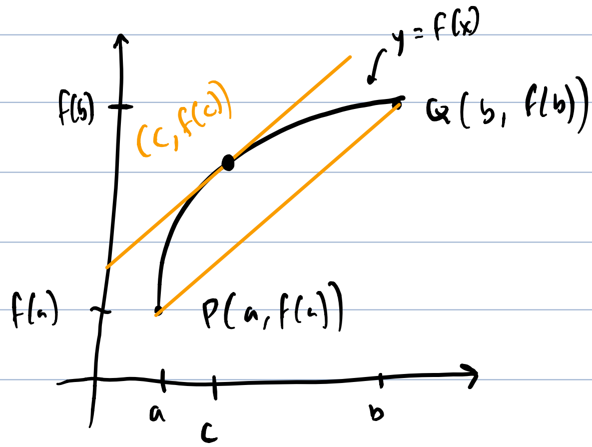
Mean Value Theorem



$$f'(c) = \frac{f(b) - f(c)}{b - a}$$

November 9, 2016

① If $y = f(x)$ is cont's on $[a, b]$

② and if $y = f(x)$ is diff'ble on (a, b)

Then there is a number 'c' in (a, b) so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

ex. $f(x) = x^3 - 3x + 2$ on $[-2, 2]$

Find all values of 'c' that satisfy the cond of the MVT.

Since $f(x)$ is a poly f'n, it is: - cont's on $[-2, 2]$

- diff'ble on $(-2, 2)$

So $y = f(x)$ satisfies conditions of MVT

By MVT, there is at least one value 'c' in $(-2, 2)$, so that

$$f'(c) = \frac{f(2) - f(-2)}{2-(-2)} = \frac{4-0}{4} = 1$$

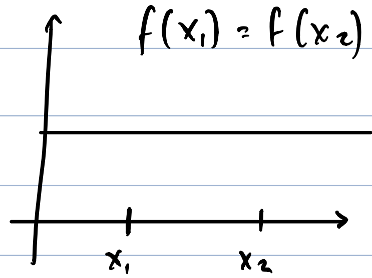
But $f'(x) = 3x^2 - 3$

$1 = 3c^2 - 3$

$c = \pm\sqrt{\frac{4}{3}}$ both lie in $(-2, 2)$

Theorem: If $f'(x) = 0$ for all x in (a, b)
then $f(x) = C$ for some constant C (true for all x's in (a, b))

Proof: $f(x_1) = f(x_2)$



Must show that $f(x_1) = f(x_2)$ for any choice of x_1, x_2 in (a,b)

Assume $f'(x) = 0$ for all x in (a, b)

Recall: If $y = f(x)$ is diff^{ble} at a number, it is cont^s at that number.

So since $y = f(x)$ is diff^{ble} on (a, b), $y = f(x)$ must also be cont^s on (a, b)

Chose any number x_1 & x_2 inside (a, b)

So ⓐ $y = f(x)$ is cont^s on $[x_1, x_2]$

and ⓑ $y = f(x)$ is diff$^{b/e}$ on (x_1, x_2)

So by MVT, there exists a value 'C' in (x_1, x_2) so that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

But $f'(c) = 0$ by assumption.

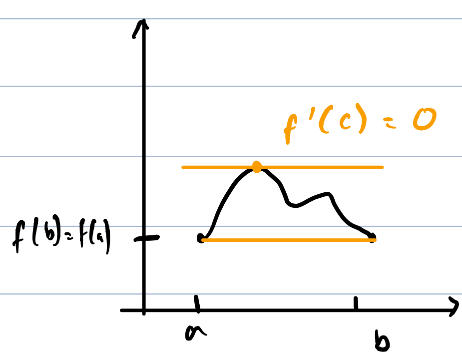
So $$0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$0 = f(x_2) - f(x_1)$$

$$f(x_1) = f(x_2)$$

Rolle's Theorem

[Special Case of MVT]



