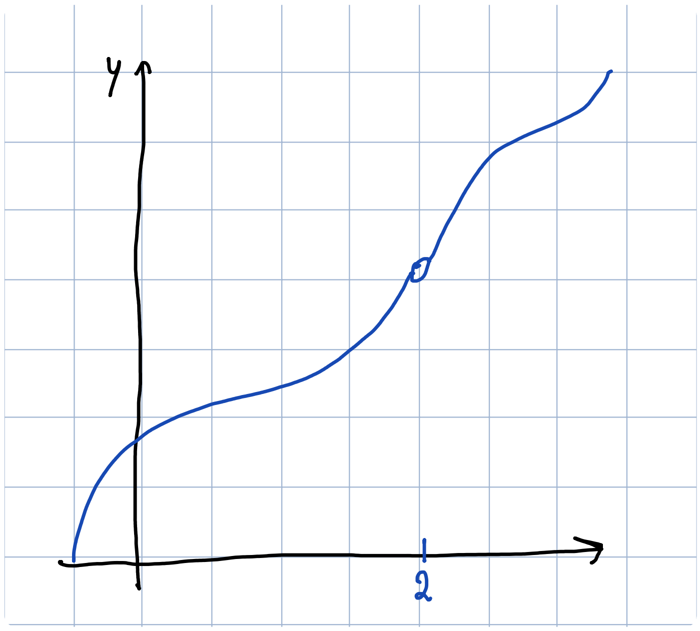
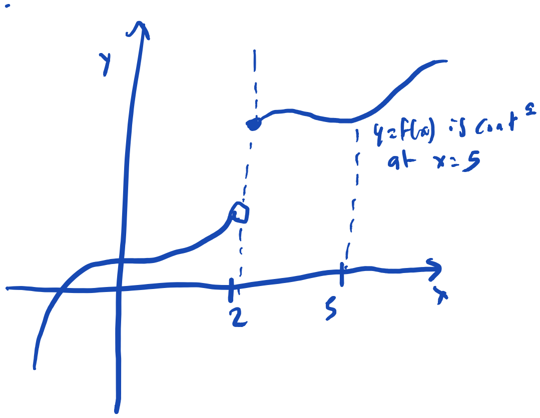
Continuity

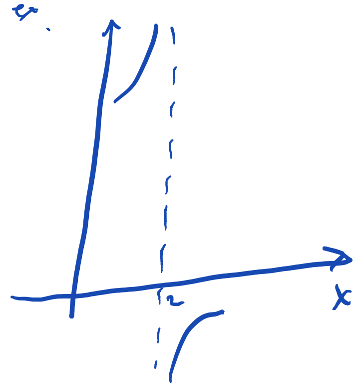


$y = f(x)$ is not continuous at $x = 2$

ex.

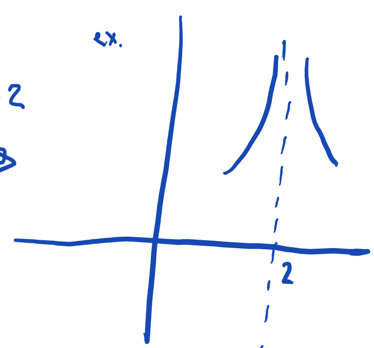


ex.

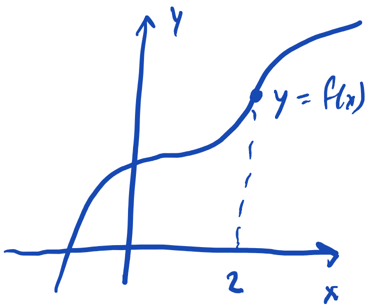


$y = f(x)$ is not continuous at $x = 2$

ex.



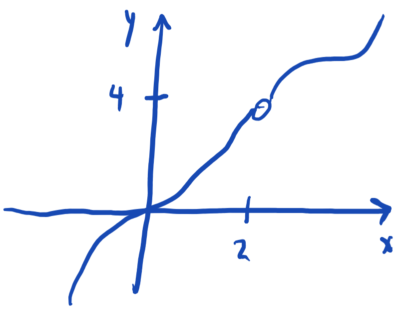
Ex.



