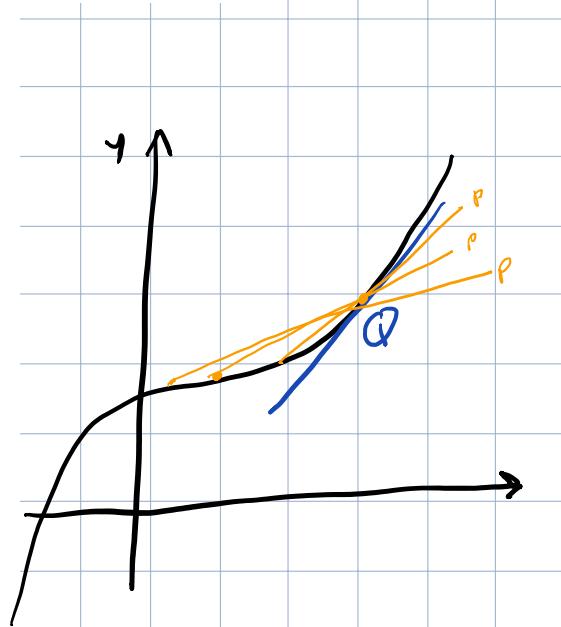
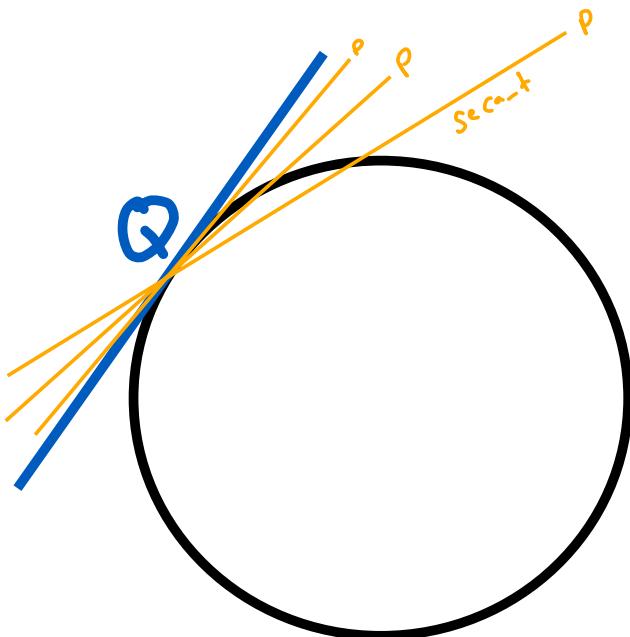
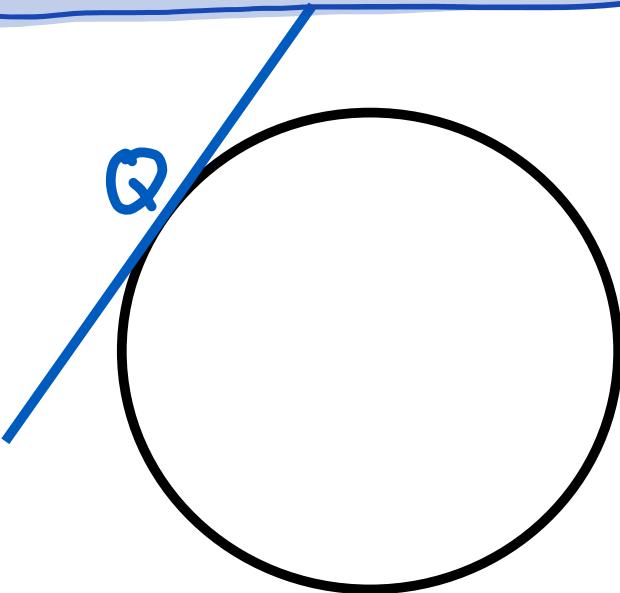


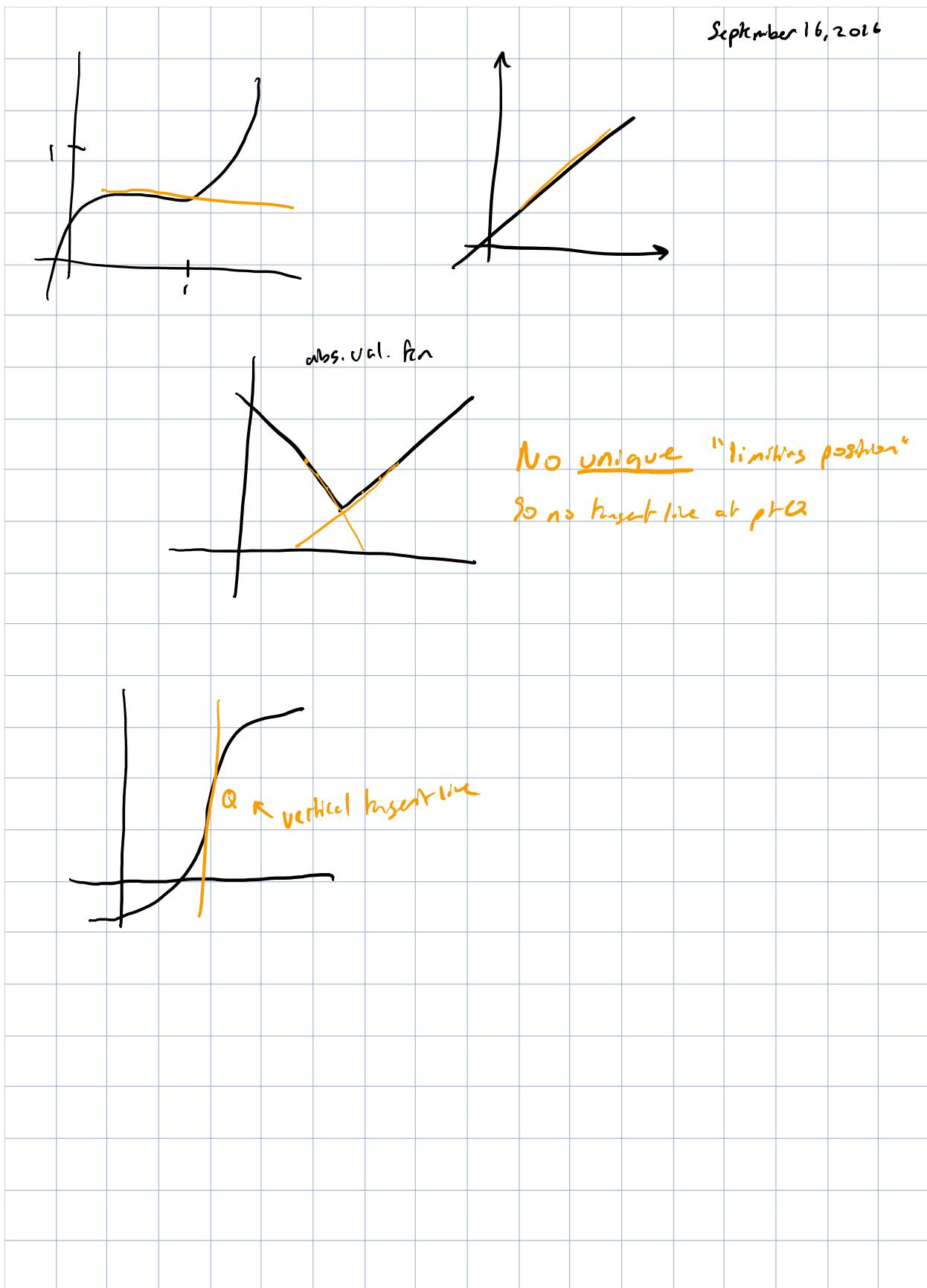
## DERIVATIVES AND TANGENT LINES



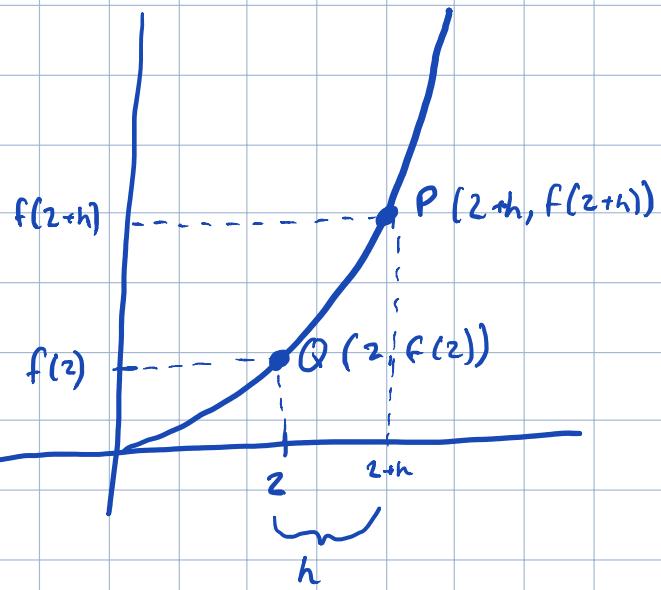
Choose pts P on the graph of  $y = f(x)$  approaching pt Q (from both sides, none touching pt Q).

