

August 26, 2016

LIMITS

Consider $y = f(x) = \frac{x^2 - 4}{x - 2}$ $\text{DOM} = \mathbb{R} \setminus \{2\}$

x	1.9	1.99	1.999	1.9999	...	
$y = f(x) = \frac{x^2 - 4}{x - 2}$	3.9	3.99	3.999	3.9999	3.99999	...

$\rightarrow 2$

$\Rightarrow 4$

x	2.1	2.01	2.001	2.0001	...	
$y = f(x) = \frac{x^2 - 4}{x - 2}$	4.1	4.01	4.001	4.0001	4.00001	...

$\rightarrow 2$

$\Rightarrow 4$

As the x -values approach '2' from both sides, but never touch '2', the $f(x)$ values approach a single value '4'. '4' is called the limiting value.

$$\lim_{x \rightarrow 2} f(x) = 4$$

i.e. $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4$

August 29, 2016

observe

$$f(x) = \frac{x^2 - 4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2$$

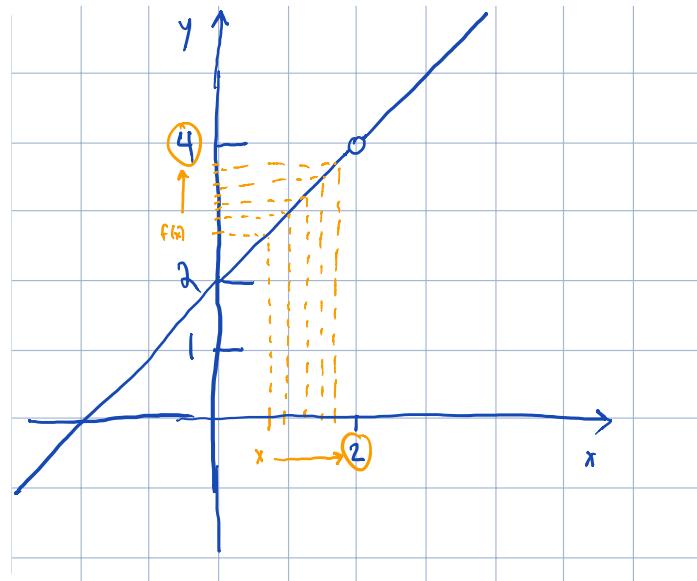
so $f(x) = x+2$ as long as $x \neq 2$

$$f(x) = x+2 \quad (x \neq 2)$$

$$g(x) = 2$$

f & g are not the same funcs, but limit is still 4.

$$\lim_{x \rightarrow 2} f(x) = 4$$



ex/ consider $g(x) = 3$

$$\text{Find } \lim_{x \rightarrow 1} g(x)$$

$$\begin{array}{c|ccc} x & 0.9 & 0.99 & 0.999 \\ \hline g(x) & 3 & 3 & 3 \end{array} \rightarrow \frac{1}{3}$$

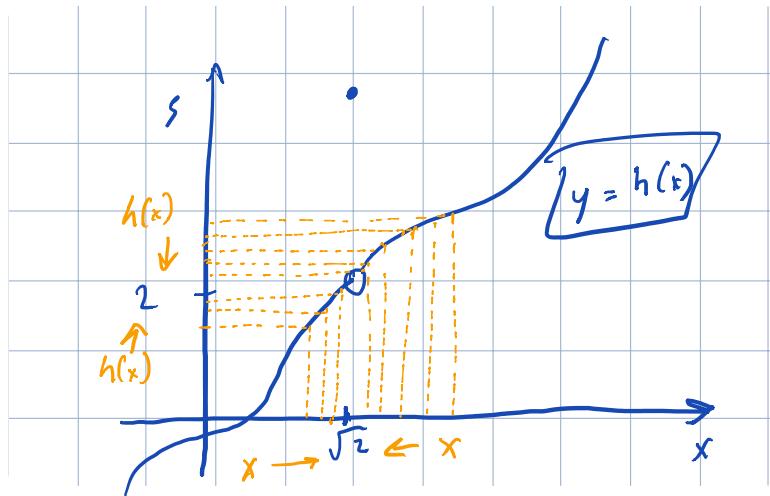
$$\begin{array}{c|ccc} x & 1.1 & 1.01 & 1.001 \\ \hline g(x) & 3 & 3 & 3 \end{array} \rightarrow \frac{1}{3}$$

$$\lim_{x \rightarrow 1} g(x) = 3$$

$x \rightarrow 1$

<u>Rule ①</u>	$\lim_{x \rightarrow a} C = C$	$\Leftrightarrow \lim_{x \rightarrow -8} 12 = 12$
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ex/



$$\lim_{x \rightarrow \sqrt{2}} h(x) = 5$$

$$h(\sqrt{2}) = 5$$

Quicker method - Substitution Rule

$$\text{ex/ } \lim_{x \rightarrow 3} (x+5) = 8$$

↑
replace 'x'
with 3

$$\text{ex/ } \lim_{x \rightarrow 5} \sqrt{\frac{x+4}{x-1}} = \sqrt{\frac{5^2+4}{5-1}} = \sqrt{\frac{29}{4}}$$

