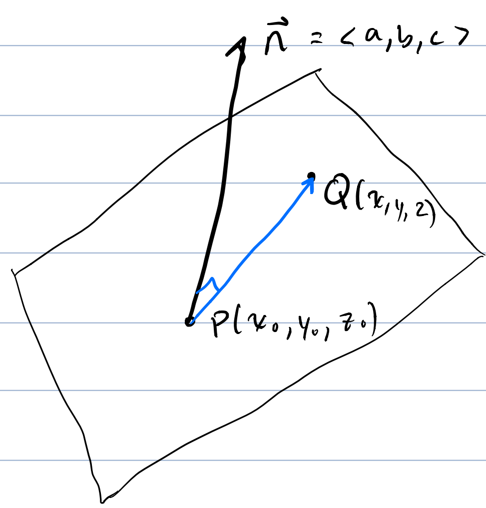
III - Planes

Given a point $P(x_0, y_0, z_0)$ and a normal vector $n = \langle a, b, c \rangle$,

we want an expression that is satisfied by the points $Q(x, y, z)$

(and only these points) that are on the plane perpendicular

to $\vec{n}$ and passing through P.



The point $Q(x, y, z)$ is on the plane if and only if

$$\vec{PQ} \cdot \vec{n} = 0$$

$$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$ ← POINT NORMAL EQUATION

$$ax + by + cz + (-ax_0 - by_0 - cz_0) = 0$$

$ax + by + cz + d = 0$ ← GENERAL EQUATION

ex. Write a general equation of the plane passing through $P(1,2,3)$ and perpendicular to $\vec{n} = \langle 0, 2, -3 \rangle$

Method ①

$$0(x-1) + 2(y-2) - 3(z-3) = 0$$

$$2y - 3z + 5 = 0$$

Method ②

$$0x + 2y - 3z + d = 0$$

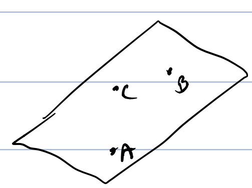
Subst $(1,2,3)$ to find d

$$2(2) - 3(3) + d = 0$$

$$d = 5$$

$$\therefore 2y - 3z + 5 = 0$$

ex. Find the equation of the plane passing through $A(1,3,1)$, $B(2,0,1)$, $C(-2,1,4)$



$\vec{n} = \vec{AB} \times \vec{AC}$

$= \langle 1, -3, 0 \rangle \times \langle -3, -2, 3 \rangle$

$$\begin{matrix} 1 & -3 & 0 \\ -3 & -2 & 3 \end{matrix}$$

$= \langle -9, -3, -11 \rangle$ or $\langle 9, 3, 11 \rangle$

$9(x-1) + 3(y-3) + 11(z-1) = 0 \leftarrow$ or stop here if it doesn't say general equation

$$\boxed{9x + 3y + 11z - 29 = 0}$$

ex. Find an equation of the plane containing the following two intersecting lines.

$$l_1 : \begin{cases} x = 11 + 2t \\ y = 6 + t \\ z = -4 - 4t \end{cases} \quad t \in \mathbb{R}$$

$$l_2 : \overbrace{\frac{x+3}{4} = \frac{y-5}{-1} = \frac{z+6}{7}}^{pt}$$

$\rightarrow$ direction vector

$\frac{y-5}{-1} = t$

$\frac{x+3}{4} = t$

$\frac{z+6}{7} = t$

$$\begin{cases} x = -3 + 4t \\ y = 5 - t \\ z = -6 + 7t \end{cases}$$

pt : $t = 0$

$(-3, 5, -6)$

$\vec{d_1} = \langle 2, 1, -4 \rangle$

$\vec{d_2} = \langle 4, -1, 7 \rangle$

$\vec{n} = \vec{d_1} \times \vec{d_2}$

$= \langle 3, -30, -6 \rangle = 3 \langle 1, -10, -2 \rangle$

$1(x+3) - 10(y-5) - 2(z+6) = 0$

$x - 10y - 2z + 41 = 0$

ex. Find an equation of the plane that contains the lines:

$\left.\begin{array}{l} l_1 : \langle x, y, z \rangle = \langle 1, 3, 2 \rangle + t\langle -1, 0, 1 \rangle \\ l_2 : \langle x, y, z \rangle = \langle 5, 7, 3 \rangle + t\langle -1, 0, 1 \rangle \end{array}\right\} t \in \mathbb{R}$

