Section 7 - Series and Convergence

<u>Objective</u>: To identify whether or not a series, $\sum_{k=1}^{\infty} a_k$, converges to a fixed sum, or diverges to ∞ (where $a_k \geq 0$)

ex. Does the series $\sum_{k=0}^{\infty} 2\left(\frac{1}{2}\right)^k$ converge or not

$$\sum_{k=0}^{\infty} 2\left(\frac{1}{2}\right)^k = 2 + 1 + \frac{1}{4} + \frac{1}{8} + \dots$$

To be discussed...

ex. Does the series $\sum_{k=1}^{\infty} \frac{1}{k}$ converge or not

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

To be discussed...

<u>Def</u>

A series is convergent if its partial sums are convergent as $n \to \infty$

<u>i.e.</u> $\sum_{k=1}^{\infty} a_k$ converges if $\underbrace{\sum_{k=1}^{n} a_k}_{S_n}$ converges as $n \to \infty$

ex.

a) Find the n^{th} partial sum of $\sum_{k=0}^{n} \left(\frac{1}{2}\right)^k$

b) Evaluate $\lim_{n \to \infty} \left(\frac{1}{2}\right)^k$

$$S_0 = \sum_{k=0}^{0} \left(\frac{1}{2}\right)^k = 1$$

$$S_1 = \sum_{k=0}^{1} \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} = \frac{3}{2}$$

$$S_2 = \sum_{k=0}^{2} \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

$$S_3 = \sum_{k=0}^{3} \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$$

Based on our observations:

$$S_n = \frac{2 \cdot 2^n - 1}{2^n}$$

i.e. $\sum_{k=0}^{n} \left(\frac{1}{2}\right)^k = \frac{2 \cdot 2^n - 1}{2^n}$

b) $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$ converges if $\lim_{n \to \infty} \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k$ converges

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{2 \cdot 2^n - 1}{2^n}$$

$$= \lim_{n\to\infty} 2 - \frac{1}{2^n}$$

$$= 2$$

Since $S_n \longrightarrow 2$ as $n \to \infty$

then $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2$

$\therefore$ Convergent series

<u>ex.</u> Consider the series $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$

Does this series converge or not?

$$S_1 = \sum_{k=1}^{1} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{2}$$

$$S_2 = \sum_{k=1}^{2} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3}$$

$$S_3 = \sum_{k=1}^{3} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right)$$

$$= 1 - \frac{1}{4}$$

$$S_n = 1 - \frac{1}{n+1}$$

as $n \to \infty$, $S_n \to 1$

Conclusion: $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1$

ex. À vous

Does $\sum_{k=1}^{\infty} \ln\left(\frac{k}{k+1}\right)$ converse or diverse?

$S_1 = \sum_{k=1}^{1} \ln\left(\frac{k}{k+1}\right) = \ln\left(\frac{1}{2}\right)$

$S_2 = \sum_{k=1}^{2} \ln\left(\frac{k}{k+1}\right) = \ln\left(\frac{1}{2}\right) + \ln\left(\frac{2}{3}\right) = \ln\left(\frac{2}{6}\right) = \ln\left(\frac{1}{3}\right)$

$S_3 = \sum_{k=1}^{3} \ln\left(\frac{k}{k+1}\right) = \ln\left(\frac{1}{2}\right) + \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3}{4}\right) = \ln\left(\frac{1}{4}\right)$

$\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{3}}{4}$

$S_n = \ln\left(\frac{1}{n+1}\right)$

$\lim_{\to\infty} S_n = \lim_{n\to\infty} \ln\left(\frac{1}{n+1}\right) \quad \left(= \ln(0^+)\right)$ $k \geq 0$

$\lim_{n\to\infty} \left[\ln 1 - \ln(n+1)\right] = -\infty$ Diverges

ex. Evaluate, if it exists, the sum of the following telescoping (collapsing) series.

