4A) SECOND ORDER (LINEAR) ODEs

First, consider the equation:

$$(A) \quad y'' + p(x)y' + q(x)y = 0$$

The general solution of (A) on an interval I is a fcn of form

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where $y_1(x)$ and $y_2(x)$ are linearly independant solutions to (A) and c_1 and c_2 are arbitrary constants. We say that these two solns form a basis for the soln space.

IVP problems with these 2^{nd} order eqns will require two initial conditions: $y(x_0) = K_0$ & $y(x_1) = K_1$.

We start with the second order linear case where $p(x)$ and $q(x)$ are constants:

$$y'' + ay' + by = 0 \quad (B)$$

How to solve?

Recall first order case: $y' + ay = 0 \Rightarrow y = e^{-ax}$ is gen'l soln

Q: Could exponentials also be the solns in the second order case?

A: Lets try!

Plug $y = e^{\lambda x}$ into (B).

$$(e^{\lambda x})'' + a(e^{\lambda x})' + be^{\lambda x} = 0$$

$$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = 0$$

$$e^{\lambda x}\left[\lambda^2 + a\lambda + b\right] = 0$$

Lets see if we can find a λ that works...

(By Fundamental Theorem of Algebra, there will always be two complex λ's as solns).

E.g. Solve the IVP : $y'' + y' - 2y = 0$ $\quad y(0) = 4$, $y'(0) = 1$

Plug $y = e^{\lambda x}$

$$(e^{\lambda x})'' + (e^{\lambda x})' - 2(e^{\lambda x}) = 0$$

$$e^{\lambda x}(\lambda^2 + \lambda - 2) = 0$$

$$\lambda^2 + \lambda - 2 = 0$$

$$(\lambda + 2)(\lambda - 1) = 0$$

$$\lambda = 1 \text{ or } \lambda = -2$$

Thus, $y_1(x) = e^x$ and $y_2(x) = e^{-2x}$ are both possible solutions

Genl soln: $y(x) = Ae^{-2x} + Be^x$

I.C. $y(0) = 4 = Ae^{-2(0)} + Be^0$

$$4 = A + B \Rightarrow \boxed{B = 4 - A} \text{ ①}$$

$$y'(x) = -2Ae^{-2x} + Be^x$$

$$y'(0) = 1 = -2Ae^{-2(0)} + Be^0$$

$$\underbrace{1 = -2A + B}_{\text{sub } B = 4-A}$$

$$1 = -2A + 4 - A \rightarrow 1 = -3A + 4$$

$$-3 = -3A \Rightarrow \boxed{A = 1} \Rightarrow \boxed{B = 3}$$

$$\therefore \boxed{y(x) = 3e^x + e^{-2x}}$$

E.g. Find real solns to the DE: $y'' - 2y' + 10y = 0$

Try $y = e^{\lambda x}$

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + 10e^{\lambda x} = 0$$

$$\lambda^2 - 2\lambda + 10 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4 - 40}}{2}$$

$$\lambda = 1 + \frac{\sqrt{-36}}{2} = 1 \pm \frac{6i}{2} = 1 \pm 3i \Rightarrow \lambda = 1 + 3i \text{ or } \lambda = 1 - 3i$$

Note: when $\lambda \in \mathbb{C} \wedge \lambda \notin \mathbb{R}$, $\lambda_2 = \overline{\lambda_1}$ (can easily be seen by quadratic formula).

$$y_1(x) = e^{(1+3i)x} = e^x \cdot e^{3ix} = e^x(\cos(3x) + i\sin(3x))$$
$$y_2(x) = e^{(1-3i)x} = e^x \cdot e^{-3ix} = e^x(\cos(-3x) + i\sin(-3x))$$

cos is even so $f(-a) = f(a)$ sin is odd so $f(-a) = -f(a)$

$$y_2(x) = e^x(\cos(3x) - i\sin(3x))$$

OK, now check this: (Its about to get $\mathbb{R}$eal)

$$\frac{y_1 + y_2}{2} = \frac{y_1 + \overline{y_1}}{2} = Re(y_1) = e^x \cos 3x$$
$$\frac{y_1 - y_2}{2i} = \frac{y_1 - \overline{y_1}}{2i} = Im(y_1) = e^x \sin 3x$$

We just used complex numbers to obtain an answer that works in real numbers!!

$$\therefore y(x) = Ae^x\cos(3x) + Be^x\sin(3x)$$

Note: $\frac{1}{2}(y_1 + y_2) = \frac{1}{2}(y_1 + \overline{y_1}) = Re(y_1)$

$\frac{1}{2i}(y_1 - y_2) = \frac{1}{2i}(y_1 - \overline{y_1}) = Im(y_1)$

General Formula:

Given $y'' + ay' + b = 0$

Try $y = e^{\lambda x}$

Get characteristic eqn: $\lambda^2 + a\lambda + b = 0$

Roots: $\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$

$\Rightarrow \lambda = -\frac{1}{2}a \pm \sqrt{\frac{a^2}{4} - b}$ suppose $a^2 - 4b < 0$

$\Rightarrow \lambda = -\frac{a}{2} \pm i\underbrace{\sqrt{b - \frac{a^2}{4}}}_{\omega}$

$\Rightarrow \lambda = -\frac{a}{2} \pm i\omega$

Complex solutions: $y_1(x) = e^{(-\frac{a}{2} + i\omega)x} = e^{-a/2}[\cos(\omega x) + i\sin(\omega x)]$

$y_2(x) = e^{(-\frac{a}{2} - i\omega)x} = e^{-a/2}[\cos(\omega x) - i\sin(\omega x)]$

Ex: Solve the IVP: $2y'' + 2y' + 5y = 0$ $\quad y(0) = 4$

sub $y = e^{\lambda x}$ $\quad y'(0) = 1$

$$2\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} + 5e^{\lambda x} = 0$$

$$2\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4-40}}{4} = \frac{-2 \pm 6i}{4}$$

$\lambda_1 = -\frac{1}{2} + 1.5i$ or $\lambda_2 = -\frac{1}{2} - 1.5i$

$\Rightarrow \omega = 1.5$

$\Rightarrow -\frac{a}{2} = -\frac{1}{2}$

$$Y_1 = e^{-\frac{1}{2}x} \sin(1.5x)$$

$$Y_2 = e^{-\frac{1}{2}x} \cos(1.5x)$$

general real soln

$$Y(x) = Ae^{-\frac{1}{2}x} \cos(1.5x) + Be^{-\frac{1}{2}x} \sin(1.5x)$$

I.C.

$Y(0) = 4 \Rightarrow A\cancel{e^{-\frac{1}{2}(0)}} \cos(0) + 0 \Rightarrow \boxed{A = 4}$

$y'(x) =$ product rule ...

plug & chug $\Rightarrow B = 2$

Soln to IVP is $Y(x) = 4e^{-\frac{1}{2}x} \cos(1.5x) + 2e^{-\frac{1}{2}x} \sin(\omega x)$

Recap: Starting with a $y'' + ay' + by = 0$, trying $y = e^{\lambda x}$.

Get characteristic eqn: $\lambda^2 + a\lambda + b = 0$

Case 1: 2 distinct real λ

Case 2: 2 distinct complex λ*

Case 3: 1 real λ

* complex roots for quadratics are conjugates so will always be 2 λ's

Idea: Rather than looking for a $y_2(x)$ directly, we actually look for how known soln and desired soln $y_2(x)$ "differ".

We try to find $u(x) = \frac{y_2(x)}{y_1(x)}$ → Note that $u(x)$ should not be a constant.

