LIMITS

August 26, 2016

Consider $y = f(x) = \dfrac{x^2-4}{x-2}$ DOM: $\mathbb{R} \setminus \{2\}$

x	1.9	1.99	1.999	1.9999	...	$\to 2$
$y = f(x) = \frac{x^2-4}{x-2}$	3.9	3.99	3.999	3.9999	...	$\to 4$

x	2.1	2.01	2.001	2.0001	...	$\to 2$
$y = f(x) = \frac{x^2-4}{x-2}$	4.1	4.01	4.001	4.0001	...	$\to 4$

As the x-values approach '2' from <u>both sides</u>, but never touch '2', the f(x) values approach a <u>single</u> value '4'. '4' is called the <u>limiting value</u>.

$$\lim_{x \to 2} f(x) = 4$$

i.e. $\displaystyle \lim_{x \to 2} \frac{x^2-4}{x-2} = 4$

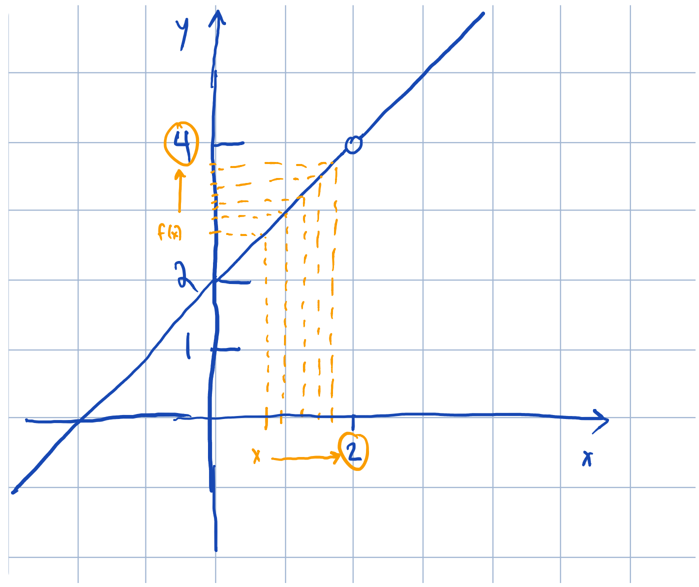
August 29, 2016

observe

$f(x) = \frac{x^2-4}{x-2} = \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = x+2$

So $f(x) = x+2$ as long as $x \neq 2$

$f(x) = x+2 \; (x \neq 2)$



$g(x) = 2$

f & g are not the same fcns, but limit is still 4.

$\lim_{x \to 2} f(x) = 4$

ex/ Consider $q(x) = 3$

Find $\lim_{x \to 1} q(x)$

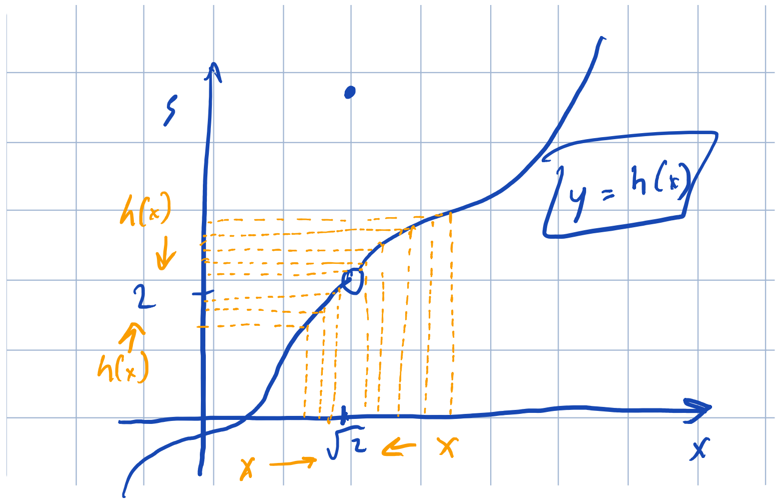
x	0.9	0.99	0.999	$\to$	1
$q(x)$	3	3	3	$\to$	3

x	1.1	1.01	1.001	$\to$	1
$q(x)$	3	3	3	$\to$	3

$\lim_{x \to 1} q(x) = 3$

Rule ① $\lim_{x \to a} c = c$ ex/ $\lim_{x \to -8} 12 = 12$

ex/



$$\lim_{x \to \sqrt{2}} h(x) = 2$$

$$h(\sqrt{2}) = 5$$

Quicker method - Substitution Rule

ex/ $\lim_{x \to 3} (x+5) = 8$

replace 'x' with 3

ex/ $\lim_{x \to 5} \sqrt{\frac{x^2+4}{x-1}} = \sqrt{\frac{5^2+4}{5-1}} = \sqrt{\frac{29}{4}}$

ex/ $\lim_{x \to 3} \frac{x^2+1}{x-2} \overset{ch}{=} \frac{3^2+1}{3-2} = 10$

So $\lim_{x \to 3} = 10$

ex/ $\lim_{x \to -1} \frac{x+1}{x^2+1} \overset{ch}{=} \frac{(-1)+1}{(-1)^2+1} = \frac{0}{2} = 0$ well defined

check

So $\lim_{x \to -1} = 0$

ex/ $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2} \overset{ch}{=} \dfrac{(2)^2-4}{2-2} = \dfrac{0}{0}$ ← indeterminate form

In this case, checking fails to offer any info.

