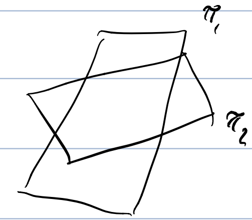
4 - Systems of Linear Equations

Finding the intersection of 2 non-parallel planes:



$$\pi_1 : a_1 x + b_1 y + c_1 z + d_1 = 0$$

$$\pi_2 : a_2 x + b_2 y + c_2 z + d_2 = 0$$

→ Find the values of x, y, z that satisfy both equations simultaneously,
i.e. solve the system of linear equations.

ex. a) $x = 5$

b) $3x + 7y - \frac{3}{2} z - 1 = 0$

c) $2x + y = 2$

d) $\frac{3x+2}{y} = 1 \rightarrow 3x + 2 = y \; (y \neq 0)$

e) $2x + \sqrt{3}\, y = 3$

f) $3x_1 - x_2 + 5x_3 - x_4 + 3x_5 = 7$

Note: An equation is linear if:
→ every term has degree 0 or 1
→ the coefficients are real numbers

In general: we can write a linear equation as:

$$a_1 x_1 + a_2 x_2 + \ldots + a_2 x_n = b$$

Solving Linear Equations

ex. $7x = 3$

$\therefore x = \frac{3}{7}$

ex. $4x - 2y - 2 = 0$

$\therefore \begin{matrix} x = 1 \\ y = 1 \end{matrix}$, $\begin{matrix} x = 2 \\ y = 3 \end{matrix}$, $\begin{matrix} x = 0 \\ y = -1 \end{matrix}$, ...

There are infinitely many solutions.

To express the solution set (set of all solutions) to this equation, we proceed as follows:

ex. $4x - 2y - 2 = 0$

(1) Isolate the first variable

$4x = 2y + 2$

$x = \frac{1}{2} + \frac{y}{2}$ free variable

leading variable

(2) Assign a parameter to every free variable and write the leading variables in terms of the parameters.

General sol'n $\begin{cases} x = \frac{1}{2} + \frac{1}{2}t \\ y = t \end{cases}$ $t \in \mathbb{R}$

* set variable as $y = t$ (independent variable)
& solve for x using it (dependent variable)

Solving Linear Systems (2 or more equations simultaneously)

ex. $2x - y = 0$ ①

$-2\,(x + 2y = 3)$ ②

$-5y = -6$

$y = \frac{6}{5}$

→ Replace y into ①

$2x - \frac{6}{5} = 0$

$x = \frac{3}{5}$

$\therefore x = \frac{3}{5},\ y = \frac{6}{5}$

This approach is not very efficient or practical when dealing with many equations & many variables. Instead, we will use the

Gauss - Jordan Algorithm.

ex. $2x - y = 0$

$x + 2y = 2$

① Write the augmented matrix of the system.

Row 1 · Leading 1 (pivot) · convention: the one you want to change goes first

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 1 & 2 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -5 & -4 \end{array}\right]$$

augmented matrix

$$\xrightarrow{-\frac{1}{5}R_2} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & \frac{4}{5} \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & \frac{6}{5} \\ 0 & 1 & \frac{4}{5} \end{array}\right]$$

echelon form

* actually in reduced row echelon form

$\therefore$ Solution: $\begin{cases} x = \frac{6}{5} \\ y = \frac{4}{5} \end{cases}$

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ex. $3x + y - z = 5$

$x + y - z = 5$

$5x + 4y + 3z = 6$

pivot

there is a "1" here, but it is not preceded by a 0, so it's not relevant.

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 3 & 1 & -1 & 5 \\ 5 & 4 & 3 & 6 \end{array}\right] \xrightarrow[R_3 - 5R_1]{R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & -2 & 2 & -10 \\ 0 & -1 & 8 & -19 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & 1 & -1 & -10 \\ 0 & -1 & 8 & -19 \end{array}\right] \xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ \boxed{0} & 1 & -1 & 5 \\ \boxed{0} & \boxed{0} & 7 & -14 \end{array}\right]$$

$$\xrightarrow{\frac{1}{7}R_3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

This matrix is in row echelon form

We can stop at Row Echelon Form and write the solutions by back substitution.

$$\begin{aligned} x + y - z &= 5 \\ y - z &= 5 \\ z &= -2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= 5 - y + z \\ y &= 5 + z \\ z &= -2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= 5 - 3 - 2 = 0 \\ y &= 5 - 2 = 3 \\ z &= -2 \end{aligned}$$

$\therefore$ Sol'n $\begin{cases} x = 0 \\ y = 3 \\ z = -2 \end{cases}$

Types of ERO's

There are 3 types of elementary row operations (ERO's) that we can use.

