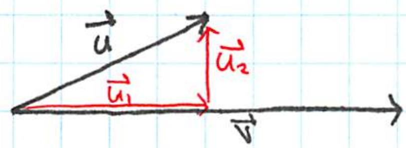
Geometry in $\mathbb{R}^3$

A) Projections

Let $\vec{u}, \vec{v} \neq \vec{0}$ in $\mathbb{R}^n$

We want to decompose $\vec{u}$ as $\vec{u} = \vec{u}_1 + \vec{u}_2$

s.t $\vec{u}_1 \parallel \vec{v}$ and $\vec{u}_2 \perp \vec{v}$



acute angle

or

obtuse angle

Since $\vec{u}_1 \parallel \vec{v}$ $\therefore$ $\boxed{\vec{u}_1 = t\vec{v}}$ for some scalar t

$\vec{u}_2 \perp \vec{v}$ $\therefore$ $\vec{u}_2 \cdot \vec{v} = 0$

$$\vec{u} = \vec{u}_1 + \vec{u}_2$$
$$\vec{u}_2 = \vec{u} - \vec{u}_1$$
$$\vec{u}_2 = \vec{u} - (t\vec{v})$$

$$(\vec{u} - t\vec{v}) \cdot \vec{v} = 0$$
$$\vec{u}\cdot\vec{v} - t\,\vec{v}\cdot\vec{v} = 0$$

$$t = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2} = t$$

Def'n: $\boxed{\vec{u}_1}$ (the one $\parallel$) is the orthogonal projection of $\vec{u}$ onto $\vec{v}$
(vector projection)

$$\boxed{\begin{array}{c} \text{proj}_{\vec{v}}\,\vec{u} = \left(\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v} \\ \vec{u}_1 = t\,\vec{v} \end{array}} = \underbrace{\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}}_{\substack{\text{scalar projection}\\ \text{of } \vec{u} \text{ onto } \vec{v}}} \underbrace{\frac{\vec{v}}{\|\vec{v}\|}}_{\substack{\text{unit vector}\\ \hat{v}}}$$

projection of $\vec{u}$ onto $\vec{v}$

Trick: "onto $\vec{v}$"
$\therefore$ $\vec{v}$ appears several times

Remarks

same thing as norm $\|\vec{u}_1\|$

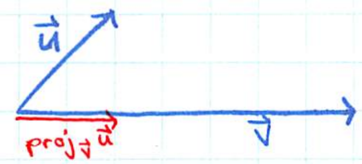
1. Norm of $\text{proj}_{\vec{v}}\vec{u}$: $\|\text{proj}_{\vec{v}}\vec{u}\|$

abs. value

$$= \left\| \underbrace{\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}}_{\text{scalar}} \vec{v} \right\| = \frac{|\vec{u}\cdot\vec{v}|}{\|\vec{v}\|^2}\|\vec{v}\| = \frac{|\vec{u}\cdot\vec{v}|}{\|\vec{v}\|} = |\text{scalar projection}|$$

Makes sense bc:

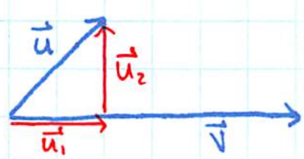
$$\text{proj}_{\vec{v}}\vec{u} = \vec{u}_1 = \underbrace{\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}}_{\substack{\text{Abs value}\\ \text{is NORM}}} \underbrace{\frac{\vec{v}}{\|\vec{v}\|}}_{\substack{\text{unit vector}\\ (\text{norm}=1)}}$$



↳ unit vector $\hat{v}$ gives direction

↳ Norm: |scalar proj.|

2. Norm $\|\vec{u}_2\|$: use pythagorial identity



$$\|\vec{u}\|^2 = \|\vec{u}_1\|^2 + \|\vec{u}_2\|^2$$

$$\|\vec{u}_2\|^2 = \|\vec{u}\|^2 - \|\vec{u}_1\|^2$$

B) Cross Product in $\mathbb{R}^3$

Def'n: 2x2 determinant : $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

ex: $\begin{vmatrix} 2 & -1 \\ 6 & 1 \end{vmatrix} = (2) - (-6) = 8$

Def'n: Let $\vec{u} = [u_1, u_2, u_3]$, $\vec{v} = [v_1, v_2, v_3]$ in $\mathbb{R}^3$,

define the cross product $\vec{u} \times \vec{v}$ to be:

$$\vec{u} \times \vec{v} = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \end{array}$$

symbolic 3x3 determinant expanded along row 1.

eliminate row and column of $\hat{\imath}$

$$= \hat{\imath} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

Always

ex: $[1, 2, 3] \times [-1, 4, -2] = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 4 & -2 \end{vmatrix} \end{array}$

$$= \hat{\imath} \begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 1 & 3 \\ -1 & -2 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix}$$

$$= [-16, -1, 6]$$

Theorem:

1. $\vec{u} \times \vec{v}$ is a vector (Note: $\vec{u} \cdot \vec{v}$ is a scalar)

2. $\vec{u} \times \vec{0} = \vec{0} \times \vec{u}$ $\qquad \begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0 = \begin{vmatrix} 0 & 0 \\ a & b \end{vmatrix}$

3. $\vec{u} \times \vec{u} = \vec{0}$ $\qquad \begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$

4. $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$ $\qquad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

5. $(K\vec{u}) \times \vec{v} = \vec{u} \times (K\vec{v}) = K(\vec{u} \times \vec{v})$ $\qquad \begin{vmatrix} Ka & Kb \\ c & d \end{vmatrix} = Kad - Kcd = K(ad - cd)$

6. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$ $\qquad \begin{vmatrix} u_1 & u_2 \\ v_1 + w_1 & v_2 + w_2 \end{vmatrix}$

$$= u_1(v_2 + w_2) - u_2(v_1 + w_1)$$

$$= u_1 v_2 + u_1 w_2 - u_2 v_1 - u_2 w_1$$

$$= \begin{vmatrix} u_1 & u_2 \\ w_1 & w_2 \end{vmatrix} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

prove both sides

7. $\vec{u} \times \vec{v} = \vec{0} \iff \vec{u} // \vec{v}$

proof: $\vec{u} // \vec{v} \quad \therefore \quad \vec{u} = t\vec{v}$

$$\vec{u} \times \vec{v} = (t\vec{v}) \times \vec{v} = t(\vec{v} \times \vec{v}) = \vec{0}$$

proof: use Lagarange's Identity

$$\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$$
$$= \|\vec{u}\|^2 \|\vec{v}\|^2 - (\|\vec{u}\| \|\vec{v}\| \cos\theta)^2$$
$$= \|\vec{u}\|^2 \|\vec{v}\|^2 - \|\vec{u}\|^2 \|\vec{v}\|^2 \cos^2\theta$$
$$= \|\vec{u}\|^2 \|\vec{v}\|^2 \underbrace{(1 - \cos^2\theta)}_{\sin^2\theta}$$

we get: $\boxed{\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 \sin^2\theta}$ $(\vec{u}, \vec{v} \neq \vec{0})$

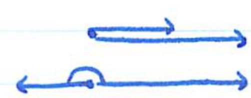
$0 \leq \theta \leq \pi$

$\therefore \sin\theta \geq 0$

if $\vec{u} \times \vec{v} = \vec{0}$

$\therefore \sin\theta = 0$

$\Rightarrow \theta = 0, \pi$



$\theta = 0$

$\theta = \pi$

$\therefore \vec{u} // \vec{v}$

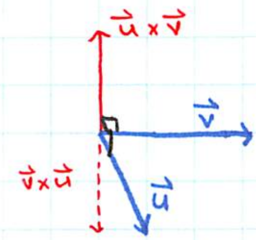
Triple Scalar Product (TSP)

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} \overset{+}{u_1} & \overset{-}{u_2} & \overset{+}{u_3} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \overbrace{u_1}^{1^{st}\text{ comp of } \vec{u}} \underbrace{\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}}_{\substack{1^{st}\text{ comp} \\ \text{of } \vec{v} \times \vec{w}}} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

* **Property**: Changing any 2 vectors of TSP, changes sign

Theorem:

1. $\vec{u} \times \vec{v}$ is $\perp$ to both $\vec{u}$ and $\vec{v}$



2. $\|\vec{u} \times \vec{v}\|$ = area of //-gram determined by $\vec{u}$ and $\vec{v}$

3. $|\vec{u} \cdot (\vec{v} \times \vec{w})|$ = volume of //-pipede (parallelipiped) determined by $\vec{u}, \vec{v}, \vec{w}$

abs value

proofs on NEXT PAGE

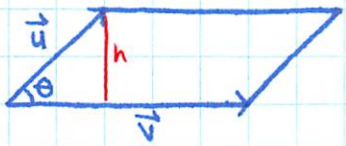
proof:

1. $\vec{u} \times \vec{v}$ $\perp$ to both $\vec{u}$ and $\vec{v}$

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = -\vec{v} \cdot (\vec{u} \times \vec{u}) = 0 \qquad \therefore \perp \text{ to } \vec{u}$$

$$\vec{v} \cdot (\vec{u} \times \vec{v}) = -\vec{u} \cdot (\vec{v} \times \vec{v}) = 0 \qquad \therefore \perp \text{ to } \vec{v}$$

