

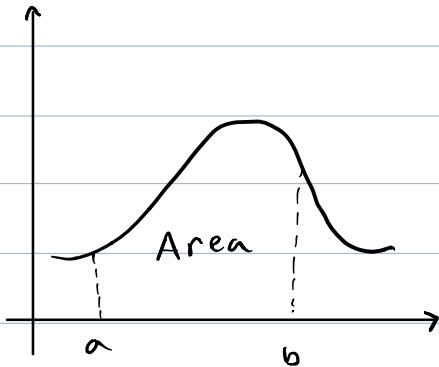
1 - Riemann Sums

Jan. 18

Objective: Find the area under a curve (and above the x-axis)

How do we find area?

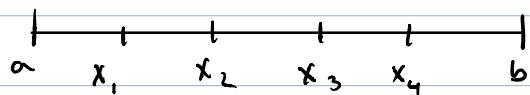
Suppose f is a cont^s fcn over $[a, b]$...

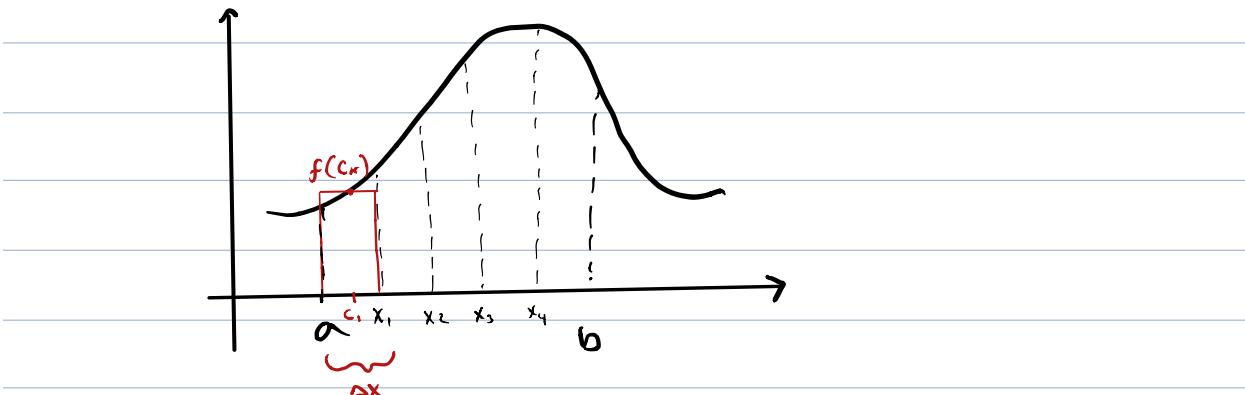


How would we evaluate this area?

Riemann approximation method

⇒ Divide $[a, b]$ into smaller intervals





→ Determine the width of each subinterval

for convenience: create equal subdivisions Δx

$$\Delta x = \frac{b-a}{n}$$

Δx will become the base of each rectangle

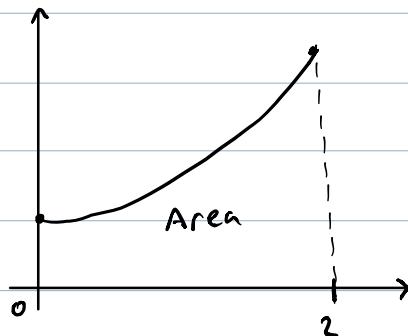
→ choose a point, c_k , within each subinterval, and use $f(c_k)$ as the rectangle's height

→ Area $\approx f(c_1) \Delta x + f(c_2) \Delta x + \dots + f(c_n) \Delta x$

$$\text{Area} \approx \sum_{\substack{\uparrow \\ \text{sigma} \\ (\text{sum of})}} f(c_k) \Delta x$$

ex. consider $f(x) = x^2 - 1$ over $[0, 2]$

use 4 rectangles to estimate the area between $f(x)$ and the x-axis within $[0, 2]$.



$$\textcircled{1} \quad \Delta x = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$$

\textcircled{2} Select pts from each subinterval

$$[0, 0.5] \rightarrow c_1 = 0 \text{ (choosing left bound)}$$

$$[0.5, 1] \rightarrow c_2 = 0.5 \text{ (")}$$

$$[1, 1.5] \rightarrow c_3 = 1$$

$$[1.5, 2] \rightarrow c_4 = 1.5$$

$$f(c_1) = f(0) = 1$$

$$f(c_2) = f(0.5) = 1.25$$

$$f(c_3) = f(1) = 2$$

$$f(c_4) = f(1.5) = 3.25$$

$$\textcircled{3} \quad \text{Area} \approx \sum f(c_k) \Delta x$$

$$= 1(0.5) + 1.25(0.5) + 2(0.5) + 3.25(0.5)$$

$\text{Area} \approx 3.75$

(underestimate of actual area)

Using right-bound : $\text{Area} \approx 5.75$

(everything gets shifted over)

If the number of subdivisions is large, the pt you pick doesn't matter

Sigma Notation

Jan. 19

= a short-hand notation to describe a sum (finite or infinite)

$$\sum_{k=1}^n a_k$$

Diagram illustrating Sigma Notation:

- The summation symbol (\sum) is labeled "Summation symbol".
- The upper bound (n) is labeled "upper bound of summation".
- The lower bound ($k=1$) is labeled "lower bound of summation".
- A red arrow points from the label "terms of the sum" to the term a_k .