terms cancel

$$\sum_{k=2}^{\infty} \frac{2}{k^2-1}$$

$$= \sum_{k=2}^{\infty} \frac{2}{(k-1)(k+1)} \rightsquigarrow \frac{2}{(k-1)(k+1)} = \frac{A}{k-1} + \frac{B}{k+1}$$

$$2 = A(k+1) + B(k-1)$$

$$k=-1: \quad 2 = -2B$$

$$B = -1$$

$$k=1: \quad 2 = 2A$$

$$A = 1$$

$$= \sum_{k=2}^{\infty} \left[\frac{1}{k-1} - \frac{1}{k+1} \right]$$

$$S_2 = \frac{1}{1} - \frac{1}{3} = 1 - \frac{1}{3}$$

$$S_3 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right)$$

2 terms before (so makes sense that 1 & $\frac{1}{2}$ won't cancel)

$$S_4 = \left(1 - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{5}\right) = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}$$

$$S_5 = \left(1 - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{5}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6}\right)$$

$$= 1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}$$

$$S_n = 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$$

as $n \to \infty$, $S_n \to 1 + \frac{1}{2}$

So,
$$\sum_{K=2}^{\infty} \frac{2}{K^2-1} = \frac{3}{2}$$

April 12

Geometric Series

A geometric series is a series whose form satisfies

$$\sum_{K=0}^{\infty} ar^K$$ where a is a constant (not a_K, so nothing changing)

r is the series' ratio

ex. Show that the following are all geometric series.

$$\boxed{4} \xrightarrow{1/2} 2 \xrightarrow{1/2} \frac{1}{2} \xrightarrow{1/2} \frac{1}{4} + \dots \quad = \sum_{K=0}^{\infty} 4\left(\frac{1}{2}\right)^K$$

$$\text{a)}\quad 6 \xrightarrow{-2/3} -4 \xrightarrow{-2/3} +\frac{8}{3} - \frac{16}{9}$$

$$\text{b)}\quad 1 \xrightarrow{5/4} +\frac{5}{4} \xrightarrow{5/4} +\frac{25}{16} + \frac{125}{64} + \frac{623}{256} + \dots$$

$$= \sum_{K=0}^{\infty} 1\left(5/4\right)^K$$

Partial sum of a geometric series

Consider the geometric series:

$$\sum_{k=0}^{\infty} ar^k$$

et S_n represent the partial sum $\sum_{k=0}^{n} ar^k$

$$S_n = a + \cancel{ar} + \cancel{ar^2} + \dots + \cancel{ar^{n-1}} + \cancel{ar^n}$$

$$S_n = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \dots + \cancel{ar^n} + ar^{n+1}$$

$$S_n - S_n = a - ar^{n+1}$$

$$= a\left(r^{n+1} - 1\right)$$

$$n(r-1) = a\left(r^{n+1} - 1\right)$$

$$S_n = a\left[\frac{r^{n+1} - 1}{r-1}\right] \quad \text{or} \quad S_n = a\left[\frac{1 - r^{n+1}}{1-r}\right]$$

f $r > 1$, $\lim_{n \to \infty} S_n$ diverses $\left(= \frac{\infty}{\text{finite}}\right)$

f $0 < r < 1$, $\lim_{n \to \infty} S_n = a \cdot \frac{1 - \frac{1}{\infty}}{1-r} = a \cdot \frac{1}{1-r}$

f $r = 1$, $\lim_{n \to \infty} S_n$ diverses unless $a = 0$

More generally, (including negative numbers)

$$\sum_{k=0}^{\infty} ar^k \to \frac{a}{1-r} \text{ if } |r|<1$$

$$\to \text{ diverges if } |r| \geq 1$$

ex. Compute

$$\sum_{k=0}^{10} 4\left(2/3\right)^k = 4\left[\frac{1-(2/3)^{11}}{1-2/3}\right]$$

$$= 11.861268$$

$$\sum_{k=0}^{\infty} 4\left(2/3\right)^k = \frac{4}{1-2/3} = \frac{4}{1/3} = 12$$

ex. $$\sum_{k=0}^{24} 500\,(1.005)^k = 500\cdot\frac{1-(1.005)^{24}}{1-1.005}$$

$$= 13279.56$$ (\$700 more than 500×25 (without compound interest))

$$\sum_{k=0}^{\infty} 500\,(1.005)^k \text{ diverges since } r = 1.005 > 1$$

ex. $$\sum_{k=4}^{\infty} 10\left(\tfrac{1}{2}\right)^k = \sum_{k=0}^{\infty} 10\left(\tfrac{1}{2}\right)^k - \sum_{k=0}^{k=3} 10\left(\tfrac{1}{2}\right)^k$$

or $$\sum_{k=4}^{\infty} 10\left(\tfrac{1}{2}\right)^k = \frac{10}{16} + \frac{10}{16}\left(\tfrac{1}{2}\right) + \dots$$

$$= \sum_{j=0}^{\infty} \frac{10}{16}\left(\frac{1}{2}\right)^{j}$$

So far... in the world of convergence / divergence tests, we have:

