2 A)

Seperable Differential Equations:

A seperable DE is a first order ODE, which through algebraic manipulations, can be written in the form: $g(y)(y') = f(x)$, the latter being the "seperated form"

Theorem:

The following satisfies this DE: $\boxed{\int g(y)\,dy = \int f(x)\,dx + C}$

N.B. Remember to add $+C$ after antiderivative has been determined.

Proof: Let $G(y) = \int g(y)\,dy$. Let $F(x) = \int f(x)\,dx$.

RTP: soln is $G(y) = F(x) + C$.

Indeed, differentiating both sides WRT x, we get

$$\frac{d}{dx}\left[G(y)\right] = \frac{d}{dx}\left[F(x) + C\right] \Rightarrow g(y)\,y' = f(x)$$

∎

In practice, we start with a DE which may not be seperated yet, so we try to seperate it: $g(y)y' = f(x)$. And we write $y' = \frac{dy}{dx}$

$\rightarrow g(y)\frac{dy}{dx} = f(x)$

$\rightarrow g(y)\,dy = f(x)\,dx$

$\rightarrow \int g(y)\,dy = \int f(x)\,dx + C$ (Just like cal II).

E.g. Solve the DE $9yy' + 4x = 0$

$$9y \frac{dy}{dx} = -4x$$

$$9y\, dy = -4x\, dx$$

$$\int 9y\, dy = -\int 4x\, dx + C$$

$$\frac{9y^2}{2} = -2x^2 + C$$

Let $2C = D$

$$4x^2 + 9y^2 = 2C$$

$$4x^2 + 9y^2 = D$$

Recap: $g(y)\, y' = f(x)$

$$g(y) \frac{dy}{dx} = f(x)$$

$$\int g(x)\, dy = \int f(x)\, dx + C$$

Example: Solve $y' = 1 + y^2$

$$\frac{dy}{dx}\left(\frac{1}{1+y^2}\right) = 1$$

$$\int \frac{dy}{1+y^2} = \int dx + C$$

$$\arctan y = x + C \Rightarrow y = \tan(x + C)$$

Example: Solve $y' = -2xy$

$$\frac{1}{y} \frac{dy}{dx} = -2x$$

$$\int \frac{dy}{y} = \int -2x\, dx + C$$

$$\ln|y| = -x^2 + C \Rightarrow$$

$$|y| = e^{-x^2+c}$$

$$y = \underbrace{\pm e^c}_{\text{Any constant}} e^{-x^2} \Rightarrow y = De^{-x^2} \quad ; D \in \mathbb{R}$$

Say that at some $x = x_0$, it is given that $y(x)$ has a prescribed value at y_0. I.e. $y(x_0) = y_0$. This is called an initial condition. A first order ODE with an IC is called an Initial Value Problem (IVP). We solve the DE and find the particular soln that fits the IC.

Example: Solve the IVP

$$y' = y\cot(x) \quad , \quad y\left(\tfrac{\pi}{3}\right) = 2$$

$$\frac{dy}{dx}\frac{1}{y} = \cot x$$

$$\int \frac{dy}{y} = \int \cot x \, dx + C$$

$$\ln|y| = \int \frac{\cos x}{\sin x} dx + C \qquad \begin{array}{l} u = \sin x \\ du = \cos x \end{array}$$

$$\ln|y| = \ln|\sin x| + C$$

$$|y| = e^{\ln|\sin x| + C}$$

$$y = \pm e^c e^{\ln|\sin x|}$$

$$y(x) = \pm e^c \sin x$$

$$y\left(\tfrac{\pi}{3}\right) = \pm D \sin\left(\tfrac{\pi}{3}\right) = 2$$

$$2 = D\frac{\sqrt{3}}{2}$$

$$\boxed{D = \frac{4}{\sqrt{3}}}$$

À vous :

$$y'(1+x^2) + y^2+1 = 0$$

$$-y'(1+x^2) = y^2+1 \qquad y(0)=1$$

$$-\frac{dy}{dx}\frac{1}{y^2+1} = \frac{1}{1+x^2}$$

$$-\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2} + C$$

$$-\arctan y = \arctan x + c$$

$$-y = \tan(\arctan x + C)$$

$$-y = \frac{\tan(\arctan x) + \tan(C)}{1-\tan(\arctan x)\tan(c)}$$

$$y(x) = -\frac{x+\tan(C)}{1-x\tan(C)}$$

$$y(0) = 1 = -\frac{0+\tan C}{1-0(\tan c)}$$

$$-1 = \tan(C)$$

$$y = -\frac{x-1}{1+x}$$

$$\boxed{y = \frac{1-x}{1+x}}$$

2B)

Applications: Modeling Involving Seperable Equations

Newtons Law of Cooling:

An object at temperature T in a surrounding medium at (constant) temperature T_M loses heat at a rate proportional to the difference of temperatures. I.e. $T'(t) \propto T - T_M$

Example: A hot copper ball at $100^{\circ c}$ is placed in water maintained at $30^{\circ c}$. After 3min, the ball is at $78^{\circ c}$ when will it be at $31^{\circ c}$

$$T'(t) = K(T - T_M)$$

$$\frac{dT}{dt} = K(T - T_M)$$

$$\int \frac{dT}{T - T_M} = \int K\,dt + C$$

$$\ln|T - T_M| = kt + c$$

$$|T - T_M| = e^{kt+c}$$

$$T - T_M = e^c e^{kt}$$

$$T(t) = De^{kt} + T_M$$

sub $T(0) = 100$

$$100 = D\cancel{e}^{1} + T_M$$

$$D = 100 - T_M = 100 - 30 = 70$$

$$T(t) = 70e^{kt} + 30$$

$$T(3) = 78 = 70e^{3k} + 30$$

$$70e^{3k} = 48$$

$$K = \frac{\ln\left(\frac{48}{70}\right)}{3} \approx -0.1258$$

$$T(t) = 70e^{-0.1258t} + 30$$

$$T(t) = 31 = 70e^{-0.1258t} + 30$$

$$\frac{1}{70} = e^{-0.1258t}$$

$$-4.25 = -0.1258t$$

$$t = 33.77 \text{ min}$$

Example: $y' = 6x(y-1)^{2/3}$

Assume $y=1$ is not a soln (otherwise dividing by zero!)

$$\int \frac{dy}{(y-1)^{2/3}} = \int 6x\,dx$$

$$3 \cdot \sqrt[3]{y-1} = 3x^2 + C$$

$$y = (x^2 + D)^3 + 1.$$

The fcn $y=1$ IS a soln to the DE not included in our general soln (i.e. it is a singular soln). We neglected the existence of f^n $y=1$ when we divided both sides by $(y-1)^{2/3}$.

Q: Can a seperable ODE have singular solutions?

A: Yes (see above example and explanation)

Q: Can seperated ODEs have singular solns?

A: No! We get $\int g(y)\,dy = \int f(x)\,dx + C$, which we know (proof in Ch1) that this must include all solns to the ODE. $\therefore$ Seperated DEs have no singular solutions.

Torricelli's Law:

First, what is the velocity of a particle falling freely (initially at rest) for a distance h?

$$x(t) = \cancel{x_0}^{0} + \cancel{v_0}^{0} t + \tfrac{1}{2} g t^2$$

$$x(t) = \tfrac{1}{2} g t^2$$

When $x(t) = h$, $t = ?$

$$h = \tfrac{1}{2} g t^2$$

$$t = \sqrt{\frac{2h}{g}}$$

What is the velocity after having fallen h meters?

$$v(t) = \frac{d}{dt} x(t) = \frac{d}{dt}\left(x_0 + v_0 t + \tfrac{1}{2} g t^2\right)$$

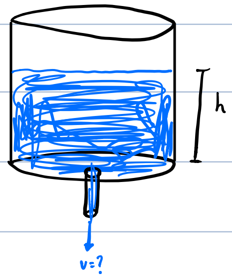
$$v(t) = \cancel{v_0}^{0} + gt$$

$$v(t) = g\sqrt{\frac{2h}{g}}$$

$$v(t) = \sqrt{2gh}$$

Now, lets consider a tank of water with an orfice from which water (vodka?) is flowing.

