Differential Equations (DE)

ex. $y' + 3x^2 y = 6x^2$

$y = 2 + 7e^{-x^3}$ ← Don't need to know how to get it

Verify that it is a sol'n to the DE above

$y = 2 + 7e^{-x^3}$

$y' = 7e^{(-x^3)}(-3x^2)$

$y' = -21x^2 e^{-x^3}$

L.S.	R.S.
$y' + 3x^2 y$	$6x^2$ ✓
$-21x^2 e^{-x^3} + 3x^2(2 + 7e^{-x^3})$	
$-\cancel{21x^2 e^{-x^3}} + 6x^2 + \cancel{21x^2 e^{-x^3}}$	
$6x^2$ ✓	

In fact $y = 2 + Ce^{-x^3}$ is a sol'n for any fixed constant C.

Verification

$y = 2 + Ce^{-x^3}$

$y' = Ce^{-x^3}(-3x^2)$

$y' = -3Cx^2 e^{-x^3}$

$\underline{LS}$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\underline{RS}$

$-3Cx^2e^{-x^3} + 3x^2\left(2 + Ce^{-x^3}\right)$ $\qquad\qquad$ $6x^2$

$\cancel{-3Cx^2e^{-x^3}} + 6x^2 + \cancel{3Cx^2e^{-x^3}}$

As it turns out $y = 2 + Ce^{-x^3}$ describes all sol'ns

So $y = 2 + Ce^{-x^2}$ is called <u>the general sol'n</u> of the DE

and $y = 2 + 7e^{-x^2}$ is " <u>a particular sol'n</u>

<u>ex.</u> Given that $y = 9$ when $x = 0$, find the particular sol'n to the DE

$$y' + 3x^2y = 6x^2$$

General sol'n: $y = 2 + Ce^{-x^3}$

$$9 = 2 + Ce^0$$

$$9 = 2 + C$$

$$C = 7$$

So particular sol'n is $y = 2 + 7e^{-x^3}$

<u>ex.</u> Show that $y = \sin(2\pi x)$ is a sol'n to the DE

$$y'' + 4\pi^2y = 0$$

$y' = 2\pi \cos(2\pi x)$

$y'' = 2\pi(-2\pi \sin 2\pi x)$

$y'' = -4\pi^2 \sin 2\pi x$

LS

$y'' + 4\pi^2 y$

$\cancel{-4\pi^2 \sin 2\pi x} \cancel{-4\pi^2 \sin 2\pi x}$

0 ✓

RS

0 ✓

ex. Show that $x = a\cos(wt) + b\sin(wt)$ is a sol'n to the DE $x'' + w^2 x = 0$ for any fixed a, b & w

$x' = -aw\sin(wt) + bw\cos(wt)$

$x'' = -aw^2\cos(wt) - bw^2\sin(wt)$

L.S.

$x'' + w^2 x$

$-aw^2\cos(wt) - bw^2\sin(wt) + w^2(a\cos(wt) + b\sin(wt))$

0 ✓

R.S.

0 ✓

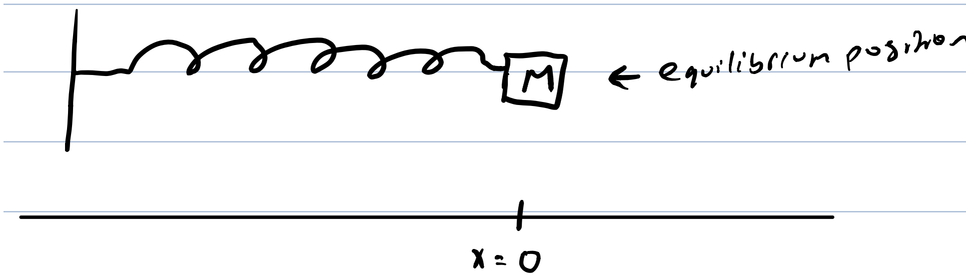
N.B. The DE above could also be written as:

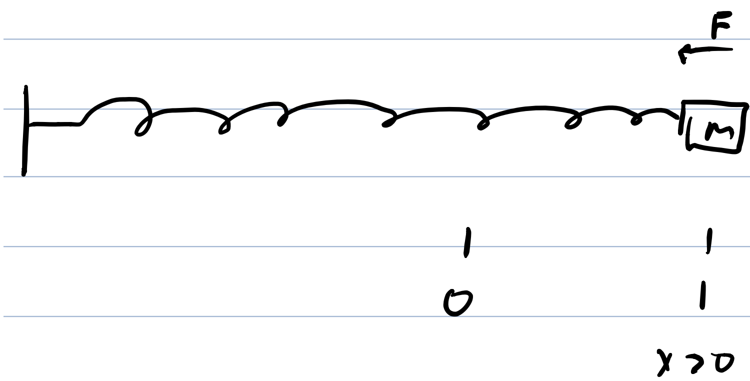
x''

