p = p \ln x$
3) $\lim_{x\to a} e^{g(x)} = e^{\lim_{x\to a} g(x)}$

ex: $\lim_{x\to 0^+} x^{\sqrt{x}}$ ($0^0$) $= \lim_{x\to 0^+} e^{\ln x^{\sqrt{x}}}$
$= \lim_{x\to 0^+} e^{\sqrt{x}\cdot \ln x}$ *
$= e^0 = 1$

* $\lim_{x\to 0^+} \sqrt{x} \ln x$ ($0\cdot -\infty$) $= \lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}}$ $\frac{\infty}{\infty}$
$\overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}}$
$= \lim_{x\to 0^+} -\frac{2x^{3/2}}{x}$ $\frac{0}{0}$
$\overset{H}{=} \lim_{x\to 0^+} \frac{-2\cdot 3/2 \cdot x^{1/2}}{1} = 0$

ex: $\lim_{x\to 0^+} x^x$ ($0^0$) $= \lim_{x\to 0^+} e^{x\ln x}$ *
$= e^0$
$= 1$

* $\lim_{x\to 0^+} x\ln x$ ($0\cdot\infty$) $= \lim_{x\to 0^+} \frac{\ln x}{x^{-1}}$ $\frac{\infty}{\infty}$
$\overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-x^{-2}}$
$= \lim_{x\to 0^+} -\frac{x^2}{x}$
$= \lim_{x\to 0^+} (-x) = 0^-$

ex: $\lim_{x\to 0} (1-x)^{1/x}$ ($1^\infty$) $= e^{\lim_{x\to 0} 1/x \ln(1-x)}$ *
$= e^{-1}$
$= 1/e$

* $\lim_{x\to 0} 1/x \ln(1-x)$ ($\infty\cdot 0$) $= \lim_{x\to 0} \frac{\ln(1-x)}{x}$ $\frac{0}{0}$
$\overset{H}{=} \lim_{x\to 0} \frac{-\frac{1}{1-x}}{1}$
$= \lim_{x\to 0} -\frac{1}{1-x} = -1$

Note: value $e$ → $\lim_{x\to 0} (1+x)^{1/x} = e$ ⇒ $\lim_{x\to 0} (1+kx)^{1/x} = e^k$
→ $\lim_{x\to\infty} (1+1/x)^x = e$

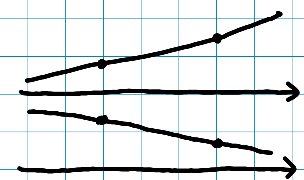
ex: $\lim_{x\to 0^+} (1+\sin 4x)^{\cot x}$ ($1^\infty$) $= \lim_{x\to 0^+} e^{\cot x \cdot \ln(1+\sin 4x)}$
$= e^4$

* $\lim_{x\to 0^+} \cot x \cdot \ln(1+\sin 4x)$ $\infty\cdot 0$
$= \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x}$ $\frac{0}{0}$
$\overset{H}{=} \lim_{x\to 0^+} \frac{\frac{4\cos 4x}{1+\sin 4x}}{\sec^2 x}$
$= \frac{\frac{4(1)}{1+0}}{1} = 4$

# Curve Sketching

## Increasing / Decreasing

Def: A fcn is said to be
- increasing (↗) on $(a, b)$ if $f(x_1) < f(x_2)$, $x_1 < x_2$
- decreasing (↘) on $(a, b)$ if $f(x_1) > f(x_2)$, $x_1 < x_2$

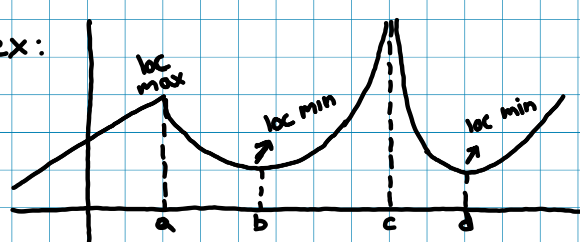


$f(x_1) < f(x_2)$

$f(x_1) > f(x_2)$

Test:
→ If $f'(x) > 0$ on interval $(a, b)$
∴ $f(x)$ ↗
→ If $f'(x) < 0$ on interval $(a, b)$
∴ $f(x)$ ↘

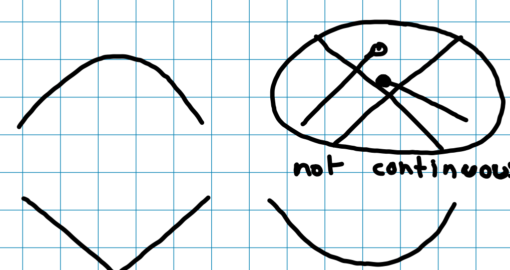
ex:




| | $(-\infty, a)$ | $(a, b)$ | $(b, c)$ | $(c, d)$ | $(d, +\infty)$ |
|---|---|---|---|---|---|
| $f'(x)$ | ⊕ | ⊖ | ⊕ | ⊖ | ⊕ |
| $f(x)$ | ↗ | ↘ | ↗ | ↘ | ↗ |

## Local Extrema

- A fcn has a local max at $x = a$
  → If it goes from ↗ to ↘ at $x = a$
  → If $f(x)$ is continuous at $x = a$
- A fcn has a local min at $x = a$
  → If it goes from ↘ to ↗ at $x = a$
  → If $f(x)$ is continuous at $x = a$