① STD (stupid test for divergence)

② Integral Test

$$\int_1^\infty f(x)\,dx \overset{\text{the behaviour is the same}}{\sim} \sum_{K=1}^{\infty} a_K$$

<u>ex.</u> Test the convergence of

$$\sum_{K=2}^{\infty} \frac{\ln K}{K^2}$$

The corresponding integral is $\displaystyle\int_2^\infty \frac{\ln x}{x^2}\,dx$

let $u = \ln x$

$du =$

$$\int_2^\infty \frac{l\ x}{x^2}\,dx$$

$$I = \int \ln x \cdot \frac{1}{x^2}\,dx$$

$f = \ln x \qquad g' = x^{-2}$

$f' = \frac{1}{x} \qquad g = -\frac{1}{x}$

$$I = -\frac{\ln x}{x} - \int \left(\frac{1}{x}\right)\left(-\frac{1}{x}\right) dx$$

$$-\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx$$

$$-\frac{\ln x}{x} - \frac{1}{x}$$ (Don't need $+C$, need <u>an</u> antiderivative)

$$= -\frac{\ln x + 1}{x}$$

$$\frac{\ln x}{x^2}\,dx$$

$$\lim_{R\to\infty} \int_2^R \frac{\ln x}{x^2}\,dx$$

$$\lim_{R\to\infty} \left[-\frac{\ln x + 1}{x}\right]_2^R$$

$$\lim_{\to\infty} \left[-\frac{\ln R + 1}{R} + \frac{\ln 2 + 1}{2}\right]$$

$\to 0$ (R grows faster than $\ln R$) $\downarrow 0$

$$= 0 + \frac{\ln 2}{2} + 1$$

Conclusion: Since $\int_2^\infty \frac{\ln x}{x^2}\,dx$ converges (to $\frac{\ln 2}{2} + 1$),

then $\sum_{k=2}^{\infty} \frac{\ln k}{k^2}$ must also converge

<u>ex.</u> À vous

Test the convergence of all the following series.

∞

a) $\sum_{K=1}^{\infty} \frac{1}{\sqrt{K}}$ c) $\sum_{K=1}^{\infty} \frac{1}{K}$

b) $\sum_{K=1} \frac{1}{K^7}$ d) $\sum_{K=1}^{\infty} \frac{1}{K^2}$

Reminder: the p-integral $\int_1^{\infty} \frac{1}{x^p}\,dx$

diverges whenever $0 < p \leq 1$, but converges whenever $p > 1$

a) $\sum_{K=1}^{\infty} \frac{1}{\sqrt{K}}$ diverges (by Integral Test)

b) $\sum_{K=1} \frac{1}{K^7}$ converges $(p = 7)$

c) $\sum_{K=1}^{\infty} \frac{1}{K}$ diverges $(p = 1)$

d) $\sum_{K=1}^{\infty} \frac{1}{K^2}$ converges $(p = 2)$

Remark

p-series is any series of form

$\sum_{K=1}^{\infty} \frac{1}{K^p}$ and ① diverges if $0 < p \leq 1$

② converges if $p > 1$

Direct Comparison Test

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ represent two (positive term) series.

① If $a_k \geq b_k$ for all k and $\sum_{k=1}^{\infty} b_k$ diverges,
then $\sum_{k=1}^{\infty} a_k$ will diverge as well.