Start over & do some algebra

$\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2} = \lim\limits_{x\to 2}\dfrac{\cancel{(x-2)}(x+2)}{\cancel{x-2}}$

$\overset{ch}{=} \lim\limits_{x\to 2}(x+2) = 4$

ex/ $\lim\limits_{x\to 9}\dfrac{\sqrt{x}-3}{x-9} \overset{ch}{=} \dfrac{\sqrt{9}-3}{x-9} = \dfrac{0}{0}$

$= \lim\limits_{x\to 9}\dfrac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)}$ } rationalize numerator $\times \left(\dfrac{\sqrt{x}+3}{\sqrt{x}+3}\right)$ SEE RATIONALIZE SECTION (p.6)

= get rid of $\sqrt{\ }$

$= \lim\limits_{x\to 9}\dfrac{x-9}{(x-9)(\sqrt{x}+3)}$

$= \lim\limits_{x\to 9}\dfrac{1}{\sqrt{x}+3}$

$\overset{ch}{=} \dfrac{1}{6}$

So $\lim\limits_{x\to 9}\dfrac{\sqrt{x}-3}{x-9} = \dfrac{1}{6}$ — don't need to include this step

ex/ $\lim\limits_{x\to 3}\dfrac{x-3}{\sqrt{x-2}-1} \quad \left(\dfrac{0}{0}\right)$

$= \lim\limits_{x\to 3}\dfrac{(x-3)(\sqrt{x-2}+1)}{(\sqrt{x-2}-1)(\sqrt{x-2}+1)}$

$= \lim\limits_{x\to 3}\dfrac{(x-3)(\sqrt{x-2}+1)}{x-3}$

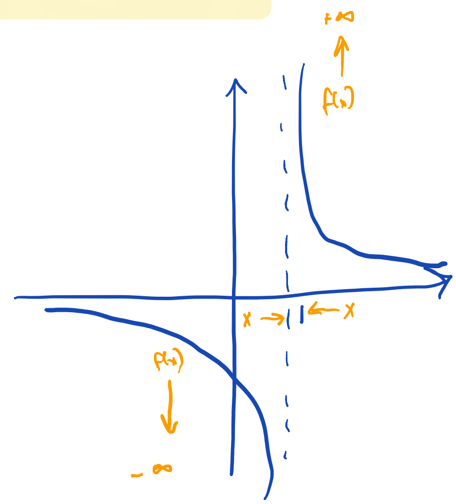
$= \lim\limits_{x\to 3}(\sqrt{x-2}+1)$ the moment you check, the 'lim' symbol gets dropped.

$\overset{ch}{=} \sqrt{3-2}+1$

$= 2$

For a limit to exist, the target value must be unique and finite.

Unique: both tables need to approach the same value.
Finite: needs to have a target.



ex/ $\lim_{x\to 1} \frac{1}{x-1} \quad \left(\text{ch } \frac{1}{0}\right)$

x	0.9	0.99	0.999	→ 1
$f(x)=\frac{1}{x-1}$	-10	-100	-1000	→ gets larger & larger in the -ve direction ($\to -\infty$)

doesn't approach any number

So $\lim_{x\to 1} \frac{1}{x-1} \quad \left(\text{ch } \frac{1}{0}\right)$ DNE - does not exist

SUMMARY OF RULES SO FAR!

① $\lim_{x\to a} c = c$

② (Cheating rule): If you have a f^n defined by a single algebraic expression (i.e. not piecewise defined or trig) of the form $y = \frac{f(x)}{g(x)}$

Sub 'x' by 'a'

If you get (A) $\frac{L}{M}$, $M \neq 0$ then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$

(B) $\frac{0}{0}$, then start over, try algebra

(C) $\frac{L}{0}$, $L \neq 0$ then $\lim_{x\to a} \frac{f(x)}{g(x)} = \text{DNE}$

ex/ $\lim_{x\to 0} \frac{x^2-x}{x+1} \overset{cn}{=} \frac{0}{1} = 0$

ex/ $\lim_{x\to 0} \frac{x+1}{x^2-x} \overset{cn}{=} \frac{1}{0} = DNE$

ex/ $\lim_{x\to -\sqrt{3}} (x+1) = -\sqrt{3}+1$

a constant

ex/ $\lim_{x\to 12} \frac{\sqrt{\pi}+1.78}{24} = \frac{\sqrt{\pi}+1.78}{24}$

Rationalize

ex/

$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\frac{\sqrt{2}+\sqrt{3}}{5} = \frac{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})}{5(\sqrt{2}-\sqrt{3})}$

