Multivariable Functions

$z = f(x,y)$ in $\mathbb{R}^3$ $\quad f: \mathbb{R}^2 \to \mathbb{R}$

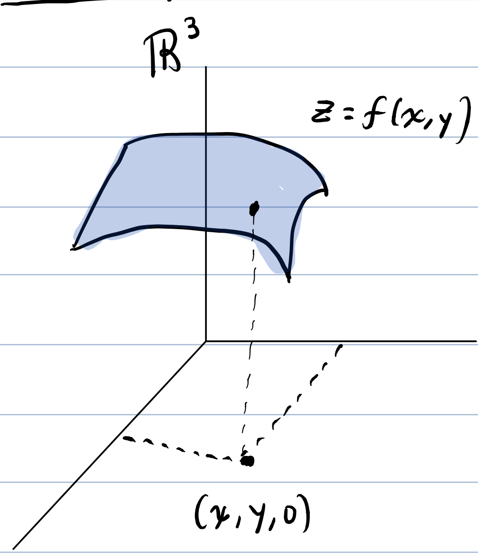
$w = f(x,y,z)$ in $\mathbb{R}^4$ $\quad f: \mathbb{R}^3 \to \mathbb{R}$

$\vdots$

$y = f(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^{n+1}$ $\quad f: \mathbb{R}^n \to \mathbb{R}$

Graphically

more in a minute



Domain

$y = f(x_1, x_2, \ldots, x_n)$ the domain of f is a subset of $\mathbb{R}^n$.

It is the set of all $(x_1, x_2, \ldots, x_n)$ where the function exists.

\# variables

in $\mathbb{R}^2$: The domain is a region of the x,y-plane

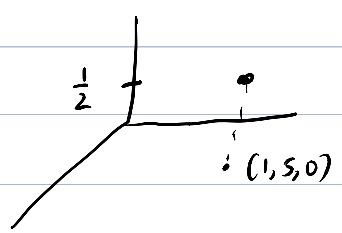
in $\mathbb{R}^3$: The " (filled) solid region of 3D-space.

ex. Find & sketch in $\mathbb{R}^2$ the domain of

$$f(x,y) = \frac{1}{\sqrt{y - x^2}}$$

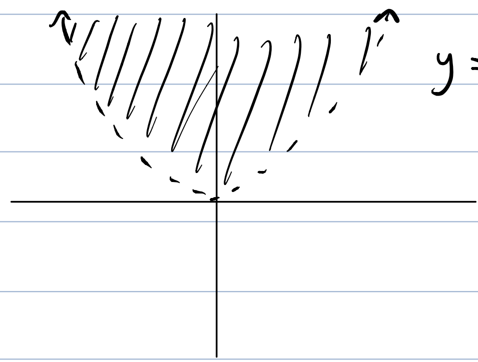
ex: $f(1,5) = \frac{1}{\sqrt{5-1}} = \frac{1}{2}$

If $z = f(x,y)$ is interpreted as height:



$y - x^2 > 0$ (sqrt + denominator)

$\therefore y > x^2$



$y = x^2$

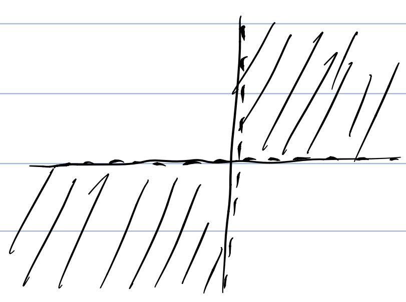
ex. For you: Find and sketch the domain of

① $z = \ln(xy)$ (sketch in $\mathbb{R}^2$)

② $w = \sqrt{x^2+y^2+z^2-1}$ (sketch in $\mathbb{R}^3$)

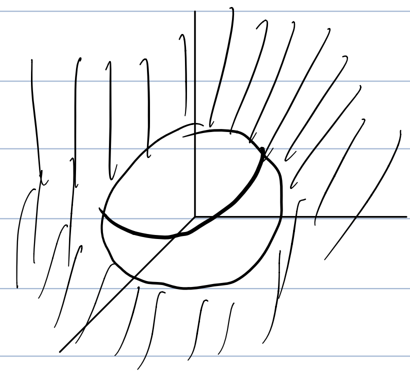
① $xy > 0$

$x > 0, y > 0$ & $x < 0, y < 0$



(2) $x^2+y^2+z^2-1 \geq 0$

$x^2+y^2+z^2 \geq 1$



Interpretation of w?

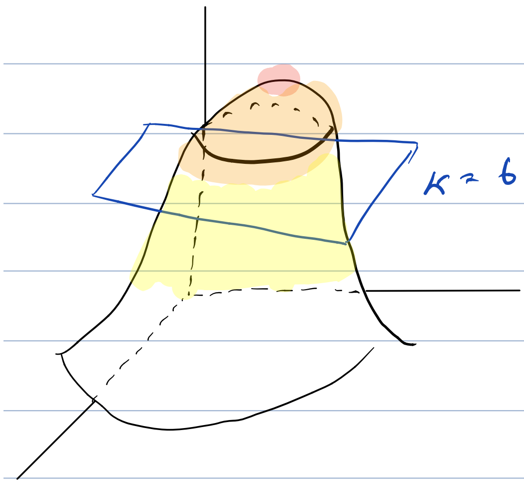
geometrically in $\mathbb{R}^3$ → not possible

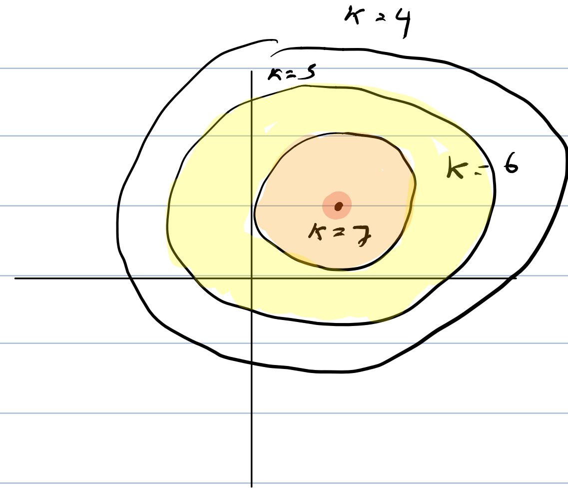
(1) - colour
- temperature } heat map

(2) time (object changing in time)

(3) density

Graphing $\mathbb{R}^3 \to \mathbb{R}^2$ | Contour Map





plot cross sections in
$z = k$ for different k-values

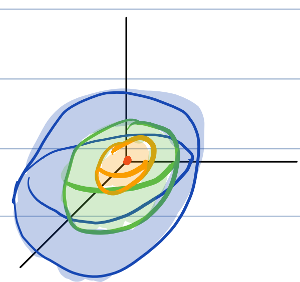
Each of these curves which results is called a "level curve"

The whole plot is called a contour-plot.

Colour → increasing height gets increasing "warmer" colours

This gives us a technique to graph $w = f(x, y, z)$ in $\mathbb{R}^3$.

ex.



← level surfaces (let $w = k$)

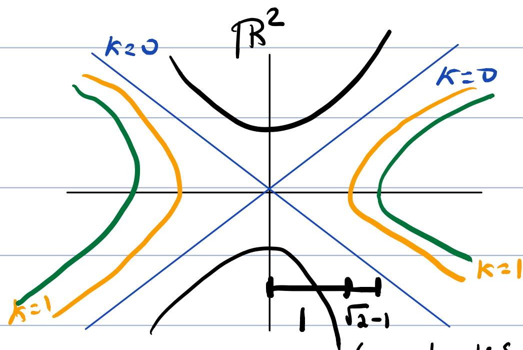
ex. Draw a contour plot (level curves) for

$z = f(x,y) = x^2 - y^2$

Examine & plot cross-sections in $z = k$

$z = x^2 - y^2$

$z = k = 0$: $0 = x^2 - y^2 \to y = \pm x$



↳ indicates steepness

$z = k = 1$: $1 = x^2 - y^2$ ... $x = 0$: $1 = -y^2$ No y-int

$y = 0$: $1 = x^2$

$x = \pm 1$

$z = k = 2$: $2 = x^2 - y^2$ No y-int, $x = \pm\sqrt{2}$

$z = k = -1$: $-1 = x^2 - y^2$ No x-int, $y = \pm 1$

April 4

Recall: Graphs

$z = f(x,y)$ 3d sketch

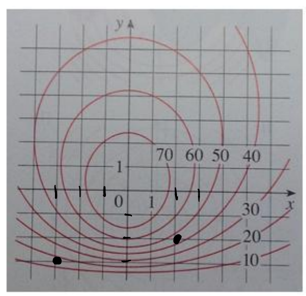
contour plot (level curves) $z=k$ → closer lines = steeper

heat map

$w = f(x,y,z)$ level surfaces ($w=k$)

heat map

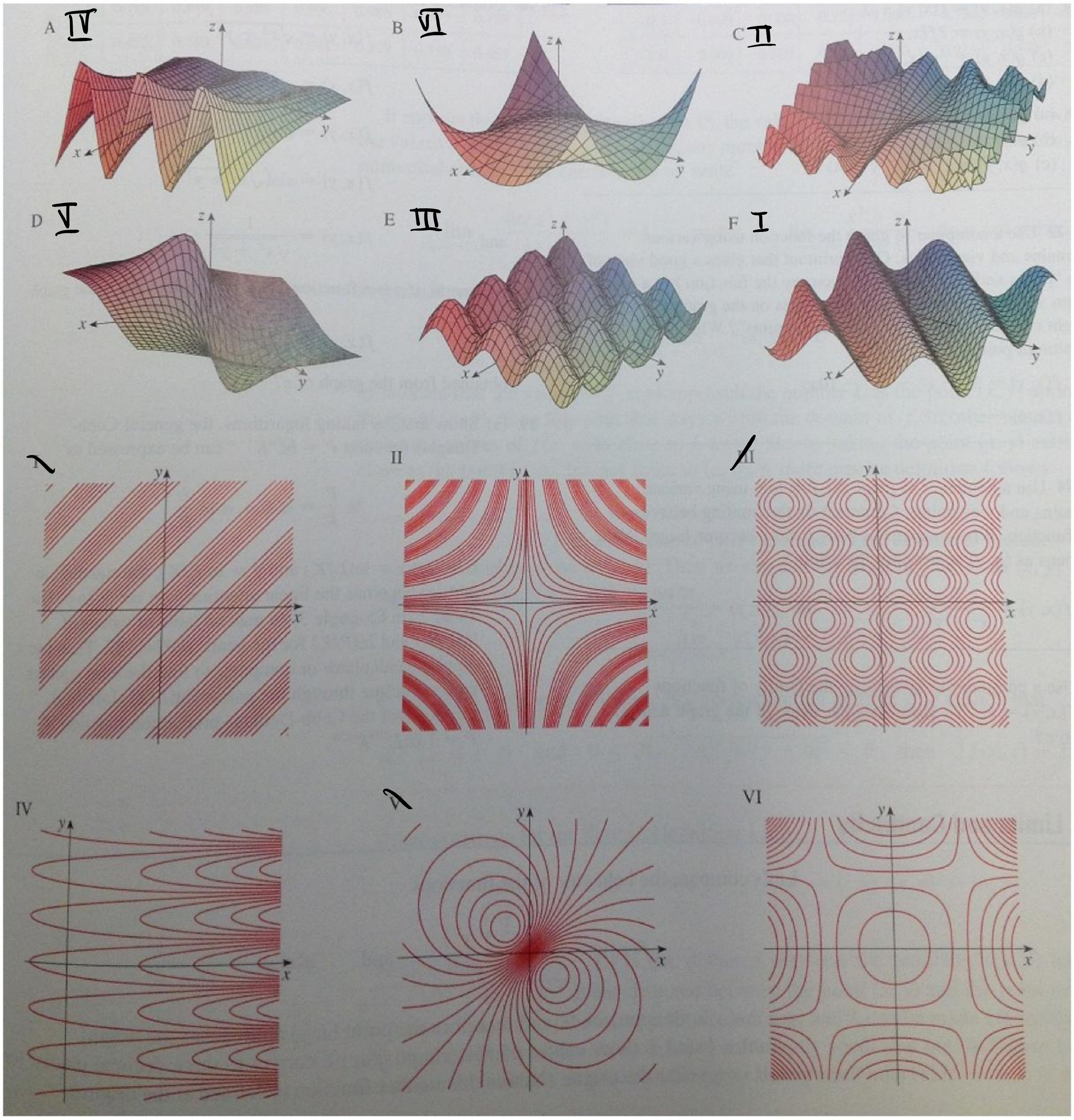
1. A contour map for a function f is shown below. Use it to estimate the values of $f(-3,3)$ and $f(3,-2)$. What can you say about the shape of the graph?



