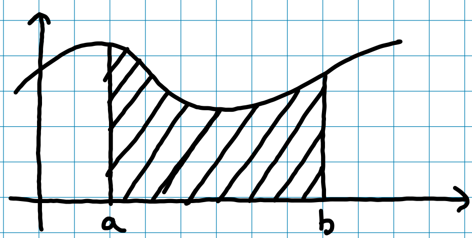
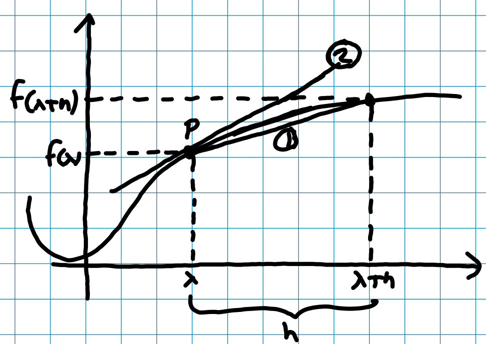
Calculus 2

Areas

→ Finding the area between the curve and the x-axis on the interval [a, b]



*Reminder: process to find slope of tangent



1 secant: $\dfrac{f(x+h) - f(x)}{x+h - x}$

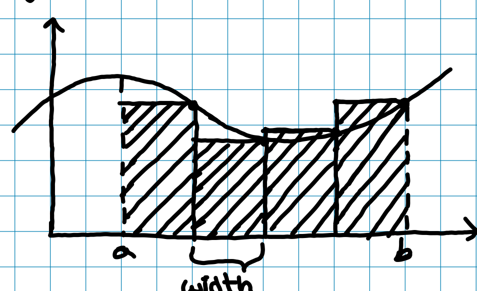
2 tangent: $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

Finding area under the curve:

* Seperate into vertical rectangles of same width but different height

→ For height, pick either left or right end point

consider:



- seperate in 4 rectangles
- use right end points

Notation: L_4

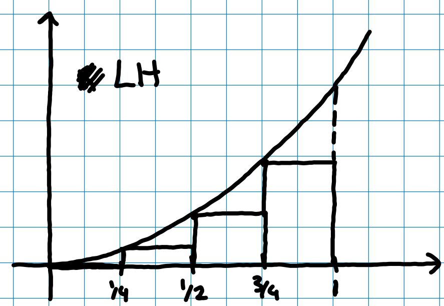
Width $= \dfrac{b-a}{4}$

The more rectangles we seperate into, the more accurate the estimation

⇒ Using limits, we try to have the number of rectangles approach ∞
→ This will give the most exact estimation
→ No difference between R and L if ∞ rectangles

ex: estimate the area between $y=x^2$ and the x-axis on interval $[0,1]$ using:
(a) 4 rectangles and LH-endpoints (notation: L_4)
(b) 4 rectangles and RH-endpoints (notation. R_4)
(c) 8 rectangles for both R and L endpoints (R_8, L_8)

(a)



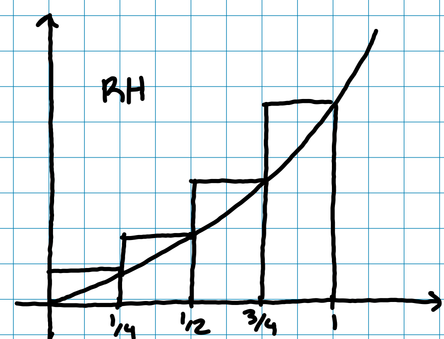
Area(S) ≈ Area(S_1) + Area(S_2) + Area(S_3) + Area(S_4)

* Area = base . height

Area(S_1) = $1/4 \cdot f(0) = 1/4 \cdot 0 = 0$
Area(S_2) = $1/4 \cdot f(1/4) = 1/4 \cdot 1/16 = 1/64$
Area(S_3) = $1/4 \cdot f(1/2) = 1/4 \cdot 1/4 = 1/16$
Area(S_4) = $1/4 \cdot f(3/4) = 1/4 \cdot 9/16 = 9/64$
+ ______
$7/32$

∴ $L_4 = 7/32 \approx 0.21875$

(b)



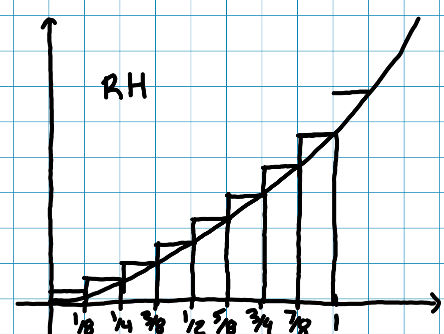
Area(S) ≈ Area(S_1) + Area(S_2) + Area(S_3) + Area(S_4)

* Area = base height

$R_4 = 1/4 \cdot f(1/4) + 1/4(f(1/2)) + 1/4\, f(3/4) + 1/4\, f(1)$
$= 1/64 + 1/16 + 9/64 + 1$
$= 15/32$

∴ $R_4 = 15/32 \approx 0.46875$

(c)



$R_8 = (1/8)(1/8)^2 + (1/8)(1/4)^2 + (1/8)(3/8)^2 + (1/4)(1/2)^2$
$+ (1/8)(5/8)^2 + (1/8)(3/4)^2 + (1/8)(7/8)^2 + (1/8)(1)^2$

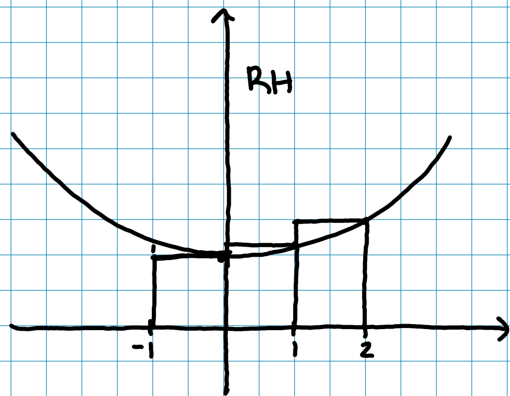
$R_8 = 51/128 \approx 0.398$

$L_8 \approx 0.273$

Note. R_8 and L_8 are closer to eachother than R_4, L_4 b/c we are getting closer to the actual value

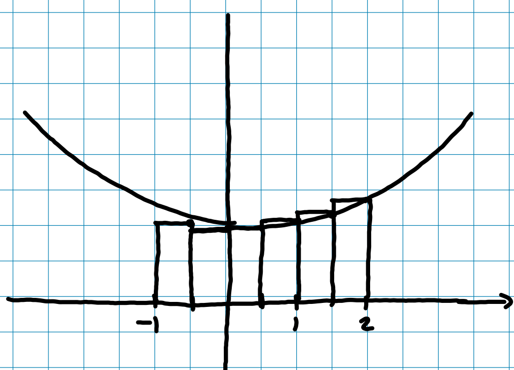
Given: $\left.\begin{array}{l} L_{1000} = 0.3328 \\ R_{1000} = 0.3338 \end{array}\right\}$ Guess that area is $1/3$

ex. Estimate the area under $y = x^2 + 1$ on $[-1, 2]$
using: R_3, L_3, R_6, L_6



$R_3 = 1 \cdot f(0) + 1 \cdot f(1) + 1 \cdot f(2)$
$= 1 + 2 + 5$
$= 8$

$L_3 = 1 \cdot f(-1) + 1 \cdot f(0) + 1 \; f(1)$
$= 2 + 1 + 2$
$= 5$



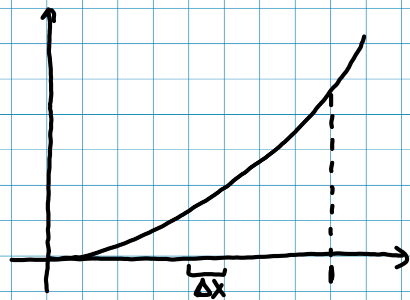
$R_6 = \frac{1}{2} \cdot f(-\frac{1}{2}) + \frac{1}{2} \cdot f(0) + \frac{1}{2} \cdot f(\frac{1}{2}) + \frac{1}{2} \; f(1)$
$+ \frac{1}{2} \cdot f(\frac{3}{2}) + \frac{1}{2} \cdot f(2)$

$= 6.875$

$L_6 = \frac{1}{2} \cdot f(-1) + \frac{1}{2} \; f(-\frac{1}{2}) + \frac{1}{2} \; f(0) + \frac{1}{2} \; f(\frac{1}{2})$
$+ \frac{1}{2} \cdot f(1) + \frac{1}{2} \cdot f(\frac{3}{2})$

$= 5.375$

ex: consider $y=x^2$ from $[0,1]$. Show that the are is actually $1/3$.



Let n = the number of rectangles

$\therefore$ Base width $= \Delta x = \frac{1}{n}$

$\therefore$ Heights (using RH) are
$f(\frac{1}{n}), f(\frac{2}{n}), f(\frac{3}{n}), \ldots, f(\frac{n}{n})$

$$\therefore R_n = \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)^2 + \left(\frac{1}{n}\right)\left(\frac{3}{n}\right)^2 + \ldots + \left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right)^2 + \left(\frac{1}{n}\right)\left(\frac{n}{n}\right)^2$$

$$\text{Area}(S) = \lim_{n\to\infty} R_n$$

* Property: $1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

$$\lim_{n\to\infty} \left(\frac{1}{n}\right)\left[\frac{1^2}{n^2} + \frac{2^2}{n^2} + \frac{3^2}{n^2} + \ldots + \frac{(n-1)^2}{n^2} + \frac{n^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^3}\left(1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + n^2\right) \quad * \text{ property}$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \qquad \frac{\deg 3}{\deg 3} \quad \frac{\infty}{\infty} \quad \therefore \text{use coefficients}$$

$$= \frac{2}{6} = \frac{1}{3}$$

Summation Notation

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n$$

end at $i=n$ → n

$$\sum_{i=1}^{n} a_i$$

start at $i=1$

term to be added

* "i" is the counter ⇒ what is changing

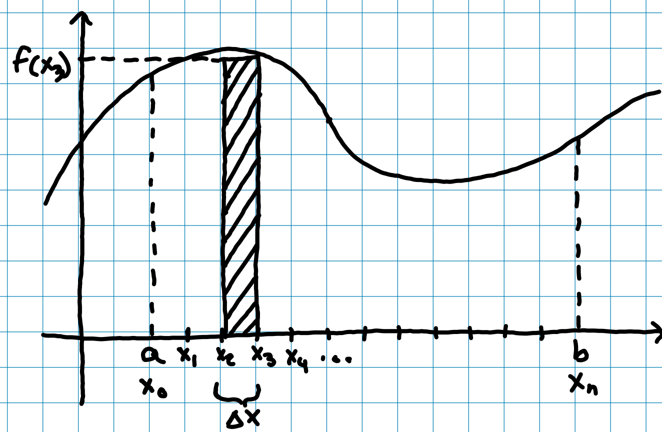
ex: $\sum_{i=1}^{n} 2i = 2(1) + 2(2) + 2(3) + \dots + 2(n)$

$i=1$ $i=2$

ex: $\left(\frac{1}{n}\right) f\left(\frac{1}{n}\right) + \left(\frac{1}{n}\right) f\left(\frac{2}{n}\right) + \dots + \left(\frac{1}{n}\right) \cdot f\left(\frac{n}{n}\right)$

$$= \boxed{\sum_{i=1}^{n} \frac{1}{n} \cdot f\left(\frac{1}{n}\right)}$$

In General: What is the area between $y=f(x)$ and the x-axis on $[a,b]$



If we us n rectangles

$\therefore$ $\boxed{\Delta x = \frac{b-a}{n}}$ is base width

and then $x_1, x_2, x_3, \ldots, x_{n-1}, x_n$ are RH endpoints of the corresponding sub-intervals $\therefore$

$$\left.\begin{aligned} x_1 &= a + \Delta x \\ x_2 &= a + 2\Delta x \\ x_3 &= a + 3\Delta x \\ x_n &= a + n\Delta x \end{aligned}\right\} \quad \therefore \quad \boxed{x_1 = a + i\Delta x}$$

and the heights are given by $f(x_1), f(x_2), \ldots, f(x_{n-1}), f(x_n)$

$$\therefore R_n = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_n)\Delta x$$

$$\therefore R_n = \sum_{i=1}^{n} \underbrace{f(x_i)}_{\text{height}} \times \underbrace{\Delta x}_{\text{base}}$$

$$\therefore \boxed{\text{Area}(S) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_1)\Delta x}$$

Setup limit representing area under $y=f(x)$ on $[a,b]$ given the function

ex: $f(x)=\frac{2x}{x^2+1}$ $\qquad 1 \le x \le 3$

$a=1,\ b=3$

① $\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$

② $x_i = a + i\Delta x = 1+\frac{2i}{n}$

③ $f(x_i) = \frac{2\left(1+\frac{2i}{n}\right)}{\left(1+\frac{2i}{n}\right)^2+1}$

$$\therefore \text{Area}(S) = \lim_{n\to\infty}\sum_{i=1}^{n} \underbrace{\frac{2\left(1+\frac{2i}{n}\right)}{\left(1+\frac{2i}{n}\right)^2+1}}_{f(x_i)} \underbrace{\left(\frac{2}{n}\right)}_{\Delta x}$$

ex: $f(x)= x^2+\sqrt{1+2x}$ $\qquad 4 \le x \le 7$

$a=4,\ b=7$

① $\Delta x = \frac{7-4}{n} = \frac{3}{n}$

② $x_i = a + i\Delta x = 4+\frac{3i}{n}$

③ $\left(4+\frac{3i}{n}\right)^2 + \sqrt{1+2\left(4+\frac{3i}{n}\right)}$

$$\therefore \text{Area}(S) = \lim_{n\to\infty}\sum_{i=1}^{n} \underbrace{\left[\left(4+\frac{3i}{n}\right)^2 + \sqrt{1+2\left(4+\frac{3i}{n}\right)}\right]}_{f(x_i)} \underbrace{\left(\frac{3}{n}\right)}_{\Delta x}$$

ex: $f(x) = x^3 + \frac{1}{2}\sqrt[3]{x}$ $\qquad [-1,4]$

① $\Delta x = \frac{4-(-1)}{n} = \frac{5}{n}$

② $x_i = -1+\frac{5i}{n}$

③ $f(x_i) = \left(-1+\frac{5i}{n}\right)^3 + \frac{1}{2}\left(-1+\frac{5i}{n}\right)^{1/3}$

$$\therefore \text{Area}(S) = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(-1+\frac{5i}{n}\right)^3 + \frac{1}{2}\left(-1+\frac{5i}{n}\right)^{1/3}\right]\left(\frac{5}{n}\right)$$

Determine a region whose area is the same to the given limit

ex: $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\left(\frac{2}{n}\right)}_{\Delta x} \underbrace{\left(5+\frac{2i}{n}\right)^{10}}_{f(x_i)}$