$= \frac{2 \cancel{-\sqrt{3}\sqrt{2}} + \cancel{\sqrt{3}\sqrt{2}} - 3}{5(\sqrt{2}-\sqrt{3})}$

ex/

$\frac{\sqrt{x+1}-7}{x+2} = \frac{(\sqrt{x+1}-7)(\sqrt{x+1}+7)}{(x+2)(\sqrt{x+1}+7)}$

$\frac{(x+1)-49}{(x+2)(\sqrt{x+1}+7)}$

ex/ $\lim_{x\to 7} \frac{\sqrt{x}-\sqrt{7}}{7-x} \stackrel{ch}{=} \frac{0}{0}$

$= \lim_{x\to 7} \frac{(\sqrt{x}-\sqrt{7})(\sqrt{x}+\sqrt{7})}{(7-x)(\sqrt{x}+\sqrt{7})}$

$= \lim_{x\to 7} \frac{x-7}{(7-x)(\sqrt{x}+\sqrt{7})}$

$= \lim_{x\to 7} \frac{-\cancel{(-x+7)}}{\cancel{(7-x)}(\sqrt{x}+\sqrt{7})}$

$= \lim_{x\to 7} \frac{-1}{\sqrt{x}+\sqrt{7}}$

$\stackrel{ch}{=} \frac{-1}{2\sqrt{7}}$

ex/ $\lim_{x\to 7} \frac{3x+1}{\sqrt{x}-\sqrt{7}}$

$\stackrel{ch}{=} \left(\frac{22}{0}\right)$ limit does not exist.

DNE

ex/ $\lim_{x\to 0} (3x^2 + hx + h)$ constant since $x \to 0$

$\stackrel{ch}{=} 3(0)^2 + h(0) + h = h$

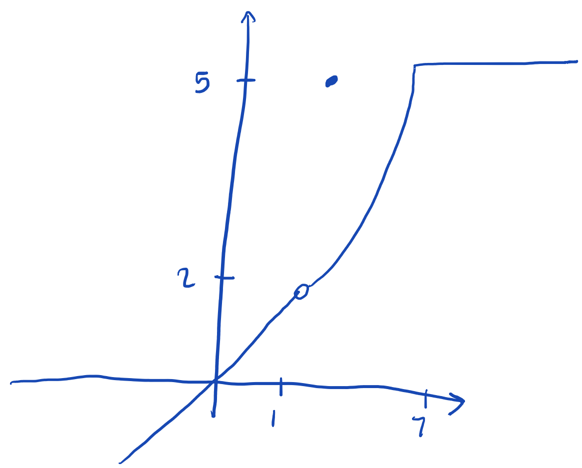
ex/ $\lim_{h\to 0} (3x^2 + hx + h)$

$\stackrel{ch}{=} 3x^2 + 0x + 0 = 3x^2$

September 1, 2016

ex/ $\lim_{h\to 0} x^2 = x^2$

Suppose $y = f(x)$ given by the graph



a) $\lim_{x\to 1} f(x) = 2$

b) $f(1) = 5$

c) $\lim_{x\to 7} = 5$

d) $d(7) = 5$

More Rules of Limits

If $\lim_{x \to a} f(x)$ & $\lim_{x \to a} g(x)$ both exist, then:

① $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

② $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

③ $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

④ $\lim_{x \to a} [k f(x)] = k \cdot \lim_{x \to a} f(x)$ (k is a constant)

⑤ $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ (provided $\lim_{x \to a} g(x) \neq 0$)

ex/ Given $\lim_{x \to 4} f(x) = -3$, $\lim_{x \to 4} g(x) = 0$ & $\lim_{x \to 4} h(x) = 5$

find $\lim_{x \to 4} \left[\frac{2 f(x)}{h(x) - g(x)}\right] = \frac{\lim_{x \to 4} [2 f(x)]}{\lim_{x \to 4} [h(x) - g(x)]}$

$= \frac{2 \lim_{x \to 4} f(x)}{\lim_{x \to 4} h(x) - \lim_{x \to 4} g(x)} = \frac{2(-3)}{5 - 0} = \frac{-6}{5}$

One-Sided Limits

(left-sided & right-sided)

ex/ $f(x) = \begin{cases} 2x-1 & \text{if } x<3 \\ 5-x & x \geq 3 \end{cases}$

Table ① from the left...

x	2.9	2.99	2.999	... → 3
$f(x) = 2x-1$	4.8	4.98	4.998	... → 5

$\lim_{x\to 3} f(x)$

Table ② from the right...

