Section 3 - Integration Techniques Feb. 15, 2017

INTEGRATION BY PARTS

When should I use IBP?

→ A u-sub fails

→ Product of
- trig × polynomial
- exp × polynomial
- ln, arctan, arcsin × polynomial
- trig × exp / trig

eg. $\int x e^{x^2+1} dx$ → by subs.

eg. $\int (x^2+1) e^x dx$ → IBP

Recall that (Product Rule)

$$(fg)' = f'g + fg'$$

$$\int (fg)' dx = \int f'g \, dx + \int f g' \, dx$$

$$fg = \int f'g \, dx + \underline{\int fg' dx}$$

$$\boxed{\int fg' dx = fg - \int f'g \, dx}$$

hope this integral is easier than $\int fg' dx$

INTEGRATION BY PARTS FORMULA

ex. $\int \underbrace{(3x+1)}_{f} \underbrace{\cos x \, dx}_{g'}$

$f = 3x+1$ $\quad$ $g' = \cos x$

$f' = 3$ $\quad$ $g = \sin x$

$= \int (3x+1) \cos x \, dx$

$= (3x+1)(\sin x) - \int 3 \sin x \, dx$

$= (3x+1) \sin x - 3 \int \sin x \, dx$

$= (3x+1) \sin x - 3(-\cos x) + C$

$= (3x+1) \sin x + 3 \cos x + C$

ex. $\int (3x+1) \ln x \, dx$

$\cancel{f = 3x+1}$ $\quad$ $\cancel{g' = \ln x}$

$\int \ln x$??

$f = \ln x$ $\quad$ $g' = 3x+1$

$f' = \frac{1}{x}$ $\quad$ $g = \frac{3}{2}x^2 + x$

$I = (\ln x)(\frac{3}{2}x^2 + x) - \int \frac{1}{x}(\frac{3}{2}x^2 + x) \, dx$

$I = (\frac{3}{2}x^2 + x) \ln x - \int (\frac{3}{2}x + 1) \, dx$

$I = (\frac{3}{2}x^2 + x) \ln x - (\frac{3}{4}x^2 + x) + K$

À VOUS

$$I = \int (x^2 + 2x + 4) e^x \, dx$$

$f = x^2 + 2x + 4 \qquad g' = e^x$

$f' = 2x + 2 \qquad g = e^x$

$$I = e^x(x^2 + 2x + 4) - \underline{\int (2x+2) e^x}$$

$f = 2x + 2 \qquad g' = e^x$

$f' = 2 \qquad g = e^x$

$$I = e^x(x^2 + 2x + 4) - \left((2x+2)e^x - \int 2 e^x \, dx\right)$$

$$= e^x(x^2 + 2x + 4) - (2x+2)e^x + 2e^x + C$$

PRIORITY LIST for "f"

- → $\ln x$, $\arcsin x$, $\arctan x$
- → polynomials
- → exp, trigs

"OUR SHORTCUT METHOD" when $f =$ polynomial

ex. $I = \int (x^2 + 2x + 4)\, e^x \, dx$

	f	g'	
$+$	$x^2 + 2x + 4$	e^x	
$-$	$2x + 2$	e^x	$\to (x^2 + 2x + 4)\, e^x$
	2	e^x	$\to -(2x+2)\, e^x$
	0	e^x	$\to +2\, e^x$

stop at 0

$$I = (x^2 + 2x + 4)e^x - (2x+2)e^x + 2e^x + C$$

ex. $I = \int (x^3 + 2x + 4) \sin x \, dx$

antiderivatives

	f	g'
$+$	$x^3 + 2x + 4$	$\sin x$
$-$	$3x^2 + 2$	$-\cos x$
$+$	$6x + 2$	$-\sin x$
$-$	6	$\cos x$
$+$	0	$\sin x$