$\sim \frac{d^2x}{dt^2} + w^2 x = 0$

also $x''(t) + w^2 x(t) = 0$

SIMPLE HARMONIC MOTION





Hooke's law: $F = -kx \quad (k > 0)$ <u>N.B.</u> x is a f^n of time
<u>i.e.</u> $x = x(t)$

Newton's 2^{nd} Law: $F = ma$

So $ma = -kx$

$$m x'' = -kx$$

$$x'' = \frac{-k}{m} x$$

$$x'' + \frac{k}{m} x = 0$$

Let $\omega = \sqrt{\frac{k}{m}}$

$x'' + \omega^2 x = 0$

we know the gen'l sol'n is

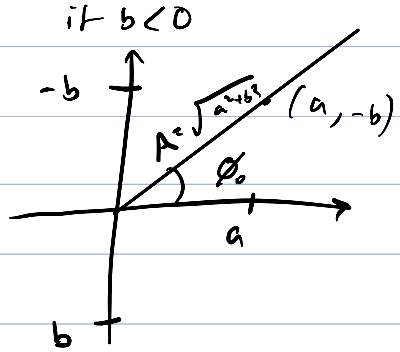
$$x(t) = a\cos(\omega t) + b\sin(\omega t)$$

Can be rewritten:

$$x(t) = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos(\omega t) - \frac{-b}{\sqrt{a^2+b^2}}\sin(\omega t)\right)$$

N.B. 'a' & 'b' determine an angle ϕ_0
where ϕ is $(-\pi, \pi]$

if $b<0$



$$\cos\phi_0 = \frac{a}{\sqrt{a^2+b^2}}$$

$$\sin\phi_0 = \frac{-b}{\sqrt{a^2+b^2}}$$

$$x(t) = A\left(\cos\phi_0\cos(\omega t) - \sin\phi_0\sin(\omega t)\right)$$
$$= A\cos(\omega t + \phi_0)$$

Recall

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

N.B.

replace

$$x(0) = A\cos(\theta_0)$$
$$x\left(\frac{2\pi}{w}\right) = A\cos(2\pi + \theta_0)$$ coterminal

Indicates that it will take

$t = \frac{2\pi}{w}$ to go through one cycle

$T = \frac{2\pi}{w}$ sec ← The period

Frequency (cycles/sec)

$f = \frac{1}{T}$ Hz (cycles/sec)

N.B. in the context of mass at end of spring,

remember $w = \sqrt{\frac{k}{m}}$

November 4, 2016

N.B. Motion of an object can be said to be simple harmonic if its acceleration $(a(t))$ is proportional to its displacement $(x(t))$

i.e. $a(t) = (\text{constant})\ x(t)$

EXAMPLES

(1) Rewrite $x(t) = \sqrt{3}\cos(3t) - \sin(3t)$ in the form $x(t) = A\cos(\omega t + \phi_0)$ and find the amplitude, period and frequency.

(2) A particle is moving along a line according to the equation of motion $s(t) = 2 - 4\cos^2(2t)$ where, at t seconds, s meters is the directed distance of the particle from the origin.
 (a) Find the velocity and acceleration at t seconds.
 (b) Show that the motion is simple harmonic.

(3) A spring with a mass of 2kg has a natural length of 0.5m. A force of 25.6N is required to maintain it stretched to a length of 0.7m. It is then released with initial velocity $v(0) = 0$ m/s. Find the position of the mass at any time t.

(4) An object passes through its equilibrium position at $t = 0, 1, 2, \ldots$ seconds. Find a position function of the form $x(t) = A\cos(\omega t + \phi_0)$ if $v(0) = -3$ m/s. What is the amplitude? What is the period?

1) $x(t) = \sqrt{3}\cos(3t) - \sin(3t)$

$x(t) = A\cos(\omega t + \theta_0)$

$w = 3$

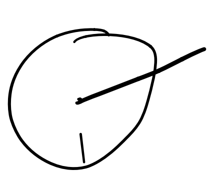
$x(t) = a\cos(\omega t) + b\sin(\omega t)$

$\left.\begin{matrix} a = \sqrt{3} \\ b = -1 \end{matrix}\right\}$ $A = \sqrt{a^2+b^2} = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$

$\cos(\theta_0) = \frac{a}{A} = \frac{\sqrt{3}}{2}$

$\sin(\theta_0) = \frac{-b}{A} = \frac{-(-1)}{2} = \frac{1}{2}$

So which θ_0 in $(-\pi, \pi]$ satisfies this?



