Jan. 29

# Mathematical Induction

- Typically used to prove that a statement is true for all natural numbers $\mathbb{N}$
$= \{1, 2, 3, \dots\}$

ex. Linear Algebra:

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(ABC)^{-1} = (A(BC))^{-1}$$
$$= (BC)^{-1}A^{-1}$$
$$= C^{-1}B^{-1}A^{-1}$$

proven by induction

$$(A_1 A_2 \dots A_n)^{-1} = A_n^{-1} \dots A_2^{-1} A_1^{-1}$$

Natural number $\mathbb{N}$

## Principle of Induction:

Let $S(n)$ be some statement that depends on a natural number $n$.

base case

If (1) $S$ is true for $n = c$ $(=1)$ often

and (2) when $S$ is true for $n = k$ (arbitrary), it follows logically that $S$ must also be true for $n = k+1$

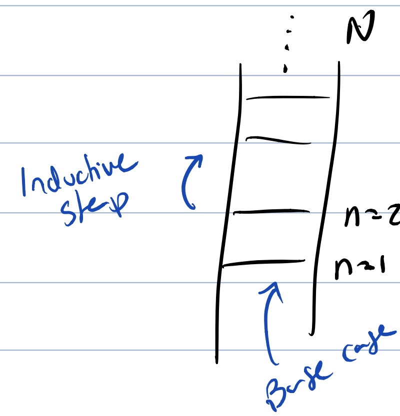
showing $S(k) \Rightarrow S(k+1)$

Inductive step

The assumption that $S(k)$ is true is the induction hypothesis $\rightsquigarrow$ hypothesis = assumption

Then, $S$ is true for all $n \in \mathbb{N}$

Pictorially:




ex. 1 Prove by induction that $\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$

$S(n)$

Proof:

1) Base case: Let $n=1$

$$\boxed{\sum_{i=1}^{1} 1 \stackrel{?}{=} \frac{1(1+1)}{2}} \quad S(1)$$

$$1 \quad\Big|\quad 1$$

✓

2) Inductive Step: Let $n = k$, assume that $S(k)$ is true & use this to prove that $S(k+1)$ follows

$$S(k): \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$ induction hypothesis

$$S(k+1): \sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$ must be shown using

$$1 + 2 + \ldots + k + k+1$$

Examine the inductive hypothesis:

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

$$(k+1) + (1 + 2 + \ldots + k) = \left(\frac{k(k+1)}{2}\right) + (k+1)$$

$$1 + 2 + \ldots + k + k+1 = \frac{k(k+1)}{2} + k+1$$

$$S(k+1): \sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{(k+2)(k+1)}{2}$$

$\therefore$ Since the base case & the inductive step are satisfied:

$$S(n): \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$ is true for all $n \in \mathbb{N}$

ex. 2 Use induction to show that $n! > 2^n$ for $n \geq 4$

Base case

Proof! (1) Base case : Let $n = 4$

$S(4) : 4! \overset{?}{>} 2^4$

| | |
|---|---|
| 24 | 16 |

✓

(2) Inductive step :

Let $n = k$,

assume $S(k): k! > 2^k$ is true (for some $k \geq 4$)

show $S(k+1) : (k+1)! > 2^{k+1}$ must therefore also be true

examine $S(k)$ : $\boxed{k! > 2^k}$

note! $k > 4 > 2$

$\therefore \boxed{k+1 > 5 > 2}$

Remember

$$\left[\begin{array}{l} a > b > 0 \\ c > d > 0 \\ \therefore ac > bd \end{array}\right]$$

$k!(k+1) > 2^k 2^1$

$(k+1)! > 2^{k+1}$

By math ind, since base case & ind step are satisfied,

$n! > 2^n$

ex. For you

Use induction to prove:

1) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

2) $(A_1 A_2 \dots A_n)^{-1} = A_n^{-1} \dots A_2^{-1} A_1^{-1}$ for square matrices of same size

3) $2^{2n+1} + 1$ is divisible by 3 for any $n$.

1) Base case: Let $n = 1$

$S(1): \sum_{i=1}^{1} i^2 \overset{?}{=} \frac{1(1+1)(2(1)+1)}{6}$

$1 \quad \Big| \quad \frac{2(3)}{6}$

$\quad\;\; \Big| \quad 1$

$\checkmark$

- Inductive step: Let $n = k$,
- assume $S(k): \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ is true for all $k$

• show $S(k+1)$ : $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$
must therefore also be true.

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

$$(k+1)^2 + 1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$1^2 + 2^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)\left(k(2k+1) + 6(k+1)\right)}{6}$$

$$= \frac{(k+1)\left(2k^2 + k + 6k + 6\right)}{6}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6} \quad : S(k+1)$$

∴ By the principle of induction, $S(n)$ is therefore true.

2) $(A_1 A_2 \ldots A_n)^{-1} = A_n^{-1} \ldots A_2^{-1} A_1^{-1}$ for square matrices of same size

Base case: Let $n = 2$

$S(2): (A_1 A_2)^{-1} \stackrel{!}{=} A_1 A_2 (A_2^{-1} A_1^{-1})$

$A_1 A_2 (A_1 A_2)^{-1}$ | $A_1 I A^{-1}$
$I$ | $A_1 A^{-1}$
$I$ ✓

Inductive step: Let $n = k$

assume $S(k): (A_1 A_2 \dots A_k)^{-1} = A_k^{-1} \dots A_2^{-1} A_1^{-1}$ for square matrices of same size

Show $S(k+1): (A_1 A_2 \dots A_k A_{k+1})^{-1} = A_{k+1}^{-1} A_k^{-1} \dots A_2^{-1} A$ for square matrices of same size

$$(A_1 A_2 \dots A_k)^{-1} = A_k^{-1} \dots A_2^{-1} A_1^{-1}$$

$$A_{k+1}^{-1} (A_1 A_2 \dots A_k)^{-1} = A_{k+1}^{-1} A_k^{-1} \dots A_2^{-1} A_1^{-1}$$

$$A_{k+1} A_k^{-1} \dots A_2^{-1} A_1^{-1} = A_{k+1}^{-1} A_k^{-1} \dots A_2^{-1} A_1^{-1} : S(k+1) \checkmark$$

∴ By the principle of induction, $S(n)$ is therefore true.