$$\text{ex/ } \lim_{x \rightarrow 3} \frac{x^2+1}{x-2} = \lim_{x \rightarrow 3} \frac{3^2+1}{3-2} = 10$$

$$\text{So } \lim_{x \rightarrow 3} = 10$$

$$\text{ex/ } \lim_{x \rightarrow -1} \frac{x+1}{x^2+1} = \underset{\text{check}}{\frac{(-1)+1}{(-1)^2+1}} = \frac{0}{2} = 0 \text{ well defined}$$

$$\text{So } \lim_{x \rightarrow -1} = 0$$

$$\text{Ex/ } \lim_{x \rightarrow 2} \frac{x^2 - 4}{x-2} \stackrel{\text{ch}}{=} \frac{(2)^2 - 4}{2-2} = \frac{0}{0} \leftarrow \text{indeterminate form}$$

In this case, cheating fails to offer any info.

Start over & do some algebra

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x-2} = \lim_{x \rightarrow 2} \frac{(x+2)(x-2)}{x-2} \stackrel{\text{ch}}{=} \lim_{x \rightarrow 2} (x+2) = 4$$

$$\text{Ex/ } \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x-9} \stackrel{\text{ch}}{=} \frac{\sqrt{9} - 3}{9-9} = \frac{0}{0}$$

$$\begin{aligned} &= \lim_{x \rightarrow 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)} \quad \left. \begin{array}{l} \text{rationalize} \\ \text{numerator} \end{array} \right\} \times \left(\frac{\sqrt{x}+3}{\sqrt{x}+3} \right) \text{ SEE RATIONALIZE SECTION} \\ &\quad \left. \begin{array}{l} = \text{get rid of} \\ \sqrt{} \end{array} \right\} \\ &= \lim_{x \rightarrow 9} \frac{x-9}{(x-9)(\sqrt{x}+3)} \\ &= \lim_{x \rightarrow 9} \frac{1}{\sqrt{x}+3} \\ &\stackrel{\text{ch}}{=} \frac{1}{6} \end{aligned}$$

$$\text{So } \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x-9} = \frac{1}{6} \quad \text{don't need to include this step}$$

$$\text{Ex/ } \lim_{x \rightarrow 3} \frac{x-3}{\sqrt{x-2}-1} \left(\frac{0}{0} \right)$$

$$= \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x-2}+1)}{(\sqrt{x-2}-1)(\sqrt{x-2}+1)}$$

$$= \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x-2}+1)}{x-3}$$

$$\begin{aligned} &= \lim_{x \rightarrow 3} (\sqrt{x-2}+1) \quad \text{the moment you cheat,} \\ &\quad \text{the 'lim' symbol gets} \\ &\stackrel{\text{ch}}{=} \sqrt{3-2} + 1 \quad \text{dropped.} \end{aligned}$$

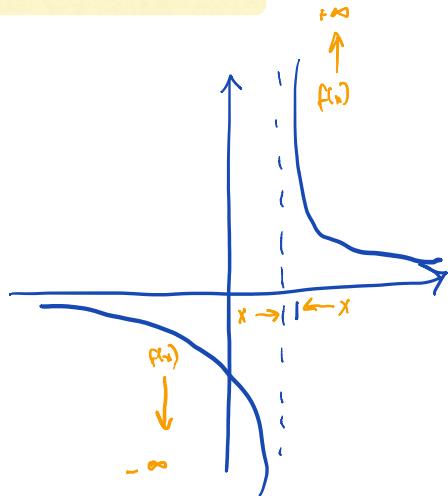
$$= 2$$

For a limit to exist, the target value must be unique and finite.

Unique: both tables need to approach the same value.
Finite: needs to have a target.

$$\text{Ex/ } \lim_{x \rightarrow 1} \frac{1}{x-1} \left(\frac{\infty}{\infty} \frac{1}{0} \right)$$

X	0.1	0.01	0.001	→ 1
$f(x) = \frac{1}{x-1}$	-10	-100	-1000	→ gets closer & closer in the -ve direction ($\rightarrow -\infty$)
				↓
				doesn't approach any number



$$\text{So } \lim_{x \rightarrow 1} \frac{1}{x-1} \left(\frac{\infty}{\infty} \frac{1}{0} \right) \text{ DNE - does not exist}$$

SUMMARY OF RULES SO FAR:

$$\textcircled{1} \lim_{x \rightarrow a} c = c$$

\textcircled{2} (Cheating rule): If you have a $f(x)$ defined by a single algebraic expression (i.e. not piecewise defined or trigs)
of the form $y = \frac{f(x)}{g(x)}$

Sub 'x' by 'a'

If you get \textcircled{A} $\frac{L}{M}$, $M \neq 0$ then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M}$

\textcircled{B} $\frac{0}{0}$, then start over, try algebra

\textcircled{C} $\frac{L}{0}$, $L \neq 0$ then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \infty$

$$\text{Ex/ } \lim_{x \rightarrow 0} \frac{x^2 - x}{x + 1} \stackrel{\text{H}}{\Rightarrow} \frac{0}{1} = 0$$

$$\text{Ex/ } \lim_{x \rightarrow 0} \frac{x+1}{x^2 - x} \stackrel{\text{H}}{\Rightarrow} \frac{1}{0} = \infty$$

$$\text{Ex/ } \lim_{x \rightarrow \sqrt{3}} (x+1) = -\sqrt{3} + 1$$

a constant

$$\text{Ex/ } \lim_{x \rightarrow 12} \frac{\sqrt{x} + 1.78}{24} = \frac{\sqrt{12} + 1.78}{24}$$

