3A) GENERAL VECTOR SPACES

Vectors as magnitude and direction in $\mathbb{R}^2, \mathbb{R}^3, \ldots, \mathbb{R}^n$ are just particular types of vectors... There are other kinds of vectors too!

Defn: A vector is an element of a "vector space"

Well, then what's a vector space?!

Defn: A vector space is a set of elements (called vectors) with 2 operations satisfying a number of properties called "axioms". These 2 operations are called addition and scalar multiplication.

* Scalars can be $\mathbb{R}$ or $\mathbb{C}$, but they could also be other things

So a vector is just anything that satisfies these axioms...
Axioms 1-5 deal with the addition
Axioms 6-10 involve scalar multiplication.

Looking carefully at these axioms, we can see that <u>functions</u> (as long as they are defined on an interval) are vectors too!

Definition of VECTOR SPACE

A vector space V is a non-empty set of elements called vectors with 2 operations, addition and scalar multiplication, satisfying the following axioms.

For all $\vec{u}$, $\vec{v}$, $\vec{w} \in V$, and all scalars[1] r, s:

A1) $\vec{u}+\vec{v} \in V$ (*Closure under addition.*)

A2) $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ ← commutativity

A3) $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$
(hence write $\vec{u}+\vec{v}+\vec{w}$ without the unnecessary parentheses).

This one implies that the vector space is non empty

A4) There is a vector $\vec{\imath} \in V$ satisfying that for all $\vec{v} \in V$:
$$\vec{v}+\vec{\imath}=\vec{v}.$$
($\vec{\imath}$ does not depend on $\vec{v}$, it is the same for any $\vec{v} \in V$. $\vec{\imath}$ is the zero-vector: $\vec{\imath}=\vec{0}$.)

A5) For all $\vec{v} \in V$, there is an $\vec{n} \in V$ such that
$$\vec{v}+\vec{n}=\vec{0}.$$
($\vec{n}$ depends on $\vec{v}$, it is the negative of $\vec{v}$: $\vec{n}=-\vec{v}$.)

S1) $r\vec{v} \in V$ (*Closure under scalar multiplication.*)

S2) $r(\vec{v}+\vec{w})=r\vec{v}+r\vec{w}$

S3) $(r+s)\vec{v}=r\vec{v}+s\vec{v}$

S4) $(rs)\vec{v}=r(s\vec{v})$

S5) $1\vec{v}=\vec{v}$

[1] In Linear Algebra I, we take the word scalar to mean *real numbers*; in Linear Algebra II and in Differential Equations, we take the word scalar to mean either *real numbers* or *complex numbers* depending on the context.

So we saw that there are many types of vectors. As long as something satisfies the 10 axioms (above) describing a vector space. Examples of vector spaces, as seen in NYC, are $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n$ (called euclidean vector spaces). But like we said, other things can be vectors too:

① The set of all $\overset{\text{fixed size}}{\overline{m \times n}}$ matrices

Note: The set M of all matrices is <u>not</u> a vector space.

Counterexample: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \in M$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in M$.

But $A+B \notin M$ ($\because$ undefined)

② The set $\mathbb{P}$ of all polynomials (any degree, including the zero polynomial $p(x) \equiv 0$ (degree is undefined or $-\infty$.)

Proof that the set $\mathbb{P}_n$ of all polynomials of degree $\leq n$ (including $p(x) \equiv 0$) is a vector space on handout.

Note: The set of all polynomials of $P_{=2}$ (degree <u>exactly</u> 2) is <u>NOT</u> a vector space. We can easily find a

Counterexample: $p(x) = -2x^2 - 7x + 5 \in \mathbb{P}_{=2}$

$$q(x) = 2x^2 + 5x - 8 \in \mathbb{P}_{=2}$$

$$p(x) + q(x) = -2x - 3 \notin \mathbb{P}_{=2}.$$

$\therefore$ Axiom A_1 does not hold $\Rightarrow$ Not a V.S.

$\mathbb{P}_2$ is a Vector Space

$$\mathbb{P}_2 = \left\{ a + bx + cx^2 \,\middle|\, a,b,c \in \mathbb{R} \right\}$$ [1]

For any $p(x) = a + bx + cx^2$,

$p_1(x) = a_1 + b_1x + c_1x^2$,

$p_2(x) = a_2 + b_2x + c_2x^2$,

$p_3(x) = a_3 + b_3x + c_3x^2$,

and any $r, s \in \mathbb{R}$, we do have:

A1) $$\begin{aligned} p_1(x) + p_2(x) &= \left(a_1 + b_1x + c_1x^2\right) + \left(a_2 + b_2x + c_2x^2\right) \\ &= a_1 + a_2 + b_1x + b_2x + c_1x^2 + c_2x^2 \\ &= \left(a_1 + a_2\right) + \left(b_1 + b_2\right)x + \left(c_1 + c_2\right)x^2 \in \mathbb{P}_2 \ \surd \end{aligned}$$

A2) $$\begin{aligned} p_1(x) + p_2(x) &= \ldots = \left(a_1 + a_2\right) + \left(b_1 + b_2\right)x + \left(c_1 + c_2\right)x^2 \\ p_2(x) + p_1(x) &= \ldots = \left(a_2 + a_1\right) + \left(b_2 + b_1\right)x + \left(c_2 + c_1\right)x^2 \\ &= \left(a_1 + a_2\right) + \left(b_1 + b_2\right)x + \left(c_1 + c_2\right)x^2 \ \surd \end{aligned}$$

A3) $$\begin{aligned} p_1(x) + \left[p_2(x) + p_3(x)\right] &= \left(a_1 + b_1x + c_1x^2\right) + \left[\left(a_2 + b_2x + c_2x^2\right) + \left(a_3 + b_3x + c_3x^2\right)\right] \\ &= a_1 + b_1x + c_1x^2 + \left[\left(a_2 + a_3\right) + \left(b_2 + b_3\right)x + \left(c_2 + c_3\right)x^2\right] \\ &= \left(a_1 + a_2 + a_3\right) + \left(b_1 + b_2 + b_3\right)x + \left(c_1 + c_2 + c_3\right)x^2 \\ \left[p_1(x) + p_2(x)\right] + p_3(x) &= \left[\left(a_1 + b_1x + c_1x^2\right) + \left(a_2 + b_2x + c_2x^2\right)\right] + \left(a_3 + b_3x + c_3x^2\right) \\ &= \left[\left(a_1 + a_2\right) + \left(b_1 + b_2\right)x + \left(c_1 + c_2\right)x^2\right] + a_3 + b_3x + c_3x^2 \\ &= \left(a_1 + a_2 + a_3\right) + \left(b_1 + b_2 + b_3\right)x + \left(c_1 + c_2 + c_3\right)x^2 \ \surd \end{aligned}$$

A4) The zero polynomial $t(x) \equiv 0$ for all x, i.e., $t(x) = 0 + 0x + 0x^2 \in \mathbb{P}_2$ does satisfy that

$$p(x) + t(x) = p(x) \quad \text{for all } p(x) \in \mathbb{P}_2.$$

A5) For any $p(x) = a + bx + cx^2 \in \mathbb{P}_2$, $n(x) = -a - bx - cx^2 \in \mathbb{P}_2$ does satisfy

$$p(x) + n(x) = 0 \quad \text{(the zero polynomial).}$$

We write $n(x) = -p(x)$.