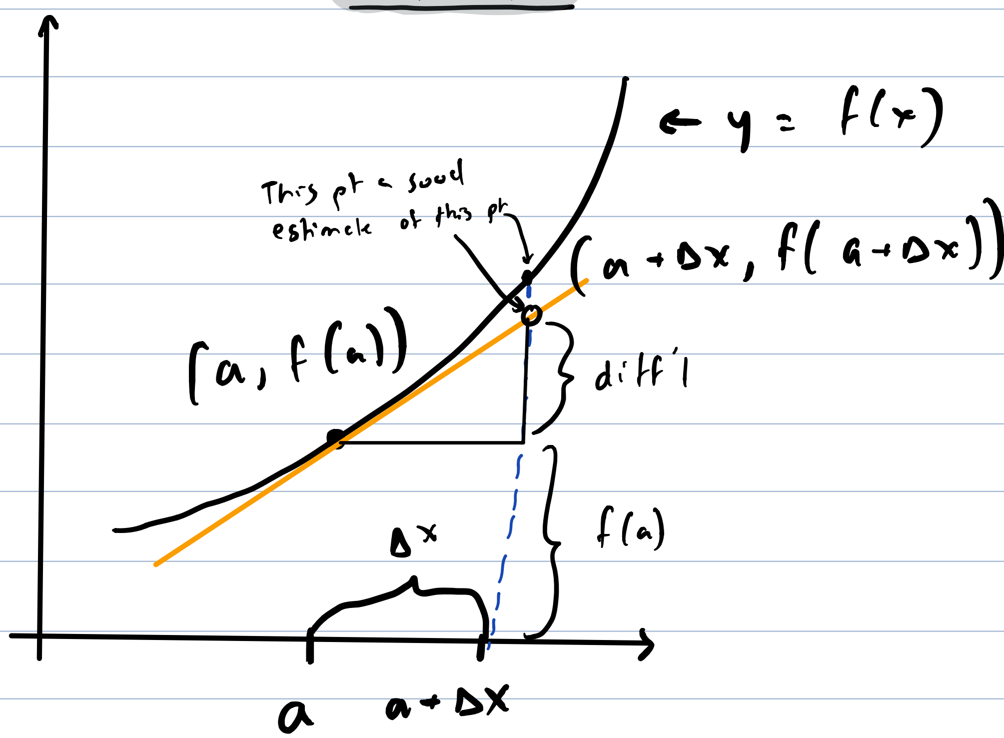
I f (a) $y = f(x)$ is cont^s on $[a,b]$

& if (b) $y = f(x)$ is diff^{ble} on (a,b)

& if (c) $f(a) = f(c)$

Then there is a value 'c' in (a,b) so that $f'(c) = 0$

Differential



For small Δx,

$$f(a+\Delta x) \approx f(a) + \text{diff'l}$$

To compute diff'l

Slope of tangent $= f'(a)$
" $= \frac{\text{rise}}{\text{run}} = \frac{\text{diff'l}}{\Delta x}$

i.e. $f'(a) = \frac{\text{diff'l}}{\Delta x}$

ex. Estimate $\sqrt{36.1}$ using a diff'l

Let $f(x) = \sqrt{x}$ $\Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$

Let $a = 36$ & $\Delta x = 0.1$

LINEAR APPROXAMATION

$$f(\underbrace{a + \Delta x}_{dx}) \approx f(a) + \underbrace{f'(a)\,\Delta x}_{dy}$$

$dy = f'(a)\,dx$

$\frac{dy}{dx} = f'(a)$

$$f(36 + 0.1) \approx f(36) + f'(36)(0.1)$$
$$\approx \sqrt{36} + \frac{1}{2\sqrt{36}}(0.1)$$

$$\approx 6 + \frac{1}{120} \approx 6.008333$$

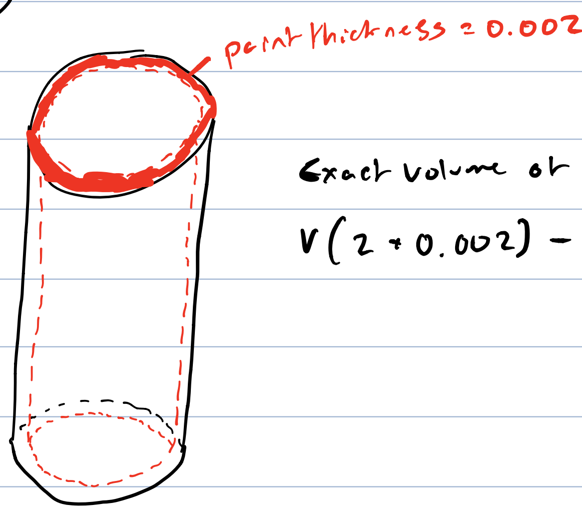
vs. Calculator: $= 6.008327$

November 10, 2016

ex. Estimate the volume of paint needed to paint the outer curved surface of a cylindrical water tank of height 5 m and radius 2 m if 2 coats are required, each $\frac{1}{10}$ cm thick.