We discuss this procedure, called Reduction of Order in the general case: $y'' + p(x)y' + q(x)y = 0$ Ⓐ

Whenever one soln $y_1(x)$ is known, we can find another soln $y_2(x)$ using reduction of order: we look for $u(x) = \frac{y_2(x)}{y_1(x)}$

Proceed (General case):

$$\text{Want } y_2(x) = u(x) \cdot y_1(x)$$

Try this as soln to DE:

$$y'' + py' + qy = 0 \quad \text{becomes}$$

$$y_2'' + py_2' + qy_2 = 0$$

$$(uy_1)'' + p(u \cdot y_1)' + q(u \cdot y_1) = 0$$

$$(u'y_1 + uy_1')' + p(u'y_1 + uy_1') + q(u \cdot y_1) = 0$$

$$u''y_1 + u'y_1' + u'y_1' + uy_1'' + pu'y_1 + puy_1' + quy_1 = 0$$

$$u''y_1 + u'(2y_1' + py_1) + u\underbrace{(y_1'' + py_1' + qy_1)}_{\text{zero! } \because y_1 \text{ is a soln of Ⓐ}} = 0$$

$$u''y_1 + u'(2y_1' + py_1) = 0$$

No u, only its derivatives. Let $u' = U$
$u'' = U'$

$$U'y_1 + U(2y_1' + py_1) = 0$$

$$U' + U\left[2\frac{y_1'}{y_1} + p(x)\right] = 0 \quad \leftarrow \text{separable}$$

$$\frac{dU}{dx} = -U\left[2\frac{y_1'}{y_1} + p(x)\right]$$

$$\int \frac{dU}{U} = -2\int \frac{y_1'}{y_1}\,dx - \int p(x)\,dx + C$$

$$\ln|U| = -2\ln|y_1| - \int p(x)\,dx + C$$

$$\boxed{U(x) = \frac{Ce^{-\int p(x)dx}}{y_1^2}}$$ ← Can use! Then get u by integrating U.

Then use $y_2(x) = u(x)\cdot y_1(x)$ to find 2^{nd} soln.

E.g. Consider the DE: $y'' - \frac{x}{x-1}y' + \frac{1}{x-1}y = 0$

a) Verify that $y = e^x$ is a possible soln

b) Find the general soln

b) $y'' - \frac{x}{x-1}y' + \frac{1}{x+1}y = 0$ Checked $y = e^x$ is a soln.

Looking for a second soln:

Set $u = \frac{y_2(x)}{y_1(x)}$ $\quad U = u' = \frac{C}{y_1^2}e^{-\int p(x)dx}$

$$\int p(x)dx = \int \frac{-x}{x-1}\,dx = \int \frac{-x+1-1}{x-1}\,dx = \int\left(-1 - \frac{1}{x-1}\right)dx$$

$$\int p(x)\,dx = -(x + \ln|x-1|)$$

$$U = \frac{C}{e^{2x}}e^{x+\ln|x-1|} = \frac{C}{e^x}(x-1)$$

$$\frac{du}{dx} = Ce^{-x}(x-1)$$

$$\int du = \int Ce^{-x}(x-1)\,dx + D$$

$$u(x) = C\left[e^{-x}(x-1) + e^{-x}\right]$$

⋮ pics of rest of example.

À la fin, $\underline{y_2(x) = x}$

Back to constant coefficients:

$$y'' + ay' + by = 0$$

Try $y = e^{\lambda x}$ to get C.E.: $\lambda^2 + a\lambda + b = 0$

Last case to look at: case only one real root, called critical case

Root of C.E. would be:

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2} \rightarrow \text{case } a^2 = 4b$$

Only one root: $\lambda = \frac{-a}{2}$

Get only one soln: $y_1(x) = e^{-a/2\, x}$.

Obtain independant $y_2(x)$ using reduction of order

Let $u(x) = \frac{y_2(x)}{y_1(x)}$ Set $U = u'$, solve $U = \frac{C}{y_1^2} e^{-\int p(x)\,dx}$

$$\int p(x)\,dx = \int a\,dx = ax$$

$$\Rightarrow U = \frac{C}{(e^{-\frac{ax}{2}})^2} e^{-ax} \Rightarrow U = \frac{C}{\cancel{e^{-ax}}} \cancel{e^{-ax}}$$

$$U = C \Rightarrow u(x) = Cx + D$$

let $C=1$, $D=0$

Take $u(x) = x$

$$\frac{y_2(x)}{y_1(x)} = x$$

$$y_2(x) = x e^{-\frac{ax}{2}}$$

Gen'l soln: $y(x) = Ae^{-\frac{ax}{2}} + Bx\, e^{-\frac{ax}{2}}$

SOLVING

$$\boxed{y''+ay'+by=0} \qquad \text{(B)}$$

when a and b are real constants.

Summary

Characteristic equation: $\boxed{\lambda^2+a\lambda+b=0}$ (C)

Depending on the type of roots that the characteristic equation has, we have the following cases.

Case	Roots of (C)	Real basis for solutions of (B)	General Solution of (B)
I	Distinct real λ_1, λ_2	$e^{\lambda_1 x}$, $e^{\lambda_2 x}$	$y=C_1e^{\lambda_1 x}+C_2e^{\lambda_2 x}$
II	Complex conjugate $\lambda_1=-\frac{a}{2}+i\omega,\ \lambda_2=-\frac{a}{2}-i\omega$ where $\omega=\sqrt{b-\frac{1}{4}a^2}$	$e^{-ax/2}\cos\omega x$, $e^{-ax/2}\sin\omega x$	$y=e^{-ax/2}\left[A\cos\omega x+B\sin\omega x\right]$
III	Real double root $\lambda=-\frac{1}{2}a$	$e^{-ax/2}$, $xe^{-ax/2}$	$y=(C_1+C_2x)e^{-ax/2}$

Important Note:

This summary is handy, but students must remain aware that they are requested to know how one obtains the real solutions in case II through the use of complex numbers and linear combinations, and how one obtains the second solution through reduction of order in case III.

E.g. Solve $y'' - 4y' + 4y = 0$

Try $y = e^{\lambda x} \Rightarrow$ C.E. $= \lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)^2 = 0$ critical case

$\boxed{\lambda = 2}$

$y = e^{2x}$

Other soln $y_2(x) = x e^{2x}$

Gen'l soln: $y(x) = A e^{2x} + B x e^{2x}$ I.C. $y(0) = 3$

$y'(0) = 1$

Trivial

Back to $y'' + p(x) y' + q(x) y = 0$

$x^2 y'' + a x y' + b y = 0$ (work on $I = (0, \infty)$)

Idea: Try $y = x^m$ ← we will try to find m that works, hopefully 2 of these m.

$x^2 m(m-1) x^{m-2} + a x m x^{m-1} + b x^m = 0$

$m(m-1) x^m + a m x^m + b x^m = 0$

$x^m \neq 0$

$m(m-1) + am + b = 0$

$\boxed{m^2 + (a-1) m + b = 0}$ Auxillary equation

Case 1: m_1 & m_2 are 2 distinct real solutions to aux. eqn.

Then $y_1(x) = x^{m_1}$ $y_2(x) = x^{m_2}$ are 2 LI solns

Case 2: If m_1 & m_2 are (non real) complex, then

$m_1 = \overline{m_2}$ (just like last time).

$m_1 = \mu + i\nu$

$m_2 = \mu - i\nu$

$$y_1(x) = x^{\mu + i\nu} = x^{\mu} \cdot x^{i\nu} = x^{\mu} e^{\ln(x^{i\nu})} = x^{\mu} e^{i\nu \ln(x)}$$

$$y_1(x) = x^{\mu}\left[\cos(\nu \ln(x)) + i \sin(\nu \ln(x))\right]$$

$$y_2(x) = x^{\mu - i\nu} = x^{\mu} \cdot x^{-i\nu} = x^{\mu} e^{\ln(x^{-i\nu})} = x^{\mu} e^{-i\nu \ln(x)}$$

$$y_2(x) = x^{\mu}\left[\cos(-\nu \ln(x)) + i \sin(-\nu \ln(x))\right]$$

$$y_2(x) = x^{\mu}\left[\cos(\nu \ln(x)) - i \sin(\nu \ln(x))\right]$$

$$Y_1 = Re(Y) = \frac{y_1 + y_2}{2} = x^{\mu} \cos(\nu \ln(x))$$

$$Y_2 = I_m(Y) = \frac{y_1 + y_2}{2i} = x^{\mu} \sin(\nu \ln(x))$$

$$y(x) = AY_1 + BY_2$$

$$y(x) = A x^{\mu} \cos(\nu \ln(x)) + B x^{\mu} \sin(\nu \ln(x))$$

general soln.