$y = f(x)$ is cont^{s} at $x = 2$

Mathematical Def^{n}

(1)

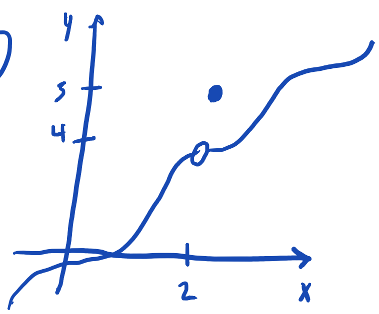


Observations:

(1) $f(2)$ UND

(2) $\lim_{x \to 2}$ exists $(=4)$

(2)

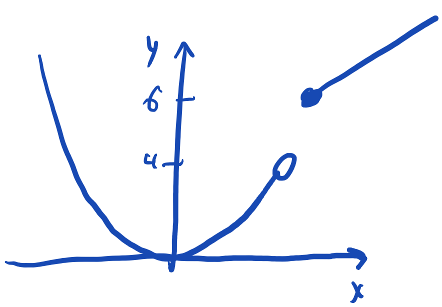


Observations!

(1) $f(2)$ defn $(=5)$

(2) $\lim_{x \to 2} f(x)$ exists $(=4)$

(3)

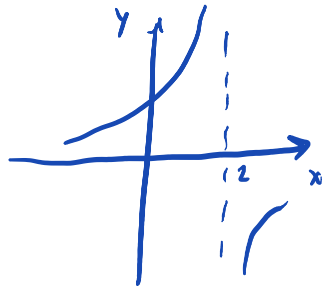


Observations:

(1) $f(2)$ defined $(=6)$

(2) $\lim_{x \to 2} f(x)$ DNE

④

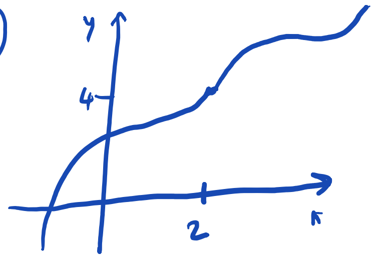


observations:

① $f(2)$ und

② $\lim_{x\to 2} f(x)$ DNE

⑤



observations:

① $f(2)$ defined $(=4)$ ✓

② $\lim_{x\to 2} f(x)$ exists $(=4)$ ✓

③ $\lim_{x\to 2} f(x) = f(2)$ ✓

Def^n

We say that a f^{cn} $y = f(x)$ is continuous at $x = a$ if:

① $f(a)$ is defined

② $\lim_{x\to a} f(x)$ exist

③ $\lim_{x\to a} f(x) = f(a)$

ex. $g(t) = \dfrac{1}{1-t^2}$ is g cont^s at $x=-1$?

Use (3pt) def^n

① $g(-1) = \dfrac{1}{1-(-1)^2} = \dfrac{1}{0}$ UND

So g is not cont^s at $x=-1$

ex. $g(t) = \frac{t^4 - 1}{1 - t^2}$ Is g cont^s at $x = -1$?

Use (3pt) def^n

① $g(-1) = \frac{1-1}{1-(-1)^2} = \frac{0}{0}$ UND

So g is not cont^s at $x = -1$

ex. Use def^n of cont^s to determine if $y = f(x)$ is cont^s at $x = 1$ where

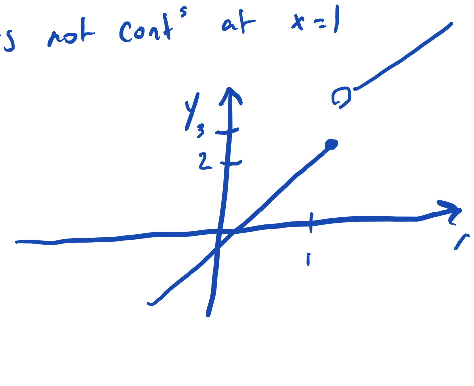
$$f(x) = \begin{cases} 3x - 1 & \text{if } 0 < x \leq 1 \\ x + 2 & \text{if } x > 1 \end{cases}$$