Ex.

$$\sum_{k=2}^5 (3k + 2) = 5 + 8 + 11 + 14 + 17$$

expanded form

ex.

$$\sum_{k=1}^5 (3a+2) = (3a+2) + (3a+2) + \dots$$

5 times

ex. Write the following series of terms in sigma form:

$$2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$$

$$\sum_{k=0}^5 2 \left(\frac{1}{2}\right)^k \quad \text{or} \quad \sum_{k=0}^5 \frac{1}{16}(2)^k$$

Summation Formulas

$$\textcircled{1} \quad \sum_{k=1}^n (1) = 1 + 1 + 1 + \dots + 1 = n$$

$$\textcircled{2} \quad \sum_{k=1}^n k = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$\textcircled{3} \quad \sum_{k=1}^n k^2 = 1 + 4 + 9 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\textcircled{4} \quad \sum_{k=1}^n k^3 = 1 + 8 + 27 + 64 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

ex. $\sum_{k=1}^{32} (1) = 32$

$$\text{Ex. } \sum_{k=1}^8 k = \frac{8(9)}{2} = 36$$

$$\text{Ex. } \sum_{k=1}^5 k^3 = \frac{5^2(6)^2}{4} = \frac{(25)(36)}{4} = 225$$

$$\text{Ex. } \sum_{k=5}^{14} k^2 = \sum_{k=1}^{14} k^2 - \sum_{k=1}^4 k^2$$

$$= \frac{(14)(15)(29)}{6} - \frac{(4)(5)(9)}{6}$$

$$= 985$$

Summation Properties

① Constant multiple rule

$$\sum_{k=1}^n c a_k = c \sum_{k=1}^n a_k$$

② Addition / subtraction rule

$$\sum_{k=1}^n (a_k \pm b_k) = \sum_{k=1}^n a_k \pm \sum_{k=1}^n b_k$$

Ex.

$$\sum_{m=1}^{200} (2m^2 - 10m)$$

$$= \sum_{m=1}^{200} 2m^2 - \sum_{m=1}^{200} 10m$$

$$= 2 \sum_{m=1}^{200} m^2 - 10 \sum_{m=1}^{200} m$$

$$= 2 \left[\frac{200(201)(401)}{6} \right] - 10 \left[\frac{200(201)}{2} \right]$$

$$= 5172400$$

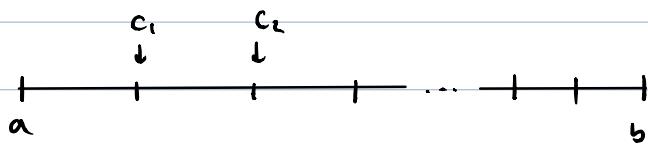
Sigma Notation and Riemann Sums

Recall that the area between a continuous function $f(x)$ and the x-axis within $[a, b]$ can be estimated by

$$\text{Area} \approx \sum_{k=1}^n f(c_k) \Delta x$$

$$\underline{\text{Step 1}} \quad \text{Find } \Delta x \dots \quad \Delta x = \frac{b-a}{n}$$

Step 2 For convenience, chose all c_k 's to be the right-hand pts
of the subintervals



$$c_1 = a + \Delta x$$

$$c_2 = a + 2\Delta x$$

$$c_3 = a + 3\Delta x$$

$$c_k = a + k \Delta x$$

Step 3 Find an expression for $f(c_k)$

Step 4 Evaluate $\sum_{k=1}^n f(c_k) \Delta x$

using summation formulas and rules

Ex. Apply Riemann Sums to $f(x) = x^2 + 1$ over $[0, 2]$ to estimate its area

$$\textcircled{1} \quad \Delta x = \frac{2-0}{n} = \frac{2}{n}$$

$$\textcircled{2} \quad c_k = 0 + k \Delta x$$

$$= k \cdot \frac{2}{n}$$

$$\textcircled{3} \quad f(x) = x^2 + 1$$

$$\begin{aligned} \text{So } f(c_k) &= c_k^2 + 1 \\ &= \left(k \cdot \frac{2}{n}\right)^2 + 1 \end{aligned}$$

$$f(c_k) = \frac{4k^2}{n^2} + 1$$

$$\textcircled{4} \quad \text{Area}_{[0,2]} \approx \sum_{k=1}^n f(c_k) \Delta x$$

$$\approx \sum_{k=1}^n \left(\frac{4k^2}{n^2} + 1 \right) \cdot \frac{2}{n}$$

$$\approx \sum_{k=1}^n \left(\frac{8}{n^3} k^2 + \frac{2}{n} \right)$$

$$\approx \sum_{k=1}^n \frac{8}{n^3} k^2 + \sum_{k=1}^n \frac{2}{n}$$

$$\approx \frac{8}{n^3} \sum_{k=1}^n k^2 + \frac{2}{n} \sum_{k=1}^n 1$$

$\textcolor{red}{\cancel{n}}$ is not
available

$$\approx \frac{8}{n^3} \frac{\pi(n+1)(2n+1)}{6} + \frac{2}{n} \cdot \pi$$

$$\approx \frac{4}{3} \left(\frac{n+1}{2} \right) \left(2 + \frac{2}{n} \right) + 2$$

$$\approx \frac{4}{3} \left(1 + \frac{1}{n} \right) \left(2 + \frac{2}{n} \right) + 2$$

$$\text{If } n = 4 \dots \quad A_{[0,2]} \approx \frac{4}{3} \left(1 + \frac{1}{4} \right) \left(2 + \frac{1}{4} \right) + 2$$

$$\approx 5.75$$

Limit of a Riemann Sum

Jan. 23

Riemann's Principle :