① $\Delta x = \frac{2}{n} = \frac{b-a}{n}$ Let $a=5$ $\therefore b=7$

② $x_i = a + i\Delta x = 5 + \frac{2i}{n}$

If $f(x_i) = \left(5+\frac{2i}{n}\right)^{10}$ $\therefore$ $f(x) = x^{10}$

$\therefore$ $\boxed{f(x) = x^{10} \text{ on } [5,7]}$

* Note: could've chosen any a, b

ex chose different a, b for same limit

① $\Delta x = \frac{2}{n} = \frac{b-a}{n}$ Let $a=0$ $\therefore b=2$

② $x_i = a + i\Delta x = \frac{2i}{n}$

If $f(x_i) = \left(5+\frac{2i}{n}\right)^{10}$ $\therefore$ $f(x) = (5+x)^{10}$

$\therefore$ $\boxed{f(x) = (5+x)^{10} \text{ on } [0,2]}$

ex: $\displaystyle \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \frac{1}{1+\frac{3}{n}} + \dots + \frac{1}{1+\frac{n-1}{n}} + \frac{1}{1+\frac{n}{n}}\right]$

$\displaystyle = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{n}\right)\left[\frac{1}{1+\frac{i}{n}}\right]$

① $\Delta x = \frac{1}{n} = \frac{b-a}{n}$ Let $a=1$ $\therefore b=2$

② $x_i = a + i\Delta x = 1 + \frac{i}{1}$

If $f(x_i) = \left(\frac{1}{1+\frac{i}{n}}\right)$ $\therefore$ $f(x) = \frac{1}{x}$

$\therefore$ $\boxed{f(x) = \frac{1}{x} \text{ on } [1,2]}$

Summation Rules

Consider if $c \in \mathbb{R}$ is any constant

(a) $\sum_{i=1}^{n} c a_i = c \cdot \sum_{i=1}^{n} a_i$

(b) $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

(c) $\sum_{i=1}^{n} c = c \cdot n$ $\Rightarrow \sum_{i=1}^{n} (1) = \underset{(n \text{ times})}{1+1+1+1+\ldots}$

(d) ① $\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$ $\Rightarrow 1+2+3+\ldots+n-1+n$

$\sum_{i=1}^{10} i = 1+2+3+4+5+6+7+8+9+10$

$\left.\begin{matrix} 10+1 = 11 = (n+1) \\ 9+2 = 11 = (n+1) \\ \vdots \end{matrix}\right\}$ can do 5 times $= \frac{n}{2}$

(e) ② $\boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$

(f) ③ $\boxed{\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2}$

ex: $\sum_{i=1}^{n} i(4i^2-3)$

$= \sum_{i=1}^{n} 4i^3 - \sum_{i=1}^{n} 3i$

$= 4\left(\sum_{i=1}^{n} i^3\right) - 3\left(\sum_{i=1}^{n} i\right)$

$= 4\left(\frac{n(n+1)}{2}\right)^2 - 3\left(\frac{n(n+1)}{2}\right)$

ex: $\sum_{i=1}^{n} (4-3i)^2$

$= \sum_{i=1}^{n} (16 - 24i + 9i^2)$

$= \sum_{i=1}^{n} 16 - 24 \sum_{i=1}^{n} i + 9 \sum_{i=1}^{n} i^2$

$= 16n - 24 \frac{n(n+1)}{2} + 9 \frac{n(n+1)(2n+1)}{6}$

Recall : $\lim_{x\to\infty}$ of type $\frac{\infty}{\infty}$

① Higher power (deg) on top $\Rightarrow \pm\infty$ (sign of highest power)

ex. $\lim_{x\to\infty} \frac{x^6+4x^4-3x^2}{3x^5+2x-1} = \infty$

proof: $\lim_{x\to\infty} \frac{x^6(1+\frac{4}{x^2}-\frac{3}{x^3})}{x^5(3+\frac{2}{x^4}-\frac{1}{x^5})} \Rightarrow \frac{x^6}{x^5} = x \therefore \infty$

② Higher power (deg) on bottom $\Rightarrow 0$

ex. $\lim_{x\to\infty} \frac{x^5+x^4-x^2}{x^6+2x-1} = 0$

proof: $\lim_{x\to\infty} \frac{x^5(1+\frac{1}{x}-\frac{1}{x^3})}{x^6(1+\frac{2}{x^5}-\frac{1}{x^6})} \Rightarrow \frac{x^5}{x^6} = \frac{1}{x} \therefore 0$

③ Same degree $\Rightarrow$ use coefficients of highest degree of each term that is being multiplied or divided

ex. $\lim_{x\to\infty} \frac{x^6+x^4-x^2}{3x^6+2x-1}$ $\frac{\text{deg } 6}{\text{deg } 6}$

$= \boxed{\frac{1}{3}}$

ex. $\lim_{x\to\infty} \frac{(2x-1)^3(3x+2)}{3x^4}$ $\frac{\text{deg } 3 \times \text{deg } 1}{\text{deg } 4} = \frac{\text{deg } 4}{\text{deg } 4}$

* multiply coefficients of the highest power in each term that is being multiplied

$\frac{2^3 \cdot 3}{3} = 8$

Evaluating limits with a summation

2 variables (n and i)

① Simplify the summation (to get rid of i)
② Solve the limit with n

* <u>Note</u>: with respect to the summation, n can be treated as a constant (only i is changing)

ex: $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4}{n}\right)\left(3+\frac{4i}{n}\right)^2$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4}{n}\right)\left(9+\frac{24i}{n}+\frac{16i^2}{n^2}\right)$

- Take Δx portion out of summation (constant)
- Simplify $f(x_i)$ parenthesis with algebra
- Distribute the summation

$= \lim_{n\to\infty} \left(\frac{4}{n}\right)\left(\sum_{i=1}^{n} 9 + \sum_{i=1}^{n} \frac{24i}{n} + \sum_{i=1}^{n} \frac{16i^2}{n^2}\right)$

$= \lim_{n\to\infty} \frac{4}{n}\left[9n + \frac{24}{n}\left(\frac{n(n+1)}{2}\right) + \frac{16}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right]$

$= \lim_{n\to\infty} \frac{36n}{n} + \frac{48\,n(n+1)}{n^2} + \frac{64\,n(n+1)(2n+1)}{6\,n^3}$

* multiply coef. of highest deg. of each <u>term</u>

$\frac{\text{deg } 1}{\text{deg } 1}$ $\qquad$ $\frac{\text{deg } 2}{\text{deg } 2}$ $\qquad$ $\frac{\text{deg } 3}{\text{deg } 3}$

$36 + 48 \qquad \frac{64\,(2)}{6}$

* <u>Note</u>. For Reiman sums, the deg. is always the same top and bottom

b/c it represents an area, which is neither 0 nor ∞

ex: $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2}{n}\right)\left(\frac{2i}{n}\right)^3 =$

$= \lim_{n\to\infty} \left(\frac{2}{n}\right) \sum_{i=1}^{n} \frac{8i^3}{n^3}$

$= \lim_{n\to\infty} \left(\frac{2}{n}\right)\left[\frac{8}{n^3}\left(\frac{n(n+1)}{2}\right)^2\right]$

$= \lim_{n\to\infty} \frac{16}{n^4}\,\frac{(n(n+1))^2}{4}$ $\quad \frac{\deg 4}{\deg 4}$

$= 16/4 = 4$

ex: $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2 =$

$= \lim_{n\to\infty} \left(\frac{1}{n}\right) \sum_{i=1}^{n} \frac{i^2}{n^2}$

$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right]$ $\quad \frac{\deg 3}{\deg 3}$

$= 2/6 = 1/3$

ex: $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^3 - 2\right] =$

$= \lim_{n\to\infty} \left(\frac{1}{n}\right)\left[\sum_{i=1}^{n} \frac{i^3}{n^3} - \sum_{i=1}^{n} 2\right]$

$= \lim_{n\to\infty} \left(\frac{1}{n}\right)\left[\frac{1}{n^3}\cdot\left(\frac{n(n+1)}{2}\right)^2 - 2n\right]$

$= \lim_{n\to\infty} \frac{[n(n+1)]^2}{n^4\cdot 2^2} - \frac{2n}{n}$

$= \quad \frac{1}{4} - 2$

$= -7/4$

Definite Integral

upper limit of integration

differential

$$\int_a^b \underbrace{f(x)}_{\text{integrand}}\, dx$$

Lower limit of integration

Note:

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x}$$

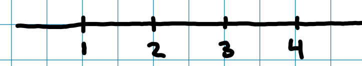
$\therefore$ definite integral also represents the area under $y = f(x)$ on $[a, b]$

ex. Consider the function $f(x) = x^2 + 2x - 5$ and the definite integral $\int_1^4 (x^2+2x-5)\,dx$

(a) Approximate the integral using 3 rectangles

$n = 3$ b/c 3 rectangles

$\Delta x = \frac{4-1}{n} = \frac{4-1}{3} = 1$



$$\int_1^4 (x^2+2x-5)\,dx \approx \text{Area}(S_1) + \text{Area}(S_2) + \text{Area}(S_3)$$

$$= f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1$$

$$= 3 + 10 + 19$$

$$= 32$$

(b) Evaluate exactly the integral $\int_1^4 (x^2+2x-5)\,dx$

① $\Delta x = \frac{3}{n}$

② $x_i = 1 + \frac{3i}{n}$

③ $f(x_i) = \left(1+\frac{3i}{n}\right)^2 + 2\left(1+\frac{3i}{n}\right) - 5$

$$\therefore \int_1^4 (x^2+2x-5)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(1+\frac{3i}{n}\right)^2 + 2\left(1+\frac{3i}{n}\right) - 5\right]\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{3}{n}\right) \sum_{i=1}^{n} \left[1 + \frac{6i}{n} + \frac{9i^2}{n^2} + 2 + \frac{6i}{n} - 5\right]$$

$$= \lim_{n\to\infty} \left(\frac{3}{n}\right) \sum_{i=1}^{n} \left[-2 + \frac{12i}{n} + \frac{9i^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \left(\frac{3}{n}\right) \left[-2n + \frac{12}{n}\,\frac{n(n+1)}{2} + \frac{9}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right]$$

$$\downarrow$$

$$= 3(-2) + 3(12/2) + 3(9\times 2/6)$$

$$= -6 + 18 + 9$$

$$= 21$$

ex: Express the following limit as a definite integral

$$\lim_{n\to\infty}\sum_{i=1}^{n} \underbrace{\left(\frac{4}{n}\right)}_{\Delta x} \underbrace{\underbrace{\left(2+\frac{4i}{n}\right)}_{x_i} \ln\left(1+\underbrace{\left(2+\frac{4i}{n}\right)^2}_{x_i}\right)}_{f(x_i)}$$

① $\Delta x = \frac{4}{n} = \frac{b-a}{n}$ Let $a=2$, $b=6$

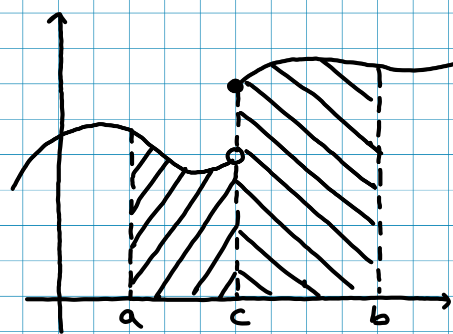
② $x_i = 2+\frac{4i}{n}$

$$\therefore \; = \int_2^6 x \ln(1+x^2)\,dx$$

Theorem: If $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$ exists

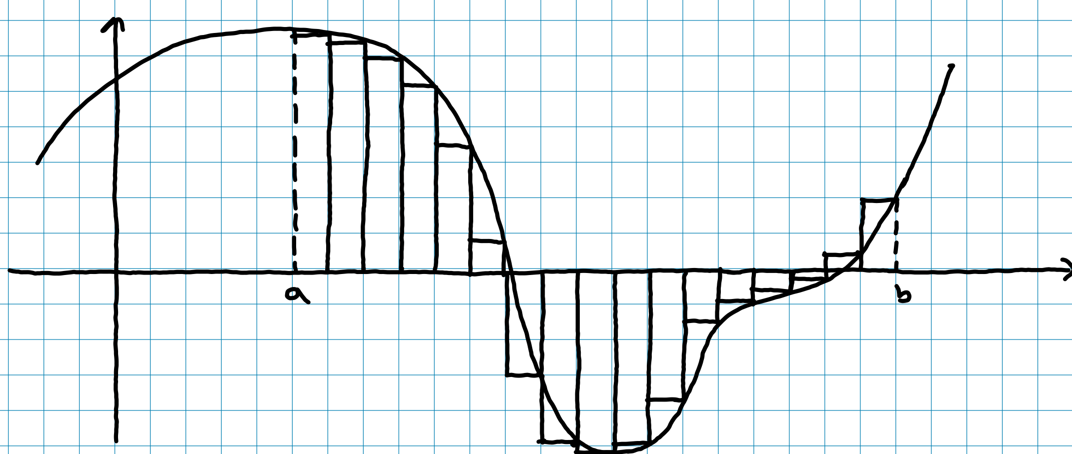
$\therefore$ f is said to be integrable on [a,b]

Note: having a discontinuity does not affect integrability



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Consider:



Area ABOVE x-axis $\Rightarrow$ pos. (+)
Area BELLOW x-axis $\Rightarrow$ neg (−)

ex: $\int_0^{2\pi} \sin x \; dx = 0$

Properties of Definite Integral

$$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$$

$a > b$

<u>proof</u>. $\displaystyle \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right)$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\left[-\left(\frac{a-b}{n}\right)\right]$$

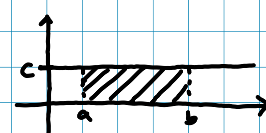
$$= -\int_b^a f(x)\,dx$$

$$\boxed{\int_a^a f(x)\,dx = 0}$$

$$\Delta x = \frac{a-a}{n} = 0$$

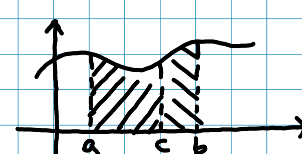
$$\boxed{\int_a^b c\ dx = c(b-a)}$$

$c \in \mathbb{R}$



$$\boxed{\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx}$$

$$* \quad \boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx}$$



Note. c does not HAVE to be between a and b

$$\boxed{\int_a^b c \cdot f(x)\,dx = c\int_a^b f(x)\,dx}$$

ex: If $\int_0^{10} f(x)\,dx = 17$ and $\int_0^{8} f(x)\,dx = 12$, find $\int_8^{10} f(x)\,dx$

$$\int_0^{10} f(x)\,dx = \int_0^{8} f(x)\,dx + \int_8^{10} f(x)\,dx$$

$$17 = 12 + \int_8^{10} f(x)\,dx$$

$$\therefore \int_8^{10} f(x)\,dx = 5$$

ex: If $\int_0^1 f(x)\,dx = 2$, $\int_1^2 f(x)\,dx = 3$, $\int_0^1 g(x)\,dx = -1$, $\int_0^2 g(x)\,dx = 4$, find:

(a) $\int_1^2 g(x)\,dx =$ $\qquad$ $\int_0^2 = \int_0^1 + \int_1^2$

$= \int_0^2 g(x)\,dx - \int_0^1 g(x)\,dx$ $\qquad$ $\int_0^2 - \int_0^1 = \int_1^2$