$f(-3,3) \approx 15$

$f(3,-2) \approx 45$

2. Match the graph (labeled A - F below) with its contour map (labeled 1 - VI).



Limits $z = f(x,y)$

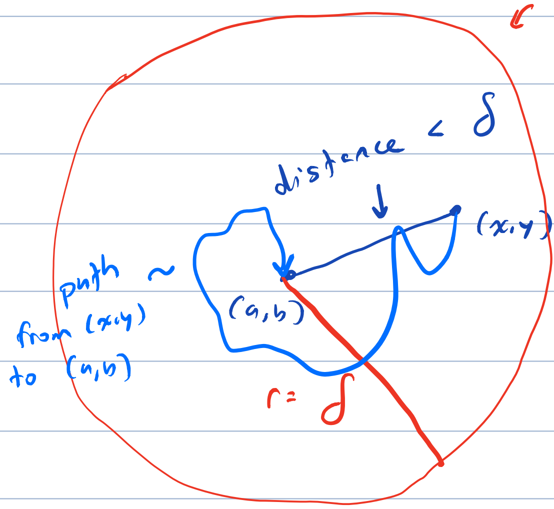
Def'n: The limit of $z = f(x,y)$ as (x,y) approaches (a,b) exists and equals L if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|f(x,y) - L| < \varepsilon$

and we write $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$

Unpacking:

$$\sqrt{(x-a)^2 + (y-b)^2} < \delta$$

whenever (x,y) is within a circle of radius δ away from (a,b) } "close" pts



then $f(x,y)$ is within ε of L

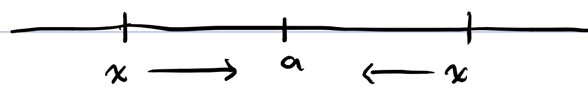
margin of error

Result: In particular, if $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists, it will be independent of the path (x,y) takes to get to (a,b)

There are infinitely many paths from (x,y) to (a,b)

Compare to Cal I : $\lim_{x\to a} f(x)$ $\begin{cases} \lim_{x\to a^-} f(x) \\ \lim_{x\to a^+} f(x) \end{cases}$ checked both paths ✓

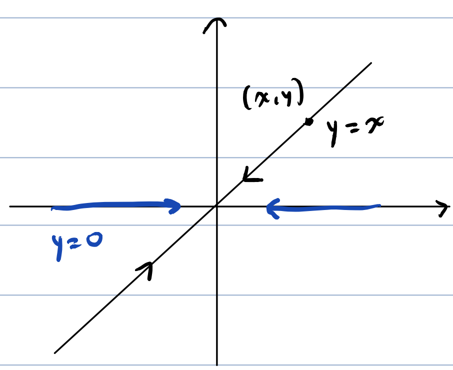
↳ only two paths



In higher dimensions, we cannot check so many paths.

__But__, if we can find two paths from $(x,y)\to(a,b)$ which yield different limit values, then we __can__ conclude that the limit DNE.

__ex.__ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$



along $y=x$

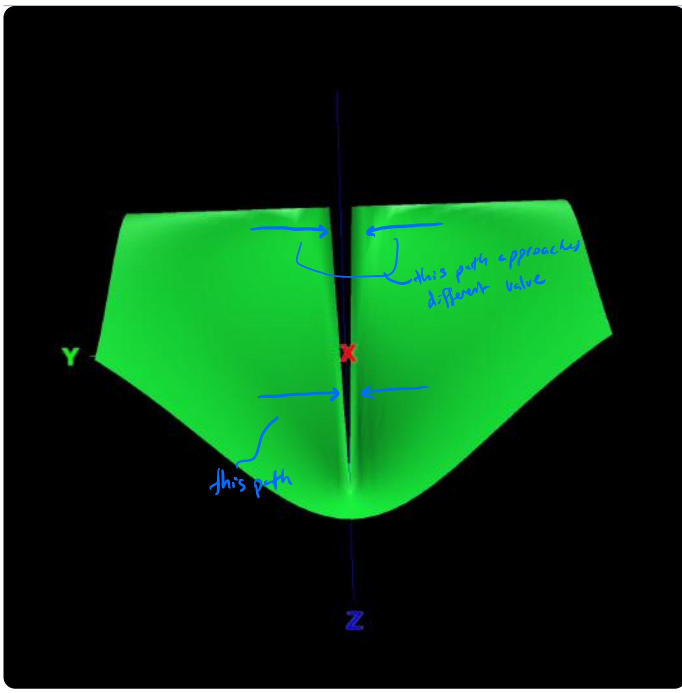
$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2-x^2}{x^2+x^2}$$

$$= \lim_{x\to 0} \frac{0}{2x^2} = 0$$

along $y=0$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

Since the limit is different along two different paths, $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE



ex, For you: Show that the following DNE

① $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

② $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

① along $y = x$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{\cancel{x^2}}{2\cancel{x^2}} = \frac{1}{2}$$

along $y = 0$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{0}{x^2} = 0$$

Since the limit is different along two different paths, $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE

② $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

along $y = \sqrt{x}$:

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{\cancel{x^2}}{2\cancel{x^2}} = \frac{1}{2}$$

along $y = 0$:

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{0}{x^2} = 0$$

Since the limit is different along 2 different paths, $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE

To show that a limit exists is much more difficult.
You need to use:

- Definition (ε, δ) → not in our course
- More sophisticated argument → ex. Squeeze Theorem
- The function is cont's (EASY)

Continuity

A function $f(x,y)$ is continuous at (a,b) if

(i) $(a,b) \in$ Domain of $f(x,y)$

(ii) $\lim_{(x,y)\to(a,b)} f(x,y)$ exists

$\boxed{\text{(iii) } \lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)}$ So just plug in if it's cont's

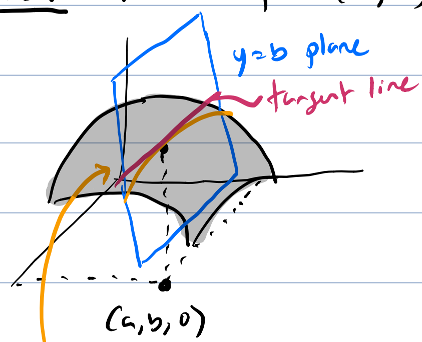
Derivative

As with limits, the concept of derivatives becomes more complicated in multivariable calculus.

We start exploring the topic with the concept of partial derivatives.

Idea: At a pt (a,b), we will define two partial derivatives

$$f_x(a,b) \ \& \ f_y(a,b)$$

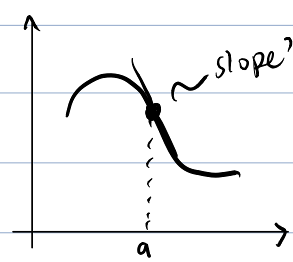


curve of intersection of $\begin{cases} z = f(x,y) \\ y = b \end{cases}$

slope of tangent line $= f_x(a,b)$, the partial derivative of $f(x,y)$ with respect to x at (a,b)

April 6

Recall: (Cal I)

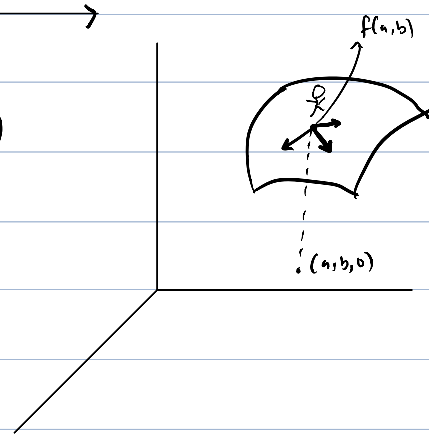


→ instantaneous rate of change

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

→ slope of tangent

$z = f(x,y)$



instantaneous rate of change?

∞ many directions to talk about slope

Later: directional derivative → specify direction

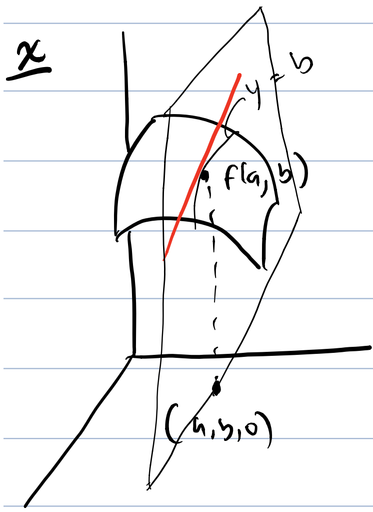
Now: prioritize two of the ∞-many directions

→ direction of coordinate axes

The instantaneous rate of change of $z = f(x,y)$ in the x-direction at (a,b) is called the partial derivative of f with respect to

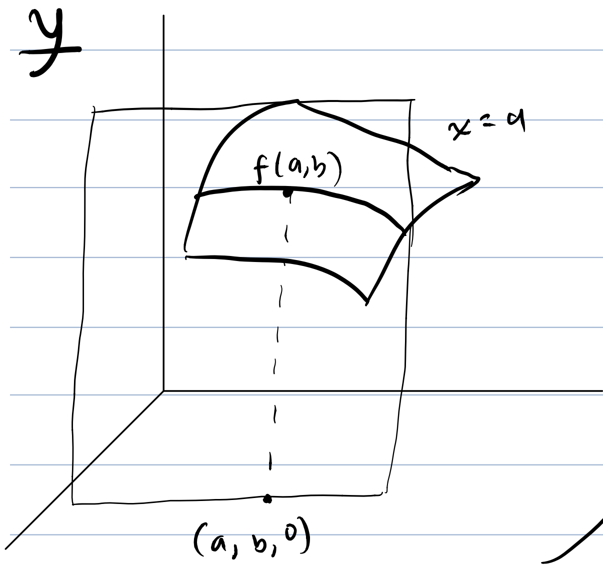
x and is noted: $z_x(a,b) = f_x(a,b) = \left.\dfrac{\partial f}{\partial x}\right|_{(a,b)} = \left.\dfrac{\partial z}{\partial x}\right|_{(a,b)} = D_x f(a,b)$

x



$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,\, b) - f(a,b)}{h}$$

y



$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

one variable stays constant, the other acts like Cal I

Technique for finding the partials of a function

$$y = f(x_1, x_2, \ldots, x_n)$$

$f_i(x_1, x_2, \ldots x_n) = \dfrac{\partial f}{\partial x_i}$ → hold all variables other than x_i constant and use the cal I rules on x_i

ex. $\omega = x^3 y^5 z + 3\cos(xz)$

instantaneous rate of change in x-direction ↳ $\dfrac{\partial w}{\partial x} = 3x^2 y^5 z - 3\sin(xz) \cdot z$

