

2A)

### Separable Differential Equations:

A separable DE is a first order ODE, which through algebraic manipulations, can be written in the form:  $g(y)(y') = f(x)$ ,  
the latter being the "separated form"

Theorem:

The following satisfies this DE:

$$\boxed{\int g(y) dy = \int f(x) dx + C}$$

N.B. Remember to add  $+C$  after antiderivative has been determined.

Proof: Let  $G(y) = \int g(y) dy$ . Let  $F(x) = \int f(x) dx$ .

RTP: soln is  $G(y) = F(x) + C$ .

Indeed, differentiating both sides WRT  $x$ , we get

$$\frac{d}{dx} [G(y)] = \frac{d}{dx} [F(x) + C] \Rightarrow g(y) y' = f(x)$$

■

In practice, we start with a DE which may not be separated yet, so we try to separate it:  $g(y)y' = f(x)$ . And we write  $y' = \frac{dy}{dx}$

$\rightarrow g(y) \frac{dy}{dx} = f(x)$

$\rightarrow g(y) dy = f(x) dx$

$\rightarrow \int g(y) dy = \int f(x) dx + C$  (Just like cal II).

E.g. Solve the DE  $9yy' + 4x = 0$

$$9y \frac{dy}{dx} = -4x$$

$$9y dy = -4x dx$$

$$\int 9y dy = -\int 4x dx + C$$

$$\frac{9y^2}{2} = -2x^2 + C$$

$$4x^2 + 9y^2 = \frac{L+C}{2}$$

$$4x^2 + 9y^2 = D$$

Recap:  $\int g(y) y' = f(x)$

$$\int g(y) \frac{dy}{dx} = f(x)$$

$$\int g(x) dy = \int f(x) dx + C$$

Example: Solve  $y' = 1+y^2$

$$\frac{dy}{dx} \left( \frac{1}{1+y^2} \right) = 1$$

$$\int \frac{dy}{1+y^2} = \int dx + C$$

$$\arctan y = x + C \Rightarrow y = \tan(x+C)$$

Example: Solve  $y' = -2xy$

$$\frac{1}{y} \frac{dy}{dx} = -2x$$

$$\int \frac{dy}{y} = \int -2x dx + C$$

$$\ln|y| = -x^2 + C \Rightarrow$$

$$|y| = e^{-x^2 + c}$$

$$y = \underbrace{\pm e^c}_{\text{Any constant}} e^{-x^2} \Rightarrow y = D e^{-x^2}; D \in \mathbb{R}$$

Say that at some  $x = x_0$ , it is given that  $y(x)$  has a prescribed value at  $y_0$ . i.e.  $y(x_0) = y_0$ . This is called an initial condition. A first order ODE with an IC is called an Initial Value Problem (IVP). We solve the DE and find the particular soln that fits the IC.

Example: Solve the IVP

$$y' = y \cot(x), y\left(\frac{\pi}{3}\right) = 2$$

$$\frac{dy}{dx} \frac{1}{y} = \cot x$$

$$\int \frac{dy}{y} = \int \cot x dx + C$$

$$\ln |y| = \int \frac{\cos x}{\sin x} dx + C \quad \begin{aligned} u &= \sin x \\ du &= \cos x \end{aligned}$$

$$\ln |y| = \ln |\sin x| + C$$

$$|y| = e^{\ln |\sin x| + C}$$

$$y = \pm e^C e^{\ln |\sin x|}$$

$$y(x) = \pm C^e \sin x$$

$$y\left(\frac{\pi}{3}\right) = \pm D \sin\left(\frac{\pi}{3}\right) = 2$$

$$2 = D \frac{\sqrt{3}}{2}$$

$$D = \boxed{\frac{4}{\sqrt{3}}}$$

A vous :

$$y'(1+x^2) + y^2 + 1 = 0$$

$$-y'(1+x^2) = y^2 + 1 \quad y(0) = 1$$

$$-\frac{dy}{dx} \frac{1}{y^2+1} = \frac{1}{1+x^2}$$

$$-\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2} + C$$

$$-\arctan y = \arctan x + C$$

$$-y = \tan(\arctan x + C)$$

$$-y = \frac{\tan(\arctan x) + \tan(C)}{1 - \tan(\arctan x)\tan(C)}$$

$$y(x) = -\frac{x + \tan(C)}{1 - x\tan(C)}$$

$$y(0) = 1 = -\frac{0 + \tan C}{1 - 0\tan(C)}$$
$$-1 = \tan(C)$$

$$y = -\frac{x - 1}{1 + x}$$

$$\boxed{y = \frac{1-x}{1+x}}$$

2B)

### Applications: Modeling Involving Separable Equations

Newton's Law of Cooling:

An object at temperature  $T$  in a surrounding medium at (constant) temperature  $T_m$  loses heat at a rate proportional to the difference of temperatures. i.e.  $T'(t) \propto T - T_m$

Example: A hot copper ball at  $100^{\circ}\text{C}$  is placed in water maintained at  $30^{\circ}\text{C}$ . After 3 min, the ball is at  $78^{\circ}\text{C}$ . When will it be at  $31^{\circ}\text{C}$

$$T'(t) = K(T - T_m)$$

$$\frac{dT}{dt} = K(T - T_m)$$

$$\int \frac{dT}{T - T_m} = \int K dt + C$$

$$\ln |T - T_m| = kt + C$$

$$|T - T_m| = e^{kt+C}$$

$$T - T_m = e^C e^{kt}$$

$$T(t) = D e^{kt} + T_m$$

$$\text{sub } T(0) = 100$$

$$100 = D e^0 + T_m$$

$$D = 100 - T_m = 100 - 30 = 70$$

$$T(t) = 70 e^{kt} + 30$$

$$T(3) = 78 = 70 e^{3k} + 30$$

$$70 e^{3k} = 48$$

$$k = \frac{\ln(\frac{48}{70})}{3} \approx -0.1258$$

$$T(t) = 70 e^{-0.1258t} + 30$$

$$T(t) = 31 = 70 e^{-0.1258t} + 30$$

$$\frac{1}{70} = e^{-0.1258t}$$

$$-4.25 = -0.1258t$$

$$t = 33.77 \text{ min}$$

Example:  $y' = 6x(y-1)^{2/3}$

Assume  $y=1$  is not a soln (otherwise dividing by zero!)

$$\int \frac{dy}{(y-1)^{2/3}} = \int 6x \, dx$$

$$3 \cdot \sqrt[3]{y-1} = 3x^2 + C$$

$$y = (x^2 + D)^3 + 1.$$

The fact  $y=1$  is a soln to the DE not included in our general soln (i.e. it is a singular soln). We neglected the existence of  $f''|_{y=1}$  when we divided both sides by  $(y-1)^{2/3}$ .

Q: Can a separable ODE have singular solutions?

A: Yes (see above example and explanation)

Q: Can separated ODEs have singular solns?

A: No! We get  $\int g(y) \, dy = \int f(x) \, dx + C$ , which we know (proof in Ch1) that this must include all solns to the ODE.  $\therefore$  Separated DEs have no singular solutions.

Torricelli's Law:

First, what is the velocity of a particle falling freely (initially at rest) for a distance  $h$ ?

$$x(t) = x_0 + v_0 t + \frac{1}{2} g t^2$$

What is the velocity after having fallen  $h$  meters?

$$x(t) = \frac{1}{2} g t^2$$

$$v(t) = \frac{dx}{dt} = \frac{d}{dt} (x_0 + v_0 t + \frac{1}{2} g t^2)$$

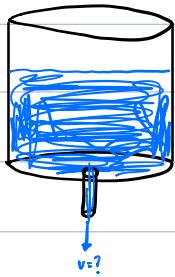
$$v(t) = v_0 + gt$$

$$v(t) = \sqrt{\frac{2h}{g}}$$

$$v(t) = \sqrt{2gh}$$

Now, let's consider a tank of water with an orifice from which water (vodka?) is flowing.