$= 4 - (-1) = 5$

(b) $\int_0^2 \left[2f(x) - 3g(x)\right] dx =$

$= 2\int_0^2 f(x)\,dx - 3\int_0^2 g(x)\,dx$ $\qquad$ $\int_0^2 f = \int_0^1 f + \int_1^2 f$

$= 2(2+3) - 3(4) = -2$

(c) $\int_1^1 g(x)\,dx =$

$= 0$

(d) $\int_1^2 f(x)\,dx + \int_2^0 g(x)\,dx =$

$= \int_1^2 f(x)\,dx - \int_0^2 g(x)\,dx$

$= 3 - 4 = -1$

(e) $\int_0^2 f(x)\,dx + \int_2^1 g(x)\,dx =$ $\qquad$ $\int_0^2 f = \int_1^2 f + \int_0^1 f$

$= (2+3) - \int_1^2 g(x)\,dx$ $\qquad$ $\int_0^2 g = \int_1^2 g + \int_0^1 g$

$= 5 - (4 - -1) = 0$ $\qquad$ $\int_1^2 g = \int_0^2 g - \int_0^1 g$

ex: $\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$, simplify

$$= \int_{-2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$$

$$= \int_{-2}^{5} f(x)\,dx + \int_{-1}^{-2} f(x)\,dx$$

Note: c not in between a and b

$a=-1$, $b=5$, $c=-2$

$$= \int_{-1}^{5} f(x)\,dx$$

ex: $\int_{0}^{2} 3f(x)\,dx + \int_{1}^{3} 3f(x)\,dx - \int_{0}^{2} 2f(x)\,dx - \int_{1}^{2} 3f(x)\,dx - \int_{2}^{3} 2f(x)\,dx$, simplify

$$= \int_{0}^{2} [3f(x) - 2f(x)]\,dx + \int_{1}^{3} 3f(x)\,dx - \int_{1}^{2} 3f(x)\,dx - \int_{2}^{3} 2f(x)\,dx$$

$$\int_{1}^{3} 3f = \int_{1}^{2} 3f + \int_{2}^{3} 3f$$

$$\int_{1}^{3} 3f - \int_{1}^{2} 3f = \int_{2}^{3} 3f$$

$$= \int_{0}^{2} f(x)\,dx + \int_{2}^{3} 3f(x)\,dx - \int_{2}^{3} 2f(x)\,dx$$

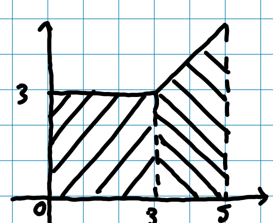
$$= \int_{0}^{2} f(x)\,dx + \int_{2}^{3} [3f(x) - 2f(x)]\,dx$$

$$= \int_{0}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx$$

$$= \int_{0}^{3} f(x)\,dx$$

ex. Find $\int_{0}^{5} f(x)\,dx$ for $f(x) = \begin{cases} 3 & , x < 3 \\ x & , x \geq 3 \end{cases}$

$$= \int_{0}^{3} f(x)\,dx + \int_{3}^{5} f(x)\,dx$$



$$\int_{0}^{3} f(x)\,dx = A_1 = 3^2 = 9$$

$$\int_{3}^{5} f(x)\,dx = A_2 = \frac{(B+b)h}{2} = \frac{(3+5)2}{2} = 8$$

$$\therefore \int_{0}^{5} f(x)\,dx = 9 + 8 = 17$$

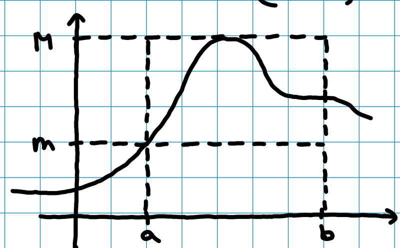
Comparaison Properties of the Definite Integral

(1) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$

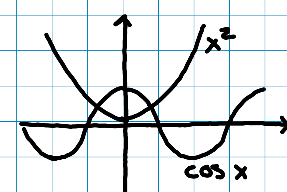
(2) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

(3) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, $m \in \mathbb{R}$, $M \in \mathbb{R}$

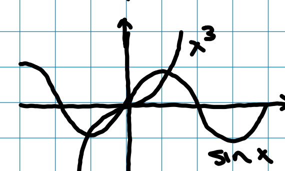
then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$



<u>Definition</u>: (a) $f(x)$ is "even" if $f(x) = f(-x)$
(ie. symetric about the y-axis)



(b) $f(x)$ is "odd" if $f(-x) = -f(x)$
(ie. symetry about the y and x axis)



<u>Theorem</u>:

(1) If $f(x)$ is "even" then

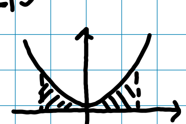
$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

symetric intervals

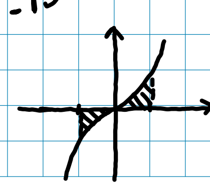
(2) If $f(x)$ is "odd" then

$$\int_{-a}^{a} f(x)\,dx = 0$$

ex: $\int_{-1}^{1} x^2\,dx = 2\int_0^1 x^2\,dx$



ex: $\int_{-1}^{1} x^3\,dx = 0$

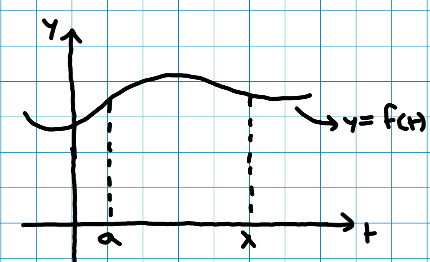


Fundamental Theorem of Calculus - I
(FTC - I)

FTC-I : If f is continuous on $[a,b]$,
then F defined by $F(x) = \int_a^x f(t)\,dt$
① is continuous on $[a,b]$
② differentiable on (a,b)
③ $F'(x) = f(x)$

consider $F(x) = \int_a^x f(t)\,dt$ $\Rightarrow$ accumulation fcn

Graph of $f(t)$



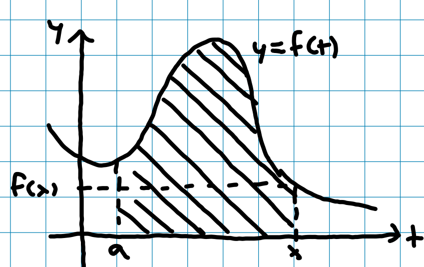
Area accumulates from a to x

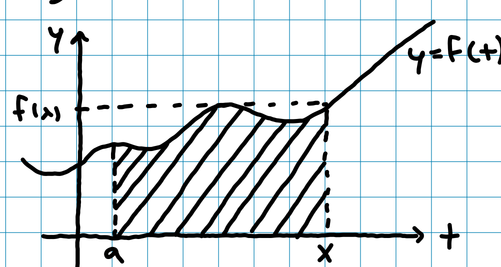
$F(x)$: Function
→ In the graph $f(t)$, it describes the area under f

$F'(x) = f(x)$: derivative of the function (graph above)
→ Describes the rate at which the area changes

Why is $F'(x) = f(x)$?

$F'(x) = \left(\text{how quickly the area is changing}\right)$





The bigger $f(x)$, $\Rightarrow$ the faster the area changes
∴ The bigger $F'(x)$

Fundamental Theorem of Calculus - I

FTC-I : If f is continuous on $[a,b]$, the F defined by $F(x) = \int_a^x f(t)\,dt$

① is continuous on $[a,b]$

② is differentiable on (a,b)

③ $F'(x) = f(x)$

Proof : Let $f(t)$ be continuous on $[a,b]$

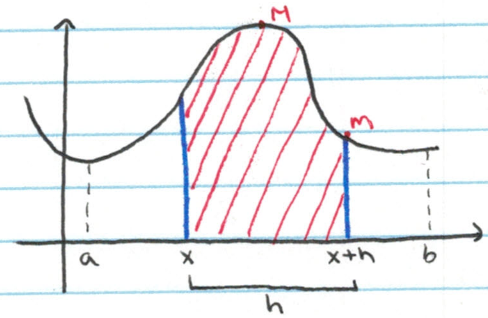
take x and $x+h$ on (a,b), $h \neq 0$

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt$$

$$\circledast \quad \therefore \boxed{\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt}$$

$\because f(t)$ is continuous on $[a,b]$ $\therefore$ continuous on $[x, x+h]$

$\therefore$ By EVT, there exists an absolute max/min for $f(t)$ on $[x, x+h]$



$$m \leq f(t) \leq M$$

$$m \cdot h \leq \int_x^{x+h} f(t)\,dt \leq M \cdot h$$

Note: As $\lim_{h \to 0}$, $m \to f(x)$, $M \to f(x)$

$$\therefore f(x) \cdot h \leq \lim_{h\to 0}\int_x^{x+h} f(t)\,dt \leq f(x) \cdot h$$

Squeeze: $\lim_{h \to 0} \int_x^{x+h} f(t)\,dt = f(x) \cdot h$

$\lim_{h \to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x)$

$\circledast \quad \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x) \quad \therefore F'(x) = f(x)$

Proof of FTC-I :

Let $f(t)$ be continuous on $[a,b]$

take x and $x+h$ on (a,b) , $h \neq 0$

Then $F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt = \int_x^{x+h} f(t)\,dt$

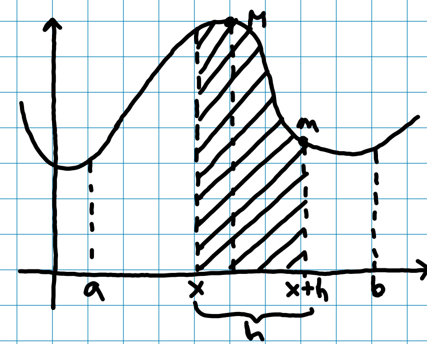
(*)

$$\therefore \boxed{\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt}$$

For $h>0$, $\because f(t)$ is continuous on $[a,b]$

$\therefore f(t)$ is continuous on $[x, x+h]$

$\therefore$ by Extreme Value Theorem, there exists an absolute max and min for $f(t)$ on $[x, x+h]$



$$\therefore \quad m \leq f(t) \leq M$$

$$\therefore \int_x^{x+h} m\,dt \leq \int_x^{x+h} f(t)\,dt \leq \int_x^{x+h} M\,dt$$

$$m\,h \leq \int_x^{x+h} f(t)\,dt \leq M.h$$

<u>Note</u>: As $\lim_{h\to 0}$, $m \to f(x)$, $M \to f(x)$ $\Rightarrow$ Area $\to$ (base) (height)
$h \cdot f(x)$

$$\therefore \quad f(x).h \leq \lim_{h\to 0}\int_x^{x+h} f(t)\,dt \leq f(x).h$$

$\therefore$ By squeeze theorem.

$$\lim_{h\to 0}\int_x^{x+h} f(t)\,dt = f(x).h$$

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x)$$

sub (*)

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = f(x)$$

$$\therefore \quad F'(x) = f(x)$$

Applying FTC–I

① $\boxed{\text{If} \quad F(x) = \int_a^x f(t)\,dt \quad \therefore \quad F'(x) = f(x)}$

② $\boxed{\text{If} \quad F(x) = \int_a^{g(x)} f(t)\,dt \quad \therefore \quad F'(x) = f(g(x)) \cdot g'(x)}$

composite fcn ⇒ chain rule

③ $\boxed{\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x)}$

* If asked to find $F'(x^2)$, x^2 is another composite function so we have to proceed by chain rule

ex: $g(x) = \int_0^{x} t^2 \cos t \cdot dt$

$\therefore g'(x) = x^2 \cos x$

ex: $h(x) = \int_1^{x^4} \sec t\, dt$

$h'(x) = \sec(x^4) \cdot 4x^3$

ex: $\ell(x) = \int_x^{x^2} e^{-t^2}\, dt$

$= \int_x^{a} e^{-t^2}\, dt + \int_a^{x^2} e^{-t^2}\, dt$

$= -\int_a^{x} e^{-t^2}\, dt + \int_a^{x^2} e^{-t^2}\, dt$

$\therefore \ell'(x) = -e^{-x^2} + e^{-(x^2)^2} \cdot (2x)$

ex: $\frac{d}{dx}\int_5^{x} (1-\sqrt{\sin t})\, dt = 1-\sqrt{\sin x}$

ex: $\frac{d}{dx}\int_1^{x^3} (1-\sqrt{\sin t})\, dt = (1-\sqrt{\sin x^3})(3x^2)$

ex: $\frac{d}{dt}\int_t^{e^{2t}} \frac{1}{\sqrt{1+x^2}}\, dx = \frac{1}{\sqrt{1+(e^{2t})^2}} \cdot e^{2t} \cdot 2 - \frac{1}{\sqrt{1+t^2}}$

ex: $\frac{d}{dx}\int_{2x-1}^{x^2} \frac{t}{1+t^3}\,dt = \frac{x^2}{1+(x^2)^3}\cdot 2x - \frac{(2x-1)}{1+(2x-1)^3}\cdot 2$

ex: $\frac{d}{d\theta}\int_{\sin\theta}^{\cos\theta} \frac{1}{\sqrt{1-x^2}}\,dx = \frac{1}{\sqrt{1-\cos^2\theta}}\cdot(-\sin\theta) - \frac{1}{\sqrt{1-\sin^2\theta}}\cdot(\cos\theta)$

$$= \frac{-\sin\theta}{\sqrt{\sin^2\theta}} - \frac{\cos\theta}{\sqrt{\cos^2\theta}} = -1-1 = -2$$

ex: $\frac{d}{dx}\int_{x\sin x}^{x^3} \frac{t}{1-t^2}\,dt = \frac{x^3}{1-(x^3)^2}\cdot 3x^2 - \frac{x\sin x}{1-(x\sin x)^2}\,(x\cos x+\sin x)$

Antiderivatives

→ A fcn F is called "antiderivative" of f on interval I, if $F'(x) = f(x)$ on I

ex: consider $f(x) = x^2$, then

$F(x) = \frac{x^3}{3}$ is an antiderivative of $f(x)$

b/c $\left(\frac{x^3}{3}\right)' = \frac{3x^2}{3} = x^2$

Note: $G(x) = \frac{x^3}{3} + 1$ is also an antiderivative

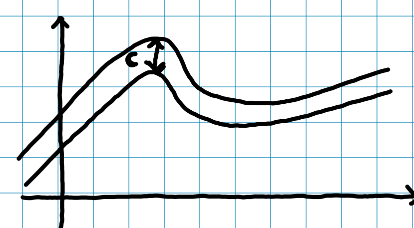
In general: the C that comes after can be any constant $C \in \mathbb{R}$
$\because$ the derivative of a constant is 0

$\therefore$ The 'general antiderivative' of $f(x) = x^2$ is $F(x) = \frac{x^3}{3} + C$

Theorem (Cal 1)

If $F'(x) = g'(x)$ on I

then $f(x) - g(x) = C$



$\therefore$ There is an infinite number of antiderivatives for a same fcn

$\therefore$ The difference between all of them is a constant.