② If $a_k \leq b_k$ for all k and $\sum_{k=1}^{\infty} b_k$ converges,
then $\sum_{k=1}^{\infty} a_k$ will converge as well.

ex. $\sum_{k=1}^{\infty} \frac{1}{k^2+7}$ Test its convergence/divergence

$$\frac{1}{k^2+7} \leq \frac{1}{k^2} \quad \text{for all } k$$

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a p-series for which $p > 1$ and therefore converges

By the comparison test, $\sum_{k=1}^{\infty} \frac{1}{k^2+7}$ must also converge

ex. Test the following series for conv/div

$$\sum_{j=3}^{\infty} \underbrace{\frac{(1+2\cos^2 j)}{5j}}_{a_j}$$

$$\frac{1+2\cos^2 j}{5^j} \geq \frac{1}{5^j} = \underbrace{\left(\frac{1}{5}\right)^j}_{\text{geometric}}$$

but... that doesn't tell us anything since $\left(\frac{1}{3}\right)^j$ converges

So let's overestimate:

$$\frac{1+2\cos^2 j}{5^j} \leq \frac{3}{5^j} = 3\left(\frac{1}{5}\right)^j$$

$\sum_{j=3}^{\infty} 3\left(\frac{1}{5}\right)^j$ is a geometric series that converges since $-1 \leq r = \frac{1}{5} \leq 1$

By the comparison test, $\sum_{j=3}^{\infty} \frac{1+2\cos^2 j}{5^j}$ converges

ex. $\sum_{k=2}^{\infty} \frac{\ln k}{k^2}$

$$\frac{\ln k}{k^2} \leq \frac{k}{k^2} = \frac{1}{k}$$

doesn't give us anything

$$\ln k << \ln k^p \rightarrow \ln k << k^{1/2}$$

$$\rightarrow \frac{\ln k}{k^2} << \frac{k^{1/2}}{k^2} = \frac{1}{k^{3/2}}$$

$\sum_{k=2}^{\infty} \frac{1}{k^{3/2}}$ is a p series where $p = 3/2$ $(p > 1)$ and thus converges

$\sum_{k=2}^{\infty} \frac{\ln k}{k^2}$ is smaller then $\sum_{k=2}^{\infty} \frac{1}{k^{3/2}}$ and therefore will also converge ex. À vous

Using tests for convergence seen so far, test the following series

a) $\sum_{n=1}^{\infty} \cos(1/n)$

$\lim_{n\to\infty} \cos(1/n) = \cos 0 = 1 \neq 0$

The series diverges in virtue of the stupid test for divergence

b) $\sum_{k=1}^{\infty} \frac{2+\cos(1/k)}{k}$

$$\underbrace{\frac{1}{k} \leq}_{\text{useful}} \frac{2+\cos 1/k}{k} \overbrace{\geq \frac{3}{k}}^{\text{useless comparison}}$$

diverges

$\sum_{k=1}^{\infty} \frac{1}{k}$ is a diverging p-series $(p=1)$, therefore $\sum \frac{2+\cos 1/k}{k}$ must also diverge

c) $\sum_{q=1}^{\infty} \left(\frac{5+\sin(q)}{3}\right)^q$

diverges (geometric)

$$\boxed{\left(\frac{4}{3}\right)^q \leq \left(\frac{5+\sin(q)}{3}\right)^q} \geq \left(\frac{6}{3}\right)^q$$

$\sum_{q=1}^{\infty} \left(\frac{4}{3}\right)^q$ is a divergent geometric series where $r = \frac{4}{3} > 1$

So $\sum_{q=1}^{\infty} \left(\frac{5 + \sin(q)}{3}\right)^q$ will diverge by the DCT

April 24, 2017

Done so far...

STD $\sum a_k$ → if they don't go to 0, series diverges

Integral Test $\int_1^{\infty} f(x)\,dx \sim \sum_{k=1}^{\infty} a_k$

Direct Comparison

<u>Limit Comparison Test</u>

ex. $\sum_{k=1}^{\infty} \frac{3k^2 + k}{5k^3 + 2k + 1}$

$\frac{3k^2}{5k^3} \times \frac{\left(1 + \frac{1}{3k}\right)}{\left(1 + \frac{2}{5k} + \frac{1}{5k^3}\right)}$

$\cancel{1} + \frac{1}{3k} \bigcirc \cancel{1} + \frac{2}{5k} + \frac{1}{5k^3}$

$\frac{1}{3k} \bigcirc \frac{2k^2 + 1}{5k^3}$

$5k^3 \bigcirc 3k(2k^2 + 1)$

$\cancel{5k^3} \bigcirc \cancel{6}k^3 + 3k$

$0 < k^3 + 3k$

So bottom > top

So multiplying by term < 1

So $\frac{3k^2}{5k^3} > \frac{3k^2 + k}{5k^3 + 2k + 1}$

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ represent two positive term series

such that $\lim_{k \to \infty} \frac{a_k}{b_k} = L$ (where L is finite and non-zero)

