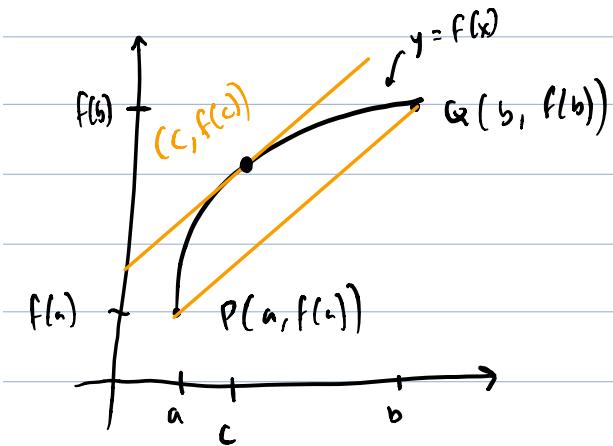


Mean Value Theorem



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

November 9, 2016

- ① If $y = f(x)$ is cont' on $[a, b]$
- ② and if $y = f(x)$ is diff'ble on (a, b)

Then there is a number 'c' in (a, b) so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

e.g. $f(x) = x^3 - 3x + 2$ on $[-2, 2]$

Find all values of 'c' that satisfy the cond of the MVT.

Since $f(x)$ is a poly f^n , it is:
 - cont' on $[-2, 2]$
 - diff'ble on $(-2, 2)$

So $y = f(x)$ satisfies conditions of MVT

By MVT, there is at least one value 'c' in $(-2, 2)$, so that

$$f'(c) = \frac{f(2) - f(-2)}{2 - (-2)} = \frac{4 - 0}{4} = 1$$

But $f'(x) = 3x^2 - 3$

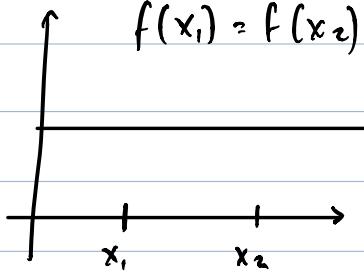
$$1 = 3c^2 - 3$$

$$c = \pm \sqrt{\frac{4}{3}} \quad \text{both lie in } (-2, 2)$$

Theorem: If $f'(x) = 0$ for all x in (a, b)

then $f(x) = C$ for some constant C (true for all x in (a, b))

Proof:



Must show that $f(x_1) = f(x_2)$ for any choice of x_1, x_2 in (a, b)

Assume $f'(x) = 0$ for all x in (a, b)

Recall: If $y = f(x)$ is diff^{ble} at a number, it is cont' at that number.

So since $y = f(x)$ is diff^{ble} on (a, b) , $y = f(x)$ must also be cont' on (a, b)

Choose any number $x_1 & x_2$ inside (a, b)

So (a) $y = f(x)$ is cont^s on $[x_1, x_2]$

and (b) $y = f(x)$ is diff^{b/c} on (x_1, x_2)

So by MVT, there exists a value 'c' in (x_1, x_2) so that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

But $f'(c) = 0$ by assumption.

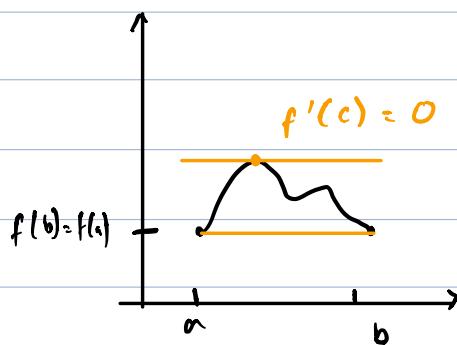
$$\text{So } 0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$0 = f(x_2) - f(x_1)$$