Fundamental Theorem of Calculus - II (FTC-II)

FTC-II: If $f(x)$ is continuous on $[a,b]$

and $F'(x) = f(x)$ ($F(x)$ is antiderivative)

$$\therefore \int_a^b f(x)\,dx = F(b) - F(a)$$

<u>Note</u>: Notation can abbreviate FTC-II

$$F(b) - F(a) = F(x)\Big|_a^b$$

$$\boxed{\int_a^b f(x)\,dx = \underbrace{F(x)}_{\text{general antiderivative}}\Big|_a^b}$$

Fundamental Theorem of Calculus - II

FTC-II : If $f(x)$ is continuous on $[a,b]$
and $F'(x) = f(x)$ ($F(x)$ is the antiderivative)

$$\therefore \int_a^b f(x)\,dx = F(b) - F(a)$$

Proof: Let $f(t)$ be continuous on $[a,b]$

✱ $\boxed{\text{Let } g(x) = \int_a^x f(t)\,dt}$

FTC-I: $\therefore g'(x) = f(x)$ ($g(x)$ is antiderivative)

Let $F(x)$ be some other antiderivative of $f(x)$

$$\therefore F'(x) = g'(x) = f(x)$$

$$\therefore F'(x) = g'(x)$$

$$\therefore F(x) = g(x) + C$$

$$F(b) - F(a) = [g(b) + \cancel{C}] - [g(a) + \cancel{C}]$$

$$F(b) - F(a) = g(b) - g(a)$$

✱
$$\begin{cases} ① \; g(b) = \int_a^b f(t)\,dt \\ ② \; g(a) = \int_a^a f(t)\,dt = 0 \end{cases}$$

$$\therefore F(b) - F(a) = \int_a^b f(t)\,dt$$

Proof of FTC-II:

Let $f(t)$ be continuous on $[a,b]$

Let $g(x) = \int_a^x f(t)\,dt$

$\therefore g'(x) = f(x)$ (From FTC-I)

$\rightarrow g(x)$ is an antiderivative of $f(x)$

Let $F(x)$ be some other antiderivative of $f(x)$

$\therefore$ Both $F(x)$ and $g(x)$ are antiderivatives of $f(x)$

$$F'(x) = f(x) = g'(x)$$

$$F'(x) = g'(x)$$

$$\therefore F(x) - g(x) = C$$

$$\therefore F(x) = g(x) + C$$

{Show that: $\int_a^b f(t)\,dt = F(b) - F(a)$}

$$F(b) - F(a) = [g(b) + C] - [g(a) + C]$$

$$\boxed{F(b) - F(a) = g(b) - g(a)}$$

① $g(b) = \int_a^b f(t)\,dt$

② $g(a) = \int_a^a f(t)\,dt = 0$

} From: $g(x) = \int_a^x f(t)\,dt$

$$\therefore F(b) - F(a) = \int_a^b f(t)\,dt - 0$$

$$\boxed{F(b) - F(a) = \int_a^b f(t)\,dt}$$

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b$$

→ $F(x)$ is the general antiderivative.

<u>Note</u>: No need to include C b/c it will cancel out

ex: $\int_1^3 e^x\,dx = e^x\Big|_1^3 = e^3 - e^1$

ex: $\int_3^6 \frac{1}{x}\,dx = \ln|x|\Big|_3^6 = \ln 6 - \ln 3 = \ln\frac{6}{3} = \ln 2$

ex: $\int_0^3 (3x^2+x-2)\,dx = \left(\frac{3x^3}{3} + \frac{x^2}{2} - 2x\right)\Big|_0^3 =$

$$= \left(3^3 + \frac{3^2}{2} - 3\cdot 2\right) - \left(0^3 + \frac{0^2}{2} - 0\right) = 51/2$$

ex: $\int_{-2}^{-1} \frac{x - x^2}{2\sqrt[3]{x}}\,dx =$ * Simplify and then integrate

$= \frac{1}{2}\int_{-2}^{-1} (x^{2/3} - x^{5/3})\,dx$

$= \frac{1}{2}\left(\frac{x^{5/3}}{5/3} - \frac{x^{8/3}}{8/3}\right)\Big|_{-2}^{-1} = \frac{1}{2}\left[\left(\frac{3}{5}(-1)^{5/3} - \frac{3}{8}(-1)^{8/3}\right) - \left(\frac{3}{5}(-2)^{5/3} - \frac{3}{8}(-2)^{8/3}\right)\right]$

ex: $\int_0^3 (t - 5^t)\,dt = \left(\frac{t^2}{2} - \frac{5^t}{\ln 5}\right)\Big|_0^3 = \left(\frac{9}{2} - \frac{5^3}{\ln 5}\right) - \left(\frac{0}{2} - \frac{5^0}{\ln 5}\right)$

ex: $\int_0^1 (1 + u^{1/3} - \frac{2}{5}u^9)\,du = \left(u + \frac{3}{4}u^{4/3} - \frac{2}{5}\frac{u^{10}}{10}\right)\Big|_0^1 = \left(u + \frac{3}{4}u^{4/3} - \frac{u^{10}}{25}\right)\Big|_0^1$

$$= \left(1 + \frac{3}{4}(1)^{4/3} - \frac{(1)^{10}}{25}\right) - \left(0 + \frac{3}{4}(0) - \frac{0}{25}\right)$$

ex: $\int_e^{2e} (\cos x - \frac{1}{x})\,dx = (\sin x - \ln|x|)\Big|_e^{2e}$

FTC-II and rational fcn

→ can't integrate numerator and denomenator seperately

⇒ Long division (if $\deg_{num} \geq \deg_{denom}$)

ex: $\displaystyle \int_1^9 \left(\frac{x^3+3x^2-x^{3/2}+3}{\sqrt{x}}\right) dx = \int_1^9 \left(x^{5/2}+3x^{3/2}-x+3x^{-1/2}\right) dx$

$$= \left(\frac{2}{7}x^{7/2} + 3\cdot\frac{2}{5}x^{5/2} - \frac{x^2}{2} + 3.2x^{1/2}\right)\Bigg|_1^9$$

$$= \underbrace{\left(\frac{2x^{7/2}}{7} + \frac{6x^{5/2}}{5} - \frac{x^2}{2} + 6x^{1/2}\right)}_{F(x)}\Bigg|_1^9 = F(9) - F(1)$$

ex: $\displaystyle \int_1^3 \frac{6x^6+3x^4+x}{2x^2+1}\, dx$

∗ Long division.

$$\begin{array}{l|l} 6x^6+3x^4+x & 2x^2+1 \\ \underline{6x^6+3x^4} & 3x^4 \\ \quad\quad 0+x & \end{array}$$

$\therefore \boxed{3x^4 + \frac{x}{2x^2+1}}$

$$= \int_1^3 \left(3x^4 + \frac{x}{2x^2+1}\right) dx$$

$$= \int_1^3 3x^4\, dx + \int_1^3 \frac{x}{2x^2+1}\, dx$$

$$= \left(\frac{3x^5}{5}\right)\Bigg|_1^3 + \underbrace{\int_1^3 \frac{x}{2x^2+1}\, dx}_{\text{can't integrate this yet}}$$

can't integrate this yet (see u-substitution)

FTC-II and absolute value fcn

$$|x| = \begin{cases} x & , \; x \geq 0 \\ -x & , \; x < 0 \end{cases}$$

$\Rightarrow$ split up definite integral into 2

ex. $\displaystyle\int_0^2 |2x-1| \, dx$

$$2x - 1 \geq 0$$
$$x \geq 1/2$$

$$|2x-1| = \begin{cases} 2x-1 & , \; x \geq 1/2 \\ -(2x-1) & , \; x < 1/2 \end{cases}$$

$$\cancel{\int_0^{1/2} (2x-1)\,dx + \int_{1/2}^{2} -(2x-1)\,dx}$$

$$= \underbrace{\int_0^{1/2} -(2x-1)\,dx}_{x < 1/2} + \underbrace{\int_{1/2}^{2} (2x-1)\,dx}_{x > 1/2}$$

$$= -(x^2 - x)\Big|_0^{1/2} + (x^2 - x)\Big|_{1/2}^{2}$$

$$= -\left[\left((1/2)^2 - (1/2)\right) - (0^2 - 0)\right] + \left[\left((2)^2 - 2\right) - \left((1/2)^2 - (1/2)\right)\right]$$

$$= -\left[-1/4\right] + \left[9/4\right]$$

$$= 5/2$$

Common Derivatives and Antiderivatives

$\frac{d}{dx} n = 0$	$\int 0 \, dx = C$
$\frac{d}{dx} x = 1$	$\int 1 \, dx = x + C$
$\frac{d}{dx} x^n = n x^{n-1}$	$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$
$\frac{d}{dx} e^x = e^x$	$\int e^x \, dx = e^x + C$
$\frac{d}{dx} \ln x = \frac{1}{x}$	$\int \frac{1}{x} \, dx = \ln\lvert x\rvert + C$
$\frac{d}{dx} a^x = a^x \ln a$	$\int a^x \, dx = \frac{a^x}{\ln a} + C$

Basic Trigonometric Antiderivatives

$\frac{d}{dx} \sin x = \cos x$	$\int \cos x \, dx = \sin x + C$
$\frac{d}{dx} \cos x = -\sin x$	$\int \sin x \, dx = -\cos x + C$
$\frac{d}{dx} \tan x = \sec^2 x$	$\int \sec^2 x \, dx = \tan x + C$
$\frac{d}{dx} \cot x = -\csc^2 x$	$\int \csc^2 x \, dx = -\cot x + C$
$\frac{d}{dx} \sec x = \sec x \tan x$	$\int \sec x \tan x \, dx = \sec x + C$
$\frac{d}{dx} \csc x = -\csc x \cot x$	$\int \csc x \cot x \, dx = -\csc x + C$
$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$
$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$	$\int -\frac{1}{\sqrt{1-x^2}} \, dx = \arccos x + C$
$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} \, dx = \arctan x + C$
$\frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2}$	$\int -\frac{1}{1+x^2} \, dx = \operatorname{arccot} x + C$
$\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} \, dx = \operatorname{arcsec} x + C$
$\frac{d}{dx} \csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}}$	$\int -\frac{1}{x\sqrt{x^2-1}} \, dx = \operatorname{arccsc} x + C$

Indefinite Integral

Def: Given a fcn $f(x)$,
an antiderivative of $f(x)$ is given by:

$$\int f(x)\,dx$$

and is called an indefinite integral.

Note: $\int f(x)\,dx = F(x) + C$, where $F'(x) = f(x)$

ex: $\int (x^2 + x^{-2})\,dx = \frac{x^3}{3} + \frac{x^{-1}}{-1} + C$

$= \frac{x^3}{3} - \frac{1}{x} + C$

ex: Verify by differentiation that the formula is correct

(a) $\int x\cos x\,dx = x\sin x + \cos x + C$

$\frac{d}{dx} x\sin x + \frac{d}{dx}\cos x + \frac{d}{dx} C =$

$= x\cos x + \sin x + (-\sin x) + 0$

$= x\cos x$

(b) $\int \frac{\sin x}{\cos^2 x}\,dx = \sec x + C$

$\frac{d}{dx}\sec x + \frac{d}{dx} C =$

$= \sec x \tan x + 0$

$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$

ex: If $\frac{dy}{dx} = 2x - 1$, what could $y = f(x)$ be?

$\int (2x-1)\,dx = \frac{2x^2}{2} - x + C = x^2 - x + C$

$\therefore y = x^2 - x + C$

If $y(1) = 0$, what is $f(x)$?

$y = x^2 - x + C$

sub $(1, 0)$

$0 = (1)^2 - (1) + C$

$\boxed{C = 0}$

$\therefore y = x^2 - x$

U-Substitution

Recall: Differential (dy)

$$\frac{dy}{dx} = f'(x) \Rightarrow \boxed{dy = f'(x)\,dx}, \text{ if } y = f(x)$$

$\therefore$ if $u = f(x)$
$du = f'(x)\,dx$

Rule: If $u = g(x)$ is a differentiable fcn then:

$$\int \underbrace{f(g(x))}_{f(u)}\,\underbrace{g'(x)\,dx}_{du} = \int f(u)\,du$$

$u = g(x)$
$du = g'(x)\,dx$

ex: $\int 2x\sqrt{1+x^2}\ dx =$

$= \int \underbrace{\sqrt{1+x^2}}_{u} \cdot \underbrace{2x\ dx}_{du}$

$u = 1+x^2$
$du = 2x\ dx$

$= \int \sqrt{u}\ du$

$= \dfrac{u^{3/2}}{3/2} + C$

$= \dfrac{2}{3}(1+x^2)^{3/2}$

ex: $\int x^3 \cos(x^4+2)\,dx =$

$u = x^4+2$
$du = 4x^3\ dx$
$\dfrac{du}{4} = x^3\ dx$

$= \int \cos(u)\ \dfrac{du}{4}$

$= \dfrac{1}{4}\int \cos(u) \cdot du$

$= \dfrac{1}{4}(\sin u) + C$

$= \dfrac{1}{4}\sin(x^4+2) + C$

ex: $\int \dfrac{e^{1/x}}{x^2}\ dx =$

$u = 1/x = x^{-1}$
$du = -1\,x^{-2}\ dx$
$-du = \dfrac{1}{x^2}\ dx$

$= \int e^u \cdot (-du)$

$= -\int e^u\ du$

$= -e^u + C$

$= -e^{1/x} + C$

Choose different u

$\int \dfrac{e^{1/x}}{x^2}\ dx =$

$u = e^{1/x}$
$du = e^{1/x} \cdot (-x^{-2})\ dx$
$-du = \dfrac{e^{1/x}}{x^2}\ dx$

$= -\int du$

$= -u + C$

$= -e^{1/x} + C$

ex: $\int \frac{x}{2x^2+1}\,dx =$

$= \frac{1}{4}\int \frac{1}{u}\,du$

$= \frac{1}{4}\ln|u| + C$

$= 1/4 \ln|2x^2+1| + C$

$u = 2x^2+1$
$du = 4x\,dx$
$\frac{1}{4}du = x\,dx$

* One can NOT have u and x
* Manipulate $u = \ldots$ to transform x into u

ex: $\int \frac{x}{\sqrt{x-4}}\,dx =$

$= \int \frac{x}{\sqrt{u}}\,du$

$= \int \frac{u+4}{\sqrt{u}}\,du$

$= \int (u^{1/2} + 4u^{-1/2})\,du$

$= \frac{u^{3/2}}{3/2} + \frac{4u^{1/2}}{1/2} + C$

$= \frac{2}{3}(x-4)^{3/2} + 8(x-4)^{1/2} + C$

$u = x-4 \Rightarrow x = u+4$
$du = 1\,dx$
$du = dx$

ex: $\int \frac{x^2}{\sqrt{1-x^6}}\,dx =$

$= \int \underbrace{\frac{1}{\sqrt{1-u^2}}}_{\frac{d}{dx}\arcsin(u)}\,\frac{du}{3}$