a) Multiplying a row by a constant

ex. $\xrightarrow{5R_2}$

b) Interchange two rows:

ex. $\xrightarrow{R_1 \leftrightarrow R_3}$

c) Adding (or subtracting) a multiple of one row to another row ex $\xrightarrow{R_3 - 2R_1}$ (this changes Row 3)

Careful: Don't write $\cancel{2R_3 - 5R_1}$ (2 operations at the same time)
↳ keep every operation separate

Row Echelon Form

ex.

a) $\begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

d) $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

A matrix is in Row Echelon Form (REF) if:

a) The 1st non-zero entry on each row is a '1'
(we call this a leading 1 (or pivot))

b) On any given row, the leading 1 is further to the right than on the previous row

c) Any row of '0's is at the bottom.

Reduced Row Echelon Form

A matrix is in reduced row echelon form (RREF) if it is in REF plus it satisfies the following fourth condition:

d) Any column with a leading 1 has '0's in every other position

(al though there will never be '0's below the leading 1 b.c of conditions in REF)

ex.

a) $\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

d) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

ROW REDUCTION (GAUSS-JORDAN ELIMINATION)

STEP 1

Begin with the leftmost nonzero column. The entry at the top of the column will serve as a pivot. The goal is to place a leading 1 at the top of the column.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 4 & -12 & 16 & -12 & 8 & 20 \\ 4 & -6 & 4 & -9 & 5 & 0 \end{bmatrix}$$

- If that entry is a zero, interchange rows to place a nonzero entry in that position.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 4 & -12 & 16 & -12 & 8 & 20 \\ 4 & -6 & 4 & -9 & 5 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 4 & -12 & 16 & -12 & 8 & 20 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 4 & -6 & 4 & -9 & 5 & 0 \end{bmatrix}$$

- Make the top entry a 1 by multiplying the top row by a suitable constant or by interchanging rows.

Step ① + Step ②

$$\begin{bmatrix} 4 & -12 & 16 & -12 & 8 & 20 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 4 & -6 & 4 & -9 & 5 & 0 \end{bmatrix} \xrightarrow{\frac{1}{4}R_1} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 4 & -6 & 4 & -9 & 5 & 0 \end{bmatrix}$$

1
0
0
0

STEP 2

Create zeros in all positions below the pivot by adding or subtracting suitable multiples of the pivot row.

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 4 & -6 & 4 & -9 & 5 & 0 \end{bmatrix} \xrightarrow[R_4 - 4R_1]{R_2 - 3R_1} \begin{bmatrix} 1 & \cancel{-3} & \cancel{4} & \cancel{-3} & \cancel{2} & \cancel{5} \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & 6 & -12 & 3 & -3 & -20 \end{bmatrix}$$

can't use this row any more

STEP 3

If the matrix is now in **REF**, then go to step 4. If not, cover (or ignore) the first row and repeat steps 1 and 2.

i.e.
$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & 6 & -12 & 3 & -3 & -20 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & 6 & -12 & 3 & -3 & -20 \end{bmatrix} \xrightarrow[R_4-6R_2]{R_3-3R_2}$$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -9 & -9 & -2 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & -9 & -9 & -2 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{-\frac{1}{9}R_3}$$

need '1' here, but it's not a diagonal anymore! so change rows

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 1 & 1 & \frac{2}{9} \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

The matrix is in **REF.** You have performed **Gaussian elimination.**

STEP 4

Beginning with the rightmost pivot (leading 1), create zeros above it by adding appropriate multiples of the pivot row.

get zeros here by subtracting R_4, then do the same with R_3, R_2

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 1 & 1 & \frac{2}{9} \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow[\substack{R_2-R_4 \\ R_1-2R_4}]{R_3-R_4} \begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 1 & 0 & \frac{-34}{9} \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

STEP 5

If the matrix is now in **RREF**, you're done! If not, cover (or ignore) the last non zero row and repeat step 4 on the smaller matrix that's left.

i.e.
$$\begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 1 & 0 & \frac{-34}{9} \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow[R_1+3R_3]{R_2-2R_3} \begin{bmatrix} 1 & -3 & 4 & 0 & 0 & \frac{-129}{9} \\ 0 & 1 & -2 & 0 & 0 & \frac{5}{9} \\ 0 & 0 & 0 & 1 & 0 & \frac{-34}{9} \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_1+3R_2}$$

column doesn't have leading '1'

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 0 & \frac{-114}{9} \\ 0 & 1 & -2 & 0 & 0 & \frac{5}{9} \\ 0 & 0 & 0 & 1 & 0 & \frac{-34}{9} \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

