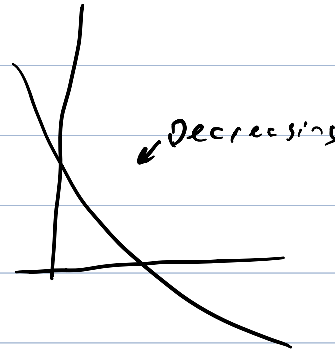
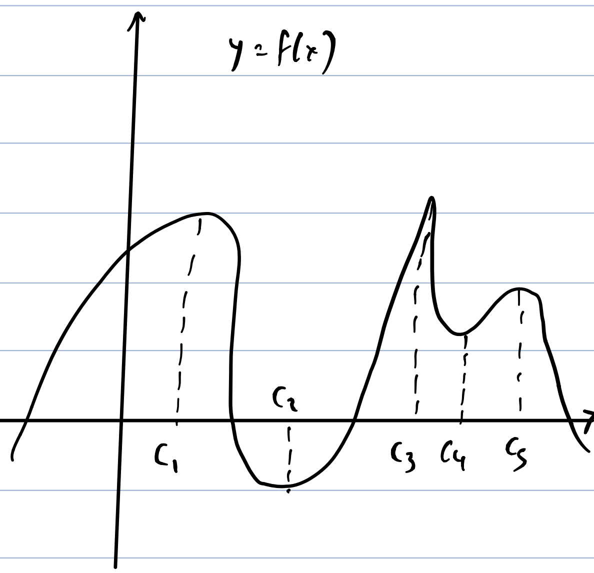
INCREASING & DECREASING F^{ns}



ex.

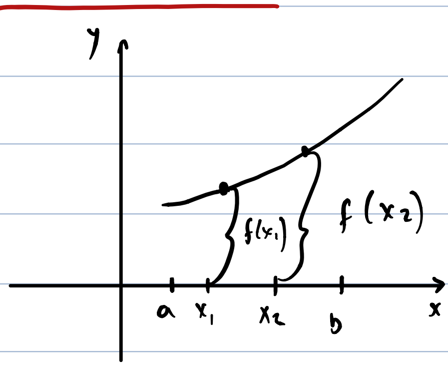


$y = f(x)$ incr over $(-\infty, c)$, (c_2, c_3) & (c_4, c_5)

decr over (c_1, c_2), (c_3, c_4) & $(c_5, +\infty)$

Doesn't matter if you use [brackets instead

Mathematical Def^n



$y = f(x)$ is inc in (a, b) when:

$$f(x_1) \leq f(x_2)$$

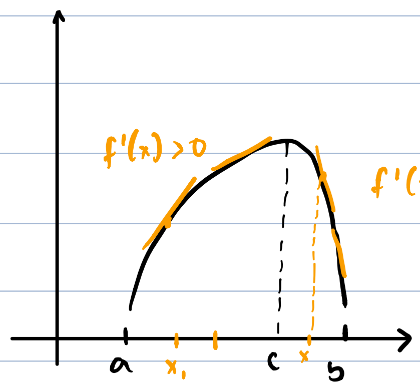
when $x_1 < x_2$ in (a, b)

$y = f(x)$ is dec in (a, b) when:

$$f(x_1) \geq f(x_2)$$

when $x_1 < x_2$ in (a, b)

Predicting where a f^{cn} $y=f(x)$ is incr/decr



① $y=f(x)$ is incr over (a,c)

$f'(x)>0$ for all x in (a,c)

② $y=f(x)$ is decr over (c,b)

$f'(x)<0$ for all x in (c,b)

Test for Incr/Decr

Suppose $y=f(x)$ is diff^{ble} over an interval I

① If $f'(x)<0$ for all x in I then $y=f(x)$ is decr over I

② If $f'(x)>0$ " incr "

ex. Find where $f(x)=x^3+3x^2-9x+4$ is incr/decr

$$f'(x)=3x^2+6x-9$$
$$=3(x^2+2x-3)$$
$$=3(x-1)(x+3)$$

Split pts:

$3(x-1)(x+3)=0$

$x=-3,\ x=1$

	$-\infty$	-3		1	$+\infty$
$f'(x)=3(x-1)(x+3)$	+	0	−	0	+
$f(x)=x^3+3x^2-9x+4$	↗		↘		↗

