7 - Subspaces

Defn: A vector space, V, is a set of vectors with two operations: addition and multiplication by a scaler ($\mathbb{R}$) satisfying a set of axioms.

We will use Euclidean vector spaces: $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \ldots, \mathbb{R}^n$

Defn: A subspace of of $\mathbb{R}^n$ is a non-empty set, W, of vectors in $\mathbb{R}^n$ that satisfies the following two conditions:

(a) If $\vec{a}$ and $\vec{b} \in W$, then $\vec{a} + \vec{b} \in W$
(closure for addition)

<u>ex.</u> set of odd numbers would not satisfy this condition

$3 + 5 = 8$ ← not an odd number, for even numbers it would satisfy this condition

(b) If $\vec{a} \in W$ and $k \in \mathbb{R}$, then $k\vec{a} \in W$
(closure for multiplication by a scaler)

Note: Any subspace of $\mathbb{R}^n$ must contain the zero vector since:

let $\vec{a} \in W$ (<u>b.c.</u> W has at least one vector)

then, $-1\vec{a} = -\vec{a} \in W$ (<u>b.c.</u> W is closed for multiplication by a scaler)

then, $\vec{a} + (-\vec{a}) = \vec{0} \in W$ (<u>b.c.</u> W is closed for addition) ex. Let $W = \{ \underbrace{\langle v_1, v_2, v_3 \rangle}_{\text{object in the set}} \mid \underbrace{v_1 + 2v_2 - v_3 = 0}_{\text{condition}} \}$

set   such that

Show that W is a subspace in $\mathbb{R}^3$

preliminary condition to check if $\vec{0}$ is included (must be)

(0) $\vec{0} = \langle 0, 0, 0 \rangle \in W$ b.c $0 + 2(0) - (0) = 0$

(1) Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle \in W$

Show that $\vec{a} + \vec{b}$ is also $\in W$

$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

i.e. Show that

$(a_1 + b_1) + 2(a_1 + b_2) - (a_3 + b_3) = 0$

$(a_1 + b_1) + 2(a_1 + b_2) - (a_3 + b_3)$

$= (a_1 + 2a_1 - a_3) + (b_1 + 2b_2 - b_3)$

$= 0 + 0$ since $\vec{a}, \vec{b}$ satisfy the condition since $a, b \in W$

$= 0$

$\therefore W$ is closed for addition

(2) Let $k \in \mathbb{R}$, $a \in \vec{w}$

show that $k\vec{a} \in W$

i.e. show that

$\vec{k} a_1 + 2ka_2 - ka_3 = 0$

$\vec{k} a_1 + 2ka_2 - ka_3$

$= k(a_1 + 2a_2 - a_3)$

$= k(0)$

$= 0$

$\therefore W$ is closed for multiplication by a scaler

$\therefore W$ is a subspace.

Another way of defining W is the following:

ex. Let's consider the same set W as in the previous exemple

$$\begin{cases} v_1 + 2v_2 - v_3 = 0 \\ \rightarrow v_3 = v_1 + 2v_2 \end{cases}$$

$$W = \{ \langle v_1, v_2, v_1 + 2v_2 \rangle \mid v_1, v_2 \in \mathbb{R} \}$$

(0) $\vec{0} \in W$

(1) Let $\vec{a} = \langle a_1, a_2, a_1 + 2a_2 \rangle$

$\vec{b} = \langle b_1, b_2, b_1 + 2b_2 \rangle$

$$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_1 + 2a_2 + b_1 + 2b_2 \rangle$$

$$= \langle a_1 + b_1, a_2 + b_2, a_1 + b_1 + 2(a_2 + b_2) \rangle \in W$$

(2) Let $k \in \mathbb{R}$, $k\vec{a} = \langle ka_1, ka_2, k(a_1 + 2a_2) \rangle$

$$= \langle ka_1, ka_2, ka_1 + 2ka_2 \rangle \in W$$

Nov. 3

Back to example.