According to Riemann, $\sum_{k=1}^n f(c_k) \Delta x$ converges towards one specific value regardless of the selected pts, c_k , as long as $\Delta x \rightarrow 0$ (or as long as $n \rightarrow \infty$)

(N.B. : Riemann's principle is true for all cont' fns over an interval $[a, b]$)

The result of the limit is denoted by ...

$$\rightarrow \text{S.A.}_{[a,b]}^{\text{Signed area}}$$

$$\rightarrow \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x$$

$$\rightarrow \int_a^b f(x) dx$$

ex. Use a limit process (as $n \rightarrow \infty$) to evaluate the exact
the exact area* between $f(x) = 4x - x^2$ over the interval
 $[0, 10]$

$$\textcircled{1} \quad \Delta x = \frac{b-a}{n} = \frac{10-0}{n} = \frac{10}{n}$$

$$\textcircled{2} \quad c_k = a + k \Delta x$$

$$= 0 + k \cdot \frac{10}{n} = \frac{10}{n} k$$

$$\textcircled{3} \quad f(c_k) = 4\left(\frac{10}{n} k\right) - \left(\frac{10}{n} k\right)^2$$

$$= \frac{40}{n} k - \frac{100}{n^2} k^2$$

$$\textcircled{4} \quad \text{Area} \approx \sum_{k=1}^n f(c_k) \Delta x$$

$$\approx \sum_{k=1}^n \left(\frac{40}{n} k - \frac{100}{n^2} k^2 \right) \cdot \left(\frac{10}{n} \right)$$

$$= \sum_{k=1}^n \left(\frac{400}{n^2} k - \frac{1000}{n^3} k^2 \right)$$

$$= \frac{400}{n^2} \sum_{k=1}^n k - \frac{1000}{n^3} \sum_{k=1}^n k^2$$

$$= \frac{400}{n^2} \cdot \frac{n(n+1)}{2} - \frac{1000}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= 200 \left(\frac{n+1}{n} \right) - \frac{500}{3} \left(\frac{n+1}{n} \right) \left(\frac{2n+1}{n} \right)$$

$$= 200 \left(1 + \frac{1}{n} \right) - \frac{500}{3} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right)$$

$$\textcircled{5} \quad SA_{[0,10]} \quad \text{or} \quad \int_0^{10} f(x) dx$$

$$= \lim_{n \rightarrow \infty} \left[200 \left(1 + \frac{1}{n} \right) - \frac{500}{3} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) \right]$$

$$= 200(1) - \frac{500}{3}(1)(2)$$

$$= -\frac{400}{3}$$

N.B. $-\frac{400}{3}$ cannot truly represent area (which "should" be positive)
 Riemann's process actually provides "Signed area" (S.A.)

Signed area is the net result from:

AREA ABOVE X-AXIS - AREA BELOW X-AXIS

N.B. we could also work that

$$\int_0^{10} \underbrace{(4x - x^2)}_{\substack{\text{height} \\ \text{sum}}} dx = -\frac{400}{3}$$

ex. Evaluate

$$\int_0^R (4x - x^2) dx$$

R represents the right-hand bound of an arbitrary interval $[0, R]$

$$\textcircled{1} \quad dx = \frac{R}{n}$$

$$\begin{aligned}\textcircled{2} \quad c_k &= 0 + k \cdot \frac{R}{n} \\ &= \frac{R}{n} k\end{aligned}$$

$$\begin{aligned}\textcircled{3} \quad f(c_k) &= 4\left(\frac{R}{n} k\right) - \left(\frac{R}{n} k\right)^2 \\ &= \frac{4R}{n} k - \frac{R^2}{n^2} k^2\end{aligned}$$

$$\textcircled{4} \quad \text{Area} \approx \sum_{k=0}^n f(c_k) \Delta x$$

$$= \sum_{k=0}^n \left(\frac{4R}{n} k - \frac{R^2}{n^2} k^2 \right) \left(\frac{R}{n} \right)$$

$$= \sum_{k=0}^n \left(\frac{4R^2}{n^2} k - \frac{R^3}{n^3} k^2 \right)$$

$$= \frac{4R^2}{n^2} \sum_{k=0}^n k - \frac{R^3}{n^3} \sum_{k=0}^n k^2$$

$$= \frac{4R^2}{n^2} \cdot \frac{\pi(n+1)}{2} - \frac{R^3}{n^3} \cdot \frac{\pi(n+1)(2n+1)}{6}$$