Toricelli's Law states that the velocity with which water issues from the orfice is proportional to $\sqrt{2gh}$ (where h is the height of the H_2O)



In the case of water, $v = 0.6\sqrt{2gh}$

Example: A cylindrical tank 1.5 m high stands on its circular base of diameter 1 m. A hole at the bottom of diameter 1 cm is opened at some instant.

a) Find the height of the water $h(t)$ at any time t.

b) Find the times at which the tank is half full, one quarter full, and empty.

At time Δt, $\overset{\text{volume}}{\Delta V}$ of the tank is $\Delta V = -v\,\Delta t\,\pi r^2$

$$\Delta V = -0.6\sqrt{2gh}\,\pi r^2\,\Delta t$$

$$\cancel{\pi} R^2\,\Delta h = -0.6\sqrt{2gh}\,\cancel{\pi} r^2\,\Delta t$$

$$\frac{\Delta h}{\Delta t} = -\frac{3\sqrt{2gh}\,r^2}{5R^2} \Longrightarrow \frac{dh}{dt} = \left(-\frac{3\sqrt{2g}}{5}\right)\left(\frac{0.01\,\cancel{m}}{1\,\cancel{m}}\right)^2\sqrt{h} \Rightarrow \frac{dh}{dt} = \overset{\gamma = -2.656\cdot 10^{-4}}{\gamma}\sqrt{h}$$

, which is an easy separable ODE

Solving this ODE,

$$\int \frac{dh}{\sqrt{h}} = \gamma \int dt$$

$$2\sqrt{h} = \gamma t$$

$$h(t) = \left(\frac{\gamma t}{2}\right)^2$$

Escape Velocity:

Consider a projectile fired upwards from the ground with initial velocity V_0. $R_E \approx 6.372 \cdot 10^6$ m

a) Write a DE for the velocity of the projectile as a fcn of its distance r from the center of the earth

b) Solve the DE to obtain velocity at a distance r (obviously $r > R_E$) in terms of the initial velocity v_0.

c) Use the above to find the value of v_0 corresponding to the escape velocity

$$F_g \propto \frac{1}{r^2} \Rightarrow a \propto \frac{1}{r^2} \Rightarrow a = \frac{k}{r^2}$$

Initial Condition: when $r = R_E$, $a_y = -g$

$$-g = \frac{k}{R_E^2} \Rightarrow \boxed{k = -g R_E^2}$$

So, $\frac{dv}{dt} = \frac{-g R_E^2}{r^2}$, but we want $v(r)$. We reparametrize velocity as $v(r(t))$.

$$\therefore \frac{dv}{dr} \underbrace{\frac{dr}{dt}}_{v} = \frac{-g R^2}{r^2} \quad \text{(by chain rule).}$$

$$v \frac{dv}{dr} = \frac{-g R^2}{r^2}$$

b) $\int v \, dv = -\int \frac{g R^2}{r^2} dr + C$

$$\frac{v^2}{2} = \frac{g R^2}{r} + C$$

$$v^2 = \frac{2 g R^2}{r} + C$$

$$v(r) = \sqrt{\frac{2 g R^2}{r} + C}$$

I.C. $v(R) = v_0$

$$v_0 = \sqrt{\frac{2 g R^2}{R} + C}$$

$$v_0^2 = 2 g R + C$$

$$\boxed{C = v_0^2 - 2 g R}$$

$$v(r) = \sqrt{\frac{2 g R^2}{r} + v_0^2 - 2 g R}$$

$$\lim_{r \to \infty} v(r) = \lim_{r \to \infty} \sqrt{\frac{2 g R^2}{r} + v_0 - 2 g R}$$

$V_{esc} = \sqrt{v_0 - 2 g R} \Rightarrow v_0 \geq 2 g R$ for V_{esc} to exist

Finishing off section 2C on reduction to seperable form.

Sometimes we may find a relevant substitution to make just by inspection...

Ex: Solve $(2x-4y+5)y' + x+2y+3=0$

$$\text{Let } u = x+2y+3 \Rightarrow u' = 1+2y' \Rightarrow y' = \frac{1-u'}{2}$$

$$(2u-1)y' + u = 0$$

Rewrite as: $(2u-1)\left(\frac{1-u'}{2}\right) + u = 0$

$$(2u-1)(1-u') = -2u$$

$$2u - 2uu' - 1 + u' = -2u$$

$$u' - 2uu' = -4u + 1$$

$$u'(1-2u) = 1-4u$$

$$u' = \frac{1-4u}{1-2u}$$

$$\frac{du}{dx} = \frac{1-4u}{1-2u}$$

$$\int \frac{1-2u}{1-4u}\,du = \int dx$$

$$\int \frac{\frac{1}{2}(2-4u)}{1-4u}\,du = x+C$$

$$\frac{1}{2}\left(\int \frac{1}{1-4u}\,du + \int \frac{1-4u}{1-4u}\,du\right) = x+C$$

$$\frac{1}{2}\left(\frac{\ln|1-4u|}{-4} + u\right) = x+C$$

$$C = \ln|1-4u| - 8u + 8x$$

$$C = \ln|1-4(x+2y+3)| - 8(x+2y+3) + 8x$$

$$\ln|1-4(x+2y+3)| - 8(x+2y+3) + 8x - C = 0$$

Disgusting. But this is the soln to the DE.

Closing section 2D on partial derivatives:

Ex: $f(x,y,z) = x\sin(xz) + y^2 z$. Show that $f_{xz} = f_{zx}$

⋮ easy shit

Ex: Suppose a fcn $f(x,y)$ satisfies $f_x = 4y^2 + 9x^2$ and $f_y = 8xy + 3\sqrt{y}$ & $f(0,4) = 14$

$$\frac{\partial f}{\partial x} = 4y^2 + 9x^2 \qquad\qquad \frac{\partial f}{\partial y} = 8xy + 3\sqrt{y}$$

$$f(x,y) = \int 4y^2 + 9x^2 \, dx + C \qquad\qquad f(x,y) = \int 8xy + 3\sqrt{y} \, dy + C$$

$$f(x,y) = 4xy^2 + 3x^3 + h(y) + C \qquad\qquad f(x,y) = 4xy^2 + 2y^{3/2} + g(x) + C$$

$$g(x) = 3x^3 ; \quad h(y) = 2y^{3/2}$$

$$f(x,y) = 4xy^2 + 3x^3 + 2y^{3/2} + C$$

$$f(0,4) = 14 = \cancel{4(0)(4)^2}^{\,0} + 3(0)^3 + 2(4)^{3/2} + C$$

$$14 = C + 16$$

$$\boxed{C = -2}$$

$$\therefore f(x,y) = 4xy^2 + 3x^3 + 2y^{3/2} - 2$$

2E) Exact Differential Equations

Consider: $M(x,y)\,dx + N(x,y)\,dy = 0$

If $M(x,y)$ and $N(x,y)$ happen to be

a $\frac{\partial u}{\partial x}$ and a $\frac{\partial u}{\partial y}$ for some $u(x,y)$,

(total differential from cal III)

Then the DE $\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$ i.e. $du = 0$

* Total diff'l corresponding to zero is a plane that runs ∥ to xy plane.

This means that $u(x,y) \equiv C$, so $u(x,y) \equiv c$ is an implicit soln in terms of x.