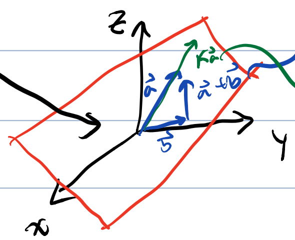
$$W = \{ \langle v_1, v_2, v_1 + 2v_2 \rangle \mid v_1, v_2 \in \mathbb{R} \}$$

i.e. $W = \{ \langle v_1, v_2, v_3 \rangle \mid v_1 + 2v_2 - v_3 = 0 \}$

i.e. $W = \{ \langle x, y, z \rangle \mid x + 2y - z = 0 \}$ eqn of plane

Geometrically:

plane passing through the origin



closure for addition, stay on plane

closure for multiplication by a scalar

* all values stay on the plane

ex. $W = \{ \langle x, y, z \rangle \mid x + 2y + 5 = 0 \}$

i.e. $\{ \langle v_1, v_2, v_3 \rangle \mid v_1 + 2v_2 + 5 = 0 \}$

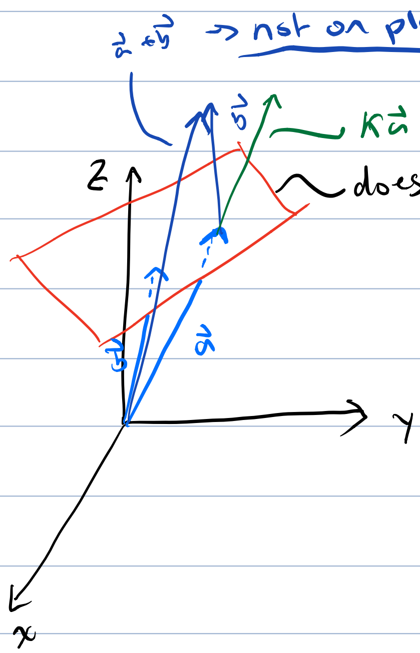
i.e. $\{ \langle -2v_2 - 5, v_2, v_3 \rangle \mid v_2, v_3 \in \mathbb{R} \}$

W is **not** a subspace of $\mathbb{R}^3$ **b.c.** $\vec{0} \notin W$

Note: W is **not** closed for addition

$\rightarrow$ ex: $\vec{a} = \langle -7, 1, 1 \rangle \in W$, $\vec{b} = \langle -5, 0, 3 \rangle \in W$

But, $\vec{a} + \vec{b} = \langle -12, 1, 4 \rangle \notin W$



$\vec{a}+\vec{b}$ $\rightarrow$ not on plane ~ not closed for addition

$k\vec{a}$ ~ not on plane ~ not closed for addition

~ does not pass through origin

Note: W from previous example can be written as

$$\vec{w} = \{ \langle -2y-5, y, z \rangle \mid y, z \in \mathbb{R} \}$$

Note: $\langle -2y-5, y, z \rangle = y\langle -2, 1, 0 \rangle + z\langle 0,0,1 \rangle + \langle -5,0,0 \rangle$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad s, t \in \mathbb{R}$$

These are all different representations of the same plane (that doesn't pass through the origin)

ex. $\vec{w} = \{ \langle 2x, x, -4x \rangle \mid x \in \mathbb{R} \}$

i.e. $W = \{ \langle 2t, t, -4t \rangle \mid t \in \mathbb{R} \}$

i.e. $\vec{w} = \{ \langle x, y, z \rangle \mid x = 2t, y = t, z = -4t \}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} \quad t \in \mathbb{R}$$

Show that W is a subspace of $\mathbb{R}^3$.

(0) $\vec{0} \in W$

(1) Closure for addition:

Let $\vec{a} = \langle 2a, a, -4a \rangle$, $b = \langle 2b, b, -4b \rangle$

$$\vec{a} + \vec{b} = \langle 2a + 2b, a + b, -4a - 4b \rangle$$
$$= \langle 2(a+b), (a+b), -4(a+b) \rangle \in W$$

(2) Closure for multiplication by a scaler:

Let $k \in \mathbb{R}$, $\vec{a} = \langle 2a, a, -4a \rangle$

$$k\vec{a} = k \langle 2a, a, -4a \rangle$$
$$= \langle k2a, ka, k(-4a) \rangle$$
$$= \langle 2(ka), (ka), -4(ka) \rangle \in W$$

$\therefore$ W is a subspace of $\mathbb{R}^3$

Subspaces

of $\mathbb{R}^2$	of $\mathbb{R}^3$
- $\{\vec{0}\}$	- $\{\vec{0}\}$
- any line through the origin	- any line through the origin
- $\mathbb{R}^2$ itself	- any plane through the origin
	- $\mathbb{R}^3$ itself

Aha! This is what we get when solving a homogeneous system.