If the "secant" lines PQ approach a unique "limiting" position, then the line that sits at this limiting position is defined to be the tangent line.

September 16, 2016



Ex. Consider  $y = f(x) = x^2$



Slope of Secant line  $\overline{PQ}$ :

$$M_{sec} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(2+h) - f(2)}{(2+h) - 2} = \frac{f(2+h) - f(2)}{h}$$

To force pt P along the graph towards pt Q, let  $h \rightarrow 0$

Then the positions of secant lines  $\overline{PQ}$  will approach the position of the tangent line.

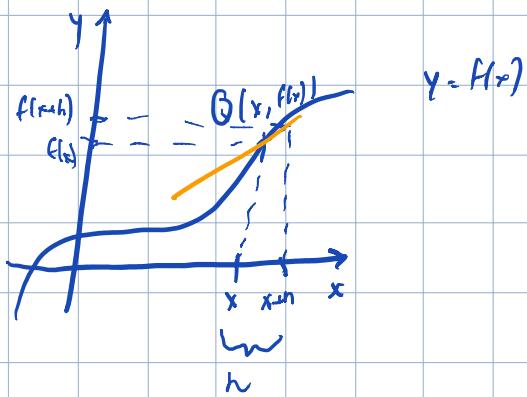
So then the slopes  $M_{sec}$  will approach  $M_{tan}$  of the tangent line

$$\text{i.e. } \lim_{h \rightarrow 0} M_{sec} = M_{tan}$$

$$\text{i.e. } M_{tan} = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(2+h)^2 - 4}{h} = \lim_{h \rightarrow 0} \frac{4+4h+h^2 - 4}{h} = \lim_{h \rightarrow 0} (4+h) \stackrel{?}{=} 4$$

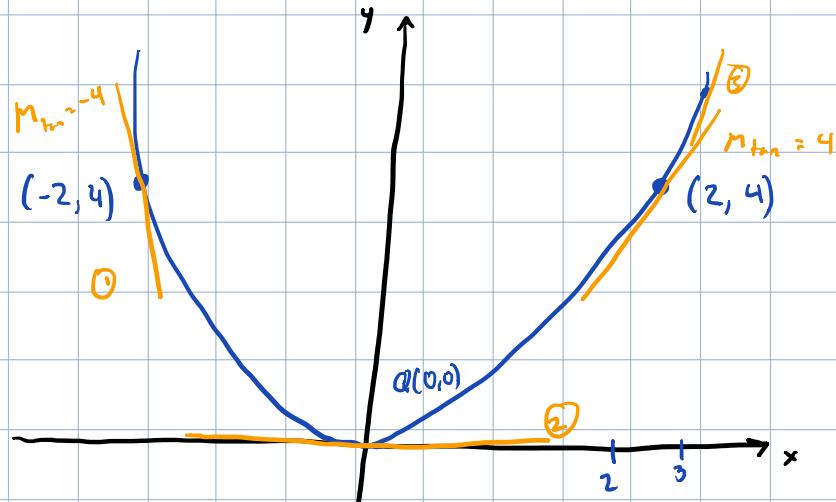
For any  $f$  in  $y = f(x)$  & at any pt  $Q(x, f(x))$  we can apply the  
same logic



$$m_{\text{sec}} = \frac{f(x+h) - f(x)}{x+h - x} = \frac{f(x+h) - f(x)}{h}$$

$$m_{\tan} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Ex.  $f(x) = x^2$



$$\textcircled{1} \quad m_{\tan} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(-2+h) - f(-2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(-2+h)^2 - 4}{h}$$

$$= \lim_{h \rightarrow 0} \frac{4 - 4h + h^2 - 4}{h}$$

$$= \lim_{h \rightarrow 0} (-4 + h)$$

$$= -4$$

$$\textcircled{2} \quad m_{\tan} = \frac{f(x+h) - f(x)}{h}$$

$$m_{\tan} = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

$$m_{\tan} = \lim_{h \rightarrow 0} \frac{h^2}{h} - 0$$

$$m_{\tan} = \lim_{h \rightarrow 0} h$$

$$m_{\tan} = 0$$

$$\textcircled{3} \quad m_{\tan} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(3+h)^2 - (3)^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{9 + 6h + h^2 - 9}{h}$$

$$= \lim_{h \rightarrow 0} (6 + h)$$

$$= \lim_{h \rightarrow 0}$$

$$= 6$$

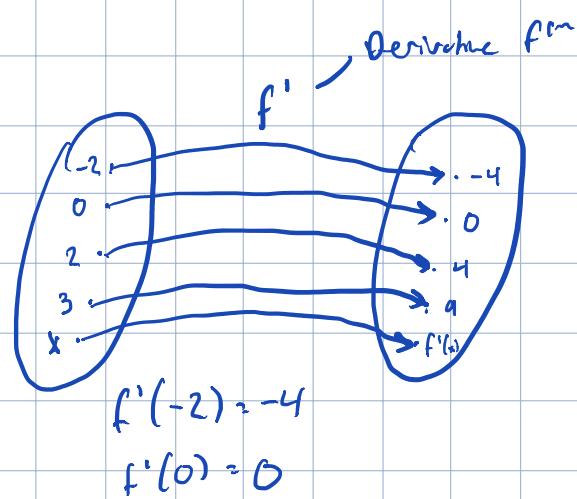
$$y = mx + b$$

$$q = 6(3) + b$$

$$q = 18 + b$$

$$b = -9$$

$$\text{eine Tangentenlinie: } y = 6x - 9$$



$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \rightarrow 0} (2x+h)$$

$$\boxed{f'(x) = 2x} \quad \leftarrow \text{derivative } f'^m$$

Defn If  $y = f(x)$  is any  $f^m$  for the derivative ( $f'$ ) is defined

by  $\rightarrow f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

Mean

Ex. If  $f(x) = \sqrt{x}$ , find  $f'(x)$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\underset{h \rightarrow 0}{\lim} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h (\sqrt{x+h} + \sqrt{x})}$$

$$\underset{h \rightarrow 0}{\lim} \frac{x+h - x}{h (\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \rightarrow 0} \frac{h}{h (\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Ex. If  $f(x) = x^3$ , find the eqn of the tangent line at  $x=2$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h}$$

$$\underset{h \rightarrow 0}{\lim} \frac{(x+h)(x+h)(x+h) - x^3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{x^3 + 2x^2h + h^2x + x^2h + 2xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

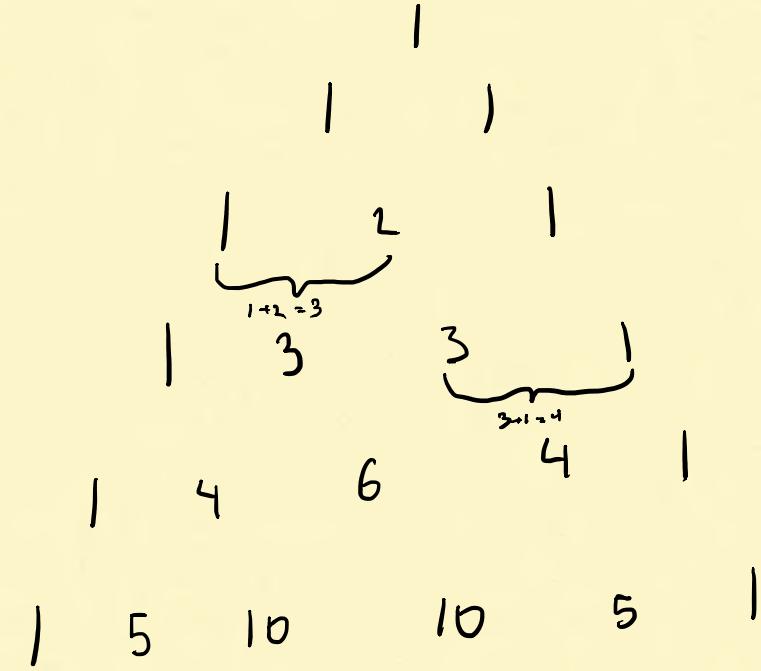
$$= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2)$$