Torricelli's Law states that the velocity with which water issues from the orifice is proportional to  $\sqrt{2gh}$  (where  $h$  is the height of the  $H_2O$ )



$$\text{In the case of water, } v = 0.6 \sqrt{2gh}$$

Example: A cylindrical tank 1.5m high stands on its circular base of diameter 1m. A hole at the bottom of diameter 1cm is opened at some instant.

a) Find the height of the water  $h(t)$  at any time  $t$ .

b) Find the times at which the tank is half full, one quarter full, and empty.

At time  $\Delta t$ ,  $\Delta V$  <sup>volume</sup> of the tank is  $\Delta V = -v \Delta t \pi r^2$

$$\Delta V = -0.6 \sqrt{2gh} \pi r^2 \Delta t$$

$$\cancel{\pi} R^2 \Delta h = -0.6 \sqrt{2g} h \cancel{\pi} r^2 \Delta t$$

$$\frac{\Delta h}{\Delta t} = -\frac{3\sqrt{2g} h r^2}{5 R^2} \Rightarrow \frac{dh}{dt} = \left(-\frac{3\sqrt{2g}}{5}\right) \left(\frac{0.01m}{1m}\right)^2 \sqrt{h} \Rightarrow \frac{dh}{dt} = 8\sqrt{h}, \text{ which is an easy separable ODE}$$

Solving this ODE,

$$\int \frac{dh}{\sqrt{h}} = 8 \int dt$$

$$2\sqrt{h} = 8t$$

$$h(t) = \left(\frac{8t}{2}\right)^2$$

### Escape Velocity:

Consider a projectile fired upwards from the ground with initial velocity  $v_0$ .  $R_E \approx 6.372 \cdot 10^6 \text{ m}$

a) Write a DE for the velocity of the projectile as a function of its distance  $r$  from the center of the earth

b) Solve the DE to obtain velocity at a distance  $r$  (obviously  $r > R_E$ )  
in terms of the initial velocity  $v_0$ .

c) Use the above to find the value of  $v_0$  corresponding to the escape velocity

$$F_g \propto \frac{1}{r^2} \Rightarrow a \propto \frac{1}{r^2} \Rightarrow a = \frac{k}{r^2}$$

Initial Condition: when  $r = R_E$ ,  $a_y = -g$

$$-g = \frac{k}{R_E^2} \Rightarrow k = -g R_E^2$$

So,  $\frac{dv}{dr} = \frac{-g R_E^2}{r^2}$ , but we want  $v(r)$ . We reparametrize velocity as  $v(r(t))$ .

$$\therefore \underbrace{\frac{dv}{dr}}_v \frac{dr}{dt} = \frac{-g R^2}{r^2} \quad (\text{by chain rule})$$

$$v \frac{dv}{dr} = \frac{-g R^2}{r^2}$$

$$b) \int v dv = - \int \frac{g R^2}{r^2} dr + C$$

$$v_0 = \sqrt{\frac{2g R^2}{r} + C}$$

$$v^2 = 2gR + C$$

$$C = v_0^2 - 2gR$$

$$\frac{v^2}{2} = \frac{g R^2}{r} + C$$

$$v^2 = \frac{2g R^2}{r} + C$$

$$v(r) = \sqrt{\frac{2g R^2}{r} + C}$$

$$v(r) = \sqrt{\frac{2g R^2}{r} + v_0^2 - 2gR}$$

$$\lim_{r \rightarrow \infty} v(r) = \lim_{r \rightarrow \infty} \sqrt{\frac{2g R^2}{r} + v_0^2 - 2gR}$$

$$\text{i.e. } v(R) = v_0$$

$$v_{\infty} = \sqrt{v_0^2 - 2gR} \Rightarrow v_0 \geq \sqrt{2gR} \text{ for } v_{\infty} \text{ to exist}$$

Finishing off section 2C on reduction to separable form.

Sometimes we may find a relevant substitution to make just by inspection...

Ex: Solve  $(2x-4y+5)y' + x+2y+3 = 0$

$$\text{Let } u = x+2y+3 \Rightarrow u' = 1+2y' \Rightarrow y' = \frac{1-u'}{2}$$

$$(2u-1)y' + u = 0$$

$$\text{Rewrite as: } (2u-1)\left(\frac{1-u'}{2}\right) + u = 0$$

$$(2u-1)(1-u') = -2u$$

$$2u - 2uu' - 1 + u' = -2u$$

$$u' - 2uu' = -4u + 1$$

$$u'(1-2u) = 1-4u$$

$$u' = \frac{1-4u}{1-2u}$$

$$\frac{du}{dx} = \frac{1-4u}{1-2u}$$

$$\int \frac{1-2u}{1-4u} du = \int dx$$

$$\int \frac{\frac{1}{2}(2-4u)}{1-4u} du = x+C$$

$$\frac{1}{2} \left( \int \frac{1}{1-4u} du + \int \frac{1-4u}{1-4u} du \right) = x+C$$

$$\frac{1}{2} \left( \frac{\ln|1-4u|}{-4} + u \right) = x+C$$

$$C = \ln|1-4u| - 8u + 8x$$

$$C = \ln|(1-4(x+2y+3))| - 8(x+2y+3) + 8x$$

$$\ln|1-4(x+2y+3)| - 8(x+2y+3) + 8x - C = 0$$

Disgusting. But this is the soln  
to the DE.

Closing section 2D on partial derivatives:

Ex:  $f(x, y, z) = x \sin(xz) + y^2 z$ . Show that  $f_{xz} = f_{zx}$   
: easy shit

Ex: Suppose a fun  $f(x, y)$  satisfies  $f_x = 4y^2 + 9x^2$  and  $f_y = 8xy + 3\sqrt{y}$   
&  $f(0, 4) = 14$

$$\frac{\partial f}{\partial x} = 4y^2 + 9x^2$$

$$\frac{\partial f}{\partial y} = 8xy + 3\sqrt{y}$$

$$f(x, y) = \int 4y^2 + 9x^2 dx + C$$

$$f(x, y) = \int 8xy + 3\sqrt{y} dy + C$$

$$f(x, y) = 4xy^2 + 3x^3 + h(y) + C$$

$$f(x, y) = 4xy^2 + 2y^{3/2} + g(x) + C$$

$$g(x) = 3x^3; \quad h(y) = 2y^{3/2}$$

$$f(x, y) = 4xy^2 + 3x^3 + 2y^{3/2} + C$$

$$f(0, 4) = 14 = 4(0)^2(4)^2 + 3(0)^3 + 2(4)^{3/2} + C$$

$$14 = C + 16$$

$$\boxed{C = -2}$$

$$\therefore f(x, y) = 4xy^2 + 3x^3 + 2y^{3/2} - 2$$

## 2E) Exact Differential Equations

Consider:  $M(x, y) dx + N(x, y) dy = 0$

If  $M(x, y)$  and  $N(x, y)$  happen to be

a  $\frac{\partial u}{\partial x}$  and a  $\frac{\partial u}{\partial y}$  for some  $u(x, y)$ , (total differential from cal III)

Then the DE  $\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$  i.e.  $du = 0$

\* Total diff'l corresponding to zero is a plane that runs // to xy plane.

This means that  $u(x, y) = C$ , so  $u(x, y) = C$  is an implicit eqn in terms of x.