S1) --> S5) on other side...

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[1] This handout actually shows that $\mathbb{P}_2(\mathbb{R})$ (polynomials with coefficients in $\mathbb{R}$) is a vector space over $\mathbb{R}$ (using the scalars r,s in $\mathbb{R}$). We could have generalized to $\mathbb{P}_2(\mathbb{C})$ is a vector space over $\mathbb{C}$.

S1) $r(p(x)) = r\left(a+bx+cx^2\right) = (ra)+(rb)x+(rc)x^2 \in \mathbb{P}_2$ √

S2) $$\begin{aligned} r\left[p_1(x)+p_2(x)\right] &= r\left[(a_1+a_2)+(b_1+b_2)x+(c_1+c_2)x^2\right] \\ &= r(a_1+a_2)+r(b_1+b_2)x+r(c_1+c_2)x^2 \\ &= (ra_1+ra_2)+(rb_1+rb_2)x+(rc_1+rc_2)x^2 \\ rp_1(x)+rp_2(x) &= \left[(ra_1)+(rb_1)x+(rc_1)x^2\right]+\left[(ra_2)+(rb_2)x+(rc_2)x^2\right] \\ &= (ra_1+ra_2)+(rb_1+rb_2)x+(rc_1+rc_2)x^2 \ \surd \end{aligned}$$

S3) $$\begin{aligned} (r+s)p(x) &= (r+s)\left[a+bx+cx^2\right] \\ &= (r+s)a+(r+s)bx+(r+s)cx^2 \\ &= (ra+sa)+(rb+sb)x+(rc+sc)x^2 \\ rp(x)+sp(x) &= r\left[a+bx+cx^2\right]+s\left[a+bx+cx^2\right] \\ &= \left[ra+rbx+rcx^2\right]+\left[sa+sbx+scx^2\right] \\ &= (ra+sa)+(rb+sb)x+(rc+sc)x^2 \ \surd \end{aligned}$$

S4) $$\begin{aligned} (rs)p(x) &= (rs)\left[a+bx+cx^2\right] = rs(a)+rs(bx)+rs(cx^2) = rsa+rsbx+rscx^2 \\ r\left[sp(x)\right] &= r\left[s\left(a+bx+cx^2\right)\right] = r\left[sa+sbx+scx^2\right] = rsa+rsbx+rscx^2 \ \surd \end{aligned}$$

S5) $1(p(x)) = 1\left(a+bx+cx^2\right) = 1a+1bx+1cx^2 = a+bx+cx^2 = p(x)$ √

③ The set $F(D)$ of all real valued function defined on a fixed domain D forms a V.S.

Remark: $\mathbb{R}^1 \cong \mathbb{R}$ (but not exactly equal)

1D euclidian vector space; is isomorphic to; The real number line

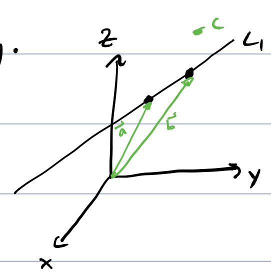
$\mathbb{R}^0$ is then $\mathbb{R}^0 = \{0\}$. This is called the <u>trivial vector space.</u>

The set E of all polynomials with even coefficients is <u>not</u> a vector space (closed under addition $\Rightarrow A_1$ ☑, but not closed under scalar multiplication $\Rightarrow S1$ ☒ e.g. multiply by $\frac{1}{2}$).

Indeed, the notions seen in linear algebra 1 with vectors in $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \dots, \mathbb{R}^n$ hold for vector spaces in general. The handout below states (or restates) these notions in a general context.

Aside: Lines not through origin are not a subspace.

E.g.



L_1 is not a subspace of $\mathbb{R}^3$ $\because$ $\vec{a}+\vec{b}=\vec{c}$, and $\vec{c}$ is not on line. Not closed under addition.

But L_1 is a <u>subset</u> of $\mathbb{R}^3$.

GENERALIZING

LINEAR ALGEBRA I

TO

GENERAL VECTOR SPACES

Definition of Vector Space

The general definition of a vector space is given on a separate handout.
For a set to be a vector space, it needs to satisfy the 10 axioms[1] A1) – A5) and S1) – S5) there.

In the end, what we realize is that there are many types of vectors apart from just arrows in $\mathbb{R}^2$ or $\mathbb{R}^3$, or even $\mathbb{R}^n$. Matrices, polynomials, certain sets of functions, etc. may be considered as *vectors*.

Definition of Subspace

Start with a known vector space *V*, and consider a *subset* $W \subseteq V$.
W is said to be a ***subspace*** of *V* if *W* itself happens to be a vector space in its own right.

Subspace Test

Start with a known vector space *V*, and consider a *subset* $W \subseteq V$.
Then *W* turns out to be a ***subspace*** of *V* if, and only if,

for any scalar *r*, and for any $\vec{u}, \vec{v} \in W$:

A4) $\vec{0} \in V$

A1) $\vec{u} + \vec{v} \in W$ (closure under vector addition)

S1) $r\vec{v} \in W$ (closure under scalar multiplication) $\boxed{\text{ST}}$

This means: When we test a completely new set for being a vector space, we need to verify 10 axioms, but when we test a subset of a previously known vector space, we need to check only 3 axioms instead of 10.

Remark: It is not necessary either to verify A4) if we know (verify) that *W* is non-empty.

Spans

Recall that the **span** of the set of vectors $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\} \subseteq V$ is the set of all linear combinations of those vectors: $\text{Span}(S) = \{\, \vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_n\vec{v}_n \mid a_1, a_2, ..., a_n \in \mathbb{R} \,\}$.

Remark: $\mathbb{R}$ could be replaced by another set of scalars.

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[1] Some of the 10 axioms of the handout are redundant; ten are given for extra clarity. One could actually define vector spaces using fewer axioms.

It is proven in Linear I that $\text{Span}(S)$ is always a *subspace* (not only a subset) of V.

Convention: $\text{Span}(\varnothing) := \{\vec{0}\}$.

Dependence and Independence.