$= 3x^2$  ← Derivative

So  $f'(2) = 3(2)^2 = 12$  ← slope of tangent line at  $x=2$

Ans  $y = 12 - 16$

PASCAL'S TRIANGLE  
 $(a+b)^n$



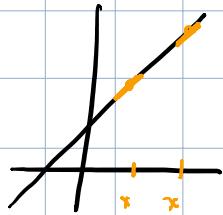
ex.  $(a+b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$

$$a^5 b^0 \quad a^4 b^1$$

$$a \quad 5 \rightarrow 4 \rightarrow 3$$

$$b \quad 0 \rightarrow 1 \rightarrow 2$$

Ex.  $f(x) = 3x + 2$  Find  $f'(x)$



Geometrically we expect  $f'(x) = 3$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{3(x+h) + 2 - (3x + 2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{3x + 3h + 2 - 3x - 2}{h}$$

$$\approx \lim_{h \rightarrow 0} 3$$

$$f'(x) = 3 \text{ as expected}$$

$$f(x) = 3x + 2$$

$$f(x+h) = 3(x+h) + 2$$

Ex.  $f(x) = 7$

$f'(x)$  should be  $= 0$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{7 - 7}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = \lim_{h \rightarrow 0} 0 = 0$$



September 14, 2016

### Constant function

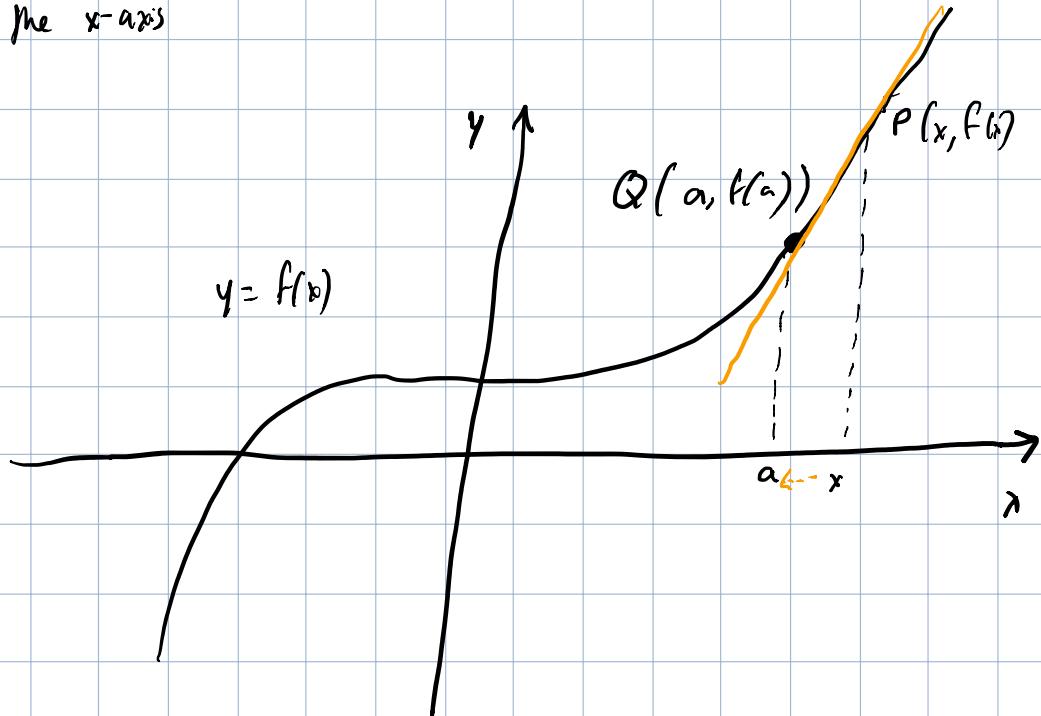
If  $y = f(x) = C$  ( $C$  is a constant), then  $f'(x) = 0$

Ex.  $g(x) = \sqrt{3} - 2.7 \Rightarrow g'(x) = 0$

### Alternative def<sup>n</sup> of the derivative

Consider a fun  $y = f(x)$  and chose any fixed number ' $c$ ' on

The  $x$ -axis



Slope of sec through  $\bar{PQ}$  is:

$$M_{sec} = \frac{f(x_p) - f(a)}{x_p - a}$$

$$M_{tan} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \quad (\text{Let } x \rightarrow a)$$

Alt def  $f'(a)$   
 $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$

\* consider  $f(x) = x^3$

choose any fixed  $x$  in  $\text{Dom } f$

$$f'(x) \stackrel{\text{original def}}{=} \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h \rightarrow 0} 3x^2 + 3xh + h^2$$

$$= 3x^2$$

choose any fixed  $a'$  in  $\text{Dom } f$

$$f'(a) \stackrel{\text{defn}}{=} \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \rightarrow a} \frac{x^3 - a^3}{x - a}$$

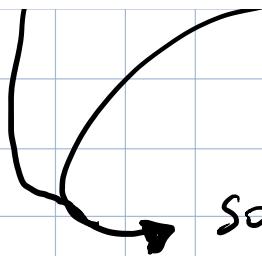
$$= \lim_{x \rightarrow a} \frac{(x-a)(x^2 + ax + a^2)}{x - a}$$

$$= \lim_{x \rightarrow a} x^2 + ax + a^2$$

$$= a^2 + a^2 + a^2$$

$$= 3a^2$$





So  $f'(x) = 3x^2$  for any  $x$  in Dom  $f$

## OBSERVING A PATTERN

### SPECIAL FACTORING

$$x^3 - a^3 = (x-a)(x^2 + ax + a^2)$$

$$x^4 - a^4 = (x-a)(x^3 + x^2a + xa^2 + a^3)$$

$$x^5 - a^5 = (x-a)(x^4 + x^3a + x^2a^2 + xa^3 + a^4)$$

:

$$x^n - a^n = (x-a) \left( x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + a^{n-1} \right)$$

$n$  terms

If  $f(x) = x^n$ , find  $f'(x)$  ( $\text{if } a \neq 0$ )

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a}$$

$$= \lim_{x \rightarrow a} \left( x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + a^{n-1} \right)$$

$$= \lim_{x \rightarrow a} /a^3 + a - a + a/$$

$$= 4a^3$$

$$\text{So } f'(x) = 4x^3$$

$$f(x) = x^2 \Rightarrow f'(x) = 2x$$

$$f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}}$$

$$f(x) = x^3 \Rightarrow f'(x) = 3x^2$$

$$f(x) = x^4 \Rightarrow f'(x) = 4x^3$$

Based on pattern

$$x^r \Rightarrow f'(x) = rx^{r-1}$$

### Power Rule

If  $f(x) = x^r$  then  $f'(x) = rx^{r-1}$  (fixed number "r")

$$\text{Ex. } g(x) = x^4 \Rightarrow g'(x) = 4x^3$$

$$\text{Ex. } h(t) = t^5 \Rightarrow h'(t) = 5t^4$$

$$\text{Ex. } y(x) = \sqrt[3]{x} = x^{\frac{1}{3}} \Rightarrow y'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$

$$\text{Ex. } f(t) = \frac{1}{\sqrt[5]{t^2}} = \frac{1}{(t^2)^{\frac{1}{5}}} \Rightarrow f'(t) = -\frac{2}{5}t^{\frac{-2}{5}-1} = -\frac{2}{5}t^{-\frac{7}{5}}$$

Ex.  $f(x) = \frac{1}{x^{10}} = x^{-10} \Rightarrow f'(x) = -10x^{-11}$  *not -9*