Rationalize

$$\begin{aligned} \text{Ex/ } \frac{1}{\sqrt{2}} &= \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2} + \sqrt{3}}{5} &= \frac{(\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3})}{5(\sqrt{2} - \sqrt{3})} \\ &= \frac{2 - \sqrt{4} + \sqrt{6} - \sqrt{3}}{5(\sqrt{2} - \sqrt{3})} \end{aligned}$$

$$\begin{aligned} \text{Ex/ } \frac{\sqrt{x+1} - 7}{x+2} &= \frac{(\sqrt{x+1} - 7)(\sqrt{x+1} + 7)}{(x+2)(\sqrt{x+1} + 7)} \\ &\quad \frac{(x+1) - 49}{(x+2)(\sqrt{x+1} + 7)} \end{aligned}$$

September 1, 2016

$$\text{ex/ } \lim_{x \rightarrow 7} \frac{\sqrt{x} - \sqrt{7}}{7-x} \stackrel{\text{oh}}{=} \frac{0}{0}$$

$$= \lim_{x \rightarrow 7} \frac{(\sqrt{x} - \sqrt{7})(\sqrt{x} + \sqrt{7})}{(7-x)(\sqrt{x} + \sqrt{7})}$$

$$= \lim_{x \rightarrow 7} \frac{x-7}{(7-x)(\sqrt{x} + \sqrt{7})}$$

$$= \lim_{x \rightarrow 7} \frac{-1}{(\cancel{7-x})(\sqrt{x} + \sqrt{7})}$$

$$= \lim_{x \rightarrow 7} \frac{-1}{\sqrt{x} + \sqrt{7}}$$

$$\stackrel{\text{oh}}{=} \frac{-1}{2\sqrt{7}}$$

$$\text{ex/ } \lim_{x \rightarrow 7} \frac{3x+1}{\sqrt{x}-\sqrt{7}}$$

$\stackrel{\text{oh}}{=} \left(\frac{22}{0} \right)$ limit does not exist.

DNE

$$\text{ex/ } \lim_{x \rightarrow 0} (3x^2 + hx + h) \quad \text{constant since } x \rightarrow 0$$

$$\stackrel{\text{oh}}{=} 3(0)^2 + h(0) + h = h$$

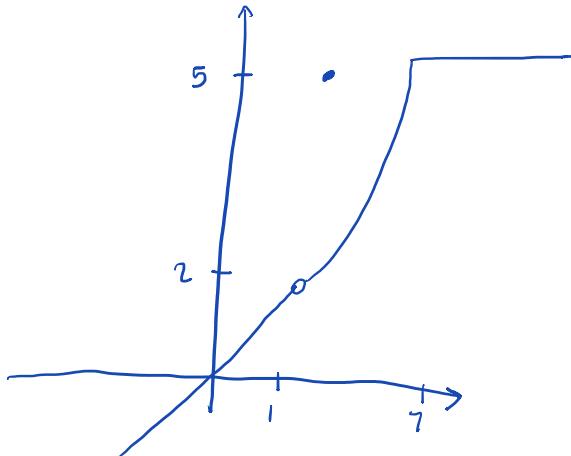
$$\text{ex/ } \lim_{h \rightarrow 0} (3x^2 + hx + h)$$

$h \rightarrow 0$

$$\stackrel{\text{oh}}{=} 3x^2 + 0x + 0 = 3x^2$$

$$\text{ex/ } \lim_{h \rightarrow 0} x^2 = x^2$$

Suppose $y = f(x)$ given by the graph



$$\text{a) } \lim_{x \rightarrow 1} f(x) = 2$$

$$\text{b) } f(1) = 5$$

$$\text{c) } \lim_{x \rightarrow 1} = 5$$

$$\text{d) } d(1) = 5$$

More Rules of Limits

If $\lim_{x \rightarrow a} f(x)$ & $\lim_{x \rightarrow a} g(x)$ both exist, then:

$$\textcircled{1} \quad \lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

$$\textcircled{2} \quad \lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

$$\textcircled{3} \quad \lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

$$\textcircled{4} \quad \lim_{x \rightarrow a} [k f(x)] = k \cdot \lim_{x \rightarrow a} f(x) \quad (k \text{ is a constant})$$

$$\textcircled{5} \quad \lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad (\text{provided } \lim_{x \rightarrow a} g(x) \neq 0)$$

e.g Given $\lim_{x \rightarrow 4} f(x) = -3$, $\lim_{x \rightarrow 4} g(x) = 0$ & $\lim_{x \rightarrow 4} h(x) = 5$

$$\text{find } \lim_{x \rightarrow 4} \left[\frac{2 f(x)}{h(x) - g(x)} \right] = \frac{\lim_{x \rightarrow 4} [2 f(x)]}{\lim_{x \rightarrow 4} [h(x) - g(x)]}$$

$$= \frac{2 \lim_{x \rightarrow 4} f(x)}{\lim_{x \rightarrow 4} h(x) - \lim_{x \rightarrow 4} g(x)} = \frac{2(-3)}{5 - 0} = \frac{-6}{5}$$