2. Recall Lagrange : $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin\theta$

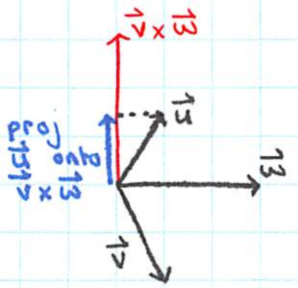


$$\text{Area} = (\text{base})\,(\text{height})$$
$$= \|\vec{v}\| \quad \|\vec{u}\| \sin\theta$$

$\frac{\text{height}}{\|\vec{u}\|} = \sin\theta$

$$= \|\vec{u} \times \vec{v}\|$$

3. Volume of //-pipede = (Area of $\underbrace{\text{base}}_{\text{//-gram}}$) (height)



$$\text{volume} = \|\vec{v} \times \vec{w}\| \, \|\text{proj}_{\vec{v} \times \vec{w}} \vec{u}\|$$

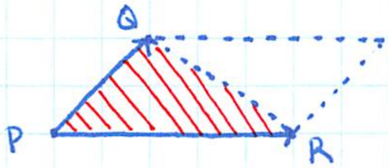
$$= \|\vec{v} \times \vec{w}\| \left| \frac{\vec{u} \cdot (\vec{v} \times \vec{w})}{\|\vec{v} \times \vec{w}\|} \right| = \left| \vec{u} \cdot (\vec{v} \times \vec{w}) \right|$$

* **Remarks:** we see that 3 vectors $\vec{u}, \vec{v}, \vec{w}$

are coplanar iff their TSP = 0

(imagine if $\vec{u}$ was coplanar to $\vec{v}, \vec{w}$ in the image above $\Rightarrow$ volume = 0)

ex: Find area of triangle PQR where P(0,1,0)
Q(-1,1,2)
R(2,1,-1)



$$\text{Area} = \frac{1}{2}\left\| \overrightarrow{PQ} \times \overrightarrow{PR} \right\|$$

$$\left.\begin{aligned} \overrightarrow{PQ} &= [-1,0,2] \\ \overrightarrow{PR} &= [2,0,-1] \end{aligned}\right\} \quad \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & 2 \\ 2 & 0 & -1 \end{vmatrix} = [0, +3, 0]$$

$$\therefore \text{Area} = \frac{1}{2}\sqrt{3^2} = 3/2$$

ex: Find the volume of the //-pipede determined by:

$2\hat{i} - \hat{k}$

$\hat{i} + \hat{j}$

$3\hat{j} + \hat{k}$

$$TSP = \begin{vmatrix} \overset{+}{2} & \overset{-}{0} & \overset{+}{-1} \\ 1 & 1 & 0 \\ 0 & 3 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} - \cancel{0\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}^{\,0} - 1\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix}$$

$$= 2(1) - 1(3) = -1$$

$$\text{volume} = |-1| = 1$$

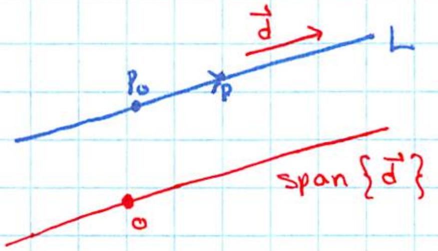
C) Lines in $\mathbb{R}^3$

To write the equation of a line in $\mathbb{R}^3$, we need:

a direction vector and a point on the line.

$\vec{d} = [a, b, c] \neq \vec{0}$

$P_0(x_0, y_0, z_0)$



Note: if L passes by origin $\therefore$ $P_0(0,0,0)$

Note: Any scalar multiple of $\vec{d}$ would work as a direction vector

Derivation of line equation

$P(x,y,z)$ is on line $L \overset{\text{iff}}{\Longleftrightarrow} \overrightarrow{P_0P} \,//\, \vec{d}$

$$\overrightarrow{P_0P} = t\vec{d} \quad , t \text{ scalar}$$

$$\vec{p} - \vec{p_0} = t\vec{d}$$

Vector eqcn of L: $\boxed{\vec{p} = \vec{p_0} + t\vec{d}}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Parametric eqcn of L: $\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$

ex: write parametric equations of L through $P_0(1,-1,0)$ and with direction vector $[2,-3,1]$

$\vec{d}$

From $\vec{d}$

$$\begin{cases} x = 1 + t(2) \\ y = -1 + t(-3) \\ z = 0 + t(1) \end{cases}$$

constant (from P_0)

ex: Give 2 pts and a direction vector for the line:

There can be ∞'ly many

$$\begin{cases} x = -1 - t \\ y = 2t \\ z = -6 \end{cases}$$

if $t=0$: point $(-1,0,-6)$

if $t=1$: point $(-2,2,-6)$

$\vec{d} = [-1,2,0]$ or any scalar multiple of it.

Now suppose that we solve for t the parametric eqcns:

$$\begin{cases} x = x_0 + ta & \Rightarrow t = \frac{x-x_0}{a} \\ y = y_0 + tb & \Rightarrow t = \frac{y-y_0}{b} \\ z = z_0 + tc & \Rightarrow t = \frac{z-z_0}{c} \end{cases}$$

Symmetric form of L:

$$\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}} = t$$

(if one of a, b, c is zero, equate the numerator to zero instead)

ex: $\begin{cases} x = -1-t \\ y = 2t \\ z = -6 \end{cases} \Rightarrow \frac{x+1}{-1} = \frac{y}{2}, \quad z = -6$

Recap: Lines (P_0, $\vec{d}$)

- vector form
- parametric form
- symmetric form

parametric → symmetric: To find $\vec{d}$ (ie. denom of symmetric) use coefficients of t.
→ if NO denom, coeff. of t is zero
To find a point, equate numerators of symmetric equation to zero.

$$\begin{cases} x = 1+t \\ y = 2 \\ z = -1-2t \end{cases} \Rightarrow \frac{x-1}{1} = \frac{z+1}{-2}, \quad y = 2$$

symmetric → parametric: $\frac{x-3}{-2} = \frac{y}{5} = \frac{z+1}{1} = t \Rightarrow \begin{cases} x = 3 - 2t \\ y = 0 + 5t \\ z = -1 + t \end{cases}$

Remarks:
1. $L_1 \parallel L_2 \Leftrightarrow \vec{d_1} \parallel \vec{d_2}$
2. Say L_1, L_2 intersect, $L_1 \perp L_2 \Leftrightarrow \vec{d_1} \perp \vec{d_2}$
3. A, B, C are collinear $\Leftrightarrow \vec{AB} \parallel \vec{AC}$

ex: Find symmetric eq. for the line through $(3,1,0)$ and $\parallel$ to the line: $\begin{cases} x = 6-2t \\ y = 4+t \\ z = 3-t \end{cases}$

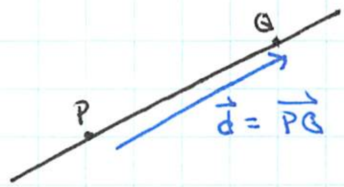
$\vec{d} = [-2, 1, -1]$

$\therefore \dfrac{x-3}{-2} = \dfrac{y-1}{1} = \dfrac{z}{-1}$

if plug in pt $(3,1,0)$, the numerators equate to zero.

ex: Find parametric eqcns for the line through $P(1,0,3)$ and $Q(2,-1,4)$

take $\vec{d} = \overrightarrow{PQ} = [1,-1,1]$



From $\vec{d}$

$\therefore L \begin{cases} x = 1+t \\ y = 0-t \\ z = 3+t \end{cases}$

From P

Lines in the xy-plane (ie. $z=0$)

$$\begin{cases} y = mx + b \\ z = 0 \end{cases} \xRightarrow[\text{symmetric form}]{} \frac{x}{1} = \frac{y-b}{m}, \quad z = 0$$

slope

$$\therefore \vec{d} = [1, m, 0]$$

So

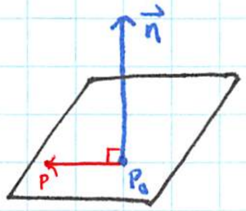
$$L_1 \parallel L_2 \text{ in xy-plane} \Leftrightarrow [1, m_1, 0] \parallel [1, m_2, 0]$$
$$\Leftrightarrow \boxed{m_1 = m_2}$$

$$L_1 \perp L_2 \text{ in xy-plane} \Leftrightarrow [1, m_1, 0] \cdot [1, m_2, 0] = 0$$
$$\Leftrightarrow 1 + m_1 m_2 + 0 = 0$$
$$\Leftrightarrow \boxed{m_1 m_2 = -1}$$

D) Planes in $\mathbb{R}^3$

A plane is completely determined by:

→ A point on it $P_0(x_0, y_0, z_0)$
→ A normal vector $\vec{n} = [a, b, c]$



Derivation of plane equation

if $P(x,y,z)$ is on plane $\Leftrightarrow$ $\overrightarrow{P_0P} \perp \vec{n}$

Note: if P NOT on plane $\therefore$ $\overrightarrow{P_0P}$ NOT $\perp$ to $\vec{n}$

$$\overrightarrow{P_0P} \cdot \vec{n} = 0$$
$$(\vec{p} - \vec{p_0}) \cdot \vec{n} = 0$$

$$\underset{[x,y,z]}{\vec{p}} \cdot \underset{[a,b,c]}{\vec{n}} = \underset{[x_0,y_0,z_0]}{\vec{p_0}} \cdot \vec{n}$$

$$ax + by + cz = ax_0 + by_0 + cz_0$$

Makes sense: 2 pts that are on plane plugged into eqcn will be sol'n to eqcn. and will give same value.
⇒ pts on plane are sol'n to eqcn.

$$\boxed{ax + by + cz = d}$$

↑ constant

Plane in $\mathbb{R}^3$ = sol'n set to eqcn. $ax + by + cz = d$
↳ All pts that satisfy eqcn. form the plane.