E.g. Solve $x^2 y'' + 7xy' + 13y = 0$

Try $y = x^m$

Get aux eqn: $m^2 + 6m + 13 = 0$

$$m = \frac{-6 \pm \sqrt{36 - 4(13)}}{2}$$

$$m = \frac{-6 \pm \sqrt{-16}}{2} = -3 \pm 2i$$

$$\therefore y(x) = x^{-3}\left(A \cos(2 \ln x) + B \sin(2 \ln(x))\right)$$

Ok but now what happens if we only get a unique root from our auxillary eqn? We know $y_1(x) = x^m$ is a soln.
Happens when we get:

$$m^2 + (a-1)m + b = 0$$

$$m = \frac{1-a \pm \sqrt{b^2-4ac}}{2} \quad \text{and } b^2 = 4ac$$

$$\Rightarrow m = \frac{1-a}{2}$$

$$\Rightarrow y_1(x) = x^{\frac{1-a}{2}}$$

Use reduction of order:

$$U = \frac{C}{y_1^2} e^{-\int p(x)\,dx}$$ works for DE of form $y'' + p(x)y' + q(x)y = 0$

But we have $x^2y'' + axy' + by = 0$

$$\Rightarrow y'' + \frac{a}{x}y' + \frac{b}{x^2}y = 0$$

$$\Rightarrow p(x) = \frac{a}{x}.$$

Now we can reduce order:

$$\int p(x)\,dx = \int \frac{a}{x}\,dx = \ln(x^a)$$

$$U = \frac{C}{(x^{\frac{1-a}{2}})^2} e^{-\ln(x^a)}$$

$$U = \frac{C}{x^{1-a}} \cdot x^{-a}$$

$$\frac{du}{dx} = \frac{C}{x}$$

$$\int du = \int \frac{C}{x}\,dx + D$$

$$u(x) = C\ln x + D$$

Take $C=0$, $D=1$

$$u(x) = \ln x$$

$$\frac{y_2(x)}{x^{\frac{1-a}{2}}} = \ln(x)$$

$$y_2(x) = x^{\frac{1-a}{2}} \ln x.$$

$\therefore$ Genl soln is $y(x) = x^{\frac{1-a}{2}}(A + B\ln x)$

E.g. Solve $x^2y'' - 3xy' + 4y = 0$

Try $y = x^m$

Get aux eqn: $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$\Rightarrow m = 2 \Rightarrow y_1(x) = x^2$ is a soln.

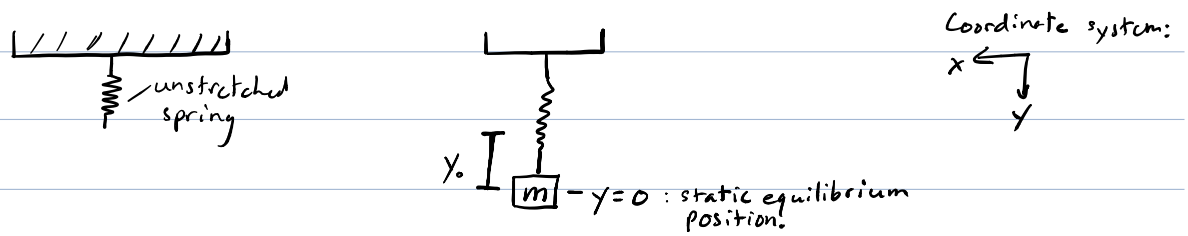
By reduction of order: $y(x) = x^2(A + B\ln(x))$

We'll now consider a very useful application of this section...

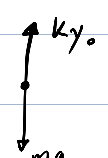
4B) Free Oscillations

Defn: Free oscillations are those that are not caused by a driving force.

Note: Friction will be considered!



FBD:



$$\sum F_y = m\cancel{a_y}^{\,0} \quad \because \text{static eq}$$

$$\boxed{mg - ky_0 = 0} \quad (1)$$

Note: knowing the y_0 allows us to determine the spring constant which would allow us to obtain the frequency of oscillation (in this frictionless environment). Indeed $\omega = \sqrt{\frac{k}{m}}$ and the mass would be in SHM.

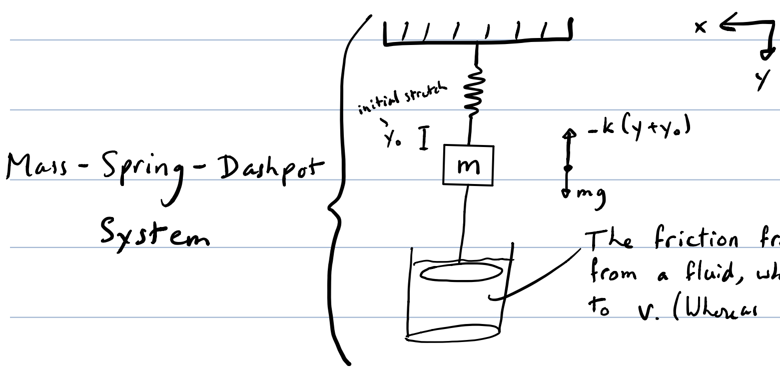
Undamped oscillations: No friction (SHM)

$$\sum F = ma$$
$$-kx = mx''$$
$$x''(t) = -\underbrace{\boxed{\tfrac{k}{m}}}_{\omega^2} x(t) \quad \Rightarrow \omega = \sqrt{\tfrac{k}{m}}$$

Does the frequency change with friction? Let's find out...

Let $\omega_0 = \sqrt{\frac{k}{m}}$ be the "natural" (angular) frequency (SHM, no friction).
Recall: $\omega_0 = 2\pi f_0$, where f_0 is the frequency in $\frac{\text{cycles}}{\text{sec}}$ or s^{-1} or Hz.
We'll compute 'ω' with friction and see if it comes out to the same thing.

Damped Oscillations: With friction



The friction from this dashpot part is the friction from a fluid, which experiments show to be roughly proportional to v. (Whereas in gas friction $\propto v^2$.)

$$\sum F_y = ma_y$$

(−) ∵ friction always opposes direction of motion.

$$mg - k(y+y_0) - bv = ma_y$$
$$\underbrace{[mg - ky_0]}_{\text{This is zero by eqn ①}} - ky - bv = ma_y \;\rightarrow\; ma_y + bv + ky = 0$$

$$\boxed{my'' + by' + ky = 0}$$

This is constant case of form we've seen in section 4A)!

We know what to do!

Lets try $y = e^{\lambda x}$

Get C.E.: $m\lambda^2 + b\lambda + k = 0$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

$$\lambda = -\frac{b}{2m} \pm \frac{1}{2m}\sqrt{b^2 - 4mk}$$

Let $\alpha = \frac{b}{2m}$; $\omega = \frac{1}{2m}\sqrt{b^2 - 4mk}$ if $b^2 > 4mk$

$\omega = \frac{1}{2m}\sqrt{4mk - b^2}$ if $b^2 < 4mk$

So,

$\lambda =$
- $-\alpha + \omega$ if $b^2 > 4mk$ (overdamped) — big friction — 2 distinct real roots
- $-\alpha + \omega i$ if $b^2 < 4mk$ (underdamped) — small friction — 2 distinct complex roots
- $-\alpha$ if $b^2 = 4mk$ (critically damped) — just enough friction — 1 real root.