$\dfrac{\partial w}{\partial y} = 5x^3 y^4 z$

$\dfrac{\partial w}{\partial z} = x^3 y^5 - 3\sin(xz) \cdot x$

ex. For you. Find all the partials

① $z = \tan(x^2 y)$

② $f(x, y, z) = x^y z$

③ $z = 3^{xy + z^2}$

1. $\frac{\partial z}{\partial x} = 2xy \sec^2(x^2 y)$

$\frac{\partial z}{\partial y} = x^2 \sec^2(x^2 y)$

2. $\frac{\partial}{\partial x} f(x, y, z) = y x^{y-1} z$

$\frac{\partial}{\partial y} f(x, y, z) = \ln x \cdot x^y z$

$\frac{\partial}{\partial z} f(x, y, z) = x^y$

3. $w = 3^{xy + z^2}$

$\frac{\partial w}{\partial x} = \ln 3 \cdot 3^{xy+z^2} \cdot y$

$\frac{\partial w}{\partial y} = \ln 3 \cdot 3^{xy+z^2} \cdot x$

$\frac{\partial w}{\partial z} = \ln 3 \cdot 3^{xy+z^2} \cdot 2z$

Higher Order Partials

$$z = f(x,y)$$

$$\begin{tikzcd}
 & & f \arrow[dl, "\frac{\partial}{\partial x}"'] \arrow[drr, "\frac{\partial}{\partial y}"] & & & \\
 & f_x = \frac{\partial f}{\partial x} \arrow[dl, "\frac{\partial}{\partial x}"'] \arrow[dr, "\frac{\partial}{\partial y}"] & & & f_y = \frac{\partial f}{\partial y} \arrow[dl, "\frac{\partial}{\partial x}"'] \arrow[dr, "\frac{\partial}{\partial y}"] & \\
\frac{\partial^2 f}{\partial x^2} = f_{xx} & & f_{xy} = \frac{\partial^2 f}{\partial y \partial x} & f_{yx} = \frac{\partial^2 f}{\partial x \partial y} & & f_{yy} = \frac{\partial^2 f}{\partial y^2}
\end{tikzcd}$$

Note: $f_{xy} = (f_x)_y$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

ex. For you

① $z = f(x,y) = \ln(x^2+y^2)$

Find z_{xx}, z_{xy}, z_{yx}, z_{yy}

$$z_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \frac{1}{x^2+y^2} \cdot 2x$$

$$= \frac{2(x^2+y^2) - 2x(2x)}{(x^2+y^2)^2} = \frac{2x^2-2y^2}{x^2+y^2}$$

$$z_{xy} = \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial y} z = \frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2}\right)$$

$$= -4xy\,(x^2+y^2)^{-2}$$

$$z_{yx} = \frac{\partial}{\partial y} \cdot \frac{\partial}{\partial y} z = \frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2}\right)$$

$$= -4xy\,(x^2+y^2)^{-2}$$

Clairaut's Theorem:
If f is defined on a disk around (a,b) and if f_{xy} and f_{yx} are conts at (a,b), then $f_{xy}(a,b) = f_{yx}(a,b)$

$$z_{yy} = \frac{2(x^2+y^2) - 4y^2}{(x^2+y^2)^2} = \frac{2y^2 - 2x^2}{x^2+y^2}$$

(2) If $u = e^{-\alpha^2 k^2 t} \sin(kx)$; $a, k \in \mathbb{R}$

Verify that

partial differential equation (PDE)
"Heat conduction Eq"

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$-\alpha^2 k^2 e^{-\alpha^2 k^2 t} \sin(kx) = \alpha^2 e^{-\alpha^2 k^2 t} \cos(kx) \cdot k$$

$$= -\alpha^2 e^{-\alpha^2 k^2 t} \sin(kx)\, k^2$$

③ Implicit differentiation

If $z^3y^2 + y^3x^2 + x^3z^2 = 4$

Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$

not constants

$\frac{\partial z}{\partial x}$: $\frac{\partial}{\partial x}\left(z^3y^2 + y^3x^3 + x^3z^2\right) = 0$

chain rule

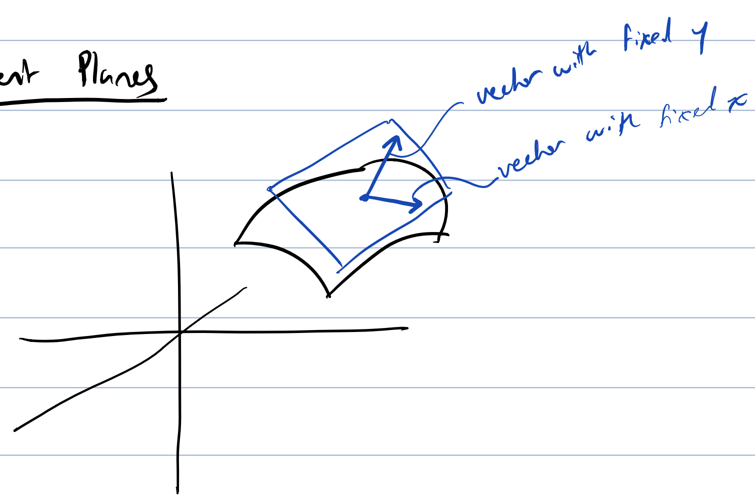
$3z^2y^2 \cdot \frac{\partial z}{\partial x} + 2xy^3 + \left(3x^2z^2 + \underbrace{x^3 \cdot 2z\frac{\partial z}{\partial x}}\right) = 0$

isolate $\frac{\partial z}{\partial x}$

$\frac{\partial z}{\partial y}$: $3z^2y^2\frac{\partial z}{\partial y} + 2z^3y + 3y^2x^2 + 2x^3z\frac{\partial z}{\partial y} = 0$

April 9

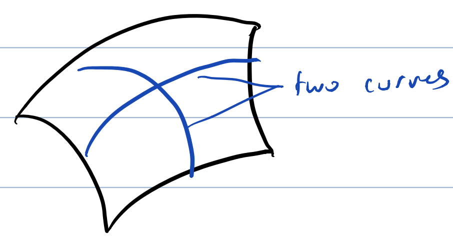
Tangent Planes



If $f(x,y)$ is a differentiable function, then the two curves $z = f(x_0, y)$, $z = f(x, y_0)$ intersect at the point $(x_0, y_0, f(x_0, y_0))$

The tangent plane should contain the point $(x_0, y_0, f(x_0, y_0))$ and the tangent vectors to the two curves.

We can parameterize $\vec{r}(x) = \langle x, y_0, f(x, y_0) \rangle$
$\vec{c}(y) = \langle x_0, y, f(x_0, y) \rangle$



Tangent vectors:

$\vec{r}\,'(x) = \langle 1, 0, f_x(x, y_0) \rangle$

$\vec{c}\,'(y) = \langle 0, 1, f_y(x_0, y) \rangle$

We can take the normal vector of the plane as

$$\vec{n} = \vec{r}\,'(x_0) \times \vec{c}\,'(y_0)$$
$$= \langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1 \rangle$$

The equation of the plane:

$$-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + (z - \overbrace{f(x_0, y_0)}^{z_0}) = 0$$

Theorem: The tangent plane exists at (x_0, y_0) if f_x and f_y are defined on a disk containing (x_0, y_0) and are continuous at (x_0, y_0)

ex. Find the eq of the tangent plane and the normal line to the given surface at the given pt.

(a) $z = xy$ at $(-1, 2, -2)$

$$\left.\begin{aligned} z_x &= y \\ z_y &= x \end{aligned}\right\} \begin{aligned} &z_x, z_y \text{ are defined and cont}^{s} \text{ everywhere} \\ &\therefore \text{ tangent plane exists at } (-1,2,-2) \end{aligned}$$

$z_x(-1,2) = 2$
$z_y(-1,2) = -1$

eqn plane:

$$2(x+1) - 1(y-2) - (z+2) = 0$$

$$2x - y - z = -2$$

normal line: $\vec{n} = \langle -2, 1, 1 \rangle$

$$\langle x, y, z \rangle = \langle -1, 2, -2 \rangle + t \langle -2, 1, 1 \rangle$$

(b) $f(x,y) = \sqrt{x^2+y^2}$ at $(3,4,5)$

$$\left.\begin{aligned} f_x &= -\frac{x}{\sqrt{x^2+y^2}} \\ f_y &= -\frac{y}{\sqrt{x^2+y^2}} \end{aligned}\right\}$$ f_x, f_y are defined & cont's at $(3,4,5)$
$\therefore$ tangent plane exists at $(3,4,5)$

$f_x(3,4) = -\frac{3}{5}$, $f_y(3,4) = -\frac{4}{5}$

eqn plane: $\frac{3}{5}(x-3) + \frac{4}{5}(y-4) + (z-5) = 0$

normal line: $\langle x,y,z \rangle = \langle 3,4,5 \rangle + t \langle \frac{3}{5}, \frac{4}{5}, 1 \rangle$

(c) $x^2y^2 + x^2z^2 + y^2z^2 = 3$ at $(1,1,1)$

f_x: $\frac{\partial}{\partial x}\left(x^2y^2 + x^2z^2 - y^2z^2 = 3\right)$

$\frac{\partial z}{\partial x}$ $2xy^2 + \left(2xz^2 + 2z\frac{\partial z}{\partial x}x^2\right) + 2z\frac{\partial z}{\partial x}y^2$

$-\frac{\partial z}{\partial x}\left(2zx^2 + 2zy^2\right) = 2xy^2 + 2xz^2$

$$\frac{\partial z}{\partial x} = -\frac{xy^2 + xz^2}{zx^2 + zy^2}$$

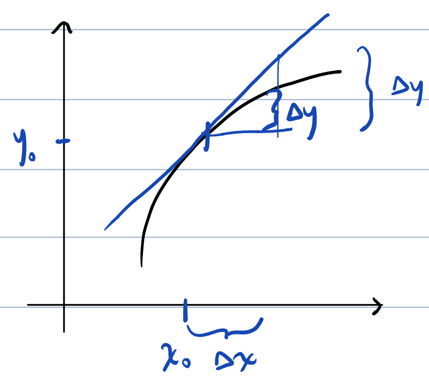
$$\frac{\partial z}{\partial y} = -\frac{yx^2 + yz^2}{zy^2 + zx^2}$$ } defined and conts at $(1,1,1)$

Tangent plane: $x + y + z = 3$

Normal line: $\langle x, y, z \rangle = \langle 1, 1, 1 \rangle + t \langle 1, 1, 1 \rangle$

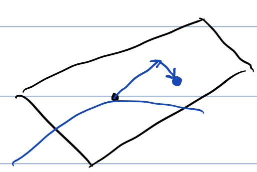
Differentials and Linearization

2D



Let $z = f(x,y)$. The differential dz is defined to be

$$dz = f_x(x,y)dx + f_y(x,y)dy$$



As in Cal I, the differential can be used to approximate the function by

$$\Delta z \approx dz$$

$$z - z_0 \approx dz$$

$$z \approx z_0 + dz = \underbrace{z_0 + f_x(x,y)dx + f_y(x_0,y)dy}_{\substack{\text{linearization of the surface} \\ z = f(x,y)}}$$

If we let $dx = x - x_0$ (or Δx)

$dy = y - y_0$ (or Δy)

Then, $z \approx z_0 + dz = \underbrace{z_0 + f_x(x,y)(x-x_0) + f_y(x_0,y)(y-y_0)}_{\text{tangent plane}}$

For $\mathbb{R}^3$: $u = f(x,y,z)$

$$du = f_x(x,y,z)dx + f_y(x,y,z)dy + f_z(x,y,z)dz$$

For $\mathbb{R}^n$: $u = f(x_1, x_2, x_3, \ldots, x_n)$

$$du = \sum_{i=1}^{n} f_{x_i}(x_1, x_2, \ldots, x_n)\, dx_i$$

$$u \approx u_0 + du$$

Ex. Use differentials to approximate the value of each of the following and compare with the value on the calculator.