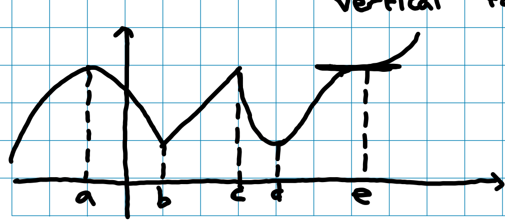

## Critical Numbers

If $f(x)$ is continuous at $x = a$
and $f'(a) = 0$ or $f'(a)$ DNE
∴ $a$ is a critical number

* $f'(x) = 0$ ∴ horizontal tangent
$f'(x)$ DNE ∴ corner (if ↗ to ↘)
vertical tangent

Note: If $f(x)$ has local extremum at $x = a$, $a$ must be a critical number

Note: If vertical asymp at $x = a$ ∴ $f'(a)$ DNE but $a \neq$ critical b/c not continuous




$a, b, c, d, e$ = critical numbers

~~xxxxxxx~~ $e \neq$ local extremum

# First derivative test

If $x=a$ is a critical number of $f(x)$

$\therefore$ $f(x)$ has local max at $x=a$ if $f'(x)$ goes $\oplus$ to $\ominus$

$\therefore$ $f(x)$ has local min at $x=a$ if $f'(x)$ goes $\ominus$ to $\oplus$

ex: Find $\nearrow$ $\searrow$ and local extrema for $y=3x^4-4x^3-12x^2+5$

Domain : Dom $f$: $x\in\mathbb{R}$

$y'=12x^3-12x^2-24x$

critical numbers

$\underline{y'=0}$

$12x(x^2-x-2)=0$

$12x(x-2)(x+1)=0$

$x=0 \mid x=2 \mid x=-1$

$\underline{y' \text{ DNE}}$

none

Table of variation (subdivide Dom into intervals using critical num)

| | $(-\infty,-1)$ | $(-1,0)$ | $(0,2)$ | $(2,+\infty)$ |
|---|---|---|---|---|
| $f'(x)$ | $\ominus$ | $\oplus$ | $\ominus$ | $\oplus$ |
| $f(x)$ | $\searrow$ | $\nearrow$ | $\searrow$ | $\nearrow$ |

$\therefore$ $f\nearrow$ on $(-1,0)\cup(2,+\infty)$

$f\searrow$ on $(-\infty,-1)\cup(0,2)$

$\therefore$ local min at $x=-1$, $2$

local max at $x=0$

ex: $f(x)=x^{3/5}(4-x)$

Dom $f$: $x\in\mathbb{R}$

$$f'(x)=3/5\,x^{-2/5}(4-x)+(-1)(x^{3/5})$$
$$=\frac{3(4-x)-x^{3/5}(5x^{2/5})}{5x^{2/5}}=\frac{12-3x-5x}{5x^{2/5}}=\frac{4(3-2x)}{5x^{2/5}}$$

critical numbers

$\underline{f'(x)=0}$

$3-2x=0$

$\boxed{x=3/2}$

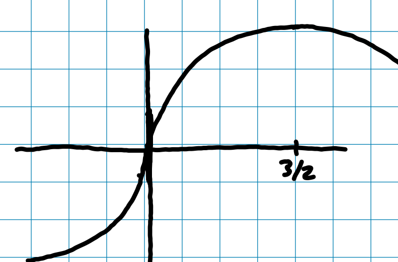
$\underline{f'(x) \text{ DNE}}$

$\boxed{x\neq 0}$

| | 0 | | 3/2 | |
|---|---|---|---|---|
| $3-2x$ | + | + | − | |
| $x^{2/5}$ | + | + | + | |
| | + | + | − | |

$x^2$ always pos

| | $(-\infty,0)$ | $(0,3/2)$ | $(3/2,+\infty)$ |
|---|---|---|---|
| $f'(x)$ | $\oplus$ | $\oplus$ | $\ominus$ |
| $f(x)$ | $\nearrow$ | $\nearrow$ | $\searrow$ |



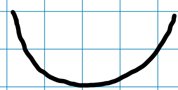
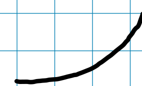
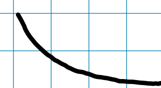

$\therefore$ $f\nearrow$ on $(-\infty,0)\cup(0,3/2)$

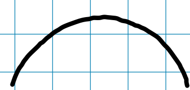
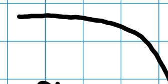
$f\searrow$ on $(3/2,+\infty)$

$\therefore$ local max at $x=3/2$

* Vertical tangent line at $x=0$

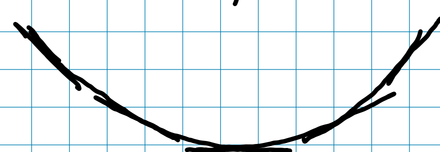
## Concavity

concave up  :  $f'(x) > 0$ ↗ $f''(x) > 0$  $f'(x) < 0$ ↘ $f''(x) > 0$

concave down  :  $f'(x) > 0$ ↗ $f''(x) < 0$  $f'(x) < 0$ ↘ $f''(x) < 0$

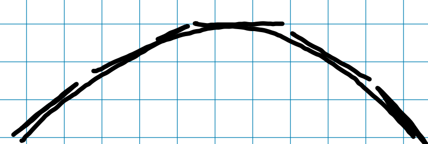
* Sign of $f''(x)$
- pos if slope of tangents getting bigger and bigger
- neg if slope of tangents getting smaller and smaller

Def: • A curve $y = f(x)$ is concave up if the tangent line to the graph of $f$ is "beneath" the graph for every value of $x$.