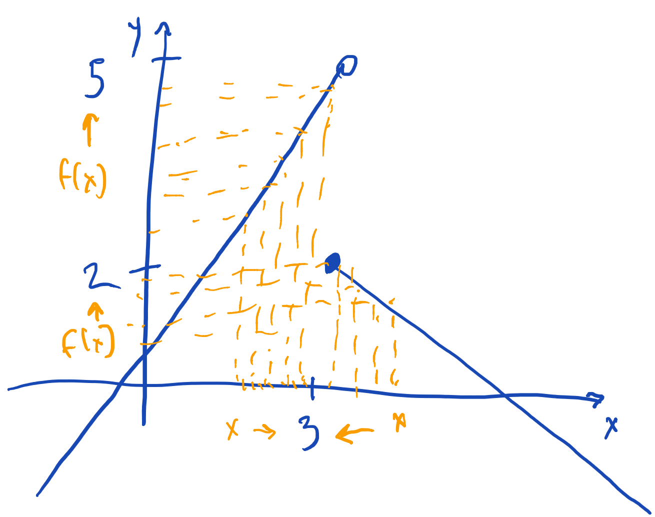
x	3.1	3.01	3.001	... → 3
$f(x) = 5-x$	1.9	1.99	1.999	... → 2

Table ①, x approaches 3 from left: $\lim_{x\to 3^-} f(x) = 5$

Table ②, x approaches 3 from right: $\lim_{x\to 3^+} f(x) = 2$

Because $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$, we can conclude that $\lim_{x\to 3} f(x)$ DNE

Sketch of Graph



In general: If $\lim_{x\to a^-} f(x) = L$ & $\lim_{x\to a^+} f(x) = L$ (L is finite)

then $\lim_{x\to a} f(x) = L$

If $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

then $\lim_{x\to a} f(x)$ DNE

The rules above apply for all three types of limits,

BUT BE CAREFUL WITH CHEATING

BACK TO:

ex/ $f(x) = \begin{cases} 2x-1 & \text{if } x<3 \\ 5-x & x \geq 3 \end{cases}$

$\lim_{x\to 3} f(x)$

1st $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (2x-1) \overset{ch}{=} 2(3)-1 = 5$

$(x<3)$

2nd $\lim_{x\to 3} f(x) = \lim_{x\to 3^+} (5-x) \overset{ch}{=} 5-3 = 2$

$(x \geq 3)$

$\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$ & so $\lim_{x\to 3} f(x)$ DNE ex/ Evaluate $\lim_{x\to 6} f(x)$ for fcn above

Method ①

$$\lim_{x\to 6^-} f(x) = \lim_{x\to 6} (5-x) = 5-6 = -1$$

$$\lim_{x\to 6^+} f(x) = \lim_{x\to 6} (5-x) = 5-6 = -1$$

So $\lim_{x\to 6} f(x) = -1$

Method ② (more efficient for cases like this)

$$\lim_{x\to 6} f(x) = \lim_{x\to 6} (5-x) \overset{cn}{=} 5-6 = -$$

ex/ $g(x) = \begin{cases} x^2+5 & \text{if } x<2 \\ 2x+5 & \text{if } x>2 \end{cases}$

$\lim_{x\to 2} (g)$?

$$\left.\begin{aligned} \lim_{x\to 2^-} g(x) &= \lim_{x\to 2^-} (x^2+5) \overset{cn}{=} (2)^2+5 = 9 \\ \lim_{x\to 2^+} g(x) &= \lim_{x\to 2} (2x+5) = 9 \end{aligned}\right\} \Rightarrow \lim_{x\to 2} g(x) = 9$$

ex/ $g(x) = \begin{cases} x^3 - 1 & \text{if } x < 4 \\ 3 & \text{if } 4 \le x < 7 \\ \sqrt{x} & \text{if } x \ge 7 \end{cases}$

$\lim_{x \to 4} g(x)$?

$$\left.\begin{array}{l} \lim_{x \to 4^-} g(x) = \lim_{x \to 4^-} (x^3 - 1) \overset{ch}{=} 63 \\ \\ \lim_{x \to 4^+} g(x) = \lim_{x \to 4^-} (3) = 3 \end{array}\right\} \Rightarrow \lim_{x \to 4} g(x) \text{ DNE}$$

ex/ $\lim_{x \to 5} \frac{3-x}{|3-x|} \overset{ch}{=} \frac{3-5}{|3-5|} = \frac{-2}{|-2|} = \frac{-2}{2} = -1$

ex/ $\lim_{x \to 3} \frac{3-x}{|3-x|} \overset{ch}{=} \frac{3-3}{|3-3|} = \frac{0}{0}$ abs. val is like a piecewise defn fcn