(a) $\sqrt{0.99}\, e^{0.02}$

$z(x,y) = \sqrt{x}\, e^{y}$ around $(1,0)$

$\Delta x = -0.01$, $\Delta y = 0.02$

$z_x = \frac{1}{2\sqrt{x}} e^{y} \rightarrow z_x(1,0) = \frac{1}{2}$

$z_y = \sqrt{x}\, e^{y} \rightarrow z_y(1,0) = 1$

$z \approx \underbrace{1}_{z_0} - 0.05 + 0.02$

≈ 1.015

$Z_{\text{calculator}} = 1.01509$

b) $f(x,y,z) = x^2 y^3 z^4$ @ $(1.05, 0.9, 3.01)$

April 11

Differentiability

{ Cont: connected, diff: smooth → see Omnivox document

The Chain rule

3 cases:

① $z = f(x,y)$ & $x = x(t)$
$y = y(t)$

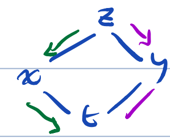
$z = \underbrace{g(t)}_{\text{single variable}} = f(x(t), y(t))$

Cal I: What is $\frac{dz}{dt}$ (not $\frac{\partial z}{\partial t}$)?
$(z'(t) = g'(t))$

Method I: Substitute $x(t)$ & $y(t)$ into $f(x,y)$,
then use Cal I rules

Method II: Chain Rule (or total derivative)

diagram:



$f(x,t)$ multi-variable

$x'(t)$ ~single variable

$$\frac{dz}{dt} = \underline{\frac{\partial z}{\partial x}\frac{dx}{dt}} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

Compare to the differential:

Idea: $\dfrac{dz}{dt} = \underset{\downarrow\ \frac{\partial z}{\partial x}}{f_x}(x,y)\dfrac{dx}{dt} + \underset{\downarrow\ \frac{\partial z}{\partial y}}{f_y}(x,y)\dfrac{dy}{dt}$

ex. A) $z = \sqrt{x - y^2}$, $x = \cos t$, $y = \sin t$

Find $\dfrac{dz}{dt}$ using both methods & compare

Method I: $z(t) = \sqrt{\cos t - \sin^2 t}$

$$z'(t) = \frac{-\sin t + 2\sin t\cos t}{2\sqrt{\cos t - \sin^2 t}}$$

Method II : $\dfrac{\partial z}{\partial x} = \dfrac{1}{2\sqrt{x-y^2}}$ $\qquad \dfrac{dx}{dt} = -\sin t$

$$\frac{\partial z}{\partial y} = \frac{-\cancel{2}y}{2\sqrt{x-y^2}} \qquad \frac{dy}{dt} = \cos t$$

$$\frac{dz}{dt} = \frac{1}{2\sqrt{x-y^2}}(-\sin t) - \frac{y}{\sqrt{x-y^2}} \cos t$$

$$= \frac{-\sin t}{2\sqrt{\cos t - \sin^2 t}} - \frac{\sin t \cos t}{\sqrt{\cos t - \sin^2 t}}$$

B) $w = x e^{y/z}$

$x = t^2$; $y = 1-t$; $z = 1+2t$

a) Guess the chain rule for $\dfrac{dw}{dt}$

$$\begin{tikzcd}[column sep=small]
 & w \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
x \arrow[dr, dash] & y \arrow[d, dash] & z \arrow[dl, dash] \\
 & t &
\end{tikzcd}$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

b) Apply it

$$\frac{\partial w}{\partial x} = e^{y/z} \qquad \frac{dx}{dt} = 2t$$

$$\frac{\partial w}{\partial y} = \frac{x}{z} e^{y/z} \qquad \frac{dy}{dt} = -1$$

$$\frac{\partial w}{\partial z} = -x e^{y/z} \cdot \frac{y}{z^2} \qquad \frac{dz}{dt} = 2$$

$$\frac{dz}{dt} = 2t\, e^{y/z} - \frac{x}{z} e^{y/z} - 2\frac{xy}{z^2} e^{y/z}$$

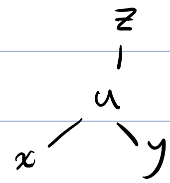
$$= 2t\, e^{\frac{1-t}{1+2t}} - \frac{t^2}{1+2t} e^{\frac{1-t}{1+2t}} + \frac{2t^2(1-t)}{(1+2t)^2} e^{\frac{1-t}{1+2t}}$$

Note: $z = f(x_1, x_2, \ldots, x_n)$ & $x_i = x_i(t)$ for all i

$$\frac{dz}{dt} = \sum_{i=1}^{n} \frac{\partial z}{\partial x_i} \frac{dx_i}{dt}$$

Case ② $z = f(u)$

$u = g(x,y)$



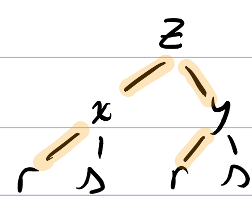
$$z = z(x,y)$$

$$\frac{\partial z}{\partial x} = \frac{dz}{du}\,\frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du}\,\frac{\partial u}{\partial y}$$

Case ③

$$\left.\begin{aligned} z &= f(x,y) \\ x &= x(r,s) \\ y &= y(r,s) \end{aligned}\right\} z = z(r,s)$$



$\frac{dz}{dr}$ or $\frac{\partial z}{\partial r}$?

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\,\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\,\frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

ex. For you

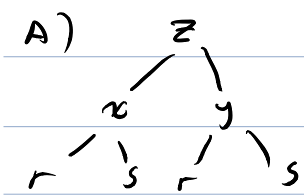
A) If $z = x^2 y^3$ and $x = r \ln s$
$y = s \ln r$

Find $\frac{\partial z}{\partial r}$ & $\frac{\partial z}{\partial s}$

B) If $W = f(x,u,z)$ and $x = x(r,s)$
$y = y(r,s)$
$z = z(r,s)$

Write the chain rule for $\frac{\partial w}{\partial s}$

A)



$\frac{\partial z}{\partial x} = 2xy^3$ $\frac{\partial x}{\partial r} = \ln s$ $\frac{\partial x}{\partial s} = \frac{r}{s}$

$$\frac{\partial z}{\partial y} = 3x^2y^2 \qquad \frac{\partial y}{\partial r} = \frac{s}{r} \qquad \frac{\partial y}{\partial s} = \ln r$$

$$\frac{\partial z}{\partial r} = 2xy^3 \ln s + 3x^2y^2 \cdot \frac{s}{r}$$

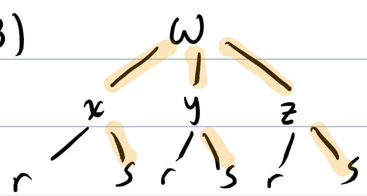
if subst. required

$$= 2r(\ln s)^2(s\ln r)^3 + 3(r\ln s)^2(s\ln r)^2 \cdot \frac{s}{r}$$

$$\frac{\partial z}{\partial s} = 2xy^3 \frac{r}{s} + 3x^2y^2 \ln r$$

$$\vdots$$

B)



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

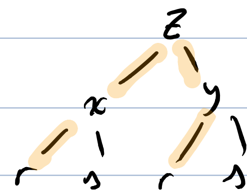
Higher Order Chain Rule

ex. $z = f(x,y) \qquad x = r^2 + s^2 \qquad y = 2rs$

$\hookrightarrow$ cont's 2^{nd} order partials $\rightarrow$ $f_{xy} = f_{yx}$ (Clairaut's theorem)

Find $\frac{\partial^2 z}{\partial r^2}$ in terms of the partials of z

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right)$$



$$= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}\right)$$

$$= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\cdot 2r + \frac{\partial z}{\partial y}\, 2s\right)$$

$$= 2\left[\underbrace{\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\, r\right)}_{\text{product rule}} + \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\, s\right)\right]$$

s is constant

$$= 2\left[\underbrace{\frac{\partial}{\partial r}\left(\overset{z_x}{\boxed{\frac{\partial z}{\partial x}}}\right)}_{\text{chain rule}} r + \frac{\partial z}{\partial x}\left(\cancelto{1}{\frac{\partial}{\partial r} r}\right) + s\,\underbrace{\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)}_{\text{chain rule}}\right]$$

$$= 2\left[r\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y\,\partial x}\frac{\partial y}{\partial r}\right) + \frac{\partial z}{\partial x}\right.$$

$$+ s\left(\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\right)\Bigg]$$

then simplify

April 13

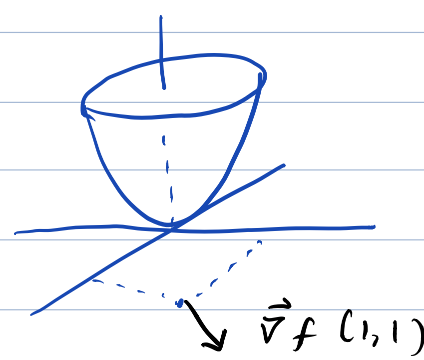
The Gradient & The Directional Derivative

Def: If $z = f(x,y)$, then the gradient of f at a point (a,b) is a vector in $\mathbb{R}^2$, notated

$$\vec{\nabla} f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle$$

nabla

ex. $z = x^2 + y^2$ ($= f(x,y)$) $\rightsquigarrow$ paraboloid



Calculate $\vec{\nabla} f(1,1)$

$f_x(x,y) = 2x$

$f_y(x,y) = 2y$

$\vec{\nabla} f(x,y) = \langle 2x, 2y \rangle$

$\{ \langle 2,2,0 \rangle$

$\vec{\nabla} f(1,1) = \langle 2, 2 \rangle$

$\vec{\nabla} f(0,2) = \langle 0, 4 \rangle$

In general if $y = f(x_1, x_2, \ldots, x_n)$

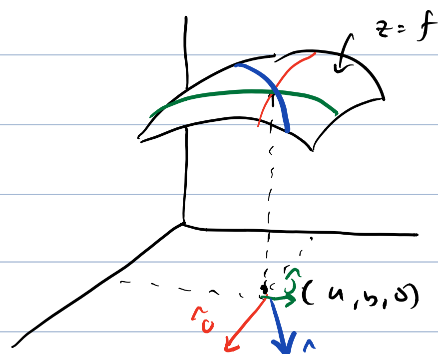
$$\vec{\nabla} f(a_1, a_2, \ldots, a_n)$$

$$= \langle f_{x_1}(a_1, \ldots, a_n), f_{x_2}(a_1, \ldots, a_n), \ldots f_{x_n}(a_1, \ldots, a_n) \rangle$$

in the space of the input variables

1 dimension down

Directional Derivative

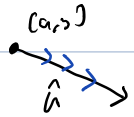


slope $= f_x(a,b)$

slope$_y = f_y(a,b)$

$\hat{u} \sim$ an arbitrary unit vector

The directional derivative $D_{\hat{u}} f(a,b)$ is the slope of the surface in the direction of $\hat{u}$



$\rightsquigarrow \langle a,b\rangle + h\langle u_1, u_2\rangle$

$\langle a + hu_1, b + hu_2\rangle$

$h \to 0$

Let $\hat{u} = \langle u_1, u_2\rangle$, $\|\hat{u}\| = 1$

$$\boxed{D_{\hat{u}} f(a,b) = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}}$$

But! Limits are cumbersome. We want a derivative rule instead.