Note: 

$f'(x)$ ↗ (slopes increasing)
ex: $-2, -1, 0, 1, 2$
$\therefore f''(x) \oplus$

• A curve $y = f(x)$ is concave down if the tangent line to the graph of $f$ is "above" the graph for every value of $x$.

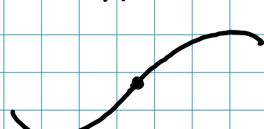
Note: 

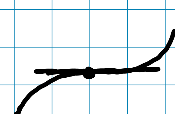
$f'(x)$ ↘ (slopes decreasing)
ex: $2, 1, 0, -1, -2$
$\therefore f''(x) \ominus$

## Concavity Test

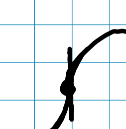
- If $f''(x) > 0$ $\therefore$ $f(x)$ is $\cup$ on $(a, b)$
- If $f''(x) < 0$ $\therefore$ $f(x)$ is $\cap$ on $(a, b)$

* Point of inflection:
→ $f$ changes from $\cap$ to $\cup$ at $(x=a, f(a))$
→ $f$ changes from $\cup$ to $\cap$ at $(x=a, f(a))$
* $f(x)$ continuous at $x=a$
* At critical number $f''(x) = 0$ or $f''(x)$ DNE

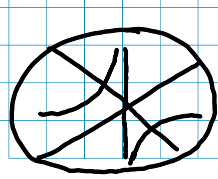


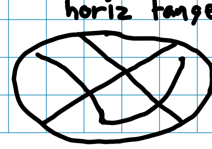


horiz tangent



vertical tangent





ex: $f(x) = x^4 - 4x^3$, sketch a reasonnable graph

1: Dom f : $x \in \mathbb{R}$

2: Find first and second derivative

$$f'(x) = 4x^3 - 12x^2$$
$$f''(x) = 12x^2 - 24x$$

3: Critical numbers:

$\underline{f'(x) = 0}$

$4x^3 - 12x^2 = 0$

$x^2(4x - 12) = 0$

$4x^2(x-3) = 0$

$x = 0 \mid x = 3$

$\underline{f'(x) \text{ DNE}}$

none

$\underline{f''(x) = 0}$

$12x^2 - 24x = 0$

$12x(x-2) = 0$

$x = 0 \mid x = 2$

$\underline{f''(x) \text{ DNE}}$

none

4: Table of variations (show where $f'(x)$ or $f''(x) = 0$)

| | $(-\infty, 0)$ | $(0, 2)$ | $(2, 3)$ | $(3, +\infty)$ |
|---|---|---|---|---|
| $f'(x)$ | $\ominus$ | $\ominus$ | $\ominus$ | $\oplus$ |
| $f''(x)$ | $\oplus$ | $\ominus$ | $\oplus$ | $\oplus$ |
| $f(x)$ | ↘ ∪ | ↘ ∩ | ↘ ∪ | ↗ ∪ |

↘ ∪

↗ ∪

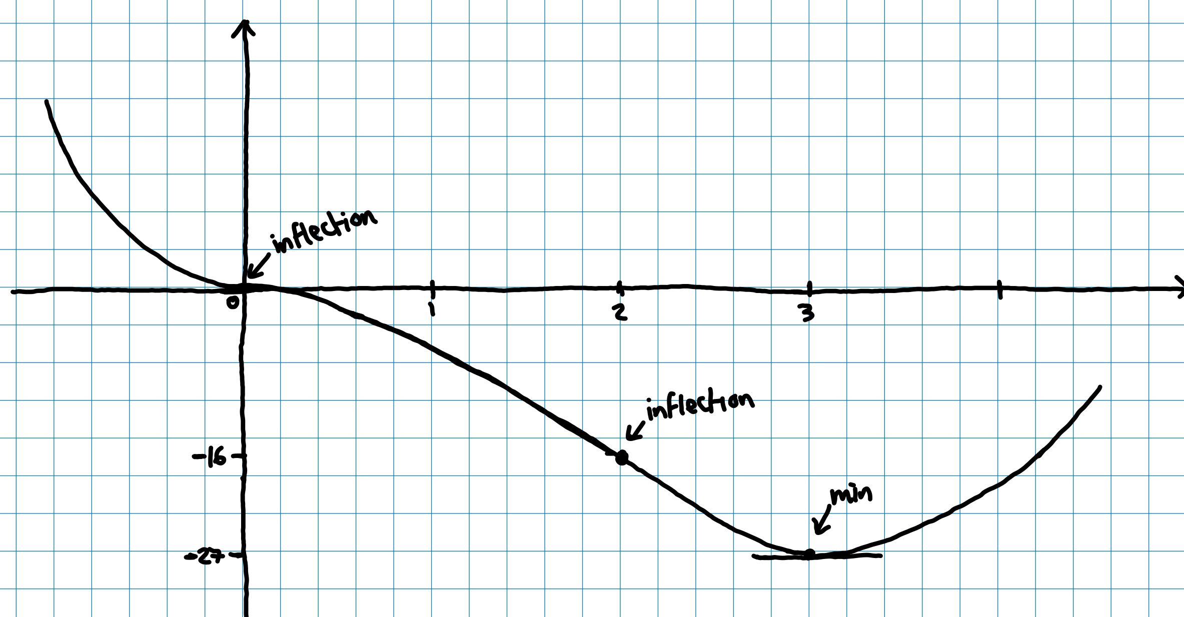
↘ ∩

↗ ∩

∴ pt of inflection : $(0, f(0)) \rightarrow (0, 0)$

$(2, f(2)) \rightarrow (2, -16)$

∴ local min : $(3, f(3)) \rightarrow (3, -27)$



ex: Let $y = \frac{x}{(2x-3)^2}$, Find domain of f
Find x and y intercepts
Find vertical / horizontal asymptotes
Find I/D intervals and local extrema
Find concavity and points of inflection