One-Sided Limits

(left-sided & right-sided)

$$\text{ex/ } f(x) = \begin{cases} 2x-1 & \text{if } x < 3 \\ 5-x & \text{if } x \geq 3 \end{cases}$$

$$\lim_{x \rightarrow 3^-} f(x)$$

Table ① from the left...

x	2.9	2.999	2.9999	$\dots \rightarrow 3$
$f(x) = 2x-1$	4.8	4.98	4.998	$\dots \rightarrow 5$

from the right...

x	3.1	3.01	3.001	$\dots \rightarrow 3$
$f(x) = 5-x$	1.9	1.99	1.999	$\dots \rightarrow 2$

Table ①, x approaches 3 from left!

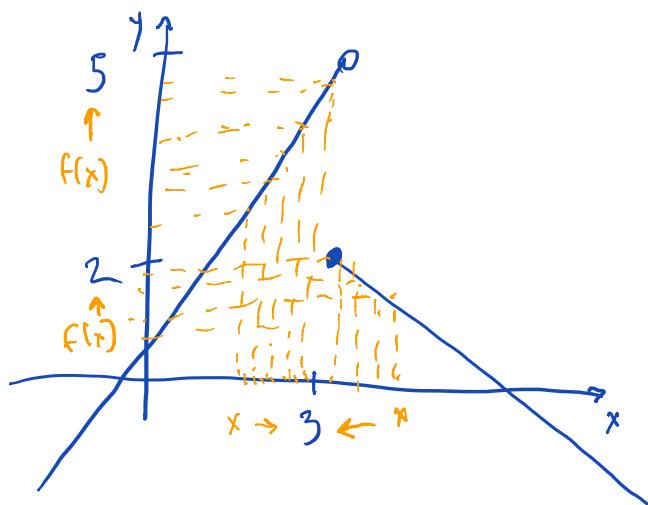
$$\lim_{x \rightarrow 3^-} f(x) = 5$$

Table ②, x approaches 3 from right!

$$\lim_{x \rightarrow 3^+} f(x) = 2$$

Because $\lim_{x \rightarrow 3^-} f(x) \neq \lim_{x \rightarrow 3^+} f(x)$, we can conclude that $\lim_{x \rightarrow 3} f(x)$ DNE

Sketch of Graph



In general: If $\lim_{x \rightarrow a^-} f(x) = L$ & $\lim_{x \rightarrow a^+} f(x) = L$ (L is finite)

then $f(a) = L$

If $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$

then $f(a)$ DNE

The rules above apply for all three types of limits,
BUT BE CAREFUL WITH CHEATING

BACK TO:

$$\text{ex/ } f(x) = \begin{cases} 2x-1 & \text{if } x < 3 \\ 5-x & x \geq 3 \end{cases}$$

$$\lim_{x \rightarrow 3} f(x) \quad \text{1st} \quad \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (2x-1) \stackrel{\text{oh}}{=} 2(3)-1 = 5$$

$(x < 3)$

$$\text{2nd} \quad \lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3^+} (5-x) \stackrel{\text{oh}}{=} 5-3 = 2$$

$(x \geq 3)$

$\lim_{x \rightarrow 3^-} f(x) \neq \lim_{x \rightarrow 3^+} f(x)$ & so $\lim_{x \rightarrow 3} f(x)$ DNE

Ex/ Evaluate $\lim_{x \rightarrow 6} f(x)$ for $f(x)$ as below

Method ①

$$\lim_{x \rightarrow 6^-} f(x) = \lim_{x \rightarrow 6} (5-x) = 5-6 = -1$$

$$\lim_{x \rightarrow 6^+} f(x) = \lim_{x \rightarrow 6} (5-x) = 5-6 = -1$$

$$\therefore \lim_{x \rightarrow 6} f(x) = -1$$

Method ② (more appropriate cases)
if the ans

$$\lim_{x \rightarrow 6} f(x) = \lim_{x \rightarrow 6} (5-x) \stackrel{H}{=} 5-6 = -1$$

Ex/ $g(x) = \begin{cases} x^2 + 5 & \text{if } x < 2 \\ 2x + 5 & \text{if } x > 2 \end{cases}$

$$\lim_{x \rightarrow 2} g(x) ?$$

$$\lim_{x \rightarrow 2^-} g(x) = \lim_{x \rightarrow 2^-} (x^2 + 5) \stackrel{H}{=} (2)^2 + 5 = 9$$

$$\lim_{x \rightarrow 2^+} g(x) = \lim_{x \rightarrow 2^+} (2x + 5) = 9$$

$$\Rightarrow \lim_{x \rightarrow 2} g(x) = 9$$

$$\text{Ex/ } g(x) = \begin{cases} x^3 - 1 & \text{if } x < 4 \\ 3 & \text{if } 4 \leq x < 7 \\ \sqrt{x} & \text{if } x \geq 7 \end{cases}$$