Given the DE:

$$\textcircled{*} M(x,y) dx + N(x,y) dy = 0$$

We hope to find some  $u(x,y)$  such that  $\frac{\partial u}{\partial x} = M$  &  $\frac{\partial u}{\partial y} = N$ .

We then say that the DE  $\textcircled{*}$  is in exact form.

So the DE is  $\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$  i.e.  $du = 0$ , which means both is

$$u(x,y) = C$$

Or simply  $u(x,y) = 0$  iff the arbitrary constant  $+C$  is zero.

Q: Given a DE  $\textcircled{*}$ , how do you know if such a  $u(x,y)$  can be found?

A: We need  $M_y = N_x$  to

Test of Exactness:  $M(x,y) dx + N(x,y) dy = 0$  is exact iff

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \text{ which shows... } \textcolor{red}{\leftarrow \text{How come?}}$$

Example:

$$\text{Solve } \underbrace{2x \sin(3y)}_{M(x,y)} dx + \underbrace{(3x^2 \cos(3y) + 2y)}_{N(x,y)} dy = 0$$

Fuck trying to make that separable. Let's test exactness.

$$\frac{\partial M}{\partial y} = 6x \cos(3y), \quad \frac{\partial N}{\partial x} = 6x \cos(3y).$$

$$\therefore \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \quad \therefore \text{We have an exact ODE.}$$

We want  $u(x,y)$  such that:

$$\frac{\partial u}{\partial x} = (2x \sin 3y) dx$$

$$\frac{\partial u}{\partial y} = (3x^2 \cos 3y + 2y) dy$$

$$\int \frac{\partial u}{\partial x} dx = \int 2x \sin 3y dx$$

$$\int \frac{\partial u}{\partial y} dy = \int (3x^2 \cos 3y + 2y) dy$$

$$u(x, y) = x^2 \sin(3y) + g(y) + c \quad | \quad u(x, y) =$$

$$u(x, y) = x^2 \sin(3y) + y^2 + c$$

$$\text{But } du=0 \Rightarrow \int du = c$$

$$\cancel{c} = x^2 \sin(3y) + y^2 + \cancel{c}$$

$$x^2 \sin(3y) + y^2 = 0$$

$$\text{Another e.g. } \frac{2y}{x} dy - \frac{y^2}{x^2} dx = 0$$

Hyperbolic Functions:

$$\sinh(x) \equiv \frac{e^x - e^{-x}}{2}; \quad \cosh(x) \equiv \frac{e^x + e^{-x}}{2}; \quad \tanh(x) \equiv \frac{\sinh(x)}{\cosh(x)}$$

$$\sinh^2(x) - \cosh^2(x) = ?$$

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \left(\frac{e^{2x} + 2e^{-x}e^x + e^{-2x}}{4}\right) - \left(\frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4}\right)$$

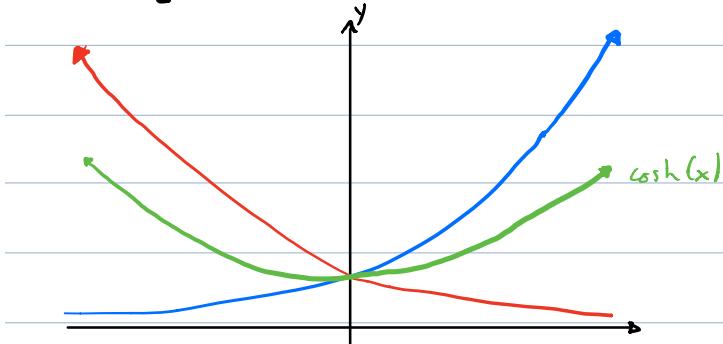
$$= \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}}}{4} + \frac{-\cancel{e^{2x}} + 2 - \cancel{e^{-2x}}}{4}$$

$$= \frac{1}{2} + \frac{1}{2} = 1. \quad \therefore \sinh(x) - \cosh(x) = 1$$

$$\frac{d}{dx} [\sinh(x)] = \cosh(x) \quad ; \quad \frac{d}{dx} [\cosh(x)] = \sinh(x)$$

$\frac{A+B}{2}$  is the average of A and B

$\cosh(x) = \frac{e^x + e^{-x}}{2}$  is the average of  $e^x$  and  $e^{-x}$ .

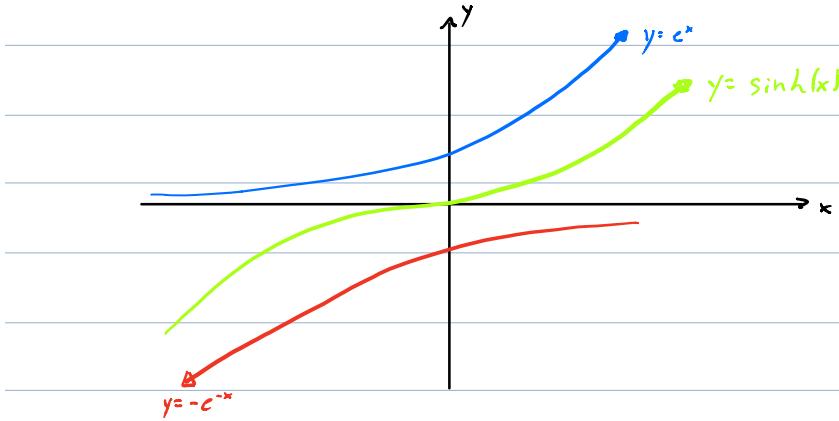


Example:



Wire hanging between lamp posts  
(or just w/ 2 fixed ends) has shape of  
 $\cosh(x)$

$y = \sinh(x) = \frac{e^x - e^{-x}}{2}$  is the average of  $e^x$  and  $-e^{-x}$ .



Hyperbolic functions "share" a lot of identities and calculus properties  
with the regular trig. funcs.

So far, we've seen that:

$$1) \cosh^2(x) - \sinh^2(x) = 1$$

$$2) \frac{d}{dx} [\sinh(x)] = \cosh(x)$$

$$3) \frac{d}{dx} [\cosh(x)] = \sinh(x)$$

But there are many others:

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

$$\frac{d}{dx} [\tanh(x)] = \operatorname{sech}^2(x)$$

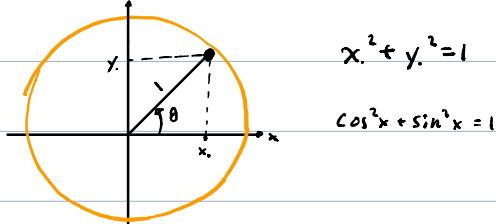
plus!

Hyperbolic Functions are linked to hyperbolas but are not hyperbolas themselves!

Regular trigonometric functions are sometimes called "circular functions"

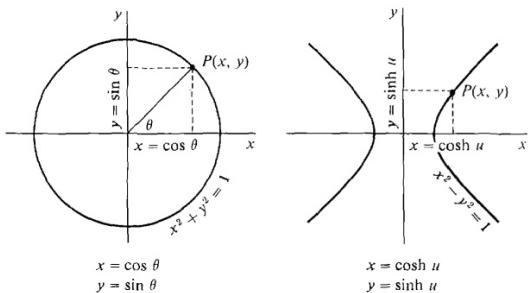
Trigonometric identities are sometimes called "circular identities"

Trig vs. circle:



Now with  $\cosh(x)$  and  $\sinh(x)$ ...

If a function has  $x = \cosh(t)$ ;  $y = \sinh(t)$ .  $x^2 - y^2 = 1$ , so  $\cosh^2 x - \sinh^2 x = 1$  and  $(\cosh(t), \sinh(t))$  must lie on a hyperbola.