Ex.  $g(x) = \pi^4 \Rightarrow g'(x) = 0$  *constant*

Ex.  $f(x) = x^1 \Rightarrow f'(x) = 1 \cdot x^{1-1} = x^0 = 1$

### Proof of Power Rule

If  $f(x) = x^r$  then  $f'(x) = rx^{r-1}$  (*fixed number "r"*)

Proof for pos. integers only ( $n = 1, 2, 3, \dots$ )

#### Proof

Assume  $n$  is some pos. integer and assume  $f(x) = x^n$

$$f'(a) \stackrel{\text{defn}}{=} \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

$$\begin{aligned} &= \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} \quad n \text{ terms} \\ &= \lim_{x \rightarrow a} \frac{(x-a)(x^{n-1} - x^{n-2}a + x^{n-3}a^2 + \dots + xa^{n-2} + a^{n-1})}{x-a} \end{aligned}$$

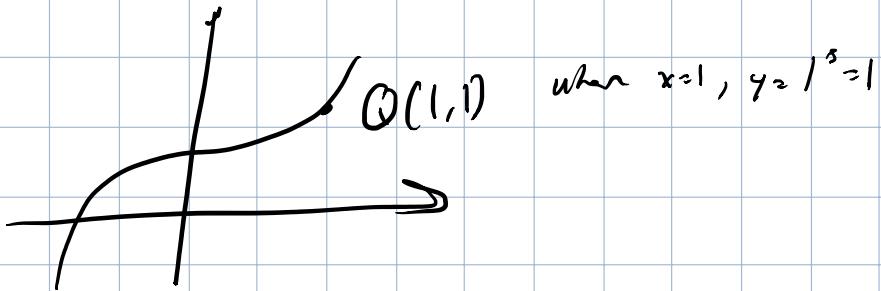
$$\stackrel{h}{=} a^{n-1} + a^{n-2} \cdot a + a^{n-3}a^2 + \dots + a \cdot a^{n-2} + a^{n-1}$$

$$= a^{n-1} + a^{n-1} + a^{n-1} + \dots + a^{n-1}$$

$$f'(a) = na^{n-1}$$

$$\text{So } f'(x) = nx^{n-1}$$

ex.  $y = g(x) = x^5$  find the eqn of tangent line at  $x=1$



$$g'(x) = 5x^4$$

slope of tangent line at  $x=1$  is  $g'(1) = 5(1)^4 = 5$

so eqn of tang. line

$$y = mx + b$$



$$y = 5x + b$$



$$x=1, y=1$$

$$1 = 5(1) + b$$

$$b = -4$$

eqn tang. line at  $x=1$ :  $y = 5x - 4$

# PRIME NOTATION / LEIBNIZ NOTATION

## Prime Notation

$$y = f(x) = x^4$$

$$y' = 4x^3$$

$$r'(x) = 4x^3$$

$$(x^4)' = 4x^3$$

## Leibnitz Notation

$$\frac{d}{dx} y = 4x^3 \Rightarrow \frac{dy}{dx}$$

*y depends on x*

$$\frac{d}{dx} (f(x)) = 4x^3$$

$$\frac{d}{dx} (x^4) = 4x^3$$

# RULES OF DERIVATIVES

## ① Constant Function Rule

$$y = f(x) = c$$

$$y' = 0$$

$$f'(x) = 0$$

$$(c)' = 0$$

$$\frac{dy}{dx} = 0$$

$$\frac{d}{dx} (f(x)) = 0$$

$$\frac{d}{dx} (c) = 0$$

$$\text{ex. } (\sqrt{2})' = 0$$

$$\text{or } \frac{d}{dx} (\sqrt{2}) = 0 \leftarrow f(x) = \sqrt{2}$$

$$\text{or } \frac{d}{dx} (\sqrt{2}) = 0 \leftarrow f(t) = \sqrt{2}$$

## ② Power Rule

$$y = f(x) = x^r$$

$$y' = rx^{r-1}$$

$$f'(x) = rx^{r-1} \quad (r \text{ is a constant})$$

$$\frac{dy}{dx} = rx^{r-1}$$

$$\frac{d}{dx} (f(x)) = rx^{r-1}$$

$$\text{ex. } \frac{d}{dx} (x^5) = 5x^4$$

$$\frac{d}{dt} (t^4) = 4t^3$$

$$\frac{d}{du} (u^5) = 0 \leftarrow \text{constant}$$

### ③ Constant Multiple Rule

$$\text{If } y = k f(x), \text{ then } y' = k f'(x)$$

$$\therefore [k \circ f(x)]' = k [f(x)]' = k f'(x)$$

Ex.  $y = 6x^4 \Rightarrow \frac{dy}{dx} = 6(x^4)' \quad ④$   
 $\quad \quad \quad \quad \quad 6(4x^3) \quad ⑤$   
 $\quad \quad \quad \quad \quad 24x^3$

Ex.  $y = \sqrt{5x}$ , find  $y'$

$$\begin{aligned}y &= \sqrt{5} \sqrt{x} \\&= \sqrt{5} x^{\frac{1}{2}} \\&= \sqrt{5} \left(x^{\frac{1}{2}}\right)' \\&= \sqrt{5} \left(\frac{1}{2} x^{-\frac{1}{2}}\right) \\&= \frac{\sqrt{5}}{2} x^{-\frac{1}{2}} \\&= \frac{\sqrt{5}}{2\sqrt{x}}\end{aligned}$$

Ex. What is the slope of the tangent line at  $x=9$ ?

$$\begin{aligned}y &= \sqrt{5x} \\y' &= \frac{\sqrt{5}}{2\sqrt{x}} = \frac{\sqrt{5}}{6}\end{aligned}$$

Note:

$$\begin{aligned}y &= \sqrt{5x} = (5x)^{\frac{1}{2}} \\y' &= \frac{1}{2} (5x)^{-\frac{1}{2}}\end{aligned}$$

$$\text{ex. } f(x) = \frac{5}{x^3}, \quad f'(x) = ?$$

$$f(x) = 5x^{-3}$$

$$f'(x) = 5(x^{-3})'$$

$$f'(x) = 5(-3x^{-4})$$

$$f'(x) = -15x^{-4}$$

### Proof of Constant Multiple Rule

$$\text{If } y = k f(x)$$

then

$$y' = k f'(x)$$

$$\text{Proof: Let } y = Q(x) = k \cdot f(x)$$

$$y' = Q'(x) = \lim_{h \rightarrow 0} \frac{Q(x+h) - Q(x)}{h}$$

$$Q'(x) = \lim_{h \rightarrow 0} \frac{k f(x+h) - k f(x)}{h}$$

$$Q'(x) = \lim_{h \rightarrow 0} \left[ k \cdot \frac{f(x+h) - f(x)}{h} \right]$$

$$= \lim_{h \rightarrow 0} k \cdot \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= k \cdot f'(x)$$

Ex.  $y = 5x$ ,  $y' = ?$

$$y' = (5x)' = 5(x)' = 5(1) = 5$$

### (4/5) Sum Rule & Difference Rule

$$\boxed{\text{If } y = f(x) \pm g(x) \text{ then } y' = f'(x) \pm g'(x)}$$

Ex.  $y = x^2 + 4$ , find  $y'$

$$y' = (x^2)' + (4)'$$

$$= 2x + 0 \\ = 2x$$

Ex.  $y = t^3 + 6t^2 + 3$ , find  $\frac{dy}{dt}$

$$\frac{dy}{dt} = (t^3)' + (6t^2)' + (3)'$$

$$= 3t^2 + 6(t^2)' + 0 \\ = 3t^2 + 12t$$

Note: Sum rule extends:

$$(f+g+h+j)' = f' + g' + h' + j'$$

### Proof of Sum Rule

Let  $y = Q(x) = f(x) + g(x)$

$$\text{So } y' = Q'(x) = \lim_{h \rightarrow 0} \frac{Q(x+h) - Q(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}$$