Given the DE:

$$\circledast \quad M(x,y)\,dx + N(x,y)\,dy = 0$$

We hope to find some $u(x,y)$ such that $\frac{\partial u}{\partial x} = M$ & $\frac{\partial u}{\partial y} = N$.

We then say that the DE $\circledast$ is in <u>exact form.</u>

So the DE is $\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = 0$ i.e. $du = 0$, which means soln is

$u(x,y) = c$

Or simply $u(x,y) = 0$ if the arbitrary constant $+C$ is zero.

Q: Given a DE $\circledast$, how do you know if such a $u(x,y)$ can be found?

A: We need $M_y = N_x$ to

Test of Exactness: $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact iff

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \text{ which shows} \ldots$$

How come??

Example:

Solve $\underbrace{2x\sin(3y)}_{M(x,y)}\,dx + \underbrace{(3x^2\cos(3y) + 2y)}_{N(x,y)}\,dy = 0$

Fuck trying to make that separable. Let's test exactness.

$$\frac{\partial M}{\partial y} = 6x\cos(3y) \; ; \; \frac{\partial N}{\partial x} = 6x\cos(3y).$$

$\because \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ $\therefore$ We have an exact ODE.

We want $u(x,y)$ such that:

$$\frac{\partial u}{\partial x} = (2x\sin 3y)\,dx \qquad \frac{\partial u}{\partial y} = (3x^2\cos(3y) + 2y)\,dy$$

$$\int \frac{\partial u}{\partial x}\,dx = \int 2x\sin(3y)\,dx \qquad \int \frac{\partial u}{\partial y} = \int (3x^2\cos(3y) + 2y)\,dy$$

$u(x,y) = x^2\sin(3y) + g(y) + c$; $u(x,y) =$

$$u(x,y) = x^2\sin(3y) + y^2 + c$$

But $du = 0 \Rightarrow \int du = c$

$$\cancel{c} = x^2\sin(3y) + y^2 + \cancel{c}$$

$$x^2\sin(3y) + y^2 = 0$$

Another e.g. $\frac{2y}{x}dy - \frac{y^2}{x^2}dx = 0$

Hyperbolic Functions:

$\sinh(x) \equiv \frac{e^x - e^{-x}}{2}$; $\cosh(x) = \frac{e^x + e^{-x}}{2}$; $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$

$$\sinh^2(x) - \cosh^2(x) = ?$$

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \left(\frac{e^{2x} + 2e^{-x}e^{x} + e^{-2x}}{4}\right) - \left(\frac{e^{2x} - 2e^{x}e^{-x} + e^{-2x}}{4}\right)$$

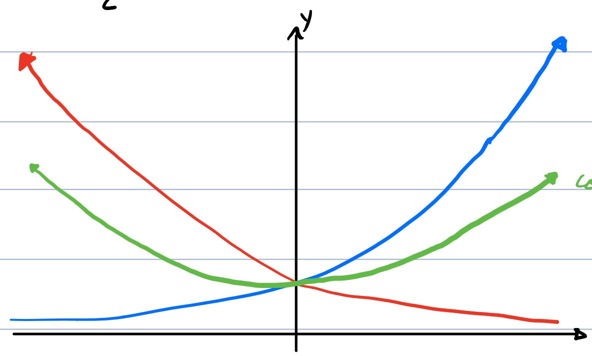
$$= \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}}}{4} + \frac{-\cancel{e^{2x}} + 2 - \cancel{e^{-2x}}}{4}$$

$$= \frac{1}{2} + \frac{1}{2} = 1. \quad \therefore \sinh(x) - \cosh(x) = 1$$

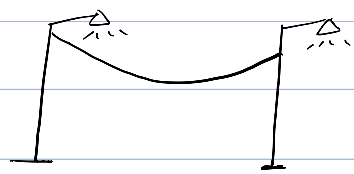
$\frac{d}{dx}[\sinh(x)] = \cosh(x)$; $\frac{d}{dx}\left[\cosh(x)\right] = \sinh(x)$

$\frac{A+B}{2}$ is the average of A and B

$\cosh(x) = \frac{e^x + e^{-x}}{2}$ is the average of e^x and e^{-x}.

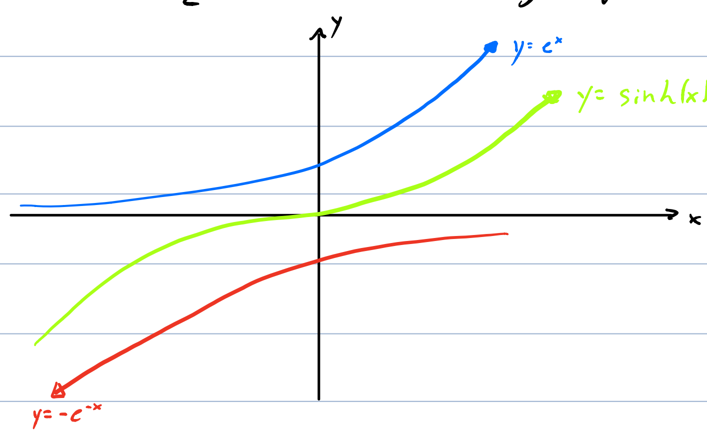


Example:



Wire hanging between lamposts (or just w/ 2 fixed ends) has shape of $\cosh(x)$

$y = \sinh(x) = \frac{e^x - e^{-x}}{2}$ is the average of e^x and e^{-x}.



Hyperbolic functions "share" a lot of identities and calculus properties with the regular trig. fcns.

So far, we've seen that:

1) $\cosh^2(x) \overset{\text{minus!}}{-} \sin^2 h(x) = 1$

2) $\frac{d}{dx}[\sinh(x)] = \cosh(x)$

3) $\frac{d}{dx}[\cosh(x)] = \sinh(x)$

But there are many others:

$\cosh(x+y) = \cosh(x)\cosh(y) \overset{\text{plus!}}{+} \sinh(x)\sinh(y)$

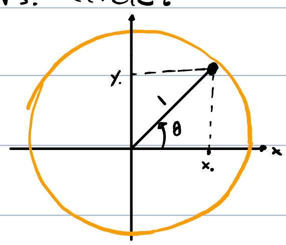
$\frac{d}{dx}[\tanh(x)] = \operatorname{sech}^2(x)$

Hyperbolic Fcns are linked to hyperbolas but are not hyperbolas themselves!

Regular trigonometric fcns are sometimes called "circular fcns"

Trigonometric identities are sometimes called "circular identities"

Trig vs. Circle:

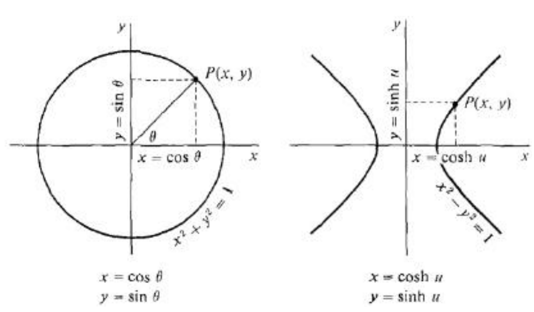


$x.^2 + y.^2 = 1$

$\cos^2 x + \sin^2 x = 1$

Now with cosh(x) and sinh(x)...

If a fcn has $x = \cosh(t)$; $y = \sinh(t)$. $x^2 - y^2 = 1$, so $\cosh^2 x - \sinh^2 x = 1$ and $(\cosh(t), \sinh(t))$ must lie on a hyperbola.