Solve the initial value problem:

$$\sin x \cosh y - y' \cos x \sinh y = 0$$

$$\sin x \cosh y - \frac{dy}{dx} \cos x \sinh y = 0 \quad \text{multiply through by } dx$$

$$\underbrace{\sin x \cosh(y)}_M dx - \underbrace{\cos x \sinh(y)}_N dy = 0$$

Let's test exactness:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} [\sin x \cosh y] = \sin x \sinh(y) \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x} [-\cos x \sinh(y)] = \sin x \sinh(y).$$

Exact ODE!

$$u_x = \sin x \cosh y$$

$$u_y = -\cos x \sinh y$$

$$u(x, y) = \int \sin x \cosh(y) dx + C$$

$$u(x, y) = \int -\cos x \sinh(y) dy + C$$

$$u(x, y) = -\cos x \cosh(y) + h(y) + C$$

$$u(x, y) = -\cos x \cosh(y) + g(x) + C$$

$$g(x) = 0, \quad h(y) = 0$$

$$u(x, y) = -\cos x \cosh(y) + C$$

$$\text{But } u(x, y) = 0 \Rightarrow 0 = -\cos x \cosh y + C$$

$$\text{l.c. } y(0) = 0$$

$$0 = -\cos(0) \cosh(0) + C$$

$$\Rightarrow C = 1$$

$\therefore 1 = \cos x \cosh y$  is soln of DE

## 2F) INTEGRATING FACTORS (Bringing into exact form)

$\textcircled{*} \sqrt{y} dx - \frac{x}{\sqrt{y}} dy = 0$  is not exact.  $\frac{\partial}{\partial y} \sqrt{y} = \frac{1}{2\sqrt{y}}$ ;  $\frac{\partial}{\partial x} \frac{x}{\sqrt{y}} = \frac{1}{\sqrt{y}}$

$$\sqrt{y} \neq \frac{1}{\sqrt{y}}! \therefore \text{Not exact}$$

However, if we multiply the DE  $\textcircled{*}$   $F(x,y) = \frac{\sqrt{y}}{x^2}$ , we obtain:

$$\frac{y}{x^2} dx - \frac{1}{x} dy = 0$$

Now test exactness:  $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} \left[ \frac{y}{x^2} \right] = \frac{1}{x^2}$ ;  $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \left[ -\frac{1}{x} \right] = \frac{1}{x^2}$   $\text{ID}$

We now have an exact ODE! We can solve accordingly.

Given the ODE of form  $P(x,y)dx + Q(x,y)dy = 0$ , we may not have an exact form  $\because P_y \neq Q_x$ . But we will try to find  $F(x,y)$  such that  $\overbrace{FP(x,y)}^M dx + \overbrace{FQ(x,y)}^N dy = 0$  becomes exact, i.e.  $M_y = N_x$ .

We will do a few more examples of guessing to build intuition and understanding. Then, we will figure out one or two formulas that may sometimes work to find integrating factors.

Example:  $x^2y^3 dx + x(1+y^2) dy = 0$  is not exact

$$\frac{\partial L_0}{\partial y} = 3x^2y^2 \quad | \quad \frac{\partial L_0}{\partial x} = 1+y^2$$

Let's try to find  $F(x,y)$  so that

$$Fx^2y^3 dx + Fx(1+y^2) dy = 0.$$

$$\text{We want } M_y = \frac{\partial}{\partial y} [Fx^2y^3] = N_x = \frac{\partial}{\partial x} [Fx(1+y^2)]$$

We can try to make the  $y$ 's go away in  $\frac{\partial}{\partial y}$  and the  $x$ 's go away in the  $\frac{\partial}{\partial x}$ , that way we would be taking derivatives of constants in both cases.  $F = \frac{1}{xy^3}$  works for this purpose

$$M_y = \frac{\partial}{\partial y} \left[ \frac{1}{xy^3} x^2 y^3 \right] = \frac{\partial}{\partial y} [x] = 0$$

$$N_x = \frac{\partial}{\partial x} \left[ \frac{1}{xy^3} x(1+y^2) \right] = \frac{\partial}{\partial x} \left[ \frac{1+y^2}{y^3} \right] = 0 \quad \square$$

$N_x = M_y \therefore$  exact ODE.

Let's solve:

$$\frac{1}{xy^3} x^2 y^3 dx + \frac{1}{xy^3} x(1+y^2) dy = 0 \cdot \frac{1}{xy^3}$$

$$x dx + (y^{-3} + y^{-1}) dy = 0$$

$$u(x,y) = \int x dx + C$$

$$u(x,y) = \int y^{-3} + y^{-1} dy + C$$

$$u(x,y) = \frac{x^2}{2} + h(y) + C$$

$$u(x,y) = \frac{y^{-2}}{-2} + \ln|y| + g(x) + C$$

$$h(y) = \frac{y^{-2}}{-2} + \ln|y| ; \quad g(x) = \frac{x^2}{2}$$

$$\therefore u(x,y) = \frac{x^2}{2} - \frac{1}{2y^2} + \ln|y| + C.$$

Put  $u=0$  and multiply through by 2.

$C = x^2 - y^{-2} + \ln(y^2)$  is an implicit soln for the DE.

$$\text{Example: Solve } y + (2x - ye^y) \frac{dy}{dx} = 0.$$

Hmm... Not separable, not exact. Integrating factor?

We want  $yF dx + F(2x - ye^y) dy = 0$ ,

$$\text{so we need } \frac{\partial}{\partial y} [yF] = \frac{\partial}{\partial x} [2x - ye^y].$$

Trick: Let's think of a simpler form than  $F(x,y)$ . Let's try to find an  $F$  that would have no  $x$ :  $F = F(y)$  only.

$$\frac{\partial}{\partial y} [y F(y)] = \frac{\partial}{\partial x} [(2x - ye^y) F(y)] \text{ expand product rule,}$$

$$F(y) + y F'(y) = 2F(y)$$

$$y F'(y) = F(y)$$

$$y \frac{dF}{dy} = F(y)$$

$$\int \frac{dF}{F} = \int \frac{dy}{y} + C \dots \text{Find } F \text{ and solve exact DE.}$$

$$\text{Solv of DE: } xy^2 + e^y(-y^2 + 2y - 2) = C$$

Sometimes, integrating factors are hard to find.

$$x + (x+2) \sin y F^? dx + x \cos y F^? dy = 0$$

Very hard to guess,  $F = xe^x$  turns out to work, as you could verify.

So we need actual techniques to find integrating factors...

Consider  $P(x, y) dx + Q(x, y) dy = 0$

or simply  $P dx + Q dy = 0$

If it is not exact, i.e.  $P_y \neq Q_x$

We need an  $F(x, y)$  |  $\underbrace{FP}_m dx + \underbrace{FQ}_n dy = 0$  becomes exact.

$$M_x = N_y : \text{We want } \frac{\partial}{\partial y} [FP] = \frac{\partial}{\partial x} [FQ] \quad \textcircled{*}$$

Let's hope to find some  $F$  that would depend on  $x$  only.

So  $F = F(x) \rightarrow \text{no } y$ . Then the requirement from  $\textcircled{*}$  becomes

$$\frac{\partial}{\partial y} \left[ \underbrace{F}_1 \underbrace{P(x,y)}_1 \right] = \frac{\partial}{\partial x} \left[ \underbrace{F}_1 \underbrace{Q(x,y)}_1 \right] \Rightarrow FP_y = F_x Q + F Q_x$$
$$FP_y - F Q_x = F' Q$$

$$\boxed{\frac{P_y - Q_x}{Q} = \frac{F'}{F}}$$

In problem sets, derive a similar formula for  $F'_y$ .