The matrix is in **RREF.** (You've performed **Gauss-Jordan elimination.**)

Now, we need to write the solution

Writing the Solution

If the final matrix is in RREF

ex. Using the final matrix from handout:

leading variable

free variable

$\therefore$ change to t

$$\begin{aligned} x_1 - 2x_3 &= -\frac{114}{9} \\ x_2 - 2x_3 &= \frac{5}{9} \\ x_4 &= \frac{-34}{9} \\ x_5 &= 4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= -\frac{114}{9} + 2x_3 \\ x_2 &= \frac{5}{9} + 2x_3 \\ x_4 &= -\frac{34}{9} \\ x_5 &= 4 \end{aligned}$$

General sol'n:

$$\begin{cases} x_1 = -\dfrac{114}{9} + 2t \\ x_2 = \dfrac{5}{9} + 2t \\ x_4 = -\dfrac{34}{9} \\ x_5 = 4 \end{cases}$$

OR in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{114}{9} \\ \frac{5}{9} \\ 0 \\ -\frac{34}{9} \\ 4 \end{bmatrix} + t \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad t \in \mathbb{R}$$

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ex. From handout, the last matrix in RREF

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & 0 & -\frac{114}{9} \\ 0 & 1 & -2 & 0 & 0 & \frac{5}{9} \\ 0 & 0 & 0 & 1 & 0 & -\frac{34}{9} \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \end{array}$$

free variable

$x_1 = -\frac{114}{9} + 2x_3$

$x_2 = \frac{5}{9} + \underline{2x_3}$ free variable

$x_4 = -\frac{34}{9}$

$x_5 = 4$

a) General sol'n:

$$\begin{cases} x_1 = -\frac{114}{9} + 2t \\ x_2 = \frac{5}{9} + 2t & t \in \mathbb{R} \\ x_3 = t \\ x_4 = -\frac{34}{9} \\ x_5 = 4 \end{cases}$$

b) Give any two particular solutions:

for $t = 0$

$x_1 = -\frac{114}{9}$

$x_2 = \frac{5}{9}$

$x_3 = 0$

for $t = 1$

$x_1 = -\frac{96}{9} \underbrace{= -\frac{32}{3}}_{\text{not necessary}}$

$x_2 = \frac{23}{9}$

$x_3 = 1$

$x_4 = -\frac{34}{9}$ $\qquad$ $x_4 = -\frac{34}{9}$

$x_5 = 4$ $\qquad$ $x_5 = 4$

ex. From the handout, the last matrix in step ③ is in REF

$$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 1 & 1 & \frac{2}{9} \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Start from bottom:

$x_5 = 4$

$x_4 = \frac{2}{9} - 4 = -\frac{34}{9}$

$x_2 = -3 - 2\left(-\frac{34}{9}\right) + 2x_3 - 4$

$= -3 + \frac{68}{9} - 4 + 2t$

$= \frac{5}{9} + 2t$

$x_1 = 5 + 3\left(\frac{5}{9} + 2t\right) - 4t + 3\left(-\frac{34}{9}\right) - 2(4)$

$= -\frac{114}{9} + 2t$

General sol'n:

$$\begin{cases} x_1 = -\frac{114}{9} + 2t \\ x_2 = \frac{5}{9} + 2t \\ x_3 = t \\ x_4 = -\frac{34}{9} \\ x_5 = 4 \end{cases} \qquad t \in \mathbb{R}$$

ex. Find the general solution, if possible:

$2x - y + 3z = 5$

$2x - 2y + 5z = 7$

$4x - 3y + 8z = 13$

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 5 \\ 2 & -2 & 5 & 7 \\ 4 & -3 & 8 & 13 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 5 \\ 0 & -1 & 2 & 2 \\ 0 & -1 & 2 & 3 \end{array}\right]$$

$$\xrightarrow[-1R_1]{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & 5 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & 2 & 3 \end{array}\right] \xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

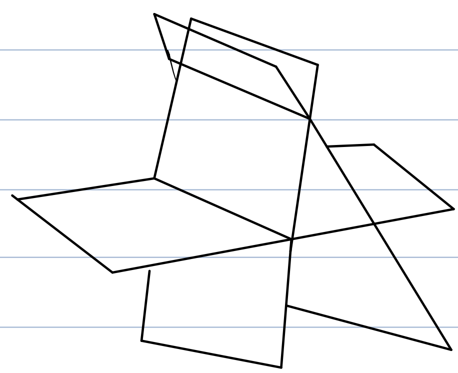
REF

$0x + 0y + 0z = 1$ No Solution

still considered a leading one

$\therefore$ The system is inconsistent.