1: Restriction
$2x-3 \neq 0$
$x \neq 3/2$

$\therefore$ Dom f : $x \in (-\infty, 3/2) \cup (3/2, +\infty)$

2: $\underline{\text{x-int}}$ (Let $y=0$)

$\frac{x}{(2x-3)^2} = 0$

$\boxed{x = 0}$

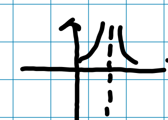
$\underline{\text{y-int}}$ (Let $x=0$)

$y = \frac{0}{(2 \cdot 0 - 3)^2}$

$\boxed{y = 0}$

$\therefore$ pt $(0,0)$

3: vertical asympt
at $x = 3/2$ $\therefore \lim_{x \to 3/2^-} \frac{x}{(2x-3)^2} = +\infty$, $\cdot \lim_{x \to 3/2^+} x \cdot \frac{1}{(2x-3)^2} = +\infty$



horizontal asympt
- $\lim_{x \to +\infty} \frac{x}{(2x-3)^2} \quad \frac{\infty}{\infty} \overset{H}{=} \lim_{x \to +\infty} \frac{1}{2(2x-3) \cdot 2} = \frac{1}{\infty} = 0$
- $\lim_{x \to -\infty} \frac{x}{(2x-3)^2} = 0$

4: $y' = \frac{(2x-3)^2 - 2(2x-3) \cdot 2 \cdot x}{(2x-3)^4} = \frac{(2x-3)(2x-3-4x)}{(2x-3)^4} = \frac{-(2x+3)}{(2x-3)^3}$

$y'' = \frac{-2(2x-3)^3 - 3(2x-3)^2 \cdot 2\,(-2x-3)}{(2x-3)^6} = \frac{(2x-3)^2\left(-2(2x-3) - 6(-2x-3)\right)}{(2x-3)^6} = \frac{8x+24}{(2x-3)^4}$

$\underline{y' = 0}$
$2x+3 = 0$
$\boxed{x = -3/2}$

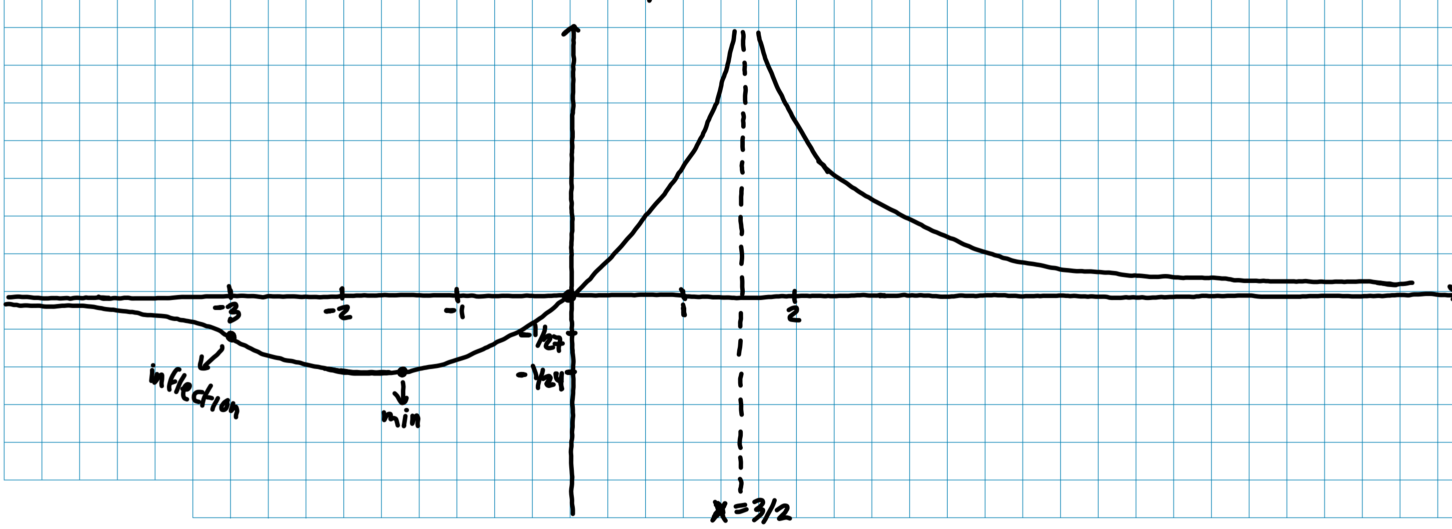
$\underline{y' \text{ DNE}}$
$2x-3 \neq 0$
$\boxed{x \neq 3/2}$ Asympt