This corresponds exactly to the 3 cases we got before.

1) Underdamping:

We saw that $y_1(t) = e^{(-\alpha+\omega i)t}$ and $y_2(t) = e^{(-\alpha-\omega i)t}$ can be recombined into $Y_1(t) = e^{-\alpha t}\cos(\omega t)$ and $Y_2(t) = e^{-\alpha t}\sin(\omega t)$. We get a real gen'l soln: $y(t) = e^{-\alpha t}[A\cos(\omega t) + B\sin(\omega t)]$. We can rewrite this as $y(t) = Ce^{-\alpha t}\cos(\omega t - \delta)$ (delta) where $C = \sqrt{A^2 + B^2}$ and $\tan(\delta) = \frac{B}{A}$.

Note: $-Ce^{-\alpha t} \leq y(t) \leq Ce^{-\alpha t}$

$$\lim_{t\to\infty} -Ce^{-\alpha t} = 0 \quad \text{and} \quad \lim_{t\to\infty} Ce^{-\alpha t} = 0$$

By squeeze theorem, $\lim_{t\to\infty} y(t) = 0$

That is, the motion is returning to stabilize at the static equilibrium position.

2) Overdamping:

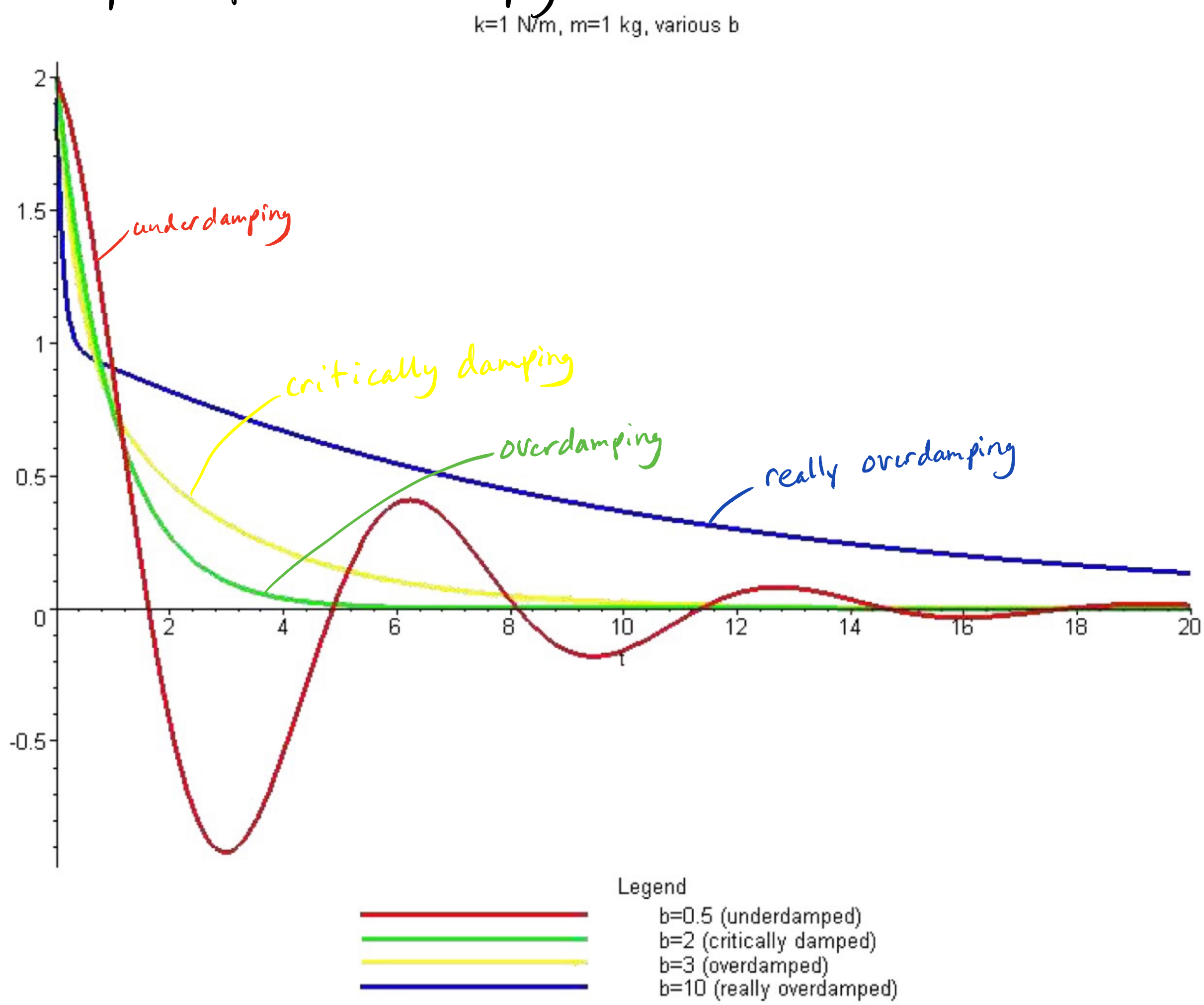
Gen'l soln is immediately obtained: $y(t) = Ae^{(-\alpha+\omega)t} + Be^{(-\alpha-\omega)t}$

$\lim_{t\to\infty} y(t) = 0$ $\therefore$ Spring returns to equilibrium.

3) Critical Damping:

Gen'l soln is the same as we found in section 4A) using reduction of order: $y(t) = e^{-\alpha t}[A + Bt]$. Also approaches 0 as $t\to\infty$, as can be shown w/ a limit.

Recap: Graph of all damping situations we've seen thus far:



4C) Existence and Uniqueness

We want to prove that a DE of degree n has exactly n linearly independant solutions.

Consider first the homogeneous linear case for order n:

(H) $y^{(n)} + p_{n-1}(x)\,y^{(n-1)} + p_{n-2}(x)\,y^{(n-2)} + \dots + p_1(x)\,y' + p_0(x)\,y = 0$

If we had an IVP, we would need: $y(x_0) = K_0,\ y'(x_1) = K_1,\ \dots,\ y^{(n-1)}(x_n) = K_n$

Note: $y'' + p(x)\,y' + q(x)\,y = 0$ (H2)

Theorem:

If $p_0(x), p_1(x), \dots, p_{n-1}(x)$ in (H) are cont' on I, then $\exists!$ soln to the IVP. (This proof is really technical and doesn't yeild much insight...)

This then turns out to be the key to the rest...

We hope to turn the fact that DE's with initial conditions have a unique soln into the fact that DE' prior to setting ICs must have exactly n linearly independant solns.

But we'll need a tool... The Wronskian

Consider n functions: $f_1(x), f_2(x), \dots, f_n(x)$. Their Wronskian or Wronski Determinant is:

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)} \end{vmatrix}$$

Remarks: ① W is not a number, it is a fcn of the variable x: $W(x)$

② Note that the order in which you enter the fcns into the Wronskian may change the sign of W. But...

③ Sign does not matter much anyway. We will really want to know if W is zero or not.

Prop'n: Let $f_1, f_2, \ldots, f_n$ be $n-1$ times diff'ble on I. If these are dependent, then $W(f_1, f_2, \ldots, f_n) = 0 \ \forall x \in I$.

Proof: $(n=2) \rightarrow$ Similar for bigger size

$$W(f_1, f_2) = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix}$$

If f_1 and f_2 are dependant, then one of these can be written as a lin. combination of other(s)

Say $kf_1 = f_2$

Then, $$W = \begin{vmatrix} kf_2 & f_2 \\ kf_2' & f_2' \end{vmatrix}^T = \begin{vmatrix} kf_2 & kf_2' \\ f_2 & f_2' \end{vmatrix} \xrightarrow{R_1 - kR_2} = \begin{vmatrix} kf_2 & kf_2' \\ 0 & 0 \end{vmatrix} = 0$$

$\therefore$ If fcns are dependant, $W = 0 \ \forall x \in I$.