3) $2^{2n+1} + 1$ is divisible by 3 for any $n$, i.e. $2^{2n+1} + 1 = 3a$, $a \in \mathbb{N}$

Base case: $n = 0$

$$2^1 + 1 \stackrel{?}{=} 3a \quad , a \in \mathbb{N}$$
$$3 = 3a \quad \text{for } a = 1 \quad \checkmark$$

Inductive Step: Let $n = k$

Assume $S(k) : 2^{2k+1} + 1 = 3m \quad , m \in \mathbb{N}$

Show $S(k+1) : 2^{2k+3} + 1 = 3q \quad , q \in \mathbb{N}$

$$\begin{aligned} S(k) : \quad 2^{k+1} + 1 &= 3m \\ 2^2 \, 2^{2k+1} &= 2^2(3m - 1) \\ 2^{2k+3} &= 4(3m - 1) \\ 2^{2k+3} &= 12m - 4 \\ 2^{2k+3} &= 12m - 3 - 1 \\ &= 3(4m-1) - 1 \\ 2^{2k+3} + 1 &= 3q \end{aligned}$$

Jan. 31

# II. Sequences

Why? underly series $\longrightarrow$ sequence of partial sums

**Def'n:** A sequence is any function whose domain is the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$

**Remark:** The range set can be anything, but for us, it will most often be $\mathbb{R}$

**Notation:** • $a_n = f(n)$ where $f$ is a function
↳ the n'th term of the sequence

• $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$ is the entire (range) of the sequence

• most often, sequences are written as an ordered list of terms:

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

$a_1 = f(1) = 1$
$a_2 = f(2) = \frac{1}{2}$
$a_3 = f(3) = \frac{1}{4}$

in this case $a_n = \left(\frac{1}{2}\right)^{n-1}$

**Representation of sequences**

① Formulas $a_n = f(n)$ = expression in terms of $n$
(closed form)

② Recursive formula $a_n = \begin{cases} 1 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ a_{n-1} + a_{n-2} & \text{if } n > 2 \end{cases}$

The Fibonacci sequence

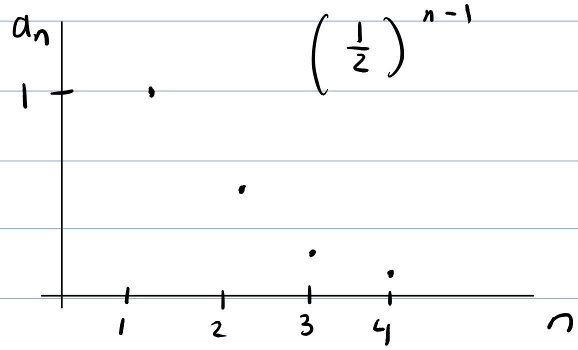
1, 1, 2, 3, 5, 8, 13, ...

③ 2, 3, 5, 7, 11, 13, ...

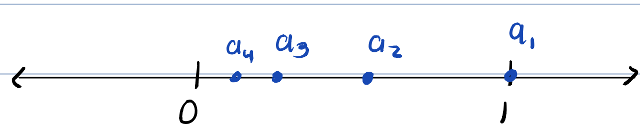
In words : "The prime numbers"

④ Graphically:

Ⓐ $\left(\frac{1}{2}\right)^{n-1}$




Ⓑ Number line:



Some terms and symbols:

$\{a_n\}$ is a sequence; $a_n$ is the $n^{\text{th}}$ term in the sequence $\{a_n\}$; $\forall$ means "for all";

$\mathbb{N}$ is the set of natural or counting numbers, $\mathbb{N} = \{1, 2, 3, ...\}$

## Definition

A sequence is any function whose domain set is the natural numbers, $\mathbb{N}$. The range set can be anything, but for us will commonly be the set of all real numbers, $\mathbb{R}$.

## Vocabulary

A sequence $\{a_n\}$ is said to be:

- **positive** if $a_n \geq 0 \; \forall n \in \mathbb{N}$.
- **negative** if $a_n \leq 0 \; \forall n \in \mathbb{N}$.
- **alternating** if $a_n \cdot a_{n+1} < 0 \; \forall n \in \mathbb{N}$. (Consecutive terms have opposite sign).
- **increasing** if $a_{n+1} \geq a_n \; \forall n \in \mathbb{N}$. (Every term is greater than or equal to all terms before it).
- **decreasing** if $a_{n+1} \leq a_n \; \forall n \in \mathbb{N}$. (Every term is less than or equal to all terms before it).
- **monotonic** if it is either entirely increasing or decreasing. → does not change
- **bounded below** by a number $m$ if $a_n \geq m \; \forall n \in \mathbb{N}$. range
- **bounded above** by a number $M$ if $a_n \leq M \; \forall n \in \mathbb{N}$.
- **bounded** if it is both bounded below and bounded above.

## Convergence

We say that a sequence $\{a_n\}$ **converges** to a number $L$, and we write $\lim_{n\to\infty} a_n = L$, if for every $\epsilon > 0$ we can find a number $N_\epsilon$ so that $|a_n - L| < \epsilon$ for all $n > N_\epsilon$.