$$f(x_1) = f(x_2)$$

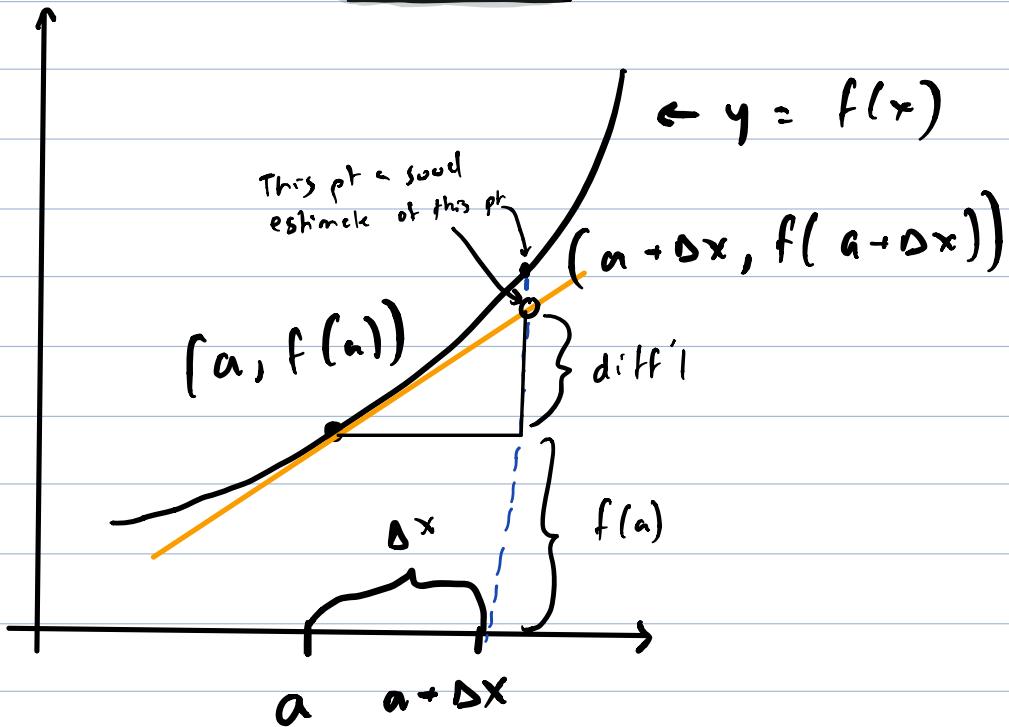
Rolle's Theorem

[Special Case of MVT]



If (a) $y = f(x)$ is cont's on $[a, b]$
 & if (b) $y = f(x)$ is diff'ble on (a, b)
 & if (c) $f(a) = f(c)$
 Then there is a value 'c' in (a, b) so that $f'(c) = 0$

Differential



For small Δx ,

$$f(a + \Delta x) \approx f(a) + \text{diff'l}$$

To compute diff'l

Slope of tangent = $f'(a)$

$$" = \frac{\text{rise}}{\text{run}} = \frac{d(f)}{dx}$$

i.e. $f'(a) = \frac{d(f)}{dx}$

Ex. Estimate $\sqrt{36.1}$ using a diff'l

$$\text{Let } f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$$

$$\text{Let } a = 36 \text{ & } \Delta x = 0.1$$

LINEAR APPROXIMATION

$$f(a + \Delta x) \approx f(a) + \frac{f'(a)}{dy} \Delta x \quad \begin{aligned} dy &= f'(a) dx \\ \frac{dy}{dx} &= f'(a) \end{aligned}$$

$$f(36 + 0.1) \approx f(36) + f'(36)(0.1)$$

$$\approx \sqrt{36} + \frac{1}{2\sqrt{36}}(0.1)$$

$$\approx 6 + \frac{1}{120} \approx 6.008333$$

$$\text{vs. Calculator: } = 6.008327$$

November 10, 2016

Ex. Estimate the volume of paint needed to paint the

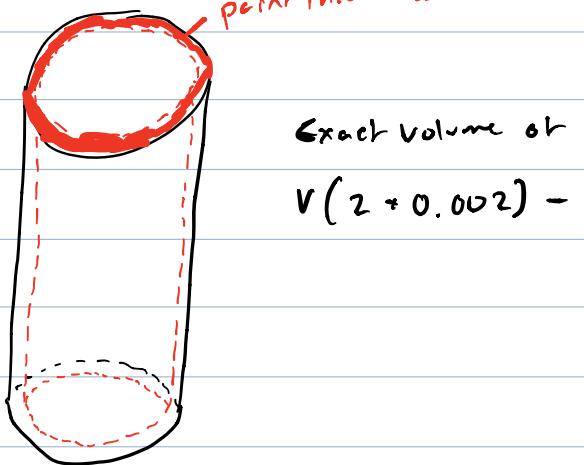
outer curved surface of a cylindrical water tank of height 5 m and radius 2 m if 2 coats are required, each $\frac{1}{10}$ cm thick.