Careful!  $\frac{F'}{F}$  has only  $x$ , so we need  $\frac{P_y - Q_x}{Q}$  to be only  $x$ .

$$\text{Then, } \int \frac{P_y - Q_x}{Q} dx = \int \frac{dF}{F}$$

:

Get one  $F$ , never mind the "C"

Example: Solve  $\overbrace{(4x + 3y^2)}^P dx + \overbrace{2xy}^Q dy = 0$

Not separable, not exact.  $P_y = 6y$ ;  $Q_x = 2y$

$$\text{Now } \frac{P_y - Q_x}{Q} = \frac{6y - 2y}{2xy} = \frac{4}{2x} = \frac{2}{x}$$

$$\therefore \frac{F'}{F} = \frac{2}{x} \rightarrow \int \frac{2}{x} dx = \int \frac{dF}{F}$$

$$2\ln|x| = \ln|F| \quad \checkmark \text{ Fuck the } C \text{ doesn't matter here}$$

$$\ln(x^2) = \ln|F|$$

$|F(x) = x^2|$  - This is our integrating factor

$$x^2(4x + 3y^2)dx + x^2(2xy)dy = 0$$

$$(4x^3 + 3x^2y^2)dx + 2x^3y dy = 0 \text{ should be exact. Test exactness:}$$

$$\frac{\partial}{\partial y}[4x^3 + 3x^2y^2] = 6x^2y ; \quad \frac{\partial}{\partial x}[2x^3y] = 6x^2y. \quad \text{It works!}$$

$$f_x = 4x^3 + 3x^2y^2 dx \quad ; \quad f_y = 2x^3y dy$$

$$f(x,y) = \int 4x^3 + 3x^2y^2 dx \quad ; \quad f(x,y) = \int 2x^3y dy$$

$$f(x,y) = x^4 + x^3y^2 + h(y) + C \quad ; \quad f(x,y) = x^3y^2 + g(x) + C$$

$$g(x) = x^4$$

$$f(x,y) = x^3y^2 + x^4 + C$$

$$\text{But } f(x,y) = 0$$

$C = x^3y^2 + x^4$  is the solution to the DE

## 2G) Linear Differential Equations

(We will deal mostly with the first order case in this section, but we first introduce arbitrary orders)

Defn: A differential equation of order  $n$  in the unknown function  $y(x)$  is said to be linear provided the func in question is a linear fun (highest power is 1) in  $y$  and its derivatives:  $y(x), y', y'', \dots, y^{(n)}$

Note: The function does not need to be linear in  $x$ , linearity is determined by power of  $y$ .

For example:

1)  $x^2 y''' + \sin x y'' + e^x y' - y + \sqrt{x} = 0$  is a linear DE

2)  $y'' + \cancel{y'} = x$  Degree higher than 1 for  $y$ , so NOT linear.

3) The first order DE:

(A)  $x y y' + y^2 + 4y = 0$  is not linear,

but we can  $\div$  through by  $y$  to get (B)

(B)  $x y' + y + 4 = 0$

(B) is separable, could also be written in exact form.

We can find the solution to be:

$$y = \frac{C}{x} - 4$$

Note that  $y=0$  is a singular soln ( $\because$  we  $\div$  by  $y$  to get from (A) to (B)) for the DE in (A).

Linear DEs of order  $n$  can be written as:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \dots + p_1(x)y' + p_0(x)y = r(x).$$

However, when  $r(x) \equiv 0$ , we say that the linear fun is homogeneous.

### Linear First Order Differential equations:

These can be written in the form:

$$y' + p(x)y = r(x)$$

In the homogeneous case,  $r(x) \equiv 0$ , so the eqn is separable.

If  $p(x) \equiv 0$ , then we can solve by integrating

So now, lets assume  $r(x)$  and  $p(x)$  are not identically zero.

$$\frac{dy}{dx} = r(x) + p(x)y$$

$$dy - [r(x) + p(x)y]dx = 0$$

Compare to  $Pdx + Qdy = 0$

$$P_y = \frac{\partial}{\partial y} [r(x) + p(x)y] = p(x) \}$$

$$Q_x = \frac{\partial}{\partial x} [1] = 0 \}$$

This is not exact, so lets find

an integrating factor:

$$\frac{F'}{F} = \frac{P_y - Q_x}{Q}$$

$$\frac{F'}{F} = \frac{p(x) - 0}{1}$$

$$\int \frac{dF}{F} = \int p(x) dx + C$$

$$\ln|F| = \int p(x) dx \quad \text{No need for } C \text{ integrating factor}$$

$$F = e^{\int p(x) dx} \text{ is a possibility.}$$

Let  $h(x) = \int p(x) dx$

$e^h$  is an integrating factor where  $h' = p$ .

$$y' + p(x)y = r(x) \text{ becomes } e^h y' + e^h p y = e^h r$$

However, this is a  $\frac{d}{dx} [e^h \cdot y]$  (product rule)

$$\therefore \frac{d}{dx} [e^h y] = e^h r(x)$$

$$e^h y = \int e^h r(x) dx + C$$

$$y = e^{-h} \left[ \int e^h r(x) dx + C \right] \text{ where } h(x) = \int p(x) dx$$

Back to  $xy' + y + 4 = 0$  with general solution  $y = \frac{C}{x} - 4$   
separable, exact, ...

Here, look at it as linear...

$$\div \text{ through by } x \Rightarrow y' + y \left( \frac{1}{x} \right) = -\frac{4}{x}$$

$$\text{Soh is: } h(x) = \int p(x) dx = \int \frac{1}{x} dx = \ln|x| + C \quad \text{no need for } +C$$

$$\therefore y(x) = e^{-h(x)} \int e^{h(x)} r(x) dx = e^{-\ln|x|} \int e^{\ln|x|} \left( -\frac{4}{x} \right) dx + C$$

$$= \frac{1}{|x|} \int |x| \left( -\frac{4}{x} \right) dx + C \quad \begin{array}{l} x \text{ both } \oplus \text{ or both } \ominus \\ \Rightarrow \text{can get rid of absolute values} \end{array}$$

$$= \frac{1}{x} \int x \cdot -\frac{4}{x} dx + C$$

$$= \frac{1}{x} \int -4 dx + C$$

$$= \frac{1}{x} \left[ (-4x) + C \right]$$

$$y(x) = \frac{C}{x} - 4 \quad \checkmark$$

Review:

A Linear ODE of order 'n' can be written as:

$$\underbrace{y^{(n)}}_{\substack{\text{degree of} \\ \text{derivative}}} + P_{n-1}(x)y^{(n-1)} + \dots + P_1(x)y' + P_0(x) = r(x)$$

\* If  $r(x) = 0 \Rightarrow$  "homogeneous" ODE

First order case:

$$y' P(x) + P_0(x) = r(x)$$

Is solved by  $y(x) = e^{-h(x)} \left[ \int e^{h(x)} r(x) dx + C \right]$  where  $h(x) = \int P(x) dx$

Ex: Solve  $y' + y \tan x = \sin 2x$  on  $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ . Initial condition:  $y(0) = 1$ .