$$I = (x^3 + 2x + 4)(-\cos x) - (3x+2)(-\sin x) + (6x+2)(\cos x) - 6 \sin x$$

ex. $I = \int \arctan x \, dx$

$f = \arctan x \qquad g' = 1$

$f' = \dfrac{1}{1+x^2} \qquad g = x$

$I = (\arctan x)\, x - \int \dfrac{1}{1+x^2}\, x \, dx$

let $u = 1 + x^2$

$du = 2x \, dx$

$= x \arctan x - \dfrac{1}{2} \int \dfrac{1}{u} \, du$

$= x \arctan x - \dfrac{1}{2} \ln(1 + x^2)$

ex. $I = \int x \sec^2 x \, dx$

$f = x \qquad g' = \sec^2 x$

$x \sec^2 x$

$f' = 1 \qquad g = \tan x$

	f	g'
$\oplus$	x	$\sec^2 x$
$\ominus$	1	$\tan x$
	0	$\ln\lvert\sec x\rvert$

antideriv

IBP

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ex. Warm up

$$\int 2x^3 \cos(x^2)\,dx$$

Let $u = x^2$

$du = 2x\,dx$

$$= \int x^2 \cos(x^2)\, 2x\,dx$$

$$= \int u \cos u \, du$$

$f = u \qquad g' = \cos u$

$f' = 0 \qquad g = \sin u$

$$= u\sin u - \int 1 \cdot \sin u$$

$$= u\sin u + \cos u + C$$

$$= x^2 \sin x^2 + \cos x^2 + C$$

ex. $\int (\ln x)^m \, dx$

$$= x(\ln x)^m - \int m(\ln x)^{m-1} \cdot \frac{1}{\cancel{x}} \cancel{x}\, dx$$

$f = (\ln x)^m \qquad g' = 1$

$f' = m(\ln x)^{m-1} \cdot \frac{1}{x} \qquad g = x$

chain rule

$$\boxed{\int (\ln x)^m \, dx = x(\ln x)^m - m\int (\ln x)^{m-1}\, dx}$$

This pattern is called a REDUCTION FORMULA

If $m = 1$ $\quad \int \ln x \, dx = x\ln x - \int 1 \, dx$

$$= x\ln x - x + C$$

If $m=2$ $\int (\ln x)^2 dx = x(\ln x)^2 - 2\int \ln x \, dx$

$= x(\ln x)^2 - 2(x\ln x - x) + C$

ex. $\int_1^e (\ln x)^3 dx$

$= x(\ln x)^3\Big|_1^e - 3\int_1^e (\ln x)^2 dx$

$= e - 3\left[x(\ln x)^2\Big|_1^e - 2\int_1^e \ln x \, dx \right]$

$= e - 3\left[e - 2\left(x\ln x\Big|_1^e - \int_1^e (\ln x)^0 dx \right) \right]$

$= e - 3e + 6\left(e - x\Big|_1^e\right)$

$= -2e + 6(\cancel{e} - (\cancel{e} - 1))$

$= -2e + 6$

À vous[2]

Find a reduction formula for

$\int x^m \cos x \, dx$ $\qquad f = x^m \qquad g' = \cos x$

$\qquad f' = mx^{m-1} \qquad g = \sin x$

$= x^m \sin x - \int mx^{m-1} \sin x \, dx$

$= x^m \sin x - \left[-mx^{m-1}\cos x + \int m(m-1)x^{m-2}\cos x \, dx \right]$ $\qquad f = x^{m-1} \qquad g' = \sin x$

$\qquad f' = (m-1)x^{m-2} \qquad g = -\cos x$

$= x^m \sin x + mx^{m-1} \cos x - m(m-1)\int x^{m-2} \cos x \, dx$ Need form to stay

Bonne chance...

$$\int_0^\pi x^4 \cos x \, dx$$

$$= \left[x^4 \sin x + 4x^3 \cos x\right]_0^\pi - (4)(3)\int x^2 \cos x \, dx$$

$$= -4\pi^3 - 12\left[\left[x^2 \sin x + 2x^1 \cos x\right]_0^\pi - (2)(1)\int_0^\pi x^0 \cos x \, dx\right]$$

$$= -4\pi^3 - 12\left(-2\pi - 2\sin x \Big|_0^\pi\right)$$

$$= -4\pi^3 + 24\pi$$

Your favourite nightmare...

Solve $I = \int e^x \sin x \, dx$

$f = \sin x \qquad g' = e^x$

$f' = \cos x \qquad g = e^x$

$$I = e^x \sin x - \int e^x \cos x \, dx$$

Need to be consistent here! otherwise you will just undo what you

$$= e^x \sin x - \left[e^x \cos x + \int e^x \sin x \, dx\right]$$

$f = \cos x \qquad g' = e^x$

$f' = -\sin x \qquad g = e^x$

$I = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$

But... $I = \int e^x \sin x \, dx$

So

$I = e^x \sin x - e^x \cos x - I$

$2I = e^x \sin x - e^x \cos x$

$I = \frac{1}{2}\left(e^x \sin x - e^x \cos x\right)$

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ex, Solve $I = \int e^{3x} \cos 2x \, dx$

