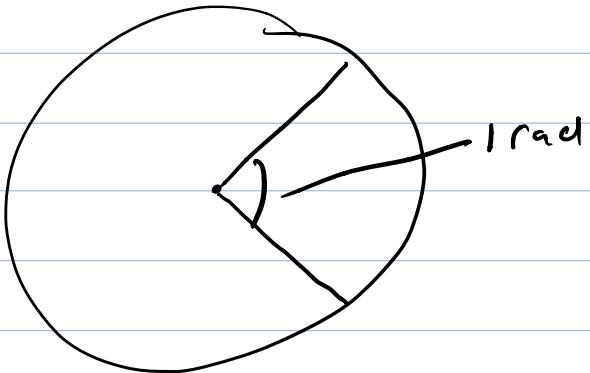


$$\pi \text{ radians} = 180^\circ$$

$$\text{rad} = \frac{180^\circ}{\pi} \approx 57.3^\circ$$



Note: Convention

$$\theta = \frac{\pi}{2} \leftarrow \text{understand it's in rad}$$

$$\beta = 30 \leftarrow \text{means } 30 \text{ rad, not } 30^\circ$$

Conversion

degrees	0°	30°	45°	60°	70°	180°	360°
radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	2π

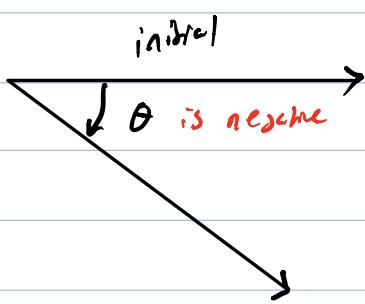
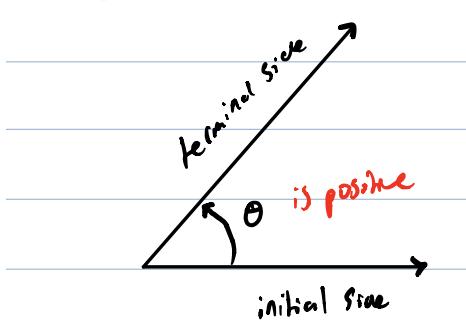
$$120^\circ = 2 \times (60^\circ) = 2 \left(\frac{\pi}{3} \right) = \frac{2\pi}{3}$$

$$-135^\circ = -3 \times 45^\circ = -\frac{3\pi}{4}$$

$$\frac{5\pi}{6} = 5 \left(\frac{\pi}{6} \right) = 5(30^\circ) = 150^\circ$$

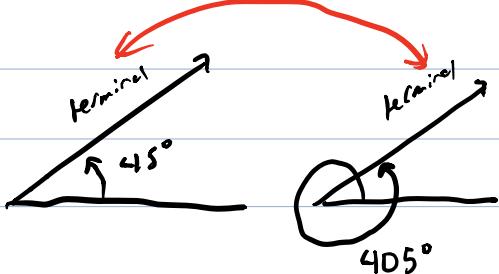
$$\therefore 5 \left(\frac{180}{6} \right) = 150^\circ$$

Positive vs. Negative angles



Coterminal angles

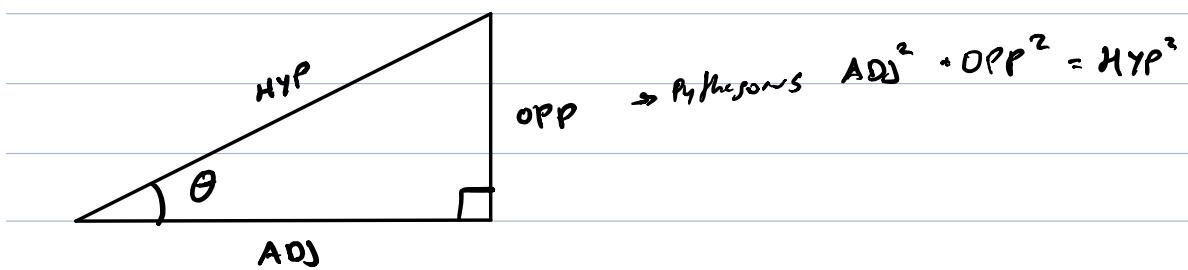
Same terminal side



45° and 405° are coterminal Δ° , but are not equivalent

$\sin(45^\circ) = \sin(405^\circ)$ Trig funcs of coterminal Δ° are always equal

Trig Funcs of Δ° ($0^\circ < \theta < 90^\circ$, $0 < \theta < \frac{\pi}{2}$)