$= \frac{1}{3}\sin^{-1}u + C$

$= 1/3 \sin^{-1}(x^3) + C$

* I want the derivative of u to be x^2 (in du) $\therefore$ u has to have x^3

$u = x^3$
$du = 3x^2\,dx$
$\frac{1}{3}du = x^2\,dx$

ex: $\int 2x(x^2-1)^4\,dx =$ $\qquad u = x^2-1$, $du = 2x\,dx$

$= \int u^4\,du$

$= \frac{u^5}{5} + C$

$= \frac{(x^2-1)^5}{5} + C$

ex: $\int \left(1+\frac{1}{x}\right)^3\left(\frac{1}{x^2}\right)dx =$ $\qquad u = 1+\frac{1}{x}$, $du = -1\cdot\frac{1}{x^2}\,dx$, $-du = \frac{1}{x^2}\,dx$

$= -\int u^3\,du$

$= -\frac{u^4}{4} + C$

$= -\frac{1}{4}\left(1+\frac{1}{x}\right)^4 + C$

ex: $\int \frac{x^2+3x+7}{\sqrt{x}}\,dx =$

$= \int\left(x^{3/2}+3x^{1/2}+7x^{-1/2}\right)dx$

$= x^{5/2}\cdot\left(\frac{2}{5}\right) + 3x^{3/2}\left(\frac{2}{3}\right) + 7x^{1/2}\cdot\left(\frac{2}{1}\right) + C$

ex: $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx =$ $\qquad u = \sqrt{x}$, $du = 1/2\, x^{-1/2}\,dx$, $2\,du = \frac{1}{\sqrt{x}}\,dx$

$= \int \cos u \cdot 2du$

$= 2\int \cos u\,du$

$= 2\sin u + C$

$= 2\sin(\sqrt{x}) + C$

ex: $\int \frac{4}{x^2}\sin\left(\frac{1}{x}\right)dx =$ $\qquad u = \frac{1}{x}$, $du = -\frac{1}{x^2}\,dx$, $-du = \frac{1}{x^2}\,dx$

$= -4\int \sin u\,du$

$= -4(-\cos u) + C$

$= 4\cos\left(\frac{1}{x}\right) + C$

ex: $\int \sin(2x)\cos(2x)\,dx =$

$\boxed{u = \sin(2x)}$

$du = 2\cos(2x)\,dx$

$\frac{1}{2}\,du = \cos(2x)\,dx$

$= \frac{1}{2}\int u\,du$

$= \frac{1}{2}\,\frac{u^2}{2} + C$

$= \frac{\sin^2(2x)}{4} + C$

Or using different u

$\int \sin(2x)\cos(2x)\,dx =$

$\boxed{u = \cos(2x)}$

$du = 2(-\sin(2x))\,dx$

$-\frac{1}{2}\,du = \sin(2x)\,dx$

$= \int u \cdot -\frac{1}{2}\,du$

$= -\frac{1}{2}\int u\,du$

$= -\frac{1}{2}\,\frac{u^2}{2} + C$

$= -\frac{\cos^2(2x)}{4} + C$

ex: $\int x^5\sqrt{3x^2-2}\;dx =$

$u = 3x^2-2 \Rightarrow x^2 = \frac{u-2}{3}$

$du = 6x\;dx$

$\frac{du}{6} = x\;dx$

$= \int x^4\sqrt{3x^2-2}\,x\,dx$

$= \int x^4\sqrt{u}\;\frac{du}{6}$

$= \frac{1}{6}\int\left(\frac{u-2}{3}\right)^2 u^{1/2}\;du$

$= \frac{1}{6}\cdot\frac{1}{9}\int(u^2-4u+4)u^{1/2}\,du$

$= \frac{1}{54}\int(u^{5/2}-4u^{3/2}+4u^{1/2})\,du$

$= \frac{1}{54}\left(\frac{2}{7}\cdot u^{7/2} - 4\cdot\frac{2}{5}u^{5/2} + 4\cdot\frac{2}{3}u^{3/2}\right) + C$

$= \frac{1}{189}\cdot(3x^2-2)^{7/2} - \frac{4}{135}\cdot(3x^2-2)^{5/2} + \frac{4}{81}(3x^2-2)^{3/2} + C$

ex: $\int \frac{(\ln x)^2}{x}\,dx$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$= \int u^2\,du$

$= \frac{u^3}{3} + C$

$= \frac{(\ln x)^3}{3} + C$

ex: $\int \sqrt{\cot x}\;\csc^2 x\;dx =$

$u = \cot x$

$du = -\csc^2 x\;dx$

$-du = \csc^2 x\;dx$

$= -\int\sqrt{u}\;du$

$= -\,u^{3/2}\cdot\frac{2}{3} + C$

$= -\frac{2}{3}(\cot x)^{3/2} + C$

ex: $\int e^{\tan 2x} \sec^2 2x \; dx$

$u = \tan(2x)$

$du = 2 \cdot \sec^2(2x)\, dx$

$\frac{1}{2} du = \sec^2(2x)\, dx$

$= \frac{1}{2} \int e^u \, du$

$= \frac{1}{2} e^u + C$

$= \frac{e^{\tan 2x}}{2} + C$

ex: $\int 3^{2x} \cot(3^{2x})\, dx$

$u = 3^{2x}$

$du = 3^{2x} \cdot \ln(3) \cdot (2)\, dx$

$\frac{du}{2 \cdot \ln 3} = 3^{2x}\, dx$

$= \frac{1}{2\ln 3} \cdot \int \underbrace{\cot(u)\, du}_{\cot(u) = \frac{\cos(u)}{\sin(u)}}$

$= \frac{1}{2\ln 3} \int \frac{\cos(u)}{\sin(u)}\, du$

$w = \sin(u)$

$dw = \cos(u)\, du$

* Always use denom.

$= \frac{1}{2\ln 3} \int \frac{1}{w}\, dw$

$= \frac{1}{2\ln 3} \cdot \ln|w| + C$

$= \frac{1}{2\ln 3} \cdot \ln|\sin(3^{2x})| + C$

ex: $\int \frac{5 - e^x}{e^{2x}}\, dx$

$= \int \frac{5}{e^{2x}}\, dx - \int \frac{e^x}{e^{2x}}\, dx$

$= \int \frac{5}{e^{2x}}\, dx - \int e^{-x}\, dx$

$= \int 5e^{-2x}\, dx - \int e^{-x}\, dx$

$u = -2x$	$w = -x$
$du = -2dx$	$dw = -dx$
$-\frac{1}{2} du = dx$	$-dw = dx$

$= -5 \cdot \frac{1}{2} \int e^u\, du - - \int e^w\, dw$

$= -\frac{5}{2} e^u + e^w + C$

$= e^{-x} - \frac{5}{2} e^{-2x} + C$

Integration by parts (IBP)

$$\boxed{\int f(x) \cdot \frac{d}{dx} g(x) \; dx = f(x) \cdot g(x) - \int g(x) \cdot \frac{d}{dx} f(x) \; dx}$$

Let $u = f(x)$, $v = g(x)$ $\quad \therefore \boxed{\int u \, dv = uv - \int v . du}$

Proof (from product rule):

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = \frac{d}{dx} f(x) \cdot g(x) + \frac{d}{dx} g(x) \cdot f(x)$$

$\therefore f(x) . g(x)$ is the antiderivative of:

$$\frac{d f(x)}{dx} \cdot g(x) + \frac{d \, g(x)}{dx} \cdot f(x)$$

or $\displaystyle \int \left(\frac{df(x)}{dx} \cdot g(x) + \frac{dg(x)}{dx} \cdot f(x) \right) dx = f(x) \cdot g(x)$

$$\int \frac{df(x)}{dx} \cdot g(x) \, dx + \int \frac{dg(x)}{dx} \cdot f(x) \, dx = f(x) \cdot g(x)$$

$$\therefore \boxed{\int f(x) \cdot \frac{dg(x)}{dx} \cdot dx = f(x) \, g(x) - \int g(x) \cdot \frac{df(x)}{dx} \cdot dx}$$

Let $u = f(x)$ | $v = g(x)$

$du = f'(x) \, dx = \frac{df(x)}{dx} \cdot dx$ | $dv = g'(x) \, dx = \frac{dg(x)}{dx} \cdot dx$

$$\int u . dv = u . v - \int v . du$$

Order to chose "u" :

- **L** ogarithmic fcn
- **I** nverse trig fcn
- **A** lgebreic fcn
- **T** rigonometric fcn
- **E** xponential fcn

LIATE order of priority

Match Left-hand side of formula:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

ex: $\int x \sin x \; dx$

$$\begin{bmatrix} u = x & dv = \sin x\,dx \\ du = dx & v = -\cos x \end{bmatrix}$$

$= -x\cos x - \int -\cos x\,dx$

$= -x\cos x - (-\sin x) + C$

$= -x\cos x + \sin x + C$

ex: $\int \frac{\ln x}{x^2}\,dx$

$u = \ln x \qquad dv = x^{-2}\,dx$

$du = \frac{1}{x}\,dx \qquad v = -\frac{1}{x}$

$= -\frac{\ln x}{x} - \int -x^{-2}\,dx$

$= -\frac{\ln x}{x} + \int x^{-2}\,dx$

$= -\frac{\ln x}{x} - x^{-1} + C$

ex: $\int (x^2+1)e^{2x}\,dx$

$$\begin{bmatrix} u = x^2+1 & dv = e^{2x}\,dx \\ du = 2x\,dx & v = \frac{1}{2}e^{2x} \end{bmatrix}$$ IBP 1

$= (x^2+1)\frac{1}{2}e^{2x} - \frac{2}{2}\underbrace{\int xe^{2x}\,dx}$

$$\begin{bmatrix} u = x & dv = e^{2x}\,dx \\ du = dx & v = \frac{1}{2}e^{2x} \end{bmatrix}$$ IBP 2

$= \frac{1}{2}e^{2x}(x^2+1) - \left[\frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx\right]$

$= \frac{1}{2}e^{2x}(x^2+1) - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$

Boomerang!

→ If $I = f(x) - I$ $\therefore$ $2I = f(x)$

$\therefore$ $I = f(x)/2$

ex: $\int e^x \sin x \, dx = I$

IBP 1 $\left[\begin{array}{ll} u = \sin x & dv = e^x dx \\ du = \cos x \, dx & v = e^x \end{array} \right.$

$= e^x \sin x - \underbrace{\int e^x \cos x \, dx}$

IBP 2 $\left[\begin{array}{ll} u = \cos x & dv = e^x dx \\ du = -\sin x \, dx & v = e^x \end{array} \right.$

$= e^x \sin x - \left[e^x \cos x - - \int e^x \sin x \, dx \right]$

$= \underbrace{e^x \sin x - e^x \cos x}_{f(x)} - \underbrace{\int e^x \sin x \, dx}_{I}$

$I = f(x) - I$

$2I = f(x)$

$I = \frac{1}{2} f(x)$

$\therefore \int e^x \sin x \, dx = \frac{1}{2} \left(e^x \sin x - e^x \cos x \right) + C$

Proof of $\int \ln x \, dx$ (IBP)

$$\int \ln x \, dx$$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = x$

$$= x \ln x - \int \frac{x}{x} dx$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + C$$

$$= x(\ln x - 1) + C$$

$$\therefore \boxed{\int \ln x \, dx = x(\ln x - 1) + C}$$

Proof of $\int \tan^{-1} x \, dx$ (All inverse trig) (IBP)

$$\int \tan^{-1} x \, dx$$

$\left[\begin{array}{ll} u = \tan^{-1} x & dv = dx \\ du = \frac{1}{1+x^2} dx & v = x \end{array} \right]$

$$= x \tan^{-1} x - \int \underbrace{\frac{x}{1+x^2}}_{\text{sub}} dx$$

$u = 1 + x^2$

$du = 2x \, dx$

$\frac{1}{2} du = x \, dx$

$$= x \tan^{-1} x - \frac{1}{2} \int \frac{1}{u} du$$

$$= x \tan^{-1} x - \frac{1}{2} \ln |u| + C$$

$$= x \tan^{-1} x - \frac{1}{2} \ln |x^2 + 1| + C$$

Definite Integral and u-sub

* Adjust limits of integration with respect to x to limits of integration with respect to u

⇒ Don't need to go back to x

ex: $\int_0^1 x e^{-x^2} dx =$

$u = -x^2$

$du = -2x\,dx$

$-\frac{1}{2} du = x\,dx$

* if $x=0$, $u=0$
  $x=1$, $u=-1$

$= -\frac{1}{2}\int_0^{-1} e^u du$

$= -1/2 \cdot e^u \Big|_0^{-1}$

$= -1/2\left[e^{-1} - e^0\right]$

$= -\frac{1}{2e} + \frac{1}{2}$

* Note: No +C b/c definite integral

Definite Integral and IBP

$$\boxed{\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du}$$

ex: $\int_0^1 x 5^x dx =$

$u = x$ $\qquad dv = 5^x dx$

$du = dx$ $\qquad v = \frac{5^x}{\ln 5}$

$= \frac{x5^x}{\ln 5}\Big|_0^1 - \int_0^1 \frac{5^x}{\ln 5} dx$

$= \frac{1}{\ln 5} \cdot x5^x \Big|_0^1 - \frac{1}{\ln 5}\int_0^1 5^x dx$

$= \frac{1}{\ln 5} x5^x \Big|_0^1 - \frac{1}{\ln 5}\left(\frac{5^x}{\ln 5}\right)\Big|_0^1$

$= \frac{1}{\ln 5}\left(5^{(1)} - 0\right) - \frac{1}{\ln 5}\left(\frac{5^1}{\ln 5} - \frac{5^0}{\ln 5}\right)$

$= \frac{5}{\ln 5} - \frac{5-1}{(\ln 5)^2}$

$= \frac{5}{\ln 5} - \frac{4}{(\ln 5)^2}$

ex: $\int_0^9 e^{\sqrt{x}}\,dx$

$u = \sqrt{x}$

$du = 1/2 \cdot x^{-1/2}\,dx$

isolate dx

$2\,du \cdot \sqrt{x} = dx$

sub $\sqrt{x} = u$

$dx = 2u\,du$

If $x = 0$, $u = 0$
$x = 9$, $u = 3$

$= \int_0^3 e^u \cdot 2u\,du$

$= 2\int_0^3 ue^u\,du$

$$\boxed{\begin{array}{ll} w = u & dv = e^u du \\ dw = du & v = e^u \end{array}}$$

$= 2\left(ue^u\Big|_0^3 - \int_0^3 e^u\,du \right)$

$= 2\left(ue^u\Big|_0^3 - e^u\Big|_0^3 \right)$

$= 2\left(3e^3 - 0 - [e^3 - e^0]\right)$

$= 2(3e^3 - e^3 + 1)$

$= 2(2e^3 + 1)$

ex: $\int \frac{\ln\sqrt{x}}{\sqrt{x}}\,dx$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2\,du = \frac{1}{\sqrt{x}}\,dx$