$$\lim_{x \rightarrow 4^-} g(x) ?$$

$$\lim_{x \rightarrow 4^-} g(x) = \lim_{x \rightarrow 4^-} (x^3 - 1) \stackrel{\text{oh}}{=} 63 \quad \left. \begin{array}{l} \\ \Rightarrow \end{array} \right\} \lim_{x \rightarrow 4} g(x) \text{ DNE}$$

$$\lim_{x \rightarrow 4^+} g(x) = \lim_{x \rightarrow 4^+} (3) = 3$$

$$\text{Ex/ } \lim_{x \rightarrow 5} \frac{3-x}{|3-x|} \stackrel{\text{oh}}{=} \frac{3-5}{|3-5|} = \frac{-2}{|-2|} = \frac{-2}{2} = -1$$

$$\text{Ex/ } \lim_{x \rightarrow 3} \frac{3-x}{|3-x|} \stackrel{\text{oh}}{=} \frac{3-3}{|3-3|} = \frac{0}{0} \quad \text{abs. val is like a piecewise defn fun}$$

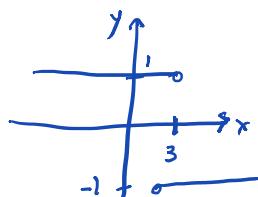
$$|w| = \begin{cases} -w & \text{if } w < 0 \\ w & \text{if } w \geq 0 \end{cases}$$

$$\begin{array}{c} \text{Left side} \\ \lim_{x \rightarrow 3^-} \frac{3-x}{|3-x|} = \lim_{x \rightarrow 3^-} \frac{3-x}{3-x} = \lim_{x \rightarrow 3^-} 1 = 1 \\ (x \leq 3) \end{array}$$

$$|3-x| = \begin{cases} -(3-x) & \text{if } 3-x < 0 \Rightarrow x > 3 \\ 3-x & \text{if } 3-x \geq 0 \Rightarrow x \leq 3 \end{cases}$$

$$\begin{array}{c} \text{Right side} \\ \lim_{x \rightarrow 3^+} \frac{3-x}{|3-x|} = \lim_{x \rightarrow 3^+} \frac{3-x}{-(3-x)} = \lim_{x \rightarrow 3^+} (-1) = -1 \\ (x > 3) \end{array}$$

$$\underline{\text{Concl}} \quad \lim_{x \rightarrow 3} \frac{3-x}{|3-x|} \text{ DNE}$$



Infinite Limits & Vertical Asymptotes

Ex/ Consider the function $f(x) = \frac{1}{x-1}$

Evaluate $\lim_{x \rightarrow 1^+} f(x)$

$$\lim_{x \rightarrow 1^+} \frac{1}{x-1} \stackrel{\text{sh}}{=} \frac{1}{0^-} \quad \text{usually } \infty \text{ resp}$$

x	1.1	1.01	1.001	1.0001...	$\rightarrow 1$
$f(x) = \frac{1}{x-1}$	10	100	1000	10000...	

As x -values approach one from right (& never touch 1), the $f(x)$ values get larger & larger

Answer: $\lim_{x \rightarrow 1^+} \frac{1}{x-1} = +\infty \leftarrow \text{says limit DNE & says why}$

If $x \rightarrow$ vertical asymptote, answer will be $\pm \infty$
 $\approx \left(\frac{-1}{0^+} \right) = +\infty$

Methods to answer this question

$$\textcircled{1} \quad \lim_{x \rightarrow 1^+} \frac{1}{x-1} \stackrel{\text{sh}}{=} \frac{1}{0^-} \begin{cases} +\infty \\ -\infty \end{cases}$$

Test say $x = 1.001$ in $f(x) = \frac{1}{x-1}$

$$= \frac{1}{0.001} = 1000$$

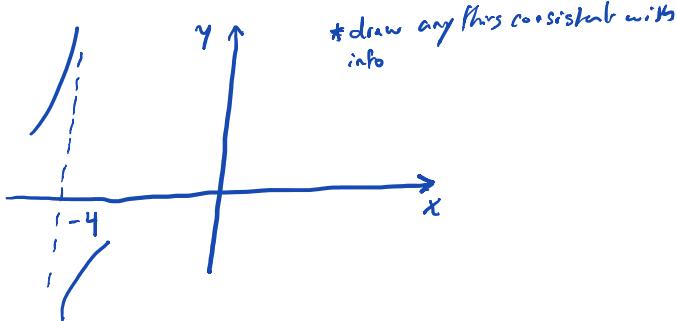
$$\text{So } \lim_{x \rightarrow 1^+} \frac{1}{x-1} = +\infty$$

$$\textcircled{2} \quad \lim_{\substack{x \rightarrow 1^+ \\ (x > 1)}} \frac{1}{x-1} \left(\stackrel{\text{sh}}{=} \frac{1}{0^+} \right) = +\infty$$

Ex/ Find all vertical asymptotes of $g(x) = \frac{3-x^2}{4+x}$ & analyse all relevant limits.