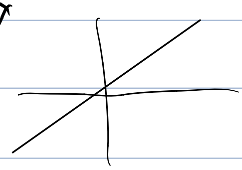
Any sequence which does not converge is said to **diverge**.

## Bounded Monotone Convergence Theorem

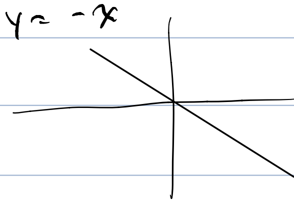
Every bounded monotonic sequence in $\mathbb{R}$ converges.

ex, monotomic

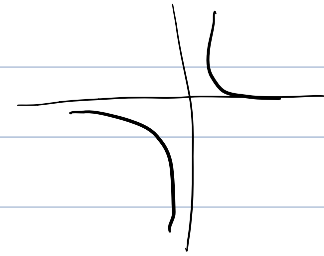
$y = x$



$y = -x$

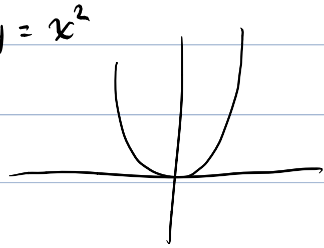


$y = \frac{1}{x}$

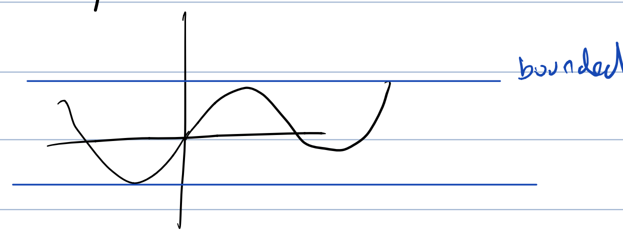


ex, non monotomic

$y = x^2$



$y = \sin x$




ex, bounded below : $\sin x$

bounded above, but not below : $x^2$

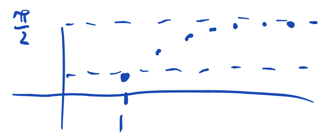
1. Given an example of the following if possible. If it is not possible, briefly explain why.

   a) an increasing sequence that is not bounded

   $a_n = n$ or $\{n\}$

   b) an increasing sequence that is bounded

   $a_n = \dfrac{1}{1+e^{-n}}$ or $\{\arctan(n)\}$

   least upper bound

   greatest lower bound

   c) a decreasing sequence that is not bounded above

   Not possible. ($a_n = -n$ is bounded above!) [There is a bound also here, but it is not the greatest lower bound

   ↳ it's not a function

   d) a bounded sequence that is not monotonic

   $a_n = \sin(n)$ or $\{(-1)^n\}$

   e) an alternating sequence that is bounded below but not bounded above

   Possible $a_n = \begin{cases} e^n & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$

   f) an alternating sequence that is increasing

   Impossible (need to go from ⊕ to ⊖ → decrease)

2. Show using **three different** methods that $\left\{\dfrac{n^2+1}{n}\right\}$ is increasing.

3. Let $a_n = \begin{cases} 1 & \text{if} \quad n = 1 \\ \sqrt{6 + a_{n-1}} & \text{if} \quad n > 1 \end{cases}$

Use induction to show that:

a) $\{a_n\}$ is increasing.

b) $\{a_n\}$ is bounded above by 3.

ex. A closed formula which can form a sequence but not a continuous function

$$\sum_{n=0}^{n=\infty} \frac{f^{(n)}(x)}{n!}(x-a)^n$$

$n! = n(n-1)\ldots 3\cdot 2\cdot 1$

$n \in \mathbb{N}$

$0! = 1$

$\pi!$ (except: gamma function is a natural extension of the $(\ )!$)

Showing that a sequence is increasing/decreasing

5 ways:

Consider $\left\{\frac{n}{5^n}\right\}$. Show that it is decreasing

① If $a_n = f(n)$ where $f$ is a differentiable function

take $f'$, show that $f' < 0$ for $x > 1$

(or for $x > c$, then we say "ultimately")

ex. Let $f(x) = \frac{x}{5^x}$

$$f'(x) = \frac{\cancel{5^x} - x\cancel{5^x}\ln 5}{(5^x)^{\cancel{2}}} = \frac{1 - x\ln 5}{5^x}$$ $\rightarrow$ (+)

$\frac{1 - x\ln 5}{5^x} < 0$ when $1 - x\ln 5 < 0$

$1 < x\ln 5$

$x > \frac{1}{\ln 5} \approx 0.6$

$\therefore$ when $x > 1$, f decreases

② Use the definition (always valid)

$a_n > a_{n+1}$

$a_n - a_{n+1} > 0$

$a_n = \frac{n}{5^n}$ , $a_{n+1} = \frac{n+1}{5^{n+1}}$

$a_n - a_{n+1} > 0$

$\left(\frac{5^1}{5^1}\right)\frac{n}{5^n} - \frac{n+1}{5^{n+1}}$ common denominator

$= \frac{5n - (n+1)}{5^{n+1}}$

$= \frac{4n-1}{5^{n+1}}$ $\frac{+}{+}$

③ Ratio

$\frac{a_n}{a_{n+1}} > \frac{a_{n+1}}{a_{n+1}} > 0$

$$\frac{a_n}{a_{n+1}} > 1$$

$$\frac{a_n}{a_{n+1}} \overset{?}{>} 1$$

$$\frac{n}{5^n} \cdot \frac{5^{n+1}}{n+1}$$

$$= 5 \cdot \frac{n}{n+1} > 5 \cdot \frac{n}{n+n}$$ Comparison

Comparison?