$= 2\int \ln(u)\,du$

$= 2\left[u\ln(u) - u\right] + C$

$= 2\left(\sqrt{x}\ln\sqrt{x} - \sqrt{x}\right) + C$

ex: $\int \frac{\sin 4x}{\tan 4x}\,dx$

$= \int \sin 4x \cdot \frac{\cos 4x}{\sin 4x}\,dx$

$= \int \cos(4x)\,dx$

$u = 4x$
$du = 4\,dx$
$\frac{1}{4}du = dx$

$= \frac{1}{4}\int \cos(u)\,du$

$= 1/4 \cdot \sin(u) + C$

$= 1/4 \ \sin(4x) + C$

ex: $\int \sin(\ln x)\,dx$

IBP 1 $\left[\begin{array}{ll} u = \sin(\ln x) & dv = dx \\ du = \cos(\ln x)\cdot\frac{1}{x}\,dx & v = x \end{array}\right]$

$= x\sin(\ln x) - \int \cos(\ln x)\cdot\frac{x}{x}\,dx$

$= x\sin(\ln x) - \int \cos(\ln x)\,dx$

IBP 2 $\left[\begin{array}{ll} u = \cos(\ln x) & dv = dx \\ du = -\sin(\ln x)\cdot\frac{1}{x}\,dx & v = x \end{array}\right]$

$= x\sin(\ln x) - \left[x\cos(\ln x) - -\int \sin(\ln x)\cdot\frac{x}{x}\,dx\right]$

$= \underbrace{x\sin(\ln x) - x\cos(\ln x)}_{f(x)} - \underbrace{\int \sin(\ln x)\,dx}_{I}$

$I = f(x) - I$
$2I = f(x)$
$I = f(x)\cdot\frac{1}{2}$

$\therefore \int \sin(\ln x)\,dx = \frac{x\sin(\ln x) - x\cos(\ln x)}{2} + C$

IBP: Reduction Formulas

* Repeated IBP

$$\boxed{\int \sin^n x\,dx = -\frac{1}{n}\cos x.\sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx}$$

Derivation:

$$\int \sin^n x\,dx = I$$

$$= \int \sin^{n-1} x.\sin x\,dx$$

$\Big[\, u = \sin^{n-1} x \qquad dv = \sin x\,dx$

$\;\; du = (n-1)\sin^{n-2} x.\cos x\,dx \qquad v = -\cos x$

$$= -\sin^{n-1} x.\cos x - -(n-1)\int \sin^{n-2} x.\underbrace{\cos^2 x}\,dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\left(\int \sin^{n-2} x\,dx - \int \sin^n x\,dx\right)$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\underbrace{\int \sin^n x\,dx}_{I}$$

$I = f(x) - (n-1)I$

$I + (n-1)I = f(x)$

$n.I = f(x)$

$I = \frac{1}{n}.f(x)$

$$\therefore \int \sin^n x\,dx = -\frac{1}{n}\cos x\sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} dx$$

ex: evaluate at $n=4$

$$\int \sin^4 x\,dx = \overset{n=4}{-\frac{1}{4}\cos x\sin^3 x} + \frac{3}{4}\left[\overset{n=2}{-\frac{1}{2}\cos x\sin^1 x} + \frac{1}{2}\int \cancel{\sin^0}\,x\,dx\right]$$

$$= -\frac{1}{4}\cos x\sin^3 x + \frac{3}{4}\left(-\frac{1}{2}\cos x\sin x + \frac{1}{2}(x)\right) + C$$

$$\boxed{\int (\ln x)^n \, dx = x(\ln x)^n - n\int (\ln x)^{n-1}\, dx}$$

Derivation:

$$\int (\ln x)^n \, dx = I \qquad \left[\begin{array}{ll} u = (\ln x)^n & dv = dx \\ du = n \cdot (\ln x)^{n-1} \cdot \frac{1}{x}\, dx & v = x \end{array}\right.$$

$$= x(\ln x)^n - n\int (\ln x)^{n-1} \cdot \frac{x}{x}\, dx$$

$$= x(\ln x)^n - n\int (\ln x)^{n-1}\, dx$$

ex: evaluate at $n = 4$

$$\int (\ln x)^4 \, dx =$$

$$= \overset{n=4}{x(\ln x)^4} - 4\Big[\overset{n=3}{x(\ln x)^3} - 3\big[\overset{n=2}{x(\ln x)^2} - 2\big(\overset{n=1}{x(\ln x)} - 1\int \overset{1}{\cancel{(\ln x)^0}}\, dx\big)\big]\Big]$$

$$= x(\ln x)^4 - 4\Big[x(\ln x)^3 - 3\big[x(\ln x)^2 - 2\big(x(\ln x) - x\big)\big]\Big] + C$$

$$\boxed{\int x^n e^x \, dx = x^n e^x - n\int x^{n-1} e^x \, dx}$$

Derivation:

$$\int x^n e^x \, dx \qquad \left[\begin{array}{ll} u = x^n & dv = e^x\, dx \\ du = n \cdot x^{n-1}\, dx & v = e^x \end{array}\right.$$

$$= x^n e^x - n\int x^{n-1} \cdot e^x \, dx$$

Trigonometric Integral

Recall: Trig identity

$$\sin^2 x + \cos^2 x = 1 \Rightarrow \boxed{\begin{aligned} \cos^2 x &= 1 - \sin^2 x \\ \sin^2 x &= 1 - \cos^2 x \end{aligned}}$$ Easy case

$$\cos 2x = \cos^2 x - \sin^2 x \Rightarrow \boxed{\begin{aligned} \cos^2 x &= \tfrac{1}{2}(1 + \cos 2x) \\ \sin^2 x &= \tfrac{1}{2}(1 - \cos 2x) \end{aligned}}$$ Hard case

$$\sin 2x = 2 \sin x \cos x$$

Note: ⊛ $\int \cos(n.x)\,dx = \frac{1}{n}\sin(n.x) + C$ (Hard case)

$$\int \sin(n.x)\,dx = -\frac{1}{n}\cos(n.x) + C$$

Evaluate: $\int \sin^n x \cdot \cos^m x \, dx$ $\quad n = 0, 1, 2, \ldots$ $\quad m = 0, 1, 2, \ldots$

Easy case: m and/or n are odd

$$\boxed{\cos^2 x + \sin^2 x = 1}$$

Hard case: Both m and n are even

$$\boxed{\begin{aligned} \cos^2 x &= 1/2\,(1 + \cos 2x) \\ \sin^2 x &= 1/2\,(1 - \cos 2x) \end{aligned}}$$

ex: $\int \sin^3 x \cos^2 x \, dx$

sin is odd
→ take out one sin
→ work with $u = \cos x$

$= \int \sin^2 x \cdot \cos^2 x \cdot \sin x \, dx$

$= \int (1-\cos^2 x) \cos^2 x \sin x \, dx$

$u = \cos x$
$du = -\sin x \, dx$
$-du = \sin x \, dx$

$= -\int (1-u^2) u^2 \, du$

$= -\int (u^2 - u^4) \, du$

$= -\left(\frac{u^3}{3} - \frac{u^5}{5} \right) + C$

$= -\left(\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5} \right) + C$

ex: $\int_0^{\pi/2} \cos^5 x \, dx$

$= \int_0^{\frac{\pi}{2}} (\cos^2 x)^2 \cdot \cos x \, dx$

$= \int_0^{\frac{\pi}{2}} (1-\sin^2 x)^2 \cos x \, dx$

$u = \sin x$
$du = \cos x \, dx$
if $x = 0$, $u = 0$
$x = \frac{\pi}{2}$, $u = 1$

$= \int_0^1 (1-u^2)^2 \, du$

$= \int_0^1 (1 - 2u^2 + u^4) \, du$

$= \left(u - 2\frac{u^3}{3} + \frac{u^5}{5} \right) \Big|_0^1 = \frac{8}{15}$

ex: $\displaystyle\int_0^{\pi} \underbrace{\sin^2 x}\, dx$ * Hard case

$$= \int_0^{\pi} \tfrac{1}{2}(1-\cos 2x)\,dx$$

$$= \tfrac{1}{2}\int_0^{\pi} (1-\cos 2x)\,dx$$

$$= \tfrac{1}{2}\left(x - \tfrac{1}{2}\sin 2x\right)\Big|_0^{\pi}$$

$$= \tfrac{1}{2}\left(\pi - \tfrac{1}{2}\cancelto{0}{\sin(2\pi)} - \left(0 - \tfrac{1}{2}\cancelto{0}{\sin(0)}\right)\right)$$

$$= \pi/2$$

ex: $\displaystyle\int \underbrace{\sin^4 x}\, dx$ $\sin^2 x = \tfrac{1}{2}(1-\cos 2x)$

$$= \int \left(\tfrac{1}{2}(1-\cos 2x)\right)^2 dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos(2x) + \underbrace{\cos^2 2x}\right) dx$$ $\cos^2 x = \tfrac{1}{2}(1+\cos 2x)$

$$= \frac{1}{4}\int \left(1 - 2\cos(2x) + \tfrac{1}{2}(1+\cos 4x)\right) dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos(2x) + \tfrac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x - 2\cdot\frac{1}{2}\sin(2x) + \frac{1}{2}\cdot\frac{1}{4}\sin(4x)\right] + C$$

ex: $\displaystyle\int \frac{\sin^2\sqrt{x}\ \cos^3\sqrt{x}}{\sqrt{x}}\,dx$

$u = \sqrt{x}$
$du = \frac{1}{2\sqrt{x}}\,dx$
$2\,du = \frac{1}{\sqrt{x}}\,dx$

$= 2\int \sin^2 u \cos^3 u \; du$

$= 2\int \sin^2 u \,.\, \cos^2 u \,.\, \cos u \, du$

$= 2\int \sin^2 u\,(1-\sin^2 u)\cos u\, du$

$w = \sin u$
$dw = \cos u\, du$

$= 2\int w^2(1-w^2)\,dw$

$= 2\int (w^2 - w^4)\,dw$

$= 2\left(\frac{w^3}{3} - \frac{w^5}{5}\right) + C$

$= 2\left(\frac{\sin^3\sqrt{x}}{3} - \frac{\sin^5\sqrt{x}}{5}\right) + C$

ex: $\int \sin^2(\pi x)\cos^2(\pi x)\,dx$

* $u = \pi x$
$du = \pi\, dx$
$\frac{1}{\pi}\,du = dx$

Hard case

$= \frac{1}{\pi}\int \underbrace{\sin^2 u}\,.\,\underbrace{\cos^2 u}\, du$

$= \frac{1}{\pi}\int \underbrace{\tfrac{1}{2}(1-\cos 2u)}_{a-b}\,.\,\underbrace{\tfrac{1}{2}(1+\cos 2u)}_{a+b}\,du$

$= \frac{1}{4\pi}\int (1-\cos^2 2u)\,du$

$= \frac{1}{4\pi}\int \left(1 - \tfrac{1}{2}(1+\cos 4u)\right)du$

$= \frac{1}{4\pi}\int \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 4u\right)du$

$= \frac{1}{4\pi}\left(\tfrac{1}{2}u - \tfrac{1}{2}\cdot\tfrac{1}{4}\sin 4u\right) + C$

$= \frac{1}{8\pi}u - \frac{1}{32\pi}\sin 4u + C$

$= \frac{1}{8\pi}\cdot\pi x - \frac{1}{32\pi}\sin(4\pi x) + C$

Recall: ⊛ $\tan^2 x + 1 = \sec^2 x$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

Evaluate: $\int \sec^n x \tan^m x \, dx$

Easy case: ① n is even ($\sec x$ even)

(i) set aside $\sec^2 x$

(ii) convert $\sec x \to \tan x$ ⊛

(iii) $u = \tan x$

$du = \sec^2 x \, dx$

② m is odd ($\tan x$ odd)

(i) set aside $\sec x \tan x$

(ii) convert $\tan x \to \sec x$ ⊛

(iii) $u = \sec x$

$du = \sec x \tan x \, dx$

ex: $\int \tan^4 x \sec^{4} x \, dx$

$= \int \tan^4 x \cdot \sec^2 x \cdot \sec^2 x \, dx$

$= \int \tan^4 x \, (\tan^2 x + 1) \sec^2 x \, dx$ $\qquad u = \tan x$

$du = \sec^2 x \, dx$

$= \int u^4 (u^2+1) \, du$

$= \int (u^6 + u^4) \, du$

$= \frac{u^7}{7} + \frac{u^5}{5} + C$

$= \frac{\tan^7 x}{7} + \frac{\tan^5 x}{5} + C$

ex: $\int \tan^{5} x \sec^7 x \, dx$

$= \int (\sec^2 x - 1)^2 \sec^6 x \cdot \sec x \tan x \, dx$ $\qquad u = \sec x$

$du = \sec x \tan x \, dx$

$= \int (u^2-1)^2 u^6 \, du$

$= \int (u^4 - 2u^2 + 1) u^6 \, du$

$= \int (u^{10} - 2u^8 + u^6) \, du$

$= \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} + C$

$= \frac{\sec^{11} x}{11} - \frac{2\sec^9 x}{9} + \frac{\sec^7 x}{7} + C$

ex: $\int \tan^{3} x \sec^3 x \, dx$

$= \int (\sec^2 x - 1) \sec^2 x \cdot \sec x \tan x \, dx$

$u = \sec x$

$du = \sec x \tan x \, dx$

$= \int (u^2 - 1) u^2 \, du$

$= \int (u^4 - u^2) \, du$

$= \frac{u^5}{5} - \frac{u^3}{3} + C$

$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$

Hard Cases

→ Try to get to an easy case

Use:

① $\boxed{\int \tan x \, dx = \ln|\sec x| + C}$

proof: $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$

$= \int \frac{\sin x}{\cos x} dx$ $\quad u = \cos x$, $du = -\sin x \, dx$, $-du = \sin x \, dx$

$= -\int \frac{1}{u} du$

$= -\ln|u| + C$

$= -\ln|\cos x| + C$ log property

$= \ln|\cos^{-1}| + C = \ln|\sec x| + C$

* Any time num. and denom are derivatives of eachother, use $u =$ "denominator"

② $\boxed{\int \sec x \, dx = \ln|\sec x + \tan x| + C}$

proof: $\int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx$

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$ $\quad u = \sec x + \tan x$, $du = (\sec x \tan x + \sec^2 x) \, d$

$= \int \frac{1}{u} du$

$= \ln|\sec x + \tan x| + C$

Odd tan, no sec   not same easy case

ex: $\int \tan^{3} x \, dx$

$= \int \tan^2 x \cdot \tan x \, dx$

$= \int (\sec^2 x - 1) \tan x \, dx$

$= \int (\sec^2 x \tan x - \tan x) \, dx$

$= \underbrace{\int \sec^{2} x \tan x \, dx}_{EC} - \underbrace{\int \tan x \, dx}_{\ln|\sec x|}$

$u = \tan x$
$du = \sec^2 x \, dx$

$= \int u \, du - \ln|\sec x|$

$= \frac{u^2}{2} - \ln|\sec x| + C$

$= \frac{\tan^2 x}{2} - \ln|\sec x| + C$

Odd sec, no tan

ex: $\int \sec^3 x\, dx = I$

$$\left[\begin{array}{ll} u = \sec x & dv = \sec^2 x\, dx \\ du = \sec x \tan x\, dx & v = \tan x \end{array}\right.$$

$= \sec x \tan x - \int \tan^2 x \sec x\, dx$

$= \sec x \tan x - \int (\sec^2 x - 1) \sec x\, dx$

$= \sec x \tan x - \underbrace{\int \sec^3 x\, dx}_{I} + \underbrace{\int \sec x\, dx}_{\ln|\sec x + \tan x|}$