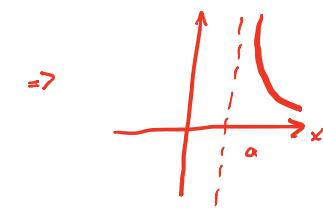
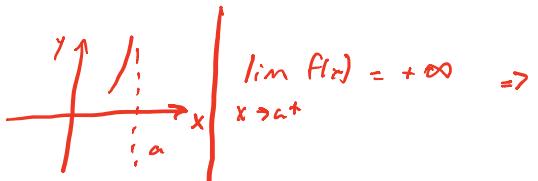
$$g(x) = \lim_{\substack{x \rightarrow -4^+ \\ (x < 4 \\ x \neq 4)}} \frac{3-x^2}{4+x} \left(\stackrel{\text{sh}}{=} \frac{-19}{0^+} \right) = +\infty$$

$$g(x) = \lim_{\substack{x \rightarrow 4^+ \\ (x > 4 \\ x \neq 4)}} \frac{3-x^2}{4+x} \left(\stackrel{\text{sh}}{=} \frac{-19}{0^+} \right) = -\infty$$

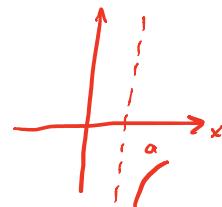
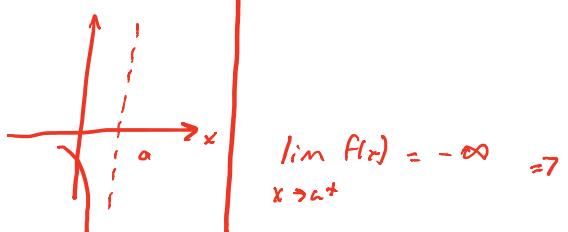


Summary

$$\lim_{x \rightarrow a^-} f(x) = +\infty \Rightarrow$$



$$\lim_{x \rightarrow a^-} f(x) = -\infty$$



$$\text{Evaluate } \lim_{x \rightarrow 3} \frac{4}{9-x^2} \left(\stackrel{\text{ch}}{=} \frac{4}{0} \right)$$

$$\lim_{x \rightarrow 3^+} \frac{4}{9-x^2} \left(\stackrel{\text{ch}}{=} \frac{4}{0^+} \right) = +\infty$$

$$\lim_{x \rightarrow 3^-} \frac{4}{9-x^2} \left(\stackrel{\text{ch}}{=} \frac{4}{0^-} \right) = -\infty$$

Since limits from both sides approaches 2 different numbers

$$\lim_{x \rightarrow 3^+} \frac{4}{(9-x)^2}$$

$$\lim_{x \rightarrow 9^-} \frac{4}{(9-x)^2} \left(\stackrel{\text{ch}}{=} \frac{4}{0^+} \right) = +\infty$$

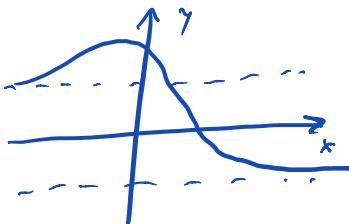
$$\lim_{x \rightarrow 9^+} \frac{4}{(9-x)^2} \left(\stackrel{\text{ch}}{=} \frac{4}{0^+} \right) = +\infty$$

LIMITS AT INFINITY & HORIZONTAL ASYMPTOTES (HA)

Consider $y = f(x) = \frac{1}{x}$

x	10	100	1000	10 000	...	$\rightarrow +\infty$
$f(x) = \frac{1}{x}$	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$	$\frac{1}{10\ 000}$...	$\rightarrow 0$

$$\lim_{x \rightarrow +\infty} f(x) = 0$$



Note
 $\lim_{x \rightarrow \infty} f(x)$
 NONSENSE

If $\lim_{x \rightarrow -\infty} f(x) = L$ (L is a finite #)

then $y = L$ is a HA in -ve direction

If $\lim_{x \rightarrow +\infty} f(x) = L$ (L is a finite #)

then $y = L$ is a HA in +ve direction

September 8, 2016

Rules / Techniques for Computing Limits at Infinity

① $\lim_{x \rightarrow +\infty} \frac{k}{x^r} = 0$ k is any constant & $r > 0$ (also a constant)

② $\lim_{x \rightarrow -\infty} \frac{k}{x^r} = 0$ k is any constant, $r > 0$,
and x^r is defined when $x < 0$

ex. $\lim_{x \rightarrow -\infty} \frac{7}{\sqrt[3]{x}}$, $x^{\frac{1}{3}}$ is defined when $x < 0$

③ All previous rules apply except clearing / substitution

ex. $\lim_{x \rightarrow +\infty} -3x^2 = -\infty$

$x \rightarrow +\infty$

No horizontal asymptote in
either direction since ' $-\infty$ ' is
not finite

ex. $\lim_{x \rightarrow +\infty} (5x^4 - 100x^3 - 1000x^2 - 100x - 10)$
 $= \lim_{x \rightarrow +\infty} 5x^4 \left(1 - \frac{100x^3}{5x^4} - \frac{1000x^2}{5x^4} - \frac{100x}{5x^4} - \frac{10}{5x^4}\right)$
 $= \lim_{x \rightarrow +\infty} 5x^4 \left(1 - \frac{20}{x} - \frac{200}{x^2} - \frac{20}{x^3} - \frac{2}{x^4}\right)$
1
 $= +\infty$