Consider the following vectors of V: $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$.
They are said to be *linearly dependent* if, and only if, at least one of these can be written as a linear combination of the others. Equivalently, we may say that they are *linearly independent* if none of these can possibly be written as a linear combination of the others.

In practice, we test the linear independence of vectors through the following:

A set $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ of vectors of V is linearly independent when

$$a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_n\vec{v}_n = \vec{0} \Rightarrow a_1 = a_2 = ... = a_n = 0.$$

(That is, all a_i equal to zero is the *only* possibility.)

In other words, the only way to obtain the zero vector by means of a linear combination of the vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$ is through the trivial linear combination.

Spanning Sets, Bases, and Dimension

A **spanning set** for V, or a **generating set** of V is a set S of vectors of V that happens to span V:
$\text{Span}(S) = V$.

A spanning set of V which is linearly independent is said to be a **basis** B of V.

The **dimension** of V is the cardinality (number of elements in the set) of any one of its possible bases B:
$\dim(V) = \#B$.

Remark: It is mentioned without proof in Linear Algebra I, and then proven in Linear Algebra II, that any two bases of V must necessarily have the same cardinality.

Recall that if $B = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ is a basis of V, then any vector $\vec{w} \in V$ can be written *uniquely* as a linear combination of the basis vectors; that is, writing $\vec{w}$ as $\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_n\vec{v}_n$ can be made with only one possible set of values for $a_1, a_2, ..., a_n$.

In a sense, a basis of V turns out to be a generating set for V that has the least number of elements possible while still spanning V.

$\mathbb{P}$ is a vector space. $\mathbb{P}_2$ is a subset of $\mathbb{P}$ and a subspace in $\mathbb{P}$.

$\mathbb{P}_{=2}$ is a subset of $\mathbb{P}$, but it is <u>not</u> a subspace (as shown above).

Remark: • Any vector space V is a subset and a subspace of itself.

• Whatever V is, it must have a zero vector (b/c of A4). Then $W = \{\vec{0}\} \subseteq V$ is a subspace of V called the trivial subspace or the zero subspace.

E.g. Let D_3 be the set of all diagonal 3x3 matrices.
Is this a subspace
(No need to check all 10 axioms)
$\hookrightarrow D_3 \subseteq M_3$, a known V.S., so we don't need to check all 10 axioms, it is sufficient to check closure under linear combinations and presence of a zero element

1) $\vec{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in D_3$ ☑ $\therefore 0 \in D_3$

2) Let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \in D_3$, let $B = \begin{bmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{bmatrix} \in D_3$. $A+B = \begin{bmatrix} a+d & 0 & 0 \\ 0 & b+e & 0 \\ 0 & 0 & d+f \end{bmatrix} \in D_3$ ☑ $\therefore$ closed under addition

3) Let $r \in S$ and $A \in D_3$. $rA = r\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} ra & 0 & 0 \\ 0 & rb & 0 \\ 0 & 0 & rc \end{bmatrix} \in D_3$ ☑ $\therefore$ closed under scalar mult.

$\therefore D_3$ is a subspace of M_3, i.e. it is indeed a vector space

E.g. Let $W = \{a+bx+cx^2+dx^3 \in \mathbb{P}_3 \mid a+3d=0\}$

1) Zero element?

$\underset{a}{0} + 0x + 0x^2 + \underset{d}{0}x^3 \qquad a+3d = 0+3\cdot 0 = 0$ ☑

$\therefore 0 \in W$ ☑

2) Closure under addition:

$p_1(x) = a_1 + b_1x + c_1x^2 + d_1x^3 \mid a_1 + 3d_1 = 0$

$p_2(x) = a_2 + b_2x + c_2x^2 + d_2x^3 \mid a_2 + 3d_2 = 0$

$p_1(x) \in W$ and $p_2(x) \in W$

$$p_1(x) + p_2(x) = a_1 + b_1x + c_1x^2 + d_1x^3 + a_2 + b_2x + c_2x^2 + d_2x^3$$

$$= \underbrace{(a_1+a_2)}_{\text{new } a} + (b_1+b_2)x + (c_1+c_2)x^2 + \underbrace{(d_1+d_2)}_{\text{new } d}x^3$$

$a_1 + a_2 + 3(d_1+d_3) = a_1 + a_2 + 3d_1 + 3d_2 = (a_1+3d_1) + (a_2+3d_2) = 0$ ☑

$\therefore p_1(x) + p_2(x) \in W$. Closed under addition

3) Let r be a scalar and let $p(x) \in W$

$$r\,p(x) = r(a+bx+cx^2+dx^3)$$

$$= \underbrace{(ra)}_{\text{new } a} + rbx + rcx^2 + \underbrace{(rd)}_{\text{new } d}x^3$$

$ra + 3rd = r(a+3d) = r(0) = 0$ ☑

$\therefore$ closed under scalar multiplication.

Let $\mathcal{C}^k(I)$ denote the set of fcns whose k^{th} derivative exists and is cont's on a given interval I. We write $\mathcal{C}(I)$ to mean $\mathcal{C}^0(I)$, i.e. the set of all cont's fcns on I (0^{th} derivative is cont's)

E.g. Is $\mathcal{C}(I)$ a vector space?

First, we note that $\mathcal{C}(I) \subseteq F(I)$, the known vector space of all fcns defined on an interval (see notes above).

$\therefore$ We may use the subspace test.

1) $f(x) \equiv 0$ is cont's $\forall x \in I$ $\therefore 0 \in S$

2) Let $f(x)$ & $g(x) \in \mathcal{C}(I)$,

then $f(x) + g(x)$ is cont's (proved in cal I $\rightarrow$ using limit laws)

$\therefore$ closed under addition

3) Let r be a scalar & $f(x) \in \mathcal{C}(I)$,

then $f(x)$ is cont' (proved in cal I)

$\therefore$ closed under scalar multiplication

Thus $\mathcal{C}(I)$ is a subspace of $F(I)$ and is itself a vector space.

E.g. Is $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ a linear combination of

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$.

Does $M = aA + bB + cC$.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} c & 2c \\ c & 2c \end{bmatrix}$$

$$\left.\begin{aligned} a + c &= 1 \\ b + 2c &= 2 \\ 3 &= b + c \\ 4 &= a + 2c \end{aligned}\right\} \text{system is inconsistent (can check)}$$

$\therefore M$ not in span $(\{A, B, C\})$

Does $f(x) = 2\sin(x+\frac{\pi}{6})$ belong to span$\{\sin x, \cos x\}$?