Solve the initial value problem:

$$\sin x \cosh y - y' \cos x \sinh y = 0$$

$$\sin x \cosh y - \frac{dy}{dx} \cos x \sinh y = 0 \qquad \text{multiply through by } dx$$

$$\underbrace{\sin x \cosh(y)}_{M} dx \underbrace{- \cos x \sinh y}_{N} dy = 0$$

Lets test exactness:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[\sin x \cosh y] = \sin x \sinh(y) \qquad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[-\cos x \sinh(y)] = \sin x \sinh(y).$$

Exact ODE!

$$u_x = \sin x \cosh y \qquad u_y = -\cos x \sinh y$$

$$u(x,y) = \int \sin x \cosh(y)\, dx + C \qquad u(x,y) = \int -\cos x \sinh(y)\, dy + C$$

$$u(x,y) = -\cos x \cosh(y) + h(y) + C \qquad u(x,y) = -\cos x \cosh(y) + g(x) + C$$

$$g(x) = 0 \,;\, h(y) = 0$$

$$u(x,y) = -\cos x \cosh(y) + C$$

But $u(x,y) = 0 \Rightarrow 0 = -\cos x \cosh y + C$

I.C. $y(0) = 0$

$$0 = -\cos(0) \cosh(0) + C$$

$$\Rightarrow C = 1$$

$\therefore\ 1 = \cos x \cosh y$ is soln of DE

2F) INTEGRATING FACTORS (Bringing into exact form)

⊛ $\sqrt{y}\,dx - \frac{x}{\sqrt{y}}\,dy = 0$ is not exact. $\frac{\partial}{\partial y}\sqrt{y} = \frac{1}{2\sqrt{y}}$; $\frac{\partial}{\partial x}\frac{x}{\sqrt{y}} = \frac{1}{\sqrt{y}}$

$\sqrt{y} \neq \frac{1}{\sqrt{y}}$! $\therefore$ Not exact

However, if we multiply the DE ⊛ $F(x,y) = \frac{\sqrt{y}}{x^2}$, we obtain:

$$\frac{y}{x^2}\,dx - \frac{1}{x}\,dy = 0$$

Now test exactness: $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[\frac{y}{x^2}\right] = \frac{1}{x^2}$; $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[-\frac{1}{x}\right] = \frac{1}{x^2}$ $\boxed{\checkmark}$

We now have an exact ODE! We can solve accordingly.

Given the ODE of form $P(x,y)\,dx + Q(x,y)\,dy = 0$, we may not have an exact form $\because P_y \neq Q_x$. But we will try to find $F(x,y)$ such that $\overbrace{FP(x,y)}^{M}\,dx + \overbrace{FQ(x,y)}^{N}\,dy = 0$ becomes exact, i.e. $M_y = N_x$. We will do a few more examples of guessing to build intuition and understanding. Then, we will figure out one or two formulas that may sometimes work to find integrating factors.

Example: $x^2y^3\,dx + x(1+y^2)\,dy = 0$ is not exact

$\frac{\partial}{\partial y} L_o = 3x^2y^2$ | $\frac{\partial}{\partial x} L_o = 1+y^2$

Lets try to find $F(x,y)$ so that

$$F x^2y^3\,dx + F\,x(1+y^2)\,dy = 0.$$

We want $M_y = \frac{\partial}{\partial y}\left[F x^2 y^3\right] = N_x = \frac{\partial}{\partial x}\left[F x(1+y^2)\right]$

We can try to make the ys go away in $\frac{\partial}{\partial y}[\ldots]$ and the xs go away in the $\frac{\partial}{\partial x}[\ldots]$, that way we would be taking derivatives of constants in both cases. $F = \frac{1}{xy^3}$ works for this purpose

$$M_y = \frac{\partial}{\partial y}\left[\frac{1}{\cancel{x}\cancel{y^3}} x^{\cancel{2}} \cancel{y^3}\right] = \frac{\partial}{\partial y}[x] = 0$$

$$N_x = \frac{\partial}{\partial x}\left[\frac{1}{\cancel{x}y^3} \cancel{x}(1+y^2)\right] = \frac{\partial}{\partial x}\left[\frac{1+y^2}{y^3}\right] = 0 \quad ☑$$

$N_x = M_y \quad \therefore$ exact ODE.

Lets solve:

$$\frac{1}{xy^3} x^2 y^3\, dx + \frac{1}{xy^3} x(1+y^2)\, dy = 0 \cdot \frac{1}{xy^3}$$

$$x\, dx + (y^{-3} + y^{-1})\, dy = 0$$

$u(x,y) = \int x\, dx + C \qquad u(x,y) = \int y^{-3} + y^{-1}\, dy + C$

$u(x,y) = \frac{x^2}{2} + h(y) + C \qquad u(x,y) = \frac{y^{-2}}{-2} + \ln|y| + g(x) + C$

$$h(y) = \frac{y^{-2}}{-2} + \ln|y| \ ; \ g(x) = \frac{x^2}{2}$$

$\therefore u(x,y) = \frac{x^2}{2} - \frac{1}{2y^2} + \ln|y| + C.$

Put $u=0$ and multiply through by 2.

$C = x^2 - y^{-2} + \ln(y^2)$ is an implicit soln for the DE.

Example: Solve $y + (2x - ye^y)\frac{dy}{dx} = 0.$

Hmmm... Not separable, not exact. Integrating factor?

We want $yF\, dx + F(2x - ye^y)\, dy = 0,$

so we need $\frac{\partial}{\partial y}[yF] = \frac{\partial}{\partial x}[2x - ye^y].$

Trick: Lets think of a simpler form than $F(x,y)$. Lets try to find an F that would have no x: $F = F(y)$ only.

$\frac{\partial}{\partial y}\left[y\,F(y)\right] = \frac{\partial}{\partial x}\left[(2x - ye^{y})\,F(y)\right]$ expand product rule,

$F(y) + yF'(y) = 2F(y)$

$yF'(y) = F(y)$

$y\,\frac{dF}{dy} = F(y)$

$\int \frac{dF}{F} = \int \frac{dy}{y} + C$ ... Find F and solve exact DE.

Soln of DE: $xy^2 + e^y(-y^2+2y-2) = C$

Sometimes, integrating factors are hard to find.

$$x + (x+2)\sin y \, F^{?} \, dx + x\cos y \, F^{?} \, dy = 0$$

Very hard to guess, $F = xe^x$ turns out to work, as you could verify.

So we need actual techniques to find integrating factors...

Consider $P(x,y)\,dx + Q(x,y)\,dy = 0$

or simply $P\,dx + Q\,dy = 0$

If it is not exact, i.e. $P_y \neq Q_x$

We need an $F(x,y) \mid \underbrace{FP}_{M}\,dx + \underbrace{FQ}_{N}\,dy = 0$ becomes exact.

$M_x = N_y$: We want $\frac{\partial}{\partial y}[FP] = \frac{\partial}{\partial x}[FQ]$ ⊛

Lets hope to find some F that would depend on x only.

So $F = F(x) \to$ no y. Then the requirement from ⊛ becomes

$$\frac{\partial}{\partial y}[\underbrace{F P}_{\text{constant multiple rule}}] = \frac{\partial}{\partial x}[\underbrace{F Q}_{\text{Product rule}}] \implies F P_y = F_x Q + F Q_x$$

($F = F(x)$, $P = P(x,y)$; $F = F(x)$, $Q = Q(x,y)$)

$$F P_y - F Q_x = F' Q$$

$$\boxed{\frac{P_y - Q_x}{Q} = \frac{F'}{F}}$$

In problem sets, derive a similar formula for $F(y)$.

Careful! $\frac{F'}{F}$ has only xs, so we need $\frac{P_y - Q_x}{Q}$ to be only x.