So $y=f(x)$ is incr over $(-\infty,-3)$, $(1,+\infty)$

decr over $(-3,1)$

ex. Find where $f(x) = 6x^{\frac{2}{3}} - 4x$ is incr/decr

$f'(x) = 4x^{-\frac{1}{3}} - 4$ could express as a single fraction

$= 4\left(\dfrac{1 - \sqrt[3]{x}}{\sqrt[3]{x}}\right)$ 0 at $x=1$, und at $x=0$

Split pts are where $f'(x) = 0$ or where $f'(x)$ UND

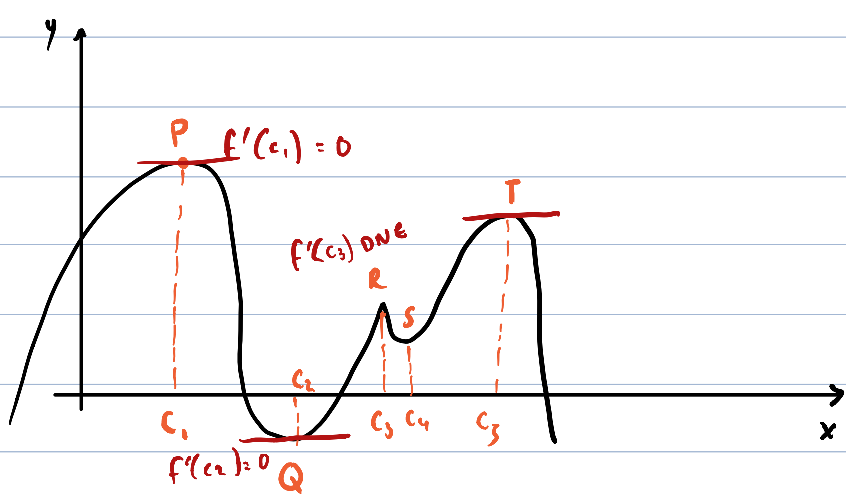
Split pts: 0, 1

	$-\infty$ to 0	0	0 to 1	1	1 to $+\infty$
$f'(x) = 4\left(\dfrac{1-\sqrt[3]{x}}{\sqrt[3]{x}}\right)$	$-$	UND (sharp pt)	$+$	0	$-$
$f(x)$	↘		↗		↘

So $f(x)$ is decr over $(-\infty, 0) \cup (1, +\infty)$

incr over $(0, 1)$

LOCAL (RELATIVE) MAX & MIN



pts P, R, T are called local max pts

pts Q, S " local min pts

pt P is an absolute max pt

No absolute min pt

observe that local max & local min occur at number 'c' in Domain f

where $f'(c) = 0$ or $f'(c)$ DNE Called critical values. (CV^s)

Implied that f(c) is defn need to check if f(c) is defn

Fact: Local max/min can only occur at critical value pts

ex. $f(x) = 2x^3 - 3x^2 - 72x + 15$

Find all CV^s

$f'(x) = 6x^2 - 6x - 72$

$f'(x) = 6(x^2 - 6 - 12)$

$6(x^2 - 6 - 12) = 0$

$6(x+3)(x-4) = 0$

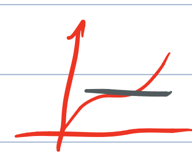
So CV^s at $x = -3, 4$

i.e. $f'(-3) = 0$

$f'(4) = 0$

So LOC EXTREMA can only occur at $x = -3, 4$ (if they occur at all)
(min & max)

ex.



not local max or " min

ex. $f(x) = (x-1)^{2/3}$

$f'(x) = \frac{2}{3}(x-1)^{-\frac{1}{3}} \cdot (1)$

$= \dfrac{2}{3\sqrt[3]{x-1}}$ → $f'(1)$ DNE and $f(1)$ is def^n!

So $x=1$ is a CV

ex. $f(x) = \dfrac{1}{x-1} = (x-1)^{-1}$

$f'(x) = -(x-1)^{-2}$

$= -\dfrac{1}{(x-1)^2}$ → $f'(1)$ DNE but $f(1)$ is not defined.

So $x=1$ is not a CV.

No local extrema

TESTING CV^s FOR LOCAL MAX/MIN

First Derivative Test

$f(x) = 2x^3 - 3x^2 - 72x + 15$

$f'(x) = 6(x+3)(x-4)$

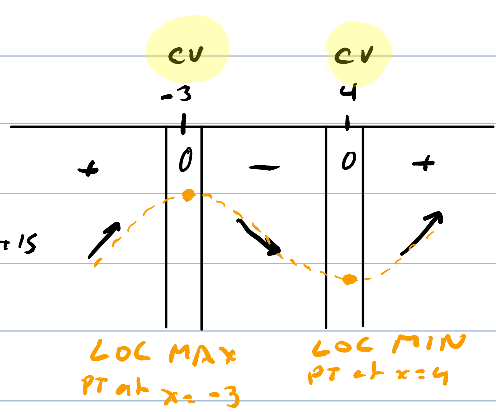
$\Rightarrow$ CV^s at $x = -3$ & $x = 4$

Split pts: -3 & 4

N.B Not all split pts for $f'(x)$ are CV^s

$f'(x) = 6(x+3)(x-4)$

$f(x) = 2x^3 - 3x^2 - 72x + 15$



First Derivative Test: Assume 'c' is a critical value for $y = f(x)$ and $y = f(x)$ is cont at $x = $ 'c'

1) If $y = f(x)$ changes from inc to decr at $x = c$ then $y = f(x)$ has a local max at $x = c$

2) If $y = f(x)$ changes from decr to inc " " local min "

3) If there is no change from incr to decr at $x = c$, then there is no local max/min at $x = c$.

ex. Split pts

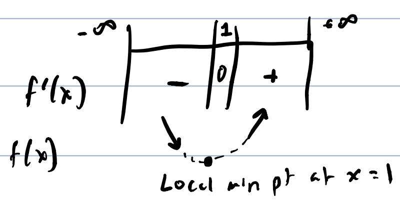
$\nearrow \; \nearrow \; \nearrow$ $\therefore$ No local max/min

ex. $f(x) = (x-1)^{\frac{2}{3}}$ Find all local max/min pts.

$$f'(x) = \frac{2}{3}(x-1)^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x-1}}$$

$f'(1)$ DNE and $f(1) = (1-1)^{\frac{2}{3}} = 0^{\frac{2}{3}} = 0$ defined ✓

So $x = 1$ is a CV for $f(x) = (x-1)^{2/3}$



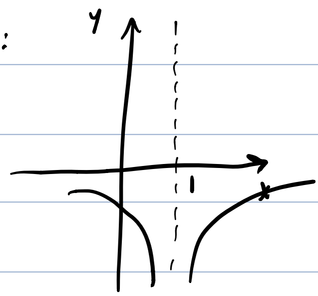
ex. $f(x) = -\dfrac{1}{(x-1)^2}$ Find all loc max/min pts