$$\frac{n}{n+n} \cdot 5 \qquad \frac{\cancel{n}}{2\cancel{n}}$$

$n > 1$

$\frac{1}{1} > \frac{1}{n}$

$$\frac{1}{2} \cdot 5$$

$$\frac{5}{2} > 1 \checkmark$$

④ Induction (most often with recursive sequences)

⑤ If $\{a_n\} \uparrow$ with $a_n > 0$

then $\left\{\frac{1}{a_n}\right\} \downarrow$

Feb. 2

Review $\{a_n\}_{n=1}^{\infty} = \{a_n\} = a_1, a_2, a_3, \ldots$

→ closed formula (function)

→ recursively (previous number)

→ words

→ graphically (dots)

↳ discrete (not connected)

(at any spot (not like step fcn which has cont's pieces)

ex. Methods to show that a sequence is increasing:

① Def: $a_{n+1} \geq a_n$ for all $n \in \mathbb{N} = \{1, 2, \ldots\}$

or

$a_{n+1} - a_n \geq 0$

② If $a_{n+1} \geq a_n \geq 0 \rightarrow \frac{a_{n+1}}{a_n} \geq 1$

③ If $a_n = f(n)$, $f$ diff'ble ; $f'(x) \geq 0$ for all $x \geq 1$

④ Induction

⑤ $\{a_n\} \downarrow$ then $\{\frac{1}{a_n}\} \uparrow$ for $a_n > 0$

2. Show using **three different** methods that $\underbrace{\left\{\frac{n^2+1}{n}\right\}}_{a}$ is increasing.

① $a_n = \dfrac{n^2+1}{n}$

$a_{n+1} = \dfrac{(n+1)^2+1}{n+1}$

$a$ is inc. if $a_{n+1} - a_n \geq 0$

$$\frac{(n+1)^2+1}{n+1}\cdot\frac{n}{n} - \frac{n^2+1}{n}\cdot\frac{n+1}{n+1}$$

$$= \frac{n\left[(n+1)^2+1\right] - (n^2+1)(n+1)}{n(n+1)}$$

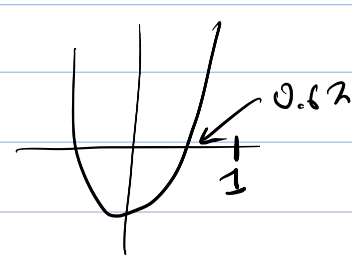
$$= \frac{n(n^2+2n+2) - (n^2+1)(n+1)}{n(n+1)}$$

$$= \frac{\cancel{n^3}+2n^2+2n-(\cancel{n^3}+n^2+n+1)}{n(n+1)} = \frac{n^2+n-1}{n(n+1)}$$

Show that $n^2+n-1 \geq 0$

$$n = \frac{-1\pm\sqrt{1^2-4(1)(-1)}}{2}$$

$$= \frac{-1\pm\sqrt{5}}{2} = \pm 0.62$$



(2) $a$ is inc. if $\dfrac{a_{n+1}}{a_n} \geq 1$

$$\frac{(n+1)^2+1}{n+1} \cdot \frac{n}{n^2+1}$$

$$= \frac{n\left[(n+1)^2+1\right]}{(n+1)(n^2+1)}$$

$$= \frac{n\left(n^2+2n+2\right)}{(n+1)(n^2+1)}$$

$$= \frac{n^3+2n^2+2n}{n^3+n^2+n+1} = \frac{\cancel{n}(n^2+2n+2)}{\cancel{n}(n^2+n+\underbrace{1+\frac{1}{n}}_{<2 \text{ for } n\geq 1})} \geq 1 \quad \text{for } n \geq 1$$

(3) $f(x) = \dfrac{x^2+1}{x}$

$f(x) = x + \dfrac{1}{x}$ $\qquad x^{-1}$

$f'(x) = 1 - \dfrac{1}{x^2} \geq 0$

$1 > \frac{1}{x^2}$ ✓ for $x^2 \geq 1 \to x \geq 1$

3. Let $a_n = \begin{cases} 1 & \text{if } n = 1 \\ \sqrt{6 + a_{n-1}} & \text{if } n > 1 \end{cases}$

Use induction to show that:

a) $\{a_n\}$ is increasing.

$\{a_n\} = 1, \sqrt{7}, \sqrt{6+\sqrt{7}}, \sqrt{6+\sqrt{6+\sqrt{7}}}, \ldots$

a) $a_n$ is increasing

Base case: $a_2 > a_1$

$\sqrt{7} > 1$ (or $\sqrt{7} > \sqrt{1} = 1$)

Inductive step: Assume: $a_{k+1} > a_k$ : $S(k)$

Show : $a_{k+2} > a_{k+1}$ : $S(k+1)$

$a_k = \sqrt{6 + a_{k-1}}$

$a_{k+1} = \sqrt{6 + a_k}$

$a_{k+2} = \sqrt{6 + a_{k+1}}$

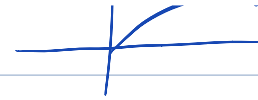
Note:

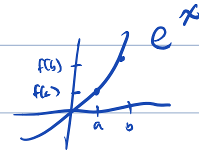
If $a < b$

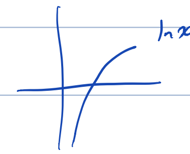
when is $f(a) < f(b)$?

$\therefore$ $f$ is monotomic increasing

$\sqrt{x}$






Examine:

$$a_{k+1} > a_k$$

$$6 + a_{k+1} > 6 + a_k$$

$$\sqrt{6 + a_{k+1}} > \sqrt{6 + a_k}$$

$$a_{k+2} > a_{k+1} \quad \therefore \{a_n\} \text{ is incr by induction}$$

b) $\{a_n\}$ is bounded above by 3.