$$y' = Q'(x) \circ f'(x) + g'(x)$$

Ex,  $y = \frac{x^3 - x^4}{3x}$ ,  $y' = ?$

$$y = \frac{x^3}{3x} - \frac{x^4}{3x}$$

$$= \frac{1}{3}x^2 - \frac{1}{3}x^3$$

$$y' = \frac{2}{3}x - x^2$$

$$y = \sqrt{3x^5} - 2x^4 + \frac{4}{x^3}$$

Find  $\frac{dx}{dy}$

$$y = (3x^5)^{\frac{1}{2}} - 2x^4 + \frac{4}{x^3}$$

$$\frac{dx}{dy} \left( 3x^5 \right)^{\frac{1}{2}} = \frac{d}{y} \left( \sqrt{3} \cdot x^{\frac{5}{2}} \right) = \sqrt{3} \cdot \frac{5}{2} x^{\frac{3}{2}}$$

$$\frac{dx}{dy} \left( \frac{4}{x^3} \right) = \frac{dx}{dy} \left( 4x^{-3} \right) = 4 \cdot -3x^{-4}$$

$$\frac{dy}{dx} = \sqrt{3} \cdot \frac{5}{2} x^{\frac{3}{2}} - 8x^3 - 12x^{-4}$$

Quick Rules so far:

$$\textcircled{1} \quad (c)' = 0$$

$$\textcircled{2} \quad (x^r)' = r x^{r-1}$$

$$\textcircled{3} \quad (c f)' = c \cdot f'$$

$$\textcircled{4} \quad (f+g)' = f' + g'$$

$$\textcircled{5} \quad (f-g)' = f' - g'$$

$$\textcircled{6} \quad (f \cdot g)' = f' \cdot g' \text{ is FALSE !!}$$

$$\text{ex. } (x^3 \cdot x^2)' = (x^3)' \cdot (x^2)' = (3x^2)(2x)$$
$$\downarrow$$
$$(x^5)$$
$$\downarrow$$
$$5x^4$$

### ⑥ Product Rule

If  $y = f(x) \cdot g(x)$  then  $y' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

i.e.

$$(f \cdot g)' = f \cdot g' + f \cdot g'$$

e.g. Back to:

$$(x^3 \cdot x^2)' = (x^3)'(x^2) + (x^3)(x^2)'$$

$$= (3x^2)(x^2) + (x^3)(2x)$$

$$= 3x^4 + 2x^4$$

$$= 5x^4$$

Ex.

$$y = (x^3 + 1)(x^4 - 5) + 7x^4 \quad \text{Find } \frac{dy}{dx}$$

$$y' = [(x^3 + 1)(x^4 - 5)]' + [7x^4]' \quad \text{Sum rule}$$

$$y' = (x^3 + 1)'(x^4 - 5) + (x^3 + 1)(x^4 - 5)' + 7[x^4]' \quad \text{Constant mult. rule}$$

$$y' = (3x^2)(x^4 - 5) + (4x^3) + 7(4x^3)$$

$$y' = (3x^2)(x^4 - 5) + (4x^3) + (28x^3)$$

$$\cancel{\text{Proof of Product rule}}$$

### STANDARD EXAM QUESTION

If  $y = f(x) \cdot g(x)$  then  $y' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Proof:

$$y = Q(x) = f(x) \cdot g(x)$$

$$y' = Q'(x) \stackrel{\text{defn}}{=} \lim_{h \rightarrow 0} \frac{Q(x+h) - Q(x)}{h}$$

$$y' = Q'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x+h) + f(x) \cdot g(x+h) - f(x) \cdot g(x)}{h} \\
 &= \lim_{h \rightarrow 0} \left[ \frac{(f(x+h) - f(x)) g(x+h)}{h} + \frac{f(x) (g(x+h) - g(x))}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[ \frac{f(x+h) - f(x)}{h} \cdot g(x+h) \right] + \lim_{h \rightarrow 0} \left[ f(x) \cdot \frac{g(x+h) - g(x)}{h} \right] \\
 &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h \rightarrow 0} g(x+h) + \lim_{h \rightarrow 0} f(x) \cdot \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}
 \end{aligned}$$

$$y' = Q'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

### ⑦ Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{f'}{g} \quad \text{FALSE}$$

$$\text{If } y = \frac{f(x)}{g(x)}, \text{ then } y' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2} \quad \text{order matters}$$

Do not need to know the proof

$$\text{ex. } \left( \frac{x^5}{x^3} \right)' = \frac{(x^5)' \cdot x^3 - x^5 \cdot (x^3)'}{(x^3)^2} = \frac{5x^4 \cdot x^3 - x^5 \cdot 3x^2}{x^6}$$

$$= \frac{5x^7 - 3x^7}{x^6} = \frac{2x^7}{x^6} = 2x$$

$$\text{ex. } y = \frac{(x^3 - 2x + 1)(\sqrt{x} - 1)}{x^2 + 4x} \Rightarrow \frac{(x^3 - 2x + 1)}{x^2 + 4x} \cdot (\sqrt{x} - 1)$$

For also use product rule I<sup>st</sup>, but not recommended

$$y' = \frac{(x^3 - 2x + 1)'(\sqrt{x} - 1) - (x^3 - 2x + 1)(\sqrt{x} - 1)'}{(x^2 + 4x)^2}$$

$$y' = \frac{[(x^3 - 2x + 1)(\sqrt{x} - 1) + (x^2 - 2x + 1)(x^{\frac{1}{2}} - 1)](x^2 + 4x) - [(x^3 - 2x + 1)(\sqrt{x} - 1)]_{(x=1)}}{(x^2 + 4x)^2}$$

$$y' = \frac{[(3x^2 - 2)(\sqrt{x} - 1) + (x^2 - 2x + 1)(\frac{1}{2}x^{-\frac{1}{2}})](x^2 + 4x) - (x^3 - 2x + 1)(6\sqrt{x} - 1)(x^2 + 4x)}{(x^2 + 4x)^2}$$

$$y = \frac{(x^3 - 2x+1)(x^{\frac{1}{2}} - 1)}{x^2 + 4x} \quad \left( \frac{f}{g} \right) \quad \frac{\partial a}{\partial c}$$

$$y' = \frac{[(x^3 - 2x+1)(x^{\frac{1}{2}} - 1)](x^2 + 4x) - [(x^3 - 2x+1)(x^{\frac{1}{2}} - 1)](x^2 + 4x)}{(x^2 + 4x)^2}$$

$$y' = \frac{[(x^3 - 2x+1)(x^{\frac{1}{2}} - 1) + (x^3 - 2x+1)(x^{\frac{1}{2}} - 1)](x^2 + 4x) - [(x^3 - 2x+1)(x^{\frac{1}{2}} - 1)](2x+4)}{(x^2 + 4x)^2}$$

$$y' = \frac{[(3x^2 - 2)(x^{\frac{1}{2}} - 1) + (x^3 - 2x+1)(\frac{1}{2}x^{-\frac{1}{2}})](x^2 + 4x) - [(x^3 - 2x+1)(x^{\frac{1}{2}} - 1)](2x+4)}{(x^2 + 4x)^2}$$

September 29, 2016

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$$

$$y = \frac{(x-3)(x^2 + 4)}{2x-3}$$

$$\frac{dy}{dx} = \frac{[(x^3 - 1)(x^2 + 4)]'(2x-3) - [(x^3 - 1)(x^2 + 4)](2x-3)'}{(2x-3)^2}$$

$$\frac{dy}{dx} = \frac{[(x^3 - 1)(x^2 + 4) + (x^3 - 1)'(x^2 + 4)](2x-3)'}{(2x-3)^2} - \frac{[(x^3 - 1)(x^2 + 4)](2x-3)'}{(2x-3)^2}$$

$$= \frac{[(x^3-1)(2x) + (3x^2)(x^2+1)] - [(2x-3) - (x^3-1)(x^2+1)]}{(2x-3)^2}$$

Differenciate = "find the derivative"