$|w| = \begin{cases} -w \text{ if } w < 0 \\ w \text{ if } w \ge 0 \end{cases}$

$|3-x| = \begin{cases} -(3-x) \text{ if } 3-x < 0 \Rightarrow \text{if } x > 3 \\ 3-x \text{ if } 3-x \ge 0 \Rightarrow \text{if } x \le 3 \end{cases}$

Left side

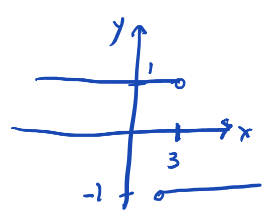
$\lim_{x \to 3^-} \frac{3-x}{|3-x|} = \lim_{x \to 3^-} \frac{\cancel{3-x}}{\cancel{3-x}} = \lim_{x \to 3^-} 1 = 1$

$(x \le 3)$

Right side

$\lim_{x \to 3^+} \frac{3-x}{|3-x|} = \lim_{x \to 3^+} \frac{3-x}{-(3-x)} = \lim_{x \to 3^+} (-1) = -1$

$(x > 3)$



Concl $\lim_{x \to 3} \frac{3-x}{|3-x|}$ DNE

Infinite Limits & Vertical Asymptotes

ex/ Consider the f^{cn} $f(x) = \frac{1}{x-1}$

Evaluate $\lim_{x \to 1^+} f(x)$

$\lim_{x \to 1^+} \frac{1}{x-1} \overset{ch}{=} \frac{1}{0}$ – usually this means lim DNE

x	1.1	1.01	1.001	1.0001 ... $\to 1$
$f(x) = \frac{1}{x-1}$	10	100	1000	10000 ...

As x-values approach one from right (& never touch 1), the $f(x)$ values get larger & larger

Answer: $\lim_{x \to 1^+} \frac{1}{x-1} = +\infty$ ← says limit DNE & says why

If $\lim_{x \to \text{vertical assymptote}^{+/-}}$, answer will $= \pm\infty$

$\left(\frac{-1}{0^-}\right) = +\infty$

Methods to answer this question

① $\lim_{x \to 1^+} \frac{1}{x-1} \overset{ch}{=} \frac{1}{0}$ $+\infty$ or $-\infty$

Test say $x = 1.001$ in $f(x) = \frac{1}{x-1}$

$= \frac{1}{0.001} = 1000$

So $\lim_{x \to 1^+} \frac{1}{x-1} = +\infty$

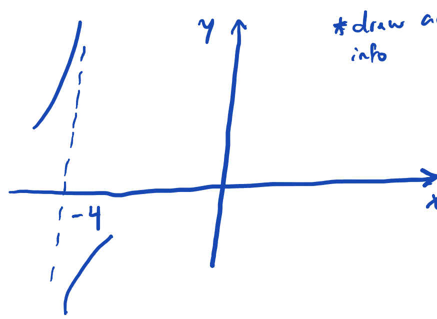
② $\lim_{x \to 1^+} \frac{1}{x-1} \left(\overset{ch}{=} \frac{1}{0^+}\right) = +\infty$

↳ $x > 1$

$x - 1 > 0$

ex/ Find all vertical asymptotes (VA's) of $g(x) = \frac{3-x^2}{4+x}$ & analyse all relevant limits.

*draw anything consistent with info



$g(x) = \lim_{x \to -4^-} \frac{3-x^2}{4+x} \left(\overset{ch}{=} \frac{-13}{0^-}\right) = +\infty$

↳ $x < 4$

$x + 4 < 0$

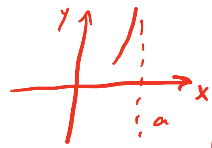
$g(x) = \lim_{x \to -4^+} \frac{3-x^2}{4+x} \left(\overset{ch}{=} \frac{-13}{0^+}\right) = -\infty$

↳ $x > 4$

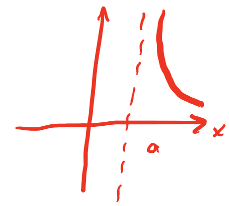
$x + 4 > 0$

Summary

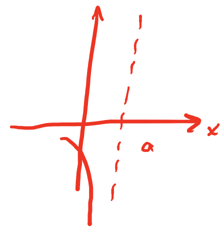
$\lim_{x \to a^-} f(x) = +\infty \Rightarrow$



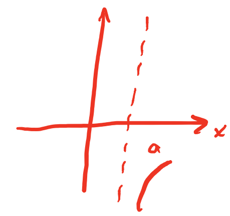
$\lim_{x \to a^+} f(x) = +\infty \Rightarrow$