By contrapositive:

If $\exists \, x \in I \mid W(f_1, f_2, \ldots, f_n) \neq 0$, then f_1, f_2, f_n are linearly independant.

E.g. Use Wronskian to check independance of $f_1 = x$, $f_2 = \sin(\omega x)$, $f_3 = \cos(\omega x)$.

$$W = \begin{vmatrix} x & \cos\omega x & \sin\omega x \\ 1 & -\omega\sin\omega x & \omega\cos\omega x \\ 0 & -\omega^2\cos\omega x & -\omega^2\sin\omega x \end{vmatrix} = x\left(\omega^3\sin^2(\omega x) + \omega^3\cos^2(\omega x)\right)$$
$$- \left(\cancel{-\omega^2\cos(\omega x)\sin(\omega x)} + \cancel{\omega^2\cos(\omega x)\sin(\omega x)}\right)$$
$$= x\omega^3$$

$W = 0$ only at one pt $(x=0) \Rightarrow$ fcns are linearly independant

We would like to have if $W=0$ then fcns are dependant.
But this doesn't always work.

E.g. Consider $f_1(x) = x^3$, $f_2(x) = |x^3|$ on $I = \mathbb{R}$

Now $W(f_1, f_2) = \begin{vmatrix} x^3 & |x^3| \\ 3x^2 & \text{sig}(x)\, 3x^2 \end{vmatrix} = 3x^5 \,\text{sig}(x) - 3|x^3|\, x^2$
$\begin{cases} x>0 : 0 \\ x=0 : 0 \\ x<0 : 0 \end{cases}$

$\therefore\ W(x^3, |x^3|) \equiv 0 \quad \forall\, x \in \mathbb{R}$

But they are <u>not</u> dependant on I.

Actually, we can do better. Let's see what happens if we take arbitrary functions such that these functions are solns of a linear ODE?

Theorem:

If $\exists\ x_0 \in I \mid W(y_1(x_0), y_2(x_0), \dots, y_n(x_0)) = 0$ <u>and</u> $y_1(x), y_2(x), \dots, y_n(x)$ are solns of (H), then $y_1, y_2, \dots, y_n$ are linearly dependant, in which case $W \equiv 0\ \forall\, x \in I$.

RTP: $W(y_1, y_2) = 0$ for some $x \in I \Rightarrow y_1$ & y_2 are dependant.

<u>Proof:</u>

We show for $n=2$ for simplicity. We'll work with

$y'' + p(x)y' + q(x)y = 0$ (H2) and let $y_1(x)$ and $y_2(x)$ be solns of (H2)

For any particular $x_0 \in I$, the following fcns are "known values":
$y_1(x_0) \quad y_2(x_0)$
$y_1'(x_0) \quad y_2'(x_0)$

Key 1: We look at the following functions given's system of eqns. to be solved for the unknowns a_1 & a_2.

$$\overbrace{y_1(x_0)}^{\text{constant}} \cdot a_1 + \overbrace{y_2(x_0)}^{\text{constant}} \cdot a_2 = 0$$
$$y_1'(x_0) \cdot a_1 + y_2'(x_0) \cdot a_2 = 0$$

Soln for a_1 unique or not?

Well: $\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0$ (by hypothesis) $\Rightarrow \exists \infty$ many solns a_1 & a_2

$\therefore \exists$ non trivial (not both zero) solns for a_1 & a_2. Pick one such soln, i.e. pick some $a_1 \neq 0$, $a_2 \neq 0$ that works in

$$a_1 y_1(x_0) + a_2 y_2(x_0) = 0$$
and $$a_1 y_1'(x_0) + a_2 y_2'(x_0) = 0$$

This doesn't yet show that $y_1(x)$ & $y_2(x)$ are dependant, we still need to show that $a_1 y_1(x) + a_2 y_2(x) = 0$ $\underline{\forall x \in I}$

Key 2: (Invoke IVP Existance & Uniqueness)

Give name: $y^*(x) = a_1 \cdot y_1(x) + a_2 y_2(x)$ is a soln of (H2)

$$y'' + p(x)y' + q(x)y = 0$$ because y_1 & y_2 are

$$y^*(x_0) = 0$$
$$y^{*\prime}(x_0) = 0$$

But soln of IVP is... unique!

But soln is known! $y(x) \equiv 0 \quad \forall x \in I$.

$\therefore y^*(x)$ & $y(x) \equiv 0$ are the same!

I.e. $y^*(x) = a_1 y_1(x) + a_2 y_2(x) \equiv 0 \quad \forall x$ $\therefore$ Dependant

Q.E.D

Back to Q_1 & Q_2 from the beginning of the section.

Q_1: Do homogeneous DEs always have a soln of the form

(*) $y(x) = C_1 y_1(x) + C_2 y_2(x) + \dots + C_n y_n(x)$

Theorem: If $p_0(x), p_1(x), \dots, p_{n-1}(x)$ are cont's on I,
then (H) always has a gen'l soln of form (*) on I.

Proof: Case $n=2$

Consider the IVP 1: $y'' + p(x) y' + q(x) y = 0$
with $y_1(x_0) = 1$, $y_2'(x_0) = 0$

& IVP 2: with $y_2(x_0) = 0$, $y_2'(x_0) = 0$

Both IVP's have a unique soln $y_1(x)$ & $y_2(x)$ respectively.

Rest of proof??

We have shown that it is indeed possible to find n linearly independant solns $y_1, y_2, \dots, y_n$ so that ① exists. We finalize showing that it is not possible to find yet more independant solns.

Theorem:

If $p_0, p_1, \dots, p_{n-1}$ (the coefficient functions) are conts on I then any soln of Ⓗ is included in the gen'l soln for some $c_1, c_2, \dots, c_n$. That is, such a general soln does include all solns. (Dim of soln space $= n$ is enough)

Proof:

Let $y_1(x)$ & $y_2(x)$ be two linearly indep. solns of (H2): $y'' + py' + qy = 0$ (we know they exist by previous thm).
Now let $Y(x)$ be any other soln of (H2). We'll show that necessarily $Y(x)$ must be a linear comb. of y_1 and y_2.
For this, we look at IVP and invoke uniqueness of its solns...
Which IVP? → Take any $x_0 \in I$, consider the initial conditions

$$y(x_0) = Y(x_0)$$

Like for example, the soln fcn $S(x)$ must assign the correct fcn value to each pt x_0.

$$y'(x_0) = Y'(x_0)$$

Then $y(x)$ satisfies the IVP $y'' + py' + qy = 0$; $y(x_0) = Y(x_0)$, $y'(x_0) = Y'(x_0)$
Could these same conditions be satisfied by a linear comb. of y_1 & y_2? If yes, then Y is same as linear comb. of y_1 and y_2 by uniqueness of solns of IVP.

Now consider the system of equations:

$$y_1(x_0) \cdot C_1 + y_2(x_0) \cdot C_2 = Y(x_0)$$
$$y_1'(x_0) \cdot C_1 + y_2'(x_0) \cdot C_2 = Y'(x_0)$$

Can we solve for some C_1, C_2?

Wronskian at x_0: $\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \neq 0$ for any $x \in I$

because y_1 and y_2 are linearly independant

Can find unique soln for C_1 & C_2 (cramers rule)...

Now $y(x) = C_1 y_1(x) + C_2 y_2(x)$ is a soln of (H2) with some IC, $y(x_0)$ $y'(x_0)$.