To do this, consider:

(I) $g(t) = f(a+tu_1, b+tu_2)$

$g(0) = f(a,b)$

$g(h) = g(0+h) = f(a+hu_1, b+hu_2)$

then $g'(0) = \lim_{h\to 0} \dfrac{g(0+h) - g(0)}{h} = \lim_{h\to 0} \dfrac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$

$= D_{\hat{u}} f(a,b)$

(II) $g(t) = f(x,y)$ where $x = a + tu_1$

$y = b + tu_2$

Apply chain rule:

$$\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt}$$

$$g'(t) = f_x(x,y)\,\overset{\frac{dx}{dt}}{u_1} + f_y(x,y)\,u_2$$

$$g'(0) = f_x(a,b)\,u_1 + f_y(a,b)\,u_2$$

$$= \langle f_x(a,b),\ f_y(a,b)\rangle \cdot \langle u_1, u_2\rangle$$

$$D_{\hat{u}} f(a,b) = \vec{\nabla} f(a,b) \cdot \hat{u}$$

$$D_{\hat{u}} f(a_1, \dots, a_n) = \vec{\nabla} f(a_1, \dots, a_n) \cdot \hat{u}$$

1. Find the rate of change of the function $f(x,y) = \arctan(x^2 - y^2)$ in the direction of the vector $\vec{u} = \langle 3, 4\rangle$ at the point $(1, -1)$.

$$\hat{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

$$f_x = \frac{2x}{1 + (x^2 - y^2)^2}$$

$$f_y = \frac{2y}{1 + (x^2 - y^2)^2}$$

$\vec{\nabla} f(1,-1) = \langle \frac{2}{5}, -\frac{2}{5} \rangle$

$D_{\hat{u}} f(1,-1) = \langle \frac{6}{25}, -\frac{8}{25} \rangle$

2. Find the rate of change of $w = xy^2 + yz^2 + zx^2$ at $P(2, -1, 2)$ heading towards the origin.

$W_x = y^2 + 2xz$
$W_y = 2yx + z^2$
$W_z = 2zy + x^2$

$\vec{u} = -\langle 2, -1, 2 \rangle$
$\hat{u} = \langle -\frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \rangle$

$\vec{\nabla} w(2, -1, 2) = \langle 9, 0, 0 \rangle$

$D_{\hat{u}} (2,-1,2) = -6$

other Geometric Interpretations of $\vec{\nabla} f$

$D_{\hat{u}} f(a,b) = \vec{\nabla} f(a,b) \cdot \hat{u}$
$= \| \vec{\nabla} f(a,b) \| \, \| \hat{u} \| \cos\theta$
$= \underbrace{\| \vec{\nabla} f(a,b) \|}_{\text{fixed at } (a,b)} \cos\theta$ ← all change comes from here

$$-1 \le \cos\theta \le 1$$

$$-\|\vec{\nabla} f(a,b)\| \le \underbrace{\|\vec{\nabla} f(a,b)\| \cos\theta}_{D_{\hat{u}} f(a,b)} \le \|\vec{\nabla} f(a,b)\|$$

min: steepest descent occurs when $\theta = \pi$

$\hat{u}$ opposite direction from gradient

max: steepest ascent from $(a,b, f(a,b))$ occurs when $\theta = 0$

$\hat{u}$ same direction as $\vec{\nabla} f(a,b)$

" $\vec{\nabla} f(a,b)$ is the direction of steepest ascent"

" $-\vec{\nabla} f(a,b)$ is the direction of steepest descent"

at (a,b)

3. Find the maximum rate of change on the surface $f(x,y) = x \sin y - y \cos x$ at the point $(0, \pi)$. In which direction is this maximum increase?

$f_x = \sin y + y \sin x$

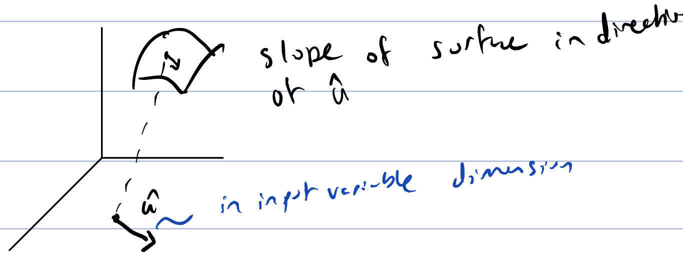
$f_y = x \cos y - \cos x$

$\vec{\nabla} f(0,\pi) = \langle 0, 1 \rangle$ ← direction

$\|\vec{\nabla} f(0,\pi)\| = 1$ ← max rate of change

April 16

Review: $D_{\hat{u}} f(a,b) = \vec{\nabla} f(a,b) \cdot \hat{u}$

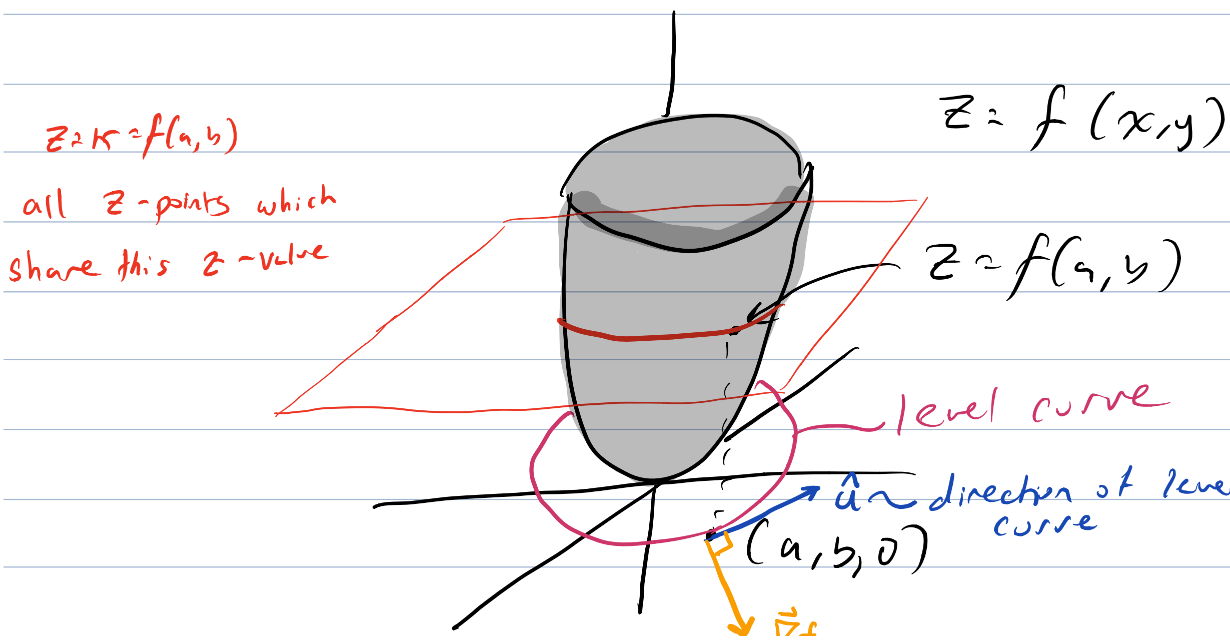


Gradient: $\vec{\nabla} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

$$\max D_{\hat{u}} f(a,b) = \| \vec{\nabla} f(a,b) \|$$

& occurs in direction of $\vec{\nabla} f(x,y)$

Proposition: ① If $\hat{u}$ is taken in the direction of a level curve at (a,b), then $D_{\hat{u}} f(a,b)$



Proof: $D_{\hat{u}} f(a,b) = \lim\limits_{h \to 0} \dfrac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$

$$= \lim_{h \to 0} \frac{f(a,b) - f(a,b)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

Proposition: ② The gradient at (a,b) is $\perp$ to the level curve at (a,b)

Proof: Take $\hat{u}$ at (a,b) in the direction of the level curve at (a,b)

Examine $D_{\hat{u}} f(a,b) = \vec{\nabla} f(a,b) \cdot \hat{u}$

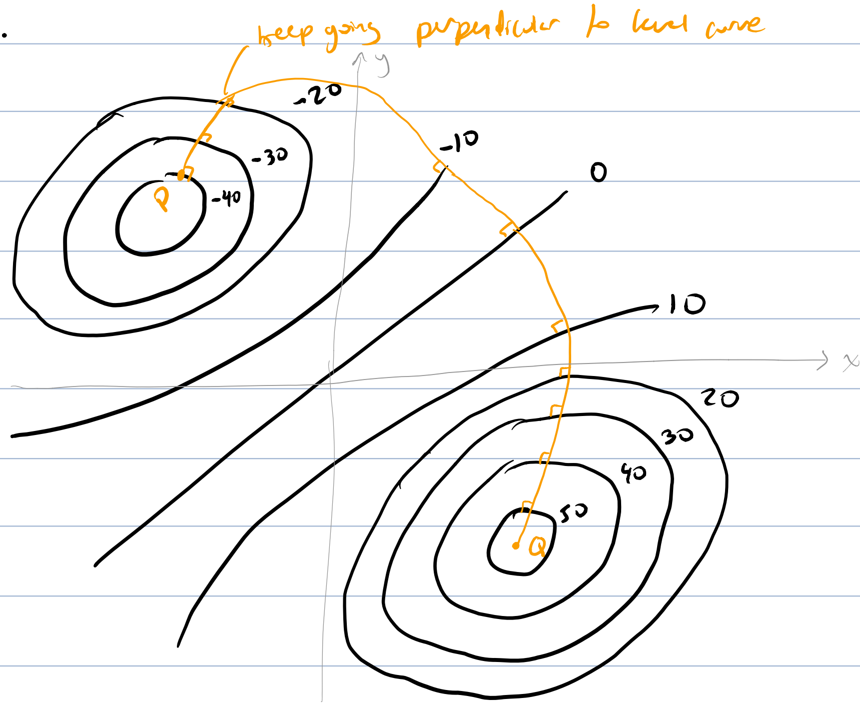
$$0 = \vec{\nabla} f(a,b) \cdot \hat{u}$$

$$\therefore \vec{\nabla} f(a,b) \perp \hat{u}$$

Summary

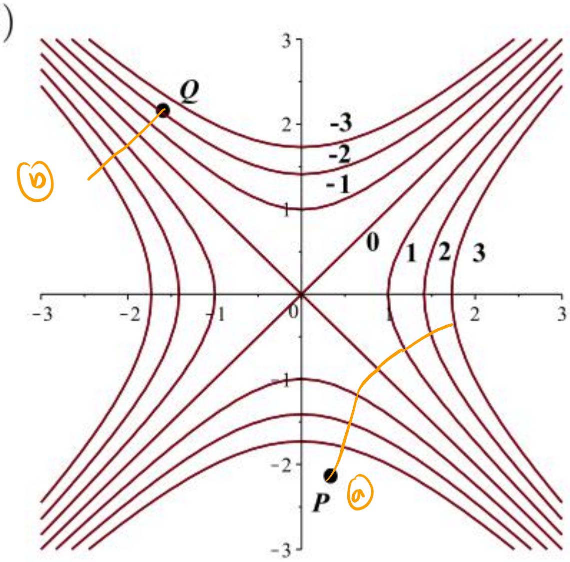
- $\vec{\nabla} f(a,b)$ gives the direction of steepest ascent
- steepest slope at $(a,b) = \|\vec{\nabla} f(a,b)\|$
- $\vec{\nabla} f(a,b) \perp$ level curve at (a,b)

ex. Draw the path of steepest ascent from P to Q.