$I = f(x) - I$

$2I = f(x)$

$I = \frac{1}{2} f(x)$

$\therefore \int \sec^3 x\, dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x| \right) + C$

Even tan, no sec

ex: $\int \tan^6 x \, dx$

$= \int (\sec^2 x - 1) \tan^4 x \, dx$

$= \underbrace{\int \sec^{2} x \tan^4 x \, dx}_{\text{easy}} - \underbrace{\int \tan^4 x \, dx}_{\text{hard}}$

$= \int \tan^4 x \sec^2 x \, dx - \int (\sec^2 x - 1) \tan^2 x \, dx$

$= \int \tan^4 x \sec^2 x \, dx - \int \sec^2 x \tan^2 x + \int \tan^2 x \, dx$

$= \int \tan^4 x \sec^2 x \, dx - \int \tan^2 x \sec^2 x + \int (\sec^2 x - 1) \, dx$

$u = \tan x$
$du = \sec^2 x \, dx$

$= \int u^4 \, du - \int u^2 \, du + \cancel{\int (u^2 - 1)\, du} + \tan x - x$

$= \dfrac{u^5}{5} - \dfrac{u^3}{3} + \tan x - x + C$

$= \dfrac{\tan^5 x}{5} - \dfrac{\tan^3 x}{3} + \tan x - x + C$

Even tan, Odd sec

ex: $\int \tan^4 x \sec^3 x \, dx$

$$= \int (\sec^2 x - 1)^2 \sec^3 x \, dx$$

$$= \int (\sec^4 x - 2\sec^2 x + 1) \sec^3 x \, dx$$

$$= \int (\sec^7 x - 2\sec^5 x + \sec^3 x) \, dx$$

solve as if odd sec, no tan

Even sec → Always easy

Even sec, odd tan → easy

ex: $\int (\tan^2 x + \tan^4 x)\, dx$

$= \int \tan^2 x \, dx + \int \tan^4 x \, dx$

$= \int (\sec^2 x - 1)\, dx + \int (\sec^2 x - 1) \tan^2 x \, dx$

$= \int (\sec^2 x - 1)\, dx + \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx$

$= \int (\sec^2 x - 1)\, dx + \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1)\, dx$

$u = \tan x$
$du = \sec^2 x$

$= (\cancel{\tan x} - x) + \int u^2 \, du - (\cancel{\tan x} - x)$

$= \dfrac{u^3}{3} + C$

$= \frac{1}{3} \tan^3 x + C$

OR

$\int (\tan^2 x + \tan^4 x)\, dx = \int \left(\dfrac{\sin^2 x}{\cos^2 x} + \dfrac{\sin^4 x}{\cos^4 x} \right) dx$

$= \int \dfrac{\sin^2 x . \cos^2 x + \sin^4 x}{\cos^4 x} \, dx$

$= \int \dfrac{\sin^2 x \,(\sin^2 x + \cos^2 x)}{\cos^4 x} \, dx$

$= \int \dfrac{\sin^2 x \, . \, (1)}{\cos^4 x} \, dx$

$= \int \tan^2 x \sec^2 x \, dx$ $\qquad u = \tan x$, $du = \sec^2 x \, dx$

$= \int u^2 \, du$

$= \dfrac{u^3}{3} + C$

$= \dfrac{\tan^3 x}{3} + C$

ex: $\int \cos^2 x \tan^3 x \, dx =$

$$= \int \cos^2 x \cdot \frac{\sin^3 x}{\cos^3 x} \, dx$$

$$= \int \frac{\sin^3 x}{\cos x} \, dx$$

odd sin → put sin aside

$$= \int \frac{(1-\cos^2 x) \cdot \sin x}{\cos x} \, dx$$

$u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$$= -\int \frac{(1-u^2)}{u} \, du$$

$$= -\int \left(\frac{1}{u} - u\right) du$$

$$= -\left(\ln|u| - \frac{u^2}{2}\right) + C = -\left(\ln|\cos x| - \frac{\cos^2 x}{2}\right) + C$$

ex: $\int \frac{\sin^2 x}{\cos x} \, dx$

odd cos → put cos aside

$$= \int \frac{\sin^2 x}{\cos x} \cdot \frac{\cos x}{\cos x} \, dx$$

$$= \int \frac{\sin^2 x \cdot \cos x}{\cos^2 x} \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$$= \int \frac{\sin^2 x}{(1-\sin^2 x)} \cos x \, dx$$

$$= \int \frac{u^2}{1-u^2} \cdot du$$

* Rational → long division

$$\begin{array}{r|l} u^2 & 1-u^2 \\ \hline u^2-1 & -1 \\ \hline 1 & \end{array}$$

$= -1 + \frac{1}{1-u^2}$

$$= \int \left(-1 + \frac{1}{1-u^2}\right) du$$

$$= -u + \int \frac{1}{1-u^2} \, du$$

$$= -\sin x + \int \frac{1}{1-u^2} \, du$$

see partial fractions

ex: $\int \frac{\tan^3 x}{\cos^4 x}\, dx$

$= \int \frac{\sin^3 x}{\cos^7 x}\, dx$

$= \int \frac{\sin^2 x}{\cos^7 x} \cdot \sin x \, dx$

$= \int \frac{(1-\cos^2 x)}{\cos^7 x} \sin x \, dx$

$u = \cos x$

$du = -\sin x\, dx$

$-du = \sin x\, dx$

$= -\int \frac{1-u^2}{u^7}\, du$

$= -\int (u^{-7} - u^{-5})\, du$

$= -\left(\frac{u^{-6}}{-6} - \frac{u^{-4}}{-4}\right) + C$

$= -\left(-\frac{1}{6\cos^6 x} + \frac{1}{4\cos^4 x}\right) + C$

ex: $\int \frac{1-\tan^2 x}{\sec^2 x}\, dx$

* split up

$= \int \frac{1}{\sec^2 x}\, dx - \int \frac{\tan^2 x}{\sec^2 x}\, dx$

$= \int \cos^2 x \, dx - \int \frac{\sin^2 x}{\cancel{\cos^2 x}} \cdot \cancel{\cos^2 x}\, dx$

$= \int \frac{1}{2}(1+\cos 2x)\, dx - \int \frac{1}{2}(1-\cos 2x)$

$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) - \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$

$= 1/2 \sin 2x + C$

Trigonometric Sub

Trig Identity to use

Case 1 : $\sqrt{a^2 - x^2}$ — $\cos^2 x = 1 - \sin^2 x$

$$\boxed{x = a \sin \theta}$$

Case 2 : $\sqrt{a^2 + x^2}$ — $\sec^2 x = 1 + \tan^2 x$

$$\boxed{x = a \tan \theta}$$

Case 3 : $\sqrt{x^2 - a^2}$ — $\tan^2 x = \sec^2 x - 1$

$$\boxed{x = a \sec \theta}$$

Case 1

ex: $\int \sqrt{16-x^2}\,dx$

$$= \int \underbrace{4\cos\theta}\,\underbrace{(4\cos\theta\ d\theta)}_{dx}$$

$$= 16\int \cos^2\theta\ d\theta$$

$$= 16\int \tfrac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= 8\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C$$

use $\boxed{\sin 2\theta = 2\sin\theta\cos\theta}$

$$= 8\left(\theta + \cancel{\tfrac{1}{2}}\cdot\cancel{2}\sin\theta\cos\theta\right) + C$$

$1 - \sin^2 x = \cos^2 x$

$a = 4$

Let $x = a\sin\theta$

$x = 4\sin\theta$

$dx = 4\cos\theta\ d\theta$

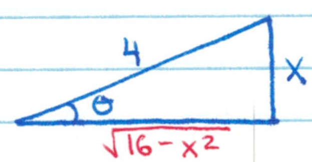
$$\sqrt{16-x^2} = \sqrt{16-(4\sin\theta)^2}$$
$$= \sqrt{16-16\sin^2\theta}$$
$$= \sqrt{16(1-\sin^2\theta)}$$
$$= \sqrt{16\cos^2\theta}$$
$$= 4\cdot|\cos\theta|$$

* No need Abs. value

$\boxed{x = 4\sin\theta}$

① $\sin\theta = \frac{x}{4}$

② $\theta = \arcsin\left(\frac{x}{4}\right)$



Note: same as original

③ $\cos\theta = \dfrac{\sqrt{16-x^2}}{4}$

$$= 8\left(\sin^{-1}\left(\frac{x}{4}\right) + \left(\frac{x}{4}\right)\cdot\frac{\sqrt{16-x^2}}{4}\right) + C$$

Case 2

ex: $\displaystyle\int \frac{dx}{(x^2+9)^{3/2}}$

$= \displaystyle\int \frac{dx}{(\sqrt{x^2+9})^3}$

$= \displaystyle\int \frac{3\sec^2\theta}{(3\sec\theta)^3}\, d\theta$

$= \displaystyle\frac{1}{3}\int \frac{1}{\sec\theta}\, d\theta$

$= \displaystyle 1/3 \int \cos\theta \, d\theta$

$= 1/3 \sin\theta \quad + C$

$1 + \tan^2 x = \sec^2 x$

$a = 3$

$x = a\tan\theta$

$x = 3\tan\theta$

$dx = 3\sec^2\theta \, d\theta$

$\sqrt{x^2+9} = \sqrt{(3\tan\theta)^2+9}$

$= \sqrt{9\tan^2\theta + 9}$

$= \sqrt{9(\tan^2\theta+1)}$

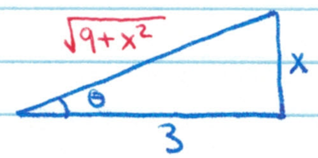
$= \sqrt{9\sec^2\theta}$

$= 3\sec\theta$

$\boxed{x = 3\tan\theta}$

$\tan\theta = \dfrac{x}{3}$

$\theta = \arctan\left(\dfrac{x}{3}\right)$



① $\sin\theta = \dfrac{x}{\sqrt{9+x^2}}$

$= 1/3\left(\dfrac{x}{\sqrt{9+x^2}}\right) + C$

Case 3

ex: $\int \frac{dx}{\sqrt{x^2-16}}$

$\sec^2 x - 1 = \tan^2 x$

$a = 4$

$x = a\sec\theta$

$x = 4\sec\theta$

$dx = 4\sec\theta\tan\theta\, d\theta$

$\sqrt{x^2-16} = \sqrt{(4\sec\theta)^2-16}$

$= \sqrt{16\sec^2\theta - 16}$

$= \sqrt{16\tan^2\theta}$

$= 4\tan\theta$

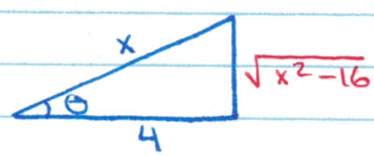
$= \int \frac{4\sec\theta\tan\theta}{4\tan\theta}\, d\theta$

$= \int \sec\theta\, d\theta$

$= \ln|\sec\theta + \tan\theta| + C$

$\boxed{x = 4\sec\theta}$

① $\sec\theta = \frac{x}{4} = \frac{hyp}{adj}$



② $\tan\theta = \frac{\sqrt{x^2-16}}{4}$

$= \ln\left|\left(\frac{x}{4}\right) + \frac{\sqrt{x^2-16}}{4}\right| + C$

ex: $\int \frac{\sqrt{1+x^2}}{x}\,dx$

$1 + \tan^2 x = \sec^2 x$

$a = 1$

$x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$\sqrt{1+x^2} = \sqrt{1+(\tan\theta)^2}$

$= \sqrt{\sec^2\theta}$

$= \sec\theta$

$$= \int \frac{\sec\theta \cdot \sec^2\theta \, d\theta}{\tan\theta}$$

$$= \int \frac{\sec^3\theta}{\tan\theta}\, d\theta$$

$$= \int \frac{\sec^2\theta \cdot \sec\theta \tan\theta \, d\theta}{\tan^2\theta}$$

$$= \int \frac{\sec^2\theta}{(\sec^2\theta - 1)} \cdot \sec\theta \tan\theta \, d\theta$$

$u = \sec\theta$

$du = \sec\theta \tan\theta \, d\theta$

$$= \int \frac{u^2}{u^2-1}\, du$$

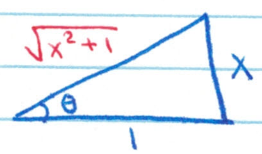
$$= \int 1\,du + \underbrace{\int \frac{1}{u^2-1}}_{\text{partial fractions}} du$$

$$\begin{array}{r|l} u^2 & u^2-1 \\ \hline \underline{u^2-1} & 1 \\ 1 & \end{array}$$

$= 1 + \frac{1}{u^2-1}$

$u = \sec\theta$

$\boxed{\tan\theta = x}$



$= \sqrt{x^2+1}$

$$= \sqrt{1+x^2} + \underbrace{\int \frac{1}{u^2-1}\, du}_{\text{partial fraction}}$$

Trinomial in Radical

→ Complete the square

$ax^2 + bx + c \qquad \pm \left(\tfrac{1}{2} b\right)^2$

→ make coefficient of x^2 1

ex: $\displaystyle\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$

complete the square:

$$-x^2 - 2x + 3 = -(\underbrace{x^2 + 2x + 1} - 1) + 3$$
$$= -\left[(x+1)^2 - 1\right] + 3$$
$$= -(x+1)^2 + 4$$

$$= \int \frac{x}{\sqrt{4-(x+1)^2}}\,dx$$

$u = x+1 \quad \Rightarrow x = u - 1$

$du = dx$

$$= \int \frac{u-1}{\sqrt{4-u^2}}\,du$$

$1 - \sin^2\theta = \cos^2\theta$

$a = 2$

$u = 2\sin\theta$

$du = 2\cos\theta\, d\theta$

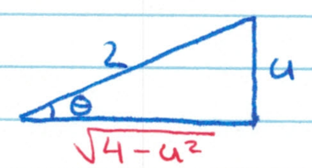
$$\sqrt{4-u^2} = \sqrt{4-(2\sin\theta)^2}$$
$$= \sqrt{4(1-\sin^2\theta)}$$
$$= \sqrt{4\cos^2\theta}$$
$$= 2\cos\theta$$

$$= \int \frac{(2\sin\theta - 1)(\cancel{2\cos\theta}\, d\theta)}{\cancel{2\cos\theta}}$$

$$= (-2\cos\theta - \theta) + C$$

$\boxed{u = 2\sin\theta}$

$\sin\theta = \dfrac{u}{2}$



① $\theta = \arcsin\left(\frac{u}{2}\right)$

② $\cos\theta = \dfrac{\sqrt{4-u^2}}{2}$

$$= -2\left(\frac{\sqrt{4-u^2}}{2}\right) - \sin^{-1}\left(\frac{u}{2}\right) + C$$

$$= -\sqrt{4-(x+1)^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C$$

Partial Fraction Decomposition

Good to integrate $\underbrace{\text{rational function}}_{\text{Ratio of 2 polynomials}}$!

$$\int \frac{P(x)}{Q(x)}\,dx$$

⇒ Seperate a "hard to integrate" fcn into smaller "easier" partial fractions.