$f = e^{3x}$ $\quad g' = \cos 2x$

$f' = 3e^{3x}$ $\quad g = \int \cos 2x \, dx$ $\quad$ Let $u = 2x$

$= \frac{1}{2} \sin(2x)$

$I = \frac{1}{2} e^{3x} \sin(2x) - \frac{3}{2} \int e^{3x} \sin(2x) \, dx$

$f = e^{3x}$ $\quad g' = \sin(2x)$

$f' = 3e^x$ $\quad g = \int \sin(2x) \, dx$

$= -\frac{1}{2} \cos 2x$

$I = \frac{1}{2} e^{3x} \sin(2x) - \frac{3}{2}\left[-\frac{1}{2} e^{3x} \cos 2x + \frac{3}{2}\int e^{3x} \cos 2x \, dx\right]$

$I = \frac{1}{2} e^{3x} \sin 2x + \frac{3}{4} e^{3x} \cos 2x - \frac{9}{4} I$

$I = \frac{4}{13}\left[\frac{1}{2} e^{3x} \sin 2x + \frac{3}{4} e^{3x} \cos 2x\right]$

$I = \frac{2}{13} e^{3x} \sin 2x + \frac{3}{13} e^{3x} \cos 2x + C$

ex. $\int 2^{2x} e^{2x}\, dx$

$= \frac{1}{2}\int 2^{u} e^{u}\, du$

$= \frac{1}{2}\int (2e)^{u}\, du$

$\vdots$

ex. $I = \int \sin^{n} x\, dx$

$I = \int \underbrace{\sin^{n-1} x}_{f}\ \underbrace{\sin x}_{g'}\, dx$

$f = \sin^{n-1} x \qquad g' = \sin x$

$f' = (n-1)(\sin x)^{n-2} \cdot \cos x$ chain rule $\qquad g = -\cos x$

$I = -(\sin x)^{n-1} \cos x + \int (n-1) \sin^{n-2} x \cos^{2} x\, dx$

$I = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, (1-\sin^{2} x)\, dx$

$I = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx - (n-1) \int \sin^{n} x\, dx$

$I = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx - (n-1)\, I$

$n I = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx$

$I = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x\, dx$

ex. solve $I = \int \sin^3 x \, dx$

$$I = -\frac{1}{3} \sin^2 x \cos x - \frac{2}{3} \cos x + C$$

ex. $I = \int \sec^n x \, dx$ Hint: $\tan^2 x = \sec^2 x - 1$

$$I = \int \sec^{n-2} x \sec^2 x \, dx$$

$f = (\sec x)^{n-2}$ $g' = \sec^2 x \, dx$

$f' = (n-2) \sec^{n-3} x \cdot \sec x \tan x$ $g = \int \sec^2 x \, dx$

$g = \tan x$

$$I = \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \left[\int \sec^n dx - \int \sec^{n-2} x \, dx \right]$$

$$I = \sec^{n-2} x \tan x - (n-2) I + (n-2) \int \sec^{n-2} x \, dx$$

$$(n-1) I = \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x \, dx$$

$$I = \frac{1}{n-1} \sec^{n-2} \tan x + \frac{n-2}{n-1} \int \sec^{n-2} dx$$

ex. Solve $I = \int \sec^4 x \, dx$

$$I = \frac{1}{3} \sec^2 \tan x + \frac{2}{3} \int \sec^2 dx$$

$$= \frac{1}{3} \sec^2 \tan x + \frac{2}{3} \tan x + C$$

--------------- END OF MATERIAL FOR TEST 1 ------- Feb. 22

IBP ✓

Trigonometric Powers

} when to use these?

→ when subst fails

→ sin/cos combos

→ sec/tan combos

Case 1 odd Powers of "cos"

ex. $\int \sin^2 x \cos^5 x \, dx$

Step 1: Take one $\cos x$ aside with dx

$$= \int \sin^2 x \cos^4 x \underbrace{\cos x \, dx}_{\text{"I want to be a } du \text{ one day..."}}$$

Step 2: Replace all remaining $\cos^2 x$ by $(1 - \sin^2 x)$

since we start with odd powers

$$= \int \sin^2 x (1 - \sin^2 x)(1 - \sin^2 x) \cos x \, dx$$

Step 3: Let $u = \sin x \rightarrow du = \cos x \, dx$ yay!

$$= \int u^2 (1 - u^2)(1 - u^2) \, du$$

Step 4: expand / distribute and antidifferentiate

$$= \int (u^2 - 2u^4 + u^6)\, du$$

$$= \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + ♡$$

$$= \frac{1}{3}\sin^2 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + ♡$$

À vous $\displaystyle \int \frac{\sin^4(\ln x)\cos^3(\ln x)\, dx}{x}$

Let $u = \ln x$

$du = \frac{1}{x}\, dx$

$$= \int \sin^4 u \cos^3 du$$

$$= \int \sin^4 u \cos^2 \cos du$$

$$= \int \sin^4 u (1 - \sin^2 u) \cos du$$

Let $w = \sin u \rightarrow dw = \cos u\, du$

$$= \int w^4 (1 - w^2)\, dw$$

$$= \int (w^4 - w^6)\, dw$$

$$= \frac{1}{5}w^5 - \frac{1}{7}w^7 +$$

Case 2 Odd powers of 'sin'