$\lim_{x \to a^-} f(x) = -\infty$



$\lim_{x \to a^+} f(x) = -\infty \Rightarrow$



Evaluate $\lim_{x \to 3} \frac{4}{9-x^2} \left(\overset{ch}{=} \frac{4}{0} \right)$

$\lim_{x \to 3^-} \frac{4}{9-x^2} \left(= \frac{4}{0^+} \right) = +\infty$

$9-2.9^2 > 0$

$\lim_{x \to 3^+} \frac{4}{9-x^2} \left(= \frac{4}{0^-} \right) = -\infty$

$9-3.1^2 < 0$

$\Rightarrow \lim_{x \to 3} \frac{4}{9-x^2}$ DNE

Since limits from both sides approaches 2 different numbers

ex/ $\lim_{x \to 9} \frac{4}{(9-x)^2}$

$\lim_{x \to 9^-} \frac{4}{(9-x)^2} \left(\overset{ch}{=} \frac{4}{0^+} \right) = +\infty$

$\lim_{x \to 9^+} \frac{4}{(9-x)^2} \left(\overset{ch}{=} \frac{4}{0^+} \right) = +\infty$

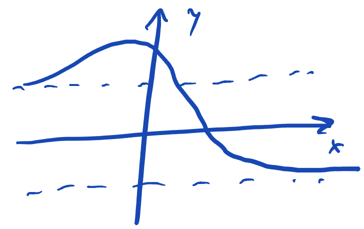
$\lim_{x \to 9} = +\infty$

LIMITS AT INFINITY & HORIZONTAL ASYMPTOTES (HA's)

Consider $y = f(x) = \frac{1}{x}$

x	10	100	1000	10 000 ...	$\to +\infty$
$f(x) = \frac{1}{x}$	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$	$\frac{1}{10\,000}$ ...	$\to 0$

$$\lim_{x \to +\infty} f(x) = 0$$



Note

$\lim_{x \to \infty} f(x)$

NONSENSE

> If $\lim_{x \to -\infty} f(x) = L$ (L is a finite #)
>
> then $y = L$ is a HA in $-$ve direction
>
> If $\lim_{x \to +\infty} f(x) = L$ (L is a finite #)
>
> then $y = L$ is a HA in $+$ve direction

September 8, 2016

Rules/Techniques for Computing limits at Infinity

① $\lim_{x \to +\infty} \frac{k}{x^r} = 0$ k is any constant & $r > 0$ (also a constant)

② $\lim_{x \to -\infty} \frac{k}{x^r} = 0$ k is any constant, $r > 0$, and x^r is defined when $x < 0$

ex. $\lim \frac{7}{\sqrt[3]{x}}$, $x^{\frac{1}{3}}$ is defined when $x < 0$

③ All previous rules apply except cheating/substitution

ex. $\lim_{x \to +\infty} -3x^2 = -\infty$

No horizontal asymptote in the direction since '$-\infty$' is not finite

ex. $\lim_{x \to +\infty} (5x^4 - 100x^3 - 1000x^2 - 100x - 10)$

$= \lim_{x \to +\infty} 5x^4 \left(1 - \frac{100x^3}{5x^4} - \frac{1000x^2}{5x^4} - \frac{100x}{5x^4} - \frac{10}{5x^4}\right)$

$= \lim_{x \to +\infty} 5x^4 \underbrace{\left(1 - \underset{\to 0}{\frac{20}{x}} - \underset{\to 0}{\frac{200}{x^2}} - \underset{\to 0}{\frac{20}{x^3}} - \underset{\to 0}{\frac{2}{x^4}}\right)}_{\to 1}$

$= +\infty$

This ex. suggest the following (assume $n \geq 0$)

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x - a_0$ $(a_n \neq 0)$

is any polynomial of degree n, then:

$$\lim_{x \to \pm\infty} (a_n x^n + a_{n-1} x^{n-1} + \ldots) = \lim_{x \to \infty} a_n x^n$$

Note poly. f^{ns} can never have HA's (also never have VA's) (except for when $n = 0$)

What is a polynomial?

ex. $P(x) = -3x^4 + 1$

$= -3x^4 + 0x^3 + 0x^2 + 0x + 1$

ex. $Q(x) = 5x^4 - 3x^3 + 7x^2 - 2\sqrt{x} + 1$

Not a polynomial

ex. $R(x) = -6$

$= -6x^0$

ex. $\lim\limits_{x \to +\infty} \dfrac{5x^2 + x + 3}{2x^2 + 5} = \dfrac{\lim\limits_{x \to +\infty} (5x^2 + x + 3)}{\lim\limits_{x \to +\infty} (2x^2 + 5)} = \dfrac{+\infty}{+\infty}$