Note: constant $\boxed{d}$
→ planes with same a, b, c but different d are parallel planes

Same a, b, c ⇔ same $\vec{n}$ $\therefore$ parallel planes.

ex: Find an equation for the plane through $\overset{P_0}{(2,3,4)}$ and $\perp$ to $\underset{\vec{n}}{[1,-4,3]}$.

$$\therefore \quad 1x - 4y + 3z = 1(2) - 4(3) + 3(4)$$

$$x - 4y + 3z = 2$$

ex: Find a unit vector normal to $2x - 3y + 7z = 3$

$$\vec{n} = [2,-3,7]$$

unit vector: $\pm \dfrac{1}{\sqrt{4+9+49}}[2,-3,7] = \pm \dfrac{1}{\sqrt{62}}[2,-3,7]$

Rules of Thumb

1. $//$ to plane $\Leftrightarrow$ $\perp$ to normal

   $\perp$ to plane $\Leftrightarrow$ $//$ to normal

2. To get a normal for a plane, either look:
   - → directly for $\perp$ vector

   OR

   - → find 2 vectors $//$ to plane BUT NOT to EACHOTHER and cross them.

Coplanarity

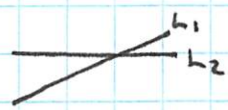
Note: 3 pts are always coplanar (form a triangle)

- A, B, C, D coplanar $\Leftrightarrow$ TSP: $\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = 0$
- L_1 and L_2 coplanar $\Leftrightarrow$ TSP: $\vec{P_1P_2} \cdot (\vec{d_1} \times \vec{d_2}) = 0$

bc if lines are coplanar ∴ These 3 vectors are ALSO all coplanar

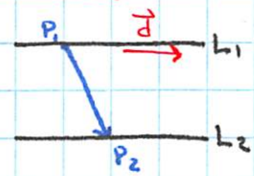
↳ To get a normal to a plane containing 2 lines (coplanar)

- $\boxed{\text{NOT } //}$ $\vec{n} = \vec{d_1} \times \vec{d_2}$



- $\boxed{//}$ $\vec{n} = \vec{P_1P_2} \times \vec{d}$

Note: $\vec{d_1} = \vec{d_2}$



Note: Lines that are NOT coplanar (ie. do not intersect and not parallel) are called "skewed".

E) Intersections

· between 2 lines

· First look at direction vector

→ if parallel, is a point on the line also on the other?

- yes: coincident (coplanar)
- NO: parallel and distinct (coplanar)

→ if NOT //, equate x, y, z of parametric forms

BUT WITH 2 DIFFERENT PARAMETERS

$$L_1 \begin{cases} x = x_{0_1} + t a_1 \\ y = y_{0_1} + t b_1 \\ z = z_{0_1} + t c_1 \end{cases}, \quad L_2 \begin{cases} x = x_{0_2} + s a_2 \\ y = y_{0_2} + s b_2 \\ z = z_{0_2} + s c_2 \end{cases}$$

find:

- 1 sol'n: ∴ 1 point (plug in parameter to find the pt.) (coplanar)
- NO sol'n: **SKEWED**

* Remember to use different parameters s, t

- between line and plane

  - Plug in parametric equations of line into plane equation

    ie. $a_p(x_0+ta_L)+b_p(y_0+tb_L)+c_p(z_0+tc_L)=d_p$

Find:

- nothing (line // to plane) — eg. $0t=2$
- one point — eg. $t=6$
- the whole line — eg. $0t=0$

- between 2 planes

  - First look at their normals

$$\begin{cases} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \end{cases}$$

→ Normals are parallel if:

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$$

- also $=\frac{d_1}{d_2}$ — coincident
- $\neq\frac{d_1}{d_2}$ — // and distinct

→ Normals NOT parallel:
intersect along a LINE

→ To find parametric eqcns for the line solve corresponding 2×3 system

Remark: The line of intersection of 2 planes is // to both planes

$\Rightarrow$ line $\perp$ to both $\vec{n}_1$ and $\vec{n}_2$

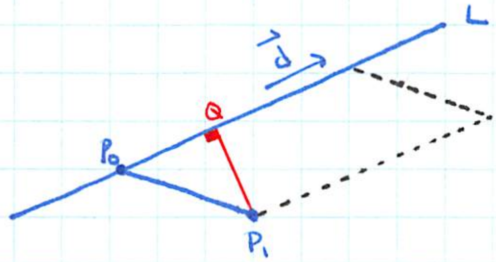
$\Rightarrow$ $\vec{d}=\vec{n}_1\times\vec{n}_2$ is direction vector of line.

F) Distances

· between 2 points

P_1 and P_2 : $\| \overrightarrow{P_1 P_2} \|$

· between point P_1 and line L (Note: shortest distance)



Q is pt on line closest to P_1

$d(P_1, L)$ = heigh of //-gram determined by $\overrightarrow{P_0P}$ and $\vec{d}$

Area = (base)(height)

$\therefore$ Height = $\dfrac{\text{Area of //-gram}}{\text{base}} = \boxed{\dfrac{\| \overrightarrow{P_0P} \times \vec{d} \|}{\| \vec{d} \|}}$

To find point Q (closest to P_1)

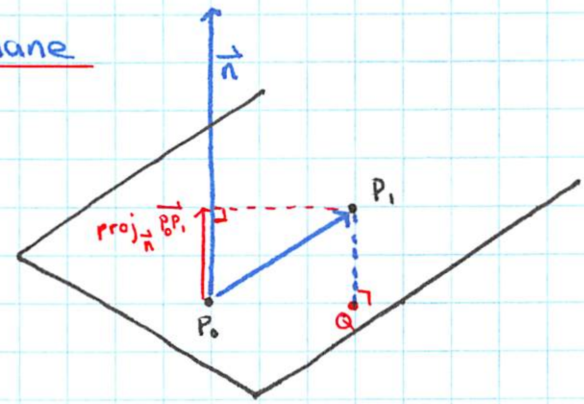
- Q is on L $\therefore$ coordinates follow parametric eqcns of L
- $\overrightarrow{PQ} \perp \vec{d}$ $\therefore$ solve for parameters in eqcn: $\overrightarrow{PQ} \cdot \vec{d} = 0$

- between point P_1 and plane

Let the plane equation be:

$ax + by + cz + d = 0$

(Note: constant on left)



Say $P_0(x_0, y_0, z_0)$ on plane (satisfies plane eqcn)

$$\therefore \underbrace{ax_0 + by_0 + cz_0}_{\vec{n} \cdot \vec{P_0}} + d = 0$$

$$\boxed{\vec{n} \cdot \vec{p_0} = -d}$$

Say $P_1(x_1, y_1, z_1)$ (Not on plane)

$$\text{distance} = \left\| proj_{\vec{n}} \vec{P_0P_1} \right\| = \frac{|\vec{P_0P_1} \cdot \vec{n}|}{\|\vec{n}\|}$$

$$= \frac{|(\vec{P_1} - \vec{P_0}) \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|\vec{P_1} \cdot \vec{n} - \vec{P_0} \cdot \vec{n}|}{\|\vec{n}\|}$$

$$= \frac{|\vec{P_1} \cdot \vec{n} - (-d)|}{\|\vec{n}\|}$$

$$= \boxed{\frac{|ax_1 + by_1 + cz_1 + d|}{\|\vec{n}\|}}$$

This value of constant d is when "d" is on the left of the eqcn.

To find pt Q on plane closest to P_1

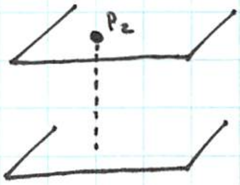
- Q is on plane
- $\vec{P_1Q} \parallel \vec{n}$ $\quad \therefore \vec{P_1Q} = t\vec{n}$

- between a line and a plane

(only relevant if line // to plane ; if NOT, they intersect)

→ Choose any pt on line
⇒ Find distance between "pt and plane"

- between 2 // planes



Assume

$$\begin{cases} ax+by+cz+d_1=0 \\ ax+by+cz+d_2=0 \end{cases}$$

Note: same a, b, c
→ if not multiply by a factor

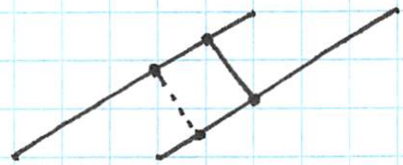
- Take any point on the 2nd plane $P_2(x_2, y_2, z_2)$ and find its distance to plane I

$$\text{distance} = \frac{|\overbrace{ax_2+by_2+cz_2}^{-d_2}+d_1|}{\|\vec{n}\|} = \boxed{\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}}$$

$$= \boxed{\frac{|d_1-d_2|}{\|\vec{n}\|}}$$

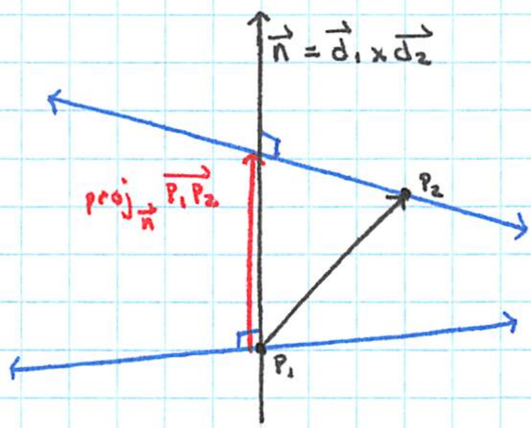
- between 2 // lines

Take any pt on one of the lines and find its distance to the other line.