ex: $\int \sin^5 x \, dx$

Step 1: Place a $\sin x$ aside with dx

Step 2: Replace all remaining $\sin^2 x$ by $(1 - \cos^2 x)$

Step 3: Let $u = \cos x \rightarrow du = -\sin x \, dx$

Step 4: Expand and integrate

$$= \int \sin^4 x \sin x \, dx$$

$$= \int (1 - \cos^2 x)^2 \sin x \, dx$$

Let $u = \cos x$

$du = -\sin x \, dx$

$$= -\int (1 - u^2)^2 \, du$$

$$= -\int (1 - 2u^2 + u^4) \, du$$

$$= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 +$$

À vous

$$\int_4^9 \frac{\sin^3(\sqrt{x})}{\sqrt{x} \, \cos^4(\sqrt{x})} \, dx$$

$$= \int_4^9 \frac{\sin^2(\sqrt{x})}{\cos^4(\sqrt{x})} \cdot \sin(\sqrt{x}) \frac{1}{\sqrt{x}} \, dx$$

$$= \int_4^9 \frac{1 - \cos^2(\sqrt{x})}{\cos^4(\sqrt{x})} \cdot \sin(\sqrt{x}) \frac{1}{\sqrt{x}} \, dx$$

Let $u = \cos(\sqrt{x})$

$du = -\sin(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \, dx$

Bounds

$x = 9 \rightarrow u = \cos(3)$

$x = 4 \rightarrow u = \cos(2)$

$$= -2 \int_{\cos(2)}^{\cos(3)} \frac{1 - u^2}{u^4} \, du$$

$$= -2 \int_{\cos(2)}^{\cos(3)} (u^{-4} - u^{-2}) \, du$$

$$= -2 \left[\left(-\frac{1}{3u^3} + \frac{1}{u} \right) \right]_{\cos(2)}^{\cos(3)}$$

$$= -2 \left[\left(-\frac{1}{3\cos^3 3} + \frac{1}{\cos 3} \right) - \left(-\frac{1}{3\cos 2} + \frac{1}{\cos 2} \right) \right]$$

Case 3 Even cos and even sin

ex. $\int \sin^2 x \, dx$ or $\int \cos^2 x \, dx$

Replace all $\sin^2 x$ by $\frac{1}{2}(1 - \cos 2x)$

and all $\cos^2 x$ by $\frac{1}{2}(1 + \cos 2x)$

ex. $\int \sin^2 x \, dx$

$= \frac{1}{2} \int (1 - \cos 2x) \, dx$ Let $u = 2x$

$du = 2dx$

$= \frac{1}{4} \int (1 - \cos u) \, dx$

$= \frac{1}{4} u - \frac{1}{4} \sin u + C$

$= \frac{1}{2} x - \frac{1}{4} \sin 2x + C$

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ex. $\int \sin^2\left(\frac{1}{4} x\right) \cos^2\left(\frac{1}{4} x\right) dx$

$= \frac{1}{4} \int \left(1 - \cos \frac{1}{2}x\right)\left(1 + \cos \frac{1}{2}x\right) dx$

$= \frac{1}{4} \int \left(1 - \cos^2 \frac{1}{2}x\right) dx$

$= \frac{1}{4} \int \left(1 - \left(\frac{1}{2}(1 + \cos x)\right)\right) dx$ OR $= \frac{1}{4} \int \sin^2\left(\frac{1}{2}x\right) dx$

$= \frac{1}{8} \int (1 - \cos x) \, dx$ $= \frac{1}{8} \int (1 - \cos x) \, dx$

$= \frac{1}{8} x - \frac{1}{8} \sin x + C$ $= \frac{1}{8} x - \frac{1}{8} \sin x + C$

Application

N.B. period $T = \frac{2\pi}{b} = \frac{2\pi}{\pi} = 2$

Let $v(t) = 8\cos(\pi t)$ represent the velocity of a mass of 1kg oscilating at the end of a spring.

a) Find $\bar{v}$ over $[0, 2]$

$$\bar{v}_{[0,2]} = \frac{1}{2}\int_0^2 8\cos \pi t \; dt$$

$$= 4\int_0^2 \cos \pi t \; dt$$

Let $u = \pi t$

$du = \pi \, dt$

Bounds

$t = 2 \Rightarrow u = 2\pi$

$t = 0 \Rightarrow u = 0$

$$= \frac{4}{\pi}\int_0^{2\pi} \cos u \; du$$

$$= \frac{4}{\pi} \sin u \Big|_0^{2\pi}$$

$$= 0$$

b) What is the average kinetic energy of this mass?