Let $\underset{n\times n}{A}\,\underset{n\times 1}{X} = \underset{n\times 1}{0}$ be a homogeneous linear system with n variables. The general solution of the system is a subspace of $\mathbb{R}^n$.

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Def'n: If $A\mathbf{x} = \mathbf{0}$ is a homogeneous linear system with m equations and n variables, then the solution set is a subspace of $\mathbb{R}^n$. This is called the null space of the matrix A, denoted by null(A).

ex. $x - 2y + 3z = 0$
$2x - 4y + 6z = 0$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 2 & -4 & 6 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Null-space:
Sol'n: $\begin{cases} x = 2s - 3t \\ y = s \\ z = t \end{cases}$ or

$$\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \overset{\vec{v}_1}{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}} + t \overset{\vec{v}_2}{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}}_{\text{a plane passing through the origin.}}$$

Can find the standard eq of the plane by taking the cross product of sec and find the normal vector

Def'n: Let $S = \{ V_1, V_2, \dots, V_r \}$ be a set of vectors in $\mathbb{R}^n$. We define the span of S as the set of all linear combinations of the vectors in S. The span is denoted by $\text{span}(S)$

ex. If $S = \{ \hat{i}, \hat{j}, \hat{k} \}$, then $\text{span}(S) = \mathbb{R}^3$ i.e. any vector in $\mathbb{R}^3$ can be written as a linear combination of $\hat{i}, \hat{j}, \hat{k}$.

ex. If $S_1 = \{ \underbrace{\langle 1, 2, 3 \rangle}_{\vec{v}_1}, \underbrace{\langle 2, -1, 0 \rangle}_{\vec{v}_2} \}$.

Then $\text{span}(S_1)$ = a plane passing through the origin.

ex. $S_2 = \{ \underbrace{\langle 1, 2, 3 \rangle}_{\vec{v}_1}, \underbrace{\langle 2, -1, 0 \rangle}_{\vec{v}_2}, \underbrace{\langle 4, 3, 6 \rangle}_{\vec{v}_3 = 2\vec{v}_1 + \vec{v}_2} \}$

linear combination of first two

Then, $\text{span}(S_2) = \text{span}(S_1)$ adding a vector that's already on the plane.

vectors are not <u>independent</u>. They can be written as a linear combination of the other vectors.

linear combination of 1 vector = just scalar multiply of one vector

ex. $S_3 = \{ \langle 3, 1 \rangle \} \longrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

span(S_3) = line through the origin in $\mathbb{R}_2$

Note: span$\left(\{1, 2, 0\}, \{3, 4, 1\} \right)$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} s + \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} t$ linear combination of vectors

also, note: no vector of constants so passes through the origin

<u>Theorem</u>: Let S be a set of vectors in $\mathbb{R}^n$.
Let $W = \text{span}(S)$.
Then, W is a subspace of $\mathbb{R}^n$

<u>Proof</u>: Let $S = \{ \vec{v}_1, \ldots, \vec{v}_r \}$, $W = \text{span}(S)$

Let $\vec{a} = k_1 \vec{v}_1 + \ldots + k_r \vec{v}_r \in W$

$\vec{b} = l_1 \vec{v}_1 + \ldots + l_r \vec{v}_r \in W$

Closure for addition:

$\vec{a} + \vec{b} = (k_1 + l_1)\vec{v}_1 + \ldots + (k_r + l_r)\vec{v}_r \in W$

closure for multiplication by a scalar

Let $K \in \mathbb{R}$, $K\vec{u} = K(k_1\vec{v}_1 + \ldots + k_r\vec{v}_r)$

$$= (Kk_1)v_1 + \ldots + (Kk_r)v_r \in W$$

So W is closed for addition and multiplication by a scalar

$\therefore$ W is a subspace of $\mathbb{R}^n$

Note: If $W = \text{span}(S)$, we say that S spans W

ex. Determine whether the set S spans $\mathbb{R}^3$.