$f'(x) = \dfrac{2}{(x-1)^3}$

	$-\infty$	1	$+\infty$
$f'(x)$	$-$		$+$
$f(x)$	↘		↗

x=1 is not a C.V. !!
f'(1) is defined, but f(1) UND
Also, no CVs anywhere, and thus
no loc max/min pts

Graph:



ex. $f(x) = (x+2)^3$ Find all loc max/min (if any)

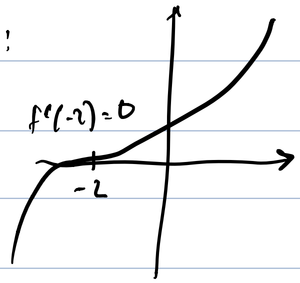
$f'(x) = 3(x+2)^2$

$f'(-2) = 0 \Rightarrow$ CV at $x = -2$ (since $f'(C.V.) = 0$, if it were und, you'd have to check dom of $f(x)$)

		-2	
$f'(x)$	$+$	0	$+$
$f(x)$	↗		↗

No local max/min at $x = -2$
So no local " anywhere

Graph:



ex. $f(x) = e^{2x} - e^x$ Find all local max/min pts

$f'(x) = 2e^{2x} - e^x = 2e^x \cdot e^x - e^x$

$= e^x(2e^x - 1)$

$e^x = 0$ $2e^x - 1 = 0$

No sol'n $e^x = \frac{1}{2}$

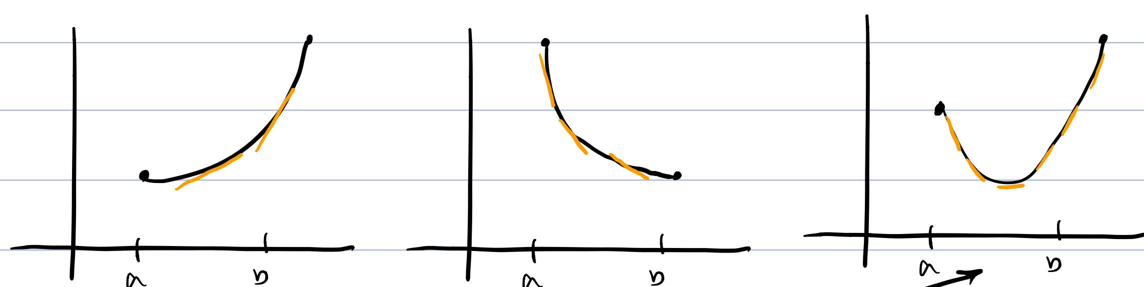
or $x = \ln 0$ DNE $x \ln e = \ln \frac{1}{2}$

$x = \ln \frac{1}{2}$ ← CV

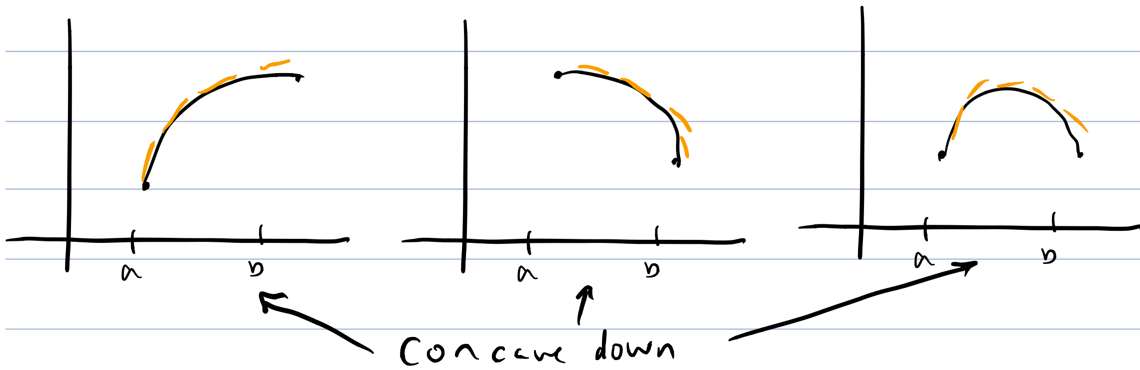
	$-\infty$	$\ln \frac{1}{2}$ ← CV	$+\infty$
$f'(x) = e^x(2e^x-1)$ $= e^{-10}(2e^{-10}-1)$ >0, <0	$-$	0	$+$
$f(x)$	↘	•	↗

local min at $x = \ln \frac{1}{2}$

CONCAVITY

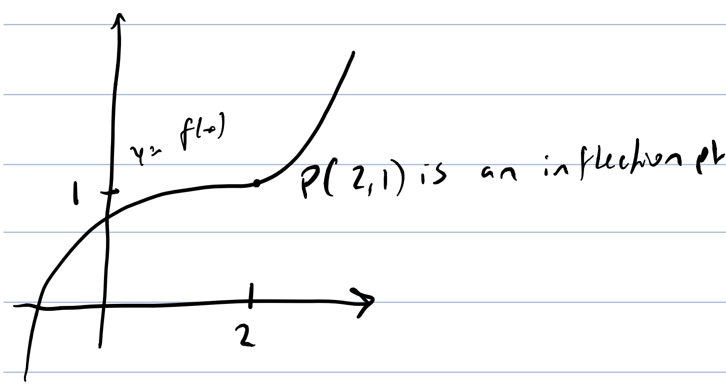


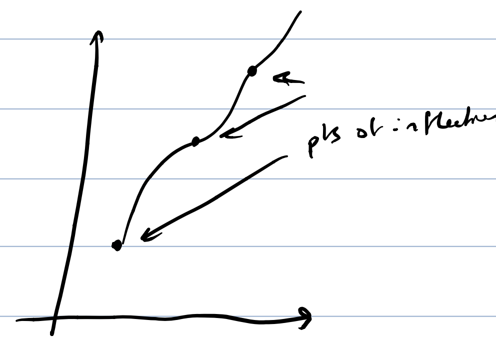
Concave up



Concavity ≠ increasing/decreasing

Where concavity changes = inflection pt (IP's)





N.B.