$$K(t) = \frac{1}{2} m\left[v(t)\right]^2$$

$$= \frac{1}{2} \cdot 1 \left(8 \cos(\pi t)\right)^2$$

$$= 32 \cos^2 \pi t$$

$$\bar{K}_{[0,2]} = \frac{1}{2}\int_0^2 32\cos^2 \pi t$$

$$= 16\int_0^2 \cos^2 \pi t \, dt$$

$$= 16 \cdot \frac{1}{2}\int_0^2 (1 + \cos 2\pi t)\, dt$$

$$= \frac{8}{2\pi}\int_0^2 (1 + \cos 2\pi t)\, 2\pi \, dt$$

Let $u = 2\pi t$

$du = 2\pi\, dt$

Bounds

$t = 2 \rightarrow u = 4\pi$

$t = 0 \rightarrow u = 0$

$$= \frac{8}{2\pi}\int_0^{4\pi} (1 + \cos u)\, du$$

$$= \frac{8}{2\pi}\Big[u + \sin u\Big]_0^{4\pi}$$

$$= \frac{8}{2\pi}\Big[(4\pi + 0) - (0 + 0)\Big]$$

$$= \frac{8}{2\cancel{\pi}} \cdot 4\cancel{\pi}$$

$$= 16$$

Case 4 Odd powers of 'tan'

ex. $\int \sec^3 x \tan^3 x \, dx$

Step 1: Move 1 $\sec x \tan x$ aside with dx

$$= \int \sec^2 x \tan^2 x \sec x \tan x \, dx$$

Step 2: Change all remaining $\tan^2 x$ into $\sec^2 x - 1$

$$= \int \sec^2 x \, (\sec^2 - 1) \sec x \tan x \, dx$$

Step 3: Let $u = \sec x$, $du = \sec x \tan x \, dx$

$$= \int u^2 (u^2 - 1) \, du$$

Step 4: Expand and integrate

$$= \int (u^4 - u^2) \, du$$

$$= \frac{1}{5} u^5 - \frac{1}{3} u^3 + C$$

$$= \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C$$

<u>Case 5</u> Even powers of 'sec'

<u>ex.</u> $\int \sec^6 x \, dx$

Step 1: Move 1 $\sec^2 x$ aside with dx

$$= \int \sec^4 x \sec^2 x \, dx$$

Step 2: Transform all remaining $\sec^2 x$ into $\tan^2 x + 1$

$$= \int (\tan^2 x + 1)^2 \sec^2 x \, dx$$

Step 3: Let $u = \tan x \rightarrow du = \sec^2 x$

Step 4: Expand, distribute and integrate.

$$= \int (u^2 + 1)^2 \, du$$

$$= \int (u^4 + 2u^2 + 1) \, du$$

$$= \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + \text{☺}$$

À vous

$$\int \sec^4 x \tan^3 x \, dx$$

As an even secant

$$= \int \sec^2 x \tan^3 x \sec^2 x \, dx$$

$= \int (1+\tan^2 x)\tan^3 x \sec^2 x \, dx$

Let $u = \tan x$

$du = \sec^2 x \, dx$

$= \int (1+u^2) u^3 \, du$

$= \int (u^3 + u^5) \, du$

$= \frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + ☺$

As an odd tangent

$= \frac{1}{6}\sec^6 x - \frac{1}{4}\sec^4 x + K$

If you use trig identities, find that they will only differ by a constant

<u>Case 6</u> OMGIDKWTS

<u>ex.</u> $\int \sec x \, dx$

$= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$

let $u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x)\, dx$

$= \int \frac{1}{u}\, du$

$= \ln|u| + C$

$= \ln|\sec x + \tan x| + C$

ex. $I = \int \sec^3 x\, dx$

$f = \sec x \qquad g' = \sec^2 x$

$f' = \sec x \tan x \qquad g = \tan x$

$I = \sec x \tan x - \int \sec x \tan^2 x\, dx$

$= \sec x \tan x - \int \sec x (\sec^2 x - 1)\, dx$

$I = \sec x \tan x - \underbrace{\int \sec^3 x\, dx}_{I} + \int \sec x\, dx$

$I = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$

Trig Substitutions

March 1

When should I use trig subs?