By uniqueness of solns to IVP: $y(x) = C_1 y_1(x) + C_2 y_2(x)$ is same as $Y(x)$. $\square$

4D) Homogeneous Linear DE's Of Order N (with constant coefficients).

The generalization from order 2 to order n is quite straight-foward. (At least, if we don't insist on writing down all the proofs....)

Consider $y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$

where $a_{n-1}, a_{n-2}, \dots, a_1, a_0$ are n constant real coefficients.

Try $y = e^{\lambda x}$ as a soln:

$$\lambda^n e^{\lambda x} + a_{n-1} \lambda^{n-1} e^{\lambda x} + \dots + a_1 \lambda e^{\lambda x} + a_0 e^{\lambda x} = 0$$

Get C.E.: $\boxed{\lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0}$

By FTA and its corrolaries, the C.E. has n roots.

These may be complex, but then they would come in conjugate pairs, so these can be recombined into sines and cosines.

E.g. Solve $y^{(v)} + y^{iv} - 4y''' - 7y'' + 3y' + 6y = 0$

Try $y = e^{\lambda x}$

Get C.E: $\lambda^5 + \lambda^4 - 4\lambda^3 - 7\lambda^2 + 3\lambda + 6 = 0$

Finding roots... RRT? Try $\lambda = \pm 1$, $\lambda = \pm 2$, $\lambda = \pm 3$, $\lambda = \pm 6$

See that $\lambda = 1$ works, $\lambda = 2$ works, $\lambda = -1$ also works.

Let's factor out: $(\lambda - 1)(\lambda + 1)(\lambda - 2) = \lambda^3 - 2\lambda^2 - \lambda + 2$

Euclidean:

$$\begin{array}{r|l}
 & \lambda^2 + 3\lambda + 3 \\
\hline
\lambda^3 - 2\lambda^2 - \lambda + 2 & \lambda^5 + \lambda^4 - 4\lambda^3 - 7\lambda^2 + 3\lambda + 6 \\
 & -(\lambda^5 - 2\lambda^4 - \lambda^3 + 2\lambda^2) \\
\hline
 & 3\lambda^4 - 3\lambda^3 - 9\lambda^2 + 3\lambda \\
 & -(3\lambda^4 - 6\lambda^3 - 3\lambda^2 + 6\lambda) \\
\hline
 & 3\lambda^3 - 6\lambda^2 - 3\lambda + 6 \\
 & -(3\lambda^3 - 6\lambda^2 - 3\lambda + 6) \\
\hline
 & 0
\end{array}$$

Thus, we can rewrite the characteristic eqn as:

$$(\lambda - 1)(\lambda + 1)(\lambda - 2)(\lambda^2 + 3\lambda + 3)$$

$$\lambda = \frac{-3 \pm \sqrt{3^2 - 4(3)}}{2} = -\frac{3}{2} \pm \frac{\sqrt{3}}{2} i$$

So the five linearly independant solutions are:

$y_1(x) = e^{x}$, $y_2(x) = e^{-x}$, $y_3(x) = e^{2x}$, $y_4(x) = e^{\left(-3/2 + \frac{\sqrt{3}}{2} i\right)x}$, $y_5(x) = e^{\left(-\frac{3}{2} - \frac{\sqrt{3}}{2} i\right)x}$

But $y_4(x)$ and $y_5(x)$ recombine as $e^{-1.5x}\cos\left(\frac{\sqrt{3}}{2}x\right)$ and $e^{-1.5x}\sin\left(\frac{\sqrt{3}}{2}x\right)$.

So the general solution is:

$$y(x) = C_1 e^{x} + C_2 e^{-x} + C_3 e^{2x} + C_4 e^{-1.5x}\cos\left(\frac{\sqrt{3}}{2}x\right) + C_5 e^{-1.5x}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

What if some of the n roots repeat?

Reduction of order can be used: Take some $y = e^{\lambda x}$ that is a solution, Put $u(x) = \frac{\tilde{y}(x)}{y(x)}$ to obtain another solution $\tilde{y}(x)$.

Multiple real roots:

If λ is a real root of the CE with multiplicity K, then reduction of order can be used to show that $e^{\lambda x}, xe^{\lambda x}, x^2 e^{\lambda x}, \ldots, x^{K-1} e^{\lambda x}$ are the K linearly independant solutions (proof is ommitted).

Multiple non-real roots:

If $\lambda \notin \mathbb{R}$ and is a root of the C.E., recall that $\bar{\lambda}$ is also a root. Actually, if λ has multiplicity K, then $\bar{\lambda}$ also has multiplicity K.

Reduction of order can be used to show that:

Real soln's
$$\begin{cases} e^{\lambda x}, xe^{\lambda x}, \ldots, x^{K-1} e^{\lambda x} \\ e^{\bar{\lambda} x}, xe^{\bar{\lambda} x}, \ldots, x^{K-1} e^{\bar{\lambda} x} \end{cases}$$
are all independant solutions.

Consequently, we can make linear recombinations of these non real λ solutions to obtain real solutions. (Writing $\lambda = a + bi$) These solutions become:

Real soln's
$$\begin{cases} e^{ax}\cos(bx), \; xe^{ax}\cos(bx), \ldots, x^{K-1}e^{ax}\cos(bx) \\ e^{ax}\sin bx, \; xe^{ax}\sin(bx), \ldots, x^{K-1}e^{ax}\sin(bx) \end{cases}$$

E.g. Solve $y^{(11)}+4y^{(10)}+7y^{(9)}+36y^{(8)}-49y^{(7)}-324y^{(vi)}+333y^{(v)}+5508y'''=0$

Try $y=e^{\lambda x}$,

Get C.E.: $\lambda^{11}+4\lambda^{10}+7\lambda^{9}+36\lambda^{8}-49\lambda^{7}-324\lambda^{6}+333\lambda^{5}+5508\lambda^{3}=0$

Roots: 0 (three times), then... Rational roots thm? Maple!

Get $\lambda^3(\lambda-2)^2(\lambda^2+8\lambda+17)(\lambda^2+9)^2=0$

$\lambda=0$ has multiplicity 3: $y_1(x)=e^{0x}=1$, $y_2(x)=xe^{0x}=x$, $y_3(x)=x^2$

$\lambda=2$ has multiplicity 2: $y_4(x)=e^{2x}$, $y_5(x)=xe^{2x}$

$\lambda^2+8\lambda+17$ has roots $\lambda=-4+i$ and $\bar{\lambda}=-4-i$, which we can make linear recombinations of to get:

$$y_6(x)=e^{-4x}\cos x \quad y_7(x)=e^{-4x}\sin x$$

Similarly, λ^2+9 has roots $\pm 3i$, which can be recombined to give: $y_8(x)=\cos 3x$ and $y_9(x)=\sin(3x)$. However, the term (λ^2+9) has multiplicity $k=2$, so $y_{10}(x)=x\cos(3x)$ and $y_{11}(x)=x\sin(3x)$.

Thus, we obtain the gen'l soln $y(x)$:

$$y(x)=C_1+C_2x+C_3x^2+C_4e^{2x}+C_5xe^{2x}+C_6e^{-4x}\cos x+C_7e^{-4x}\sin x+C_8\cos 3x$$
$$+C_9x\cos 3x+C_{10}\sin 3x+C_{11}x\sin 3x.$$

4E) Non-Homogeneous Linear Differential Equations

Consider the DE: NH $P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \ldots + P_1(x)y' + P_0 y = r(x)$

Its corresponding homogeneous DE is:

$$\text{H} \quad P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \ldots + P_1(x)y' + P_0 y = 0$$

The idea to solving nonhomogeneous linear DEs is a lot like solving non-homogeneous linear systems in linear 1. In linear, we were able to solve the homogeneous system, then add particular soln to the nonhomogeneous system to obtain the general soln to the nonhomogeneous system.
Similarly, in nonhomogeneous DEs, we can obtain a general solution by first solving the corresponding homogeneous system, then adding a particular soln to the nonhomogeneous DE to the general soln of the homogeneous DE to obtain the general soln to the nonhomogeneous DE.
In short, the steps are:

1. Find gen'l soln of H (we'll call it $y_h(x)$)
2. Find one particular soln of NH (let's call it $y_p(x)$)
3. Gen'l soln of NH is $y(x) = y_h(x) + y_p(x)$

We can actually verify that $y(x) = y_h(x) + y_p(x)$ will satisfy NH in general.