Then, the series have the same (diverging or converging) behavior

(Hint: rationals + radicals)

ex, $\sum_{k=1}^{\infty} \frac{3k^2+k}{5k^3+2k+1}$

$$\frac{3k^2+k}{5k^3+2k+1} \sim \frac{3k^2}{5k^3} \sim \frac{k^2}{k^3} \sim \frac{1}{k}$$

Let's compare, if we are allowed to,

$\sum_{k=1}^{\infty} \frac{3k^2+k}{5k^3+2k+1}$ with $\sum_{k=1}^{\infty} \frac{1}{k}$

Allowed?

$$\lim_{k\to\infty} \frac{\frac{3k^2+k}{5k^3+2k+1}}{\frac{1}{k}}$$

$$= \lim_{k\to\infty} \frac{3k^3+k^2}{5k^3+2k+1}$$

$$= \lim_{k\to\infty} \frac{\cancel{k^3}\left(3+\frac{1}{k}\right)}{\cancel{k^3}\left(5+\frac{2}{k^2}+\frac{1}{k^3}\right)}$$

$$= \frac{3}{5}$$

$\sum_{k=1}^{\infty} \frac{1}{k}$ diverges (p-series, $p \leq 1$)

By the limit comparison test,

$$\sum_{k=1}^{\infty} \frac{3k^2+k}{5k^3+2k+1} \text{ will also diverge}$$

ex. À vous

$$\sum_{k=1}^{\infty} \frac{3k-1}{\sqrt{k^5-2k^2+17k}}$$

$$\frac{3k-1}{\sqrt{k^5-2k^2+17k}} \sim \frac{k}{\sqrt{k^5}} = \frac{k}{k^{5/2}} \sim \frac{1}{k^{3/2}}$$

Allowed?

$$\lim_{k\to\infty} \frac{\frac{3k-1}{\sqrt{k^5-2k^2+17k}}}{\frac{1}{k^{3/2}}} = \lim_{k\to\infty} \frac{3k-1}{\sqrt{k^5-2k^2+17k}} \cdot k^{3/2}$$

$$= \lim_{k\to\infty} \frac{3k^{5/2} - k^{3/2}}{\sqrt{k^5-2k^2+17k}}$$

$$= \lim_{k\to\infty} \frac{\cancel{k^{5/2}}\left(3 - \frac{1}{k}\right)}{\cancel{k^{5/2}}\sqrt{1 - \frac{2}{k^3} + \frac{17}{k^4}}} = 3$$

$$\sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \text{ is a convergent } p\text{-series } (p>1)$$

By LCT, $\sum_{k=1}^{\infty} \frac{3k-1}{\sqrt{k^5-2k^2+17k}}$ converges ex. $\displaystyle\sum_{k=2}^{\infty} \frac{k\sin(1/k)}{k^2+1}$

$\int$

Back to Cal 1 $\displaystyle\lim_{\heartsuit \to 0} \frac{\sin \heartsuit}{\heartsuit} = 1$

use $\sin(1/k) \sim 1/k$

$$\frac{k\sin(1/k)}{k^2+1} \sim \frac{k \cdot 1/k}{k^2+1} \sim \frac{1}{k^2+1} \sim \frac{1}{k^2}$$

Validation: $\displaystyle\lim_{k\to\infty} \frac{\frac{k\sin(1/k)}{k^2+1}}{1/k^2}$

$$= \lim_{k\to\infty} \frac{k^3 \sin 1/k}{k^2+1}$$

$$= \lim_{k\to\infty} \frac{k^2}{k^2+1} \cdot \frac{\sin 1/k}{1/k}$$

$$= \lim_{k\to\infty} \frac{k^2}{k^2(1+\frac{1}{k^2})} \cdot \lim_{k\to\infty} \frac{\sin 1/k}{1/k}$$

$$= 1$$

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ Converses (p-series $p > 1$)

So $\displaystyle\sum_{k=1}^{\infty} \frac{k\sin(1/k)}{k^2+1}$ must also converse

April 26, 2017

Today ... Ratio and Root Tests

Ratio Test

Consider a series $\sum_{k=1}^{\infty} a_k$

① If $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1$, the series converges

② If $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| > 1$, the series diverges

③ If $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = 1$, the test is inconclusive

When do we use the ratio test?