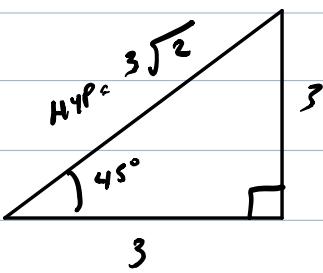
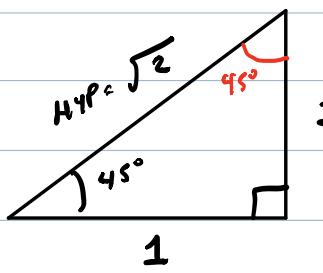
$$\sin \theta = \frac{\text{opp}}{\text{hyp}} \quad \csc \theta = \frac{\text{hyp}}{\text{opp}} - \cancel{\csc} \frac{\text{hyp}}{\text{opp}} \quad \text{- need } \csc \theta$$

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} \quad \sec \theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan \theta = \frac{\text{opp}}{\text{adj}} \quad \cot \theta = \frac{\text{adj}}{\text{opp}}$$

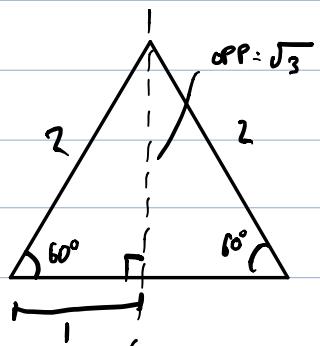
reciprocals

Trig fns of Special Δ' ($30^\circ, 45^\circ, 60^\circ$)



$$\sin 45^\circ = \frac{o}{H} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \sin 45^\circ = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{a}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

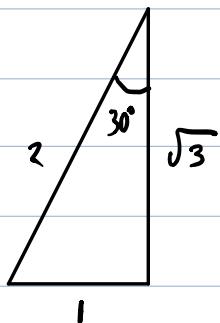


$$\sin 60^\circ = \frac{o}{H} = \frac{\sqrt{3}}{2}$$

$$\cot 60^\circ = \frac{a}{H} = \frac{1}{\sqrt{3}}$$

$$\tan 60^\circ = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\text{opp} = \sqrt{1^2 + 2^2} = \sqrt{3}$$

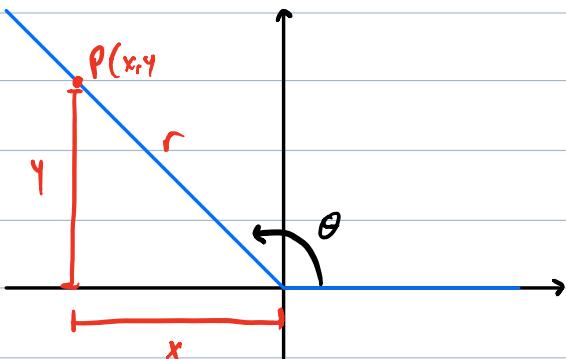


$$\sin 30^\circ = \frac{O}{H} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{A}{H} = \frac{\sqrt{3}}{2}$$