$$= \frac{4R^2}{2} \cdot \left(1 + \frac{1}{n}\right) - \frac{R^3}{6} \cdot \left(1 + \frac{1}{n}\right) \cdot \left(2 + \frac{1}{n}\right)$$

$$\textcircled{5} \quad \int_0^R (4x - x^2) dx$$

$$= \lim_{n \rightarrow \infty} \left[\frac{4R^2}{2} \cdot \left(1 + \frac{1}{n}\right) - \frac{R^3}{6} \cdot \left(1 + \frac{1}{n}\right) \cdot \left(2 + \frac{1}{n}\right) \right]$$

$$= \frac{4R^2}{2} \cdot (1) - \frac{R^3}{6^3} (1) (2)$$

$$= 2R^2 - \frac{1}{3} R^3$$

N.B.

$$\int_3^4 (4x - x^2) dx = \int_0^4 (4x - x^2) dx - \int_0^3 (4x - x^2) dx$$

Jan. 25, 2017

Fundamental Theorem of Calculus

Let $f(x)$ represent a continuous function over $[a, b]$

and let $F(x)$ represent an antiderivative of $f(x)$

i.e. $F'(x) = f(x)$

$$\text{Then } \int_a^b f(x) dx = F(x) \Big|_a^b \\ = F(b) - F(a)$$

ex. $\int_1^2 \frac{1}{x} dx = \ln x \Big|_1^2 = \ln 2 - \ln 1^0 = \ln 2$

$f(x)$ ← which must be continuous!
(in this case, it is)

ex. $\int_0^3 (10 - 9.8x) dx = (10x - 4.9x^2) \Big|_0^3$

$$= 30 - 4.9(9) - 0$$

$$= -14.1$$

$$\text{ex. } \int_0^{\ln 7} e^x dx = e^x \Big|_0^{\ln 7}$$

$$= e^{\ln 7} - e^0$$

$$= 7 - 1$$

$$= 6$$

Proof of the FTC

Reminder: Mean Value Theorem

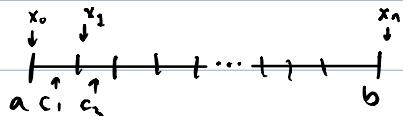
If $g'(x)$ is conts over $[a, b]$ and diff^{ntc} within (a, b) ,

then there exists at least one value $x=c$ within (a, b)

such that $g'(c) = \frac{g(b) - g(a)}{b-a}$ or $g(b) - g(a) = g'(c)(b-a)$

Proof of FTC:

Suppose $[a, b]$ is divided into n subintervals



and let $x_0, x_1, x_2, \dots, x_n$ be the extremities of each subinterval

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= \underline{F(x_n)} - F(x_{n-1}) + F(x_{n-1}) - \dots - F(x_1) + \boxed{F(x_1) - F(x_0)} \end{aligned}$$

$$\text{Applying MVT} = F'(c_n)(x_n - x_{n-1}) + F'(c_{n-1})(x_n - x_{n-1}) + \dots \\ + F'(c_2)(x_2 - x_1) + F'(c_1)(x_1 - x_0)$$

But... F needs to be cont! But... $F' = f$ so since F is diff^{ble}, then F is cont!

(Note: $c_1, c_2, c_3, \dots, c_n$ are selected within each subinterval)

Now, $F' = f$, and $x_n - x_{n-1} = \Delta x$

$$\text{so } F(b) - F(a) = f(c_n) \Delta x + f(c_{n-1}) \Delta x + \dots + f(c_1) \Delta x$$

$$= \sum_{k=1}^n f(c_k) \Delta x$$

The number of subdivisions is arbitrary and so, as $n \rightarrow \infty$

$$F(b) - F(a) = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x$$

$$= \int_a^b f(x) dx \quad \text{"L"}^{\circ}$$

ON TEST: Might write proof with blanks, or give justifications & proof parts (e.g. use MVT, assume $\Delta x \rightarrow 0, \dots$)

Do you remember your derivatives / anti-derivatives

$$\textcircled{1} (x)' = 1$$

$$\int 1 dx = x \quad \text{or} \quad \int 1 dq = q$$

$$\textcircled{1} \quad (x^n)' = nx^{n-1} \quad \int x^n dx = \frac{x^{n+1}}{n+1} \quad (\text{for } n \neq -1)$$

$$\textcircled{2} \quad (\ln x)' = \frac{1}{x} \quad \int \frac{1}{x} dx = \ln |x|$$

$$\textcircled{3} \quad (\sin x)' = \cos x \quad \int \cos x dx = \sin x$$

$$\textcircled{4} \quad (\cos x)' = -\sin x \quad \int \sin x dx = -\cos x$$

$$\textcircled{5} \quad (\tan x)' = \sec^2 x \quad \int \sec^2 x dx = \tan x$$

$$\textcircled{6} \quad (\cot x)' = -\csc^2 x \quad \int \csc^2 x dx = -\cot x$$

$$\textcircled{7} \quad (\sec x)' = \sec x \tan x \quad \int \sec x \tan x dx = \sec x$$

$$\textcircled{8} \quad (\csc x)' = -\csc x \cot x \quad \int \csc x \cot x dx = -\csc x$$