## Higher Order Derivatives

ex.  $y = f(x) = x^5$

Prime notation

1<sup>st</sup> deriv.  $f'(x) = 5x^4$

2<sup>nd</sup>  $f''(x) = 20x^3$

3<sup>rd</sup>  $f'''(x) = 60x^2$

4<sup>th</sup> deriv.  $f^{(4)}(x) = 120x$

5<sup>th</sup> deriv.  $f^{(5)}(x) = 120$

$f^{(6)}(x) = 0$

$f^{(7)}(x) = 0$

⋮

ex.  $y = f(x) = x^5$

Leibnitz Notation

$$\frac{dy}{dx} = 5x^4$$

2nd deriv.  $\Rightarrow \frac{d^2y}{dx^2} = 20x^3$

$$(f'(x))' = f''(x)$$

$\frac{d}{dx} \left( \frac{dy}{dx} \right)$   
2nd

## Differentiability

Ex. Consider  $f(x) = |x|$  Find  $f'(0)$ , if possible

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

$$f'(0) = \lim_{h \rightarrow 0} \frac{|h| - 0}{h}$$

$$= \lim_{h \rightarrow 0} \frac{|h|}{h}$$

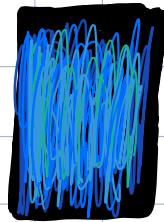
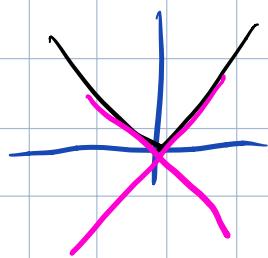
$$|h| = \begin{cases} -h & \text{if } h < 0 \\ h & \text{if } h \geq 0 \end{cases}$$

$$\lim_{h \rightarrow 0^-} \frac{|h|}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = \lim_{h \rightarrow 0^-} (-1) = -1$$

$$\lim_{h \rightarrow 0^+} \frac{|h|}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = \lim_{h \rightarrow 0^+} (1) = 1$$

$$f'(0) = \lim_{h \rightarrow 0} \frac{|h|}{h} \text{ DNE}$$

Concl:  $f(x) = |x|$  is not differentiable at  $x = 0$



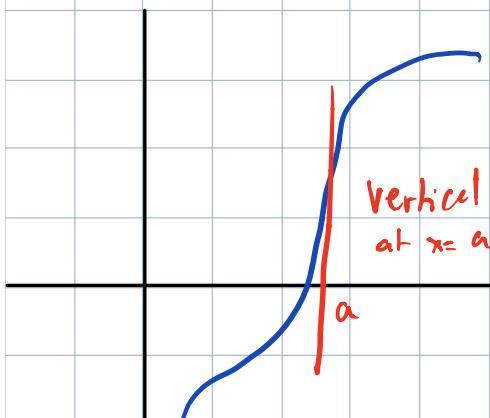
In general, a  $f^n$  that has a "sharp" pt at  $x=a$  will not be diff<sup>bic</sup> at  $x=a$ .

Def<sup>n</sup>: A fc<sup>n</sup>  $y=f(x)$  is said to be diff<sup>bic</sup> at  $x=a$  if:

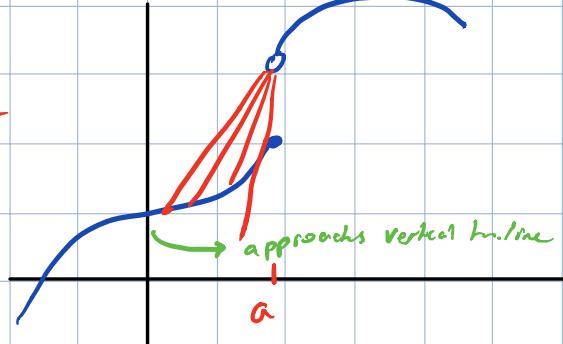
$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \text{ exists}$$

$$\text{i.e. } f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x-a} \text{ exists}$$

### Other Geometric Indicators of Non-Differentiability



$y=f(x)$  is not diff<sup>bic</sup> at  $x=a$

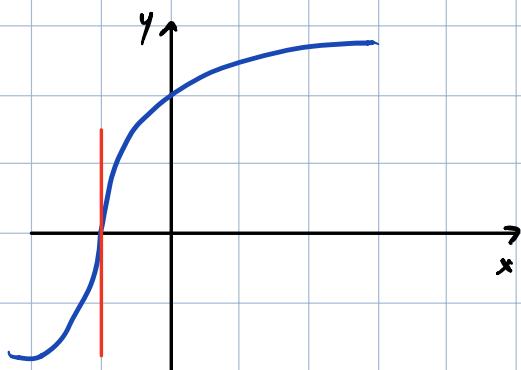


$y=f(x)$  is not diff<sup>bic</sup> at  $x=a$

Ex. Determine whether  $f(x) = \sqrt[3]{x+1}$  is diff<sup>uc</sup> at  $x=-1$

$$\begin{aligned}
 f'(-1) &= \lim_{h \rightarrow 0} \frac{f(-1+h) - f(-1)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt[3]{-1+h+1} - \sqrt[3]{-1+1}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt[3]{h}}{h} \\
 &= \lim_{h \rightarrow 0} h^{-\frac{2}{3}} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h^{\frac{2}{3}}} \stackrel{0^0}{=} \frac{1}{0^+} = +\infty \\
 &\quad (\text{DNE})
 \end{aligned}$$

Concl  $f(x) = \sqrt[3]{x+1}$  is not diff<sup>uc</sup> at  $x=-1$



Ex.  $f(x) = x^3$  Is  $f$  diff<sup>buc</sup> at  $x=2$ ?

Method A

$$f'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(2+h)^3 - 2^3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2^3 + 3(2^2)h + 3(2)h^2 + h^3 - 8}{h}$$

$$= \lim_{h \rightarrow 0} \frac{12h + 6h^2 + h^3}{h}$$

$$= \lim_{h \rightarrow 0} 12 + 6h + h^2 \stackrel{oh}{=} 12$$

Concl  $f(x)$  is diff<sup>buc</sup> at  $x=2$

Method B

By power rule, since  $f(x) = x^3$ ,  $f'(x) = 3x^2$

$$f'(2) = 3(2)^2 = 12$$

So  $f(x)$  is diff<sup>buc</sup> at  $x=2$

ex. Is  $f(x) = \sqrt[3]{x}$  diff'ble at  $x=0$ ?

We can use method A or B.

### Method B

$$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$$

$$f'(0) = \frac{1}{0}$$

cond: f is not diff'ble at  $x=0$

If a  $f^{on}$   $\overset{y=f(x)}{\text{is not cont' at } x=a}$  then  $y=f(x)$  is not diff'ble  $\leftarrow$  TRUE

If a  $f^{on}$   $\overset{y=f(x)}{\text{is cont' at } x=a}$  then  $y=f(x)$  is diff'ble  $\leftarrow$  FALSE

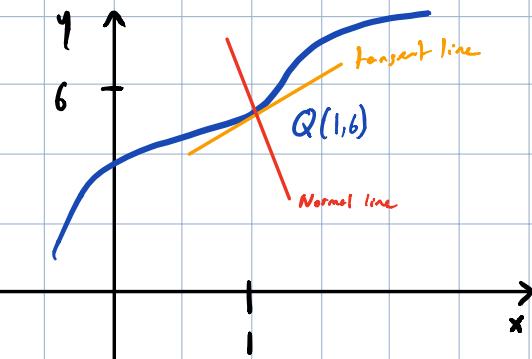
If a  $f^{on}$  is diff'ble at  $x=a$  then  $y=f(x)$  is cont' at  $x=a$   $\leftarrow$  TRUE

PROOF?