$\underline{y'' = 0}$
$8x+24 = 0$
$\boxed{x = -3}$

$\underline{y'' \text{ DNE}}$
$2x-3 \neq 0$
$\boxed{x \neq 3/2}$ Asympt

| | $(-\infty, -3)$ | $(-3, -3/2)$ | $(-3/2, 3/2)$ | $(3/2, +\infty)$ |
|---|---|---|---|---|
| $f'(x)$ | ⊖ | ⊖ | ⊕ | ⊖ |
| $f''(x)$ | ⊖ | ⊕ | ⊕ | ⊕ |
| $f(x)$ | ↘ | ↘ | ↗ | ↘ |

$\therefore$ increasing on $(-3/2, 3/2)$
decreasing on $(-\infty, -3/2) \cup (3/2, \infty)$
$\therefore$ local min : $(-3/2, f(-3/2)) \Rightarrow (-3/2, -1/24)$
$\therefore$ inflection : $(-3, f(-3)) \Rightarrow (-3, -1/27)$



ex: $f(x) = \sqrt{\frac{x}{x+4}}$

Note: can't seperate $\frac{\sqrt{x}}{\sqrt{x+4}}$ b/c they may both be negative on their own

1: Restrictions

$x+4 \neq 0$

$x \neq -4$

$\frac{x}{x+4} \geq 0$

$-4 \quad 0$

| | | | |
|---|---|---|---|
| $x$ | − | − | + |
| $x+4$ | − | + | + |
| | + | − | + |

∴ Dom f: $x \in (-\infty,-4) \cup [0,+\infty)$

2: x-int

$0 = \sqrt{\frac{x}{x+4}}$

$\boxed{0 = x}$

y-int

$y = \sqrt{\frac{0}{0+4}}$

$\boxed{y = 0}$

∴ pt $(0,0)$

3: Vertical Asymptotes

at $x=-4$

• $\lim_{x\to -4^-} \sqrt{\frac{x}{x+4}} = \lim_{x\to -4^-} \sqrt{\underset{\to -4}{x} \cdot \underset{\to -\infty}{\frac{1}{x+4}}} = \sqrt{\infty} = \infty$

• $\lim_{x\to -4^+}$ DNE on the right of $-4$ (see domain)

$\lim_{x\to -4^+} \sqrt{\underset{\to -4}{x} \cdot \underset{\to +\infty}{\frac{1}{x+4}}} = \sqrt{-\infty}$ = DNE

Horizontal Asymptotes

• $\lim_{x\to -\infty} \sqrt{\frac{x}{x(1+4/x)}} = \sqrt{1} = 1$

• $\lim_{x\to +\infty} \sqrt{\frac{x}{x(1+4/x)}} = \sqrt{1} = 1$

4: $f'(x) = \frac{1}{2\sqrt{\frac{x}{x+4}}} \cdot \left(\frac{1(x+4) - 1x}{(x+4)^2}\right) = \frac{1}{2}\sqrt{\frac{x+4}{x}} \cdot \frac{4}{(x+4)^2} = \frac{2}{\sqrt{x}\,(x+4)^{3/2}}$

$f''(x) = 2\left((x(x+4)^3)^{-1/2}\right)' = \cancel{2} \cdot -\frac{1}{\cancel{2}}(x(x+4)^3)^{-3/2} \cdot \left((x+4)^3 + 3(x+4)^2 \cdot x\right) = \frac{-\left((x+4)^3 + 3x(x+4)^2\right)}{(x(x+4)^3)^{3/2}}$

$= -\frac{(x+4)^2(x+4+3x)}{(x(x+4)^3)^{3/2}} = -\frac{4(x+1)(x+4)^2}{(x(x+4)^3)^{3/2}}$