$$\textcircled{9} \quad (e^x)' = e^x \quad \int e^x dx = e^x$$

$$\textcircled{10} \quad (a^x)' = a^x \ln a \quad \int a^x dx = \frac{a^x}{\ln a}$$

$$\textcircled{11} \quad (\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \quad \int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin\left(\frac{x}{a}\right)$$

$$\textcircled{12} \quad (\arctan x)' = \frac{1}{1+x^2} \quad \int \frac{1}{a^2+x^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right)$$

$$\textcircled{13} \quad (\text{arcsec } x)' = \frac{1}{|x|\sqrt{x^2-1}} \quad \int \frac{1}{|x|\sqrt{x^2-a^2}} dx = \frac{1}{a} \text{arcsec}\left(\frac{x}{a}\right)$$

$$\begin{aligned}
 \text{ex. } \int_0^2 \frac{1}{4+x^2} dx &= \frac{1}{2} \arctan\left(\frac{x}{2}\right) \Big|_0^2 \\
 &= \frac{1}{2} \arctan\left(\frac{2}{2}\right) - \frac{1}{2} \arctan\left(\frac{0}{2}\right) \\
 &= \frac{1}{2} \cdot \frac{\pi}{4} \\
 &= \frac{\pi}{8}
 \end{aligned}$$

FTC ... suite

Jan. 26

$$F(b) - F(a) = \int_a^b f(x) dx$$

where f is cont's over $[a, b]$

and where F is an antiderivative of f (or $F' = f$)

$$\begin{aligned}
 \text{ex. } \int_1^3 \frac{1}{x^3} dx &= \int_1^3 x^{-3} dx \\
 &= \frac{x^{-2}}{-2} \Big|_1^3 \\
 &= -\frac{1}{2x^2} \Big|_1^3 \\
 &= \left(-\frac{1}{18}\right) - \left(-\frac{1}{2}\right) \\
 &= \frac{4}{9}
 \end{aligned}$$

$$\underline{\text{ex.}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \, dx = (-\cos x) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 0 - 0$$

$$= 0$$

$$\underline{\text{ex.}} \int_0^4 \frac{1}{\sqrt{25-x^2}} \, dx \quad \int \frac{1}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right)$$

$$= \left[\arcsin\left(\frac{x}{5}\right) \right] \Big|_0^4$$

$$= \arcsin\left(\frac{4}{5}\right) - \arcsin 0$$

$$= \arcsin\left(\frac{4}{5}\right) \approx 0.927$$

$$\underline{\text{ex.}} \int_{-2}^1 \frac{1}{x^2} \, dx$$

$$= \int_{-2}^1 x^{-2} \, dx$$

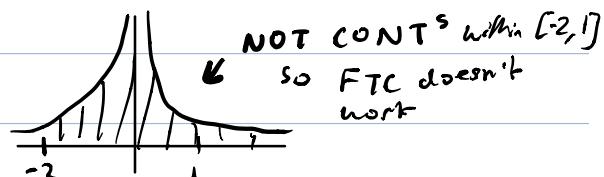
$$= \left[\frac{x^{-1}}{-1} \right] \Big|_{-2}^1$$

$$= \left(-\frac{1}{1} \right) - \left(-\frac{1}{-2} \right)$$

$$= -1 - 0.5$$

$$= -\frac{3}{2} \leftarrow \text{DOESN'T MAKE SENSE}$$

(can't be negative)



Ex. Find R such that

$$\int_0^R \sqrt[3]{x} dx = 50$$

$$\int_0^R x^{\frac{1}{3}} dx = \frac{x^{4/3}}{4/3} \Big|_0^R$$

$$= \frac{3}{4} x^{4/3} \Big|_0^R$$

$$= \frac{3}{4} (R)^{4/3} - 0$$

$$\text{So } \frac{3}{4} R^{4/3} = 50$$

$$R^{4/3} = \frac{200}{3}$$

$$R = \left(\frac{200}{3}\right)^{3/4}$$

Properties of the definite integral

① If $f(x) \geq 0$ over $[a, b]$, then $\int_a^b f(x) dx \geq 0$ (assuming that $b > a$)

② If $f(x) \geq g(x)$ over $[a, b]$ then $\int_a^b f(x) dx \geq \int_a^b g(x) dx$

③ $\int_a^a f(x) dx = 0$

④ $\int_b^a f(x) dx = - \int_a^b f(x) dx$
 $F(c) - F(b) \downarrow_{F \text{ is C}} = -(F(b) - F(a))$

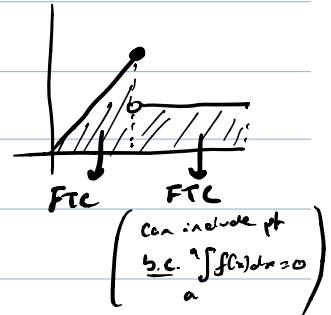
$$\textcircled{5} \int_a^b K f(x) dx = K \int_a^b f(x) dx \quad (\text{allowed in sigma notation})$$

$$\textcircled{6} \int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$

(also allowed in sigma notation)

$$\textcircled{7} \int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx \quad \text{Additive interval property}$$

WHY
as long as f is integrable over both intervals ex.