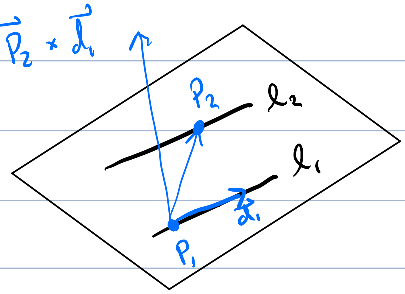
direction vectors are the same

$\therefore$ lines are parallel

$\therefore$ can't use method above

b.c. $\vec{d_1} \times \vec{d_2} = \vec{0}$

$\vec{n} = \overrightarrow{P_1P_2} \times \vec{d_1}$



$\overrightarrow{P_1P_2} = \langle 4, 4, 1 \rangle$ (chose $t=0$ on both lines)

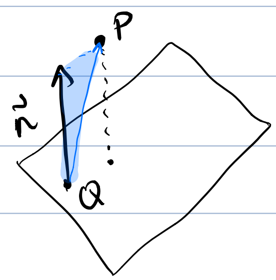
$\vec{d_1} = \langle -1, 0, 1 \rangle$

$\vec{n} = \overrightarrow{P_1P_2} \times \vec{d_1} = \langle 4, -5, 4 \rangle$

$4(x-1) - 5(y-3) + 4(z-2) = 0$

$\boxed{4x - 5y - 4z + 3 = 0}$ ← GENERAL EQUATION

Distance from a point to a plane



$\pi_1 : ax + by + cz + d = 0$

$\vec{n} = \langle a, b, c \rangle$

Chose a random pt Q on the plane.

$$D = \left\| \text{proj}_{\vec{n}} \overrightarrow{QP} \right\|$$

$$\boxed{D = \frac{|\overrightarrow{QP} \cdot \vec{n}|}{\|\vec{n}\|}}$$

ex. Find the distance from the pt $P(1,2,3)$ to the plane

$x + y - z - 2 = 0$

$\vec{n} = \langle 1, 1, -1 \rangle$

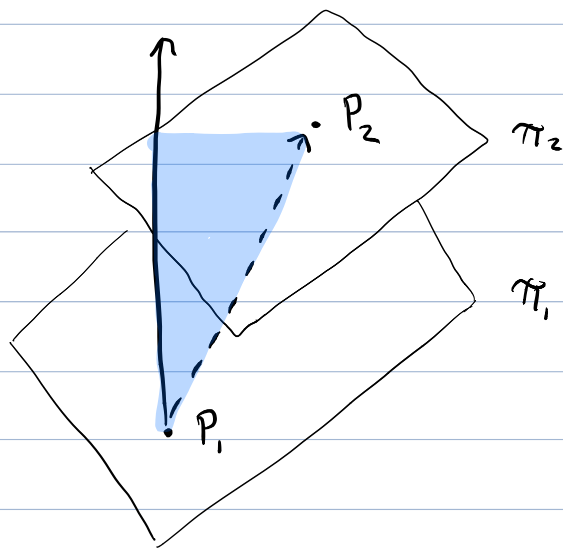
Chose a random pt Q

$x = 0 \quad , \quad y = 0 \quad , \quad z = -2$

$\overrightarrow{QP} = \langle 1, 2, 5 \rangle$

$$D = \frac{|\langle 1,2,5\rangle \cdot \langle 1,1,-1\rangle|}{\sqrt{1^2+1^2+(-1)^2}} = \frac{|-2|}{\sqrt{3}} = \boxed{\frac{2}{\sqrt{3}}}$$

Distance between two parallel planes



Choose random points P_1, P_2 on π_1 and π_2 respectively

$$D = \frac{|\overrightarrow{P_1P_2} \cdot \vec{n}|}{\|n_1\|}$$

ex Show that the two planes are parallel and find the shortest distance between them.

$$\pi_1 : x + 2y - z = 2$$
$$\pi_2 : -2x - 4y + 2z = 0$$

$\vec{n}_1 = \langle 1, 2, -1 \rangle$

$\vec{n}_2 = \langle -2, -4, 2 \rangle = -2 \langle 1, 2, -1 \rangle$

$\therefore \vec{n}_1 = k\vec{n}_2$

$\therefore$ The two planes are parallel since they have the same normal vector.