$f(x) = 2[\sin x \cos\frac{\pi}{6} + \sin\frac{\pi}{6}\cos x]$

$f(x) = 2[\sin x \frac{\sqrt{3}}{2} + \frac{1}{2}\cos x]$

$f(x) = \underbrace{\sqrt{3}}_{a}\sin x + \underbrace{1}_{b}\cos x$

$\therefore\ a\sin x + b\cos x = 2\sin(x+\frac{\pi}{6}) \Rightarrow$

E.g. Describe the span of $S = \{p_1(x), p_2(x), p_3(x)\} \subseteq \mathbb{P}$

where $p_1(x) = x^2 - x \quad p_2(x) = x^3 + x + 2 \quad p_3(x) = 2x^3 + 4$

I.e. Find conditions describing when a vector of $\mathbb{P}$ is in span or not.

Let $p(x) = ax^3 + bx^2 + cx + d$. Let r, s, t be scalars

$p(x) = r p_1(x) + s p_2(x) + t p_3(x)$.

$ax^3 + bx^2 + cx + d = rx^2 - rx + sx^3 + sx + 2\cdot s + 2tx^3 + 4t$

$ax^3 + bx^2 + cx + d = (2t+s)x^3 + (r)x^2 + (s-r)x + (2s+4t)$

↑ what is this equality? "Equality of vectors"

$\therefore$ left must be same polynomial as right.

$a = 2t + s$

$b = r$

$c = s - r$

$d = 2s + 4t$

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & a \\ 0 & 1 & 0 & b \\ -1 & 1 & 0 & c \\ 0 & 2 & 4 & d \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & b \\ 0 & 1 & 0 & b+c \\ 0 & 0 & 1 & \frac{a-b-c}{2} \\ 0 & 0 & 0 & d-2a \end{array}\right]$$

consistent if $d - 2a = 0$. r, s, t can be found if $2a = d$

I.e. A vector $p(x) = ax^3 + bx^2 + cx + d$ is in span$(\{P_1, P_2, P_3\})$ when $d = 2a$

E.g. Consider a homogeneous linear D.E

$$\circledast \quad y^{n} + p_{n-1} y^{n-1} + \dots + p_2 y'' + p_1 y' + p_0 y = 0$$

The set of all its solutions form a vector space. We then call the "solution space" of the DE "S" for set of all solutions

Note: Doesn't really mean anything to say that differential equations themselves form vector spaces...

Proof: We have that S is a subset of the known vector space $C^{(n)}(I)$ (cont's fcns n times differentiable on an interval I).

We prove using the subspace test.

1) 0 element. $f(x) \equiv 0$ solves $y^{n} + p_{n-1} y^{n-1} + \dots + p_2 y'' + p_1 y' + p_0 y = 0$

$\therefore \; 0 \in S$ ☑

2) Closure under addition?

Let $y = f(x)$ and $y = g(x)$ be solutions to $y^{n} + p_{n-1} y^{n-1} + \dots + p_1 y' + p_0 y = 0$

meaning $f^{(n)} + \sum_{i=0}^{n-1} f^{(i)} p_i(x) = 0$ and $g^{(n)} + \sum_{i=0}^{n-1} g^{(i)} p_i(x) = 0$

$$f^{(n)} + g^{(n)} + \sum_{i=0}^{n-1} (f^{(i)} + g^{(i)})\, p_{(i)}$$

$$= f^{(n)} + g^{(n)} + \sum_{i=0}^{n-1} f^{(i)} p_{(i)} + g^{(i)} p_{(i)}$$

$$= f^{(n)} + g^{(n)} + \sum_{i=0}^{n-1} f^{(i)} p_{(i)} + \sum_{i=0}^{n-1} g^{(i)} p_{(i)}$$

$$= \left(g^{(n)} + \sum_{i=0}^{n-1} g^{(i)} p_{(i)}\right) + \left(f^{(n)} + \sum_{i=0}^{n-1} f^{(i)} p_{(i)}\right)$$

$= 0 + 0 = 0 \quad \therefore$ closed under addition.

3) Closed under scalar multiplication?

Let $f(x)$ be a soln to ⊛. i.e. $f^{(n)} + \sum_{i=0}^{n-1} f^{(i)} p_i = 0$. Let r be a scalar

Then $rf(x)$ in homogeneous eqn gives

$$r f^{(n)} + \sum_{i=0}^{n-1} (r f^{(i)}) p_i$$

$$= r f^{(n)} + r \sum_{i=0}^{n-1} f^{(i)} p_i = r\left(f^{(n)} + \sum_{i=0}^{n-1} f^{(i)} p_i\right) = r \cdot (0) = 0$$

∴ Closed under scalar malitplication

So, solns to linear homogeneous ODEs are a vector spaces.

Corollary: The Superposition Principle

If $f_1(x), f_2(x), \ldots, f_k(x)$ are known solns of a homogeneous linear ODE, then all linear combinations of these solns

$$f = a_1 f_1 + a_2 f_2 + \ldots + a_n f_n.$$

In sexy words, $\mathrm{span}(\{f_1, f_2, \ldots, f_n\}) \subseteq$ of soln space of the DE.

Note that linearity and homogeneity are truly required for this.

Example: $\underbrace{y'' + y = 1}_{\text{linear, not homogeneous}}$ has solns $y_1 = 1 + \cos x$ and $y_2 = 1 + \sin x$

$$(1+\cos x)'' + 1 + \cos x = -\cancel{\cos x} + 1 + \cancel{\cos x} = 1 \ ☑$$

AND $(1+\sin x)'' + 1 + \sin x = -\sin x + 1 + \sin x = 1$ ☑

BUT $y = y_1 + y_2 = (1 + \cos x + 1 + \sin x)'' + (\sin x + 1 + \cos x + 1)$

$$= -\cos x - \sin x + \sin x + \cos x + 2 = 2 \neq 1$$

Example: $y''y - xy' = 0$ has $y_1 = 1$ and $y_2 = x^2$ as solns.

$y_1 = 1$: $(1)'' \cdot 1 - 1(1)'$ $= 0 - 0 = 0$

y_2: $(x^2)''(x^2) - x(x^2)'$ $= 2x^2 - 2x^2 = 0$

However, $y_1 + y_2$ is NOT a soln.

$(x^2+1)''(x^2+1) - x(x^2+1)'$

$\cancel{2x^2}+2-\cancel{2x^2} = 2 \neq 0$ $\therefore$ NOT a soln

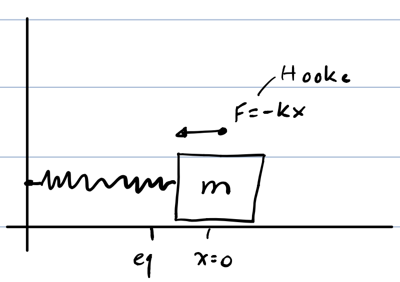
Corollary 2:

A linear ODE has no singular solns, i.e. its general soln contains <u>all</u> its possible solutions.