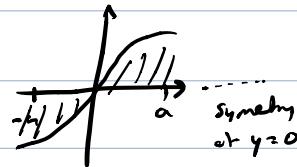
\textcircled{8} If $f(x)$ is an even function (symmetric: $f(-x) = f(x)$)

$$\text{then } \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$



\textcircled{9} If $f(x)$ is an odd function ($f(-x) = -f(x)$)

$$\text{then } \int_{-a}^a f(x) dx = 0$$



ex. Consider the function $f(x) = \begin{cases} \cos x, & x < 0 \\ 2x-1, & x \geq 0 \end{cases}$

$$\text{Evaluate } \int_{-\frac{\pi}{2}}^3 f(x) dx$$

$$= \int_0^3 2x-1 dx + \int_{-\frac{\pi}{2}}^0 \cos x dx$$

$$= \left(x^2 - x \right) \Big|_0^3 + \sin x \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

Could also split it up,
 Factor out 2,... but
 not necessary

$$= (9 - 3) + (0 - (-1))$$

$$= 6 + 1$$

$$= 7$$

$$\text{ex. } \int_1^2 \frac{(x+1)^2 - 3x}{x^2} dx$$

$$= \int_1^2 \frac{x^2 + 2x + 1 - 3x}{x^2} dx$$

$$= \int_1^2 \frac{x^2 - x + 1}{x^2} dx$$

$$= \int_1^2 \left(1 - \frac{1}{x} + \frac{1}{x^2} \right) dx$$

$$= \left(x - \ln|x| + \frac{x^{-1}}{-1} \right) \Big|_1^2$$

$$= \frac{3}{2} - \ln 2$$

Jan. 30

ex. Riemann Sum \rightarrow Integral

Find an exact value for the following Riemann sum:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{4 + \frac{4k^2}{n^2}} \cdot \frac{2}{n}$$

$$\Delta x = \frac{2}{n}$$

$$\frac{b-a}{n} = \frac{2}{n}$$

$$a = b - a \quad \text{Chose } a=0, b=2$$

$$c_k = a + k\Delta x$$

$$= 0 + k\Delta x$$

$$= k \frac{2}{n}$$

$$= \frac{2}{n} k$$

Notice that $\frac{4k^2}{n^2} = \left(\frac{2k}{n}\right)^2 = c_k^2$

Therefore, $\frac{1}{4 + \frac{4k^2}{n^2}} = \frac{1}{4 + c_k^2} = f(c_k)$

Then $f(x) = \frac{1}{4 + x^2}$

Oooo... oh! That means that

$$\begin{aligned}\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{4 + \frac{4k^2}{n^2}} \cdot \frac{2}{n} &= \int_0^2 \frac{1}{4 + x^2} dx \\ &= \frac{1}{2} \arctan\left(\frac{x}{2}\right) \Big|_0^2 \\ &= \frac{1}{2} \arctan(1) - \frac{1}{2} \arctan(0) \\ &= \frac{1}{2} \cdot \frac{\pi}{4} \\ &= \frac{\pi}{8}\end{aligned}$$

À vous: Convert $\int_1^3 \ln(2+x) dx$ into a Riemann sum:

$$b = 3$$

$$a = 1$$

$$\textcircled{1} \quad \Delta x = \frac{b-a}{n} = \frac{2}{n}$$

$$\textcircled{2} \quad c_k = 1 + k \Delta x$$

$$= 1 + \frac{2}{n} k$$

$$\textcircled{3} \quad f(c_k) = \ln(2 + c_k)$$

$$= \ln\left(2 + \frac{2}{n} k\right)$$

$$\textcircled{4} \quad \text{Area} = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x$$

$$= \lim_{n \rightarrow \infty} \sum_{k=1}^n \ln\left(2 + 1 + \frac{2}{n} k\right) \frac{2}{n}$$

$$= \lim_{n \rightarrow \infty} \sum_{k=1}^n \ln\left(3 + \frac{2}{n} k\right) \frac{2}{n}$$

$$\textcircled{1} \text{ or } F = \int_3^x \ln x \, dx$$

FTC Part I

Let $f(x)$ represent a continuous function over $[a, b]$, then

$$\frac{d}{dx} \left[\int_a^x f(t) \, dt \right] = f(x) \quad \text{where } a \leq x \leq b$$

Recall (from FTC part 2) that if f is continuous over $[a, b]$ and if F is an antiderivative of f (i.e. $F' = f$)

$$\text{then } \int_a^b f(t) \, dt = F(b) - F(a)$$

$$\begin{aligned} \frac{d}{dx} \left[\int_a^x f(t) \, dt \right] &\stackrel{\text{FTC}}{=} \frac{d}{dx} \left[F(t) \Big|_a^x \right] \\ &= \frac{d}{dx} [F(x) - F(a)] \end{aligned}$$

$$= F'(x) - 0 \quad a \text{ is a constant}$$

$$= f(x)$$

$$\underline{\text{ex.}} \quad \frac{d}{dx} \left[\int_0^x \cos t \, dt \right]$$