WARNING: Given $\sin 12^\circ$ ← Degrees
 $\sin 12^\circ$ ← RAD

Trig fns of General A's



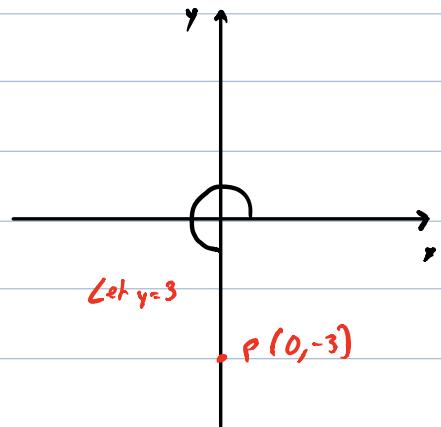
$$x^2 + y^2 = r^2$$

$$\sin \theta = \frac{y}{r} \quad \csc \theta = \frac{r}{y}$$

$$\cos \theta = \frac{x}{r} \quad \sec \theta = \frac{r}{x}$$

$$\tan \theta = \frac{y}{x} \quad \cot \theta = \frac{x}{y}$$

$\cos 270^\circ$

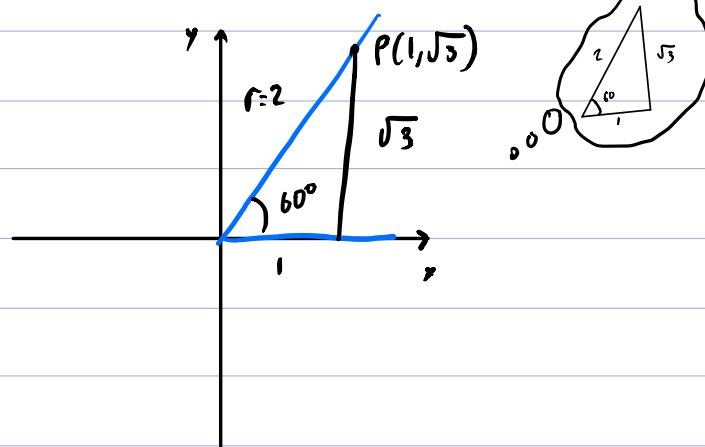


$$\cos 270^\circ = \frac{x}{r} = \frac{0}{\sqrt{3}} = 0$$

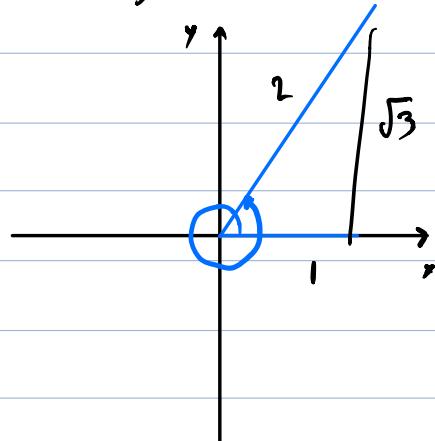
$$\sin 270^\circ = \frac{y}{r} = \frac{-3}{\sqrt{3}} = -1$$

$$\tan 270^\circ = \frac{y}{x} = \frac{-3}{0} = \text{UND}$$

$\sin 60^\circ$



$\sin(420^\circ) \rightarrow$ for tri, same ans. as 60° in tri



Proof of $\tan \theta = \frac{\sin \theta}{\cos \theta}$

$$\frac{\sin \theta}{\cos \theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{r} = \tan \theta$$

Trig Identities You Need: *Nobody will ask you to prove it*

$$1) \tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$2) \cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$3) \sec \theta = \frac{1}{\cos \theta}$$

$$4) \csc \theta = \frac{1}{\sin \theta}$$

$$5) \sin^2 \theta + \cos^2 \theta = 1$$

Proof of 5) *-Not needed*

$$\sin^2 \theta + \cos^2 \theta = (\sin \theta)^2 + (\cos \theta)^2$$

$$= \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2$$

$$= \frac{y^2 + x^2}{r^2}$$

$$= \frac{r^2}{r^2}$$

$$= 1$$

N.B.