Check this out:

$$P_n y^{(n)} + P_{n-1} y^{(n-1)} + \ldots + P_1 y' + P_0 y$$

sub $y(x) = y_p + y_h$

$$= P_n (y_h + y_p)^{(n)} + P_{n-1} (y_h + y_p)^{(n-1)} + \cdots + P_1 (y_h + y_p)' + P_0 (y_h + y_p)$$

$$= P_n y_h^{(n)} + P_n y_p^{(n)} + P_{n-1} y_h^{(n-1)} + P_{n-1} y_p^{(n-1)} + \ldots + P_1 y_h' + P_1 y_p' + P_0 y_p + P_0 y_h$$

$$= \underbrace{\left(P_n y_h^{(n)} + P_{n-1} y_h^{(n-1)} + \ldots + P_1 y_h' + P_0 y_h\right)}_{0 \text{ by virtue of being soln of } (H)} + \underbrace{\left(P_n y_p^{(n)} + P_{n-1} y_p^{(n-1)} + \ldots + P_1 y_p' + P_0 y_p\right)}_{r(x) \text{ by virtue of being particular soln of } (NH)}$$

$= r(x)$ ✓ $\Rightarrow$ Solves (NH) in general!

So thats pretty awesome but we don't know how to just find a particular soln of (NH). If only there was a way...

The Method of Undetermined Coefficients:

This method works when the DE is linear with constant coefts:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = r(x) \rightarrow {}^{*}(NH_c)$$ c is for constant

and when $r(x)$ is a fcn whose derivatives remain of a similar type to the original function, namely,

$$\underline{r(x)} = \begin{cases} \text{sines, cosines, or hyperbolic sines and cosines} \\ e^x \text{ (or exponentials in general)} \\ \text{polynomials (including constants)} \end{cases}$$

Or $r(x)$ can be the sum or product of these functions.

$r(x)$ can't be something whose derivatives don't resemble itself, e.g.: $\tan x$, $\arctan x$, $\frac{1}{x}$ etc.

* For simplicity, we'll denote the associated homogeneous eqn by (H_c)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

Basic Idea behind the Method of Undetermined Coefficients:

↳ Say we start with a DE:

$$2y'' + 5y' + 10y = 2e^{3x} \quad (*)$$

Trying $y = e^{3x}$ as a solution makes sense, but LHS may not simplify to $2e^{3x}$. So we can try $y = Ae^{3x}$ and then trying to determine for what A this would work.

Trying $y_p = Ae^{3x}$ in $(*)$:

$$2(Ae^{3x})'' + 5(Ae^{3x})' + 10(Ae^{3x}) = 2e^{3x}$$

$$[2A \cdot 9 + 5 \cdot 3A + 10A]\cancel{e^{3x}} = 2\cancel{e^{3x}}$$

$$18A + 15A + 10A = 2$$

$$43A = 2 \Rightarrow \boxed{A = \tfrac{2}{43}}$$

Thus, one <u>particular</u> soln to NH_c is $y_p = \frac{2}{43}e^{3x} \ldots$

This type of idea leads to the following "reasonable guesses" for one particular soln of NH_c using undetermined coefficients.

Note:

The Method of Undetermined Coefficients

$$\boxed{a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = r(x)} \qquad (D_c)$$

where $a_0, a_1, \ldots, a_n$ are constants.

Summary

We are trying to find a particular solution $y_p(x)$ for the non-homogeneous differential equation.

1) "Basic Rule"

When $r(x)$ is of an "allowed form", try the following for y_p:

Given form of $r(x)$	Trial y_p
$C e^{\gamma x}$	$K e^{\gamma x}$
$C \cos \omega x$ or $C \sin \omega x$	$K \cos \omega x + L \sin \omega x$
$C e^{\gamma x} \cos \omega x$ or $C e^{\gamma x} \sin \omega x$	$K e^{\gamma x} \cos \omega x + L e^{\gamma x} \sin \omega x$
$C x^n$	$K_n x^n + K_{n-1} x^{n-1} + \ldots + K_1 x + K_0$
C (case $n = 0$)	K
$C x^n e^{\gamma x}$	$(K_n x^n + K_{n-1} x^{n-1} + \ldots + K_1 x + K_0)\, e^{\gamma x}$
$C x^n \cos \omega x$ or $C x^n \sin \omega x$	$(K_n x^n + \ldots + K_1 x + K_0) \cos \omega x + (L_n x^n + \ldots + L_1 x + L_0) \sin \omega x$
$C x^n e^{\gamma x} \cos \omega x$ or $C x^n e^{\gamma x} \sin \omega x$	$(K_n x^n + \ldots + K_1 x + K_0) e^{\gamma x} \cos \omega x + (L_n x^n + \ldots + L_1 x + L_0) e^{\gamma x} \sin \omega x$

Please Turn Over

2) "<u>Sum Rule</u>"

If: $r(x)$ is the sum or difference of functions listed in the above table,
then: choose for y_p the sum of the listed trial functions.

3) "<u>Modification Rule</u>" (Ensuring that all terms in y_p are useful)

Denote by (H_c) the associated homogeneous equation to (D_c).

If: a part of y_p constains one or more terms which happen to be a solution of (H_c),
then: multiply this part by x,
or: more generally by x^k as necessary when there is multiplicity k in the solution of (H_c).

E.g. Solve $y'' + 9y = 17e^{-5x}$

First solve corresponding homogeneous eqn:

$y'' + 9y = 0$

Try $y = e^{\lambda x}$

Get C.E. : $\lambda^2 + 9 = 0 \Rightarrow \lambda = \pm 3i$

$\Rightarrow y_1(x) = \cos(3x)$, $y_2(x) = \sin(3x)$ are real solns.

Gen'l soln of H : $y_h(x) = A\cos(3x) + B\sin(3x)$

Now lets obtain one particular soln to NH :

Try $y_p(x) = Ae^{-5x}$,

$\cancel{e^{-5x}}[25A + 9A] = 17\cancel{e^{-5x}}$

$34A = 17$

$\boxed{A = 0.5}$ Determine that fucking coefficient $\Rightarrow y_p(x) = \frac{1}{2}e^{-5x}$

$\therefore$ Gen'l soln of NH is $y(x) = \frac{1}{2}e^{-5x} + A\cos(3x) + B\sin(3x)$

E.g. Solve $y'' + 4y = 8x^2$ NH

First, solve H: $y'' + 4y = 0$

Try $y = e^{\lambda x}$

Get C.E: $\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$

$\Rightarrow y_h(x) = C_1\cos(2x) + C_2\sin(2x)$

Find particular soln to NH

Put $y_p(x) = Ax^2 + Bx + C$ into NH :

$2A + 4Ax^2 + 4Bx + 4C = 8x^2 + 0x + 0$

Comparing powers:

x^2: $4A = 8$ $\Rightarrow \boxed{A=2}$

x: $4B = 0$ $\Rightarrow \boxed{B=0}$

1: $2A + 4C = 0$, $2(2) + 4C = 0 \Rightarrow \boxed{C=-1}$

$\therefore$ Gen'l soln of (NH) is: $y(x) = 2x^2 - 1 + A\cos(2x) + B\sin(2x)$

Now what if $r(x)$ has more than one term?
Then include all possibilities (see "sum rule" from handout)