$$S = \{\underbrace{\langle 1,1,2\rangle}_{\vec{v}_1}, \underbrace{\langle 1,0,1\rangle}_{\vec{v}_2}, \underbrace{\langle 2,1,4\rangle}_{\vec{v}_3}\}.$$

i.e. determine whether any vector in $\mathbb{R}^3$ can be written as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$,

i.e. Let $\vec{b} = \langle b_1, b_2, b_3\rangle$. Determine whether there exists k_1, k_2, k_3

such that $k_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

i.e. Determine whether there exists k_1, k_2, k_3 such that (for any b)

$$\begin{aligned} k_1 + k_2 + 2k_3 &= b_1 \\ k_1 \phantom{{}+k_2} + k_3 &= b_2 \\ 2k_1 + k_2 + 4k_3 &= b_3 \end{aligned}$$

i.e. . . . the linear system $AX = B$ has a solution for any B.

$$\underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{B}$$

For it to always have a sol'n : rank must be n.

Otherwise, could have

$$\left[\begin{array}{ccc|c} & & & b_1 \\ & & & b_2 \\ 0 & 0 & 0 & b_3 \neq 0 \end{array}\right] \xrightarrow{} \text{no solution.}$$

i.e. . . . $\text{rank}(A) = 3$ (in general, $\text{rank}(A) = n$)

i.e. If A is square, determine whether $\det(A) \neq 0$ (i.e. A is invertible)

to add to invertibility theorem : consistent iff $\text{rank}(A) = n$

So... back to the example...

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 4 \end{vmatrix} = -1 \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$$

$$= -2 + 1$$

$$= -1 \neq 0$$

$\therefore$ S spans $\mathbb{R}^3$

<u>Note:</u> One vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ spans a line (through the origin)

<u>Note:</u> Two non-parallel vectors in $\mathbb{R}^3$ span a plane

<u>Note:</u> To span all of $\mathbb{R}^n$, we need at least n vectors

November 14

<u>ex.</u> Let $S = \{ \langle 1, 0, 2 \rangle, \langle 1, 1, 5 \rangle, \langle 0, -1, -3 \rangle, \langle 2, 1, 7 \rangle \}$

Does S span $\mathbb{R}^3$?

Note: Can't use the determinant since it's a 4×3 (not square)

- Place vectors $\vec{v}_1, \dots, \vec{v}_4$ in the columns of a matrix
- Reduce the matrix to REF
- verify that the rank = num_variables (3) (In general, in $\mathbb{R}^n$, verify that rank = n)

$$k_1 \begin{bmatrix} \\ \\ \\ \end{bmatrix} + k_2 \begin{bmatrix} \\ \\ \\ \end{bmatrix} + k_3 \begin{bmatrix} \\ \\ \\ \end{bmatrix} + k_4 \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \overset{\vec{b}}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$$

$$\underset{3 \times 4}{\begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix}} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 2 & 5 & -3 & 7 \end{bmatrix} \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 3 & -3 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 - 3R_2} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note: Vectors put in columns to conserve relationship between vectors (RREF preserves relationship between columns)

Note (cont'd):

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} \vec{v}_1 & \vec{v}_2 & \uparrow & \uparrow \end{matrix} \end{array}$$

$\vec{v}_4 = 1(\vec{v}_1) + 1(\vec{v}_2)$

Note $\vec{v}_3 = 1(\vec{v}_1) - 1(\vec{v}_2)$

Note: S is not linearly independent.

b.c. $\vec{v}_3 = \vec{v}_1 - \vec{v}_2$ and $\vec{v}_4 = \vec{v}_1 + \vec{v}_2$

i.e. Some of the vectors in the set S are combinations of other vectors in the set.

Note: If a vector is a combination of other vectors, we can obtain the vector $\vec{0}$ by combining all the vectors.

$$\vec{v}_3 = \vec{v}_1 - \vec{v}_2$$
$$\vec{0} = \vec{v}_1 - \vec{v}_2 - \vec{v}_3$$

Def'n: Let $S = \{\vec{v}_1, \ldots, \vec{v}_r\}$ be a set of vectors in $\mathbb{R}^n$. We say that S is linearly independent if the linear combination $k_1\vec{v}_1 + \ldots + k_r\vec{v}_r = \vec{0}$ has only the trivial solution $k_1 = 0, \ldots, k_r = 0$.

ex. (a) $S = \{\langle 1,3,5 \rangle\}$ is linearly independent

(b) $S = \{\langle 0,0,0 \rangle\}$ is linearly dependent

(c) $S = \{\langle 0,0,0 \rangle, \langle 3,2,5 \rangle, \langle 1,1,7 \rangle\}$ is linearly dependent

$754\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 = \vec{0}$

Moral: Any set containing the zero vector is dependent.

ex. (d) $S = \{\underset{\vec{v}_1}{\langle 1,1,2 \rangle}, \underset{\vec{v}_2}{\langle 3,0,5 \rangle}\}$ is independent b.c.