Then, $\int \frac{P_y - Q_x}{Q}\,dx = \int \frac{dF}{F}$

Get one F, never mind the "C"

Example: Solve $\overbrace{(4x + 3y^2)}^{P}\,dx + \overbrace{2xy}^{Q}\,dy = 0$

Not separable, not exact. $P_y = 6y$; $Q_x = 2y$

Now $\frac{P_y - Q_x}{Q} = \frac{6y - 2y}{2xy} = \frac{4}{2x} = \frac{2}{x}$

$\therefore \frac{F'}{F} = \frac{2}{x} \longrightarrow \int \frac{2}{x}\,dx = \int \frac{dF}{F}$

$2\ln|x| = \ln|F|$ Fuck the C doesn't matter here

$\ln(x^2) = \ln|F|$

$\boxed{F(x) = x^2}$ — This is our integrating factor

$x^2(4x + 3y^2)\,dx + x^2(2xy)\,dy = 0$

$(4x^3 + 3x^2y^2)\,dx + 2x^3y\,dy = 0$ should be exact. Test exactness:

$\frac{\partial}{\partial y}[4x^3 + 3x^2y^2] = 6x^2y$; $\frac{\partial}{\partial x}[2x^3y] = 6x^2y$. It works!

$f_x = 4x^3 + 3x^2y^2\,dx$	$f_y = 2x^3y\,dy$
$f(x,y) = \int 4x^3 + 3x^2y^2\,dx$	$f(x,y) = \int 2x^3y\,dy$
$f(x,y) = x^4 + x^3y^2 + h(y) + C$	$f(x,y) = x^3y^2 + g(x) + C$

$g(x) = x^4$

$f(x,y) = x^3y^2 + x^4 + C$

But $f(x,y) = 0$

$C = x^3y^2 + x^4$ is the solution to the DE

2.6 Linear Differential Equations

(We will deal mostly with the first order case in this section, but we first introduce arbitrary orders)

Defn: A differential equation of order n in the unknown function $y(x)$ is said to be <u>linear</u> provided the fcn in question is a linear fcn (highest power is 1) in y and its derivatives: $y(x), y', y'', \ldots, y^{(n)}$

Note: The function does not need to be linear in x, linearity is determined by power of y.

For example:

1) $x^2 y^{(iv)} + \sin x\, y''' + e^x y' - y + \sqrt{x} = 0$ is a linear DE

2) $y'' + yy' = x$ Degree higher than 1 for y, so NOT linear.

3) The first order DE:

Ⓐ $xyy' + y^2 + 4y = 0$ is not linear,

but we can $\div$ through by y to get Ⓑ

Ⓑ $xy' + y + 4 = 0$

Ⓑ is separable, could also be written in exact form.

We can find the solution to be:

$y = \frac{C}{x} - 4$

Note that $y \equiv 0$ is a singular soln ($\because$ we $\div$ by y to get form Ⓐ to Ⓑ) for the DE in Ⓐ.

Linear DEs of order n can be written as:

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + p_{n-2}(x)\, y^{(n-2)} + \dots + p_1(x) y' + p_0(x) y = r(x).$$

However, when $r(x) \equiv 0$, we say that the linear fcn is <u>homogeneous.</u>

<u>Linear First order Differential equations:</u>

These can be written in the form:

$$y' + p_0(x) y = r(x)$$

In the homogeneous case, $r(x) \equiv 0$, so the eqn is seperable.

If $p(x) \equiv 0$, then we can solve by integrating

So now, lets assume $r(x)$ and $p(x)$ are not identically zero.

$$\frac{dy}{dx} = r(x) + p(x) y$$

$$dy - [r(x) + p(x) y]\, dx = 0$$

Compare to $P\,dx + Q\,dy = 0$

$$\left.\begin{aligned} P_y &= \frac{\partial}{\partial y}[r(x) + p(x) y] = p(x) \\ Q_x &= \frac{\partial}{\partial x}[1] = 0 \end{aligned}\right\}$$

This is not exact, so lets find an integrating factor:

$$\frac{F'}{F} = \frac{P_y - Q_x}{Q}$$

$$\frac{F'}{F} = \frac{p(x) - 0}{1}$$

$$\int \frac{dF}{F} = \int p(x)\, dx + C$$

$\ln|F| = \int p(x)\, dx$ ~~+C~~ No need for C integrating factor

$F = e^{\int p(x)\,dx}$ is a possibility.

Let $h(x) = \int p(x)\, dx$

e^h is an integrating factor where $h' = p$.

$y' + p(x)\, y = r(x)$ becomes $e^h y' + e^h \underbrace{p}_{h'} y = e^h r$

However, this is a $\frac{d}{dx}[e^h \cdot y]$ (product rule)

$\therefore \frac{d}{dx}[e^h y] = e^h r(x)$

$e^h y = \int e^h r(x)\, dx + C$

$y = e^{-h}\left[\int e^h r(x)\, dx + C\right]$ where $h(x) = \int p(x)\, dx$

Back to $\underbrace{xy' + y + 4 = 0}_{\text{separable, exact,...}}$ with general solution $y = \frac{c}{x} - 4$

Here, look at it as linear...

$\div$ through by $x \Rightarrow y' + y\underbrace{\left(\frac{1}{x}\right)}_{p(x)} = \underbrace{\frac{-4}{x}}_{r(x)}$

Soln is: $h(x) = \int p(x)\, dx = \int \frac{dx}{x} = \ln|x| + C$ ← no need for +C

$\therefore y(x) = e^{-h(x)} \int e^{h(x)} r(x)\, dx = e^{-\ln|x|} \int e^{\ln|x|}\left(-\frac{4}{x}\right) dx + C$

$= \frac{1}{|x|} \int |x|\left(-\frac{4}{x}\right) dx + C$ → x both ⊕ or both ⊖ ⇒ can get rid of absolute values

$= \frac{1}{x} \int x \cdot -\frac{4}{x}\, dx + C$

$= \frac{1}{x} \int -4\, dx + C$

$= \frac{1}{x}\left[(-4x) + c\right]$

$y(x) = \frac{c}{x} - 4$ ✓

Review:

A Linear ODE of order 'n' can be written as:

degree of derivative → $y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \dots + P_1(x)\, y' + P_0(x) = r(x)$

*If $r(x) = 0 \Rightarrow$ "homogeneous" ODE

First order case:

$$y' P(x) + P_0(x) = r(x)$$

Is solved by $y(x) = e^{-h(x)}\left[\int e^{h(x)} r(x)\, dx + C\right]$ where $h(x) = \int P(x)\, dx$

Ex: Solve $y' + y\tan x = \sin 2x$ on $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Initial condition: $y(0) = 1$.

Linear DE! Form $y' + y\,P(x) = r(x)$; $h(x) = \int P(x)\,dx$

Here, $h(x) = \int \tan x\, dx \Rightarrow h(x) = -\ln|\cos x|$

Soln: $e^{-h}\left[\int e^{-h}\, r(x)\,dx + C\right]$

$e^{+\ln|\cos x|} \int e^{-\ln|\cos x|} \sin(2x)\, dx + C$

$\cos x \left[\int \frac{2\sin x \cancel{\cos x}}{\cancel{\cos x}}\, dx + C\right]$

$\cos x\left[-2\cos x + C\right]$

$y(x) = -2\cos^2 x + C\cos x$

I.C. $y(0) = 1$

$1 = -2\cos^2(0) + C\cos(0)$

$1 = -2 + C$

$C = 3$

$$\therefore y(x) = -2\cos^2 x + 3\cos x$$

Reduction to linear form:

Substitution may sometimes be found to reduce a 1^{st} order ODE to linear form (in $u(x)$ and its derivatives).

Sometimes, we may guess substitutions by inspection.

A standard case of substitution reducing to linear form is the following:

ODE's of the form $y' + p(x)y = r(x)y^a$ $(a \neq \{0,1\} \Rightarrow$ non linear ODE$)$

are called Bernoulli equations.