$$= \frac{d}{dx} \left[\sin t \Big|_0^x \right]$$

$$= \frac{d}{dx} \left[\sin x - \sin 0 \right] \quad \text{const.}$$

$$= \cos x$$

$$\underline{\text{ex.}} \quad \frac{d}{dx} \left[\int_1^x \sqrt{\sin t + \frac{x^2}{5}} \, dt \right]$$

$$= \sqrt{\sin^2 x + \frac{x^3}{5}}$$

$$\text{A vous} \quad \frac{d}{dx} \left[\int_x^3 \cos(t^2) \, dt \right]$$

$$= \frac{d}{dx} \left[- \int_3^x \cos(t^2) \, dt \right]$$

$$= - \frac{d}{dx} \left[\int_3^x \cos(t^2) \, dt \right]$$

$$= -\cos(x^2)$$

$$\underline{\text{ex.}} \quad \frac{d}{dx} \left[\int_{\ln x}^{x^2} \underbrace{\sqrt{4+t^3}}_f \, dt \right]$$

$$= \frac{d}{dx} \left[F(t) \Big|_{\ln x}^{x^2} \right]$$

$$= \frac{d}{dx} \left[F(x^2) - F(\ln x) \right]$$

$$= F'(x^2) \cdot (x^2)' - F'(\ln x) \cdot (\ln x)'$$

$$= f(x^2) \cdot 2x - f(\ln x) \cdot \frac{1}{x}$$

$$= \sqrt{4+x^6} \cdot 2x - \sqrt{4+(\ln x)^2} \cdot \frac{1}{x}$$

"Our general FTC part I"

$$\frac{d}{dx} \left[\int_{u_1(x)}^{u_2(x)} f(t) dt \right]$$

$$= f(u_2(x)) \cdot u_2'(x) - f(u_1(x)) \cdot u_1'(x)$$

Application

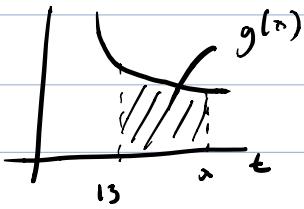
$$\text{Let } g(x) = \int_{13}^x \frac{1}{t+1^3} dt$$

a) Find $g(13)$

c) Find the eq of the line that is tangent to $g(x)$ at $x=13$

b) Find $g'(13)$

$$g(x) = \int_{13}^x \frac{1}{1+t^3} dt$$



a) $g(13) = \int_{13}^{13} \frac{1}{1+t^3} dt$
 $= 0$

b) $g'(x) = \frac{d}{dx} \left[\int_{13}^x \frac{1}{1+t^3} dt \right]$

$$= \frac{1}{1+x^3}$$

$$g'(13) = \frac{1}{1+13^3} = \frac{1}{2198}$$

c) Let $y = \text{eqn tangent line of } g(x) \text{ at } x=13$

$$y = \frac{1}{2198} x + b$$

$$0 = \frac{1}{2198} (13) + b$$

$$b = -\frac{13}{2198}$$

$$\text{So } y = \frac{1}{2198} x - \frac{13}{2198}$$

Ex: Suppose

$$x^2 + \cos x = \int_{\pi}^x \frac{f(\underline{x})}{\sqrt{1+x^2}} dx + k$$

a) If this eqn is true for all x , what must k be?

Chose $x = \pi$

$$\pi^2 - 1 = 0 + k \rightarrow k = \pi^2 - 1$$

b) Can you find $f(x)$ that ensures this eqn is true for all x .

$$(x^2 + \cos x)' = \left(\int_{\pi}^x \frac{f(x)}{\sqrt{1+x^2}} dx + k \right)'$$

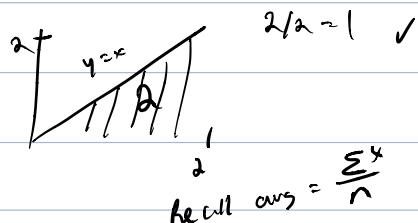
$$2x - \sin x = \frac{f(x)}{\sqrt{1+x^2}}$$

$$f(x) = \sqrt{1+x^2} (2x - \sin x)$$

Feb. 1

Def: Let $f(x)$ represent a function defined over $[a, b]$. Its average value is denoted by $\bar{f}_{[a,b]}$ and is obtained by

$$\bar{f}_{[a,b]} = \frac{1}{b-a} \int_a^b f(x) dx \quad \frac{\text{Area}}{\text{width}}$$



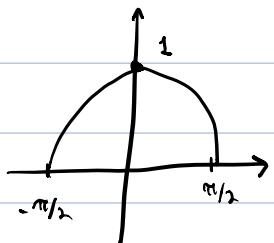
ex. Let $f(x) = \cos x$. Find the average value of f over $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\bar{f}_{[-\frac{\pi}{2}, \frac{\pi}{2}]} = \frac{1}{\frac{\pi}{2} - (-\frac{\pi}{2})} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x) dx$$

$$= \frac{1}{\pi} [\sin x] \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi} (1 - (-1)) = \frac{2}{\pi}$$

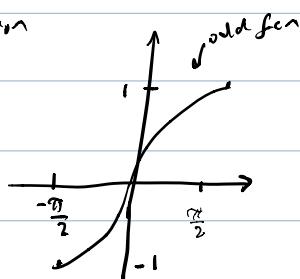
Visualization:



ex. for $\sin x$...