4. Draw the path of steepest ascent a) from P b) from Q.

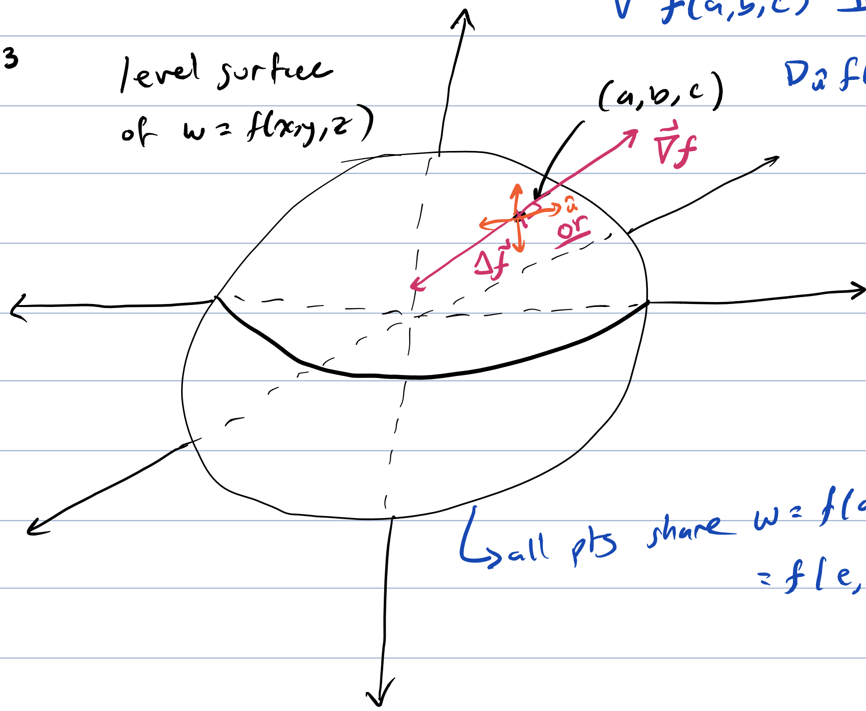
(What type of surface is this?)



$\vec{\nabla} f(a,b,c) \perp \hat{u}$

$D_{\hat{u}} f(a,b,c) = 0$

$\Rightarrow$ In $\mathbb{R}^3$ level surface of $w = f(x,y,z)$



↳ all pts share $w = f(a,b,c)$
$= f(e,f,g)$

The gradient gives us a normal vector to the surface at (a,b,c) → Method 2: use gradient on a function

$w = 2x^2y^3z^4 + xy^2z^3$ at level surface $w = k = 3$

ex. $2x^2y^3z^4 + xy^2z^3 = 3$

Find the tangent plane to this surface at $(1,1,1)$

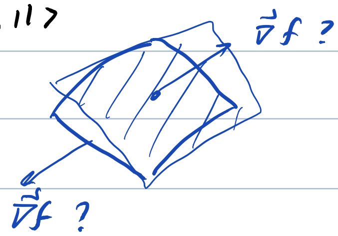
Method 1: Implicit differentiation to find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial x}$

Method 2: ← treat it as a level surface

$\vec{\nabla} f(x, y, z) = \langle f_x(x,y,z),\ f_y(x,y,z),\ f_z(x,y,z) \rangle$

$= \langle 4xy^3z^4 + y^2z^3,\ 6x^2y^2z^4 + 2xyz^3,\ 8x^2y^3z^3 + 3xy^2z^2 \rangle$

$\vec{\nabla} f(1,1,1) = \langle 5, 8, 11 \rangle$



Tangent plane: $5(x-1) + 8(y-1) + 11(z-1) = 0$

5. Find the equation of the tangent plane and normal line to $xy^2 + 2z^2 = 12$ at $(1, 2, 2)$.

$f = xy^2 + 2z^2$

$\vec{\nabla} f = \langle y^2, 2y, 4z \rangle$

$\vec{\nabla} f(1,2,2) = \langle 4, 4, 8 \rangle$

Tangent plane: $4(x-1) + 4(y-2) + 8(z-2)$

6. Consider the surface $x^2 + y^2 + z^2 - 8x - 8y - 6z + 24 = 0$ and $3x^2 + 2y^2 - 2z = 12$.

a) Show that the surfaces have a common point $P(2, 1, 1)$.

b) Find the angle between their tangent planes at P.

c) Find a vector equation for the tangent line to their curve of intersection at P.

a) $x^2 + y^2 + z^2 - 8x - 8y - 6z + 24$

$0 \stackrel{?}{=} \cancel{4 + 1 + 1} - 16 - 8 \cancel{-6} + 24$

$= 0$ ✓

$12 = 3x^2 + 2y^2 - 2z$

$12 \stackrel{?}{=} 12 + \cancel{2 - 2}$

$= 12$ ✓

b) Let $f_1 = x^2 + y^2 + z^2 - 8x - 8y - 6z$

$\vec{\nabla} f_1 = \langle 2x - 8, 2y - 8, 2z - 6 \rangle$

$\vec{\nabla} f_1 (2,1,1) = \langle -4, -6, -4 \rangle$

Let $f_2 = 3x^2 + 2y^2 - 2z$

$\vec{\nabla} f_2 = \langle 6x, 4y, -2 \rangle$

$\vec{\nabla} f_2 (2,1,1) = \langle 12, 4, -2 \rangle$

$\vec{\nabla} f_1 \cdot \vec{\nabla} f_2 = -48 - 24 + 8 = -64 = \overbrace{\sqrt{68}\ \sqrt{164}\ \cos\theta}^{\|\vec{\nabla} f_1\|\ \|\vec{\nabla} f_2\|\ \cos\theta}$

$\theta = \arccos\left(\left| \frac{-64}{\sqrt{68}\ \sqrt{164}} \right| \right)$ can use abs value to avoid $180° - \theta$

$= 127°$

want $\alpha \in [0, 90°] \rightarrow \alpha = 180° - \theta = 53°$

$$\begin{matrix} 2 & 3 & 2 \\ 6 & 2 & -1 \end{matrix}$$

c) $\vec{n} = \vec{\nabla} f_1 \times \vec{\nabla} f_2 = \langle -4, -6, -4 \rangle \times \langle 12, 4, -2 \rangle$

$= -2 \langle 2, 3, 2 \rangle \times 2 \langle 6, 2, -1 \rangle$

$= \langle -7, 14, -14 \rangle$

$= \langle -1, 2, -2 \rangle$

Tangent line: $\langle x, y, z \rangle = \langle 2, 1, 1 \rangle + t \langle -1, 2, -2 \rangle$

7. Consider the surface $x^2 + y^2 - z^2 = 1$ and the curve $\vec{r}(t) = \langle t, t^2, t^2 \rangle$.

a) Find their point(s) of intersection.

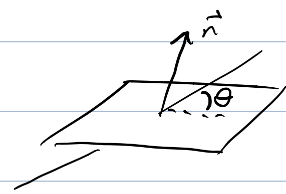
b) Find the angle of their intersection at the point where $x, y > 0$.

a) $t^2 + \cancel{t^4 - t^4} = 1$

$t = -1, \; t = 1$

pts: $(-1, 1, 1)$ & $(1, 1, 1)$

b) pt $(1, 1, 1)$



Let $f = x^2 + y^2 - z^2$

$\vec{\nabla} f = \langle 2x, 2y, -2z \rangle$

$\vec{n} = \vec{\nabla} f(1,1,1) = \langle 2, 2, -2 \rangle$ ← normal vector

$\vec{r}'(t) = \langle 1, 2t, 2t \rangle$

$\vec{r}'(-1) = \langle 1, -2, -2 \rangle = \vec{t}$

$\vec{n} \cdot \vec{t} = \|\vec{n}\| \, \|\vec{t}\| \cos\theta$

$\theta = \arccos\left(\frac{|2 \cancel{-4+4}|}{\sqrt{12} \cdot 3}\right)$

$\alpha = 90 - \theta$

End of Test 2

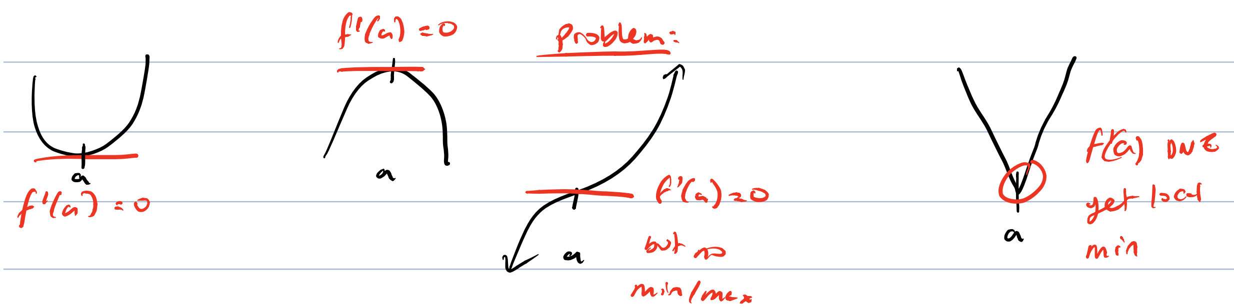
Extrema

Local max, min

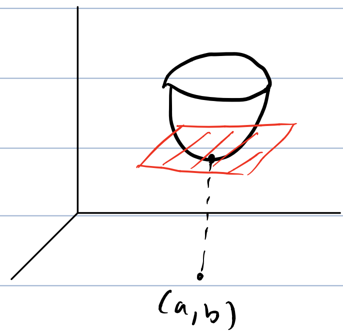
Abs max, min

In Cal I:

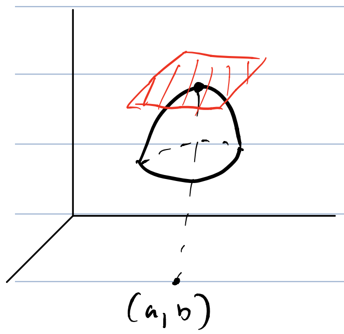
Local extrema

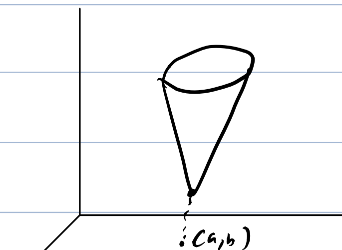


In Cal III:

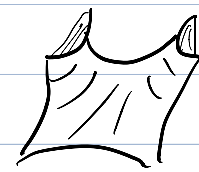


$$\left.\begin{aligned} f_x(a,b) &= 0 \\ f_y(a,b) &= 0 \end{aligned}\right\} \quad \vec{\nabla} f(a,b) = \langle 0,0 \rangle = \vec{0}$$





$\vec{\nabla} f$ DNE



Saddle

$\vec{\nabla} f = \vec{0}$

In Cal I => 2^{nd} derivative test

next class: 2^{nd} derivative test for $z = f(x,y)$

April 18

Extrema (Cont'd)

We want to distinguish between the above cases algebraically.

We will use a 2^{nd} derivative test.