Linear DE! Form  $y' + y P(x) = r(x)$ ;  $h(x) = \int P(x) dx$

$$\text{Here, } h(x) = \int \tan x dx \Rightarrow h(x) = -\ln |\cos x|$$

$$\text{Solt: } e^{-h} \left[ \int e^{-h} r(x) dx + C \right]$$

$$e^{\ln |\cos x|} \int e^{-\ln |\cos x|} \sin(2x) dx + C$$

$$\cos x \left[ \int \frac{2 \sin x \cos x}{\cos x} dx + C \right]$$

$$\cos x [-2 \cos x + C]$$

$$y(x) = -2 \cos^2 x + C \cos x$$

$$\text{l.c. } y(0) = 1$$

$$1 = -2 \cos^2(0) + C \cos(0)$$

$$1 = -2 + C$$

$$C = 3$$

$$\therefore y(x) = -2 \cos^2 x + 3 \cos x$$

Reduction to linear form:

Substitution may sometimes be found to reduce a 1<sup>st</sup> order ODE to linear form (in  $u(x)$  and its derivatives).

Sometimes, we may guess substitutions by inspection.

A standard case of substitution reducing to linear form is the following:

ODE's of the form  $y' + p(x)y = r(x)y^a$  ( $a \neq \{0, 1\} \Rightarrow$  non linear ODE)  
are called Bernoulli equations.

It turns out that the substitution  $u = y^{1-a}$  brings Bernoulli equations into linear form in  $u(x)$ .

Example:  $y' = x^3y^2 + xy$

$$\Rightarrow y' + y \underbrace{\frac{(-x)}{p(x)}}_{r(x) y^a} = \underbrace{x^3 y^2}_{(Bernoulli\ equation)} \quad a=2$$

$$\text{Let } u = y^{1-2} \Rightarrow u = \frac{1}{y}$$

$$-\frac{u'}{u^2} - \frac{x}{u} = \frac{x^3}{u^2}$$

$$u' + \frac{p(x)}{r(x)}u = \frac{r(x)}{-x^3}, \quad \text{linear ODE!}$$

$$u' = -\frac{1}{y^2} \cdot y' \Rightarrow u' = -u^2 y'$$

$$\Rightarrow y' = -\frac{u'}{u^2}$$

$$h(x) = \int p(x) dx = \int x dx$$

$$h(x) = \frac{x^2}{2}$$

$$\text{Sln to DE: } u = e^{-h} \left[ \int e^h r(x) dx + C \right]$$

$$u = e^{-\frac{x^2}{2}} \left[ \int e^{\frac{x^2}{2}} \cdot x^3 dx + C \right] \quad \begin{matrix} t = \frac{x^2}{2} \\ dt = x dx \end{matrix}$$

$$u = e^{-\frac{x^2}{2}} \left[ - \int e^t \cdot \frac{x^2}{2} \frac{dt}{x} + C \right]$$

$$u = e^{-\frac{x^2}{2}} \left[ \int e^t \cdot 2t dt + C \right]$$

$$u = e^{-\frac{x^2}{2}} \left[ C - 2 \underbrace{\int t e^t dt}_I \right]$$

$$I = \int t e^t dt \quad u=t \quad v=e^t \\ du=dt \quad dv=e^t dt$$

$$= t e^t - \int e^t dt$$

$$= t e^t - e^t$$

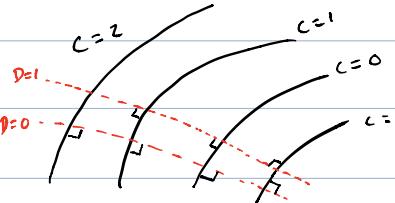
$$\therefore u = e^{-\frac{x^2}{2}} \left[ C - 2 \left( \frac{x^2}{2} e^{\frac{x^2}{2}} - e^{\frac{x^2}{2}} \right) \right]$$

$$u = C e^{-\frac{x^2}{2}} - x^2 + 2$$

$$\frac{1}{y(x)} = C e^{-\frac{x^2}{2}} - x^2 + 2$$

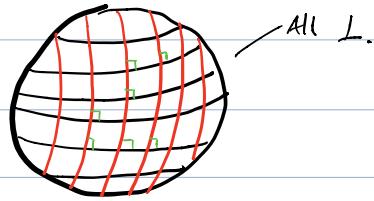
$$y(x) = \frac{1}{C e^{-\frac{x^2}{2}} - x^2 + 2}$$

## 2H) Application To Orthogonal Trajectories:



All  $C^s$  are  $\perp$  to all  $D^s$ .

E.g. On earth, the longitude geodesics are  $\perp$  to latitude's geodesics.



E.g. In a topographical map (level curve plot), the trajectory is a curve that is  $\perp$  to all the equipotential lines (or level curves)

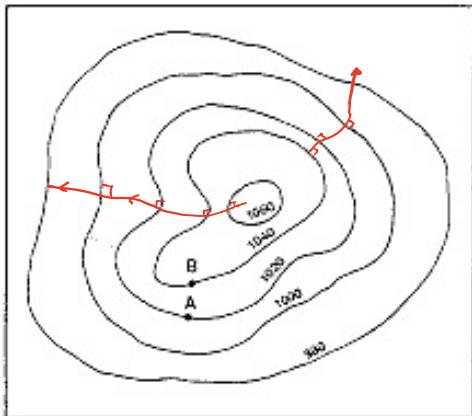
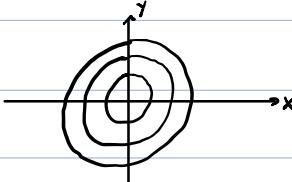


Figure E-1: Isolated Hill

Recall that first order DEs have one parameter family of curves as their soln. What we want to do is go the other way...

Given a family of curves, find the DE for which this family is the soln.

$$\text{For example: } x^2 + y^2 = c^2$$



Given this family of curves, we can find the desired DE by implicit diffn of the fam

$y' = -\frac{x}{y}$  is the DE solved by the circle...

$$\text{Ansatz: } \frac{dy}{dx} = -\frac{x}{y} \quad \Rightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$\int y dy = -\int x dx + C \quad x^2 + y^2 = C \quad \text{ID}$$

General strategy:

- ① Given a family of curves
- ② Find a DE for which this family of curves is the general soln
- ③  $y'$  of first family  $\mapsto$   $(-\frac{1}{y'})$   $y'$  of second family
- ④ Find the family of orthogonal trajectories

$$\text{E.g. } x^2 + y^2 = c^2 \rightarrow y' = -\frac{x}{y} \Rightarrow -\frac{1}{y'} = \frac{y}{x} \stackrel{\text{④}}{\Rightarrow} y'_{\text{second fam}} = \frac{x}{y}$$

$$\text{Solve ④: } \int \frac{dy}{y} = \int \frac{dx}{x} + C$$

$$\ln|y| = \ln|x| + C$$

$$y = Cx \quad \text{— all straight lines through the origin!}$$

Example: Consider the family of parabolas  $y = Cx^2$ . Find its orthogonal trajectories.

$$\text{Given } y = Cx^2 \rightarrow C = \frac{y}{x^2}$$

$$y' = 2Cx \quad (\text{get rid of the } C!) \\ \text{sub } C = \frac{y}{x^2}$$

$$y' = 2\left(\frac{y}{x^2}\right)x$$

$$y' = 2\frac{y}{x}$$

$$\cdot \perp : y' = -\frac{x}{2y} \quad \text{To find orthogonal family:}$$

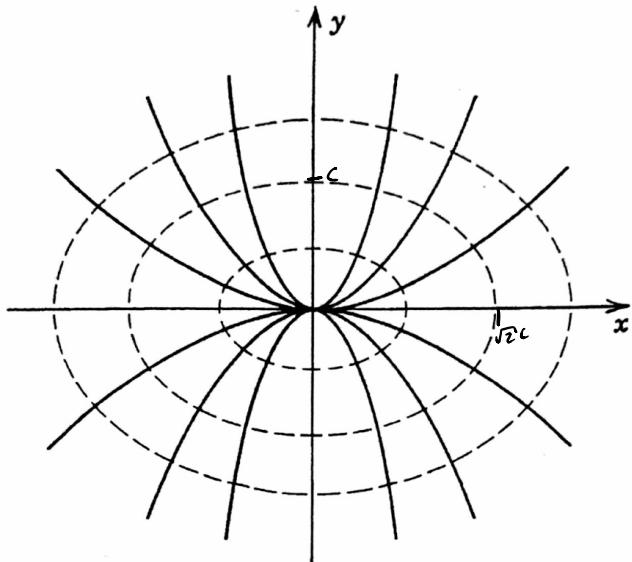
$$\text{Solve } \frac{dy}{dx} = -\frac{x}{2y}$$

$$\int 2y \, dy = -\int x \, dx + C$$

$$y^2 = -\frac{x^2}{2} + C = -\frac{x^2}{2} + C^2$$

$$\frac{x^2}{(2C)^2} + \frac{y^2}{C^2} = 1$$

Sketch:



E.g. Find the orthogonal trajectories of the circles

$$\textcircled{1} \quad x^2 + (y - c)^2 = c^2$$

$$\textcircled{1} \quad 2x + 2(y - c)y' = 0$$

eliminate this piece

Go back to original family

$$x^2 + (y - c)^2 = c^2$$

$$x^2 + (y^2 - 2cy + c^2) = c^2$$

$$x^2 - 2cy + y^2 = 0$$

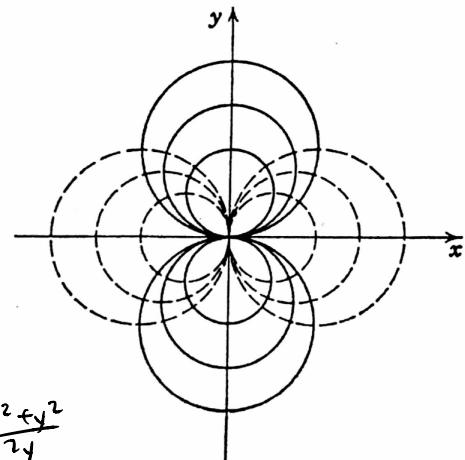
$$x^2 + y^2 = 2cy$$

$$c = \frac{x^2 + y^2}{2y}$$

$$y - c = y - \frac{x^2 + y^2}{2y}$$

$$y - c = \frac{2y^2}{2y} - \frac{y^2 + x^2}{2y}$$

$$\boxed{y - c = \frac{y^2 - x^2}{2y}}$$



$$\therefore x + \left(\frac{y^2 - x^2}{2xy}\right)y' = 0$$

$$y' = \frac{-x \cdot 2y}{y^2 - x^2}$$

$$\boxed{y' = \frac{2xy}{x^2 - y^2}}$$

is the DE solved by the family of curves  $\textcircled{1}$   
(circles)

② Orthogonal trajectories:

$$\frac{-1}{y'} = \frac{y^2 - x^2}{2xy}$$

$y'$  of new family.

$$\text{Thus for orth. traj: } y' = \frac{y^2 - x^2}{2xy}$$

$$\frac{dy}{dx} = \frac{1}{2} \left( \frac{y}{x} - \frac{x}{y} \right) \quad \text{Let } u = \frac{y}{x}$$

$$u'x + u = \frac{1}{2} \left( u - \frac{1}{u} \right) \Rightarrow u' = u'x + u$$

$$2u'x + 2u = \frac{u^2 - 1}{u}$$

$$2u'x = -\frac{2u^2}{u} + \frac{u^2 - 1}{u} - \frac{1}{u}$$

$$2u'x = -\frac{u^2 + 1}{u}$$

$$2x \frac{du}{dx} = -\frac{u^2 + 1}{u}$$

$$\int \frac{u}{u^2 + 1} du = \int -2x dx$$

$$-x^2 + C = \int \frac{u}{u^2 + 1} \quad t = u^2 + 1 \quad dt = 2u du$$

$$C - x^2 = \frac{1}{2} \ln|u^2 + 1|$$

$$C - x^2 = \ln \sqrt{u^2 + 1}$$

$$Ce^{-x^2} = \sqrt{u^2 + 1}$$

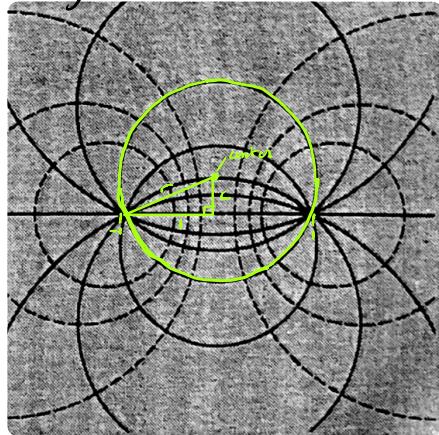
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$$\text{Ans: } y^2 + x^2 - 2Cx = 0$$

Ex: Consider 2 opposite charges of the same strength at  $P(-1, 0)$  and  $Q(1, 0)$ . An experiment suggests that the lines of force can be approximated by circles passing through  $P$  and  $Q$ .

- Express the lines of force as a family of curves (with a parameter  $C$ )
- Compute the equipotential lines.

Diagram:



Let center of circle be  $(0, c)$ . In a given circle (e.g. green circle in diagram),  $r^2 = C^2 + 1$ .