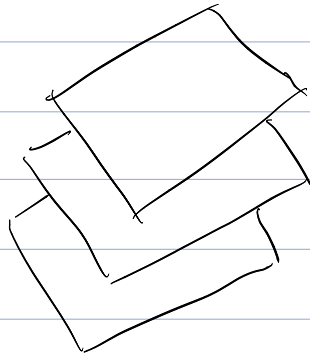
example geometric representation:



Note: In theory could also have 3 parallel planes or 2 parallel with 1 cutting through both

ex. Give an example of a linear system corresponding to the following diagrams:

a)

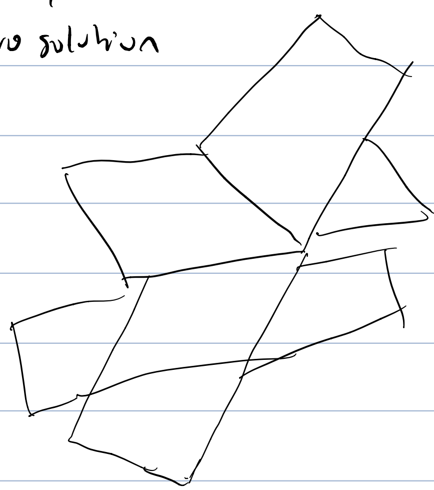


$$x + y + z = 1$$
$$2x + 2y + 2z = 1$$
$$3x + 3y + 3z = 1$$

Note if ① = 1, ② = 2, ③ = 3 then they'd be the same planes and would have infinite solutions

∴ No solution

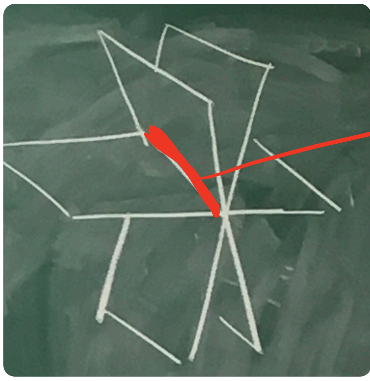
b)



$$x + y + z = 1$$
$$2x + 2y + 2z = 1$$
$$3x + 5y + 8z = 1$$

∴ No solution

c)



intersection is a line

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 6 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

2 leading '1's but 3 variables!

So sol'n is a line

ex: $x + y + z = 1$

$2x - 3y - z = 5$

$3x - 2y = 6 \leftarrow R_3 - (R_2 - R_1) = 0$ which gives you the third row as 0s