$y = f(x)$ is conc Down

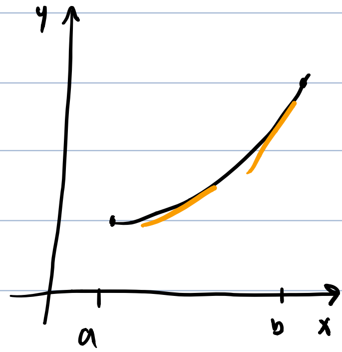
over $(-\infty, 2)$

conc up

over $(2, +\infty)$

$f''(x) > 0$?

November 17, 2016



$y = f(x)$ is conc ↑ over (a,b)

N.B.

$f'(x_1) < f'(x_2)$ when $x_1 < x_2$ in (a,b)

Let $h(x) = f'(x)$

$h(x_1) < h(x_2)$ —— " ——

So h is an incr f^{cn} over (a,b)

So $h'(x) > 0$ for all x in (a,b)

So $f''(x) > 0$ —— " ——

TEST FOR CONC ↑/↓

① If $f''(x) < 0$ for all x in an interval I

then $y = f(x)$ is conc ↓ over I

② If $f''(x) > 0$ for all x in an interval I

then $y = f(x)$ is conc ↑ over I

ex. Find where $f(x) = x^3 - 3x^2 + 5x - 4$ is conc ↑/↓.

$f'(x) = 3x^2 - 6x + 5$

$f''(x) = 6x - 6 = 6(x-1)$

Split pts: ← Note '1' is not a C.V. (C.V. → $f'(0)$)

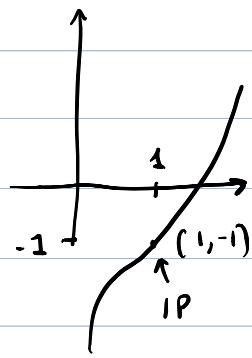
	$-\infty$		1		$+\infty$
$f''(x)$		−	0	+	
$f(x)$		conc ↓		conc ↑	

So $x=1$ is an inflection pt

Concl: $f(x)$ is conc ↓ in $(-\infty, 1)$

conc ↑ in $(1, +\infty)$

Observe graph:



ex. Find where $f(x) = x^4 - 18x^2 + 5$ is conc ↑/↓

$$f'(x) = 4x^3 - 36x$$

$$f''(x) = 12x^2 - 36 = 12(x^2 - 3) = 12(x - \sqrt{3})(x + \sqrt{3})$$

↓

Reminder, can also do:

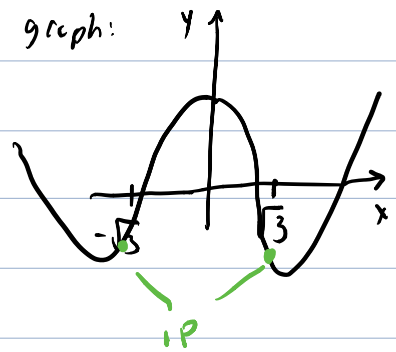
$$12x^2 - 36 = 0$$

	$-\infty$	$-\sqrt{3}$		$+\sqrt{3}$	$+\infty$
$f''(x)$	$+$		$-$		$+$
$f(x)$	conc ↑		conc ↓		conc ↑
		IP		IP	

So $y = f(x)$ is conc↑ over $(-\infty, -\sqrt{3})$ & $(\sqrt{3}, +\infty)$

conc↓ over $(-\sqrt{3}, +\sqrt{3})$

Observe graph:



GRAPH SKETCHING

ex. Sketch graph of $y = f(x) = x^3 - 6x^2 + 9x - 4$

① DOM $= \mathbb{R}$

② VA - None

③ HA - None

④ y-int : $x=0$, $y=-4$

⑤ x-int: Use Rational Root Theorem, but for us, not absolutely viable

not required "Cannot be easily obtained"

⑥ Split pts for $f'(x)$

$f'(x) = 3x^2 - 12x + 9$

$= 3(x^2 - 4x + 3)$

$= 3(x-1)(x-3)$

$\Rightarrow$ Split pts for $f'(x)$ are $1, 3$

Split pts for $f''(x)$

$f''(x) = 6x - 12$

$= 6(x-2)$

$\Rightarrow$ Split pt for $f''(x)$ is 2

	$-\infty$	CV 1		IP 2		CV 3	$+\infty$
$f'(x) = 3(x-1)(x-3)$	+	0	−		−	0	+
$f''(x) = 6(x-2)$	−		−		+		+
$f(x) = x^3 - 6x^2 + 9x - 4$		loc max at $x=1$				loc min at $x=3$	

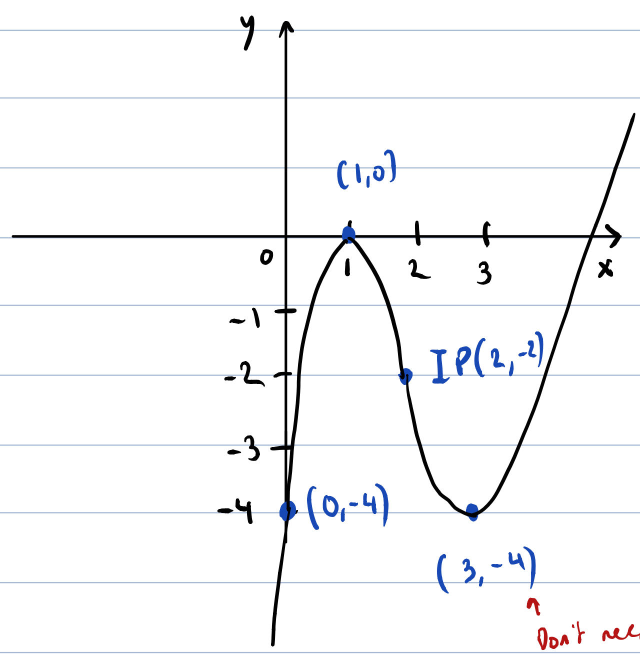
Need to write

i.e. conc ↓ & inc.