use : distance $= \dfrac{\| \overrightarrow{P_0 P_1} \times \vec{d} \|}{\| \vec{d} \|}$

- between 2 skewed lines



P_2 : any pt on L_2

P_1 : any pt on L_1

$$\text{distance} = \left\| \text{proj}_{\vec{d}_1 \times \vec{d}_2} \overrightarrow{P_1 P_2} \right\| = \boxed{\frac{\left| \overrightarrow{P_1 P_2} \cdot (\vec{d}_1 \times \vec{d}_2) \right|}{\| \vec{d}_1 \times \vec{d}_2 \|}}$$

Note : if lines intersect numerator $= 0$

$\therefore$ distance $= 0$ ✓

Square Matrices

A) Introduction

Let A be $n \times n$

- A is symmetric iff $\boxed{A^T = A}$

  ex: $\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$

- A is antisymmetric OR skew-symmetric iff $\boxed{A^T = -A}$

  (This implies that all entries on main diagonal $= 0$)

  ex: $\begin{bmatrix} 0 & 2 & -5 \\ -2 & 0 & 1 \\ 5 & -1 & 0 \end{bmatrix}$

- A is diagonal if $a_{ij} = 0$ for all $i \neq j$

  (only non-zero entries are on main diagonal)

  ex: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- A is scalar if $A = a I_n$

  ex: $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3I$

· A is <u>upper triangular</u> if $a_{ij} = 0$ for $i > j$

$$\begin{bmatrix} \ddots & * \\ 0 & \ddots \end{bmatrix}$$

ex: $\begin{bmatrix} 1 & 6 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

· A is <u>lower triangular</u> if $a_{ij} = 0$ for $i < j$

$$\begin{bmatrix} \ddots & 0 \\ * & \ddots \end{bmatrix}$$

ex: $\begin{bmatrix} 1 & 0 & 0 \\ 6 & -2 & 0 \\ 3 & 5 & 0 \end{bmatrix}$

<u>Note</u>: Transpose upper Tr.
$\Downarrow$
lower Tr.

<u>Note</u>: scalar $\Rightarrow$ diagonal

diagonal $\Rightarrow$ symmetric
$\Rightarrow$ both upper and lower Tr.

Also, when A is square, we can define:

$A^2 = AA$

$A^3 = AAA$

$A^0 = I$

ex: $p(x) = x^2 + x + 3$

$p(A) = A^2 + A + 3I$

bc $3x^0 = 3$

Def'n: An $n \times n$ matrix is called elementary if its obtained from I_n by one (only one) elementary row operation called its associated elt. row operation.

↳ "What you do to I, to get your elt matrix"

ex: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $R_1 \leftrightarrow R_2$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $R_3 \leftarrow 3R_3$ $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_1 \leftarrow R_1 - R_2$

Theorem: Let A be $m \times n$ (not necessarily square)

Let E be $m \times m$ elt. matrix with associated operation "op".

Then EA is the matrix obtained by performing the same operation on A.

$$\boxed{I \overset{op}{\sim} E \implies A \overset{op}{\sim} EA}$$

ex: $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$

$$\therefore A \overset{R_2 \leftarrow -2R_2}{\sim} \begin{bmatrix} 2 & 3 \\ 2 & -8 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}}_{A}$$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \overset{R_2 \leftarrow -2R_2}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

$\therefore$ Performing an elt. op corresponds to multiplication on the left by the associated elt. matrix.

Note: op, is the associated operation associated with E, More generally, say A, B are $m \times n$ matrices linked by a series of operations $op_1 \ldots op_k$

$$A \overset{op_1}{\sim} E_1 A \overset{op_2}{\sim} E_2 E_1 A \, [\ldots] \overset{op_k}{\sim} B = \underbrace{E_k \ldots E_1}_{U} A$$

<u>Note</u>: we compute U by performing the same operations on I:

$$I \overset{op_1}{\sim} E_1 \overset{op_2}{\sim} E_2 E_1 \, [\ldots] \overset{op_k}{\sim} \underbrace{E_k \ldots E_2 E_1}_{U} = U$$

<u>In conclusion</u>:

$$[\,A \mid I\,] \overset{op_1}{\sim} \cdots \overset{op_k}{\sim} [\,\underset{= B}{UA} \mid U\,]$$

ex: Let $A = \begin{bmatrix} 0 & 2 & -4 \\ 1 & 3 & 2 \end{bmatrix}$ and let R be its RREF

Find U s.t $R = UA$ and also express U as a product of elt. matrices.

$$[A \mid I] = \left[\begin{array}{ccc|cc} 0 & 2 & -4 & 1 & 0 \\ 1 & 3 & 2 & 0 & 1 \end{array}\right] \quad \underset{op_1}{R_1 \leftrightarrow R_2}$$

$$\left[\begin{array}{ccc|cc} 1 & 3 & 2 & 0 & 1 \\ 0 & 2 & -4 & 1 & 0 \end{array}\right] \quad R_2 \xleftarrow{op_2} 1/2\, R_2$$

$$\left[\begin{array}{ccc|cc} 1 & 3 & 2 & 0 & 1 \\ 0 & 1 & -2 & 1/2 & 0 \end{array}\right] \quad \underset{op_3}{R_1 \leftarrow R_1 - 3R_2}$$

$$\underbrace{\begin{array}{ccc} 1 & 0 & 8 \\ 0 & 1 & -2 \end{array}}_{UA = R} \;\Big|\; \underbrace{\begin{array}{cc} -3/2 & 1 \\ 1/2 & 0 \end{array}}_{U}$$

$$\therefore R = \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -2 \end{bmatrix} = UA = \underbrace{\begin{bmatrix} -3/2 & 1 \\ 1/2 & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 0 & 2 & -4 \\ 1 & 3 & 2 \end{bmatrix}}_{A}$$

Also $U = E_3 E_2 E_1 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -3/2 & 1 \\ 1/2 & 0 \end{bmatrix}$

we get E_1, E_2, E_3 by applying op_1, op_2, op_3 on the identity matrix: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

* **Remark**: For large matrices, it is much faster to perform a row operation than to multiply on the left by assoc. elt. matrix.

$$\underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 6 & 5 & 2 & 1 \\ 2 & 3 & 1 & -1 \\ 3 & 0 & 1 & 5 \end{bmatrix}} = \begin{bmatrix} 6 & 5 & 2 & 1 \\ 4 & 6 & 2 & -2 \\ 3 & 0 & 1 & 5 \end{bmatrix}$$

elt. matrix

op: $R_2 \leftarrow 2R_2$

instead of multiplying on left by E, perform op. onto A.

B) Invertible Matrices

$1 \leftrightarrow I_n$ related

$\frac{1}{2} \cdot 2 = 1$

inverse

Def'n: we say that an $n \times n$ matrix A is invertible if we can find an $n \times n$ matrix B s.t.:

$$\boxed{AB = BA = I_n}$$

They commute

If NOT, we say that A is singular.

Theorem: $AB = I$ and $CA = I \Rightarrow B = C$

proof: compute:

$$CAB$$

$$\begin{aligned} (CA)B &\qquad C(AB) \\ = IB &\qquad = CI \\ = B &\qquad = C \end{aligned}$$

* Consequently, if A has an inverse, it must be unique.

$\Rightarrow$ we denote it by $\boxed{A^{-1}}$ Do NOT write $\frac{1}{A}$

$$\boxed{AA^{-1} = A^{-1}A = I}$$

1. 2×2 case

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff $ad - bc \neq 0$

↳ its determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

and

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

ex: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ determinant $= 4 - 4 = 0$ $\therefore$ <u>singular</u>

ex: $\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ determinant $= 6 - 5 = 1$ $\therefore$ <u>invertible</u>

$$\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}^{-1} = \frac{1}{1} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

2. Now let E be an elementary matrix associated to op : $I \overset{op}{\sim} E$

And let E' be the elt. matrix associated to the reverse operation op^{-1} :

$$\left.\begin{array}{l} I \overset{op}{\sim} E \overset{op^{-1}}{\sim} E'E = I \\ I \overset{op^{-1}}{\sim} E' \overset{op}{\sim} EE' = I \end{array}\right\} \quad E' = E^{-1}$$

ie. the inverse of E

Theorem: Every elt. matrix is invertible and its inverse is the elt matrix associated to its reverse operation. More precisely :

(a) if $E: R_i \leftrightarrow R_j \quad \therefore \quad E^{-1} : E : R_i \leftrightarrow R_j$ → matrix is its own inverse

(b) if $E: R_i \leftarrow kR_i \quad \therefore \quad E^{-1} : \frac{1}{k} R_i$

(c) if $E: R_i \leftarrow R_i + kR_j \quad \therefore \quad E^{-1} : R_i \leftarrow R_i - kR_i$

ex: $$\underset{R_1 \leftrightarrow R_2}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1}} = \underset{R_1 \leftrightarrow R_2}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$ same thing.