- → when a u-sub fails
- → ① $x^2 - a^2$ form
  - ② $a^2 - x^2$ form
  - ③ $a^2 + x^2$ form

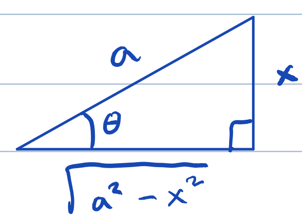
Case 1: $a^2 - x^2$ form

ex. $\displaystyle\int \frac{1}{(9-x^2)^{3/2}}\,dx$

Survival Guide:

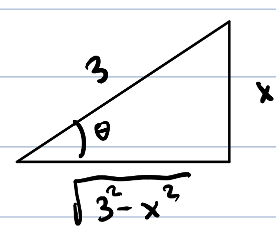
Step 1: Let $x = a\sin\theta \longrightarrow \sin\theta = \frac{x}{a} = \frac{\text{opp}}{\text{hyp}}$

$dx = a\cos\theta\,d\theta$



Let $x = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$



$= \sqrt{9-x^2}$

$$\int \frac{1}{(9-x^2)^{3/2}}\,dx$$

$$= \int \frac{1}{(9-9\sin^2\theta)^{3/2}} \cdot 3\cos\theta\, d\theta$$

Step 2: Replace all $1-\sin^2\theta$ by $\cos^2\theta$ and simplify

$$= \int \frac{3\cos\theta}{(9\cos^2\theta)^{3/2}}\,d\theta$$

$$= \int \frac{3\cos\theta}{27\cos^3\theta}\,d\theta$$

$$= \int \frac{1}{9\cos^2\theta}\,d\theta$$

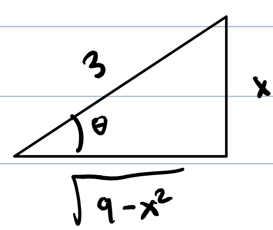
Step 3: Good luck! Trig powers ...

$$= \int \frac{1}{9}\sec^2\theta\, d\theta$$

$$= \frac{1}{9}\tan\theta + K$$

Step 4: Return to x's (using your)

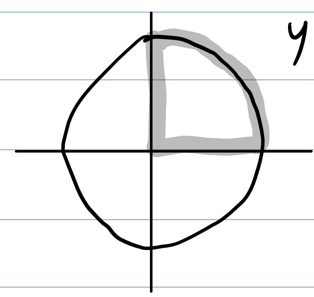
$$= \frac{1}{9}\,\frac{x}{\sqrt{9-x^2}} + K$$



$$= \frac{x}{9\sqrt{9-x^2}} + k$$

ex. Show that the area of a circle is πr^2.

$x^2 + y^2 = r^2$



Area of a circle = 4 x area of a quarter circle

$$\text{Area} = 4\int_0^r \sqrt{r^2 - x^2}\, dx$$

Let $x = r\sin\theta$

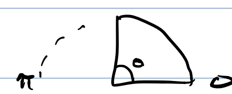
$dx = r\cos\theta\, d\theta$

Bounds

$x = 0 \rightarrow 0 = \sin\theta$

$\theta = 0$ and $\cancel{\pi}$

not in domain



$x = r \rightarrow 1 = \sin\theta$

$\theta = \frac{\pi}{2}$

$$A = 4\int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2\theta}\; r\cos\theta\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{r^2(1-\sin^2\theta)}\ \ r\cos\theta\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} r\cos\theta\ \ r\cos\theta\, d\theta$$

Doesn't have to be $|r|$ since $r \geq 0$ in domain we're integrating

$$= 4r^2\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

(even $\cos\theta$)

$$= 4r^2\int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \frac{2r^2}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\, 2d\theta$$

$u = 2\theta$ $\quad\theta = 0 \rightarrow u = 0$

$du = 2d\theta$ $\quad\theta = \frac{\pi}{2} \rightarrow u = \pi$

$$= r^2\int_0^{\pi} (1+\cos u)\, du$$

$$= r^2\left[u + \sin u\right]_0^{\pi}$$

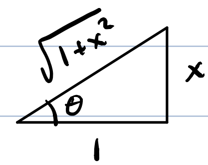
$$= r^2(\pi)$$

$$= \pi r^2$$

À vous

$$\int \frac{1}{\sqrt{x^2+1}}\,dx \qquad \text{Let } x = \tan\theta, \quad dx = \sec^2\theta\, d\theta$$

$$= \int \frac{1}{\sqrt{\tan^2\theta+1}} \sec^2\theta\, d\theta$$



$$= \int \frac{1}{\cancel{\sec\theta}} \sec^{\cancel{2}}\theta\, d\theta \qquad \text{assume } \sec\theta \geq 0$$

$$= \int \sec\theta\, d\theta$$

New formula (case 6)

$$= \ln|\sec\theta + \tan\theta| + K$$

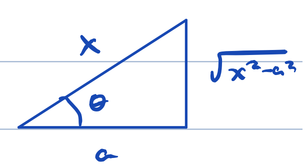
$$= \ln\left|\frac{\sqrt{1+x^2}}{1} + \frac{x}{1}\right| + K \qquad \text{Subs from triangle}$$