① $f(1)\ 3(1) - 1 = 2$ (defined) ✓

② $\lim_{x \to 1} f(x)$ exist?

$$\left.\begin{aligned} \lim_{x \to 1^-} f(x) &= 3x - 1 \overset{ch}{=} 3(1) - 1 = 2 \\ \lim_{x \to 1^+} f(x) &= x + 2 \overset{ch}{=} 1 + 2 = 3 \end{aligned}\right\} \lim_{x \to 1} f(x) \text{ DNE}$$

Concl f is not cont^s at $x = 1$



Called a jump discontinuity at $x = 1$ because $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$

ex. $f(x) = \begin{cases} 1-x^2 & \text{if } x < -3 \\ 4 & \text{if } x = -3 \\ x-5 & \text{if } x > -3 \end{cases}$

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① $f(-3) = 4$ (defined) ✓

② $\lim_{x \to -3} f(x)$ exist?

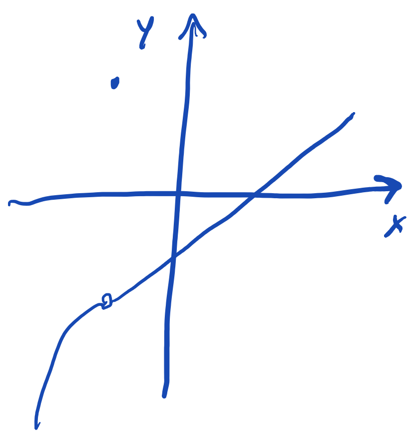
$$\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (1-x^2) \overset{ch}{=} -8$$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} (x-5) = -8$$

So $\lim_{x \to -3} f(x) = -8$ (exists) ✓

③ $\lim_{x \to -3} f(x) \neq f(-3) \Rightarrow$ So f is not cont' at $x = -3$

Called removable discontinuity at $x = -3$ (since $\lim_{x \to -3} f(x)$ exists)



Summary

① A f^{cn} $y = f(x)$ has a jump discontinuity at $x = a$

if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

② A f^{cn} $y = f(x)$ has a removable discontinuity at $x = a$
if $y = f(x)$ is not cont^s at $x = a$ but $\lim_{x \to a} f(x)$ exists.

Def^n

- A <u>polynomial expression</u> is an algebraic expression that can be written in the form:

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_0 \quad \begin{pmatrix} n = 0, 1, 2, \dots \\ \text{and } a_n \neq 0 \end{pmatrix}$$

- A <u>polynomial f^{cn}</u> is a f^{cn} defined by a <u>single polynomial expression</u>. (i.e. not piecewise defined involving poly expressions) and with DOMAIN $= \mathbb{R}$ (i.e. no restrictions in the domain)

ex. $f(x) = -3x^4 + 5x + 1$ is a poly f^{cn}

ex. $h(x) = \begin{cases} x^2 + 2x - 1 & \text{if } x < 2 \\ x^3 - 4x + 5 & \text{if } x \geq 2 \end{cases}$ is not considered to be a poly fcn

ex. $q(x) = -3x^5 + 7x + 2 \ (0 \leq x \leq 3)$ is not a poly fcn ex. Is the poly f^{cn} $f(x) = -3x^4 + 5x + 1$ cont^s at $x = 1$?