$f'(x)=0$

none

$f'(x)$ DNE

$\sqrt{x}\,(x+4)^{3/2} \neq 0$

$x=0 \mid x=-4$

$f''(x)=0$

$(x+1)(x+4)^2 = 0$

$x=-1 \mid x=-4$

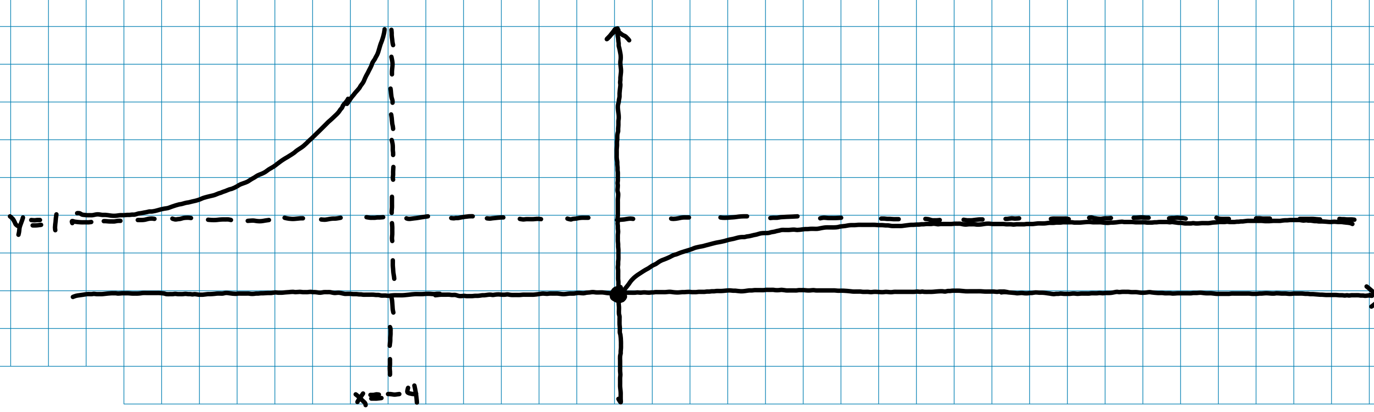
not in domain

$f''(x)$ DNE

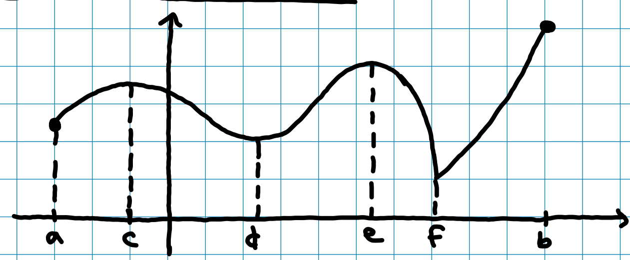
$x(x+4)^3 \neq 0$

$x \neq 0 \mid x \neq -4$

| | $(-\infty,-4)$ | $(-4,0)$ | $(0,+\infty)$ |
|---|---|---|---|
| $f'(x)$ | ⊕ | | ⊕ |
| $f''(x)$ | ⊕ | | ⊖ |
| $f(x)$ | ↗ | | ↗ |



# Absolute Extrema




- $f(x)$ continuous on $[a,b]$
- $a,b \Rightarrow$ end points of closed interval
- $c,d,e,f \Rightarrow$ critical numbers

Local min : $x=d$ , $x=f$ (Not a b/c no ↘ to ↗)
Local max : $x=c$ , $x=e$ (Not b b/c no ↗ to ↘)
Absolute min : $x=f$
Absolute max : $x=b$ (largest value)

**Note 1** : The absolute extrema (greatest or smallest) of a continuous fcn on a closed interval can occur at one or more critical numbers or at one of the end points

**Note 2**: Extreme Value Theorem
If $y=f(x)$ is continuous on an interval $[a,b]$, then there exists $c\in[a,b]$ and $d\in[a,b]$ s.t. $f(c)=$ abs. min , $f(d)=$ abs. max.

Closed Interval Method
- find each critical number (and end points)
- → evaluate the fcn at each critical number and end point

  - Biggest = MAX
  - Smallest = MIN

⊛ Method can be used in optimization if closed interval

ex: $f(x)=x^3-3x^2+1$ on $[-\frac{1}{2},4]$ , find abs. extrema

$f'(x)=3x^2-6x$

$f'(x)=0$
$3x(x-2)=0$
$x=0 \mid x=2$

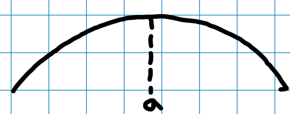
$x=0 \Rightarrow y=1$
$x=2 \Rightarrow y=-3$
$x=-\frac{1}{2} \Rightarrow y=\frac{1}{8}$
$x=4 \Rightarrow y=17$

∴ abs max is $17=f(4)$
abs min is $-3=f(2)$

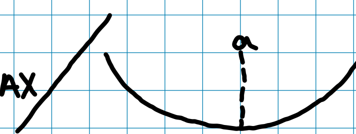
# Optimization

- Write the function you are trying to minimize or maximize
- Using resources available, write a relationship between the 2 variables
- Find max or min

2nd derivative test

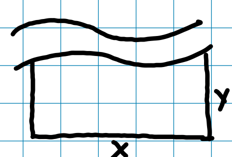



$f'(a)=0$
$f''(a)<0$ $\cap$ $\therefore$ MAX




$f'(a)=0$
$f''(a)>0$ $\cup$ $\therefore$ MIN

ex: Farmer Bob has 2400 m of fencing to build a rectangular coral that borders a river. He doesn't need fencing on the river side.




What are the dimensions of the river that maximize the Area?

Area: $A = x \cdot y$ (to maximize)

$x + 2y = 2400$
$x = 2400 - 2y$
sub into A
$A(y) = (2400 - 2y) \cdot y$
$= -2y^2 + 2400y$
$A(y) = 2y(1200 - y)$

Find maximum value of $A(y)$
$A'(y) = 2(1200 - y) + 2y(-1)$
$= 2400 - 4y$

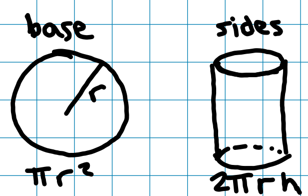
$\underline{A'(y) = 0}$
$2400 - 4y = 0$
$\boxed{y = 600}$

| | (0, 600) | (600, 1200) |
|---|---|---|
| A'(y) | ⊕ | ⊖ |
| A(y) | ↗ | ↘ |

$\therefore$ A(y) has a max at $y = 600$
$\therefore$ $x = 1200$

ex: Farmer Bob wants to build a metal cylinder with an open top with a volume of $1m^3$. Find the dimensions that minimize material used (surface area).