Same as $\frac{0}{0}$

Indeterminate form

$= \lim\limits_{x \to +\infty} \dfrac{\cancel{x^2}\left(5 + \overset{\to 0}{\frac{1}{x}} + \overset{\to 0}{\frac{3}{x^2}}\right)}{\cancel{x^2}\left(2 + \underset{\to 0}{\frac{5}{x^2}}\right)} = \dfrac{5 + 0 + 0}{2 + 0} = \dfrac{5}{2}$

Factor highest power in denominator

(use for limits of $\dfrac{f(x)}{g(x)}$

graph of $y = \dfrac{5x^2 + x + 3}{2x^2 + 5}$ has $y = \frac{5}{2}$ as a HA in +ve direction

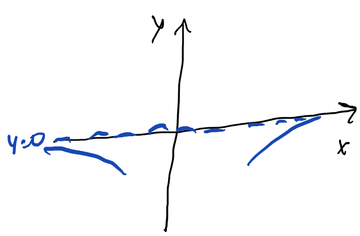
ex. Find all HA's of $f(x) = \dfrac{2x - 8}{3x^2 - 1}$

$\lim\limits_{x \to \pm\infty} \dfrac{\cancel{x^2}\left(\overset{\to 0}{\frac{2}{x}} - \overset{\to 0}{\frac{8}{x^2}}\right)}{\cancel{x^2}\left(3 - \underset{\to 0}{\frac{1}{x^2}}\right)} = \dfrac{0}{3} = 0$

Can start with both infinities

<u>Concl</u>

$y = 0$ is a HA in both +ve & −ve directions



ex. Find all HA's of $f(x) = \frac{2x}{\sqrt{x^2+4}}$

$$\lim_{x\to-\infty} \frac{2x}{\sqrt{x^2+4}} = \lim_{x\to-\infty} \frac{2x}{\sqrt{x^2\left(1+\frac{4}{x^2}\right)}} = \lim_{x\to-\infty} \frac{2x}{|x|\sqrt{1+\frac{4}{x^2}}}$$

$$= \frac{2x}{-x\sqrt{1+\underbrace{\frac{4}{x^2}}_{\to 0}}} = \frac{2}{-\sqrt{1+0}} = -2$$

replace $|x|$ by $-x$

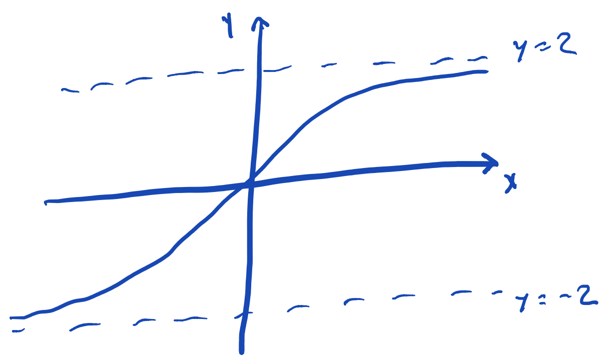
Since $\lim_{x\to-\infty}$

$$\lim_{x\to+\infty} \frac{2x}{\sqrt{x^2+4}} = \lim_{x\to+\infty} \frac{2x}{\sqrt{x^2\left(1+\frac{4}{x^2}\right)}} = \lim_{x\to+\infty} \frac{2x}{|x|\sqrt{1+\frac{4}{x^2}}}$$

$$= \frac{2x}{+x\sqrt{1+\underbrace{\frac{4}{x^2}}_{\to 0}}} = \frac{2}{\sqrt{1+0}} = 2$$

<u>Concl</u> $y=-2$ is a HA in -ve direction

$y=+2$ is a HA in +ve direction



ex. $\lim\limits_{x \to +\infty} \dfrac{\sqrt[3]{x^2+8}}{x+2}$ HMwk

Method 1

$= \lim\limits_{x\to\infty} \dfrac{\sqrt[3]{x^2\left(1+\frac{8}{x^2}\right)}}{x+2}$

$= \lim\limits_{x\to\infty} \dfrac{\sqrt[3]{x^2}\,\sqrt[3]{1+\frac{8}{x^2}}}{x+2} \to 0$

$\cancel{= \lim\limits_{x\to\infty} \dfrac{\sqrt[3]{x^2}}{x+2}}$ Can't set only one 'x' = 0

$= \lim\limits_{x\to\infty} \dfrac{x^{2/3}\sqrt[3]{1+\frac{8}{x^2}}}{x\left(1+\frac{2}{x}\right)}$

$= \lim\limits_{x\to\infty} \dfrac{\sqrt[3]{1+\frac{8}{x^2}}}{\sqrt[3]{x}\left(1+\frac{2}{x}\right)}$

$= \lim\limits_{x\to\infty} \left[\dfrac{1}{x^{\frac{1}{3}}} \cdot \dfrac{\sqrt[3]{1+\frac{8}{x^2}}}{1+\frac{2}{x}}\right]$