CASE 1

The denominator $Q(x)$ factors into linear terms, none of which repeat.

→ each linear term seperated into fraction

ex: $$\frac{3x^3+x^2-7}{(x+1)(x-3)(x+4)} = \frac{A}{(x+1)} + \frac{B}{(x-3)} + \frac{C}{(x+4)}$$

CASE 2

The denominator $Q(x)$ factors into linear terms, but some repeat.

→ if linear term is repeated (deg), it gets that many fractions (Note: one for each degree)

ex: $$\frac{x^3-x+1}{x^2(x-1)^3} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{(x-1)^3} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)}$$

* Note: Always start with long division if possible:

degree (num) $\geq$ degree (denom)

Q. How to find A and B.

ex: $$\frac{8x+29}{\underbrace{(x+7)(x-2)}_{\text{LCD}}} = \frac{A}{(x+7)} + \frac{B}{(x-2)}$$

Multiply both sides by LCD

$$8x+29 = \left[\frac{A}{(x+7)} + \frac{B}{(x-2)}\right](x+7)(x-2)$$

$$8x+29 = A(x-2) + B(x+7)$$

①

$$8x+29 = A\underbrace{(x-2)}_{x=2} + B\underbrace{(x+7)}_{x=-7}$$

Let:

$\boxed{x=2}$ $8(2)+29 = 0 + B(2+7)$

$45 = 9 \cdot B$

$\boxed{B=5}$

$\boxed{x=-7}$ $8(-7)+29 = A(-7-2)+0$

$-27 = -9 \cdot A$

$\boxed{A=3}$

② : Expand

$$8x+29 = Ax - 2A + Bx + 7B$$

$$8x+29 = (A+B)x + (-2A+7B)$$

coefficient of x must be equal

i $8 = A+B$

ii $29 = -2A+7B$

$A = 8 - B$

sub in (ii)

$29 = -2(8-B) + 7B$

$29 = -16 + 2B + 7B$

$45 = 9B$

$\boxed{B=5} \Rightarrow \boxed{A=3}$

$$\therefore \frac{8x+29}{(x+7)(x-2)} = \frac{3}{x+7} + \frac{5}{x-2}$$

ex: $\int \frac{x^4 - 2x^2 + 4x + 1}{(x-1)^2(x+1)}$ $\frac{\deg 4}{\deg 3} \Rightarrow$ long division

$x^3 - x^2 - x + 1$

$$\begin{array}{r|l} \cancel{x}^4 - 2x^2 + 4x + 1 & x^3 - x^2 - x + 1 \\ -\ \underline{x^4 - x^3 - x^2 + x} & x + 1 \\ x^3 - x^2 + 3x + 1 & \\ -\ \underline{x^3 - x^2 - x + 1} & \\ 4x & \end{array}$$

$$= \int (x+1)\,dx + \underbrace{\int \frac{4x}{(x-1)^2(x+1)}\,dx}_{\text{partial frac}}$$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{(x+1)}$$

$$4x = A(x+1) + B(x-1)(x+1) + C(x-1)^2$$

$\boxed{x=1}$ C, B out — $4(1) = A(1+1)$, $\boxed{A = 2}$

$\boxed{x=-1}$ A, B out — $4(-1) = C(-1-1)^2$, $\boxed{C = -1}$

Pick any other x-value

$\boxed{x=0}$ $0 = A(1) + B(-1) + C$

$0 = 2 - B + (-1)$

$\boxed{B = 1}$

$$= \int (x+1)\,dx + \int \frac{2}{(x-1)^2}\,dx + \int \frac{1}{(x-1)}\,dx + \int \frac{-1}{(x+1)}\,dx$$

$$= \frac{x^2}{2} + x + 2\left(-(x-1)^{-1}\right) + \ln|x-1| - \ln|x+1| + C$$

$$= \frac{x^2}{2} + x - \frac{2}{x-1} + \ln|x-1| - \ln|x+1| + C$$

ex: $\displaystyle\int \frac{x^2+2x-1}{x^3-x}\,dx$

$$x^3 - x = x(x^2-1) = x(x-1)(x+1)$$

$$= \int \frac{x^2+2x-1}{x(x-1)(x+1)}\,dx$$

$$\frac{x^2+2x-1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$x^2+2x-1 = A(x-1)(x+1) + B(x)(x+1) + C(x)(x-1)$$

$\boxed{x=0}$ $\quad -1 = A(-1)(1)$

$$\boxed{A = 1}$$

$\boxed{x=1}$ $\quad 1+2-1 = B(1)(2)$

$$\boxed{B = 1}$$

$\boxed{x=-1}$ $\quad 1-2-1 = C(-1)(-2)$

$$\boxed{C = -1}$$

$$= \int \frac{1}{x}\,dx + \int \frac{1}{x-1}\,dx + \int \frac{-1}{x+1}\,dx$$

$$= \ln|x| + \ln|x-1| - \ln|x+1| + C$$

CASE 3

The denominator $Q(x)$, when factored contains an irreducible quadratic, that is not repeated.

ex: $$\frac{x}{(x-2)^3(x^2+4)} = \underbrace{\frac{A}{(x-2)^3} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)}}_{\text{usual}} + \underbrace{\frac{Dx+E}{x^2+4}}_{*}$$

* Q. How to integrate?

$$\int \frac{Dx+E}{x^2+a^2}\,dx$$

→ split up

$$= \underbrace{D\int \frac{x}{x^2+a^2}\,dx}_{\text{sub: } u = x^2+a^2,\ du = 2x\,dx} + \underbrace{E\int \frac{1}{x^2+a^2}\,dx}_{\text{Trig sub: } x = a\tan\theta}$$

↓

Note: if $a=1$

$$\boxed{\int \frac{1}{x^2+1}\,dx = \arctan(x) + C}$$

CASE 4

The denominator $Q(x)$ contains irreducible quadratics, some of which repeat.

ex: $$\frac{x^4}{(x^2-x+1)(x^2+2)^2} = \frac{Ax+B}{(x^2-x+1)} + \frac{Cx+D}{(x^2+2)^2} + \frac{Ex+F}{(x^2+2)}$$

ex: Killer!

$$\int \frac{1}{x^3-1}\,dx$$

Recall: $a^3-b^3=(a-b)(a^2+ab+b^2)$
$a^3+b^3=(a+b)(a^2-ab+b^2)$

$$=\int \frac{1}{(x-1)(x^2+x+1)}\,dx$$

$$\frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

CASE 3

$$1 = A(x^2+x+1) + (Bx+C)(x-1)$$

$\boxed{x=1}$ $1 = A(3)$

$\boxed{A = 1/3}$

$\boxed{x=0}$ $1 = A(1) + C(0-1)$

$1 - 1/3 = -C$

$\boxed{C = -2/3}$

$\boxed{x=-1}$ $1 = A(1) + (-B+C)(-2)$

any other value

$1 = 1/3 + (-B - 2/3)(-2)$

$-1/3 = -B - 2/3$

$B = -2/3 + 1/3$

$\boxed{B = -1/3}$

$$\therefore \int \frac{1}{(x-1)(x^2+x+1)}\,dx = \frac{1}{3}\int \frac{1}{x-1}\,dx + \int \frac{-\frac{1}{3}x - \frac{2}{3}}{(x^2+x+1)}\,dx$$

Killer! (cont'd)

$$\frac{1}{3}\int \frac{1}{x-1}\,dx + \int \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2+x+1}\,dx$$

$$= \frac{1}{3}\ln|x-1| + \boxed{\int \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2+x+1}\,dx} \quad (*)$$

Trig Sub

∴ complete the square (denom)

* Do before "split up"

$$\left\{ \begin{aligned} x^2+x+1 &= x^2+x+\tfrac{1}{4}-\tfrac{1}{4}+1 \\ \pm\left(\tfrac{1}{2}b^2\right) \quad &= \left(x+\tfrac{1}{2}\right)^2 + \tfrac{3}{4} \end{aligned} \right.$$

(*) $$\int \frac{-\frac{1}{3}x - \frac{2}{3}}{(x+\frac{1}{2})^2 + \frac{3}{4}}\,dx$$

$u = x + \frac{1}{2} \Rightarrow x = u - \frac{1}{2}$

$du = dx$

$$= \int \frac{-\frac{1}{3}(u-\frac{1}{2}) - \frac{2}{3}}{u^2 + \frac{3}{4}}\,du$$

$$= \int \frac{-\frac{1}{3}u + \frac{1}{6} - \frac{2}{3}}{u^2 + \frac{3}{4}}\,du$$

split $$= -\frac{1}{3}\int \frac{u}{u^2+\frac{3}{4}}\,du - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\,du$$

$$\boxed{\begin{aligned} &u^2 + a^2 \\ &\therefore a = \tfrac{\sqrt{3}}{2} \end{aligned}}$$

$$\boxed{\begin{aligned} w &= u^2 + \tfrac{3}{4} \\ dw &= 2u\,du \end{aligned}}$$

$u = \frac{\sqrt{3}}{2}\tan\theta$

$du = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$

$u^2 + \frac{3}{4} = (\frac{3}{4}\tan^2\theta) + \frac{3}{4}$

$= \frac{3}{4}\sec^2\theta$

$$= -\frac{1}{3}\cdot\frac{1}{2}\int \frac{1}{w}\,dw - \frac{1}{2}\int \frac{\frac{\sqrt{3}}{2}\cancel{\sec^2\theta}}{\frac{3}{4}\cancel{\sec^2\theta}}\,d\theta$$

$$= -\frac{1}{6}\ln|w| - \frac{\sqrt{3}}{3}\int d\theta$$

Killer! (cont'd)

$$-\tfrac{1}{6}\ln|w| - \frac{\sqrt{3}}{3}\int d\theta$$

$$= -\tfrac{1}{6}\ln\left|(x+\tfrac{1}{2})^2 + \tfrac{3}{4}\right| - \frac{\sqrt{3}}{3}\,\theta \quad + C$$

$u = \frac{\sqrt{3}}{2}\tan\theta$

$\theta = \arctan\left(\frac{2u}{\sqrt{3}}\right)$

$\theta = \arctan\left(\frac{2(x+\frac{1}{2})}{\sqrt{3}}\right)$

$$= -\tfrac{1}{6}\ln\left|(x+\tfrac{1}{2})^2 + \tfrac{3}{4}\right| - \frac{\sqrt{3}}{3}\arctan\left(\frac{2(x+\frac{1}{2})}{\sqrt{3}}\right) + C = \circledast$$

$$\therefore \int \frac{1}{x^3-1}\,dx = \tfrac{1}{3}\ln|x-1| + \circledast$$

ex: $\displaystyle\int \frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2}\,dx$ CASE 4

$$\frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2} = \frac{Ax+B}{x^2+2x+2} + \frac{Cx+D}{(x^2+2x+2)^2}$$

$$x^3+2x^2+3x-2 = (Ax+B)(x^2+2x+2) + (Cx+D)$$

$$x^3+2x^2+3x-2 = Ax^3+2Ax^2+2Ax+Bx^2+2Bx+2B+Cx+D$$

$$x^3+2x^2+3x-2 = x^3(A)+x^2(2A+B)+x(2A+2B+C)+(2B+D)$$

equate coefficients

① $\boxed{A=1}$

② $2A+B=2 \Rightarrow \boxed{B=0}$

③ $2A+2B+C=3 \Rightarrow 2+C=3 \Rightarrow \boxed{C=1}$

④ $2B+D=-2 \Rightarrow \boxed{D=-2}$

$$= \int \frac{x}{x^2+2x+2}\,dx + \int \frac{x-2}{(x^2+2x+2)^2}\,dx$$

complete the square

$$\begin{cases} x^2+2x+2 = x^2+2x+1-1+2 \\ \qquad\qquad\;\; = (x+1)^2+1 \end{cases}$$

$$= \int \frac{x}{(x+1)^2+1}\,dx + \int \frac{x-2}{\left[(x+1)^2+1\right]^2}\,dx$$

(cont'd)

$$= \int \frac{x}{(x+1)^2+1}\,dx + \int \frac{x-2}{[(x+1)^2+1]^2}\,dx \qquad \boxed{\begin{array}{l} u = x+1 \Rightarrow x = u-1 \\ du = dx \end{array}}$$

$$= \int \frac{(u-1)}{u^2+1}\,du + \int \frac{u-1-2}{(u^2+1)^2}\,du$$

$$= \int \frac{u}{u^2+1}\,du - \underbrace{\int \frac{1}{u^2+1}\,du}_{\arctan(u)} + \int \frac{u}{(u^2+1)^2}\,du - \underbrace{3\int \frac{1}{(u^2+1)^2}\,du}_{\text{Trig sub}}$$

$$\boxed{\begin{array}{l} w = u^2+1 \\ dw = 2u\,du \\ \frac{1}{2}\,dw = u\,du \end{array}}$$

Trig sub

$a = 1$

$u = 1 \cdot \tan\theta$

$du = \sec^2\theta\, d\theta$

$u^2+1 = \tan^2\theta + 1$

$= \sec^2\theta$

$$= \frac{1}{2}\int \frac{1}{w}\,dw - \arctan(u) + \frac{1}{2}\int \frac{1}{w^2}\,dw - 3\int \frac{\cancel{\sec^2\theta}\,d\theta}{(\sec^2\theta)^{\cancel{2}}}$$

$$= \frac{1}{2}\ln|w| - \arctan(u) + \frac{1}{2}\left(-w^{-1}\right) - \underbrace{3\int \cos^2\theta\, d\theta}$$

$$= \int \cos^2\theta\, d\theta$$

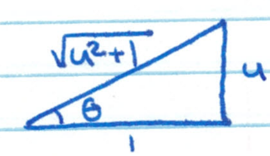
$$= \int 1/2\,(1 + \cos 2\theta)\,d\theta$$

$$= 1/2\left(\theta + \frac{1}{2}\sin 2\theta\right)$$

$$= \frac{1}{2}\ln|w| - \arctan(u) + \frac{1}{2}\left(-\frac{1}{w}\right) - 3/2\left(\theta + \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right) + C$$

$\tan\theta = u$

$\theta = \arctan(u)$



$\sin\theta = \frac{u}{\sqrt{u^2+1}}$

$\cos\theta = \frac{1}{\sqrt{u^2+1}}$

$$= 1/2\ln\left|(x+1)^2+1\right| - \arctan(x+1) - \frac{1}{2((x+1)^2+1)}$$

$$- \frac{3}{2}\left(\arctan(x+1) + \frac{(x+1)}{\sqrt{(x+1)^2+1}} \cdot \frac{1}{\sqrt{(x+1)^2+1}}\right) + C$$

Turn into rational fcn

ex: $\int \frac{\sqrt{x+4}}{x}\,dx$

$u=\sqrt{x+4} \Rightarrow u^2-4=x$

$du = \frac{1}{2\sqrt{x+4}}\,dx$

$du = \frac{1}{2u}\,dx$

$2u\,du = dx$

$= \int \frac{u}{u^2-4}(2u\,du)$

$= 2\int \frac{u^2}{u^2-4}\,du$

* use partial fractions