ex: $$\underset{R_2 \leftarrow -1/2 R_2}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}} = \underset{R_2 \leftarrow -2R_2}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

ex: $$\underset{R_1 \leftarrow R_1 + 3R_3}{\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}} = \underset{R_1 \leftarrow R_1 - 3R_3}{\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

Theorem:

1. A invertible $\Rightarrow$ A^{-1} invertible and $(A^{-1})^{-1} = A$

   proof: $AA^{-1} = A^{-1}A = I$

2. A, B invertible $\Rightarrow$ AB invertible and $(AB)^{-1} = B^{-1}A^{-1}$

   inverse of AB is $B^{-1}A^{-1}$

   proof: Recall: $AA^{-1} = A^{-1}A = I$

   $\therefore$ $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$

   and $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$

3. $A_1, A_2, \dots, A_k$ invertible $\Rightarrow$ $A_1A_2 \dots A_k$ is invertible and

   $$(A_1A_2 \dots A_k)^{-1} = A_k^{-1} \dots A_2^{-1}A_1^{-1}$$

   proof

4. A invertible $\Rightarrow$ A^k invertible and $(A^k)^{-1} = (A^{-1})^k$

   proof: Apply #3 with all $A_i = A$

5. A invertible $\Rightarrow$ kA invertible and $(kA)^{-1} = \frac{1}{k}A^{-1}$ $(k \neq 0)$

   proof: $(kA)(\frac{1}{k}A^{-1}) = k(\frac{1}{k})AA^{-1} = I$ bc $k \cdot \frac{1}{k} = 1$

   $(\frac{1}{k}A^{-1})(kA) = \frac{1}{k}(k)A^{-1}A = I$ inverse

6. A invertible $\Rightarrow$ A^T invertible and $(A^T)^{-1} = (A^{-1})^T$

   proof: $(A^T)(A^{-1})^T = (A^{-1}A)^T = I^T = I$

   $(A^{-1})^T(A)^T = (AA^{-1})^T = I^T = I$

7. Cancellation Laws: if A invertible $\therefore$ $AB = AC \Rightarrow B = C$

   $BA = CA \Rightarrow B = C$

   hit both on left

   proof: $AB = AC \Rightarrow$ $A^{-1}AB = A^{-1}AC$

   $IB = IC$

   $BA = CA \Rightarrow$ $BAA^{-1} = CAA^{-1}$

   $BI = CI$

MAT-NYC INVERTIBILITY THEOREM

Theorem Let A be an $n \times n$ matrix. Then the following statements are equivalent.

1. A is invertible.
2. The reduced row-echelon form of A is I_n.
3. The rank of A is n.
4. The system $A\vec{x} = \vec{b}$ is consistent for all $\vec{b}$.
5. The system $A\vec{x} = \vec{b}$ has exactly one solution for all $\vec{b}$.
6. The homogeneous system $A\vec{x} = \vec{0}$ only has the trivial solution.
7. A is a product of elementary matrices.
8. The determinant of A is non zero.
9. The column space of A is $\mathbb{R}^n$.
10. The column vectors of A span $\mathbb{R}^n$.
11. The column vectors of A are linearly independent.
12. The column vectors of A form a basis for $\mathbb{R}^n$.
13. the nullity of A is 0.
14. The nullspace of A is $\{\vec{0}\}$.

For MAT-NYC Enriched:

15. The row space of A is $\mathbb{R}^n$.
16. The row vectors of A span $\mathbb{R}^n$.
17. The row vectors of A are linearly independent.
18. The row vectors of A form a basis for $\mathbb{R}^n$.
19. The orthogonal complement of the null space of A is $\mathbb{R}^n$.
20. The orthogonal complement of the row space of A is $\{\vec{0}\}$.

Invertibility Theorem (only square matrices)

→ We can now prove #1 and #7 of the invertibility theorem.

#1 ⇒ #5 : A is invertible ⇒ $A\vec{x} = \vec{b}$ has exactly one sol'n for all $\vec{b}$

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}\vec{b}$$ (only one sol'n) ✓

#5 ⇒ #2 : $A\vec{x} = \vec{b}$ has exactly one sol'n for all $\vec{b}$ ⇒ RREF of A is I_n

pivot in every column and row (square)

#2 ⇒ #7 : RREF of A is I_n ⇒ A is a product of elt. matrices

Gaus Jordan: $A \overset{op_1}{\sim} E_1 A \overset{op_2}{\sim} E_2 E_1 A \; \dots \overset{op_K}{\sim} \boxed{\underbrace{(E_K \dots E_2 E_1)}_{\text{invertible}} A = I}$

$$(E_K \dots E_2 E_1) A = I$$
$$A = (E_K \dots E_2 E_1)^{-1} I$$
$$A = E_1^{-1} E_2^{-1} \dots E_K^{-1}$$

(elt. matrices are invertible and their inverse is also an elt. matrix)

∴ A is a product of elt. matrices

ex: Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, express A and A^{-1} as a product of elt. matrices.

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{op_1} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \xrightarrow[R_2 \leftarrow R_2 - 2R_1]{op_2} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \xrightarrow[R_2 \leftarrow -R_2]{op_3} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow[R_1 \leftarrow R_1 - 2R_2]{op_4}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$E_4 E_3 E_2 E_1 A = I$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A = I$$

① $E_4 E_3 E_2 E_1 A A^{-1} = I A^{-1}$

$$A^{-1} = E_4 E_3 E_2 E_1$$

$$A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

② $E_4 E_3 E_2 E_1 A = I$

$$A = (E_4 E_3 E_2 E_1)^{-1} I$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$$

perform reverse operation op_1^{-1}

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Theorem: Let A and B be $n \times n$ matrices

Then if $AB = I \Leftrightarrow BA = I$

ie. $\underbrace{AB = I \quad \text{OR} \quad BA = I}_{\text{only need to check one of them from now on.}} \Rightarrow A$ is invertible with $B = A^{-1}$

proof:

Given $AB = I$, take $\vec{b}$ in $\mathbb{R}^n$, then

$B\vec{b}$ is a sol'n of $A\vec{x} = \vec{b}$ since

$$A(B\vec{b}) = AB(\vec{b}) = I\vec{b} = \vec{b}$$

$\therefore$ $A\vec{x} = \vec{b}$ is consistent for all $\vec{b}$

$\Rightarrow$ pivot in every row (always consistent)

$\Rightarrow$ RREF $= I$

$\Rightarrow$ A is invertible

But then,

$$AB = I \Rightarrow A^{-1}AB = A^{-1}I$$

$$B = A^{-1}$$

$$\boxed{B = A^{-1}}$$

$\therefore$ $BA = I$ also

How to compute A^{-1}

Take A to be any $n \times n$ matrix and let R be its RREF.

We have 2 possibilities:

- R has zero row $\Rightarrow rk(A) < n \Rightarrow A$ is singular
- R has No zero row $\Rightarrow R = I$

↳ Apply Gaus-Jordan

$$[A \mid I] \sim [\underset{\overset{\parallel}{UA}}{I} \mid U]$$

$$\therefore UA = I \overset{\text{previous}}{\underset{\text{theorem}}{\Longrightarrow}} \boxed{U = A^{-1}}$$

$$[A \mid I] \xrightarrow{G.J} [I \mid A^{-1}]$$

Determinants

A) The Determinant Function

Let A be square (ie. A is $n \times n$)

we define a function

$$\det : \left\{ \begin{array}{l} \text{set of all} \\ \text{square matrices} \end{array} \right\} \to \mathbb{R}$$

$$A \longmapsto \det A = |A|$$

- A is 1×1 (ie $A = [a]$)

$$\det A = a$$

- 2×2 determinant (recall cross product)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- 3×3 determinant (recall TSP)

$$\begin{vmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} & \overset{+}{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{vmatrix}$$

Let A be $n \times n$ square

Def'n: A_{ij} = minor matrix of entry a_{ij} of A

= submatrix obtained from A by deleting its i^{th} row and j^{th} column.

ex: $A = \begin{array}{cc} & \begin{array}{cccc} & & 3 & \end{array} \\ \begin{array}{c} \\ 2 \\ \\ \\ \end{array} & \begin{bmatrix} 6 & 2 & \cancel{1} & -1 \\ \cancel{0} & \cancel{1} & \cancel{5} & \cancel{2} \\ 2 & 5 & \cancel{6} & 0 \\ 3 & 1 & \cancel{3} & -2 \end{bmatrix} \end{array}$, $A_{23} = \begin{bmatrix} 6 & 2 & -1 \\ 2 & 5 & 0 \\ 3 & 1 & -2 \end{bmatrix}$

Def'n: c_{ij} = cofactor of entry a_{ij} of A

$= (-1)^{i+j} \det(A_{ij})$

even: +
odd: −

$$\begin{bmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{bmatrix}$$

ex: $A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$

$$c_{12} = (-1)^3 \begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix} = (-1)(16-6) = -10$$

$$c_{32} = (-1)^5 \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = (-1)(18--8) = -26$$

Def'n: For $n \geq 2$

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \dots + a_{1n}C_{1n}$$

only looking at row $i=1$

cofactor expansion along row 1

Theorem: The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion along any row OR column.