ex. $2w + 3x - y + 4z = 1$

$3w - x + z = 1$

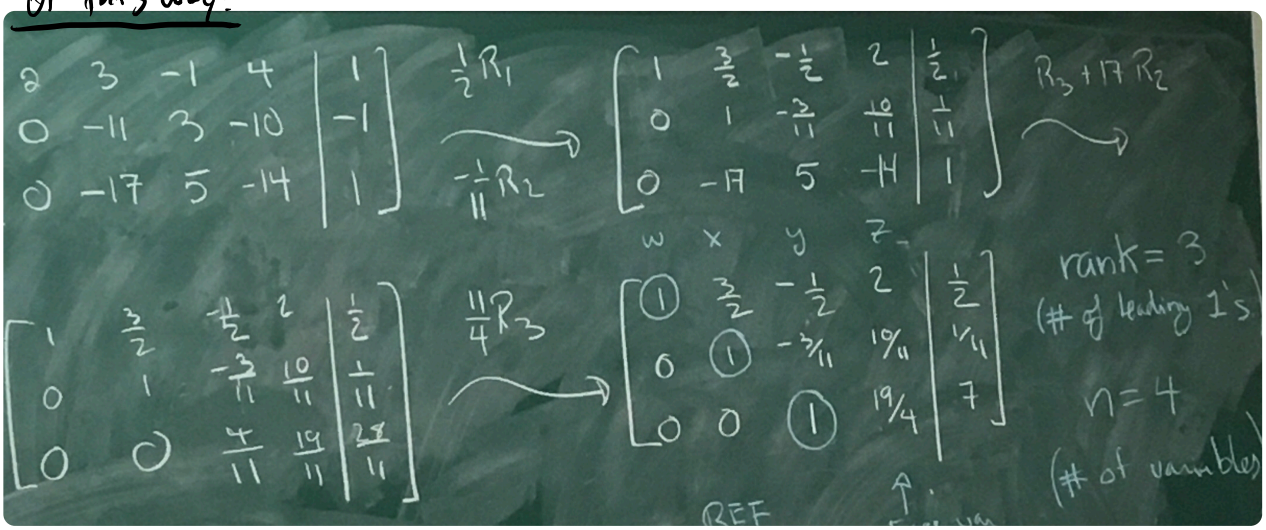
$3w - 4x + y - z = 2$

$$\left[\begin{array}{cccc|c} 2 & 3 & -1 & 4 & 1 \\ 3 & -1 & 0 & 1 & 1 \\ 3 & -4 & 1 & -1 & 2 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{cccc|c} 3 & -1 & 0 & 1 & 1 \\ 2 & 3 & -1 & 4 & 1 \\ 3 & -4 & 1 & -1 & 2 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{cccc|c} 1 & -4 & 1 & -3 & 0 \\ 2 & 3 & -1 & 4 & 1 \\ 3 & -4 & 1 & -1 & 2 \end{array}\right]$$

$$\xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & -4 & 1 & -3 & 0 \\ 0 & 11 & -3 & 10 & 1 \\ 0 & 8 & -2 & 8 & 2 \end{array}\right] \xleftrightarrow{R_2}$$

Or this way:



Def'n: The <u>rank</u> of a matrix is the number of leading '1's once we reduce the matrix to REF

ex.

$$\frac{1}{3}x_1 + x_2 - x_3 - 6x_4 = 2$$
$$\frac{1}{6}x_1 + \frac{1}{2}x_2 - 3x_4 + x_5 = -1$$
$$\frac{1}{3}x_1 - 2x_3 - 4x_5 = 8$$

⋮

$$\left[\begin{array}{ccccc|c} 1 & 0 & -6 & 0 & -12 & 24 \\ 0 & 1 & 2 & -6 & 6 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

rank = 2 (# leading 1 s)

$n = 5$ (# variables)

$\therefore$ # parameters = n − rank

$= 5 - 2 = 3$

$$\begin{cases} x_1 = 24 + 6s + 12\mu \\ x_2 = -10 - 2s - 6\mu + 6t \\ x_3 = s \\ x_4 = t \\ x_5 = \mu \end{cases}$$

or in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 24 \\ -10 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 6 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \mu\begin{bmatrix} 12 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Note: Linear System

- Inconsistent → No solution
- Consistent
  - One sol'n (rank = n)
  - Infinitely many soln's (rank < n)

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ex.
$$\begin{aligned} 2x + y - 2z &= 10 \\ 3x + 2y + 2z &= 1 \\ 5x + 3y &= 11 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & -2 & 10 \\ 3 & 2 & 2 & 1 \\ 5 & 3 & 0 & 11 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -1 & 5 \\ 3 & 2 & 2 & 1 \\ 5 & 3 & 0 & 11 \end{array}\right]$$

$$\xrightarrow[R_3 - 5R_1]{R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -1 & 5 \\ 0 & \frac{1}{2} & 5 & -14 \\ 0 & \frac{1}{2} & 5 & -14 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -1 & 5 \\ 0 & \frac{1}{2} & 5 & -14 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{2R_2} \underbrace{\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -1 & 5 \\ 0 & 1 & 10 & -28 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{REF}} \xrightarrow{R_1 - \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 0 & -6 & 19 \\ 0 & 1 & 10 & -28 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

∴ rank < n
∴ infinitely many sol'ns

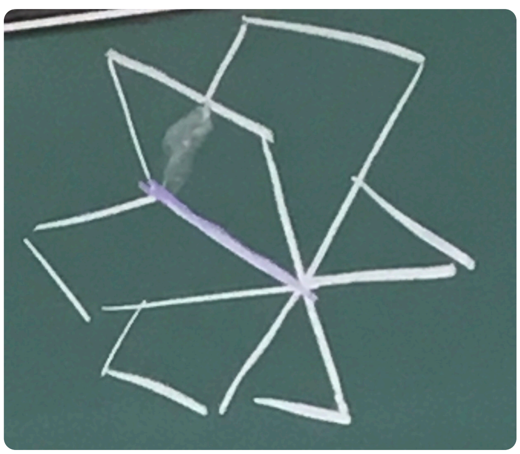
rank (# leading 1s) = 2 ; n < 3

General sol'n : $\begin{cases} x = 14 + 6t \\ y = -28 - 10t \\ z = t \end{cases}$ $t \in \mathbb{R}$

or in vector form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ -28 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -10 \\ 1 \end{bmatrix} \quad t \in \mathbb{R}$$

Representation:



Homogeneous Linear Equations / Systems

- A linear equation is called homogenous if the constant term is zero.
- A system of homogeneous equations is called a homogeneous system

ex.