E.g. Solve $y'' + 2y' + 5y = 16e^x + \sin(2x)$ (NH)

First solve (H):

$$y'' + 2y' + 5y = 0$$

try $y = e^{\lambda x}$

Get C.E.: $\lambda^2 + 2\lambda + 5 = 0$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4(5)}}{2} = \frac{-2 \pm 4i}{2}$$

$$\lambda = -1 \pm 2i$$

$$\Rightarrow y_h(x) = C_1 e^{-x}\cos 2x + C_2 e^{-x}\sin 2x$$

Now find particular soln to (NH):

Put $y_h(x) = A\cos x + B\sin x + Ce^x$ in (NH):

$$(A\cos(2x) + B\sin(2x) + Ce^x)'' + 2(A\cos(2x) + B\sin(2x) + Ce^x)'$$
$$+ 5(A\cos(2x) + B\sin(2x) + Ce^x) = 16e^x + \sin(2x)$$

$$-4A\cos(2x) - 4B\sin(2x) + Ce^x - 4A\sin(2x) + 4B\cos(2x) + 2Ce^x + 5A\cos(2x)$$
$$+ 5B\sin 2x + 5Ce^x = 16e^x + \sin(2x)$$

$$A\cos(2x) - 4A\sin(2x) + B\sin(2x) + 4B\cos(2x) + 8Ce^x = 16e^x + \sin(2x)$$

Now compare different functions

e^x: $16 = 8C$ $\quad$ $\boxed{C=2}$

$\cos(2x)$: $A + 4B = 0$
$A = -4B$

$\sin(2x)$: $B - 4A = 1$
$-\frac{1}{4}A - 4A = 1$
$A = \frac{-4}{17}$
$\Rightarrow \boxed{B = \frac{1}{17}}$

$\therefore$ Gen'l soln to (NH) is

$$y(x) = 2e^x - \frac{4}{17}\cos(2x) + \frac{1}{17}\sin(2x) + Ae^{-x}\cos(2x) + Be^{-x}\sin(2x)$$

Example with forced oscillations:

Suppose a mass-spring-dashpot system is subject to an external sinusoidal driving force $r(t) = 170\sin t$

Find the eqn of motion if $m = 1\text{kg}$, $c = 2\frac{\text{kg}}{\text{s}}$, $k = 10\frac{\text{kg}}{\text{s}^2}$ and if the mass is initialized at rest from the static equilibrium position.

Using Newton's 3rd law, we find that

$$my'' + cy' + ky = F_{app}$$

(NH) $y'' + 2y' + 10y = 170\sin t$; $y(0) = 0$, $y'(0) = 0$

First, solve (H):

Try $y(t) = e^{\lambda t} \Rightarrow$ Get C.E.: $\lambda^2 + 2\lambda + 10 = 0$

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4(10)}}{2} = \frac{-2 \pm 6i}{2}$$

$$\lambda = -1 \pm 3i$$

$$\Rightarrow y_h(t) = e^{-t}\left[A\cos(3t) + B\sin(3t)\right]$$

Finding particular soln to (NH).

Put $y_p(t) = C\cos t + D\sin t$,

Get: $(C\cos t + D\sin t)'' + 2(C\cos t + D\sin t)' + 10(C\cos t + D\sin t) = 170\sin t$

$-C\cos t - D\sin t - 2C\sin t + 2D\cos t + 10C\cos t + 10D\sin t = 170\sin t$

$9C\cos t - 2C\sin t + 9D\sin t + 2D\cos t = 170\sin t$

Grouping and comparing:

$\cos(t)$: $9C + 2D = 0$ $\qquad$ $\sin(t)$: $-2C + 9D = 170$

sub $D = -4.5C$

$D = -4.5C$ $\qquad$ $-2C - 40.5C = 170$

$C = \frac{170}{-42.5} \Rightarrow \boxed{C = -4}$

$\Rightarrow \boxed{D = 18}$

Gen'l soln: $y(x) = -4\cos t + 18\sin t + e^{-t}[A\cos(3t) + B\sin(3t)]$

I.C. $y(0) = 0$

$-4 + A = 0 \Rightarrow \boxed{A = 4}$

$y'(t) = 4\sin t + 18\cos t - e^{-t}[A\cos(3t) + B\sin(3t)] + e^{-t}[-3A\sin(3t) + 3B\cos(3t)]$

$y'(0) = 18 - A + 3B = 0$

$18 - 4 = -3B$

$\boxed{B = -\frac{14}{3}}$

$\therefore$ The eqn of motion is $y(t) = e^{-t}\left[4\cos(3t) - \frac{14}{3}\sin(3t)\right] + 18\cos t - 4\sin t$

$\square$

Now consider the case where a piece of $r(x)$ happens to be a soln of the associated homogeneous equation...

E.g. Solve $y'' + 3y' - 10y = 4e^{2x}$ (NH)

First, solving (H): Try $y = e^{\lambda x}$

Get C.E.: $\lambda^2 + 3\lambda - 10 = 0$

$(\lambda + 5)(\lambda - 2) = 0$

$y_h(x) = Ae^{-5x} + Be^{2x}$

Uh Oh... Trying $y_p(x) = Ce^{2x}$ won't give $4e^{2x}$ ever! Because its a soln to (H), it will always give 0!

This sort of problem leads to the "modification rule" on the handout. In short, y_p is obtained by multiplying the trial y_p (which is already a soln to (H)) by x^k, where k is the multiplicity of the trivial soln.

Applying this to our problem,

Put $y_p(x) = Cxe^{2x}$ into (NH)

$(Cxe^{2x})'' + 3(Cxe^{2x})' + 10Cxe^{2x} = 4e^{2x}$

$2Ce^{2x} + 4Cxe^{2x} + 2Ce^{2x} + 3(Ce^{2x} + 2Cxe^{2x}) + 10Cxe^{2x} = 4e^{2x}$

$4Ce^{2x} + 4Cxe^{2x} + 3Ce^{2x} + 2Cxe^{2x} + 10Cxe^{2x} = 4e^{2x}$

$7Ce^{2x} + 16xe^{2x} = 4e^{2x}$

Compare:

e^x: $7C = 4$

$\boxed{C = \frac{4}{7}} \Rightarrow y_p(x) = \frac{4}{7}xe^{2x}$

$\therefore$ Gen'l soln: $y(x) = \frac{4}{7}xe^{2x} + Ae^{-5x} + Be^{2x}$

E.g. Solve $y'' - 2y' + y = e^x + x$

First solve (H):

Try $y = e^{\lambda x}$, Get C.E: $\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0 \longrightarrow y_h(x) = Ae^x + Bxe^x$

$\Rightarrow \lambda = 1$

Put $y_p(x) = Cx^2e^x + Dx + E$

$$y_p'' - 2y_p' + y_p = e^x + x$$

$$(Cx^2e^x)'' - 2[(Cx^2e^x)' + D] + Cx^2e^x + Dx + E = e^x + x$$

$$(2Cxe^x + Cx^2e^x)' - 2[(2Cxe^x + Cx^2e^x) + D] + Cx^2e^x + Dx + E = e^x + x$$

$$\underline{2Ce^x} + 2Cxe^x + 2Cxe^x + Cx^2e^x - 4Cxe^x - 2Cx^2e^x - 2D + Cx^2e^x + Dx + E = e^x + x$$

Compare

$e^x: 2C = 1$ $\Rightarrow \boxed{C = \frac{1}{2}}$

$x: \boxed{D = 1}$

$1: -2D + E = 0$ $\Rightarrow \boxed{E = 2}$

$$\therefore y_p(x) = \frac{1}{2}e^x + x + 2$$

Gen'l soln: $y(x) = \frac{1}{2}e^x + x + 2 + Ae^x + Bxe^x$