It turns out that the substitution $u = y^{1-a}$ brings Bernoulli equations into linear form in $u(x)$.

Example: $y' = x^3y^2 + xy$

$$\Rightarrow y' + y\underbrace{(-x)}_{p(x)} = \underbrace{x^3}_{r(x)}\underbrace{y^2}_{y^a} \quad \text{(Bernoulli equation)} \quad a=2$$

Let $u = y^{1-2} \Rightarrow u = \frac{1}{y}$

$u' = -\frac{1}{y^2} \cdot y' \Rightarrow u' = -u^2 y'$

$\Rightarrow y' = -\frac{u'}{u^2}$

$$-\frac{u'}{u^2} - \frac{x}{u} = \frac{x^3}{u^2}$$

$$u' + \overbrace{x}^{p(x)}u = \overbrace{-x^3}^{r(x)}$$ linear ODE!

$$h(x) = \int p(x)\,dx = \int x\,dx$$

$$h(x) = \frac{x^2}{2}$$

Soln to DE: $u = e^{-h}\left[\int e^{h} r(x)\,dx + C\right]$

$$u = e^{-\frac{x^2}{2}}\left[\int e^{x^2/2} \cdot x^3\,dx + C\right] \qquad t = \frac{x^2}{2}, \quad dt = x\,dx$$

$$u = e^{-x^2/2}\left[-\int e^{t}\, x^{\cancel{3}\,2}\, \frac{dt}{\cancel{x}} + C\right]$$

$$u = e^{-\frac{x^2}{2}}\left[-\int e^{t} \cdot 2t\,dt + c\right]$$

$$u = e^{-\frac{x^2}{2}}\left[c - 2\underbrace{\int te^{t}\,dt}_{I}\right]$$

$$I = \int t e^t \, dt \qquad u = t \quad du = dt \qquad v = e^t \quad dv = e^t dt$$

$$= te^t - \int e^t dt$$

$$= te^t - e^t$$

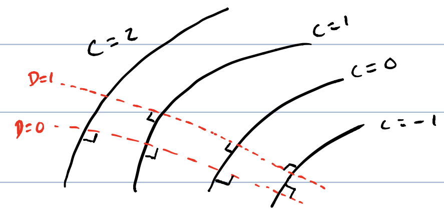
$$\therefore u = e^{-\frac{x^2}{2}} \left[C - 2 \left(\frac{x^2}{2} e^{\frac{x^2}{2}} - e^{\frac{x^2}{2}} \right) \right]$$

$$u = Ce^{-\frac{x^2}{2}} - x^2 + 2$$

$$\frac{1}{y(x)} = Ce^{-\frac{x^2}{2}} - x^2 + 2$$

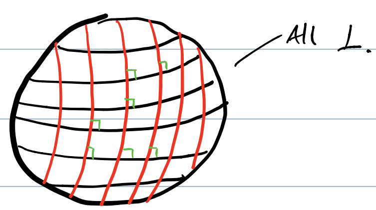
$$y(x) = \frac{1}{Ce^{-\frac{x^2}{2}} - x^2 + 2}$$

2H) Application To Orthogonal Trajectories:



All C's are $\perp$ to all D's.

E.g. On earth, the longitude geodesics are $\perp$ to latitude's geodesics.



E.g. In a topographical map (level curve plot), the trajectory is a curve that is $\perp$ to all the equipotential lines (or level curves)

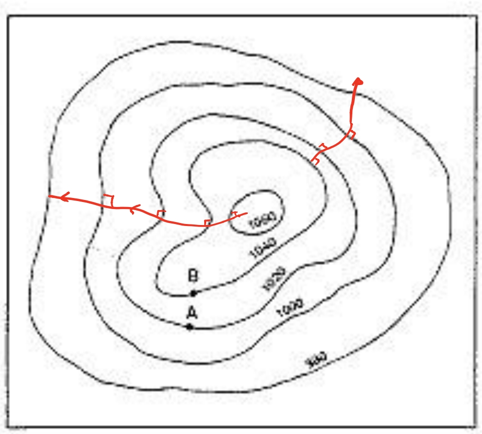
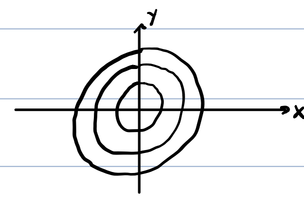


Figure E-1: Isolated Hill

Recall that first order DEs have one parameter families of curves as their soln. What we want to do is go the other way...
Given a family of curves, find the DE for which this family is the soln.

For example: $x^2+y^2=c^2$



Given this family of curves, we can find the desired DE by implicit diff'n of the fcn

$$y' = -\frac{x}{y}$$ is the DE solved by the circle...

Aside: $\frac{dy}{dx} = -\frac{x}{y}$

$$\int y\,dy = -\int x\,dx + C \quad \rightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$x^2+y^2=C \quad \boxed{\checkmark}$$

General strategy:

1. Given a family of curves
2. Find a DE for which this family of curves is the general soln
3. y' of first family $\longmapsto -\frac{1}{y'}$ y' of second family
4. Find the family of orthogonal trajectories

E.g. $x^2+y^2=c^2 \longrightarrow y' = \frac{-x}{y} \Rightarrow -\frac{1}{y'} = \frac{y}{x} \overset{(*)}{\Rightarrow} y'_{\text{second fam}} = \frac{y}{x}$

Solve (*): $\int \frac{dy}{y} = \int \frac{dx}{x} + C$

$$\ln|y| = \ln|x| + C$$

$$y = Cx$$ — all straight lines through the origin!

Example: Consider the family of parabolas $y = Cx^2$. Find its orthogonal trajectories.

Given $y = Cx^2 \longrightarrow C = \frac{y}{x^2}$

$y' = 2Cx$ Get rid of the C!

sub $C = \frac{y}{x^2}$

$y' = 2\left(\frac{y}{x^2}\right)x$

$y' = 2\frac{y}{x}$

• $\perp$: $y' = -\frac{x}{2y}$. To find orthogonal family:

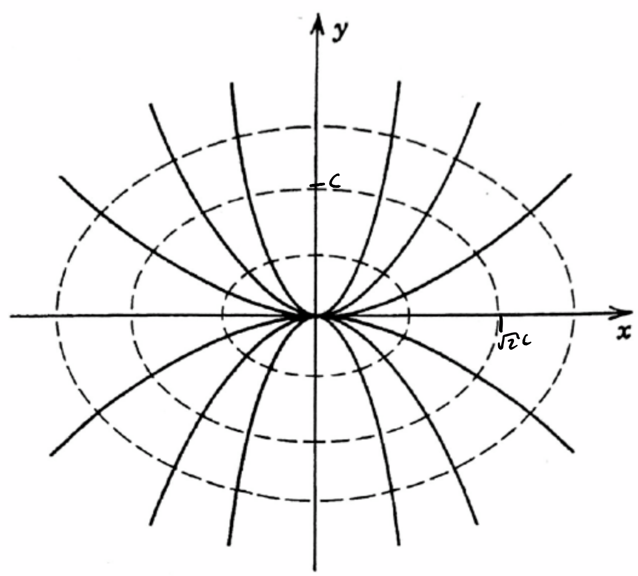
Solve $\frac{dy}{dx} = \frac{-x}{2y}$

$\int 2y\,dy = -\int x\,dx + C$

$y^2 = \frac{-x^2}{2} + C = \frac{-x^2}{2} + C^2$

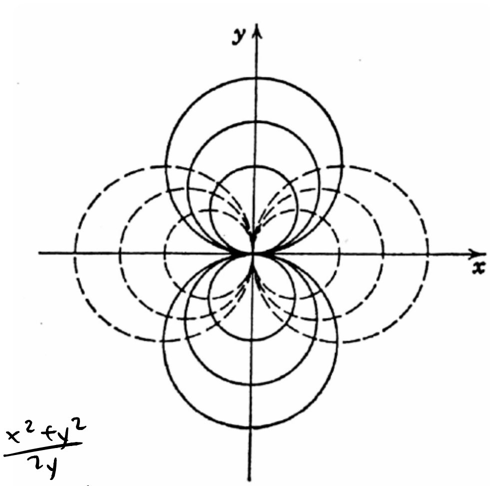
$\frac{x^2}{(\sqrt{2}\,C)^2} + \frac{y^2}{C^2} = 1$