⑦

x	$y = f(x) = x^3 - 6x^2 + 9x - 4$	
0	−4	y-INT
1	0 ←	LOC MAX
2	−2 ← IP	Sub in to find y
3	−4 ←	LOC MIN

⑧



Label your pts

Don't need to write loc min/max

ex. $f(x) = \dfrac{2x^2}{x^2-1} = \dfrac{2x^2}{(x-1)(x+1)}$

① DOM $\mathbb{R} \setminus \{-1, 1\}$

② VA at $x = -1, 1$ (Don't need the $\lim\limits_{x \to 1^-} f(x)$ analysis)

③ HA $\lim\limits_{x \to \pm\infty} \dfrac{2x^2}{x^2-1} = \lim\limits_{x \to \pm\infty} \dfrac{\cancel{x^2}(2)}{\cancel{x^2}(1-\frac{1}{x^2})} = 2$

HA $y=2$ in +ve and -ve directions

④ ⑤ $\Big\}$ $x=0, y=0$ Interesting...

⑥ $f'(x) = \dfrac{-4x}{(x^2-1)^2}$ => Split pts: $-1, 0, 1$

UND UND

$f''(x) = \dfrac{12x^2+4}{(x^2-1)^3}$ => Split pts: $-1, 1$

will never = 0

$\dfrac{12x^2+4}{\cancel{(x^2-1)^3}} = 0$

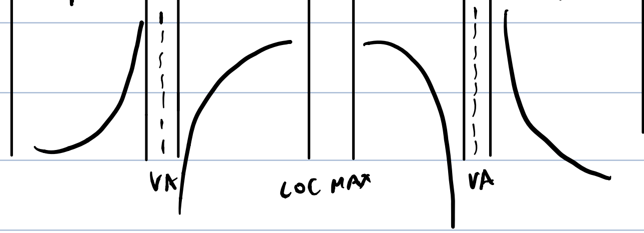
$x^2 = \dfrac{-4}{12}$ UND

$$f'(x) = \frac{-4x}{(x^2-1)^2}$$

$$f''(x) = \frac{12x^2+4}{(x^2-1)^3}$$

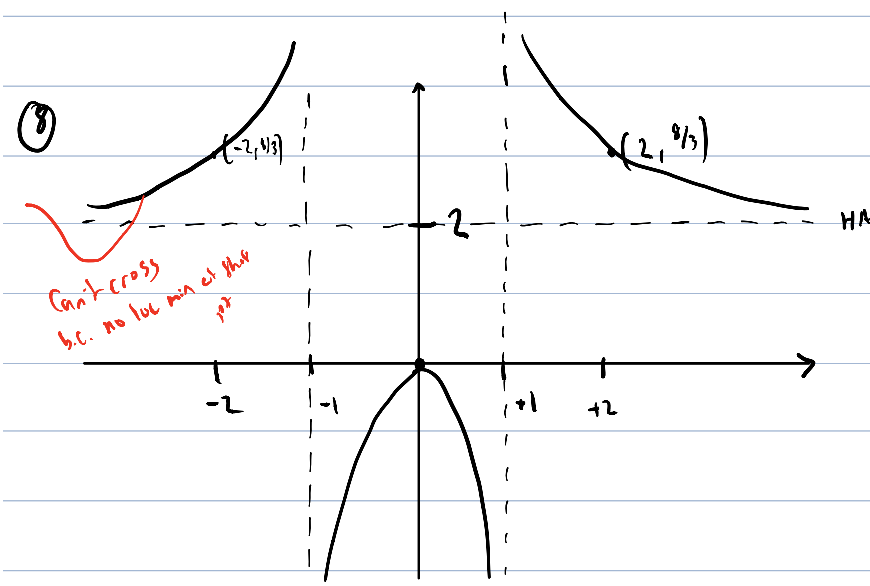
	$-\infty$ to -1 (VA)	-1 to 0 (CV)	0 to $+1$ (VA)	$+1$ to $+\infty$
$f'(x)$	+	+	−	−
$f''(x)$	+	−	−	+

$f(x)$



⑦ x	$y = f(x) = \frac{2x^2}{x^2-1}$	
-2	$8/3$	← extrm pts
-1	UND	
0	0	
$+1$	UND	
$+2$	$8/3$	← extrm pts

⑧



ex.