$P_1(0, 0, 2)$

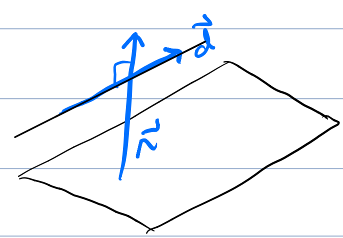
$P_2(0, 0, 0)$

$$D = \frac{|\langle 0,0,-2 \rangle \cdot \langle 1,2,-1 \rangle|}{\|\langle 1, 2, -1 \rangle\|}$$

$$= \frac{|2|}{\sqrt{6}}$$

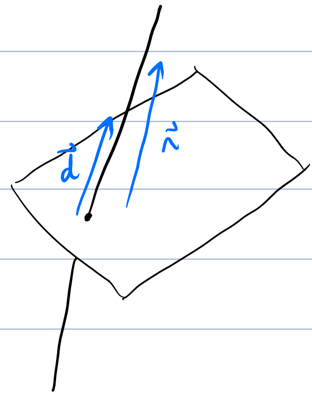
$$= \frac{2}{\sqrt{6}}$$

Relative positions of a line and a plane:

a) 

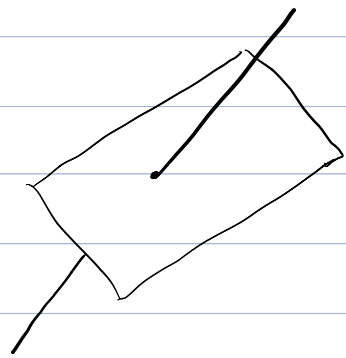
A line is parallel to a plane iff $\vec{n}$ and $\vec{d}$ are orthogonal (i.e. $\vec{n} \cdot \vec{d} = 0$)

b)



A line is <u>perpendicular</u> to a plane iff $\vec{n}$ and $\vec{d}$ are parallel (i.e. $\vec{n} \times \vec{d} = \vec{0}$ or $\vec{n} = k\vec{d}$)

c)



A line that is <u>not</u> parallel to the plane intersects the plane at one point.

September 22, 2017

<u>Finding the intersection between a line and a plane:</u>

<u>ex.</u> Given the line $\ell : \begin{cases} x = t \\ y = 3 + t \\ z = 1 - t \end{cases} \quad t \in \mathbb{R}$

and the plane $\Pi: x + 2y - z = 0$. Find the intersection.

→ substitute the parametric expressions for x, y, z into the general equation of the plane and solve for t.

$$t + 2(3 + t) - (1 - t) = 0$$

$$t + 6 + 2t - 1 + t = 0$$

$$4t = -5$$

$t = -\frac{5}{4}$

Note: If line // plane $\therefore$ no solution

line on plan $\therefore$ solution for all $\mathbb{R}$

$\rightarrow$ substitute back into ℓ

$x = -\frac{5}{4}$

$y = \frac{7}{4}$

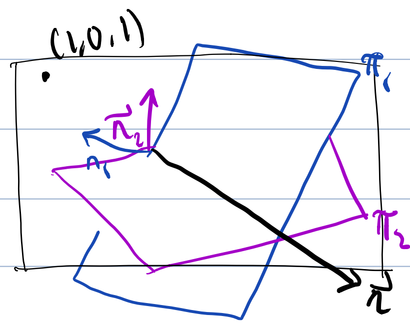
$z = \frac{9}{4}$

$\therefore$ The pt of intersection is $\left(-\frac{5}{4}, \frac{7}{4}, \frac{9}{4}\right)$.

ex. Find a plane $\perp$ to the 2 following plane and that passes through the pt $(1,0,1)$.

$\pi_1 : x + 2y - z = 0 \rightarrow \vec{n_1} = \langle 1, 2, -1 \rangle$

$\pi_2 : x - y = 5 \longrightarrow \vec{n_2} = \langle 1, -1, 0 \rangle$



$\vec{n} = \vec{n_1} \times \vec{n_2} = \langle -1, -1, -3 \rangle$

$= -1 \langle 1, 1, 3 \rangle$

$(x-1) + y + 3(z-1) = 0$