Show that $a_n \leq 3$ for all $n$

By induction:

Base case: $a_1 \leq 3$

$1 \leq 3$ ✓

Inductive step: Assume: $a_k \leq 3$

Show: $a_{k+1} \leq 3$

$$a_{k+1} = \sqrt{6 + \underbrace{a_k}_{\leq 3}} \leq \sqrt{6 + 3} = \leq \sqrt{9} \leq 3$$

or

$$a_k \leq 3$$

$$a_k + 6 \leq 3 + 6 = 9$$

$$\sqrt{a_k + 6} \leq 3$$

$a_{k+1} \leq 3$ ✓

## 3 - Limits of Sequences

$$\lim_{n \to \infty} a_n = L \in \mathbb{R} \quad (L \neq \pm\infty)$$

Intuitively: as $n$ gets larger & larger, $a_n$ gets closer & closer to $L$

### Formal Definition

$\lim_{n \to \infty} a_n = L$ if $\forall \, \varepsilon > 0 \; \exists \, N_\varepsilon \ni n \geq N_\varepsilon$

$\Rightarrow |a_n - L| < \varepsilon$

In words: Let $\{a_n\}$ be a sequence and let $L \in \mathbb{R}$,

we say that $\{a_n\}$ converges to $L$ and we

write $\lim_{n \to \infty} a_n = L$ if for any $\varepsilon > 0$, there

exists a corresponding number $N_\varepsilon$ such that

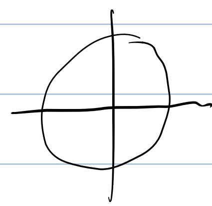
for all $n \geq N_\varepsilon$, we are guaranteed that all the subsequent values

$|a_n - L| < \varepsilon$

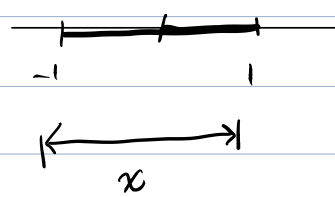
Feb. 5

Unpacking

$\|\vec{v}\| < 1$ $\quad \mathbb{R}^2$

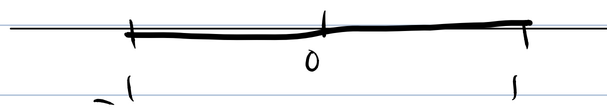


$\|\vec{v}\| < 1$ $\quad \mathbb{R}^1$




same as $|x| < 1$

① So, $\underbrace{|a_n - L|}_{\text{distance}} < \varepsilon$




$x \in (-1, 1)$

$|x| < 1$

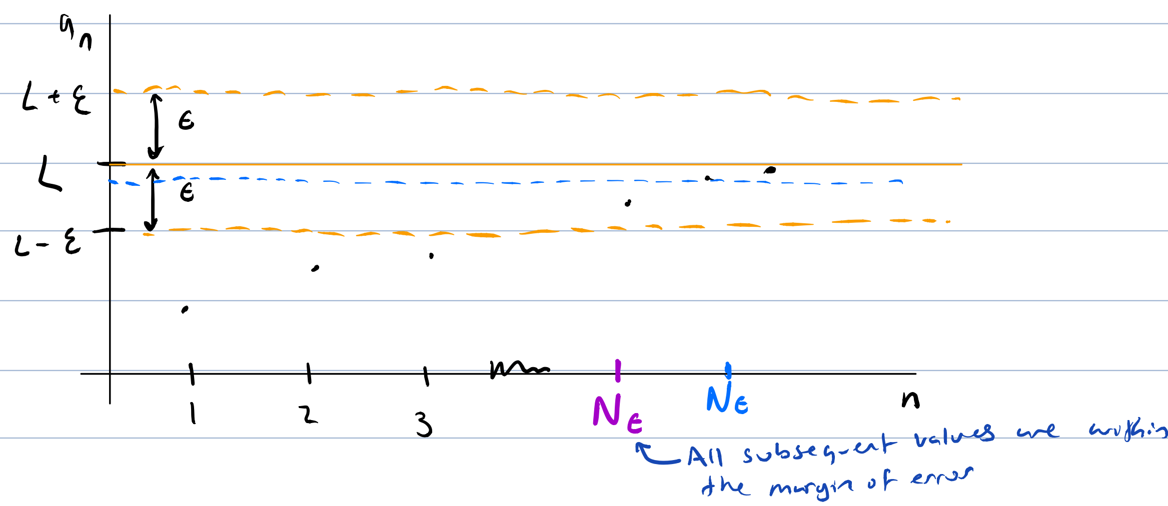
② $-1 < x < 1$

Putting ① and ② together

$$|a_n - L| < \varepsilon$$

$$-\varepsilon < a_n - L < \varepsilon$$

$$L - \varepsilon < a_n < L + \varepsilon$$




think of $\varepsilon$ as a margin of error around $L$

ex. Consider the sequence $\left\{ a_n = 1 - \frac{1}{\sqrt{n}} \right\}$

From Cal I, we expect that $\lim\limits_{n\to\infty} a_n = 1 = L$

a) Let $\varepsilon = 0.001$, find a corresponding $N_\varepsilon = N_{0.001}$ so that the sequence remains in the interval $(0.999, 1.001)$ for all terms past $N_{0.001}$ (for all $n > N_\varepsilon$)