Sketch:



E.g. Find the orthogonal trajectories of the circles

(*) $x^2+(y-c)^2=c^2$



(1)

$$2x + 2\underbrace{(y-c)}_{\text{eliminate this piece}}y' = 0$$

Go back to original family

$$x^2+(y-c)^2=c^2$$

$$x^2+(y^2-2cy+\cancel{c^2})=\cancel{c^2}$$

$$x^2-2cy+y^2=0$$

$$x^2+y^2=2cy$$

$$c=\frac{x^2+y^2}{2y}$$

$$y-c = y-\frac{x^2+y^2}{2y}$$

$$y-c=\frac{2y^2}{2y}-\frac{y^2+x^2}{2y}$$

$$\boxed{y-c=\frac{y^2-x^2}{2y}}$$

$$\therefore\ x+\left(\frac{y^2-x^2}{2y}\right)y'=0$$

$$y'=\frac{-x\cdot 2y}{y^2-x^2}$$

$$\boxed{y'=\frac{2xy}{x^2-y^2}}$$

Is the DE solved by the family of curves (*) (circles)

(2) Orthogonal trajectories:

$$-\frac{1}{y'}=\frac{y^2-x^2}{2xy}$$

y' of new family.

Thus for orth. traj: $y'=\dfrac{y^2-x^2}{2xy}$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{y}{x}-\frac{x}{y}\right) \qquad \text{Let } u=\frac{y}{x} \Rightarrow u'=u'x+u$$

$$u'x+u=\frac{1}{2}\left(u-\frac{1}{u}\right)$$

$$2u'x+2u=\frac{u^2-1}{u}$$

$$2u'x=-\frac{2u^2}{u}+\frac{u^2}{u}-\frac{1}{u}$$

$$2u'x=-\frac{u^2+1}{u}$$

$$2x\frac{du}{dx}=-\frac{u^2+1}{u}$$

$$\int\frac{u}{u^2+1}du=\int -2x\,dx$$

$$-x^2+C=\int\frac{u}{u^2+1} \qquad t=u^2+1,\ dt=2t\,dt$$

$$C-x^2=\frac{1}{2}\ln|u^2+1|$$

$$C-x^2=\ln\sqrt{u^2+1}$$

$$Ce^{-x^2}=\sqrt{u^2+1}$$

⋮ Copy.

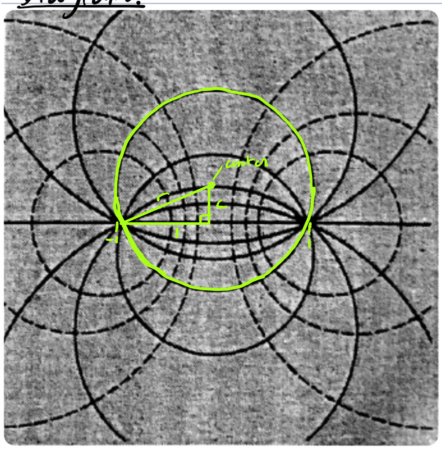
Ans: $y^2+x^2-\underset{?}{2C}x=0$

Ex: Consider 2 opposite charges of the same strength at $P(-1,0)$ and $Q(1,0)$. An experiment suggests that the lines of force can be approximated by circles passing through P and Q.

a) Express the lines of force as a family of curves (with a parameter C)

b) Compute the equipotential lines.

Diagram:



Let center of circle be $(0,c)$. In a given circle (e.g. green circle in diagram), $r^2 = C^2+1$.

Eqn of circles are $x^2+(y-c)^2=r^2$

$\Rightarrow x^2+(y-C)^2 = C^2+1$. Now find family of tangent curves.

Imp. diffn $\frac{d}{dx}[x^2+(y-C)^2] = \frac{d}{dx}[C^2+1]$

$$2x+2(y-c)\cdot y' = 0$$

$$x + y'\underbrace{(y-c)}_{\text{get rid of this!}} = 0$$

$$x^2+(y-C)^2 = C^2+1$$

$$x^2+y^2-2Cy+\cancel{C^2} = \cancel{C^2}+1$$

$$2Cy = x^2+y^2-1$$

$$C = \frac{x^2+y^2-1}{2y}$$

$$\Rightarrow y-C = y - \frac{x^2+y^2-1}{2y}$$

$$y-C = \frac{2y^2}{2y} + \frac{1-y^2-x^2}{2y}$$

$$\boxed{y-C = \frac{1+y^2-x^2}{2y}}$$

$$x+y'\left(\frac{y^2-x^2+1}{2y}\right) = 0$$

$$y' = \frac{-x\cdot 2y}{y^2-x^2+1}$$

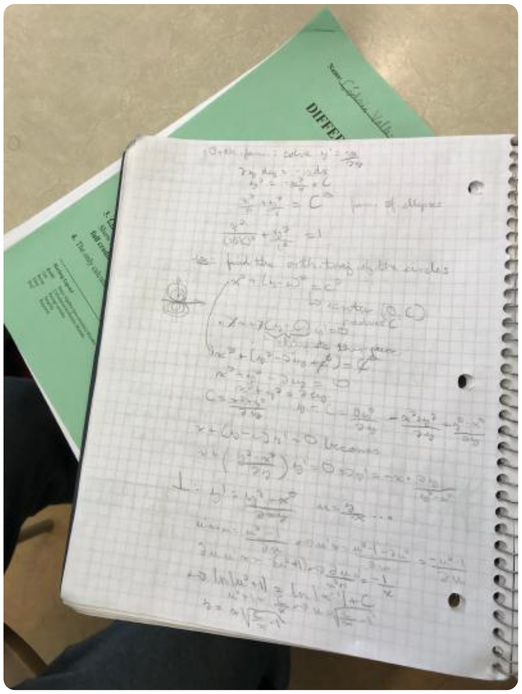
$$y' = \frac{2xy}{x^2-y^2-1}$$

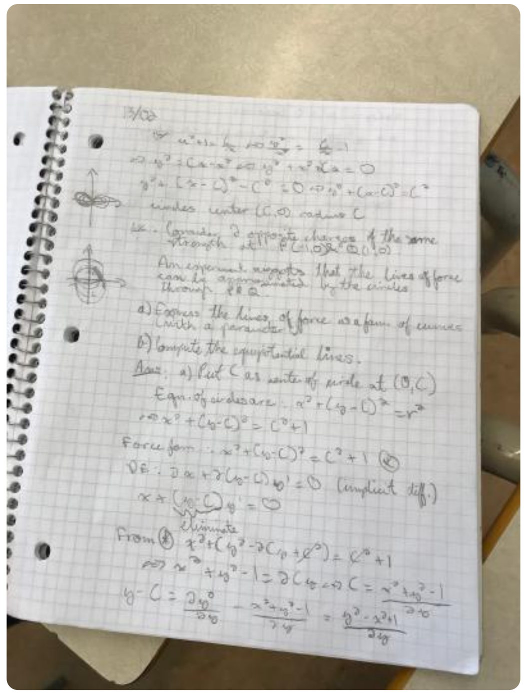
$$-\frac{1}{y'} = \frac{y^2-x^2+1}{2xy}$$