$$\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = 1$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$6) 1 + \tan^2 \theta = \sec^2 \theta$$

$$\frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta}$$

$$7) 1 + \cot^2 \theta = \csc^2 \theta$$

8) $\sin(A+B) = \sin A \cdot \cos B + \sin B \cdot \cos A$

IF this is a '-' also a '-' minus

$$\therefore \sin(75^\circ) = \sin(30^\circ + 45^\circ)$$

$$\begin{aligned} &= \sin 30^\circ \cdot \cos 45^\circ + \sin 45^\circ \cdot \cos 30^\circ \\ &= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} \\ &= \frac{\sqrt{2} + \sqrt{6}}{4} \end{aligned}$$

9) $\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$

N.B.

$$\sin(A+B) = \sin A \cos B + \sin B \cos A$$

Suppose $\theta = A = B$

$$\sin(0+\theta) = \sin \theta \cos \theta + \sin \theta \cos \theta$$

$$10) \sin(2\theta) = 2 \sin \theta \cos \theta$$

$$9) \cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$$

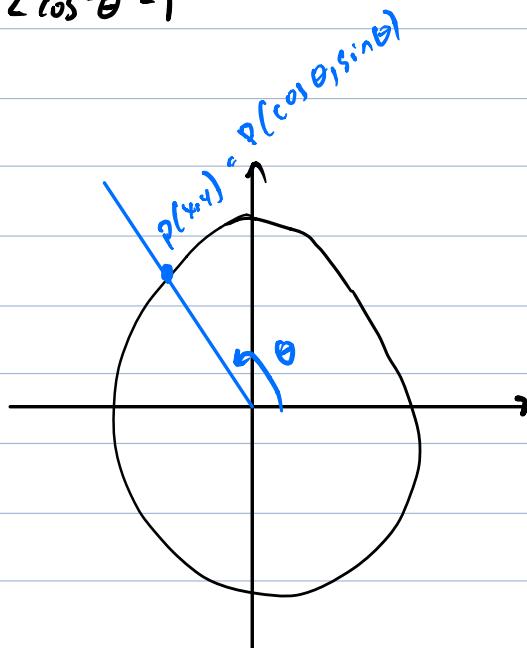
Let $\theta = A + B$

$$\cos(\theta) = \cos B \cos \theta - \sin B \sin \theta$$

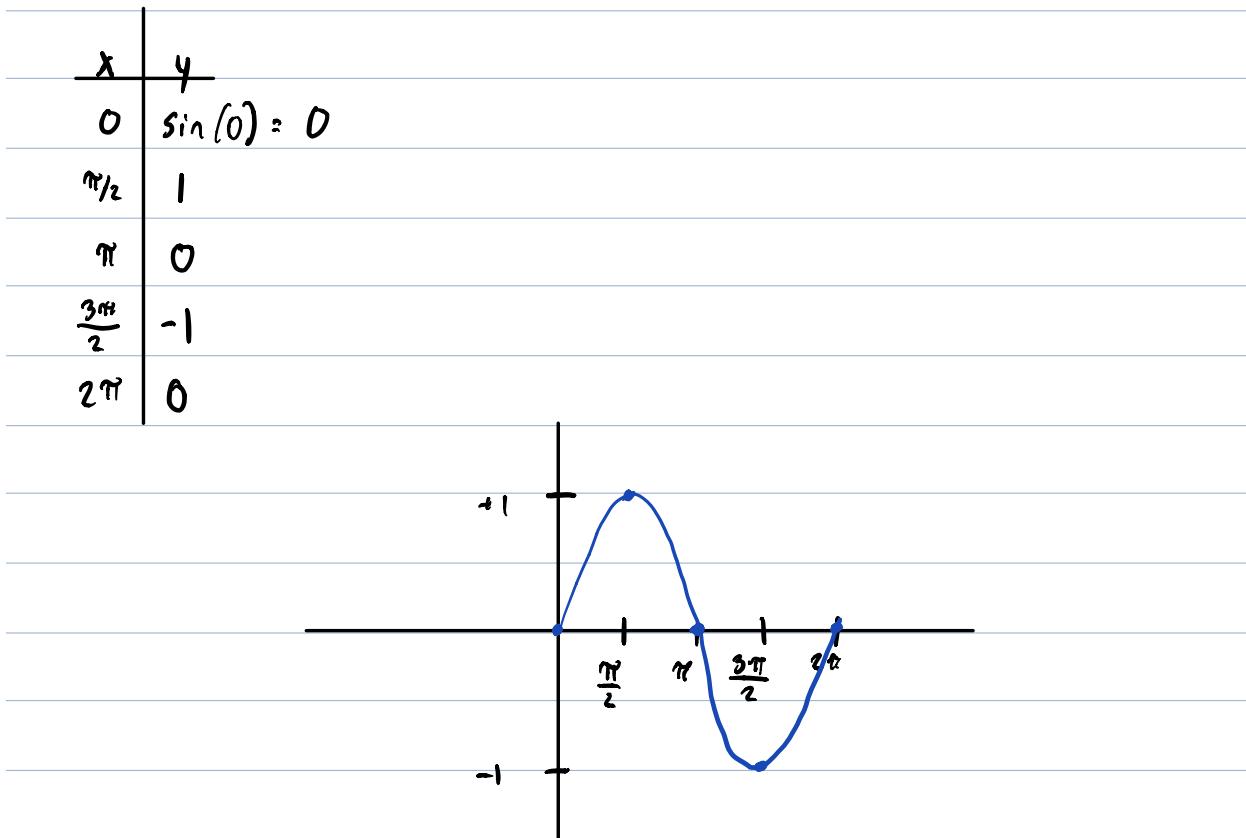
$$11a) \cos(2\theta) = \cos^2 \theta - \sin^2 \theta \quad \text{---} \quad \sin^2 \theta + \cos^2 \theta = 1$$