# NORMAL LINES

ex. What is the eqn of the NORMAL LINE to the graph of  $f(x) = 4x^4 + x + 1$  at  $x=1$

normal = perpendicular



$$\text{when } x=1, y = 4(1)^4 + (1) + 1 = 6$$

$$f'(x) = 16x^3 + 1$$

$$f'(1) = 17 \leftarrow \text{slope of fun line}$$

$$\text{so slope of NORMAL LINE is } -\frac{1}{17}$$

$$y = mx + b$$

$$y = -\frac{1}{17}x + b$$

$$\text{Subst } x=1, y=6$$

$$6 = -\frac{1}{17}(1) + b$$

$$6 + \frac{1}{17} = b$$

$$b = \underline{\underline{103}}$$

September 30, 2016

Proof: Assume  $y = f(x)$  is diff'ble at  $x=a'$

i.e.  $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$  exists      ②  $\lim_{x \rightarrow a} f(x) = f(a)$

(Note this implies  $f(x)$  is def')

$$\textcircled{1} \quad \lim_{x \rightarrow a} [f(x) - f(a)]$$

$$= \lim_{x \rightarrow a} \left[ \frac{f(x) - f(a)}{x - a} \cdot (x - a) \right]$$

$$= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \rightarrow a} (x - a)$$

$$= f'(a) \cdot 0$$

$$= 0$$

$$\textcircled{2} \quad \lim_{x \rightarrow a} [f(x) - f(a)] = 0$$

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} [(f(x) - f(a)) + f(a)]$$

constant  
↓

$$= \lim_{x \rightarrow a} [f(x) - f(a)] + \lim_{x \rightarrow a} f(a)$$

$$= 0 + f(a)$$

$$= f(a)$$

i.e.  $\lim_{x \rightarrow a} f(x) = f(a)$

so  $f$  is cont'd at  $x=a'$

## CHAIN RULE

Consider  $y = (x^2 + 1)^3$

Let  $g(x) = x^2 + 1$   
 $\& f(y) = y^3$

$$(f \circ g)(x) = f(g(x)) = [g(x)]^3 = (x^2 + 1)^3 = y$$

So  $y = (x^2 + 1)^3$  is a composition of two "simpler"  $f$ 's

$$f(x) = x^3 \quad \& \quad g(x) = x^2 + 1$$

Chain rule let's us find the deriv. of a composition of two  $f$ 's

⑧ Chain Rule

$$\text{If } y = f(g(x)) \text{ then } \frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

$$\text{Back to } y = (x^2 + 1)^3 = f(g(x))$$

$$\text{where } g(x) = x^2 + 1 \rightarrow 2x$$

$$\text{& } f(u) = u^3 \rightarrow 3u^2 \rightarrow f'(g(x)) = 3(g(x))^2$$

By chain rule

$$\begin{aligned}\frac{dy}{dx} &= f'(g(x)) \cdot g'(x) \\ &= 3(g(x))^2 \cdot g'(x) \\ &= 3(x^2 + 1)^2 \cdot 2x\end{aligned}$$

$$\underline{\text{i.e.}} \quad \frac{dy}{dx} = 6x(x^2 + 1)^2$$

### Summary

Started with  $y = (x^2 + 1)^3$

$$\frac{dy}{dx} = 3(x^2 + 1)^2 \cdot (x^2 + 1)'$$

It appears that the "structure" of result looks like:

$$y(\text{inside})^r \rightarrow \frac{dy}{dx} = r(\text{inside})^{r-1} (\text{inside})'$$

$$\text{e.g. } y = \sqrt{3x-1} \quad \frac{dy}{dx} = \frac{1}{2}(3x-1)^{-\frac{1}{2}} \cdot (3x-1)'$$

$$\text{ex. } f(t) = \left(\frac{t-1}{t+1}\right)^6, \quad f'(t) = ?$$

$$\begin{aligned}f'(t) &= 6 \left(\frac{t-1}{t+1}\right)^5 \left(\frac{t-1}{t+1}\right)' \quad \text{chain rule} \\&= 6 \left(\frac{t-1}{t+1}\right)^5 \cdot \frac{(t-1)'(t+1) - (t-1)(t+1)'}{(t+1)^2} \\&\Rightarrow 6 \left(\frac{t-1}{t+1}\right)^5 \cdot \frac{(t+1) - (t-1)}{(t+1)^2} \\&= 6 \left(\frac{t-1}{t+1}\right)^5 \cdot \frac{2}{(t+1)^2} \\&= \left(\frac{t-1}{t+1}\right)^5 \cdot \frac{12}{(t+1)^2}\end{aligned}$$

WARNING

$$\text{ex. } y = (x^3 - 2x + 1)^4, \quad y' = ?$$

$$y' = (\cancel{3x^2 - 2})^3 (3x^2 - 2)$$

$$y' = 4(x^3 - 2x + 1)^3 (x^3 - 2x + 1)'$$

$$= 4(x^3 - 2x + 1)^3 (3x^2 - 2)$$

$$\text{ex. } y = -5(3x^2-1)^7 ; \frac{dy}{dx} = ?$$

$$\frac{dy}{dx} = -5 \left[ (3x^2-1)^7 \right]' \quad \text{Constant mult. rule}$$

$$= -5 \left[ 7(3x^2-1)^6 (6x) \right]$$

$$= -5 \left[ 7(3x^2-1)^6 (6x) \right]$$

$$= -210x(3x^2-1)^6$$

$$\text{ex. } y = \frac{1}{\sqrt[3]{(x^2-1)^5}} = \frac{1}{[(x^2-1)^5]^{\frac{1}{3}}} = (x^2-1)^{-\frac{5}{3}}$$

$$y' = -\frac{5}{3}(x^2-1)^{-\frac{8}{3}}(x^2-1)'$$

$$= -\frac{5}{3}(x^2-1)^{-\frac{8}{3}}(2x)$$

$$= -\frac{10}{3}x(x^2-1)^{-\frac{8}{3}}$$

Chain Rule using only Leibniz Notation

$$y = f(x) \Rightarrow \frac{dy}{dx} = f'(x)$$

$$y = f(t) \Rightarrow \frac{dy}{dt} = f'(t)$$

$$y = f(u) \Rightarrow \frac{dy}{du} = f'(u)$$

### Chain Rule - Notation

If  $y = f(g(x))$  then  $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$

$$\text{Let } u = g(x) \Rightarrow \frac{du}{dx} = g'(x)$$

$$\text{So } f'(g(x)) = f'(u) = \frac{dy}{du}$$

So chain rule says:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Ex.  $y = (x^5 + 1)^4$

$$\text{Let } u = x^5 + 1 \rightarrow \frac{du}{dx} = 5x^4$$

$$\text{So } y = u^4 \rightarrow \frac{dy}{du} = 4u^3$$

$$\text{So } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$4u^3 (5x^4)$$

$$= 4(x^5 + 1)^3 (5x^4)$$

Ex. Find  $\frac{dy}{dx}$  if  $y = \sqrt{7 - 4x^3}$

using ①  $y = (\text{inside})^r$

$$\textcircled{2} \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\textcircled{1} \quad y = (7 - 4x^3)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2} (7 - 4x^3)^{-\frac{1}{2}} \cdot (7 - 4x^3)'$$

$$\frac{dy}{dx} = \frac{1}{2} (7 - 4x^3)^{-\frac{1}{2}} \cdot (-12x^2)$$

$$\frac{dy}{dx} = -\frac{6x^2}{\sqrt{7 - 4x^3}}$$

$$\textcircled{2} \quad \text{Let } u = 7 - 4x^3 \Rightarrow \frac{du}{dx} = -12x^2$$

$$\text{so } y = \sqrt{u} \Rightarrow \frac{dy}{du} = \frac{1}{2} u^{-\frac{1}{2}}$$

$$\begin{aligned} \text{so } \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= (-12x^2) \left( \frac{1}{2} u^{-\frac{1}{2}} \right) \\ &= (-12x^2) \frac{1}{2} (7 - 4x^3)^{-\frac{1}{2}} \\ &= \underline{\underline{-6x^2}} \\ &\qquad \qquad \qquad \sqrt{7 - 4x^3} \end{aligned}$$

$$y = f \sqrt{3t+4}$$

$$\begin{aligned}\frac{dy}{dt} &= (f)'(3t+4)^{\frac{1}{2}} + f\left[(3t+4)^{\frac{1}{2}}\right]' \\ &= (3t+4)^{\frac{1}{2}} + f\left[\frac{1}{2}(3t+4)^{-\frac{1}{2}}(3t+4)'\right] \\ &\quad ;\end{aligned}$$

October 6, 2016

$$y = Q(x) = f(g(x))$$

$$Q'(x) = f'(g(x)) \cdot g'(x)$$

If we let  $v = g(x)$

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$$

$$\text{Recall } y = Q(x) = (x^2+1)^3 = f(g(x))$$

$$\text{where } g(x) = x^2+1 \rightarrow g'(x) = 2x$$

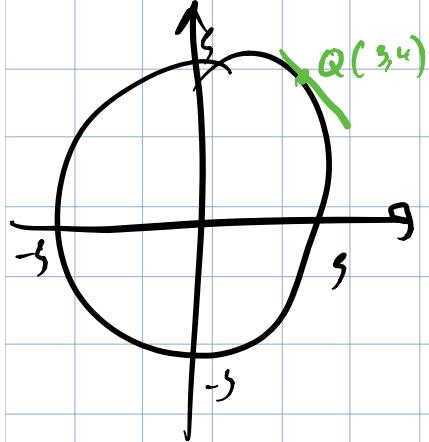
$$f(x) = x^3 \rightarrow f'(x) = 3x^2$$

$$Q'(x) = f'(g(x)) \cdot g'(x)$$

$$\begin{aligned}Q'(3) &= f'(g(3)) \cdot g'(3) \\ &= f'(10) \cdot g'(3)\end{aligned}$$