This ex. suggest the following (assume $n \geq 0$)

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0$ ($a_n \neq 0$)

is any polynomial of degree n , then:

$$\lim_{x \rightarrow \pm\infty} (a_n x^n + a_{n-1} x^{n-1} + \dots) = \lim_{x \rightarrow \infty} a_n x^n$$

Note: poly. funcs can never have HA's (also never have VA's)
(except for when $n=0$)

What is a polynomial?

ex. $P(x) = -3x^4 + 1$
 $= -3x^4 + 0x^3 + 0x^2 + 0x + 1$

ex. $Q(x) = 5x^4 - 3x^3 + 7x^2 - 2\sqrt{x} + 1$
 Not a polynomial

ex. $R(x) = -6$
 $= -6x^0$

ex. $\lim_{x \rightarrow \infty} \frac{5x^2 + x + 3}{2x^2 + 5} = \frac{\lim_{x \rightarrow \infty} (5x^2 + x + 3)}{\lim_{x \rightarrow \infty} (2x^2 + 5)} = \frac{+\infty}{+\infty}$ Same as $\frac{\infty}{\infty}$
 Indeterminate form

$$= \lim_{x \rightarrow \infty} \frac{x^2 (5 + \frac{1}{x} + \frac{3}{x^2})}{x^2 (2 + \frac{5}{x^2})} = \frac{5 + 0 + 0}{2 + 0} = \frac{5}{2}$$

Factor highest power in denominator
 Use for limits of $\frac{f(x)}{g(x)}$

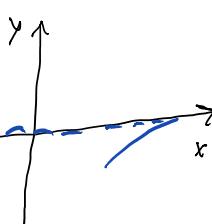
graph of $y = \frac{5x^2 + x + 3}{2x^2 + 5}$ has $y = \frac{5}{2}$ as a HA in +ve direction

ex. Find all HAs of $f(x) = \frac{2x-8}{3x^2-1}$

$$\lim_{x \rightarrow \pm\infty} \frac{x^2 (\frac{2}{x} - \frac{8}{x^2})}{x^2 (3 - \frac{1}{x^2})} = \frac{0}{3} = 0$$

Run out with both infinities

Concl
 $y = 0$ is a HA in both +ve & -ve directions



Ex. Find all HA's of $f(x) = \frac{2x}{\sqrt{x^2+4}}$

$$\lim_{x \rightarrow -\infty} \frac{2x}{\sqrt{x^2+4}} = \lim_{x \rightarrow -\infty} \frac{2x}{\sqrt{x^2(1+\frac{4}{x^2})}} = \lim_{x \rightarrow -\infty} \frac{2x}{|x|\sqrt{1+\frac{4}{x^2}}}$$

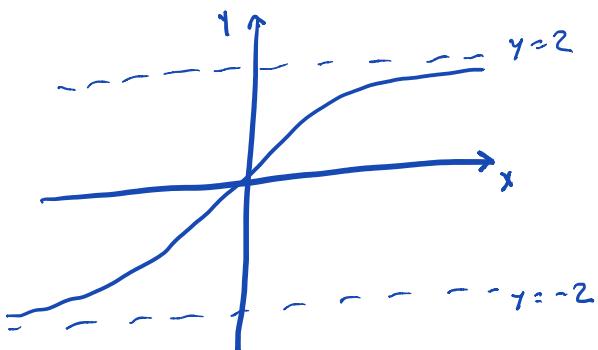
$$= \frac{2x}{-x\sqrt{1+\frac{4}{x^2}}} = \frac{2}{-\sqrt{1+0}} = -2$$

replace
|x| by -x
Since $\lim_{x \rightarrow -\infty}$

$$\lim_{x \rightarrow +\infty} \frac{2x}{\sqrt{x^2+4}} = \lim_{x \rightarrow +\infty} \frac{2x}{\sqrt{x^2(1+\frac{4}{x^2})}} = \lim_{x \rightarrow +\infty} \frac{2x}{|x|\sqrt{1+\frac{4}{x^2}}}$$

$$= \frac{2x}{+x\sqrt{1+\frac{4}{x^2}}} = \frac{2}{\sqrt{1+0}} = 2$$

Concl $y = -2$ is a HA in -ve direction
 $y = +2$ is a HA in +ve direction



Ex.

$$\lim_{x \rightarrow +\infty} \frac{\sqrt[3]{x^2+8}}{x+2}$$

HMWK

Method 1

$$= \lim_{x \rightarrow \infty} \frac{\sqrt[3]{x^2(1 + \frac{8}{x^2})}}{x+2}$$

$$= \lim_{x \rightarrow \infty} \frac{\sqrt[3]{x^2} \sqrt[3]{1 + \frac{8}{x^2}}}{x+2} \quad \cancel{x}$$

$$= \cancel{\lim_{x \rightarrow \infty} \frac{\sqrt[3]{x^2}}{x+2}} \quad \text{Can't cancel only one 'x' = 0}$$

$$= \lim_{x \rightarrow \infty} \frac{x^{2/3} \sqrt[3]{1 + \frac{8}{x^2}}}{x(1 + \frac{2}{x})}$$