$$\bar{f}_{[-\frac{\pi}{2}, \frac{\pi}{2}]} = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x dx$$

Visualization



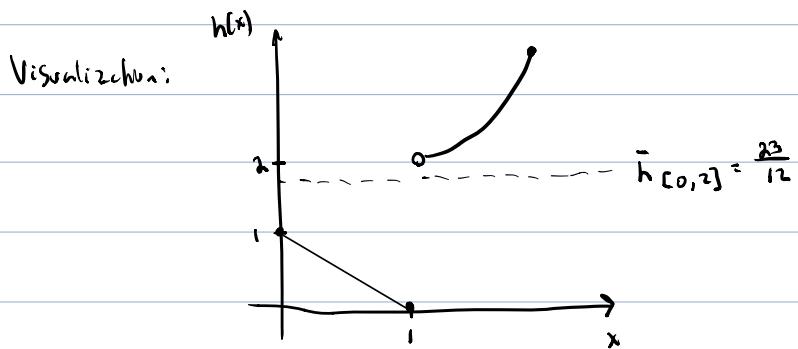
$$= 0$$

ex. Find $\bar{h}_{[0,2]}$ if $h(x) = \begin{cases} 1-x; & x \leq 1 \\ x^2+1; & x > 1 \end{cases}$

$$\bar{h}_{[0,2]} = \frac{1}{2} \int_0^2 h(x) dx$$

$$= \frac{1}{2} \left[\int_0^1 (1-x) dx + \int_1^2 (x^2+1) dx \right]$$

$$\begin{aligned}
 &= \frac{1}{2} \left[\left(x - \frac{x^2}{2} \right) \Big|_0^1 + \left(\frac{x^3}{3} + x \right) \Big|_0^1 \right] \\
 &= \frac{1}{2} \left[\frac{1}{2} + \frac{8}{3} + 2 - \left(\frac{1}{3} + 1 \right) \right] \\
 &= \frac{23}{12}
 \end{aligned}$$



MEAN VALUE THM OF INTEGRAL CALCULUS

Let $f(x)$ represent a cont's fcn over $[a,b]$, then there exists at least one value of c in $]a,b[$ such that

$$f(c) = \bar{f}_{[a,b]}$$

or

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx$$

Proof: Let $g(x) = \int_a^x f(t) dt$ where f is cont'

① What is $g(a)$?

$$g(a) = \int_a^a f(t) dt = 0$$

② What is $g(b)$?

$$g(b) = \int_a^b f(t) dt$$

According to MVT (CAL I):

$$\frac{g(b) - g(a)}{b - a} = g'(c) \quad \text{for at least one value 'c' in } [a, b]$$

③ What is $g'(c)$?

$$g'(x) = \frac{d}{dx} g(x)$$

$$g'(x) = \frac{d}{dx} \left[\int_a^x f(t) dt \right] = f(x)$$

$$\text{so } g'(c) = f(c)$$

Substitute all accumulated into ①, ②, ③

$$\frac{\int_a^b f(t) dt - 0}{b - a} = f(c)$$

$$\underbrace{\phantom{f_{[a,b]}}}_{\bar{f}_{[a,b]}}$$

Conclusion: $f(c) = \frac{1}{b-a} \int_a^b f(t) dt$

Application:

Let $f(x) = \frac{1}{x^2}$. Find $\bar{f}_{[1,7]}$. Find, if it exists, the value of c such that $f(c) = \bar{f}_{[1,7]}$

$$\bar{f}_{[1,7]} = \frac{1}{6} \int_1^7 f(x) dx$$

$$= \frac{1}{6} \int_1^7 \frac{1}{x^2} dx$$

$$= \frac{1}{6} \left(-\frac{1}{x} \right) \Big|_1^7$$

$$= \frac{1}{6} \left(-\frac{1}{7} + 1 \right)$$

$$= \frac{1}{7}$$

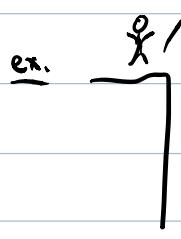
$$f(c) = \frac{1}{c^2}$$

$$\frac{1}{7} = \frac{1}{c^2}$$

$$c^2 = 7$$

$$c = \sqrt{7} \quad \text{or} \quad c = -\sqrt{7}$$

Not in $[1,7]$



Let $v(t) = 10 - 9.8t$

a) Find $\bar{v}_{[0,2]}$

b) Find t_c such that $v(t_c) = \bar{v}_{[0,2]}$

$$a) \overline{V}_{[0,2]} = \frac{1}{2} \int_0^2 v(t) dt$$

$$= \frac{1}{2} (10t - 4.9t^2) \Big|_0^2$$

$$= \frac{1}{2} \left(\frac{2}{5} \right)$$

$$= \frac{1}{5}$$

$$b) v(t) = 10 - 9.8t_c$$

$$\frac{1}{5} = 10 - 9.8t_c$$

$$\frac{10 - \frac{1}{5}}{9.8} = t_c$$

$$\boxed{t_c = 1}$$