→ factorials are involved $(k!)$

→ geometrics are involved (r^k)

→ combinations of the above with k^k or k^p

ex. Test the following series for convergence / divergence

a) $\sum_{k=1}^{\infty} \underbrace{\frac{1}{k!}}_{a_k}$ $\qquad a_k = \frac{1}{k!}$

$a_{k+1} = \frac{1}{(k+1)!}$

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}}$$

$$= \lim_{k \to \infty} \frac{k!}{(k+1)!}$$

$$= \lim_{k \to \infty} \frac{\cancel{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (k-1) \cdot k}}{\cancel{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (k-1) k}\,(k+1)}$$

$$= \lim_{k \to \infty} \frac{1}{k+1}$$

$$= 0 \quad < 1$$

Conclusion $\sum_{k=1}^{\infty} \frac{1}{k!}$ converges

b) $\sum_{k=1}^{\infty} \frac{k^2}{2^k}$

$$\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \frac{\frac{(k+1)^2}{2^{k+1}}}{\frac{k^2}{2^k}}$$

$$= \lim_{k \to \infty} \frac{(k+1)^2}{2^{k+1}} \cdot \frac{2^k}{k^2}$$

$$= \lim_{k \to \infty} \frac{(k+1)^2}{2k^2}$$

$$= \lim_{k \to \infty} \frac{1}{2} \left(\frac{k+1}{k} \right)^2$$

$$= \lim_{k \to \infty} \frac{1}{2} \left(1 + \frac{1}{k} \right)^2$$

$= \frac{1}{2} < 1$

Conclusion: $\sum_{k=1}^{\infty} \frac{k^2}{2^k}$ converges

c) $\sum_{j=2}^{\infty} \frac{(-1)^j 2^j}{(2j)!}$

$[2(j+1)]! = (2j+2)!$

not even necessary since abs. val (just takes care of sign)

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{j\to\infty} \left| \frac{(-1)^{j+1} 2^{j+1}}{(2j+2)!} \cdot \frac{(2j)!}{(-1)^j 2^j} \right|$$

$$= \lim_{j\to\infty} \left| 2(-1) \cdot \frac{1 \cdot 2 \cdot 3 \cdot \cancel{\ldots \cdot 2(j-1) \cdot 2j}}{1 \cdot 2 \cdot 3 \cdot \cancel{\ldots \cdot 2(j-1) 2j} (2j+1)(2j+2)} \right|$$

$$= \lim_{j\to\infty} \frac{2}{(2j+1)(2j+2)} = 0 < 1$$

Therefore, the series

$\sum_{j=2}^{\infty} \frac{(-1)^j 2^j}{(2j)!}$ converges

ex. Mr. Nashy

Part 1: Show that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$

Let $L = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ $\quad$ (1^∞ form ← indeterminate)

$\ln L = \lim_{n\to\infty} \ln\left(1 + \frac{1}{n}\right)^n$ $\quad$ (ln can be added inside)

$= \lim_{n\to\infty} n \ln\left(1 + \frac{1}{n}\right)$ $\quad$ $(\infty \cdot 0)$

$= \lim_{n\to\infty} \dfrac{\ln\left(1 + \frac{1}{n}\right)}{1/n}$

$\overset{L'H}{=} \lim_{n\to\infty} \dfrac{\frac{1}{1+\frac{1}{n}} \cdot \cancel{\left(-\frac{1}{n^2}\right)}}{\cancel{-\left(1/n^2\right)}}$

$= 1$

So $\ln L = 1$

$\rightarrow L = e^1$

$= e$

Part 2: Test the following series:

$$\sum_{k=1}^{\infty} \frac{n!}{n^n}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}\right|$$

n

$$= \lim_{n\to\infty} \cancel{(n+1)} \cdot \frac{n^{n}}{\cancel{(n+1)}(n+1)^n}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n \qquad (1^{\text{any power}} = 1)$$

$$= \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n}$$

$$= \frac{1}{e} \quad < 1$$

Conclusion: $\sum_{k=1}^{\infty} \frac{n!}{n^n}$ converges

Root Test

$\sum \sqrt[n]{a \cdot r^n}$ geometric

Consider a series $\sum_{n=1}^{\infty} a_n$ $\sim$

① If $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, the series converges

② If $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$, the series diverges

③ If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the test is inconclusive

Use the root test if all terms of the series have "n" as a common exponent

ex. $\sum_{w=1}^{\infty} \left(\frac{2w+1}{5w+3} \right)^{w}$

$\lim_{w \to \infty} \sqrt[w]{\left(\frac{2w+1}{5w+3} \right)^{w}}$

$= \lim_{w \to \infty} \frac{2w+1}{5w+3}$

$= \lim_{w \to \infty} \frac{\cancel{w}(2 + \frac{1}{w})}{\cancel{w}(5 + \frac{3}{w})}$

$= \frac{2}{5} < 1$

By the root test, the series converges.