Idea of proof:

If f_1 were claimed to be a singular solution, then f_1 together with the other known solns $f_2, f_3, \ldots, f_k$ would satisfy that $a_1 f_1 + a_2 f_2 + \ldots + a_k f_k$ are also solutions. If that is the case $a_1 = 1$, and $a_2 = a_3 = \ldots = a_k = 0$. Therefore, the soln f_1 is already contained in the general soln.

Application: Simple Harmonic Motion



$\sum F_x = ma_x$

$-kx = ma$

$-\frac{k}{m}x(t) = x''(t)$ let $\omega = \sqrt{\frac{k}{m}}$

$\boxed{x'' + \omega^2 x = 0}$ linear homogeneous ODE

Can easily guess two solns: $y_1 = \cos(\omega x)$, $y_2 = \sin(\omega x)$.

All linear combinations are solutions: $y = A\sin(\omega x) + B\cos(\omega x)$

and there are no other solutions ($\because$ no singular solutions)

E.g. Consider the set A_3 of M_3 (3×3 matrices) consisting of all the antisymmetric (aka skew-symmetric) matrices in M_3 (i.e. $M^T = -M$).

Recall that matrices in A_3 are of the form

$$A_3 = \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$$

The 3 antisymmetric matrices $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

These 3 matrices span A_3: Any matrix $\begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$ in A_3 is $aA + bB + cC$.

Check independance: $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = 0$ has only trivial soln

$$a\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -b \\ 0 & 0 & 0 \\ b & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -c \\ 0 & c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Only works w/ $a=b=c=0$. I.e. $\exists!$ trivial soln $\Rightarrow$ Independant!

$\therefore$ A, B, C are independant and the set $\{A, B, C\}$ spans A_3

$\Rightarrow \{A, B, C\}$ is a basis for A_3 and $\dim(A_3) = 3$.

E.g. What is the dimension of $\mathbb{P}_3$ (all polynomials of the form $a + bx + cx^2 + dx^3$ | a, b, c, d are scalars.

Clearly, the 4 polynomials

$p_1(x) = 1$
$p_2(x) = x$
$p_3(x) = x^2$
$p_4(x) = x^3$

are 4 polynomials in $\mathbb{P}_3$ that clearly span $\mathbb{P}_3$. $\because$ Any $a + bx + cx^2 + dx^3 = a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) + a_4 p_4(x)$

Moreover: Put $a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) + a_4 p_4(x) = 0$ ← equality of vectors must be satisfied $\forall x$

$\therefore \exists!$ soln (trivial soln). $a_1 = a_2 = a_3 = a_4 = 0$

$\therefore$ Linearly independant

$\Rightarrow$ Basis$(\mathbb{P}_3) = \{p_1(x), p_2(x), p_3(x), p_4(x)\}$ and Dim$(\mathbb{P}_3) = 4$

Aside: Dim$(\mathbb{P}_2) = 3$

Dim$(\mathbb{P}_{10}) = 11$

Dim$(\mathbb{P}) = \infty$!!! (countably infinite)

E.g. Consider the vector space $F(I)$ of all cont's fcns defined on a fixed interval I. Are $f(x) = \sin x$, $g(x) = \cos x$, $h(x) = \tan x$ on $I = (-\frac{\pi}{2}, \frac{\pi}{2})$ linearly dependant or independant

Try: Put $a f(x) + b g(x) + c h(x) = 0$

$a \sin x + b \cos x + c \tan x = 0$ ← equality of vectors, so true $\forall x$.

If $x = 0$ $\quad a\sin(0) + b\cos(0) + c\tan(0) = 0 \Rightarrow b = 0$

If $x = \frac{\pi}{4}$ $\quad a\sin(\frac{\pi}{4}) + b\cos(\frac{\pi}{4}) + c\tan(\frac{\pi}{4}) = 0 \Rightarrow \frac{1}{\sqrt{2}}a + \frac{1}{\sqrt{2}}b + c = 0$

If $x = \frac{\pi}{6}$ $\quad a\sin(\frac{\pi}{6}) + b\cos(\frac{\pi}{6}) + c\tan(\frac{\pi}{6}) = 0 \Rightarrow \frac{1}{2}a + \frac{3}{\sqrt{2}}b + \frac{1}{\sqrt{3}}c = 0$

$\frac{1}{\sqrt{2}}a + c = 0$, $b = 0$, $\frac{1}{2}a + \frac{1}{\sqrt{3}}c = 0$

$-\sqrt{2}c = a$; $\quad \frac{1}{2}a + \frac{1}{\sqrt{3}}c = 0$

$-\frac{\sqrt{2}c}{\sqrt{2}} + \frac{1}{\sqrt{3}}c = 0$

$c\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}\right) = 0 \Rightarrow c = 0 \Rightarrow a = 0.$

$\therefore \exists!$ soln (trivial soln). $\Rightarrow \{\sin(x), \cos(x), \tan(x)\}$ is independant.

3B) Complex Numbers

By defn, i is a soln to $x^2+1=0$. I.e. $i^2=-1$

$\mathbb{C}=\{a+bi \mid a,b\in\mathbb{R}\}$.

Note that the vector space of dimension 2 over the scalar $\mathbb{R}$. $\vec{v}_1=1$, $\vec{v}_2=i$. Any $\vec{v}\in\mathbb{C}=a\cdot 1+b\cdot i$. (If scalar is $\mathbb{C}$, one dimension vector space)

Let $z=a+bi$ $\quad Re(z)=a$

$\underline{\underline{Im(z)=b}} \rightarrow \underline{\underline{\text{NOT } bi}}$

If $a=0$, z is purely "imaginary". If $b=0$, z is "real"

Multiplication requires distribution:

$z_1=a+bi$, $z_2=c+di$

$$z_1\cdot z_2=(a+bi)\cdot(c+di)=ac+adi+bci+bdi^2$$
$$=\overbrace{(ac-bd)}^{\text{real}}+\overbrace{(ad+bc)}^{\text{imaginary}}i.$$

Given $z=a+bi$

Its complex conjugate is $\bar{z}=a-bi$. The norm/modulus/magnitude of z is $\|z\|=\sqrt{a^2+b^2}\in\mathbb{R}$.

Note that $z\bar{z}=(a+bi)(a-bi)=a^2-bi^2=a^2+b^2=\|z\|^2$ $\left(z\bar{z}=\|z\|^2\right)$.

We use the conjugate for division. $\dfrac{z_1}{z_2}=\dfrac{z_1}{z_2}\cdot\dfrac{\bar{z}_2}{\bar{z}_2}=\dfrac{z_1\bar{z}_2}{z_2\bar{z}_2}=\dfrac{z_1\bar{z}_2}{\|z_2\|^2}$

$\left(\dfrac{z_1}{z_2}=\dfrac{z_1\bar{z}_2}{\|z_2\|^2}\right)$.