$\vec{v}_2$ is not a scalar multiple of $\vec{v}_1$.

(e) $S = \{\langle 1,1,2 \rangle, \langle 3,0,5 \rangle, \langle 0,0,1 \rangle, \langle 3,-1,17 \rangle\}$ is linearly dependent

b.c. $\begin{bmatrix} 1 & 3 & 0 & 3 \\ 1 & 0 & 0 & -1 \\ 2 & 5 & 1 & 17 \end{bmatrix} \cdots \rightsquigarrow$ REF

rank ≤ 3

But there are 4 vectors!

So the set is dependent.

Moral: When determining whether a set S of vectors in $\mathbb{R}^n$ is linearly independent, we place the vectors in the columns of a matrix, A, and reduce the matrix to RREF.

- If the matrix is $n \times n$ (square), then S is independent iff $\det(A) \neq 0$

- If the matrix is not square, then S is independent iff rref...

Def'n: A set B of vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$ if B satisfies both of the following:

(1) B is linearly independent

(2) B spans $\mathbb{R}^n$

ex. (a) $B = \{\hat{i}, \hat{j}, \hat{k}\}$ is a basis of $\mathbb{R}^3$

(b) $B = \{\langle 1,1\rangle, \langle 2,0\rangle, \langle 17,5\rangle\}$ is not a basis of $\mathbb{R}^3$ b.c. it is not independent (too many vectors to be linearly independent)

(c) $B = \{\langle 1,0,2\rangle, \langle 3,4,5\rangle\}$ is not a basis of $\mathbb{R}^3$ (too little vectors to span all of $\mathbb{R}^3$)

Note: The same definition applies to a basis of a subspace ex. $W = \{\vec{a} \in \mathbb{R}^3 \mid \vec{a} = s\langle 1,1,5\rangle + t\langle 2,-1,0\rangle\}$

subspace

The set $B = \{\langle 1,1,5\rangle, \langle 2,-1,0\rangle\}$ is a basis of W

Subspace of $\vec{0}$ has no basis

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Remember: A set B of vectors in $\mathbb{R}^n$ (or in a subspace of $\mathbb{R}^n$) is a basis of $\mathbb{R}^n$ (or of the subspace W) if B

- is linearly independent
- spans $\mathbb{R}^n$ (or W)

ex. (a) $B = \{\hat{i}, \hat{j}, \hat{k}\}$ is a standard basis of $\mathbb{R}^n$

(b) $B = \{\langle 1,2\rangle, \langle 0,1\rangle, \langle 3,2\rangle\}$ is not a basis of $\mathbb{R}^2$

b.c. B is not linearly independent.

(c) Let $W = \{\vec{a} \in \mathbb{R}^3 \mid \vec{a} = t\langle 1,5,0\rangle, t \in \mathbb{R}\}$

W is a subspace of $\mathbb{R}^3$.
$B = \{\langle 1,5,0\rangle\}$ is a basis of W.

Note: The number of vectors in a basis of a given subspace of $\mathbb{R}^n$ is unique. We call this the dimension of the subspace, denoted

ex. (a) $W = \{ \vec{a} \in \mathbb{R}^3 \mid \vec{a} = s\langle 1,3,0 \rangle + t\langle 0,1,1 \rangle, s,t \in \mathbb{R} \}$

$B = \{ \langle 1,3,0 \rangle, \langle 0,1,1 \rangle \}$ is a basis of W

So $\dim(W) = 2$

(b) $W = \mathbb{R}^n$, $\dim(W) = \dim(\mathbb{R}^n) = n$

ex. Let $S = \{ \underset{\vec{v}_1}{\langle 1,0,2 \rangle}, \underset{\vec{v}_2}{\langle 3,2,1 \rangle}, \underset{\vec{v}_3}{\langle 4,2,3 \rangle}, \underset{\vec{v}_4}{\langle -5,-4,0 \rangle} \}$