April 27, 2017

Reminder on Taylor Polynomials

Let f represent an infinitely differentiable function at $x=a$. Then, the taylor polynomial of degree n associated to f near $x=a$ is:

$T_n(x) = f(a) + f'(a) \cdot (x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$

$$+ \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$T(x)$ is a Taylor series when $\lim\limits_{n\to\infty} T_n(x)$

ex. Consider $f(x) = e^x$ near $x=0$

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$f^{(n)}(0)/n!$
0	e^x	1	1
1	e^x	1	1
2	e^x	1	$1/2! = \frac{1}{2}$
3	e^x	1	$1/3! = \frac{1}{6}$

So, $T_n(x)$, the n^{th} order Taylor Polynomial for $f(x) = e^x$ centered at $x=a$ is...

$$T_n(x) = 1 + 1x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots + \frac{1}{n!}x^n$$

The taylor series expansion for $f(x) = e^x$ when $x=0$ is

$$T(x) = 1 + 1x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$$

$$T(x) = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$$

What values of x guarantees this series will converge?

Ratio test: $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{\frac{1}{(k+1)!} x^{k+1}}{\frac{1}{k!} x^k} \right|$

$$= \lim_{k\to\infty} \left| \frac{1}{(k+1)!} x^{k+1} \cdot \frac{k!}{x^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{\overset{\text{constant}}{x}}{k+1} \right| \qquad \left(\frac{x^{k+1}}{x^k} = \frac{\cancel{x^k} \cdot x}{\cancel{x^k}} \right)$$

$$= 0 < 1$$ This series converges for <u>all</u> x

Application:

According to our work, we can predict that

$$\sum_{k=0}^{\infty} \frac{1}{k!} \overset{\text{for } x=1}{=} e^1 \quad \text{or} \quad e^x = \underbrace{\sum_{k=0}^{\infty} \frac{x^k}{k!}}_{T(x)}$$

Check: (for $x=1$)

$$\sum_{k=0}^{\infty} \frac{1}{k!} = \underbrace{1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120}}_{2.7166} + \ldots$$

$e = 2.71828$

ex, Let $f(x) = \sin x$

Find $T(x)$, the taylor series expansion for $f(x)$ around $x=0$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$f^{(n)}(0)/n!$	
0	$\sin x$	0	0	
1	$\cos x$	1	1	x
2	$-\sin x$	0	0	x^2
3	$-\cos x$	-1	$-\frac{1}{3!} = -\frac{1}{6}$	x^3
4	$\sin x$	0	0	x^4
5	$\cos x$	1	$\frac{1}{5!} = \frac{1}{120}$	x^5

$$T(x) = 1x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

$$T(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

Ratio test: $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{\frac{(-1)^{k+1} x^{2(k+1)+1}}{(2(k+1)+1)!}}{(-1)^k \frac{x^{2k+1}}{(2k+1)!}} \right|$

$$= \lim_{k\to\infty} \left| \frac{(-1)^{k+1} x^{2k+3}}{(2k+3)!} \times \frac{(2k+1)!}{(-1)^k \cdot x^{2k+1}} \right|$$

$(2k+1)! = \cancel{1 \cdot 2 \cdot 3 \cdots (2k)(2k+1)}$

$(2k+3)! = \cancel{1 \cdot 2 \cdot 3 \cdots (2k+1)}(2k+2)(2k+3)$

can disregard

$$= \lim_{k\to\infty} \left| \frac{\cancel{-1}\, x^2}{(2k+2)(2k+3)} \right|$$

$x^2 \rightsquigarrow$ constant

$$= \lim_{k\to\infty} \frac{x}{\underbrace{(2k+2)(2k+3)}_{\to \infty}} = 0 < 1 \quad \text{for all } x$$