$$= \ln\left|\sqrt{1+x^2} + x\right| + K$$

Recall $\int \frac{1}{\sqrt{x^2-1}}\,dx = \operatorname{arcsec} x$

Case 3 $\quad x^2 - a^2$ forms

Steps: ① Let $x = a\sec\theta \rightarrow \sec\theta = \frac{x}{a} \rightarrow$



$dx = a\sec\theta\tan\theta\, d\theta$

② Replace all $\sec^2\theta - 1$ by $\tan^2\theta$ and simplify

③ Solve the trig power that results from ②

④ Go back to x's

ex. $\int \frac{2e^x}{\sqrt{4e^{2x}-25}}\,dx$

$= \int \frac{2e^x}{\sqrt{(2e^x)^2-25}}\,dx$

Let $u = 2e^x$

$du = 2e^x\,dx$

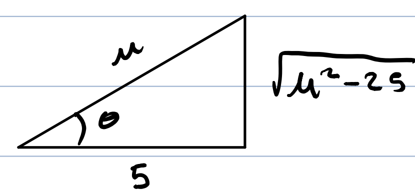
$= \int \frac{1}{\sqrt{u^2-25}}\,du$

$= \int \frac{1}{\sqrt{25\sec^2\theta-25}}\,5$

Let $u = 5\sec\theta$

$du = 5\sec\theta\tan\theta\,d\theta$

$= \frac{1}{\cancel{5}}\int \frac{1}{\sqrt{\sec^2\theta+1}}\,\cancel{5}\sec\theta\tan\theta\,d\theta$



$= \int \frac{1}{\cancel{\tan\theta}}\sec\theta\,\cancel{\tan\theta}\,d\theta$

$= \int \sec\theta\,d\theta$

$= \ln|\sec\theta+\tan\theta| + K$

$= \ln\left|\frac{u}{5} + \frac{\sqrt{u^2-25}}{5}\right| + K$

$$= \ln \left| \frac{2e^x + \sqrt{4e^{2x}-25}}{5} \right| + K$$

N.B. also $= \ln \left| 2e^x + \sqrt{4e^{2x}-25}$

March 13, 2017

Partial Fraction Decomposition

When to use?

→ not basic integral

→ u-sub fails

→ proper rational fcn (power on top is less), where the denominator has more than one factor

ex. $\displaystyle \int \frac{2x^2+x-1}{\underbrace{x^3-3x^2+3x-9}} \, dx$

$x^2(x-3)+3(x-3)$

$(x^2+3)(x-3)$ ← denominator has 2 factors (linear + quadratic)

The decomposition of a proper fraction depends on the nature of its denominator's factors

→ linear factor $(\ell(x)) \longrightarrow \dfrac{A}{\ell(x)}$

→ repeated linear factor $((\ell(x))^k) \longrightarrow \dfrac{A_1}{\ell(x)} + \dfrac{A_2}{(\ell(x))^2}$

$+ \dots + \dfrac{A_k}{(\ell(x))^k}$

→ quadratic factors $(q(x))$ → $\dfrac{Ax+B}{q(x)}$

→ repeated quadratic factors $((q(x))^n)$

$$\longrightarrow \frac{A_1x+B}{q(x)} + \frac{A_2x+B_2}{(q(x))^2} + \dots + \frac{A_nx+B_n}{(q(x))^n}$$

ex. What form would the decomposed version of

$$\frac{3x^2+2x-1}{x^3(x^2+1)} \quad \text{have?}$$

repeated linear

quadratic

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+1}$$

À vous

Decompose

$$\frac{4x^5+3x^2-7x+1}{x^2(x-1)(x-2)^2(x^2+4)^2}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x-2} + \frac{E}{(x-2)^2} + \frac{Fx+G}{x^2+4} + \frac{Hx+I}{(x^2+4)^2}$$

Now the mechanics

ex, Find the partial fraction decomposition for

$$\frac{2x+3}{x^2-9x-10} = \boxed{\frac{2x+3}{(x-10)(x+1)} = \frac{A}{x-10} + \frac{B}{x+1}}$$

What A and B satisfy this identity?