$$= \lim_{x \rightarrow \infty} \frac{\sqrt[3]{1 + \frac{8}{x^2}}}{\sqrt[3]{x} (1 + \frac{2}{x})}$$

$$= \lim_{x \rightarrow \infty} \left[\frac{1}{x^{1/3}} \cdot \frac{\sqrt[3]{1 + \frac{8}{x^2}}}{1 + \frac{2}{x}} \right]$$

$$= \lim_{x \rightarrow \infty} \left(\frac{1}{x^{1/3}} \underset{\rightarrow 0}{\circlearrowright} \cdot \lim_{1 + \frac{2}{x} \rightarrow 0} \frac{\sqrt[3]{1 + \frac{8}{x^2}} \rightarrow 0}{1 + \frac{2}{x} \rightarrow 0} \right)$$

$$= \frac{\sqrt[3]{1}}{1} = 0 \cdot 1 = 0$$

Method 2

$$= \lim_{x \rightarrow \infty} \frac{\sqrt[3]{x^3 \left(\frac{1}{x} + \frac{8}{x^2} \right)}}{x+2}$$

$$= \lim_{x \rightarrow \infty} \frac{\cancel{\sqrt[3]{x^3}} \sqrt[3]{\frac{1}{x} + \frac{8}{x^2}}}{x(1 + \frac{2}{x})}$$

$$= \frac{\sqrt[3]{\frac{1}{x} + \frac{8}{x^2}}}{\underset{\rightarrow 0}{\cancel{x}}} \underset{\rightarrow 0}{\cancel{x}}$$

$$1 \rightarrow \frac{2}{x} \rightarrow 0$$

$$= \frac{\sqrt[3]{0}}{1 \rightarrow 0} = \frac{0}{1} = 0$$

Squeeze Theorem

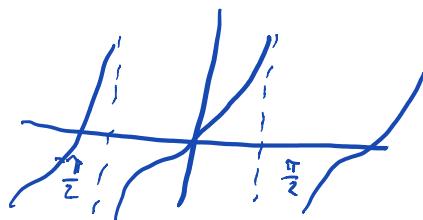
September 9, 2016

Cheating works with my fins as long as the answer is well-defn.

ex. $\lim_{x \rightarrow 0} \sin x \stackrel{oh}{=} \sin 0 = 0$

ex. $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\cos x}{x} \stackrel{oh}{=} \frac{\cos(\frac{\pi}{4})}{\frac{\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{\frac{\pi}{4}} = \frac{2\sqrt{2}}{\pi}$

ex. $\lim_{x \rightarrow \frac{\pi}{2}^+} \tan x = \tan \frac{\pi}{2} = -\infty$



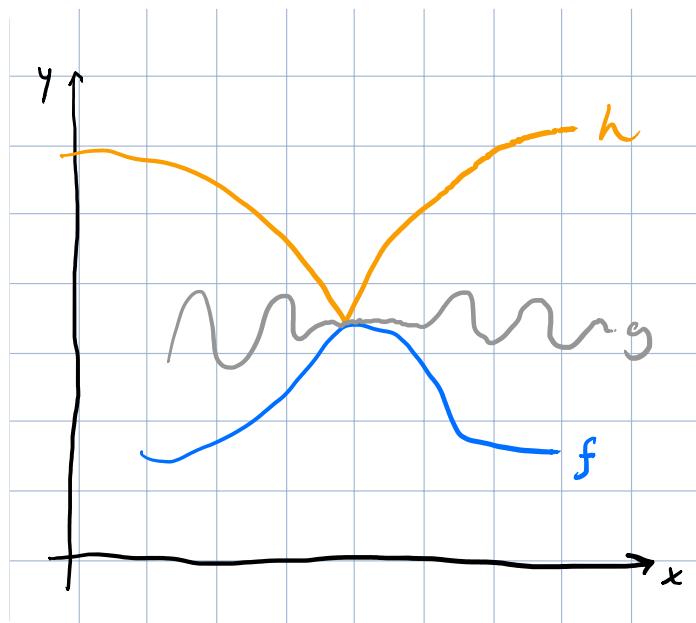
SQUEEZE THEOREM

If $f(x) \leq g(x) \leq h(x)$ for all x near "a"

The graph of f is under graph of g , ...

& if $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} h(x) = L$ (L is a finite ∞)

then $\lim_{x \rightarrow a} g(x) = L$



ex. $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right) \stackrel{\text{oh}}{=} 0 \sin\frac{1}{0}$??

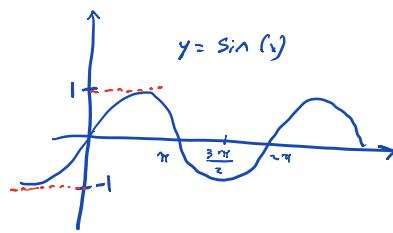
Cannot determine directly

$$-1 \leq \sin \frac{1}{x} \leq 1$$

$$-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2 \quad \text{Multiply all sides by } x^2$$

This is squeezed between $-x^2$ & x^2

$$\lim_{x \rightarrow 0} (-x^2) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (x^2) = 0$$



By Squeeze Thm

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$