$= 0.001$m

$$V(a) = h\pi r^2$$
$$= 5\pi x^2$$



Exact volume of paint needed:

$$V(2 + 0.002) - V(2)$$

$$V(a + \Delta x) \approx V(a) + V'(a)\Delta x$$

Let $a = 2$ & $\Delta x = 0.002$

$$V(2.002) \approx V(2) + V'(2)(0.002)$$
$$V(2.002) - V(2) \approx V'(2)(0.002)$$
$$\approx 10\pi(2)(0.002)$$
$$\approx 0.04\pi \text{ m}^3$$

Factorial Notation

$5! = 5 \times 4 \times 3 \times 2 \times 1$

$2! = 2 \times 1$

$1! = 1$

- $0! = 1$

ex. $\frac{10!}{8!} = \frac{10 \times 9 \times \cancel{8!}}{\cancel{8!}} = 90$

Taylored Polynomials

Given a f^n $y = f(x)$ that has 4 deriv^s, the 4^{th} degree Taylor Polynomial $T_4(x)$ of $y = f(x)$ centered at 'a' is by def^n:

$$T_4(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4$$

Assume $y = f(x)$ has 'n' derivatives $(n = 1, 2, 3, 4, \ldots)$, then the n^{th} degree Taylor Polynomial $T_n(x)$ centered at $x = a$ is defined by

$$T_4(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots \frac{f^{(n)}(a)}{n!}(x-a)^n$$

ex. Find $T_4(x)$ centered at $x=1$ for $f(x) = \ln(x)$

$$T_4(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$
$$+ \frac{f^{(4)}(1)}{4!}(x-1)^4$$

$f(x) = \ln x \rightarrow f(1) = \ln(1) = 0$

$f'(x) = \frac{1}{x}\ x^{-1} \rightarrow f'(1) = 1$

$f''(x) = -\frac{1}{x^2} \rightarrow f''(1) = -1$

$f'''(x) = 2x^{-3} = \frac{2}{x^3} \rightarrow f'''(1) = 2$

$f^{(4)} = -6x^{-4} = -\frac{6}{x^4} \rightarrow f^{(4)}(1) = -6$

So

$$T_4(x) = 0 + \frac{1}{1!}(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{6}{4!}(x-1)^4$$
$$= 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \frac{1}{4}(x-1)^4$$

Use above to estimate $\ln(1.1)$

$T_4(1.1) = (1.1-1) - \frac{1}{2}(1.1-1)^2 \dots$

$= 0.04530833$

Calculator $= 0.04531$

ex. Find $T_3(x)$ centered at $x=0$ for $f(x)=\sin x$

N.B. a Taylor poly centred at $x=0$ is called a Maclaurin poly

$$T_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$f(x) = \sin x \rightarrow f(0) = 0$

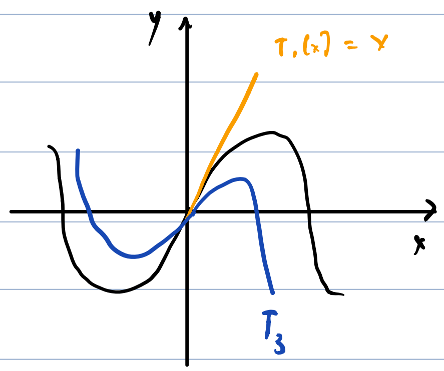
$f'(x) = \cos x \rightarrow f'(0) = \cos(0) = 1$

$f''(x) = -\sin x \rightarrow f''(0) = -\sin(0) = 0$

$f'''(x) = -\cos x \rightarrow f'''(0) = -\cos(0) = -1$

$$T_3(x) = x - \frac{1}{6}x^3$$

$T_3(0.01) \approx \sin(0.01)$



Connection between Taylor & Linear Approx

November 11, 2016

$$T_1(x) = f(a) + \frac{f'(a)}{1!}(x-a)$$

$$= f(a) + f'(a)(x-a)$$

Recall Linear Approx: $f(a+\Delta x) \approx f(a)\,\Delta x$

Let $x = a+\Delta x$ so $\Delta x = x-a$

$f(x) \approx f(a) + f'(a)(x-a) = T_1(x)$