Let $W = \text{span}(S)$

a) Find a basis of W using the vectors in S

b) Write the deleted vectors in terms of the vectors of B.

a)
$$\begin{array}{c} \begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{array} \\ \begin{bmatrix} 1 & 3 & 4 & -5 \\ 0 & 2 & 2 & -4 \\ 2 & 1 & 3 & 0 \end{bmatrix} \end{array} \xrightarrow[\frac{1}{2}R_2]{R_3 - 2R_1} \begin{bmatrix} 1 & 3 & 4 & -5 \\ 0 & 1 & 1 & -2 \\ 0 & -5 & -5 & 10 \end{bmatrix}$$

$$\xrightarrow{R_3+5R_2} \begin{bmatrix} 1 & 3 & 4 & -5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1-3R_2} \begin{array}{c} \begin{array}{cccc} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 & \vec{c}_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

So $\mathcal{B} = \{ \underbrace{\langle 1,0,2 \rangle}_{\vec{v}_1}, \underbrace{\langle 3,2,1 \rangle}_{\vec{v}_2} \}$

W would be a plane through the origin

b) $\vec{v}_3 = 1 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2 = \vec{v}_1 + \vec{v}_2$

$\vec{v}_4 = \vec{v}_1 - 2\vec{v}_2$

<u>Def'n:</u> Let A be a <u>$m \times n$</u> matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Columns could be n vectors in $\mathbb{R}^m$

$$\vec{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \vec{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \dots, \quad \vec{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

These are vectors in $\mathbb{R}^m$. We define the column space of A as the subspace of $\mathbb{R}^m$ spanned by these vectors

$$\text{col}(A) = \text{span}\{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\}$$

<u>Def'n:</u> Consider the m rows of A.

$$\vec{r}_1 = [a_{11}, a_{12}, \dots, a_{1n}], \quad \vec{r}_2 = [a_{21}, a_{22}, \dots, a_{2n}],$$

$$\vec{r}_m = [a_{m1}, a_{m2}, \dots, a_{mn}].$$

These are vectors in $\mathbb{R}^n$. We define the row space of A as the subspace of $\mathbb{R}^n$ spanned by these m vectors.

$$\text{Row}(A) = \text{span}\{\vec{r}_1, \vec{r}_2, \dots, \vec{r}_m\}$$

<u>Def'n:</u> We define the null space of A as the subspace of $\mathbb{R}^n$ consisting of the general solution of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ ← dimension is number of free variables

<u>Remarks:</u>

(1) The $\dim(\text{Nul}(A))$ is called the nullity(A)

(2) The $\dim(\text{Col}(A))$ is " the rank(A)

(3) If A is $m \times n$, $\underbrace{\text{nullity}(A)}_{\text{free variables}} + \underbrace{\text{rank}(A)}_{\text{leading variables}} = \underbrace{n}_{\text{total num variables}}$

i.e. consider the system $AX = \vec{0}$: # free variables
+ # leading variables $= n$

November 17

Remember: If A is $n \times n$,

- $\dim(\mathrm{Col}(A)) = \mathrm{rank}(A)$
- $\dim(\mathrm{Nul}(A)) = n - \mathrm{rank}(A)$ = # free variables in $AX = \vec{0}$

(nullity (A))

So $\dim(\mathrm{Nul}(A)) + \dim(\mathrm{Col}(A)) = n$

ex. $A = \begin{array}{c} \begin{array}{ccccc} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 & \vec{c}_4 & \vec{c}_5 \end{array} \\ \begin{bmatrix} 2 & 3 & -1 & 0 & 7 \\ -1 & 0 & -1 & -3 & 1 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 2 & -2 & 1 & 6 \end{bmatrix} \end{array}$

a) Find a basis and the dim of $\mathrm{Nul}(A)$

b) ———— " ———— $\mathrm{Col}(A)$

Give a dependency relationship for the deleted vectors.