$$\begin{aligned} 3x + y &= 0 \\ 2y + z &= 0 \\ x - y + z &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 3 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 - 3R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 4 & -3 & 0 \end{array}\right] \xrightarrow{R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{5}R_3]{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

rank = 3

n = 3

Sol'n : $\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$ This system has only the <u>trivial solution</u>

<u>Note</u>: A homogeneous system is never inconsistent. It always has <u>at least</u> the trivial solution.

ex. $\begin{aligned} 2x + y - z &= 0 \\ -x - y + 2z &= 0 \\ 2x \quad\quad + 2z &= 0 \end{aligned}$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 2 & 1 & -1 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & -2 & 6 & 0 \end{array}\right]$$

* can start from this matrix (multiplied second row by -1, and flipped R_1 and R_2)

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ General solution: $\begin{cases} x = -t \\ y = 3t \\ z = t \end{cases} \quad t \in \mathbb{R}$

This is a line through the origin

Homogeneous Systems

One solution
(the trivial solution)

$n = \text{rank}$

Infinitely many sol's
(The system has non-trivial sol's)
$\text{rank} < n$

ex. $x + y = 1$ where $k \in \mathbb{R}$

$3x + 3y = k$

For what value(s) of k will the system have:

a) infinitely many sol's b) no sol c) one sol

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 3 & 3 & k \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & k-3 \end{array}\right]$$

a) $k - 3 = 0$

$\boxed{k = 3}$

b) $k - 3 \neq 0$

$\boxed{k \neq 3}$

c) Never

ex.

$$\begin{aligned} x \qquad\quad - 3z &= -3 \\ 2x + ky - z &= -2 \\ x + 2y + kz &= 1 \end{aligned}$$

For what value(s) of k does the system have:

a) infinitely many sol's

b) no sol

c) one sol

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 2 & k & -1 & -2 \\ 1 & 2 & k & 1 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & k & 5 & 4 \\ 1 & 2 & k & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & k & 5 & 4 \\ 0 & 2 & k+3 & 4 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & 2 & k+3 & 4 \\ 0 & k & 5 & 4 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2} R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & 1 & \frac{k+3}{2} & 2 \\ 0 & k & 5 & 4 \end{array}\right] \xrightarrow{R_3 - kR_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & 1 & \frac{k+3}{2} & 2 \\ 0 & 0 & 5 - \frac{k^2+3k}{2} & 4-2k \end{array}\right]$$

a) Need the last row to be a row of 0s

$$5 - \frac{k^2+3k}{2} = 0 \qquad \text{and} \qquad 4 - 2k = 0$$

$$\underline{k = 2}$$

$$k^2 + 3k - 10 = 0$$

$$(k-2)(k+5) = 0$$

$$\underline{k=2} \qquad \cancel{\underline{k=-5}}$$

$$\therefore \boxed{k=2}$$

b) Need the last row to be: $\left[0 \ \ 0 \ \ 0 \mid \neq 0\right]$:

$5 - \frac{k^2+3k}{2} = 0$ and $4 - 2k \neq 0$

$k = 2 \mid k = -3$ $k \neq 2$

$\therefore \boxed{k = -5}$

c) Need last row to be: $\left[0 \ \ 0 \ \ \neq 0 \mid \neq 0\right]$

$k \neq 2$ and $k \neq -5$ or $k \in \mathbb{R} \setminus \{-5, 2\}$

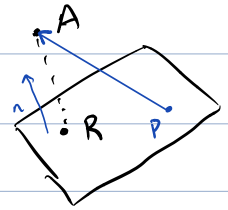
ICP 4

1) Consider the plane $\pi : x + 2y + 3z - 9 = 0$ and the pt $A(2, 5, -3)$ not on the plane. Find the pt on the plane closest to A.

2) Find a plane that is parallel to $\pi : x + 2y + 3z - 9 = 0$ passing through

3) Find the intersection between the line $\langle x, y, z \rangle = \langle 2, -1, 0 \rangle + t\langle 1, 1, 0 \rangle$ and the plane $x - y + z = 0$

1)



$\vec{RA} = \text{proj}_{\vec{n}} \vec{PA}$

$\vec{OR} = \vec{OA} + \vec{AR}$ or $\vec{OR} = \vec{OP} + \text{orth}_{\vec{n}} \vec{PA}$

$= \vec{OA} - \vec{RA}$

$= \langle 2, 5, -3 \rangle - \text{proj}_{\vec{n}} \vec{PA}$

$= \langle 2, 5, -3 \rangle - \dfrac{\vec{PA} \cdot \vec{n}}{\|\vec{n}\|^2} \vec{n}$