$= \lim\limits_{x\to\infty} \left(\dfrac{1}{x^{\frac{1}{3}}}\right)_{\to 0} \cdot \lim \dfrac{\sqrt[3]{1+\frac{8}{x^2}}_{\to 0}}{1+\frac{2}{x}_{\to 0}}$

$= \dfrac{\sqrt[3]{1}}{1} = 0 \cdot 1 = 0$

Method 2

$= \lim\limits_{x\to\infty} \dfrac{\sqrt[3]{x^3\left(\frac{1}{x}+\frac{8}{x^3}\right)}}{x+2}$

$= \lim\limits_{x\to\infty} \dfrac{\cancel{\sqrt[3]{x^3}}\,\sqrt[3]{\frac{1}{x}+\frac{8}{x^3}}}{\cancel{x}\left(1+\frac{2}{x}\right)}$

$= \dfrac{\sqrt[3]{\underbrace{\frac{1}{x}}_{\to 0}+\underbrace{\frac{8}{x^3}}_{\to 0}}}{1+\underbrace{\frac{2}{x}}_{\to 0}}$

$= \dfrac{\sqrt[3]{0}}{1+0} = \dfrac{0}{1} = 0$

Squeeze Theorem

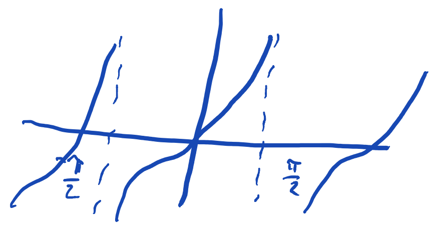
September 9, 2016

Cheating works with trig fcns as long as the answer is well defn

ex. $\lim_{x\to 0} \sin x \overset{ch}{=} \sin 0 = 0$

ex. $\lim_{x\to \frac{\pi}{4}} \frac{\cos x}{x} \overset{ch}{=} \frac{\cos\left(\frac{\pi}{4}\right)}{\pi/4} = \frac{\frac{\sqrt{2}}{2}}{\frac{\pi}{4}} = \frac{2\sqrt{2}}{\pi}$

ex. $\lim_{x\to \frac{\pi}{2}^+} \tan x = \tan \frac{\pi}{2} = -\infty$



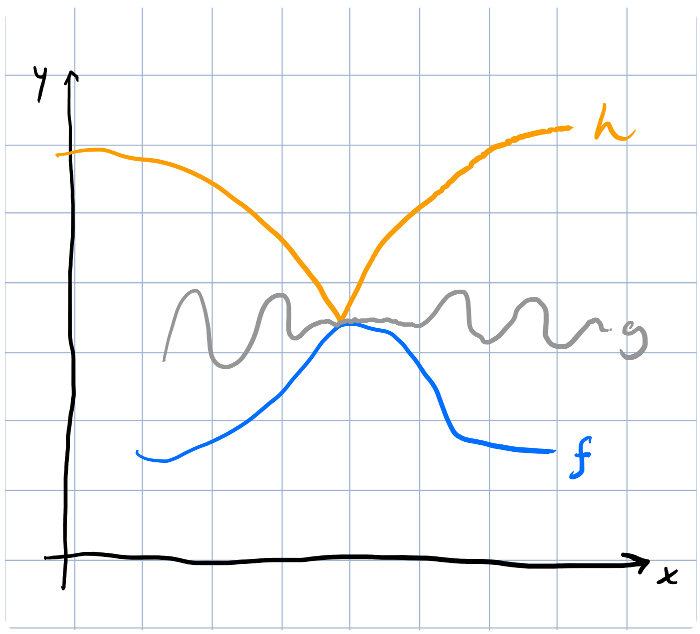
SQUEEZE THEOREM

If $f(x) \leq g(x) \leq h(x)$ for all x near "a"

The graph of f is under graph of g, ...

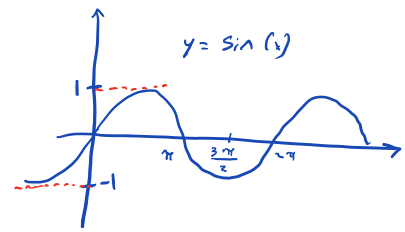
& if $\lim f(x) = L$ and $\lim\limits_{x \to a} h(x) = L$ (L is a finite #)

then $\lim\limits_{x \to a} g(x) = L$



ex. $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) \overset{ch}{=} 0 \sin \frac{1}{0}$??

Cannot get answer directly



$-1 \le \sin \frac{1}{x} \le 1$

$-x^2 \le \underbrace{x^2 \sin \frac{1}{x}} \le x^2$ Multiply all sides by x^2

This is squeezed between $-x^2$ & x^2

$\lim_{x \to 0} (-x^2) = 0$ & $\lim_{x \to 0} (x^2) = 0$

By Squeeze Th^m

$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$