__Ingredient:__ Hessian Matrix

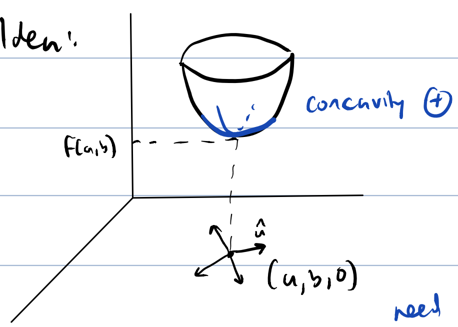
often symmetric

If $z = f(x,y)$, $H(x,y) = \begin{bmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y \end{bmatrix}$

$\det(H)(x,y) = f_{xx} f_{yy}(x,y) - f_{xy}(x,y) f_{yx}(x,y)$

__Proof:__ 2^{nd} derivative test part (i)

Idea:



We know that $D_{\hat{u}} f$ is the slope at (a,b)

need expression

We want the 2^{nd} derivative at (a,b) in any $\hat{u}$ direction and show that it is +

Let $\hat{u} = \langle u_1, u_2 \rangle$ be an arbitrary unit vector.

Then, $D_{\hat{u}} f(a,b) = \vec{\nabla} f(a,b) \cdot \hat{u}$
$= f_x(a,b)\, u_1 + f_y(a,b)\, u_2$

We can make a directional derivative function.

$D_{\hat{u}} f(x,y) = f_x(x,y)\, u_1 + f_y(x,y)\, u_2$

Let $g(x,y) = D_{\hat{u}} f(x,y)$ the 2^{nd} derivative in the direction of $\hat{u}$

$$D^2_{\hat{u}} f(x,y) = D_{\hat{u}}\left(D_{\hat{u}} f(x,y)\right) = D_{\hat{u}}\left(g(x,y)\right)$$

$$= g_x(x,y)u_1 + g_y(x,y)u_2$$

<u>Aside:</u> $g_x(x,y) = \frac{\partial}{\partial x} g(x,y)$

$$= \frac{\partial}{\partial x}\left(f_x(x,y)u_1 + f_y(x,y)u_2\right)$$

$$= f_{xx}(x,y)u_1 + f_{yx}(x,y)u_2$$

$$D^2_{u} f(x,y) = \left(f_{xx}(x,y)u_1 + f_{yx}(x,y)u_2\right)u_1 + \left(f_{yx}(x,y)u_1 + f_{yy}(x,y)u_2\right)u_2$$

$$= f_{xx}(x,y)u_1^2 + f_{yx}(x,y)u_1u_2 + f_{xy}(x,y)u_1u_2 + f_{yy}(x,y)u_2^2$$

$$= f_{xx}(x,y)u_1^2 + 2f_{xy}(x,y)u_1u_2 + f_{yy}(x,y)u_2^2$$ assume $f_{xy} = f_{yx}$

Complete the square

$$= f_{xx}(x,y)\left[u_1^2 + 2\frac{f_{xy}(x,y)}{f_{xx}(x,y)}u_1u_2 + \frac{f_{yy}(x,y)}{f_{xx}(x,y)}u_2^2\right.$$

$$\left. + \overbrace{\frac{f_{xy}(x,y)^2}{f_{xx}(x,y)^2} - \frac{f_{xy}(x,y)^2}{f_{xx}(x,y)^2}}^{\frac{1}{2}(\text{mid})^2}\right]$$

determinant

$$= f_{xx}(x,y)\left[\underbrace{\left(u_1 + \frac{f_{xy}(x,y)}{f_{xx}(x,y)}u_2\right)^2}_{+} + \underbrace{\frac{f_{xx}(x,y)\,f_{yy}(x,y) - f_{xy}(x,y)^2}{f_{xx}(x,y)^2}u_2^2}_{+}\right]$$

If $D>0$ & $f_{xx}>0$, $D^2_{\hat{u}} f > 0 \Rightarrow$ U the function is concave up in the direction of $\hat{u}$

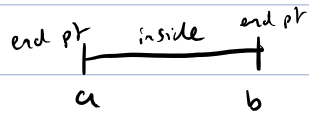
But $\hat{u}$ is <u>arbitrary</u>, so it is concave up in all directions.

Then, we have a local min!

<u>Absolute Extrema</u>

extreme value theorem

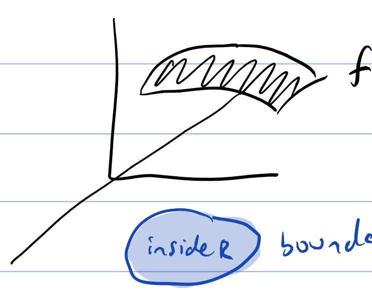
Cal I: EVT: If $f(x)$ is conts on $[a,b]$, then f has both an abs. min & abs. max on $[a,b]$



} result: extrema happened <u>inside</u> or at end pts

critical points

Cal II:



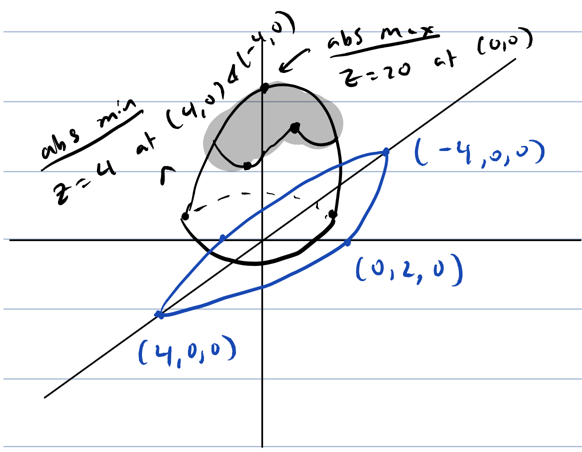
inside R boundary

Result: Extrema must occur inside <u>the region</u> or on the boundary

critical points

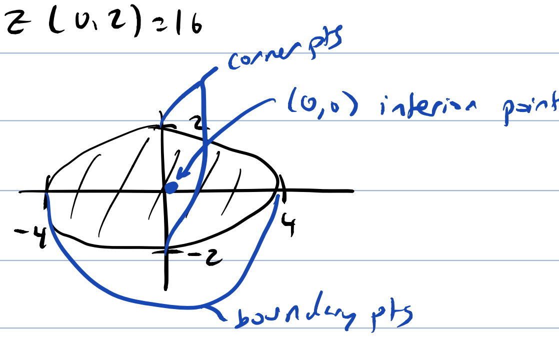
$z = f(x,y)$ is cont's on closed, bounded R, f will have both abs max & abs min on R.

ex. Find the abs max & abs min of $z = 20 - x^2 - y^2$ on region bounded by $x = y^2 - 4$, $x = 4 - y^2$



$z(4,0) = 4$

$z(0,2) = 16$



strategy: 1) Find the critical pts in interior
2) Find critical pts on boundary
3) Include any corners

} conditions for extrema

Since EVT guarantees abs extrema, we don't need the Hessian in this case. All we need to do is test all the candidates in $f(x,y)$. The largest is the abs. max. The smallest is the abs. min.

ex. (cont'd) $\rightarrow$ $z = 20 - x^2 - y^2$ on region: $x = y^2 - 4$, $x = 4 - y^2$

1) Interior: Let $\vec{\nabla} f = \vec{0}$

$f_x(x,y) = -2x \overset{\text{let}}{=} 0$

$f_y(x,y) = -2y \overset{\text{let}}{=} 0$

$\Rightarrow x = 0, y = 0 \quad \therefore (0,0)$

2 (i) $x = 4 - y^2$

$z = 20 - (4-y^2)^2 - y^2$

$z(y) = 20 - (16 - 8y + y^4) - y^2$

$= -y^4 + 7y^2 + 4$

$\frac{dz}{dy} = -4y^3 + 14y$

Let $\frac{dz}{dy} = 0 = 2y(7 - 2y^2$

$y = 0$, $y = \sqrt{\frac{7}{2}}$, $y = -\sqrt{\frac{7}{2}}$

$\downarrow$ $(4,0)$ $\quad \downarrow$ $\left(\frac{1}{2}, \sqrt{\frac{7}{2}}\right)$ $\quad \downarrow$ $\left(\frac{1}{2}, -\sqrt{\frac{7}{2}}\right)$

(ii) $x = y^2 - 4$

$z = 20 - (y^2-4)^2 - y^2$

$= 20 - (y^4 - 8y^2 + 16) - y^2$

$= 4 - y^4 - 7y^2$

$\frac{dz}{dy} = -4y^3 - 14y \overset{\text{let}}{=} 0$

$= -2y(2y^2 + 7)$

$y=0 \qquad y=\sqrt{\frac{7}{2}} \qquad y=-\sqrt{\frac{7}{2}}$

$(-4,0) \qquad \left(-\frac{1}{2}, \sqrt{\frac{7}{2}}\right) \qquad \left(-\frac{1}{2}, -\sqrt{\frac{7}{2}}\right)$

③ Corner pts $(0,2)$, $(0,-2)$

Candidates:

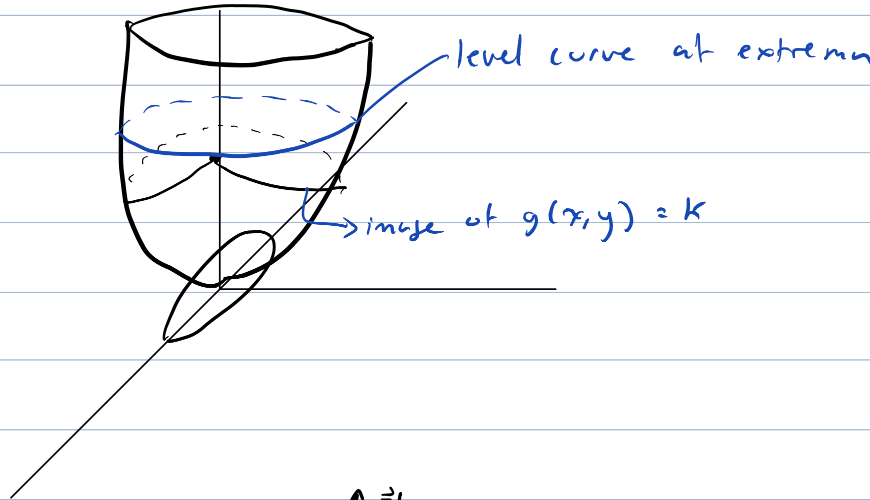
$(0,0)$	$\rightarrow 20$	$z=20$!
$(4,0)$	$\rightarrow 4$	
$\left(\frac{1}{2}, \sqrt{\frac{7}{2}}\right)$		
$\left(-\frac{1}{2}, \sqrt{\frac{7}{2}}\right)$		
$\left(\frac{1}{2}, -\sqrt{\frac{7}{2}}\right)$		
$\left(-\frac{1}{2}, -\sqrt{\frac{7}{2}}\right)$		
$(-4,0)$	4	
$(0,2)$		
$(0,-2)$		