ex:
$$\begin{vmatrix} -2 & -4 & 3 \\ 3 & 1 & 0 \\ -5 & 4 & -2 \end{vmatrix} = (-3)\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} + (1)\begin{vmatrix} -2 & 3 \\ -5 & -2 \end{vmatrix} + \cancel{(0)\begin{vmatrix} -2 & -4 \\ -5 & 4 \end{vmatrix}}$$

expansion along row 2

$$= -3(-4) + (19) = 31$$

ex:
$$\begin{vmatrix} 3 & 2 & 0 & 1 & 3 \\ -2 & 4 & 1 & 2 & 1 \\ 0 & -1 & 0 & 1 & -5 \\ -1 & 2 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix} \underset{\text{along C3}}{=} (-1)\begin{vmatrix} 3 & 2 & 1 & 3 \\ 0 & -1 & 1 & -5 \\ -1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 2 \end{vmatrix} \underset{\text{along R4}}{=} (-1)(2)\begin{vmatrix} 3 & 2 & 1 \\ 0 & -1 & 1 \\ -1 & 2 & -1 \end{vmatrix}$$

$$\underset{\text{along R2}}{=} (-2)\Bigg[\underbrace{(-1)\begin{vmatrix} 3 & 1 \\ -1 & -1 \end{vmatrix} - (1)\begin{vmatrix} 3 & 2 \\ -1 & 2 \end{vmatrix}}_{-6}\Bigg]$$

$$= (-2)(-6)$$

$$= 12$$

Alternative 3x3

$$\begin{vmatrix} 3 & 2 & 1 \\ 0 & -1 & 1 \\ -1 & 2 & -1 \end{vmatrix}\begin{matrix} 3 & 2 \\ 0 & -1 \\ -1 & 2 \end{matrix}$$

+ + +

− − −

$$= +(3)(-1)(-1) + (2)(1)(-1) + (1)(0)(2)$$
$$- (1)(-1)(-1) - (3)(1)(2) - (2)(0)(-1)$$

$$= 3 - 2 - 1 - 6 = -6$$

Theorem: If A is triangular (upper or lower) then $\det A$ is the product of the entries on the main diagonal.

ex: $\begin{vmatrix} 2 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 7 & 2 & 3 & 0 \\ 5 & 1 & 2 & -1 \end{vmatrix}$ lower triangular

$= (2)(1)(3)(-1) = -6$

ex: $\det(I) = (1)(1)\dots(1) = 1$

$\begin{vmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{vmatrix}$

ex: $\det(4 I_3) = \begin{vmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{vmatrix} = 4^3 = 64$

Note: $\det(K I_n) = K^n$

B) Properties of the Determinant

Theorem: Effect of row operations

1. $A \xrightarrow{R_i \leftarrow K R_i \ (K \neq 0)} B \Rightarrow \det B = K \det A$

2. $A \xrightarrow{R_i \leftrightarrow R_j} B \Rightarrow \det B = - \det A$

3. $A \xrightarrow{R_i \leftarrow R_i + K R_j} B \Rightarrow \det B = \det A$

$\therefore \det B = m \det A$, where $m = \begin{cases} K \neq 0 & \text{in \#1} \\ -1 & \text{in \#2} \\ 1 & \text{in \#3} \end{cases}$

$\therefore m \neq 0$

ex: we can factor a constant from a single row (#1)

$$\underbrace{\begin{vmatrix} 2 & 4 & 6 \\ 1 & 2 & 1 \\ 7 & 3 & 1 \end{vmatrix}}_{B} = 2 \underbrace{\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 7 & 3 & 1 \end{vmatrix}}_{A}$$

$A \xrightarrow{R_1 \leftarrow 2R_1} B \quad \therefore \quad \det B = 2 \det A$

ex: Say $\det A = 10$

$$A \xrightarrow{R_1 \leftrightarrow R_3} \xrightarrow{R_1 \leftarrow R_1 - 5R_2} \xrightarrow{R_2 \leftarrow 7R_2} B$$

$\therefore \det B = (-1)(1)(7) \cdot \det A$

$= (-7)(10)$

$= -70$

Theorem: A invertible $\Leftrightarrow$ $\det A \neq 0$

(statement #8 of invertibility theorem)

proof: Let A be any square matrix and let R be its RREF

$$A \sim \sim \sim R$$

$$\therefore \det R = \underbrace{m_1 m_2 m_3 \ldots}_{c \neq 0} \det A$$

$$\det R = c \cdot \det A$$

1. A is invertible $\therefore R = I \Rightarrow \det(I) = 1 = c \det A$

$$\therefore \boxed{\det A \neq 0}$$

2. A is singular $\therefore$ R has zero row $\Rightarrow \det R = 0 = \underbrace{c}_{c \neq 0} \det A$

$$\therefore \boxed{\det A = 0}$$

Theorem: If a square matrix has 2 proportional rows, then its determinant is zero.

proof: we will get a zero row in REF

since $R_i = KR_j$ (proportional)

$\therefore$ $R_i \leftarrow R_i - KR_j = 0$ (zero rows)

$\therefore$ rank $< n \Leftrightarrow$ singular $\Leftrightarrow \det = 0$
(by invertibility theorem)

ex: $\begin{vmatrix} 3 & -1 & 4 & -5 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ -9 & 3 & -12 & 15 \end{vmatrix} = 0$ proportional

Theorem: $\det A = \det(A^T)$

ex: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \Rightarrow \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$

This is an important result since it means any determinant property stated about $\boxed{\text{rows}}$ is also true about $\boxed{\text{columns}}$.

ex: ① Interchanging 2 columns changes the sign (+/−) of det
② we can factor out a constant from a single column
③ If 2 columns are proportional, then det = 0

ex $\begin{vmatrix} 1 & -2 & 7 \\ -4 & 8 & 5 \\ 2 & -4 & 3 \end{vmatrix} = 0$ $\quad C_2 = -2C_1$ (proportional)

So we may perform column as well as row operations when computing determinants. (BUT ONLY FOR THIS!)

Since cofactor expansion is at its most efficient when all but one entry in a row or column are zeros, chose a row or column with a "1" entry and "0"s if possible.

Before expanding along that row or column, make all other entries zero by:

→ column operations for a row OR

→ row operations for a column

$$\text{ex:}\quad \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} & & C_3 & \\ & & \downarrow & \end{array}$$

$$\text{ex:}\quad \begin{vmatrix} 2 & 0 & -1 & 7 \\ 6 & 1 & 0 & 4 \\ 8 & -2 & 1 & 0 \\ 4 & 1 & 0 & 2 \end{vmatrix}$$

want to eliminate -1 from column
$\therefore$ Row operation

$$R_1 \leftarrow R_1 + R_3$$

$$= \begin{vmatrix} 10 & -2 & 0 & 7 \\ 6 & 1 & 0 & 4 \\ 8 & -2 & 1 & 0 \\ 4 & 1 & 0 & 2 \end{vmatrix}$$

$$\overset{\text{along } C_3}{=} (1)\begin{vmatrix} 10 & -2 & 7 \\ 6 & 1 & 4 \\ 4 & 1 & 2 \end{vmatrix} \leftarrow R_3$$

want to eliminate entries from R_3
$\therefore$ column operations

$$C_1 \leftarrow C_1 - 4C_2$$
$$C_3 \leftarrow C_3 - 2C_2$$

$$= (1)\begin{vmatrix} 18 & -2 & 11 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \end{vmatrix}$$

$$\overset{\text{along } R_3}{=} (-1)(1)\begin{vmatrix} 18 & 11 \\ 2 & 2 \end{vmatrix}$$

$$= (-1)(2)\begin{vmatrix} 18 & 11 \\ 1 & 1 \end{vmatrix} = -2(18-11) = \boxed{-14}$$

Factor out from R_2

Theorem: Let A be $n \times n$

$$\boxed{\det(KA) = K^n \det(A)}$$

(since each row of A got multiplied by K)

Theorem: $\boxed{\det(AB) = \det(A)\det(B)}$

CAREFUL: $\det(A+B) \neq \det(A) + \det(B)$

proof: 1. Say A is singular, then $\det(A) = 0$

Then AB is singular as well; otherwise if its inverse was C $\therefore$ $\underbrace{(AB)C = I \Rightarrow A(BC) = I \Rightarrow A \text{ inv.}}_{\text{proof by contradiction}}$ $\times$

$\therefore$ AB singular $\therefore$ $\det(AB) = 0$

2. Say $A = E$ elementary where $I \overset{op}{\rightsquigarrow} E \Rightarrow B \overset{op}{\rightsquigarrow} EB$

by our previous theorem on row operations

$$\det(EB) = m\det(B), \quad \text{where } m = \begin{cases} K \neq 0 & : R_i \leftarrow KR_i \\ 1 & : R_i \leftarrow R_i + KR_j \\ -1 & : R_i \leftrightarrow R_j \end{cases}$$

same op $\therefore$ same m

Also, $\det(E) = \det(EI) = m\det(I) = m$

$\therefore$ $\det(EB) = m\det(B) = \det(E)\det(B)$

3. Say A is invertible, then $A = E_K \dots E_2 E_1$

$\Rightarrow \det(AB) = \det(E_K \dots E_2 E_1 B) = \det(E_K)\det(E_{K-1} \dots E_2 E_1)$

From #2

$\dots = \det(E_K)\det(E_{K-1}) \cdots \det(E_2)\det(E_1)\det(B)$

Also, $\det(A) = \det(AI) = \det(E_K) \dots \det(E_2)\det(E_1)\det(I)$

$\therefore$ $\det(AB) = \det(A)\det(B)$

Corollary

(i) $\det(A_1 A_2 \dots A_k) = \det(A_1)\det(A_2)\det(A_3)\dots\det(A_k)$

(ii) $\det(A^n) = [\det(A)]^n$

(iii) If A invertible, $\det(A^{-1}) = \dfrac{1}{\det(A)}$

c) Cramer's Rule

Let $A\vec{x} = \vec{b}$ be an $n \times n$ system with A invertible.