Interlude on Roots and Factors

Consider $p_1(x) = x^3 + 4x^2 - 11x - 30$

Could $x=4$ be a root? No! Doesn't divide 30 into integer

Could $x=2$ be a root? Maybe... Does divide 30 into an integer

Could $x=-6$ be a root? Maybe... Does divide 30 into an integer

Could $x=-\frac{1}{2}$ be a root? Nope

Ok this is weird... What is this rational roots thm anyways?

Rational Roots Theorem:

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n $(a_n \neq 0)$ defined over $\mathbb{Z}$ (i.e. $a_0, a_1, \ldots, a_{n-1}, a_n$ are all integers). If $p(x)$ has a rational root $\pm \frac{r}{s}$ in lowest terms, then necessarily

r must be a divisor of a_0

and s must be a divisor of a_n.

Remark: $p(x)$ could have other roots... but these would be irrational.

Reconsidering $p_1(x) = x^3 + 4x^2 - 11x - 30$. Possible rational roots are: $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30$. $-2, 3, -5$ are roots of this polynomial.

E.g. $q(x) = 2x^2 - x - 6$. Possible rational roots are $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$. (Recall that numerator must divide integer coefficient of lowest power and denominator must divide the integer coefficient of highest power).

Factor Theorem:

If r is a root of a polynomial $p(x)$, then $(x-r)$ is a factor.

Note: There may be other factors and/or an extra constant as well.

For example, $p(x) = 2x^2 - x - 6$ has $p(-2) = -8 + 16 + 22 - 30 = 0$. This means $(x+2)$ is a factor of $p(x)$.

Indeed, $p(x) = x^3 + 4x^2 - 11x - 30 = (x+2)(x+5)(x-3)$.

As we can see, factors and roots are very much related.

Back to Complex Numbers

$z = a+bi$, $\bar{z} = a - bi$, $||z|| = \sqrt{a^2+b^2}$, Saw $z\bar{z} = \ldots = ||z||^2$

Properties of Complex Conjugation

1. $\bar{\bar{z}} = z$
2. $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$
3. $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$
4. $\overline{z^n} = (\bar{z})^n$
5. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

If $\bar{z} = z$, then z is Real

If $\bar{z} = -z$, then z is imaginary

Think $z = a+bi$

$Re(z) = \frac{1}{2}(z + \bar{z}) = a$

$Im(z) = \frac{1}{2i}(z - \bar{z}) = b$

Suppose $z \in \mathbb{C}$,

Then consider the polynomial $(x-z)(x-\bar{z})$

It expands as $x^2 - x\bar{z} - xz + z\bar{z}$

$x^2 - x(z+\bar{z}) + \|z\|^2$

$x^2 - 2Re(z) + \|z\|^2$ — This whole thing is real

Roots of complex numbers

- If I know z^n, then what is/are the values of z?

Suppose $z^n = r\,cis\,\theta = r\,cis(\theta + 2\pi k)$; $k \in \mathbb{N}$ satisfies...

Recall that raising a complex number to n^{th} power raises r to the n^{th} power and multiplies θ by n.

Going backwards means we take n^{th} (positive) root of r and divide angle by n.

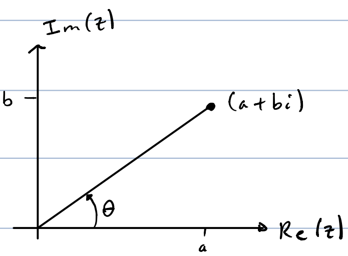
Formally:

If $z = r\,cis\,\theta = r\,cis(\theta + 2\pi k)$ (for $k \in \mathbb{N}$)

then, $z^{1/n} = \sqrt[n]{r} \cdot cis\left(\frac{\theta + 2\pi k}{n}\right)$ k takes values from 0 to $n-1$

*The above formula always gives exactly n distint roots

Recap:



$z = a + bi$

$z = r[\cos\theta + i\sin\theta]$

DeMoivre: $z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$

Roots of cmplx numbers: Given z^n, $z = r^{1/n}\,cis\left(\frac{\theta + 2\pi k}{n}\right)$ where k takes on values from 0 to $n-1$

DeMoivre showed that any non zero number (real or complex) has n distinct n^{th} roots.

E.g. Find all 3^{rd} roots of unity i.e. all solns to

$z^3 = 1$ where $z \in \mathbb{C}$.

$\Rightarrow z = \sqrt[3]{1} \Rightarrow \|z\| = \sqrt{1^2} \Rightarrow \|z\| = \boxed{r = 1} \rightsquigarrow$ when $z = 1$, $\theta = 0$ in cmplx plane

$\therefore z = 1^{1/3} \operatorname{cis}\left(\frac{0 + 2\pi k}{3}\right)$

$k = 0 \Rightarrow z_0 = 1 \operatorname{cis}\left(\frac{0 + 2\pi(0)}{3}\right) = \operatorname{cis}(0) = 1 \Rightarrow z_0 = 1$

$k = 1 \Rightarrow z_1 = 1 \operatorname{cis}\left(\frac{0 + 2\pi(1)}{3}\right) \Rightarrow z_1 = \operatorname{cis}\left(\frac{2\pi}{3}\right)$

$k = 2 \Rightarrow z_2 = 1 \operatorname{cis}\left(\frac{0 + 2\pi(2)}{3}\right) \Rightarrow z_2 = \operatorname{cis}\left(\frac{4\pi}{3}\right)$

Thus, the 3 cube roots of unity are:

$z_0 = 1$, $z_1 = \operatorname{cis}\left(\frac{2\pi}{3}\right)$, $z_2 = \operatorname{cis}\left(\frac{4\pi}{3}\right)$

$z^{1/n} = w \Rightarrow w^n = z$

Suppose w, an n^{th} root of a complex number, z, is known.

I.e. $z^{1/n} = w$. Let $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}$ be the n^{th} roots of unity.

We claim that all n^{th} roots of z are:

$w_0 = w\,\varepsilon_0$

$w_1 = w\,\varepsilon_0$

$\vdots$

$w_{n-1} = w\,\varepsilon_{n-1}$

Proof: Let $w_i \mid 0 \leq i \leq n-1$ be a root of z and let $\{\varepsilon_i\}_{i=0}^{i=n-1}$ be the n roots of unity of z. If w is a know root of z, then $w_i = w\,\varepsilon_i$ is also a root of z.

Because $(w_i)^n = (w_i \cdot \varepsilon_i)^n = w_i^n\,\varepsilon_i^n = z \cdot 1 = z$.