→ Multiply both sides by $(x-10)(x+1)$

$$2x+3 = A(x+1) + B(x-10)$$

if $x = 10$:

$$23 = 11A$$

$$A = \frac{23}{11}$$

if $x = -1$:

$$1 = -11B$$

$$B = -\frac{1}{11}$$

Therefore

$$\frac{2x+3}{(x-10)(x+1)} = \frac{23/11}{x-10} + \frac{-1/11}{x+1}$$

À vous

Solve $\int \frac{2x+3}{x^2-9x-10}\,dx$

$$= \int \frac{23/11}{x-10}\,dx - \int \frac{1/11}{x+1}\,dx$$

$$= \frac{23}{11} \ln|x-10| - \frac{1}{11} \ln|x+1| + C$$

ex. Decompose and integrate

$$\frac{4x^2+7x+9}{x^3(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1}$$

$$4x^2+7x+9 = Ax^2(x+1) + Bx(x+1) + C(x+1) + Dx^3$$

$x=-1$:

$$4-7+9 = -D$$

$$D = -6$$

$x=0$:

$$9 = C$$

$$C = 9$$

→ Let's compare powers

$$4x^2+7x+9 = Ax^3 + Ax^2 + Bx^2 + Bx + Cx + C + Dx^3$$

$x^3: \quad 0 = A + D$

$0 = A - 6$

$A = 6$

$x^2: \quad 4 = A + B$

$4 = 6 + B$

$B = -2$

$x: \quad 7 = B + C$

$7 = 7 \ \checkmark$

$x^0 = 9 = C$

$C = 9 \ \checkmark$

March 16, 2017

Mr. Memy

$$I = \int \frac{3x^4 + 2x^3 + 6x^2 + 3x + 3}{x(x^2+1)^2}\, dx$$

$$I = \int \left(\frac{A}{x} + \frac{Bx + C}{x+1} + \frac{Dx + E}{(x^2+1)^2} \right) dx$$

$$3x^4 + 2x^3 + 6x^2 + 3x + 3 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$$

if $x = 0$

$$\boxed{A = 3}$$

Expand and compare powers

$$3x^4 + 2x^3 + 6x^2 + 3x + 3 =$$

$$Ax^4 + 2Ax^2 + A + Bx^4 + Cx^3 + Bx^2 + Cx + Dx^2 + Ex$$

x^4:

$$3 = A + B \rightarrow \boxed{B = 0}$$

x^3:

$$2 = C \rightarrow \boxed{C = 2}$$

x^2:

$$6 = 2A + B + D$$

$$6 = 2(3) + 0 + D$$

$$6 = 6 + D \rightarrow \boxed{D = 0}$$

x:

$3 = C + E$

$3 = 2 + E \rightarrow \boxed{E = 1}$

x^0:

$3 = A$

$$I = \underbrace{\int \frac{3}{x}\,dx}_{I_1} + \underbrace{\int \frac{2}{x^2+1}\,dx}_{I_2} + \underbrace{\int \frac{1}{(x^2+1)^2}\,dx}_{I_3}$$

$I_1 = 3\ln|x|$

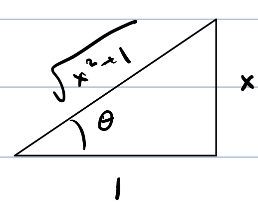
$I_2 = 2\arctan x$

$a = 1$

$I_3 = \int \frac{1}{(x^2+1)^2}\,dx$ Trig sub: Let $\frac{x}{1} = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$= \int \frac{1}{(\tan^2\theta+1)^2}\sec^2\theta\,d\theta$



$= \int \frac{1}{\sec^{\cancel{4}\,2}\theta}\cancel{\sec^2\theta}\,d\theta$

$= \int \frac{1}{\sec^2\theta}\,d\theta$

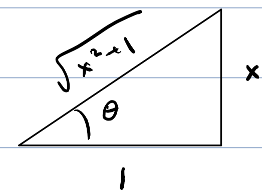
$= \int \cos^2\theta\,d\theta$ even cos

$$= \frac{1}{2}\int (1 + \cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\int 1\, d\theta + \frac{1}{2}\cdot\frac{1}{2}\int \cos \underbrace{2\theta}_{u}\, \overbrace{2\, d\theta}^{du}$$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta$$

$$= \frac{1}{2}\arctan x + \frac{1}{4}\cdot 2\sin\theta\cos\theta$$



$$= \frac{1}{2}\arctan x + \frac{1}{2}\cdot\frac{x}{\sqrt{x^2+1}}\cdot\frac{1}{\sqrt{x^2+1}}$$

$$\sin\theta = \frac{x}{\sqrt{x^2+1}}$$

$$\cos\theta = \frac{1}{\sqrt{x^2+1}}$$

So $\quad I = 3\ln|x| + 2\arctan x + \frac{1}{2}\arctan x + \frac{1}{2}\cdot\frac{x}{x^2+1} + C$

$$= 3\ln|x| + \frac{5}{2}\arctan x + \frac{1}{2}\cdot\frac{x}{x^2+1} + C$$