True or false?

$$\underbrace{\sin x = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^{2k+1}}{(2k+1)!}}_{T(x)} \quad \text{for all } x$$

ex. Let $f(x) = \frac{1}{x}$

Find $T(x)$, the taylor series expansion for $f(x)$ around $x=1$

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$f^{(n)}(1)/n!$	
0	$\frac{1}{x}$	1	1	$(x-1)^0$
1	$-1/x^2$	-1	-1	$(x-1)^1$
2	$2/x^3$	2	1	$(x-1)^2$
3	$-6/x^4$	-6	-1	$(x-1)^3$
4	$24/x^5$	24	1	$(x-1)^4$

So,

$$T(x) = 1(x-1)^0 - 1(x-1)^1 + 1(x-1)^2 - 1(x-3)^3 + \dots$$

$$= \sum_{k=0}^{\infty} (-1)^k (x-1)^k = \sum_{k=0}^{\infty} \left[-1(x-1)\right]^k \quad \text{geometric}$$

This geometric series converges as long as

$-1 < -1(x-1) < 1$

$1 > (x-1) > -1$ $\rightarrow 0 < x < 2$

$-1 < x-1 < 1$

May 3, 2017

Reminder: $T(x)$, the taylor series expansion around $x=a$ associated to $f(x)$ is obtained from:

$$T(x) = f(a) + \frac{f'(a)}{2!}(x-a) + \frac{f''(a)}{3!}(x-a)^2 \cdots \frac{f^{(n)}(a)}{n!}(x-a)^n$$

In sigma notation: $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

ex. $f(x) = \ln x$ centered at $x=1$

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$f^{(n)}(1)/n!$	
0	$\ln x$	0	0	$(x-1)^0$
1	$\frac{1}{x} = x^{-1}$	1	1	$(x-1)^1$
2	$-x^{-2}$	-1	$-\frac{1}{2}$	$\vdots$
3	$2x^{-3}$	2	$\frac{2}{3!} = \frac{2}{6} = \frac{1}{3}$	
4	$-6x^{-4}$	-6	$-\frac{6}{4!} = -\frac{6}{24} = -\frac{1}{4}$	
5	$24x^{-5}$	24	$\frac{24}{5!} = \frac{24}{120} = \frac{1}{5}$	

$$T_n(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 + \cdots$$

$$T_n(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^n}{n}$$

Ratio Test

$$\lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{n+1} \cdot \frac{n}{\cancel{(x-1)^n}} \right|$$

$$= \lim_{n\to\infty} \left| (x-1) \cdot \frac{n}{n+1} \right|$$

$$= \lim_{n\to\infty} \left| (x-1) \cdot \frac{1}{1+1/n} \right|$$

$$= |x-1|$$

The geometric series converges as long as

$$|x-1| > 1$$

$$-1 > x-1 > 1$$

$$0 > x > 2$$

ex. Mr. Nashy

Around what values of x is the taylor series of $\frac{1}{\sqrt{x}}$ is a good approximation

Centered around $x=1$

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$f^{(n)}(1)/n!$
0	$x^{-1/2}$	1	1
1	$-\frac{1}{2}x^{-3/2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
2	$-\frac{1}{2}\cdot\left(-\frac{3}{2}\right)x^{-5/2} = \frac{3}{4}x^{-5/2}$	$\frac{3}{4}$	$\frac{3}{8}$

3	$-\frac{15}{8}x^{-7/2}$	$-\frac{15}{8}$	$-\frac{5}{16}$
4	$\frac{105}{16}x^{-9/2}$	$\frac{105}{16}$	$\frac{35}{128}$

$$T_n(x) = 1 - \frac{1}{4}(x-1) + \frac{3}{8}(x-1)^2 - \frac{5}{16}(x-1)^3 + \frac{35}{128}(x-1)^4 \dots$$

$$T(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(\dots)}{2^n}(x-1)^n$$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{(2n+3)}{2^{\cancel{n+1}}} \cdot (x-1)^{\cancel{n+1}} \; \frac{\cancel{2^n}}{(2n+1)\cancel{(x-1)^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1}{2} \cdot \frac{2n+3}{2n+1} \cdot (x-1) \right|$$

$$= \frac{1}{2}(x-1)$$