$(w_i)^n = z \Rightarrow z^{1/n} = w_i$.

$\therefore w_i$ is also an n^{th} root of z. $\blacksquare$

Are these n roots always distinct?

Ex: Find a cube root of $z = \frac{-27}{\sqrt{2}} + \frac{27}{\sqrt{2}} i$. Use the n roots of unity to find all cube roots of z.

$$z = -\frac{27}{\sqrt{2}} + \frac{27}{\sqrt{2}} i \Rightarrow \|z\| = \sqrt{\left(-\frac{27}{\sqrt{2}}\right)^2 + \left(\frac{27}{\sqrt{2}}\right)^2}$$

$$\|z\| = \sqrt{\frac{27^2}{2} + \frac{27^2}{2}} = \sqrt{27^2}$$

$$\|z\| = 27$$

$$z = 27\left[-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right]$$

$$\left.\begin{array}{l} \cos\theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{3\pi}{4} \text{ or } -\frac{3\pi}{4} \\ \sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} \text{ or } \frac{3\pi}{4} \end{array}\right\} \therefore \theta = \frac{3\pi}{4}$$

$$\Rightarrow z = 27 \operatorname{cis}\left(\frac{3\pi}{4}\right)$$

$$z^{1/3} = 27^{1/3} \operatorname{cis}\left(\frac{3\pi}{3 \cdot 4}\right) \Rightarrow z^{1/3} = 3 \operatorname{cis}\left(\frac{\pi}{4}\right) = \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}} i$$

$\therefore \omega = \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}} i$ is <u>a</u> cube root of z.

Cube roots of unity? $z^3 = 1 = \cos(0) + i\sin(0)$

$$\Rightarrow z^{1/3} = \operatorname{cis}\left(\frac{0 + 2\pi k}{3}\right)$$

If $k=0$	If $k=1$	If $k=2$
$\varepsilon_0 = \operatorname{cis}(0)$	$\varepsilon_1 = \operatorname{cis}\left(\frac{2\pi}{3}\right)$	$\varepsilon_2 = \operatorname{cis}\left(\frac{4\pi}{3}\right)$
$\varepsilon_0 = 1$	$\varepsilon_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$	$\varepsilon_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

$$\omega_0 = \omega \cdot \varepsilon_1 = \left(\frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}} i\right)(1) \Rightarrow \omega_0 = \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}} i$$

$$\omega_1 = \omega \cdot \varepsilon_1 = \left(\frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}} i\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$$

$$\omega_1 = \frac{-3}{2\sqrt{2}} + \frac{3\sqrt{3}}{2\sqrt{2}} i - \frac{3}{2\sqrt{2}} i + \frac{3\sqrt{3}}{2\sqrt{2}} i^2 = \frac{-3}{2\sqrt{2}}\left((\sqrt{3}+1) + (1-\sqrt{3})i\right) \Rightarrow \omega_1 = \frac{-3}{2\sqrt{2}}\left((\sqrt{3}+1) + (1-\sqrt{3})i\right)$$

$$\omega_2 = \omega\,\varepsilon_2 = \left(\frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}} i\right)\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)$$

$$\omega_2 = \cancel{\frac{-3}{2\sqrt{2}}} - \frac{3\sqrt{3}}{2\sqrt{2}} i - \frac{3}{2\sqrt{2}} i - \cancel{\frac{3\sqrt{3}}{2\sqrt{2}} i^2} = \frac{-3}{2\sqrt{2}}(\sqrt{3}+1) i \Rightarrow \omega_2 = \frac{-3}{2\sqrt{2}}(\sqrt{3}+1) i$$

We could also have found all the cube roots of $z = \frac{-27}{\sqrt{2}} + \frac{27}{\sqrt{2}} i$ using polar coordinates.

Recall that $z = 27 \text{ cis}\left(\frac{3\pi}{4}\right)$.

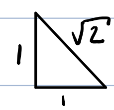
$$\Rightarrow z^{1/3} = 27^{1/3} \text{ cis}\left(\frac{\frac{3\pi}{4} + 2\pi k}{3}\right)$$

$$\Rightarrow z^{1/3} = 3 \text{ cis}\left(\frac{\pi}{4} + \frac{2\pi k}{3}\right)$$

If $k=0$	If $k=1$	If $k=2$
$w_0 = 3 \text{ cis}\left(\frac{\pi}{4}\right)$	$w_1 = 3 \text{ cis}\left(\frac{11\pi}{12}\right)$	$w_2 = 3 \text{ cis}\left(\frac{19\pi}{12}\right)$

$w_0 = \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}} i$ * As obtained on previous page.

These have messed up θs, but plugging the numbers into the calculator shows that w_0, w_1 and w_2 are the same whether we obtain them by multiplying a known n^{th} root by each n^{th} root of unity or whether we use the DeMoivre method in polar coordinates.

Interlude: 

$c^2 = a^2 + b^2 \Rightarrow c^2 = 2 \Rightarrow c = \sqrt{2}$.

Pythagoras' cult believed that the only numbers that really existed were the rational numbers $(\mathbb{Q} = \frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{Z})$.
However, a mathematican of the pythagorean cult showed that the hypotenuse of the triangle with unit edge lengths $(h = \sqrt{2})$ was NOT a ratio of integers. This was a problem, because how could the hypotenuse of this triangle not be a number?! They needed to expand their notion of what a number was...

E.g. $x^3 + 3x + 2 = 0$ has 2 real roots $(-1 \;\&\; 2)$

$x^2 + 4x + 4$ has 1 real root (-2)

$x^2 + 1$ has no real roots...

$\mathbb{C}$ "$i \equiv \sqrt{-1}$" $\longrightarrow$ Back to the heart

The Fundamental Theorem of Algebra:

Any non constant polynomial over $\mathbb{C}$ has a complex root.

Let $p(x)$ be a polynomial of degree n over $\mathbb{C}$ $(n \neq 0)$.

If $p(x)$ has a root, call it $r_1 \Rightarrow (x-r_1)$ is a factor of $p(x)$.

So $p(x) = (x-r_1)\underbrace{q(x)}_{\text{degree } n-1}$. $q(x)$ will also have a root, call it r_2, and $(x-r_2)$ would then be a factor. $p(x) = (x-r_1)(x-r_2)\underbrace{A(x)}_{\text{degree } n-2}$

We can repeat this process until $p(x)$ is factored into n linear terms.

So, $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 = \overset{\text{constant}}{K}(x-r_1)(x-r_2)\cdots(x-r_n)$

where $r_1, r_2, \dots, r_n$ are the n complex roots of the polynomial.

Note: These roots do not need to be complex.