① $f(1) = -3(1)^4 + 5(1) + 1 = 3$ (defined)

② $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} (-3x^4 + 5x + 1) \overset{ch}{=} -3(1)^4 + 5(1) + 1 = 3$ (exists)

③ $\lim\limits_{x \to 1} f(x) = f(1)$

So f is cont^s at $x = 1$

In fact, the poly f^{cn} $f(x) = -3x^4 + 5x + 1$ cont^s at $x =$ 'a'
for any choice of a

If y = P(x) is a poly fcn & 'a' is any fixed no.
then y = P(x) is conts at x = 'a'

So a **poly fcn has no pts of discountinuity** or breaks in its graph

ex. Consider

$$g(x) = \begin{cases} x^3 + 4x + 1 & \text{if } x < 3 \\ x^2 + 2x + 8 & \text{if } x \geq 3 \end{cases}$$

- This fc^n is cont^s at every $x =$ 'a' for any $a \neq 3$
  Since each "piece" is built from a poly expression
- The only place where g may not be cont^s is at $x = 3$. Here we apply 3 pt def^n.

① $g(3) = 3^2 + 2(3) + 8 = 23$

② $\lim\limits_{x \to 3^-} g(x) = \lim\limits_{x \to 3^-} (x^3 + 4x + 1) \overset{ch}{=} 3^3 + 4(3) + 1 = 40$

$\lim\limits_{x \to 3^+} g(x) = \lim\limits_{x \to 3^+} (x^2 + 2x + 8) \overset{ch}{=} (3)^2 + 2(3) + 8 = 23$

So g is not cont^s at $x = 3$

ex/ Consider $f(x) = x^2 \ (0 \le x \le 2)$

Is $f(x)$ cont^s at $x=2$?

① $f(2) = 2^2 = 4$

② $\lim_{x \to 2} f(x)$ exist?

$\lim_{x \to 2^-}$

$\lim_{x \to 2^+} f(x) = $ DNE

Technically $f(x) = x^2 \ (0 \le x \le 2)$ is not cont^s at $x=2$

However, note:

① ✓

② $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} = x^2 \stackrel{ch}{=} 4$ (exists)

③ $\lim_{x \to 2^-} f(x) = f(2)$

$y = f(x)$ is left cont^s at $x=2$

So $y = f(x) = x^2 \ (0 \le x \le 2)$ is <u>not</u> cont^s at $x=2$, but it is LEFT-CONT^s at $x=2$

In gen'l, any f^{n} is said to be LEFT-CONT^s at $x = 'a'$

if: ① $f(a)$ is defined

② $\lim_{x \to a^-} f(x)$ exists

③ $\lim_{x \to a^-} f(x) = f(a)$

Same for RIGHT-CONT^s

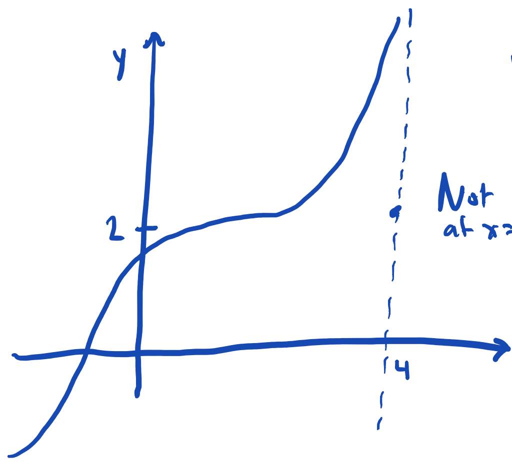
$y = f(x) = x^2$ $(0 \le x \le 2)$ is RIGHT-CONTS at $x = 0$

since: ① $f(0) = 0$ defined

② $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0$

③ $\lim_{x \to 0^+} f(x) = f(0)$

ex.



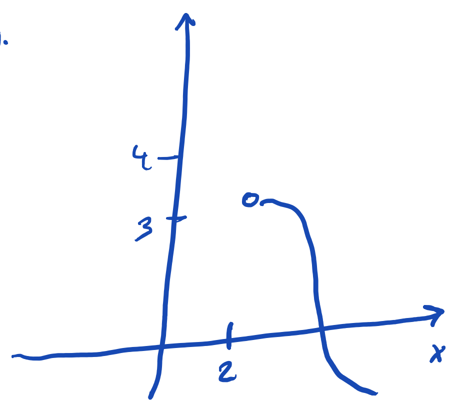
$y = g(x)$

Not LEFT CONTS at $x = 4$

① $g(4) = 2$ ✓

② $\lim_{x \to 4^-} g(x) = \infty$ (=> DNE) X

ex.