# IMPLICIT FCNS

e.g.  $x^2 + y^2 = 25$   $\leftarrow$  Not a fcn

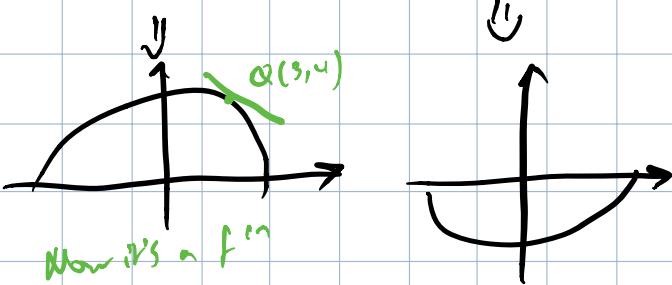


$$x^2 + y^2 = 25$$

$$y^2 = 25 - x^2$$

$$y = \pm \sqrt{25 - x^2}$$

$$y = +\sqrt{25 - x^2} \text{ or } y = -\sqrt{25 - x^2}$$

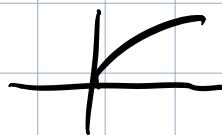
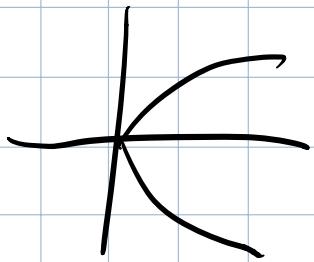


We say that the eqn  $x^2 + y^2 = 25$  implicitly defines 2 fns:

$$y = +\sqrt{25 - x^2} \quad \text{and} \quad y = -\sqrt{25 - x^2}$$

Ex.  $x = y^2$  implicitly defines two fns

$$y = +\sqrt{x} \quad y = -\sqrt{x}$$



Ex.  $x + y = 5$  implicitly defines  $y = 5 - x$

Ex. Back to  $x^2 + y^2 = 25$

$$\begin{aligned} y &= \sqrt{25 - x^2} \\ y &= (25 - x^2)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2} (25 - x^2)^{-\frac{1}{2}} (25 - x^2)' \\ &= \frac{1}{2} \cdot \frac{1}{\sqrt{25 - x^2}} (-2x) \end{aligned}$$

$$= -\frac{x}{\sqrt{25 - x^2}} = -\frac{x}{y}$$

$$\begin{aligned} y &= -\sqrt{25 - x^2} \\ y &= -(25 - x^2)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= -\frac{1}{2} (25 - x^2)^{-\frac{1}{2}} (25 - x^2)' \\ &= -\frac{1}{2} \cdot \frac{1}{\sqrt{25 - x^2}} (-2x) \end{aligned}$$

$$= -\frac{x}{\sqrt{25 - x^2}} = -\frac{x}{y}$$

We have found 1 formula  $\frac{dy}{dx} = -\frac{x}{y}$  which describes the slope of the tangent line at any  $(x, y)$  on the circle

$$x^2 + y^2 = 25$$

## IMPLICIT DIFFERENTIATION

$$x^2 + y^2 = 25$$

assume that it implicitly defines  $y$  as a function of  $x$ .  
i.e. assume  $y = f(x)$

$$x^2 + (f(x))^2 = 25$$

Take deriv. of both sides

$$\frac{d}{dx} \left[ x^2 + (f(x))^2 \right] = \frac{d}{dx} [25]$$

$$\frac{d}{dx} [x^2] + \frac{d}{dx} [(f(x))^2] = \frac{d}{dx} [25]$$

$$2x + 2[f(x)][f(x)'] = 0$$

$$2x + 2y \cdot y' = 0$$

$$y' = -\frac{x}{y}$$

Shorter Method...

$$\frac{d}{dx} [x^2 + (y)^2] = \frac{d}{dx} [2s]$$

$$\frac{d}{dx} [x^2] + \frac{d}{dx} [y^2] = \frac{d}{dx} [2s]$$

↓  
chain

$$2x + 2y \cdot y' = 0$$

;

Even Shorter...

$$[x^2 + y^2]' = [2s]'$$

$$(x^2)' + (y^2)' = (2s)'$$

$$2x + 2yy' = 0$$

not 2y

$$\text{Ex. } 5x^2 + 4y^3 = y, \text{ find } \frac{dy}{dx}$$

$$(5x^2 + 4y^3)' = y'$$

$$(5x^2)' + (4y^3)' = y'$$

$$10x + 4[3y^2(y')] = y'$$

$$10x + 12y^2y' = y'$$

$$10x = y' - 12y^2 \cdot y'$$

$$10x = y'(1 - 12y^2)$$

$$y' = \frac{10x}{1 - 12y^2}$$

$$\text{Ex. } x^2 + 1 = y^3$$

Find  $\frac{dy}{dx}$  by finding the derivative

① Explicitly

② Implicitly

$$\textcircled{1} \quad y = (x^2 + 1)^{\frac{1}{3}} \leftarrow \text{explicit form}$$

$$y' = \frac{1}{3} (x^2 + 1)^{-\frac{2}{3}} (x^2 + 1)'$$

$$= \frac{2x}{3(x^2 + 1)^{\frac{4}{3}}}$$

$$\textcircled{2} \quad (x^2 + 1)' = (y^3)'$$

$$2x = 3y^2 \cdot y'$$

$$y' = \frac{2x}{3y^2}$$

ex. Find the eqn of the tangent line

to the graph of  $x^3 + y^3 = x + 1$

at pt  $Q(0,1)$

$$x^3 + y^3 = x + 1$$

$$(x^3 + y^3)' = (x+1)'$$

$$3x^2 + 3y^2 \cdot y' = 1 \leftarrow \text{could have solved } x=0, y=1 \text{ in here}$$

$$y' = \frac{1 - 3x^2}{3y^2}$$

At pt  $Q(0,1)$

$$y' = \frac{1 - 3(0)^2}{3(1)^2}$$

$$m = y' = \frac{1}{3}$$

Tangent line

$$y = \frac{1}{3}x + b$$

$$1 = \frac{1}{3}(0) + b$$

$$b = 1$$

$$y = \frac{1}{3}x + 1$$

$$\text{Ex. } 3y^2 + x^2y^3 = 2 - y$$

$$\frac{dy}{dx} = ?$$

$$\begin{aligned} (3y^2 + x^2y^3)' &= (2 - y)' \\ 6y \cdot y' + (x^2)y^3 + x^2(y^3)' &= 2' - y' \\ 6y \cdot y' + 2x^2y^3 + x^2(3y^2y') &= -y' \end{aligned}$$

$$\begin{aligned} 6y \cdot y' + 2x^2y^3 + 3x^2y^2y' &= -y' \\ 6y \cdot y' + 3x^2y^2y' + y' &= -2x^2y^3 \\ y'(6y + 3x^2y^2 + 1) &= -2x^2y^3 \\ y' &= \frac{-2x^2y^3}{6y + 3x^2y^2 + 1} \end{aligned}$$

$$y = \sqrt[3]{x^4 + x}$$

OK

Impunkt

$$\begin{aligned} y &= (x^4 + x)^{\frac{1}{3}} \\ y' &= \frac{1}{3}(x^4 + x)^{-\frac{2}{3}} (4x^3 + 1) \end{aligned}$$

$$\begin{aligned} y^3 &= x^4 + x \\ (y^3)' &= (x^4 + x)' \\ 3y^2y' &= 4x + 1 \\ y' &= \frac{4x + 1}{3y^2} \end{aligned}$$

$$y' = \frac{4x + 1}{3\sqrt[3]{x^4 + x}})$$