We want

$$|a_n - L| < \varepsilon$$

$$\left|\cancel{1} - \frac{1}{\sqrt{n}} - \cancel{1}\right| < 0.001$$

$$\left|-\frac{1}{\sqrt{n}}\right| < 0.001$$

$$\frac{1}{\sqrt{n}} < 0.001$$

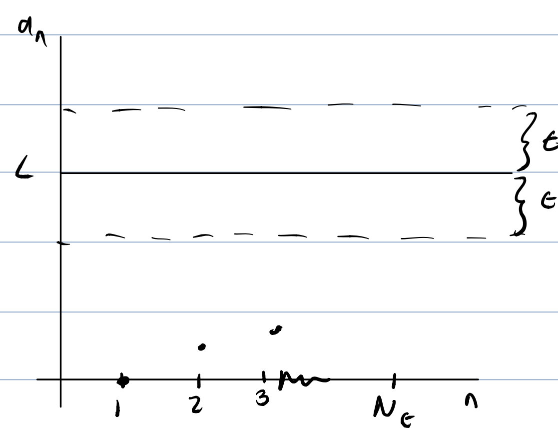
$$\frac{1}{0.001} < \sqrt{n}$$

$$(10^3)^2 < (\sqrt{n})^2$$

→ you can square since

for positive $n$, monotonic increasing

$$10^6 < n$$




but this $\varepsilon$ isn't special, we can use any $\varepsilon$!

b) Use the definition of a limit to show that

$$\lim_{n\to\infty}\left(1-\frac{1}{\sqrt{n}}\right)=1$$

Let $\varepsilon>0$ be given

$$|a_n - L| < \varepsilon$$

$$\left|\cancel{1}-\frac{1}{\sqrt{n}}\cancel{-1}\right|<\varepsilon$$

$$\left|-\frac{1}{\sqrt{n}}\right|<\varepsilon$$

$$\frac{1}{\sqrt{n}}<\varepsilon$$

$$\frac{1}{\varepsilon}<\sqrt{n}$$

$$\frac{1}{\varepsilon^2}<\left(\sqrt{n}\right)^2$$

$$\frac{1}{\varepsilon^2}<n$$

From the aside, take $n > N_\varepsilon$ & show that

$$|a_n - L| < \varepsilon$$

$$\left|1-\frac{1}{\sqrt{n}}-1\right|<\varepsilon$$

If $n > N_\varepsilon = \frac{1}{\varepsilon^2}$,

$$\sqrt{n} > \frac{1}{\varepsilon}$$

$$\varepsilon > \frac{1}{\sqrt{n}}$$

$$\varepsilon > \left| \frac{-1}{\sqrt{n}} \right|$$

$$\left| 1 - \frac{1}{\sqrt{n}} - 1 \right| < \varepsilon$$

$|a_n - L| < \varepsilon$ as required

ex, For you:

① $\left\{ a_n = \frac{n}{3n-1} \right\}$

a) Find $N_\varepsilon$ if $\varepsilon = 0.01$

b) sketch

c) prove the limit using the definition

a) $\lim_{n \to \infty} a_n = \frac{1}{3} = L$

$|a_n - L| < \varepsilon$

$\left| \frac{n}{3n-1} - \frac{1}{3} \right| < 0.01$

$\left| \frac{\cancel{3n} - (\cancel{3n}-1)}{3(3n-1)} \right| < 0.01$

$\left| \frac{1}{9n-3} \right| < 0.01$

$\frac{1}{9n-3} < 0.01$ since $n > 0$, $9n-3 > 0$

$\frac{1}{0.01} < 9n-3$

$\frac{10^2+3}{9} < n$

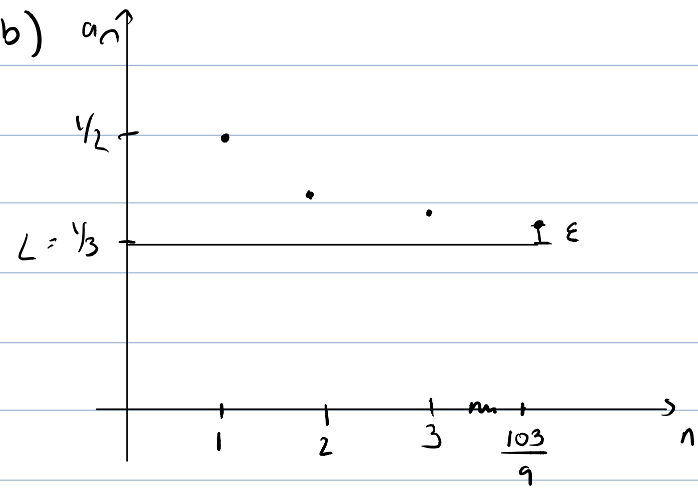
$N_\varepsilon : n > \frac{103}{9}$

↳ or next integer (if $n \geq N_\varepsilon$)

$\frac{2}{6-1} = \frac{2}{5} = \frac{4}{10}$

$\frac{3}{9-1} = \frac{3}{8}$

b)




c) Take $n > N_\varepsilon$ and show that

$$|a_n - L| < \varepsilon$$

If $n > N_\varepsilon = \dfrac{\frac{1}{\varepsilon} + 3}{9}$ :

$$9n > \frac{1}{\varepsilon} + 3$$

$$9n - 3 > \frac{1}{\varepsilon}$$

$$\varepsilon > \frac{1}{9n-3}$$

$$\varepsilon > \frac{1}{9n-3} \rightsquigarrow = \frac{1}{3n-1} - \frac{1}{3}$$

$$\varepsilon > \frac{1}{3(3n-1)}$$

$$\varepsilon > \frac{3n - 3n + 1}{3(3n-1)}$$

$$\varepsilon > \frac{3n - (3n-1)}{3(3n-1)}$$

$$\varepsilon > \frac{\cancel{3}n}{\cancel{3}(3n-1)} - \frac{\cancel{3n-1}}{3\cancel{(3n-1)}}$$

$$\varepsilon > \frac{n}{3n-1} - \frac{1}{3}$$

$$\varepsilon > \left| \frac{n}{3n-1} - \frac{1}{3} \right| \text{ for } n \geq 1$$

$\varepsilon > |a_n - L|$ as required.