$V = \pi r^2 \cdot h = 1$




$SA = \pi r^2 + 2\pi r h$ (to minimize)

$\pi r^2 h = 1$

$h = \frac{1}{\pi r^2}$

sub into SA

$SA(r) = \pi r^2 + 2\cancel{\pi} \cancel{r} \cdot \frac{1}{\cancel{\pi} r^{\cancel{2}}}$

$A(r) = \pi r^2 + 2/r$

$A(r) = \frac{\pi r^3 + 2}{r}$

Find min value of $A(r)$

$A'(r) = \frac{3\pi r^2 \cdot r - 1(\pi r^3 + 2)}{r^2}$

$A'(r) = \frac{2\pi r^3 - 2}{r^2}$

$\underline{A'(r) = 0}$

$2\pi r^3 - 2 = 0$

$\boxed{r = \frac{1}{\sqrt[3]{\pi}}}$

$\underline{A'(r) \text{ DNE}}$

$r \neq 0$

ommit b/c length

| | $\left(0, \frac{1}{\pi^{1/3}}\right)$ | | $\left(\frac{1}{\pi^{1/3}}, +\infty\right)$ |
|---|---|---|---|
| $A'(r)$ | $-$ | 0 | $+$ |
| $A(r)$ | $\searrow$ | | $\nearrow$ |

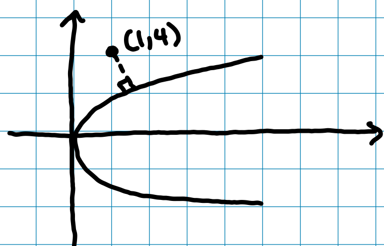
$\therefore A(r)$ has a minimum at $r = \frac{1}{\sqrt[3]{\pi}}$

$h = \frac{1}{\pi r^2}$

$= \frac{1}{\pi \frac{1}{\pi^{2/3}}}$

$\boxed{h = \frac{1}{\pi^{1/3}}}$

ex: Consider the curve $y^2 = 2x$. Find the point on the curve that is closest to the point $(1,4)$




$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(1-x)^2 + (4-y)^2} \quad \text{(to minimize)}$$

$y^2 = 2x$

$x = y^2/2$

sub into d

$$d(y) = \sqrt{\left(1 - \frac{y^2}{2}\right)^2 + (4-y)^2}$$

$$= \sqrt{1 - \frac{2y^2}{2} + \frac{y^4}{4} + 16 - 8y + y^2}$$

$$d(y) = \sqrt{\frac{y^4}{4} - 8y + 17 \cancel{-y^2} \cancel{+y^2}}$$

Note: minimizing $d(y)$ is same as minimizing $(d(y))^2 = D(y)$

$$D(y) = \frac{y^4}{4} - 8y + 17$$

Find minimum value

$$D'(y) = \frac{4y^3}{4} - 8$$

$$\boxed{D'(y) = y^3 - 8}$$

$\underline{D'(y) = 0}$

$y^3 - 8 = 0$

$y^3 = 8$

$\boxed{y = 2}$

Second derivative test

$D''(y) = 3y^2$

$D'(2) = 0$

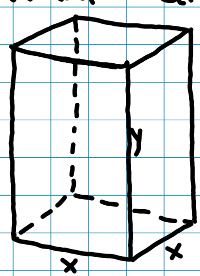
$D''(2) = +$ $\quad \therefore$ minimum at $y = 2$

$y^2 = 2x$

$(2)^2 = 2x$

$\boxed{x = 2}$ $\quad \therefore$ closest point is point $(2,2)$

ex: Farmer Bob wants to design an open box that has a square base and a volume of $10\ m^3$.
The sides of the box are made of a material that costs 2\$ per $m^2$, while the bottom is made of a material that costs $3\ \$/m^2$.
What dimensions will minimize cost of production?




$$\text{cost} = 3x^2 + 2(4xy)$$
$$= 3x^2 + 8xy \quad \text{(to minimize)}$$

$$V = x^2 y = 10$$
$$y = \frac{10}{x^2}$$

sub into $C(x)$

$$C(x) = 3x^2 + 8x\,\frac{10}{x^2}$$

$$C(x) = 3x^2 + \frac{80}{x}$$

Find minimum

$$C'(x) = 6x + \frac{-80}{x^2}$$

$$C'(x) = \frac{6x^3 - 80}{x^2}$$

$\underline{C'(x) = 0}$
$6x^3 - 80 = 0$
$x^3 = 80/6$
$\boxed{x = \sqrt[3]{40/3}}$

$\underline{C'(x)\ \text{DNE}}$
$x \neq 0$
ommit b/c dimension

Second derivative test

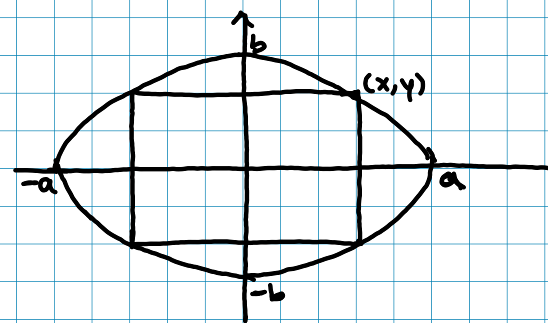
$$C''(x) = 6 + \frac{160}{x^3}$$

$$\left.\begin{aligned} C'\left(\sqrt[3]{40/3}\right) &= 0 \\ C''\left(\sqrt[3]{40/3}\right) &= \oplus \end{aligned}\right\} \therefore \text{min at } \boxed{x = \sqrt[3]{40/3}}$$

$$y = \frac{10}{x^2}$$
$$= \frac{10}{(40/3)^{2/3}} \Rightarrow \boxed{y = 10\left(\frac{3}{40}\right)^{2/3}}$$

ex. Find the area of the largest rectangle that can be inscribed in the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$