Then the unique solution is given by:

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_i is the matrix obtained from A by replacing its i^{th} column by $\vec{b}$. $(1 \le i \le n)$

ex: $\begin{cases} 8x - 3y = 2 \\ -5x + 4y = 1 \end{cases} \Rightarrow \begin{array}{c} \left[\begin{array}{cc|c} 8 & -3 & 2 \\ -5 & 4 & 1 \end{array}\right] \\ \begin{array}{ccc} x & y & \; \end{array} \end{array}$, $\det A = \begin{vmatrix} 8 & -3 \\ -5 & 4 \end{vmatrix} = 17$

$\therefore \; x = \dfrac{\begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix}}{17} = 11/17$, $\; y = \dfrac{\begin{vmatrix} 8 & 2 \\ -5 & 1 \end{vmatrix}}{17} = 18/17$

<u>proof:</u> $\vec{b} = A\vec{x} = x_1\vec{c_1} + x_2\vec{c_2} + \ldots x_n\vec{c_n}$, with $A = [\vec{c_1} \mid \vec{c_2} \mid \ldots \mid \vec{c_n}]$

$$\det(A_i) = \det[\vec{c_1} \mid \vec{c_2} \mid \ldots \mid \underset{i^{th}\text{ column}}{\vec{b}} \mid \ldots \mid \vec{c_n}]$$

$$= \det[\vec{c_1} \mid \vec{c_2} \mid \ldots \mid \underset{i^{th}\text{ column}}{x_1\vec{c_1} + x_2\vec{c_2} + \ldots + x_i\vec{c_i} + \ldots + x_n\vec{c_n}} \mid \ldots \mid \vec{c_n}]$$

column operations:

operations where $\det(B) = m \det(A)$ $\Rightarrow m = 1$

i^{th} column $\leftarrow$ i^{th} column $- x_1\vec{c_1}$
i^{th} column $\leftarrow$ i^{th} column $- x_2\vec{c_2}$
$\vdots$
i^{th} column $\leftarrow$ i^{th} column $- x_n\vec{c_n}$
(all except $x_i\vec{c_i}$)

$$= \det[\vec{c_1} \mid \vec{c_2} \mid \ldots \mid x_i\vec{c_i} \mid \ldots \mid \vec{c_n}]$$

Factor x_i $\quad = x_i \underbrace{\det[\vec{c_1} \mid \vec{c_2} \mid \ldots \mid \vec{c_i} \mid \ldots \mid \vec{c_n}]}_{A}$

$$\det(A_i) = x_i \det(A) \Rightarrow x_i = \frac{\det(A_i)}{\det(A)}$$

Complex Numbers

A) Definition

$$\boxed{\mathbb{C} = \{a + bi\}}$$

where $a, b \in \mathbb{R}$

and $\boxed{i}$ is a symbol used to represent a solution to $x^2 = -1$
ie. $i^2 = -1$

Standard Form : $\underbrace{a}_{\text{real \#}} + \underbrace{bi}_{\text{imaginary part}}$

ex: $z = 1/4 - \sqrt{2}i$ $\therefore$ $\text{Re } z = 1/4$, $\text{Im } z = -\sqrt{2}$

ex: (i) $5 + 2i$, (ii) 0, (iii) 6, (iv) πi

(i) ↳ Re = 5, Im = 2
(ii) ↳ Re = 0, Im = 0
(iii) ↳ Re = 6, Im = 0
(iv) ↳ Re = 0, Im = π

Equality : $a + bi = c + di \Leftrightarrow \begin{cases} a = c \\ b = d \end{cases}$ same real part and same imaginary part

B) Operations

$+, -, \times, \div$ same rules as before,
but with additional rule $\boxed{i^2 = -1}$

ex: $(3+2i) - (2-i) = 1 + 3i$

ex: $(3+2i)(2-i) = 6 - 3i + 4i - 2\overset{-1}{i^2} = 8 + i$

ex: $(2-3i)^2 - (4i)^2 = 4 - 12i + 9\underset{-1}{i^2} - 16\underset{-1}{i^2} = 11 - 12i$

High powers of i

$\boxed{i^0 = 1}$, $i^1 = i$, $i^2 = -1$, $i^3 = i^2 . i = -i$, $i^4 = i^2 . i^2 = 1$

$i^m = i$ ^ (remainder of the division of m by 4)

ex: $i^{101} = i^{(25 \times 4 + 1)} = i^1 = i$

OR

$i^{101} = i^{100} \cdot i^1 = (i^2)^{50} . i = (-1)^{50} i = i$

isolate even power

even

To write a <u>quotient</u> in standard form, we need:

<u>Complex Conjugate</u> : $\overline{z} = \overline{a+bi} = a - bi$

ex: $\overline{6-3i} = 6+3i$

ex: $\overline{i} = \overline{0+i} = -i$

ex: $\overline{5} = \overline{5+0i} = 5$

ex: $\overline{2i-1} = \overline{-1+2i} = -1-2i$

Add neg. (-1) to imaginary part.

<u>Rules</u>: (i) $\overline{z+w} = \overline{z} + \overline{w}$

(ii) $\overline{zw} = \overline{z}\,\overline{w}$, so $\overline{z^n} = (\overline{z})^n$

(iii) $\overline{(z/w)} = \overline{z}/\overline{w}$

* $\boxed{z\overline{z} = a^2 + b^2}$

<u>proof</u>: $z = a+bi \Rightarrow z\overline{z} = (a+bi)(a-bi)$

$= a^2 - b^2 i^2$

$= a^2 + b^2$

ex: $\dfrac{1}{6-3i} = \dfrac{(6+3i)}{(6-3i)(6+3i)} \underset{*}{=} \dfrac{6+3i}{(6)^2+(-3)^2} = \dfrac{6}{45} + \dfrac{3}{45} i$

ex: $\dfrac{3+2i}{2-i} = \dfrac{(3+2i)(2+i)}{(2-i)(2+i)} \underset{*}{=} \dfrac{6+3i+4i+2i^2}{(2)^2+(-1)^2} = \dfrac{4+7i}{5} = \dfrac{4}{5} + \dfrac{7}{5} i$

Def'n: Set $\boxed{\sqrt{-a} = i\sqrt{a}}$, where "a" is a positive real number

ex: $\sqrt{-4} = i\sqrt{4} = 2i$

ex: $\sqrt{-7} = i\sqrt{7}$

ex: $\sqrt{-2}\cdot\sqrt{-3} = i\sqrt{2}\ i\sqrt{3}$
$= i^2\sqrt{6}$
$= -\sqrt{6}$

CAREFUL: $\sqrt{-2}\sqrt{-3} \neq \sqrt{6}$

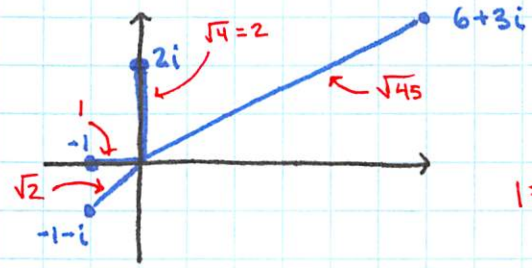
But: $\sqrt{(-2)(-3)} = \sqrt{6}$

* To be safe, always switch $\sqrt{-a}$ to $\sqrt{a}$ before doing any computation.

C) Complex Plane

$\begin{cases} x\text{-axis is real axis} \\ y\text{-axis is imaginary axis} \end{cases}$ so $\boxed{z = a+bi \Rightarrow \text{point } (a,b)}$

ex: plot: (i) $6+3i$
(ii) -1
(iii) $-1-i$
(iv) $2i$



$|z|$ = distance

Note: $\bar{z}$ is symmetric of z w.r.t x-axis (neg. imaginary)

Modulus:

$|z| = \sqrt{a^2+b^2}$ = distance from origin

ex: $|2-i| = \sqrt{5}$
ex: $|-3i| = 3$

Rules: (i) $|zw| = |z||w|$, so $|z^n| = |z|^n$
(ii) $|\bar{z}| = |z|$
(iii) $|z/w| = |z|/|w|$

ex: $\left| \dfrac{5(1-i)^2}{(3+2i)^3} \right| = \dfrac{|5|\,|1-i|^2}{|3+2i|^3} = \dfrac{5(\sqrt{2})^2}{(\sqrt{13})^3}$

ex: $\left| (7-3i)^{100} \right| = |7-3i|^{100} = (\sqrt{58})^{100} = (58)^{50}$

* **Theorem:** $\boxed{z\bar{z} = |z|^2}$

$z\bar{z} \to a^2+b^2$, $|z|^2 \to (\sqrt{a^2+b^2})^2$

ex: $A = \begin{bmatrix} i & 1+2i \\ 0 & -5 \end{bmatrix}$, Find A^{-1}

Recall: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\therefore$ $A^{-1} = \frac{1}{\underbrace{ad-bc}_{\det A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\det A = -5i$

$$A^{-1} = \frac{1}{-5i} \begin{bmatrix} -5 & -1-2i \\ 0 & i \end{bmatrix} = \begin{bmatrix} \frac{1}{i} & \frac{-1-2i}{-5i} \\ 0 & -\frac{1}{5} \end{bmatrix}$$