The number of times the same root appears in the complex linear factorization of $p(x)$ is called the <u>multiplicity</u> of $p(x)$.

Recap: Any polynomial of degree n has n roots* in the complex plane and therefore can be factored into n linear terms.

I.e. $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 = (x-r_1)(x-r_2)\cdots(x-r_n)$

where $r_1, r_2, \dots, r_n$ are the roots of $p(x)$. These roots are not necessarily distinct. (* If we count complex roots and multiplicities)

There is a very important corrolary of the FTA to the real polynomials.

Theorem: Any non constant polynomial over $\mathbb{R}$ factors completely as a product of linear (degree 1) and quadratic (degree 2) factors that are all defined over $\mathbb{R}$.

Proof: Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ be a polynomial of degree n. Let $r_1, r_2, \dots, r_R$ be the real roots of $p(x)$. Let $s_1, s_2, s_3, \dots, s_C$ be the complex roots of $p(x)$. By the conjugate roots theorem, if $s_i \in \mathbb{C}$ is a root $\overline{s_i} \in \mathbb{C}$ is also a root. $\therefore$ The complex roots of $p(x)$ are $s_1, \overline{s_1}, s_2, \overline{s_2}, \dots, s_{\frac{C}{2}}, \overline{s_{\frac{C}{2}}}$. The factor theorem tells us that

$$p(x) = (x-r_1)\cdot(x-r_2)\cdot\dots\cdot(x-r_R)\cdot(x-s_1)(x-\overline{s_1})\cdot(x-s_2)\cdot(x-\overline{s_2})\cdot\dots\cdot(x-s_{C/2})(x-\overline{s_{C/2}})$$

$$p(x) = (x-r_1)\cdot(x-r_2)\cdot\dots\cdot(x-r_R)\cdot\left[x^2 - x\overline{s_1} - xs_1 + s_1\overline{s_1}\right]\cdot\dots\cdot\left[x^2 - x\overline{s_{C/2}} - xs_{C/2} + s_{C/2}\overline{s_{C/2}}\right]$$

Recall: $Re(z) = \frac{1}{2}(z+\overline{z}) \Rightarrow -2Re(z) = -(z+\overline{z})$

Recall: $z\cdot\overline{z} = \|z\|^2$ and $\|z\|^2 \in \mathbb{R}$.

$$\Rightarrow p(x) = \underbrace{(x-r_1)(x-r_2)\cdot\dots(x-r_R)}_{\text{linear terms}}\cdot\underbrace{\left[x^2 - 2Re(s_1)x + \|s_1\|^2\right]\cdot\dots\cdot\left[x^2 - 2Re(s_{C/2}) + \|s_{C/2}\|^2\right]}_{\text{Quadratic terms (all have real coefficients)}}$$

where $R + C = n$ (by FTA).

This theorem is what we used in partial fraction decomposition.

Ex: Factor $x^4 + 16$ completely over $\mathbb{R}$.

Solve $x^4 + 16 = 0$

$x^4 = -16$

$z^4 = 16(\cos(\pi) + i\sin(\pi))$

$z = 2(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}))$

$\Rightarrow \omega = \sqrt{2} + \sqrt{2}i$ is a fourth root of -16

Now let's find the fourth roots of unity.

$z^4 = 1$

$z^4 = cis(0)$

$z = cis\left(\frac{0+2\pi k}{4}\right)$

$z = cis\left(\frac{k\pi}{2}\right)$

If $k=0$	If $k=1$	If $k=2$	If $k=3$
$\varepsilon_0 = cis(0)$	$\varepsilon_1 = cis(\frac{\pi}{2})$	$\varepsilon_2 = cis(\pi)$	$\varepsilon_3 = cis(\frac{3\pi}{2})$
$\varepsilon_0 = 1$	$\varepsilon_1 = i$	$\varepsilon_2 = -1$	$\varepsilon_3 = -i$

$$\left.\begin{aligned}
\omega_0 &= \omega\,\varepsilon_0 = (\sqrt{2}+\sqrt{2}\,i)(1) \Rightarrow \omega_0 = \sqrt{2}+\sqrt{2}\,i \\
\omega_1 &= \omega\,\varepsilon_1 = (\sqrt{2}+\sqrt{2}\,i)(i) \Rightarrow \omega_1 = -\sqrt{2}+\sqrt{2}\,i \\
\omega_2 &= \omega\,\varepsilon_2 = (\sqrt{2}+\sqrt{2}\,i)(-1) \Rightarrow \omega_2 = -\sqrt{2}-\sqrt{2}\,i \\
\omega_3 &= \omega\,\varepsilon_3 = (\sqrt{2}+\sqrt{2}\,i)(-i) \Rightarrow \omega_3 = \sqrt{2}-\sqrt{2}\,i
\end{aligned}\right\}$$

$\therefore \omega_0, \omega_1, \omega_2, \omega_3$ are the four roots of $p(x) = x^4+16$.

Note that $\overline{\omega_0} = \omega_3$ and that $\overline{\omega_1} = \omega_2$. (fkn conjugate roots thm...)

$$\Rightarrow x^4+16 = (x-\omega_0)(x-\overline{\omega_0})\cdot(x-\omega_1)(x-\overline{\omega_1})$$

$$x^4+16 = (x^2 - 2\,Re(\omega_0)x + \|\omega_0\|^2)(x^2 - 2\,Re(\omega_1)x + \|\omega_1\|^2)$$

$$x^4+16 = (x^2 - 2\sqrt{2}\,x + 4)(x^2 + 2\sqrt{2}\,x + 4)$$

Lo and behold, this polynomial factors completely as a product of terms of degree $n \leq 2$ over $\mathbb{R}$. This was predicted by above theorem.

Ex: Suppose $f(x) = x^2 + x$. What is $f(1+i)$?

$$f(1+i) = (1+i)^2 + (1+i)$$

$$f(1+i) = (\cancel{1} + 2i + \cancel{i^2}) + (1+i)$$

$$f(1+i) = 2i + 1 + i$$

$$f(1+i) = 1 + 3i$$

Now, what if $f(x)$ was more complicated? Like if we had $f(x) = \operatorname{arccot}(x)$. What would $\operatorname{arccot}(1+i)$ be?
We need to mix complex numbers with... Power Series!!

<u>FACT</u>: The power series expansions of many important fcns like e^x, $\sin x$, $\cos x$ can be used when $x \in \mathbb{C}$.

Important power series are:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$$

We can have e^x with $x = i\theta$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} \cdots$$

$$e^{i\theta} = \underbrace{\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right]}_{\cos\theta} + i\underbrace{\left[\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right]}_{\sin\theta}$$

$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ Euler's Formula!

* If $\theta = \pi$ $\quad e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0$

$\therefore e^{i\pi} + 1 = 0$