$$= 0.001 \text{ m}$$



$$V(a) = h\pi r^2$$

$$= 5\pi x^2$$



Exact volume of paint needed:

$$V(2 + 0.002) - V(2)$$

$$V(a + \Delta x) \approx V(a) + V'(a) \Delta x$$

$$\text{Let } a = 2 \text{ & } \Delta x = 0.002$$

$$V(2.002) \approx V(2) + V'(2)(0.002)$$

$$V(2.002) - V(2) \approx V'(2)(0.002)$$

$$\approx 10\pi(2)(0.002)$$

$$\approx 0.04\pi \text{ m}^3$$

Factorial Notation

$$5! = 5 \times 4 \times 3 \times 2 \times 1$$

$$2! = 2 \times 1$$

$$1! = 1$$

$$\bullet 0! = 1$$

$$\text{Ex: } \frac{10!}{8!} = \frac{10 \times 9 \times 8!}{8!} = 90$$

Taylor Polynomials

Given a f^m $y = f(x)$ that has 4 deriv's,
 the 4^{th} degree Taylor Polynomial $T_4(x)$ or $y = f(x)$
 centered at ' a ' is by defn:

$$\begin{aligned} T_4(x) &= f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 \\ &\quad + \frac{f^{(4)}(a)}{4!} (x-a)^4 \end{aligned}$$

Assume $y = f(x)$ has ' n ' derivatives ($n = 1, 2, 3, 4, \dots$),

then the n^{th} degree Taylor Polynomial $T_n(x)$ centered at $x=a$
 is defined by

$$T_4(x) = f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!} (x-a)^n$$

ex. Find $T_4(x)$ centered at $x=1$ for $f(x) = \ln(x)$

$$\begin{aligned} T_4(x) &= f(1) + \frac{f'(1)}{1!} (x-1) + \frac{f''(1)}{2!} (x-1)^2 + \frac{f'''(1)}{3!} (x-1)^3 \\ &\quad + \frac{f^{(4)}(1)}{4!} (x-1)^4 \end{aligned}$$

$$f(x) = \ln x \rightarrow f(1) = \ln(1) = 0$$

$$f'(x) = \frac{1}{x} x^{-1} \rightarrow f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \rightarrow f''(1) = -1$$

$$f'''(x) = 2x^{-3} = \frac{2}{x^3} \rightarrow f'''(1) = 2$$

$$f^{(4)} = -6x^{-4} = -\frac{6}{x^4} \rightarrow f^{(4)}(1) = -6$$

S_0

$$\begin{aligned} T_4(x) &= 0 + \frac{1}{1!} (x-1) - \frac{1}{2!} (x-1)^2 + \frac{2}{3!} (x-1)^3 - \frac{6}{4!} (x-1)^4 \\ &= 0 + (x-1) - \frac{1}{2} (x-1)^2 + \frac{1}{3} (x-1)^3 - \frac{1}{4} (x-1)^4 \end{aligned}$$

use above to estimate $\ln(1.1)$

$$\begin{aligned} T_4(1.1) &= (1.1-1) - \frac{1}{2} (1.1-1)^2 \dots \\ &= 0.04530833 \end{aligned}$$

calculator = 0.04531

ex. Find $T_3(x)$ centered at $x=0$ for $f(x)=\sin x$

N.B. a Taylor poly centered

$$T_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \quad \text{at } x=0 \text{ is called a MacLaurin poly}$$

$$f(x) = \sin x \rightarrow f(0) = 0$$

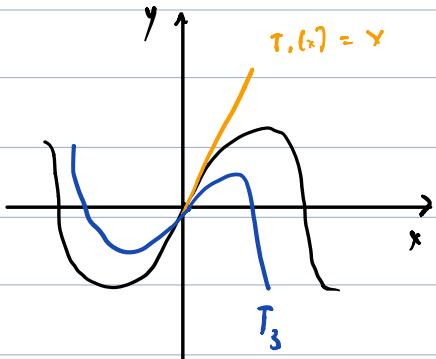
$$f'(x) = \cos x \rightarrow f'(0) = \cos(0) = 1$$

$$f''(x) = -\sin x \rightarrow f''(0) = -\sin(0) = 0$$

$$f'''(x) = -\cos x \rightarrow f'''(0) = -\cos(0) = -1$$

$$T_3(x) = x - \frac{1}{6}x^3$$

$$T_3(0.01) \approx \sin(0.01)$$



Connection between Taylor & Linear Approx November 11, 2016

$$T_1(x) = f(a) + \frac{f'(a)}{1!}(x-a)$$

$$= f(a) + f'(a)(x-a)$$

Recall Linear Approx: $f(a + \Delta x) \approx f(a) + f'(a)\Delta x$

Let $x = a + \Delta x$ so $\Delta x = x - a$

$$f(x) \approx f(a) + f'(a)(x-a) = T_1(x)$$