Not RIGHT-CONTS at $x = 2$

① $f(2) = 4$ ✓

② $\lim_{x \to 2^+} f(x) = 3$ ✓

③ $\lim_{x \to 2^+} f(x) \neq f(2)$ X

RULES OF CONT^S at pt $x = $ 'a'

Assume $y = f(x)$ & $y = g(x)$ are both cont^s at $x = $ 'a'

1. $y = f(x) + g(x)$ is also cont^s at $x = $ 'a'
2. $y = f(x) - g(x)$ "
3. $y = f(x) \cdot g(x)$ "
4. $y = K f(x)$ " (K is a constant)
5. $y = \dfrac{f(x)}{g(x)}$ " (provided $g(a) \neq 0$)

ex. let $f(x) = \sqrt{x}$ & $g(x) = x - 4$

Both f & g are cont^s at $x = 9$

so

1. $y = f(x) + g(x) = \sqrt{x} + (x-4)$ is also cont^s at $x = 9$
2. $y = f(x) - g(x) = \sqrt{x} - (x-4)$ is "
3. $y = f(x) \cdot g(x) = (\sqrt{x})(x-4)$ is "
4. $y = -\sqrt{3}\, f(x) = -\sqrt{3}\sqrt{x}$ is "
5. $y = \dfrac{f(x)}{g(x)}$ is "

ex. $y = \dfrac{x^2 + 4x + 1}{x^3 + 8}$ Where is this f^n cont^s?

$y = f(x) = x^2 + 4x + 1$ is a poly fcn & so cont^s anywhere

$y = g(x) = x^3 + 8$ is "

$y = \dfrac{x^2+4x+1}{x^3+8}$ is not defined at $x = -2$

$y = \dfrac{f(x)}{g(x)} = \dfrac{x^2+4x+1}{x^3+8}$ is cont^s everywhere except at $x = -2$

Review ex.

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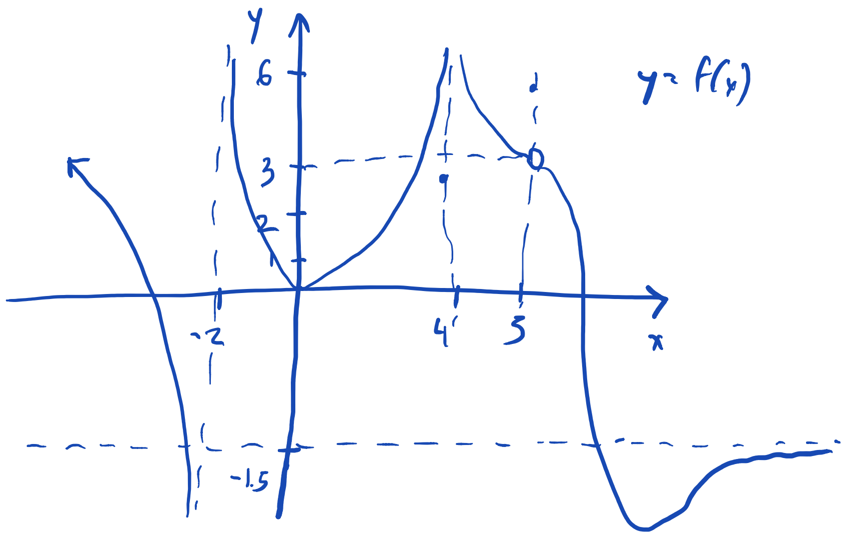


① $\lim_{x \to -\infty} f(x) = +\infty$ Not DNE

② $\lim_{x \to +\infty} f(x) = -1.5$

③ $\lim_{x \to -2^-} f(x) = -\infty$

④ $\lim_{x \to -2^+} f(x) =$

⑤ $\lim_{x \to 2} f(x) = \text{DNE}$

⑥ $\lim_{x \to 4^-} f(x) = +\infty$

⑦ $\lim_{x \to 4^+} f(x) = +\infty$

⑧ $\lim_{x \to 4} f(x) = +\infty$ ← not a finite no. DNE

⑨ $\lim_{x \to 5} f(x) = 3$

⑩ $f(5) = 6$

⑪ $f(4) = 3$

⑫ Is f cont^s at $x = 4$?
If not, what fails in def^n
" , is the discont^y removable