② repeat with $\{a_n = r^n\}$ with $|r| < 1$

$$\lim_{n \to \infty} r^n = 0$$

a) if $|r| < 1$. Let $\varepsilon = 0.01$

$$|a_n - L| < \varepsilon$$

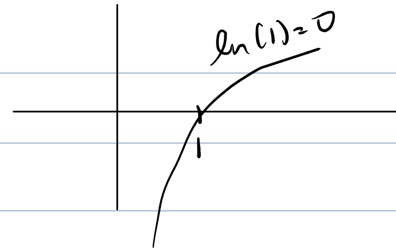
$$|r^n| < 0.01 \quad |r^n| = |r|^n$$

$$|r|^n < 0.01$$

monotonic increasing

$$n \ln|r| < \ln(0.01)$$

$$n > \frac{\ln(0.01)}{\ln|r|} \quad -$$




# Proof of a Limit Law

If $\{a_n\}$ and $\{b_n\}$ are sequences and if

$\lim\limits_{n\to\infty} a_n = L_a$ and $\lim\limits_{n\to\infty} b_n = L_b$.

$a_n$ - or just call it $c_n$

Then, $\lim\limits_{n\to\infty} (a_n + b_n) = L_a + L_b$

$L$

**Proof:** Let $\varepsilon > 0$ be given. Find $N_\varepsilon$ so that

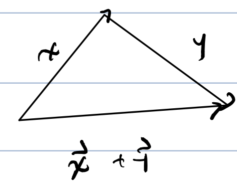
$$|c_n - L| < \varepsilon$$

$$|(a_n + b_n) - (L_a + L_b)| < \varepsilon$$

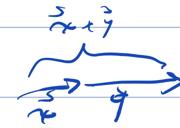
**ingredient:** $|x + y| \leq |x| + |y|$

triangle inequality

$$\|\vec{x} + \vec{y}\| = \|\vec{x}\| + \|\vec{y}\|$$




note: equal when




triangle ineq $\leq \left|(a_n - L_a) + (b_n - L_b)\right| < \varepsilon$

triangle ineq.

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ take $N_\varepsilon = \text{Max}\left(N_{\frac{\varepsilon}{2},a}, N_{\frac{\varepsilon}{2},b}\right)$

**Note:** All the limit laws are provable from the definition

A consequence of this is that all your Cal I rules apply.

- squeeze theorem
- l'Hopital's rule (need $a_n = f(n)$ where $f$ is diff)
- comparison rules

etc.

## Growth Hierarchy

rates of growth of famous sequences/functions

in order $\ln n \ll n^p \ll e^n \ll n! \ll n^n$

$p > 1$

grows much faster

We say that $\{a_n\}$ approaches $\infty$ slower than $\{b_n\}$

and we write $a_n \ll b_n$ if $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ $\left( \text{or } \displaystyle\lim_{n\to\infty} \frac{b^n}{a^n} = \infty \right)$

<u>ex.</u> For you: Show: a) $\ln(n)$ , $n^p$ (l'hopitals)

b) $e^n \ll n!$ (no l'hopitals $\to$ Comparison & Squeeze)

a) Show that $\displaystyle\lim_{n\to\infty} \frac{\ln(n)}{n^p} = 0$

$$\lim_{n\to\infty} \frac{\ln(n)}{n^p}$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{p\,n^{p-1}}$$

$$= \lim_{n\to\infty} \frac{1}{n^1 p\, n^{p-1}}$$

$$= \lim_{n\to\infty} \frac{1}{p\,n^p}$$

$$= \frac{1}{\infty} = 0 \quad \checkmark$$

So $\ln n \ll n^p$

$-n$

b) Show that $\lim_{n\to\infty} \frac{e}{n!} = 0$

$$\lim_{n\to\infty} \frac{e^n}{n!}$$

$$= \lim_{n\to\infty} \underbrace{\frac{e}{1} \cdot \frac{e}{2}}_{K} \cdot \overbrace{\left| \frac{e}{3} \cdots \frac{e}{n-1} \right|}^{<1} \cdot \frac{e}{n}$$

$$0 < \frac{e^n}{n!} < \frac{Ke}{n}$$

$\lim_{n\to\infty} 0 = 0$ & $\lim_{n\to\infty} \frac{ke}{n} = 0$

$\therefore \lim_{n\to\infty} \frac{e^n}{n!} = 0$ also by the squeeze theorem

Feb. 8

Famous limit: $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$

From finance experiment:

compound interest: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

$P = \$1$, $r = 100\%$

$t = 1$ year, $n = \#$ compounds per year

or $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$

ex. $\lim\limits_{n\to\infty} \dfrac{n^n}{\ln n} = \infty$ by hierarchy

ex. $\lim\limits_{n\to\infty} \dfrac{n^7}{n!} = 0$ by hierarchy

ex. $\lim\limits_{n\to\infty} \left(\dfrac{n}{n+1}\right)^n = \dfrac{1}{e}$ by definition of e