Sketch: $e^{2x} - e^{x}$

① DOM = $\mathbb{R}$

② VA - None

factoring trick

③ HA: $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (e^{2x} - e^{x}) = \lim_{x\to\infty} e^{x}(e^{x} - 1) = \infty \cdot \infty = \infty$

$\therefore$ No HA in +ve direction

$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} e^{-\infty} - e^{-\infty} = 0 - 0 = 0$ $\therefore y = 0$ is a HA in -ve direction

④ y-int: $x = 0$, $y = e^{2(0)} - e^{0} = 0$

⑤ x-int: $y = 0 \rightarrow 0 = e^{2x} - e^{x} = e^{x}(e^{x} - 1)$

$e^{x} = 0$ No soln

$e^{x} - 1 = 0$

$x = 0$ ✓

So $x = 0, y = 0$ is only x-int

⑥ $f'(x) = 2e^{2x} - e^{x}$

$= e^{x}(2e^{x} - 1)$

Split pts: $0 = e^{x}(2e^{x} - 1)$

$2e^{x} - 1 = 0$

$e^{x} = \frac{1}{2}$

$x \cancel{\ln e}^{1} = \ln \frac{1}{2}$

So $x = \ln \frac{1}{2}$ is split pt (also CV)

$f''(x) = 4e^{2x} - e^{x} = e^{x}(4e^{x} - 1)$

<u>Split pts:</u> $0 = e^{x}(4e^{x} - 1)$

$4e^{x} - 1 = 0$

$e^{x} = \frac{1}{4}$

$x\cancel{\ln e}^{1} = \ln\frac{1}{4}$

So $x = \ln\frac{1}{4}$ is split pt

	$-\infty$	$\ln\frac{1}{4}$		$\ln\frac{1}{2}$	$+\infty$
$f'(x) = e^{x}(2e^{x} - 1)$	$-$		$-$ ①	0	$+$
$f''(x) = e^{x}(4e^{x} - 1)$	$-$	0	$+$		$+$
$f(x)$					
		$\uparrow$ IP		$\uparrow$ LOC MIN	

① $\ln\frac{3}{8}$ ← test

$e^{\ln\frac{3}{8}}(2e^{\ln 3/8} - 1)$

$\frac{3}{8}\left(2\left(\frac{3}{8}\right) - 1\right)$

$+ \cdot - \rightarrow -$

⑦

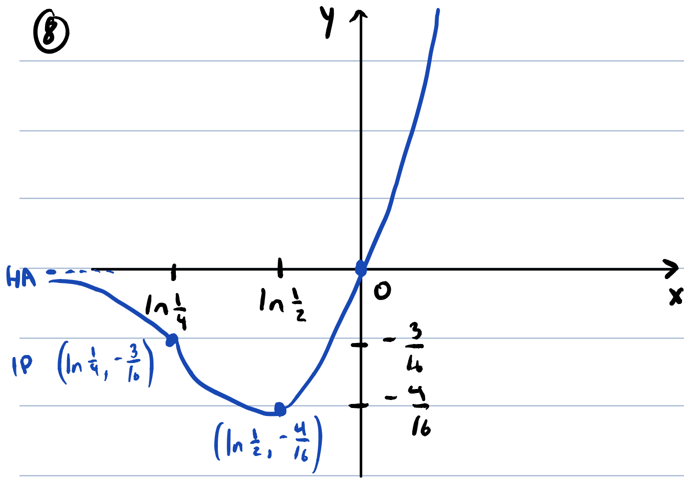
x	$y = f(x) = e^{x}(e^{x} - 1)$
$\ln\frac{1}{4}$	② $-\frac{3}{16}$
$\ln\frac{1}{2}$	$-\frac{1}{4} = -\frac{4}{16}$
0	0

② $y = e^{\ln\frac{1}{4}}\left(e^{\ln\frac{1}{4}} - 1\right)$

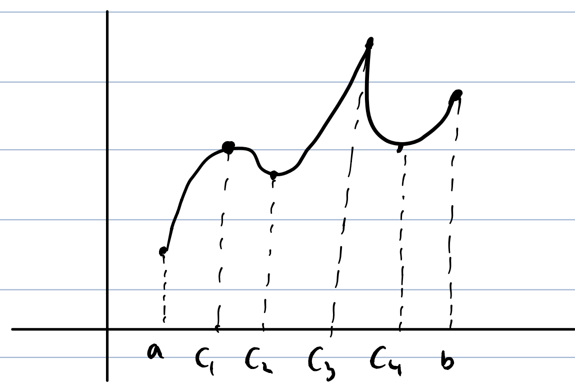
$= \frac{1}{4}\left(\frac{1}{4} - \frac{4}{4}\right)$ $\qquad$ $\frac{1}{2}\left(\frac{1}{2} - \frac{2}{2}\right)$

$= -\frac{3}{16}$ $\qquad$ $= -\frac{1}{4}$

8



ABSOLUTE MAX/MIN



$y = f(x)$ is cont's on $[a,b]$

Abs min pt at $x = a$

" max pt at $x = c_3$

EXTREME VALUE THEOREM

If $y = f(x)$ is cont's on $[a,b]$
then $y = f(x)$ has a max pt and min pt on $[a,b]$

① **FACT:** Abs max / Abs min in theorem above can only occur at a critical value in (a,b) or at $x = a$, $x = b$