$$\text{Eqn of circles are } x^2 + (y - c)^2 = r^2$$

$$\Rightarrow x^2 + (y - C)^2 = C^2 + 1. \text{ Now find family of curves.}$$

$$\text{Imp. diffn } \frac{d}{dx} [x^2 + (y - C)^2] = \frac{d}{dx} [C^2 + 1]$$

$$2x + 2(y - C) \cdot y' = 0$$

$$x + y' \underbrace{(y - C)}_{\text{get rid of it!}} = 0$$

$$x^2 + (y - C)^2 = C^2 + 1$$

$$x + y' \left( \frac{y^2 - x^2 + 1}{2y} \right) = 0$$

$$x^2 + y^2 - 2Cy + C^2 = C^2 + 1$$

$$y' = \frac{-x - 2y}{y^2 - x^2 + 1}$$

$$2Cy = x^2 + y^2 - 1$$

$$y' = \frac{2xy}{x^2 - y^2 - 1}$$

$$C = \frac{x^2 + y^2 - 1}{2y}$$

$$\Rightarrow y - C = y - \frac{x^2 + y^2 - 1}{2y}$$

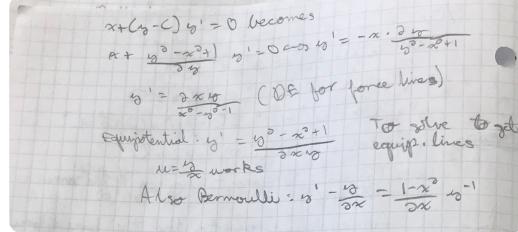
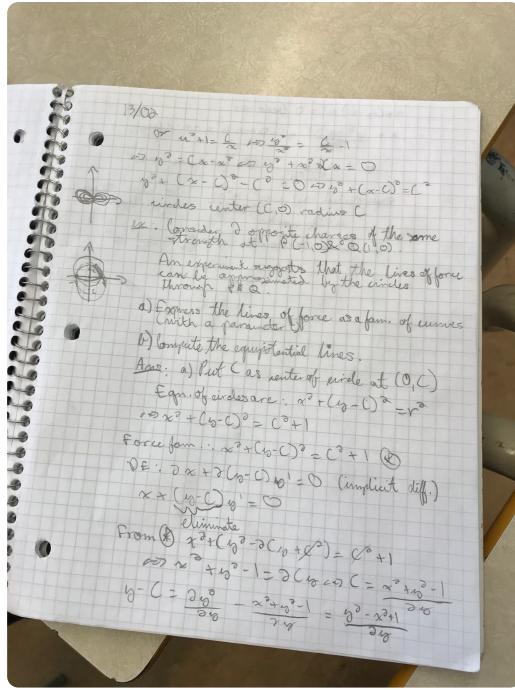
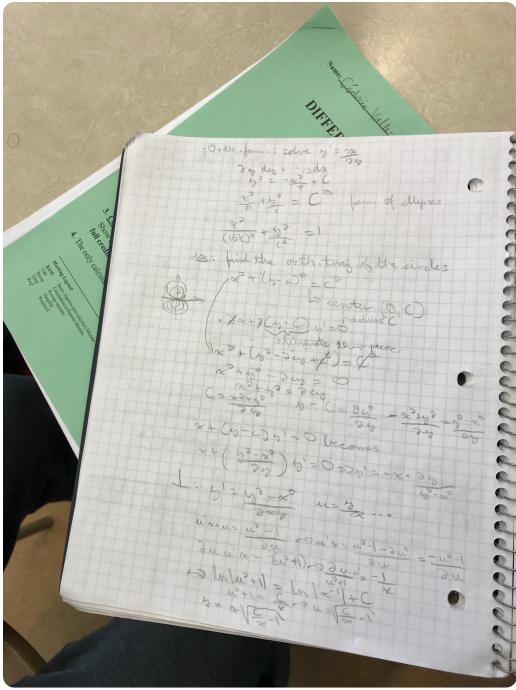
$$\left(\frac{1}{y'}\right) = \frac{y^2 - x^2 + 1}{2xy}$$

$$y - C = \frac{2y^2 + 1 - y^2 - x^2}{2y}$$

$y'$  of  
orthog.  
family

$$y - C = \frac{1 + y^2 - x^2}{2y}$$

$$y' = \frac{1}{2} \left[ \frac{y}{x} - \frac{x}{y} + \frac{1}{xy} \right] u =$$



On above problem, Found lines of force:  $x^2 + (y - c)^2 = c^2 + 1$  which solves the DE:  $y' = \frac{2xy}{x^2 - y^2 - 1} \therefore$  Equipotential lines satisfy DE.

For orth. trajectory:  $y' = \frac{y^2 - x^2 + 1}{2xy} \leftarrow$  solve to get family of L curves

$$y' - \frac{y}{2x} = \frac{1-x^2}{2x} \cdot \frac{1}{y} \Rightarrow$$

Bernoulli:  $a = -1$

$$y' + y\left(-\frac{1}{2x}\right) = \left(\frac{1-x^2}{2x}\right) \cdot y^{-1}$$

$$\text{Let } u = y^{1-a} \Rightarrow u = y^{1-(-1)}$$

$$\frac{u'}{2\sqrt{u}} - \frac{\sqrt{u}}{2x} = \left(\frac{1-x^2}{2x}\right) \sqrt{u}$$

$$u = y^2 \Rightarrow y = \sqrt{u}$$

$$u' = 2y \cdot y'$$

$$u' - \frac{u}{x} = \frac{1-x^2}{x}$$

$$\Rightarrow \frac{u'}{2y} = y' \Rightarrow y' = \frac{u'}{2\sqrt{u}}$$

$$u' + u \left(-\frac{1}{x}\right) = \frac{1-x^2}{x}$$

$$h(x) = \int p(x) dx = \int -\frac{dx}{x}$$

$$h(x) = -\ln|x|$$

$$\text{Solv: } u = e^{-h} \left[ \int e^h r(x) dx + C \right]$$

$$u = \left( e^{\ln|x|} + e^{\ln|x|} \int e^{\ln|\frac{1}{x}|} \cdot \frac{1-x^2}{x} dx \right)$$

$$u = Cx + x \int \frac{1}{x} \cdot \frac{1-x^2}{x} dx$$

$$u = Cx + x \int \left( \frac{1}{x^2} - 1 \right) dx$$

$$u = Cx + x \left[ -\frac{1}{x} - x \right]$$

$$y^2 = Cx - 1 - x^2$$

$$y^2 + x^2 - 2Cx + C^2 - C^2 = -1$$

$$(x-C)^2 + y^2 = C^2 - 1$$

$$\text{Note: } r^2 = C^2 - 1 \quad (r < C)$$

$$r = \sqrt{C^2 - 1}$$

## 2I) A First Attempt At Higher Order: Reduction To First Order!

Consider a 2<sup>nd</sup> order ODE in the unknown fcn  $y(x)$  that involves  $y'', y'$ , and  $x$  but not  $y$ , that is a DE of the form  $F(y'', y', x) = 0$ .

The substitution  $z = y'$  turns it into a first order DE in the unknown fcn  $z(x)$ .

E.g. Solve  $3xy'' = 2y'$

$$\text{Let } z = y'$$

$$3x z' = 2z$$

$$3x \frac{dz}{dx} = 2z \quad \begin{array}{l} \text{separable} \\ \text{ODE} \end{array}$$

$$\int \frac{dz}{z} = \frac{2}{3} \int \frac{dx}{x} + C$$

$$\ln|z| = \ln|x^{2/3}| + C \quad \text{no need for absolute value brackets} \because x^{2/3}$$

$$z = e^{\ln x^{2/3} + C}$$

$$z = C x^{2/3}$$

sub  $z = y'$

$$y' = C x^{2/3}$$

$$y = C \int x^{2/3} dx$$

$$y = C \underbrace{x^{5/3}}_{\text{absorb}} + D \Rightarrow y = C x^{5/3} + D$$

Now consider a DE in the unknown fcn  $y(x)$  that involves  $y'', y'$ ,  $y'$  but not  $x$  (explicitly). That is a DE of the form  $F(y, y', y'') = 0$ . Then  $z = y'$  also works, but in a trickier way.

Think of  $z = z(y)$  instead of  $z = z(x)$  (to avoid  $x \dots$ )

$$y' = \frac{d}{dx}[y] = z \quad \xrightarrow{\text{really}} z(y(x))$$

$$y'' = \frac{d}{dx}[y'(x)] = \frac{d}{dx}\left[z(y(x))\right] = z'(y(x)) \cdot \underbrace{y'(x)}_{z} \cdot x' \quad \xrightarrow{=} \\ * \Rightarrow y'' = z' \cdot z$$

\* Equivalent to doing  $\frac{d}{dy}[z(y)] = z' \cdot z$   $\xleftarrow{\text{same result}}$

So  $F(y, y', y'') = 0$

$\uparrow \quad \uparrow$   
 $z \quad z' \cdot z$  gives a DE involving  $y, z, z'$  for  $z(y)$ .

E.g. Solve  $yy'' + (y')^2 = 0$  by reduction to first order

$$\text{Let } z = y', \quad y'' = z' \cdot z \quad \rightarrow \int \frac{dz}{z} = - \int \frac{dy}{y} + C$$

$$y z' \cdot z + z^2 = 0$$

$$y z' \cdot z + z^2 = -z^2$$

$$y \frac{dz}{dy} = -z$$

$$\ln|z| = \ln|\frac{1}{y}| + C$$

$$z = e^{\ln|\frac{1}{y}| + C}$$

$$\boxed{z = \frac{C}{y}}$$

Finding  $y(x) \dots$  sub  $z = y'$

$$y' = \frac{C}{y}$$

$$\frac{dy}{dx} = \frac{C}{y}$$

$$\int y dy = C/dx + D$$

$$\frac{y^2}{2} = Cx + D$$

$$y = \pm \sqrt{Cx + D}$$