$= \langle 2, 5, -3 \rangle - \dfrac{2 + 10 - 18}{14} \langle 1, 2, 3 \rangle$

$\vdots$

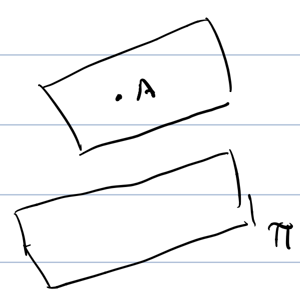
$A(2, 5, -3)$

chose pt $P(0, 0, 3)$

$\vec{PA} = \langle 2, 5, -6 \rangle$

$\vec{n} = \langle 1, 2, 3 \rangle$

2)



Same normal vectors. Find d

$x + 2y + 3z + d = 0$

Subst. $(1, 1, 4)$

Assignment 5 October 3

3. b)

$$\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\
\left[\begin{array}{ccccc|c}
1 & 2 & 1 & 3 & 1 & 1 \\
0 & 1 & -1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}$$

rank $= 3$

$n = 5$

$x_5 = t$

$x_4 = x_5 = t$

$x_3 = s$

$x_2 - x_3 + x_5 = 1$

$x_2 - s + t = 1$

$x_2 = 1 - t + s$

$x_1 + 2x_2 + x_3 + 3x_4 + x_5 = 1$

$x_1 = 1 - 2(1 - t + s) - s - 3t - t$

To go to RREF

$$\left[\begin{array}{ccccc|c}
1 & 2 & 1 & 3 & 1 & 1 \\
0 & 1 & -1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\xrightarrow{R_1 - 3R_3}
\left[\begin{array}{ccccc|c}
1 & 2 & 1 & 0 & 4 & 1 \\
0 & 1 & -1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

get rid of this one

$$\xrightarrow{R_1 - 2R_2} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 2 & -1 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_5 = t$

$x_4 = t$

$x_2 = 1 - x_3 - x_5 = 1 - s - t$

$x_1 = -1 - 3s - 2t$

$x_3 = s$

b) Give two particular sol'ns = pick the parameter values

e.g. For $s = 0;\ t = 0$

$x_1 = -1$

$x_2 = 1$

$\vdots$

5. Find intersection of two planes

$$\left[\begin{array}{ccc|c} 1 & -1 & -4 & 8 \\ 3 & 1 & -6 & 0 \end{array}\right]$$

$n = 3$

rank ≤ 2 since you can have at most 2 leading 1s

$$\xrightarrow{R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & -1 & -4 & 8 \\ 0 & 4 & 6 & -24 \end{array}\right] \xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{ccc|c} 1 & -1 & -4 & 8 \\ 0 & 1 & \frac{3}{2} & -6 \end{array}\right]$$

$$\xrightarrow{R_1 + R_2} \begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -5/2 & 2 \\ 0 & 1 & 3/2 & -6 \end{array}\right] \end{array}$$

General sol: $\begin{cases} x = 2 + \frac{5}{2}t \\ y = -6 - 3/2\, t \\ z = t \end{cases}$

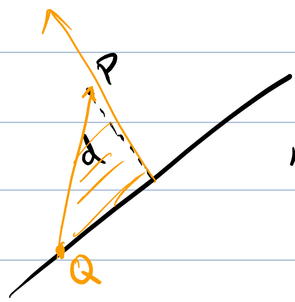
$$\langle x, y, z \rangle = \langle 2, -6, 0 \rangle + t \langle \tfrac{5}{2}, -\tfrac{3}{2}, 1 \rangle$$

Note: $\vec{d} = \langle \frac{5}{2}, -\frac{3}{2}, 1 \rangle$
$= \frac{1}{2} \langle 5, -3, 2 \rangle$

Represents a line.

Extra Practice & Review

In $\mathbb{R}^2$ - Point to line



$ax + by + c = 0$

$n = \langle a, b$

ex. Convert to parametric

$2x - y - 3 = 0$

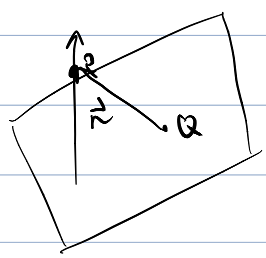
$x = \frac{3}{2} + \frac{1}{2}y$

$y = t$

$\therefore \begin{cases} x = \frac{3}{2} + \frac{1}{2}t \\ y = t \end{cases}$

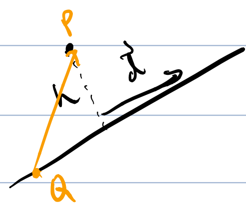
$d = \left\| \text{proj}_{\vec{n}} \overrightarrow{QP} \right\|$