$\frac{\pi}{6}\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

So $\phi_0 = \frac{\pi}{6}$

So $x(t) = 2\cos\left(3t + \frac{\pi}{6}\right)$

Period : $T = \frac{2\pi}{w} = \frac{2\pi}{3}$

Frequency: $\frac{1}{T} = \frac{3}{2\pi}$

2) $s(t) = 2 - 4\cos^2(2t)$

a) $v(t) = s'(t) = -4\left(2\cos 2t\right)\left(-2\sin 2t\right)$

$= 16\cos 2t \sin 2t$

$a(t) = v'(t) = 16\left[-\sin(2t)\cdot\sin(2t) + 2\cos(2t)\cdot\cos(2t)\right]$

$= -32\left[\sin^2(2t) - \cos^2(2t)\right]$

b) $a(t) = -32\left[1 - \cos^2(2t) - \cos^2(2t)\right]$

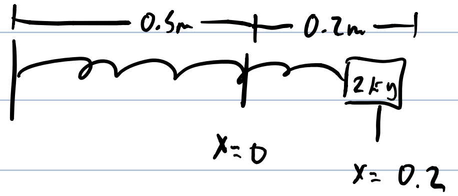
$= -32\left[1 - 2\cos^2(2t)\right]$

$= -16\underbrace{\left[2 - 4\cos^2(2t)\right]}_{\text{displacement } f^n}$

$= -16\left[s(t)\right]$

So this motion is simple harmonic.

3) $x(t) = A\cos(\omega t + \theta_0)$



So $x(0) = 0.2 \leftarrow$ 1 $\qquad v(0) = 0 \leftarrow$ 2

f^{n} of time

Restoring force: $F = -25.6\,N \leftarrow$ 3

$$x(t) = A\cos(\underbrace{\omega}_{\sqrt{\frac{k}{m}}} t + \theta_0)$$

Hooke's law: $F = -kx$

$$-23.6 = -k(0.2)$$

$$k = 128$$

So $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{128}{2}} = 8$

So $x(t) = A\cos(8t + \theta_0) \leftarrow$ 4

$v(t) = v'(t) = -A\sin(8t + \theta_0)\cdot 8$

$\qquad = -8A\sin(8t + \theta_0) \leftarrow$ 5

From 2, 5

$v(0) = 0 = -8A\sin(\theta_0)$

$\qquad \Rightarrow \sin(\theta_0) = 0 \quad (\theta_0 \text{ in } (-\pi, \pi])$

$\Rightarrow$ two possibilities $\phi_0 = 0$, or $\phi_0 = \pi$ from unit circle

$\cos \pi = \underline{-1} < 0$

from ①, ④

$0.2 = x(0) = A\cos(\phi_0)$

$\Rightarrow \cos\phi_0 = \dfrac{0.2}{A}$ $\left(A > 0, \text{ therefore } \dfrac{0.2}{A} > 0\right)$

So $\phi_0 \neq \pi$

$\phi_0 = 0$

So $x(t) = A\cos(8t)$

Since $A\cos(\phi_0) = 0.2$

$A\cos(0) = 0.2$

$A = 0.2$

So $x(t) = 0.2\cos(8t)$

⋮

$t=2$

$t=1$

$t=0$

4)

$x(0) = 0 \leftarrow$ ①

$v(0) = -3 \text{ m/s} \leftarrow$ ②

$T = 2$

$T = \dfrac{2\pi}{\omega}$

$$2 = \frac{2\pi}{\omega}$$

$$\omega = \pi$$

So

$$x(t) = A\cos(\pi t + \theta_0) \leftarrow ③$$

$$v(t) = x'(t) = -A\sin(\pi t + \theta_0)\cdot \pi$$

$$= -A\pi\sin(\pi t + \theta_0) \leftarrow ④$$

from ①, ③

$$x(0) = 0 = A\cos(\theta_0)$$

$$\cos\theta_0 = 0$$

$$\theta_0 = \frac{\pi}{2} \text{ or } -\frac{\pi}{2}$$

from ②, ④

$$v(0) = \cancel{-}3 = \cancel{-}A\pi\sin(\theta_0)$$

$$0 < \frac{3}{A\pi} = \sin(\theta_0)$$

$$\text{so } \theta_0 = \frac{\pi}{2}$$

$$\text{So } x(t) = A\cos\left(\pi t + \frac{\pi}{2}\right)$$

$$v(t) = -A\pi\sin\left(\pi t + \frac{\pi}{2}\right)$$

$$-3 = -A\pi\sin\left(\frac{\pi}{2}\right)$$

$$A = \frac{3}{\pi}$$

$$\boxed{\text{So } x(t) = \frac{3}{\pi}\cos\left(\pi t + \frac{\pi}{2}\right)}$$