y' of orthog. family

$$y_\perp' = \frac{1}{2}\left[\frac{y}{x} - \frac{x}{y} + \frac{1}{xy}\right]$$

$u =$

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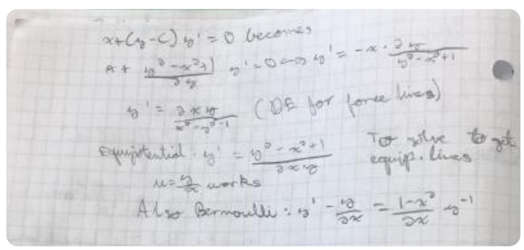
circles center $(C,0)$ radius C
Ex: Consider 2 opposite charges of the same strength at $P(-1,0)$ & $Q(1,0)$
An experiment suggests that the lines of force can be approximated by the circles through P & Q
a) Express the lines of force as a fam. of curves (with a parameter)
b) Compute the equipotential lines.
Ans: a) Put C as center of circle at $(0,C)$
Eqn. of circles are: $x^2+(y-C)^2=r^2$
$\Leftrightarrow x^2+(y-C)^2=C^2+1$
Force fam: $x^2+(y-C)^2=C^2+1$ (*)
DE: $2x+2(y-C)y'=0$ (implicit diff.)
$x+(y-C)y'=0$
eliminate
From (*) $x^2+(y^2-2Cy+\cancel{C^2})=\cancel{C^2}+1$
$\Leftrightarrow x^2+y^2-1=2Cy \Leftrightarrow C=\frac{x^2+y^2-1}{2y}$
$y-C=\frac{2y^2}{2y}-\frac{x^2+y^2-1}{2y}=\frac{y^2-x^2+1}{2y}$


$x+(y-C)y'=0$ becomes
$x+\frac{y^2-x^2+1}{2y}\,y'=0 \Leftrightarrow y'=-x\cdot\frac{2y}{y^2-x^2+1}$
$y'=\frac{2xy}{x^2-y^2-1}$ (DE for force lines)
Equipotential: $y'=\frac{y^2-x^2+1}{2xy}$
To solve to get equip. lines
$\mu=\frac{y}{x}$ works
Also Bernoulli: $y'-\frac{y}{2x}=\frac{1-x^2}{2x}y^{-1}$

On above problem, Found lines of force: $x^2+(y-C)^2=C^2+1$ which solves the DE: $y'=\frac{2xy}{x^2-y^2-1}$ $\therefore$ Equipotential lines satisfy DE.

For orth. trajectory: $y'=\frac{y^2-x^2+1}{2xy}$ ← solve to get family of $\perp$ curves

$$y'-\frac{y}{2x}=\frac{1-x^2}{2x}\cdot\frac{1}{y} \Rightarrow$$

Bernoulli: $a=-1$

$$y'+y\left(-\frac{1}{2x}\right)=\left(\frac{1-x^2}{2x}\right)\cdot y^{-1}$$

Let $u=y^{1-a} \Rightarrow u=y^{1-(-1)}$

$u=y^2 \Rightarrow y=\sqrt{u}$

$u'=2y\cdot y'$

$\Rightarrow \frac{u'}{2y}=y' \Rightarrow y'=\frac{u'}{2\sqrt{u}}$

$$\frac{u'}{2\sqrt{u}}-\frac{\sqrt{u}}{2x}=\left(\frac{1-x^2}{2x}\right)\sqrt{u}$$

$$u'-\frac{u}{x}=\frac{1-x^2}{x}$$

$$u'+u\overbrace{\left(-\frac{1}{x}\right)}^{p(x)}=\overbrace{\frac{1-x^2}{x}}^{r(x)}$$

$$h(x)=\int p(x)\,dx=\int -\frac{dx}{x}$$

$$h(x)=-\ln|x|$$

Soln: $u=e^{-h}\left[\int e^{h}\,r(x)\,dx+C\right]$

$$u=Ce^{\ln|x|}+e^{\ln|x|}\int e^{\ln|\frac{1}{x}|}\cdot\frac{1-x^2}{x}\,dx$$

$$u=Cx+x\int\frac{1}{x}\cdot\frac{1-x^2}{x}\,dx$$

$$u=Cx+x\int\left(\frac{1}{x^2}-1\right)dx$$

$$u=Cx+x\left[-\frac{1}{x}-x\right]$$

$$y^2=Cx-1-x^2$$

$$y^2+\underbrace{x^2-\overset{C\text{ can be }2C}{2Cx}+\overset{\text{clever }1}{C^2}}_{\text{perfect square}}-C^2=-1$$

$$(x-C)^2+y^2=C^2-1$$

Note: $r^2=C^2-1 \quad (r<C)$

$$\boxed{r=\sqrt{C^2-1}}$$

2I) A First Attempt At Higher Order: Reduction To First Order!

Consider a 2^{nd} order ODE in the unknown fcn $y(x)$ that involves y'', y', and x but <u>not</u> y, that is a DE of the form $F(y'', y', x) = 0$.

The substitution $z = y'$ turns it into a first order DE in the unknown fcn $z(x)$.

E.g. Solve $3xy'' = 2y'$

Let $z = y'$

$$3xz' = 2z$$

$$3x\frac{dz}{dx} = 2z$$ separable ODE

$$\int \frac{dz}{z} = \frac{2}{3}\int \frac{dx}{x} + C$$

no need for absolute value brackets $\because x^{2k}$

$$\ln|z| = \ln|x^{2/3}| + C$$

$$z = e^{\ln x^{2/3} + C}$$

$$z = Cx^{2/3}$$

sub $z = y'$

$$y' = Cx^{2/3}$$

$$y = C\int x^{2/3}\,dx$$

$$y = \underbrace{C\frac{x^{5/3}}{5/3}}_{\text{absorbed}} + D \Rightarrow y = Cx^{5/3} + D$$

Now consider a DE in the unknown fcn $y(x)$ that involves y'', y', y but <u>not</u> x (explicitly). That is a DE of the form $F(y, y', y'') = 0$. Then $z = y'$ also works, but in a trickier way.

Think of $z = z(y)$ instead of $z = z(x)$ (to avoid x ...)

$y' = \frac{d}{dx}[y] = z$ ↳ really $z(y(x))$

$y'' = \frac{d}{dx}[y'(x)] = \frac{d}{dx}\left[z(y(x))\right] = z'(y(x)) \cdot \underbrace{y'(x)}_{z} \cdot \overbrace{x'}^{=1}$

$* \Rightarrow y'' = z' \cdot z$

$*$ Equivalent to doing $\frac{d}{dy}[z(y)] = z' \cdot z$ ← same result

So $F(y, y', y'') = 0$

$\quad\quad\ \ \uparrow\ \ \uparrow$

$\quad\quad\ \ z\ \ z' \cdot z$ gives a DE involving y, z, z' for $z(y)$.

E.g. Solve $yy'' + (y')^2 = 0$ by reduction to first order

Let $z = y'$, $y'' = z' \cdot z$

$y z' \cdot z + z^2 = 0$

$y \cdot \cancel{z} \cdot z' = -z^{\cancel{2}}$

$y \frac{dz}{dy} = -z$

$\int \frac{dz}{z} = -\int \frac{dy}{y} + C$

$\ln|z| = \ln\left|\frac{1}{y}\right| + C$

$z = e^{\ln\left|\frac{1}{y}\right| + C}$

$\boxed{z = \frac{C}{y}}$

Finding $y(x)$ ... sub $z = y'$

$y' = \frac{C}{y}$

$\frac{dy}{dx} = \frac{C}{y}$

$\int y\,dy = C\int dx + D$

$\frac{y^2}{2} = Cx + D$

$y = \pm\sqrt{Cx + D}$