$\mathbb{R}^3$ - point to plane



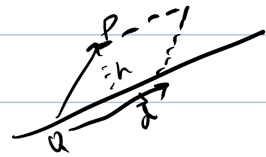
$d = \left\| \text{proj}_{\vec{n}} \overrightarrow{QP} \right\|$

$\mathbb{R}^3$ - point to line



$d = \left\| \text{orth}_{\vec{d}} \overrightarrow{QP} \right\|$

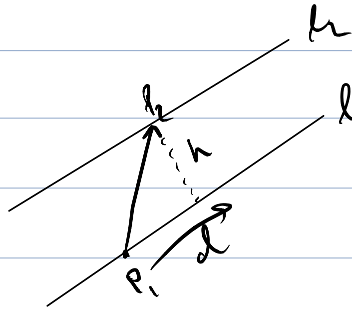
or



$$h = \frac{\text{Area}}{\text{base}}$$

$$= \frac{\|\vec{d} \times \overrightarrow{QP}\|}{\|\vec{d}\|}$$

Distance between parallel planes



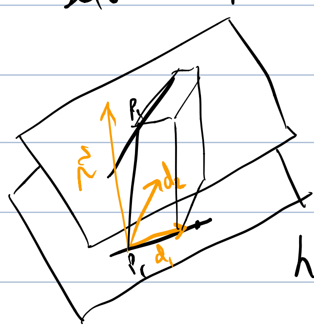
$$h = \left\|\operatorname{orth}_{\vec{d}_1} \overrightarrow{P_1P_2}\right\|$$

or

$$= \frac{\text{Area}}{\text{base}}$$

$$= \frac{\|\overrightarrow{P_1P_2} \times \vec{d}\|}{\|\vec{d}\|}$$

Distance between skew lines



$$\vec{n} = d_1 \times d_2$$

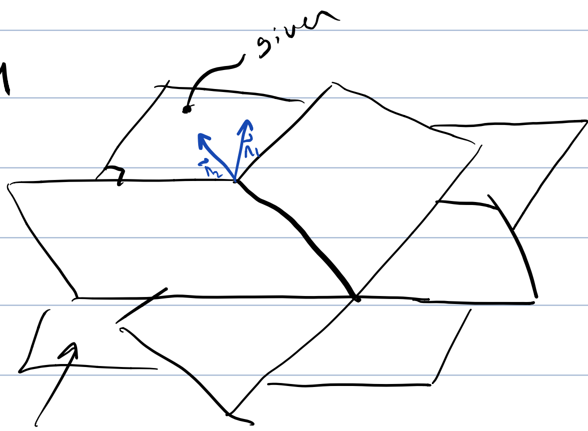
$$h = \frac{\left|\overrightarrow{P_1P_2} \cdot (\vec{d}_1 \times \vec{d}_2)\right|}{\|\vec{d}_1 \times \vec{d}_2\|}$$ Volume of parallelepiped

base

or $h = \left\|\operatorname{proj}_{\vec{d}_1 \times d_2} \overrightarrow{P_1P_2}\right\|$

p. 47

#39

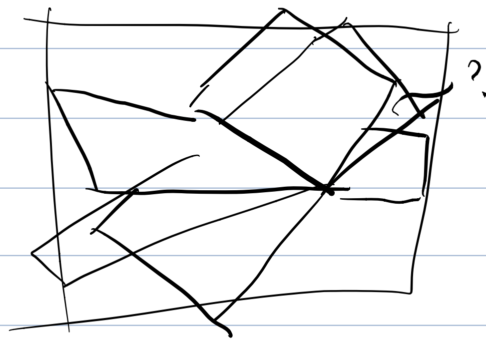


method ① $\vec{n} = \vec{n}_1 \times \vec{n}_2$

method ② Find intersection of 2 planes which is a line. The direction vector of the line is your normal vector.

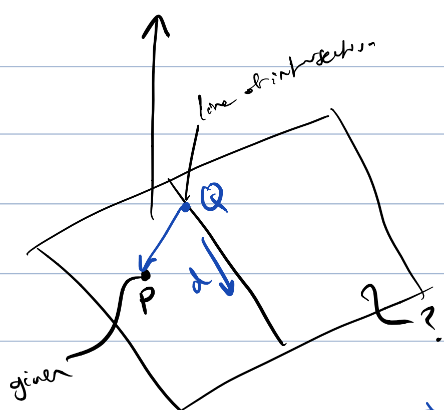
#40 ~harder one

a plane that passes through the line of intersection of 2 planes & perpendicular to a 3rd plane.



2 vectors on your plane: the direction vector of line & normal vector

#37



$$\vec{n} = \overrightarrow{QP} \times \bar{d}$$

$\vdots$