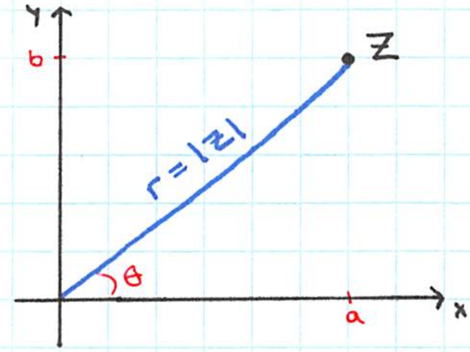
Note: $\frac{1}{i} = \frac{1}{i^2} \cdot i = -i$

Note: $\frac{1}{5i} + \frac{2}{5} = \frac{1}{5i^2} \cdot i + \frac{2}{5} = \frac{-i}{5} + \frac{2}{5}$

$$A^{-1} = \begin{bmatrix} -i & \frac{-i}{5} + \frac{2}{5} \\ 0 & -1/5 \end{bmatrix}$$

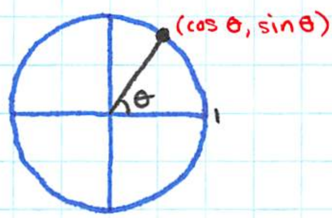
D) Polar (or Trigonometric) form of a Complex Number

Let θ = argument
= angle that a ray from the origin to z makes with the positive x-axis.



$$\boxed{\begin{aligned} a &= r\cos\theta \\ b &= r\sin\theta \end{aligned}}$$

Recall:



Here $r = 1$

Polar form: $z = \underbrace{r\cos\theta}_{a} + \underbrace{r\sin\theta}_{b}\, \underbrace{i}_{i}$

$= r(\cos\theta + i\sin\theta)$

$= r\,\text{cis}\,\theta$

Note: Polar form / Trig form vs Standard form / Rectangular form

$\boxed{z = r\,\text{cis}\,\theta}$ $\qquad$ $\boxed{z = a + bi}$

modulus (r), argument (θ)

$a + bi$

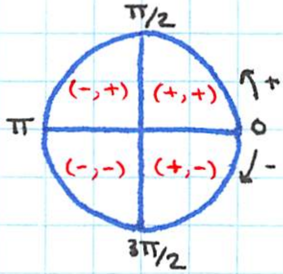
x-axis   y-axis

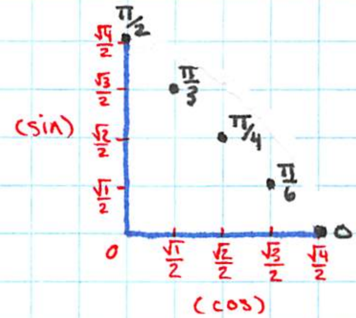
$\Downarrow$

$r(\cos\theta + \sin\theta\, i)$

x   y

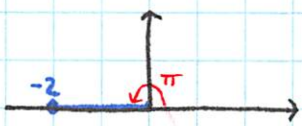
$(\cos\theta, \sin\theta)$





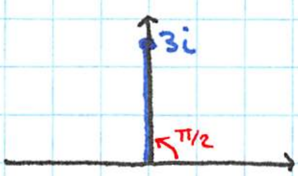
$\pi/4 : \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\pi/3 : \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\pi/6 : \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

ex: $-2 = ?$



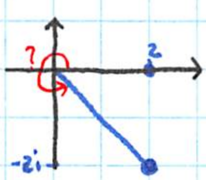
$r = 2$ $\therefore$ $-2 = 2 \text{ cis}(\pi)$

ex: $3i = ?$



$r = 3$ $\therefore$ $3i = 3 \text{ cis}(\pi/2)$

ex: $2 - 2i = ?$



① Factor out modulus (first find modulus)

$$r = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$$

$$\therefore 2 - 2i = \underbrace{2\sqrt{2}}_{r}\Big(\underbrace{\tfrac{1}{\sqrt{2}}}_{(\cos\theta} - \underbrace{\tfrac{1}{\sqrt{2}}}_{+ \sin\theta\, i)} i\Big)$$

$$= 2\sqrt{2} \text{ cis}(-\pi/4)$$

$\left.\begin{array}{l}\cos\theta = \frac{\sqrt{2}}{2}\\ \sin\theta = -\frac{\sqrt{2}}{2}\end{array}\right\}\ \theta = -\frac{\pi}{4}$

ex: $2\sqrt{3} + 2i = ?$

① Factor out modulus $\quad r = \sqrt{(2\sqrt{3})^2 + (2)^2} = \sqrt{16} = 4$

$$\therefore 2\sqrt{3} + 2i = 4\Big(\underbrace{\tfrac{\sqrt{3}}{2}}_{\cos} + \underbrace{\tfrac{1}{2}}_{\sin} i\Big)$$

$$= 4 \text{ cis}(\pi/6)$$

$\left.\begin{array}{l}\cos\theta = \frac{\sqrt{3}}{2}\\ \sin\theta = \frac{1}{2}\end{array}\right\}\ \theta = \frac{\pi}{6}$

Multiplication and Division

$z = r \operatorname{cis} \alpha = r(\cos\alpha + i\sin\alpha)$

$w = s \operatorname{cis} \beta = s(\cos\beta + i\sin\beta)$

(i) $zw = rs \operatorname{cis}(\alpha+\beta)$

(ii) $\frac{1}{z} = \frac{1}{r} \operatorname{cis}(-\alpha)$

(iii) $\frac{z}{w} = \frac{r}{s} \operatorname{cis}(\alpha-\beta)$

proof:

(i)
$$\begin{aligned} zw &= rs(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= rs(\cos\alpha\cos\beta + i\cos\alpha\sin\beta + i\sin\alpha\cos\beta + \cancel{i^2}^{-1}\sin\alpha\sin\beta) \\ &= rs(\underbrace{\cos\alpha\cos\beta - \sin\alpha\sin\beta}_{\cos(\alpha+\beta)} + i(\underbrace{\cos\alpha\sin\beta + \sin\alpha\cos\beta}_{\sin(\alpha+\beta)})) \\ &= rs \operatorname{cis}(\alpha+\beta) \end{aligned}$$

(ii)
$$\begin{aligned} \frac{1}{z} &= \frac{1}{r(\cos\alpha + i\sin\alpha)} \\ &= \frac{1}{r}\frac{1}{\cos\alpha + i\sin\alpha} \\ &= \frac{1}{r}\frac{(\cos\alpha - i\sin\alpha)}{(\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)} \\ &= \frac{1}{r}\frac{\cos\alpha - i\sin\alpha}{\underbrace{(\cos^2\alpha + \sin^2\alpha)}_{=1}} \\ &= \frac{1}{r}(\cos(-\alpha) + i\sin(-\alpha)) \\ &= \frac{1}{r}\operatorname{cis}(-\alpha) \end{aligned}$$

Recall: $z\bar{z} = a^2 + b^2$

↳ here: $a = \cos\alpha$, $b = \sin\alpha$

Note: $\cos\theta = \cos(-\theta)$

$\sin\theta = -\sin(-\theta)$

(iii) From (i) and (ii)

$\frac{z}{w} = z \cdot \frac{1}{w}$

DeMoivre's Theorem

$$(r \operatorname{cis} \theta)^n = r^n \operatorname{cis}(n\theta) \quad , \quad n \text{ any integer}$$

proof:

n positive integer

$$(r \operatorname{cis} \theta)^n = \underbrace{r \operatorname{cis}\theta \cdot r \operatorname{cis}\theta \ldots r \operatorname{cis}\theta}_{n \text{ times}} = \overbrace{r \cdot r \ldots r}^{n} \cdot \operatorname{cis}(\overbrace{\theta + \theta + \ldots + \theta}^{n})$$

$$= r^n \operatorname{cis}(n\theta)$$

$-n$ instead, but with n positive

From division rules: $\frac{1}{z} = \frac{1}{r} \operatorname{cis}(-\alpha)$

$$(r \operatorname{cis} \theta)^{-n} = \frac{1}{(r \operatorname{cis} \theta)^n} = \frac{1}{r^n \operatorname{cis}(n\theta)} = \frac{1}{r^n} \operatorname{cis}(-n\theta)$$

$$= r^{-n} \operatorname{cis}(-n\theta)$$

$\therefore$ also works for negative n

Recap:

Polar (Trigonometric) : $z = r \operatorname{cis} \theta$

r = modulus = $\sqrt{a^2 + b^2}$

θ = argument

$$\begin{cases} \cos\theta = a/r \\ \sin\theta = b/r \end{cases}$$ "Factor out modulus"

Standard (Rectangular) : $z = a + bi$

$a = r\cos\theta$
$b = r\sin\theta$

$$r \operatorname{cis}\theta = \underbrace{r\cos\theta}_{a} + \underbrace{i}_{i}\,\underbrace{r\sin\theta}_{b}$$