ex. Find the abs max/min pts for $f(x) = (x-1)^4$

for $0 \le x \le 4$

$f(x)$ is cont's on $[0,4]$

$f'(x) = 4(x-1)^3 \cdot 1$

$f'(1) = 0 \leftarrow$ CV at $x=1$

$f(1) = 0 \rightarrow$ abs. min

$f(0) = 1$

$f(4) = 81 \rightarrow$ abs. max pt

ex. Find the abs max/min pts for $f(x) = x^3 - 12x - 3$ on $[-3,3]$

$f'(x) = 3x^2 - 12$

$= 3(x^2 - 4)$

$= 3(x-2)(x+2)$

$f'(2) = 0 \leftarrow$ CV at $x=2$

$f'(-2) = 0 \leftarrow$ CV at $x=-2$

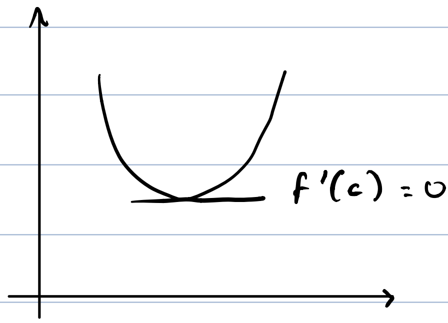
so $f(-3) = 6$

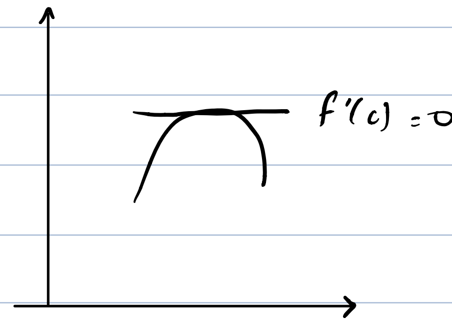
$f(-2) = 13 \leftarrow$ ABS MAX pt at $(-2, 13)$

$f(2) = -19 \leftarrow$ ABS MIN pt at $(2, -19)$

$f(3) = -12$

SECOND DERIV. TEST





$f'(c) = 0$	$f'(c) = 0$
$f(x)$ is conc ↑ at 'c'	$f(x)$ is conc ↓ at 'c'
$f''(c) > 0$	$f''(c) < 0$
So local min	So local max

SECOND DERIV TEST

Suppose 'c' is a CV of type $f'(c) = 0$

If $f''(c) < 0$, then $y = f(x)$ has a local max at $x = c$

If $f''(c) > 0$, — " ———— min — " —

If $f''(c) = 0$, then 2nd Deriv test fails to give any info

So go back to 1st Deriv test

WARNING: If the CV 'c' is of type $f'(c)$ DNE (but $f(c)$ is def'n)

STAY AWAY FROM 2nd DERIV TEST

ex. Find loc max/min pts of $f(x) = x^4 - 8x^2$

$f'(x) = 4x^3 - 16x$

$= 4x(x^2-4)$

$= 4x(x-2)(x+2)$

So CV's at $-2, 0, 2$ $\qquad f''(x) = 12x^2 - 16$

$f'(0) \longrightarrow f''(0) = -16 < 0 \rightarrow$ LOC MAX at $x=0$

$f'(-2) \longrightarrow f''(-2) = 32 > 0 \rightarrow$ LOC MIN at $x=-2$

$f'(2) \longrightarrow f''(2) = 32 > 0 \rightarrow$ LOC MIN at $x=2$

Back to abs max/min ...

ex. Does $f(x) = \dfrac{2}{x-1}$ have an abs max/min on $[0,2]$?

N.B. $f(x)$ is not cont's on $[0,2]$

$\lim_{x \to 1^-} \dfrac{2}{x-1} = \dfrac{2}{0^-} = -\infty \rightarrow$ No abs min pt

$\lim_{x \to 1^+} \dfrac{2}{x-1} = \dfrac{2}{0^+} = +\infty \rightarrow$ No abs max pt

—— OTHER TYPES OF F^{cns} RE: ABS MAX/MIN ——

ex. Find abs min pt of $f(x) = (x-1)^{2/3}$

$f'(x) = \frac{2}{3}(x-1)^{-\frac{1}{3}} \cdot 1$

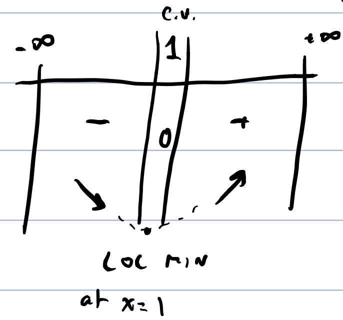
$= \dfrac{2}{3\sqrt[3]{x-1}}$ $\qquad$ N.B. $f'(1)$ DNE and $f(1) = 0$ defn ✓

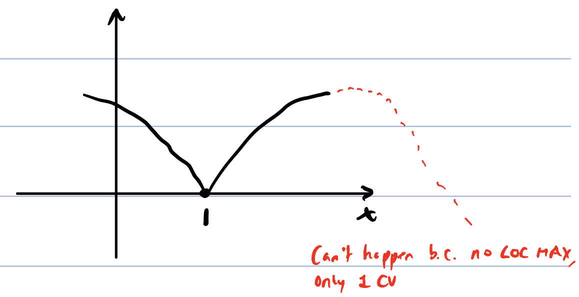
So 1 C.V. at $x=1$

Test using 1^{st} deriv test (2^{nd} doesn't work)

$f'(x) = \frac{2}{3\sqrt[3]{x-1}}$

$f(x)$





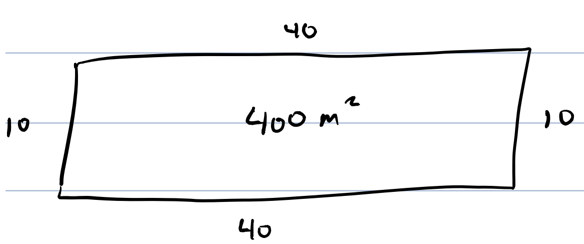
③ **FACT:** Suppose $y=f(x)$ has only one CV at $x='c'$

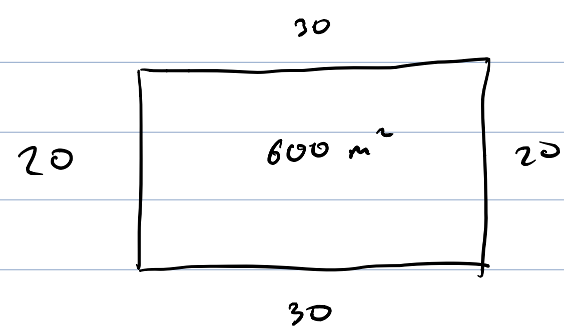
If $y=f(x)$ has a LOC MAX at $x=c$ then $y=f(x)$ has an ABS MAX at $x=c$