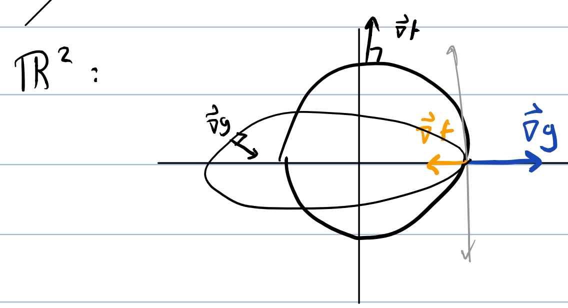
Method of Lagrange Multipliers

$z=f(x,y)$ subject to a constraint

$g(x,y)=K$ ... boundary only

ex $z=x^2+y^2$ subject to $2x^2+y^2=4$ ellipse





At an extremum, $\vec\nabla f \parallel \vec\nabla g$

$$\vec\nabla f = \lambda \vec\nabla g$$

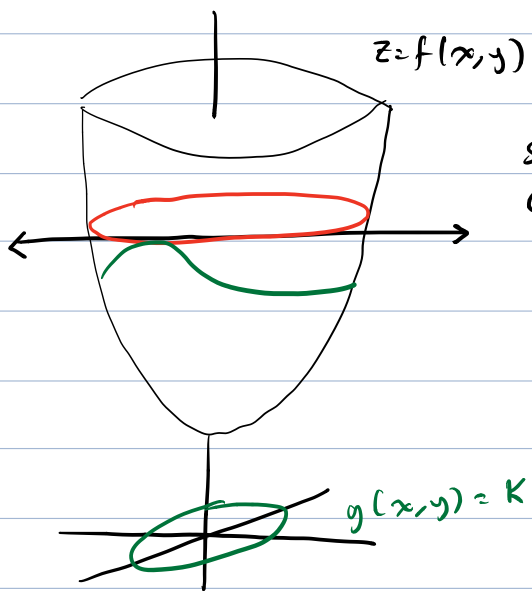
Find extrema by solving

$$\begin{cases} \vec\nabla f(x,y) = \lambda \vec\nabla g \\ g(x,y) = K \end{cases}$$

April 25

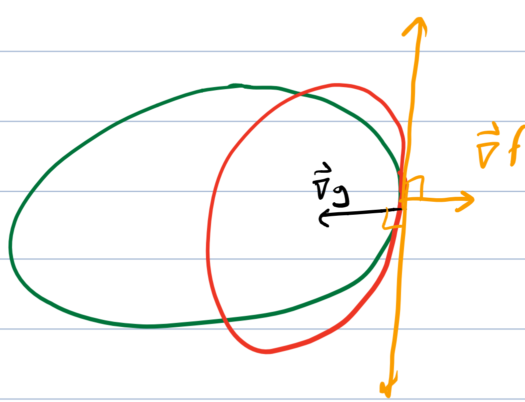
Recall: Method of Lagrange Multipliers (for Extrema)

extrema of $z = f(x,y)$ subject to constraint $g(x,y) = K$



shared tangent between level curve & image of $g(x,y) = K$

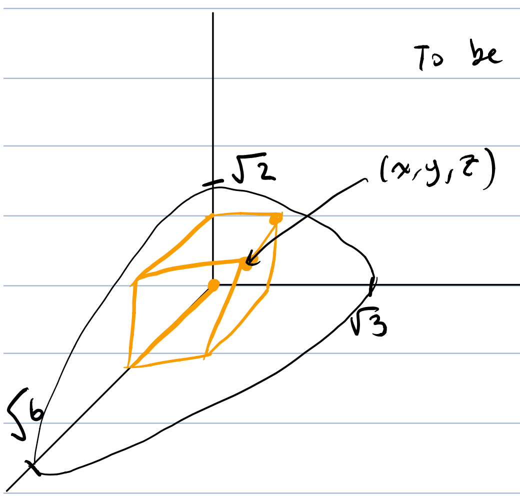
At extremum, gradients are parallel $\vec{\nabla} f = \lambda \vec{\nabla} g$



Solve: $\begin{cases} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \\ g(x,y) = K \end{cases}$ to find extrema

In general: $\begin{cases} \vec{\nabla} f(x_1, \dots, x_n) = \lambda \vec{\nabla} g(x_1, \dots, x_n) \\ g(x_1, \dots, x_n) = K \end{cases}$

2. Maximize the volume of a box in the first octant that has one corner at the origin, and the diagonally opposite corner on the ellipsoid $x^2 + 2y^2 + 3z^2 = 6$.



To be optomized: objective function

V = length × width × height

$V(x,y,z) = xyz$

subject to the constraint: (x,y,z) must satisfy

$x^2 + 2y^2 + 3z^2 = 6$

$w = f(x,y,z)$
$g(x,y,z) = K$

$\frac{\partial}{\partial x}\left(x^2+2y^2+3z^2\right)$

$$\begin{cases} \vec{\nabla} V = \lambda \vec{\nabla} g \\ g(x,y,z) = K \end{cases} \Rightarrow \begin{cases} V_x = \lambda g_x \Rightarrow yz = \lambda 2x & ① \\ V_y = \lambda g_y \Rightarrow xz = \lambda 4y & ② \\ V_z = \lambda g_z \Rightarrow xy = \lambda 6z & ③ \\ x^2 + 2y^2 + 3z^2 = 6 & ④ \end{cases}$$

Solve the system: one way: Isolate λ

① $\lambda = \frac{yz}{2x}$

② $\lambda = \frac{xz}{4y}$

③ $\lambda = \frac{xy}{6z}$

(if $z \neq 0$) → but not possible : from picture, there would be no volume (min)

$\frac{y\cancel{z}}{2x} = \frac{x\cancel{z}}{4y} \rightarrow 2x^2 = 4y^2 \rightarrow x^2 = 2y^2$

(if $y \neq 0$)

$\frac{\cancel{y}z}{2x} = \frac{x\cancel{y}}{6z} \rightarrow 2x^2 = 6z^2 \rightarrow x^2 = 3z^2$

subst in ④

$x^2 + x^2 + x^2 = 6$

$x^2 = 2$

$x = \sqrt{2}$ (can't be ⊖, since box is in first quadrant)

$2y^2 = (\sqrt{2})^2$

$y = 1$

$3z^2 = (\sqrt{2})^2$

$z^2 = \frac{2}{3}$

$z = \sqrt{\frac{2}{3}}$

justification

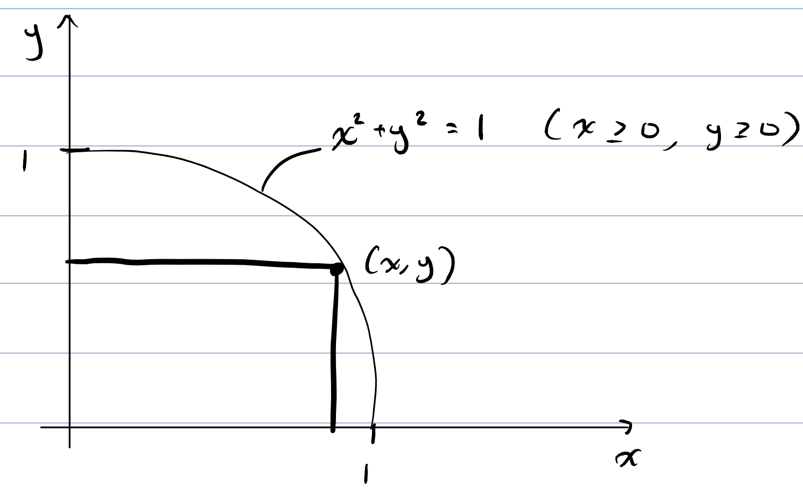
$(x,y,z) = \left(\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$ from our picture, this is obviously a maximum.

$$V_{max} = \sqrt{2} \cdot 1 \cdot \sqrt{\frac{2}{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \text{ units}^3$$

1. a) Use the method of Lagrange multipliers to maximize the area of a rectangle in the 1st quadrant inscribed above the $x-$axis and below the unit semi-cricle.

   b) Use Cal I methods to solve the same problem. → on your own time

a)



$A(x,y) = xy$

$$\begin{cases} A_x = y \rightarrow y = \lambda 2x \rightarrow \lambda = \frac{y}{2x} \\ A_y = x \rightarrow x = \lambda 2y \rightarrow \lambda = \frac{x}{2y} \\ x^2 + y^2 = 1 \end{cases}$$

$$\frac{y}{\cancel{2}x} = \frac{x}{\cancel{2}y} \rightarrow x^2 = y^2$$

$$x^2 + x^2 = 1 \rightarrow 2x^2 = 1 \rightarrow x = \frac{1}{\sqrt{2}} \rightarrow y = \frac{1}{\sqrt{2}}$$

$(x,y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ → obviously max from picture

$A = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$ units^2

b) Cal I: $A(x,y) = xy$

$x^2 + y^2 = 1$

harder step with many variables → $y = \pm\sqrt{1 - x^2}$

$A(x) = x\sqrt{1-x^2}$ then maximize

Note: Extrema

- Hessian: 2^{nd} derivative test → classify local extrema
- EVT → Absolute extrema on closed bounded region
- Lagrange → optimizing given a constraint

Critical Point A point (x_0, y_0) is a critical point of a function $f(x, y)$ if:

i) (x_0, y_0) is in the domain of f

ii) Either
   a) $\vec{\nabla} f(x_0, y_0) = \vec{0}$ (both $f_x = 0$ and $f_y = 0$)
   b) $\vec{\nabla} f(x_0, y_0)$ does not exist (at least one of f_x, f_y d.n.e)

Hessian Determinant

$$D(x, y) = \begin{vmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{vmatrix} = f_{xx} f_{yy} - f_{xy} f_{yx} \left[= f_{xx} f_{yy} - (f_{xy})^2 \text{ if Clairaut's Theorem applies}\right]$$

The Second Derivatives Test

If the second order partials are continuous at and near a critical point (x_0, y_0) of a function $f(x, y)$:

i) If $D > 0$ and $f_{xx} > 0$ at (x_0, y_0), then $f(x_0, y_0)$ is a local minimum.

ii) If $D > 0$ and $f_{xx} < 0$ at (x_0, y_0), then $f(x_0, y_0)$ is a local maximum.

iii) If $D < 0$ at (x_0, y_0), then $f(x_0, y_0)$ is a saddle point (neither min nor max).

iv) If $D = 0$ the test fails (anything is possible).

Exercises: Find and classify all the critical points for the following functions.

1. $f(x, y) = x^3 + y^3 - 3xy$

2. $f(x, y) = x^4 + y^4 - 4xy$

3. $f(x, y) = \sqrt{x^2 + y^2}$

Extreme Value Theorem

If $f(x, y)$ is continuous on a closed, bounded region R, then f attains both an absolute minimum and an absolute maximum on R.

Procedure for finding absolute extrema:

I. Collect all the critical points in the interior of R.

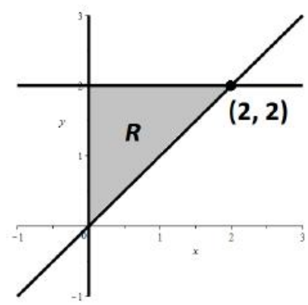
II. Collect all the critical points on the boundary curves of R.

III. Collect the vertices of R, if any.

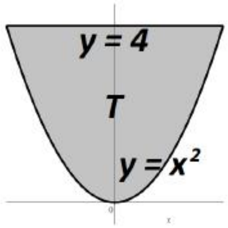
IV. Test all the points from $1 - 3$ above. The largest value will be the absolute maximum and the smallest will be the absolute minimum.

Exercises: Find the absolute extrema of the given functions on the given regions.

4. $f(x, y) = 2x^4 + y^2 - x^2 - 2y$ on R



5. $g(x, y) = 1 + xy - x - y$ on T.



Answers

1. $(0, 0)$ - saddle point; $(1, 1)$ - local minimum
2. $(0, 0)$ - saddle point; $(1, 1)$ - local minimum; $(-1, -1)$ - local minimum
3. $\vec{\nabla} f$ DNE at $(0, 0)$ so second derivatives test does not apply. But we know this is the apex of an upward facing cone, therefore a local mimimum.
4. Abs Min: $-\frac{9}{8}$ at $(\frac{1}{2}, 1)$; Abs Max: 28 at $(2, 2)$
5. Abs Min: -9 at $(-2, 4)$; Abs Max: 3 at $(2, 4)$