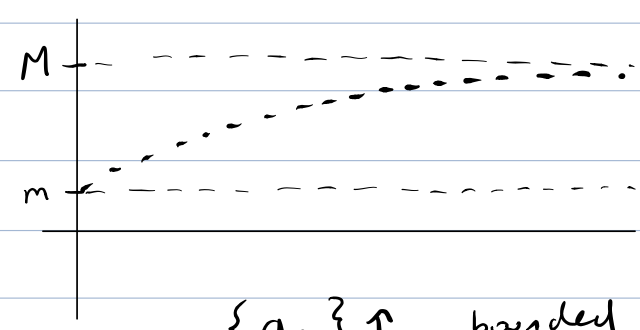
$$= \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n$$

$$= \frac{1}{\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$$

# Bounded Monotone Convergence Theorem

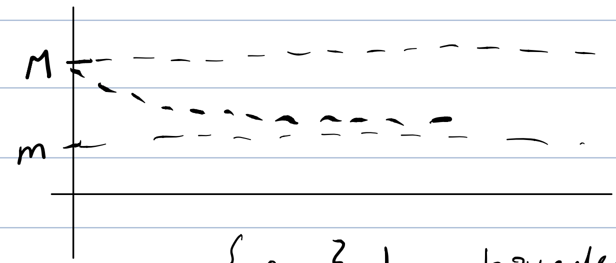
Every bounded monotone sequence converges

ex.




$\{a_n\}\uparrow$, bounded

ex.




$\{a_n\} \downarrow$, bounded

Note: This tells us that there is a limit, but not what ~~the value~~ the value of the limit is.

Proof: Case $\{a_n\} \uparrow$, bounded

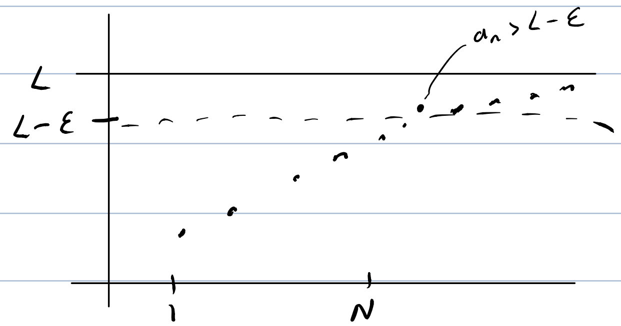
Since $\{a_n\}$ bounded, it is bounded above.

Let $L$ be the least of its upper bounds. guaranteed as a property of $\mathbb{R}$

Claim: $\lim\limits_{n\to\infty} a_n = L$ (Recall: For all $\varepsilon > 0$, there exists an $N_\varepsilon$ such that $n \geq N_\varepsilon$ implies that $|a_n - L| < \varepsilon$)

Let $\varepsilon > 0$ be given.

Consider $L - \varepsilon < L$




bounded

Since $L$ was the least upper bound, there must be an $N$ so that $a_N > L - \varepsilon$

increasing

But $\{a_n\} \uparrow$

For all $n > N$:

$$a_n > a_N > L - \varepsilon$$

$$a_n > L - \varepsilon$$

$$\varepsilon > L - a_n$$

$$L - a_n < \varepsilon$$

$|a_n - L| < \varepsilon$ as required.

ex. Recall previous example.

$$a_n = \begin{cases} 1 & \text{if } n=1 \\ \sqrt{6+a_{n-1}} & \text{if } n \geq 2 \end{cases}$$

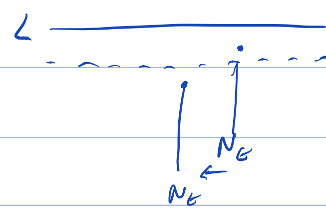
We showed: $\{a_n\} \uparrow$ (by induction)

$a_n < 3$ (by induction)

$\therefore$ BMCT $\lim_{n\to\infty} a_n$ exists

Let $\lim_{n\to\infty} a_n = L$. Find $L$. (Fact: if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} a_{n-1} = L$)

$N_\varepsilon$ just gets shifted back




examine: $a_n = \sqrt{6+a_{n-1}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{6+a_{n-1}}$$

$$\lim_{n\to\infty} a_n = \sqrt{6 + \lim_{n\to\infty} a_{n-1}}$$

$$(+)\ L = (+)\sqrt{6+L}$$

$$L^2 = 6 + L$$

$$L^2 - L - 6 = 0$$

$$(L-3)(L+2)$$

$L = \boxed{3}$ and $-2$

since $a_n \geq 1$ $\quad a_n \uparrow$

$5^2 = (-5)^2$

$\cancel{5 \overset{?}{=} -5}$

ex. For you:

Let $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)}$

a) Show that $\{a_n\}$ is bounded below

$a_n > 0$ for all $n$

b) Show that $\{a_n\} \downarrow$

Ratio Test: show that $\dfrac{a_{n+1}}{a_n} < 1$

$$\frac{\cancel{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}\,(2(n+1)-1)}{\cancel{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)}\,(2(n+1))} \cdot \frac{\cancel{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)}}{\cancel{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}}$$

$= \dfrac{2n+1}{2n+2} < 1 \quad (2n+2 > 2n+1)$

c) Find a (i) Recursive formula for $a_n$

(ii) Closed-form

(i) Recursive

$$a_n = \begin{cases} \frac{1}{2} & \text{if } n=1 \\ a_{n-1} \cdot \frac{2n-1}{2n} & \text{if } n>1 \end{cases}$$

(ii) Closed-form $\quad 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \ldots \cdot (2n-1)(2n)$

$$a_n = \frac{(2n)!}{{}^{?}\sim 2^{2n} \cdot 2n!}$$

d) Does $\lim_{n\to\infty} a_n$ exist? Can you find it?

Yes. By the BMCT, since $\{a_n\}$ is bounded below and is $\downarrow$, then $\lim_{n\to\infty} a_n$ exists

It's hard to find though