$$\begin{bmatrix} 2 & 3 & -1 & 0 & 7 \\ -1 & 0 & -1 & -3 & 1 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 2 & -2 & 1 & 6 \end{bmatrix} \xrightarrow{\cdots} \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b) $B = \{ \overset{\vec{c}_1}{\langle 2, -1, 1, 0 \rangle}, \overset{\vec{c}_2}{\langle 3, 0, 1, 2 \rangle}, \overset{\vec{c}_4}{\langle 0, -3, 0, 1 \rangle} \}$

is a basis of $Col(A)$

$\dim(Col(A)) = 3$

$\vec{c}_3 = \vec{c}_1 - \vec{c}_2$

$\vec{c}_5 = -\vec{c}_1 + 3\vec{c}_2$

a)

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{cases} x_1 = -s + t \\ x_2 = s - 3t \\ x_3 = s \\ x_4 = 0 \\ x_5 = t \end{cases} \quad s, t \in \mathbb{R}$$

So,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad s, t \in \mathbb{R}$$

2 basis vectors

$B_1 = \{ \langle -1, 1, 1, 0, 0 \rangle, \langle 1, -3, 0, 0, 1 \rangle \}$ is a basis of $Nul(A)$

$\dim(Nul(A)) = 2$ which we already knew...

Note: Any vector in the subspace can be written as a lin. combination of the basis vectors, ~ unique combination

Theorem: Let $B = \{ \vec{v}_1, \ldots, \vec{v}_r \}$ be a basis of a subspace W. Let $\vec{w} \in W$.

The expression $k_1 \vec{v}_1 + k_2 \vec{v}_2 + \ldots + k_r \vec{v}_r = \vec{w}$ is unique.

Proof: Suppose there exists $l_1, l_2, \ldots, l_r \in \mathbb{R}$ such that:

say it's not unique and that there's another combo.

$$l_1 \vec{v}_1 + l_2 \vec{v}_2 + \ldots + l_r \vec{v}_r \in \vec{w}$$

So
$$k_1 \vec{v}_1 + k_2 \vec{v}_2 + \ldots + k_r \vec{v}_r = \vec{w}$$
$$-(l_1 \vec{v}_1 + l_2 \vec{v}_2 + \ldots + l_r \vec{v}_r = \vec{w})$$

$$(k_1 - l_1) \vec{v}_1 + (k_2 - l_2) \vec{v}_2 + \ldots + (k_r - l_r) \vec{v}_r = \vec{0}$$

So $k_1 - l_1 = 0 \rightarrow k_1 = l_1$

$\vdots$

$k_r - l_r = 0 \rightarrow k_r = l_r$ b.c B is linear indep.

If it were dependent, then:

$k_1\vec{v}_1 + k_2\vec{v}_2 = \vec{0}$

So there are multiple ways of making it = 0.

An update on the invertibility theorem...

Invertibility Theorem:

If A is an $n \times n$ matrix, the following statements are equivalent:

a) A is invertible

b) The RREF of A is I (i.e. $A \sim I$)

c) The rank of A is n

d) The linear system $AX = B$ is consistent for any B.

e) The linear system $AX = B$ has exactly one sol for any B.

f) The homogeneous linear system $AX = 0$ has only the trivial sol'n.

g) A can be written as a product of elem matrices

h) $\det(A) \neq 0$

i) $\dim(\text{Col}(A)) = n$

j) $\dim(\text{Nul}(A)) = 0$

Prove the following:

- $(a) \Rightarrow (e)$

If A is invertible and $AX = B$ is a linear system,

then, $A^{-1}AX = A^{-1}B$

$$IX = A^{-1}B \quad \therefore \quad X = A^{-1}B$$

• (g) => (h)

If there exists $E_1, \dots, E_k$ elem matrices such that $A = E_1 E_2 \dots E_k$

then $\det(A) = \det(E_1 \dots E_k) = \underbrace{\det(E_1)}_{\neq 0} \underbrace{\det(E_2)}_{\neq 0} \dots \underbrace{\det(E_k)}_{\neq 0}$

So $\det(A) \neq 0$

• (h) => (a)

Let $\det(A) \neq 0$. Let R be the RREF of A.

There exists elem. matrices $E_1, \dots, E_k$ such that

$$E_k \dots E_1 A = R$$

$$\underbrace{\det(E_k)}_{\neq 0} \dots \underbrace{\det(E_1)}_{\neq 0} \underbrace{\det(A)}_{\neq 0} = \det(R)$$

So $\det(R) \neq 0$

So R has no row of zeros.

So $R = I$

$\therefore A \sim I$

$\therefore A$ is invertible.