Area = base . height

$= 2x \cdot 2y$

$A = 4xy$ (to maximize)

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$y^2 = \left(1 - \frac{x^2}{a^2}\right) b^2$

$y^2 = \left(\frac{a^2-x^2}{a^2}\right) b^2$

$y = \sqrt{\frac{b^2(a^2-x^2)}{a^2}} = \frac{b}{a}\sqrt{a^2-x^2}$

sub into $A(x)$

$A(x) = 4x\left(\frac{b}{a}\sqrt{a^2-x^2}\right)$ on Dom $[0, a]$

$A(x) = \frac{4b}{a} x\sqrt{a^2-x^2}$

Find maximum

$A'(x) = \frac{4b}{a}\left(\sqrt{a^2-x^2} + \frac{1}{2\sqrt{a^2-x^2}} \cdot -2x \cdot x\right)$

$= \frac{4b}{a}\left(\sqrt{a^2-x^2} - \frac{x^2}{\sqrt{a^2-x^2}}\right)$

$= \frac{4b}{a}\left(\frac{a^2-x^2-x^2}{\sqrt{a^2-x^2}}\right)$

$A'(x) = \frac{4b}{a}\left(\frac{a^2-2x^2}{\sqrt{a^2-x^2}}\right)$

$\underline{A'(x)=0}$

$a^2-2x^2=0$

$\boxed{x = \frac{a}{\sqrt{2}}}$

$\underline{A'(x) \text{ DNE}}$

$a^2-x^2 \neq 0$

$\boxed{x \neq a}$

Closed Interval method

$A(0) = 0$

$A(a) = 4(a)(0) = 0$

$A\left(\frac{a}{\sqrt{2}}\right) = 4x \frac{b}{a}\sqrt{a^2-x^2}$

$= 4\frac{a}{\sqrt{2}} \cdot \frac{b}{a}\sqrt{a^2-x^2} = \frac{4b}{\sqrt{2}}\sqrt{a^2-\left(\frac{a}{\sqrt{2}}\right)^2} = \frac{4b}{\sqrt{2}}\sqrt{\frac{2a^2-a^2}{2}} = \frac{4b \cdot \sqrt{a^2}}{\sqrt{2} \cdot \sqrt{2}}$

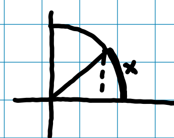
$= \frac{4ba}{2} = \boxed{2ab} \rightarrow$ max area.

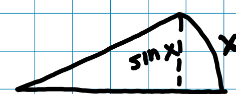
# Theory

## Special Limit (proof)

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

proof: if $0 \le x \le \pi/2$




- $\sin x < x$

  $\therefore \frac{\sin x}{x} < 1$

- $\tan x > x$

  $\frac{\sin x}{\cos x} > x$

  $\therefore \frac{\sin x}{x} > \cos x$

Squeeze: $\boxed{\cos x < \frac{\sin x}{x} < 1}$

$$\left.\begin{array}{l} \lim_{x\to 0^+} \cos x = 1 \quad \therefore \lim_{x\to 0^+} \frac{\sin x}{x} = 1 \\ \text{and since it's even} \therefore \lim_{x\to 0^-} \frac{\sin x}{x} = 1 \end{array}\right\} \therefore \lim_{x\to 0} \frac{\sin x}{x} = 1$$

## Theorem

If $f$ is differentiable at $a$ $\therefore$ $f$ is continuous at $a$

Proof: Show that if $f'(a)$ exists:

$$\lim_{x\to a} f(x) = f(a)$$

$$\lim_{x\to a} (f(x) - f(a)) = 0$$

$$\lim_{x\to a} \frac{f(x)-f(a)}{(x-a)} \cdot (x-a) = 0$$

* Want $f'(a)$ (slope) so divide by $(x-a)$

$$\lim_{x\to a} \frac{f(x)-f(a)}{(x-a)} \cdot \lim_{x\to a} (x-a) = 0$$

$$\underbrace{f'(a)}_{\substack{\text{If } f'(a) \\ \text{did not exist,} \\ \text{can't compute}}} \cdot 0 = 0$$

So yes

## Power Rule (Proof)

$$\frac{d}{dx} x^n = n x^{n-1}$$

Proof: We know that:

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a^1 + x^{n-3}a^2 + \ldots + x^1 a^{n-2} + a^{n-1})$$

Powers of x go down until 0
Powers of a go up until n-1

If $f(x) = x^n$, show that $f'(a) = n a^{n-a}$ for $x=a$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{x^n - a^n}{x - a}$$

$$= \lim_{x \to a} \frac{\cancel{(x-a)}(x^{n-1} + x^{n-2}a + \ldots + x a^{n-2} + a^{n-1})}{\cancel{(x-a)}}$$

sub $x=a$

$$= a^{n-1} + a^{n-1} + \ldots + a^{n-1} + a^{n-1}$$

$$= n a^{n-1}$$

## Exponential Rule (Proof)

$$\frac{d}{dx} e^x = e^x$$

Special Limit

$$\boxed{\lim_{x \to 0} \frac{e^x - 1}{x} = 1}$$

Proof: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x (e^h - 1)}{h}$$

$$= \underbrace{\lim_{h \to 0} e^x}_{} \cdot \underbrace{\lim_{h \to 0} \frac{e^h - 1}{h}}_{}$$

$$= e^x \cdot 1$$

$$= e^x$$