Not cont^s.
$\lim_{x \to 4} = \text{DNE}$

Not removable, not jump discontinuity

⑬ Does f have any removable discont^y?

yes at $x =$

Terminology

① $y=f(x)$ is cont^s everywhere on the inverval $(-\infty, \infty)$

$\therefore$ cont^s according to 3-pt def^n at any 'a' chosen in $(-\infty, \infty)$

② $y=f(x)$ is cont^s everywhere on the inverval $(3, 7)$

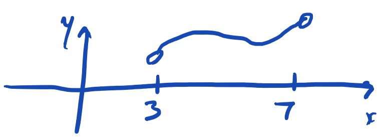
$\therefore$ cont^s according to 3-pt def^n at any 'a' chosen in $(3, 7)$

③ $y=f(x)$ is cont^s everywhere on the **closed** interval $[3,7]$

$\therefore$ cont^s according to 3-pt def^n at any 'a' chosen in $(3, 7)$

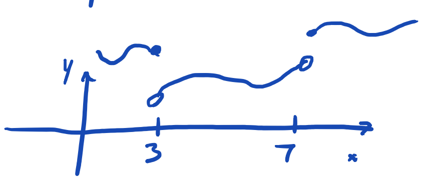
and must be right-cont^s at $x=3$ **and** left-cont^s at $x=7$

ex.



$y=f(x)$ is cont^s on $(3,7)$

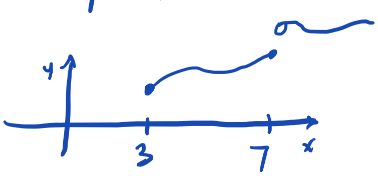
ex.



Is $y=f(x)$ is cont^s on open interval $(3,7)$?

yes

ex.



$y=f(x)$ is cont^s on $[3,7]$

INTERMEDIATE VALUE THEOREM (IVT)

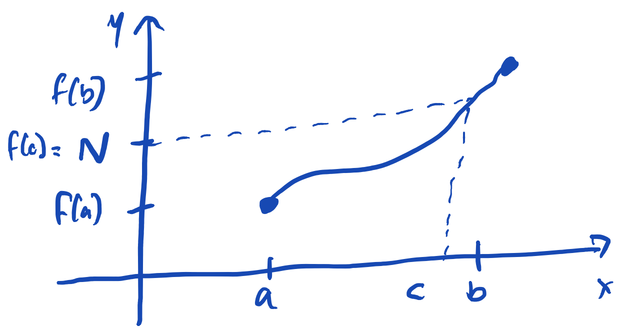
If ① $y = f(x)$ is conts on $[a,b]$

② $f(a) \neq f(b)$

③ N is any number between $f(a)$ and $f(b)$

~~then~~

there is a no. 'c' in (a,b) so that $f(c) = N$



*can have more than one value for 'c'

ex. Show that the eqn $4x^2 - 6x^2 + 3x - 2 = 0$ has a root between (i.e. a zero) between 1 & 2.

Let $f(x) = 4x^3 - 6x^2 + 3x - 2$

① $f(x)$ is conts on $[1,2]$ (since $f(x)$ is a poly fcn)

② $f(1) = 4(1)^3 - 6(1)^2 + 3(1) - 2 = -1$

$f(2) = 4(2)^3 - \ldots = 12$

$f(1) \neq f(2)$

③ We can chose $N = 0$ (falls in between $f(1) = -1$ and $f(2) = 12$)

By concl. of IVT, there is a value c in $(1,2)$

$f(c) = 0 (= N)$

i.e. $4c^2 - 6c^2 + 3c - 2 = 0$