If $y=f(x)$ — " — LOC MIN — " ———— ABS MIN — " —

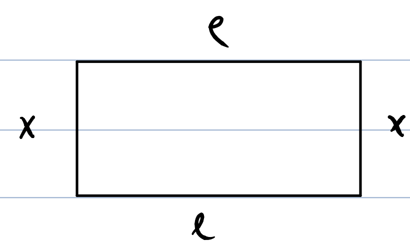
OPTIMIZATION PROBLEMS

ex. Suppose you wish to build a rectangular enclosure using exactly 100 m of fencing. What dimensions would give you the largest area?





Sol'n



Let A = enclosed area   Need to define variables

$A = lx$

But $2x + 2l = 100$

$x + l = 50$

$l = 50 - x$

Restriction

So $A(x) = (50 - x)x \quad (\underbrace{0 \le x \le}_{} 50)$

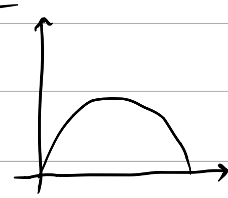
Convenient to include since $50 \times 0 = 0$

So we can use fact 1

$A(x)$ is cont^s in $[0, 50]$

$A(x) = 50x - x^2$

N.D.



$A'(x) = 50 - 2x = 2(25 - x)$

$A'(25) = 0 \Rightarrow$ C.V. at $x = 25$

$A(0) = 0$

$A(25) = 625 \leftarrow$ ABS MAX occurs at $x = 25$

$A(50) = 0$

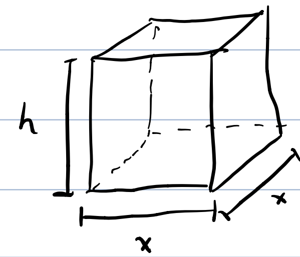
when $x = 25$, $l = 50 - x = 50 - 25 = 25$

Concl: The dim. which gives the largest area are: $25m \times 25m$.

ex. Given an open top box with a square base and volume 32 000 cm^3.

Find the minimum surface area.

Let A = Surface area



$A = 4xh + x^2$

But $x^2 h = 32000$

$h = \frac{32000}{x^2}$

So $A(x) = 4\cancel{x}\left(\frac{32000}{x^{\cancel{2}}}\right) + x^2$

$A(x) = \frac{128\,000}{x} + x^2 \quad (0 < x)$

Can have an infinitely large x, and infinitely small h since $h = \frac{32000}{x^2}$, but $x > 0$, but cannot $= 0$.

$A(x) = 128\,000x^{-1} + x^2$

$A'(x) = -128\,000x^{-2} + 2x$

$= \frac{-128\,000}{x^2} + 2x$

when x=0, it's undefined, but that's not in the DOM!

$0 = \frac{-128\,000}{x^2} + 2x$

$2x = \frac{128\,000}{x^2}$

$x^3 = 64\,000$

$x = 40$ => only one CV

Use 2nd deriv test

$A''(x) = 256\,000x^{-3} + 2$

$$= \frac{256\,000}{x^3} + 2$$

$$A''(40) = \frac{256\,000}{40^3} + 2 > 0$$

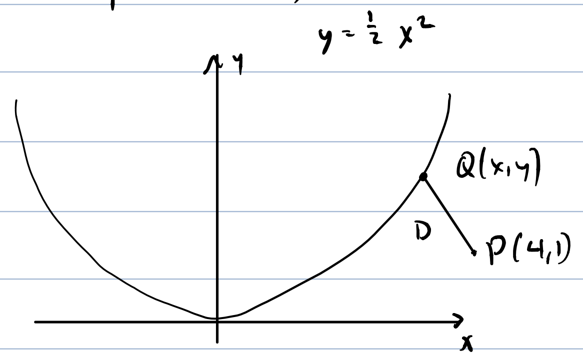
So loc min at $x = 40$

So abs min at $x = 40$

<u>Concl:</u> $A(40) = \frac{128\,000}{40} + 40$

$= 4800 \text{ cm}^2$

<u>ex.</u> Find the pt on the graph of $y = \frac{1}{2}x^2$ that is closest to the pt $P(4,1)$



Let D = distance between $P(4,1)$ & $Q(x,y)$

$$D = \sqrt{(x-4)^2 + (y-1)^2}$$

$$D(x) = \sqrt{(x-4)^2 + \left(\frac{1}{2}x^2 - 1\right)^2} \quad (-\infty < x < +\infty)$$

Let $S = D^2$ Since the value of 'x' which will minimize S will also minimize D

$$S(x) = (x-4)^2 + \left(\frac{1}{2}x^2 - 1\right)^2 \quad (-\infty < x < +\infty)$$

$$S'(x) = 2(x-4) + 2\left(\frac{1}{2}x^2 - 1\right)x$$

$S'(x) = 2x - 8 + x^3 - 2x$

$= x^3 - 8$

So $S'(2) = 0$

$x = 2$ is the only C.V.

First deriv. test

	$-\infty$	C.V. 2	$+\infty$
$S'(x) = x^3 - 8$	$-$	0	$+$
$S(x)$	↘		↗

LOC MIN at $x=2$

So ABS min at $x = 2$

Concl: when $x = 2$, $y = \frac{1}{2}(2)^2 = 2$

So pt $(2,2)$ is closest to $P(4,1)$