$$b) \cos(2\theta) = 1 - 2 \sin^2 \theta$$

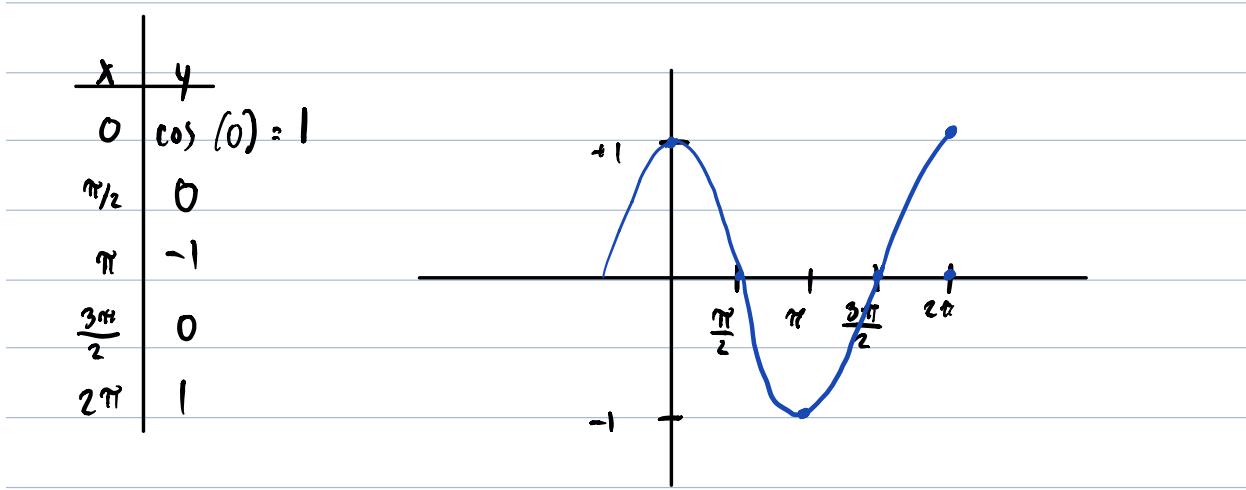
$$c) \cos(2\theta) = 2 \cos^2 \theta - 1$$



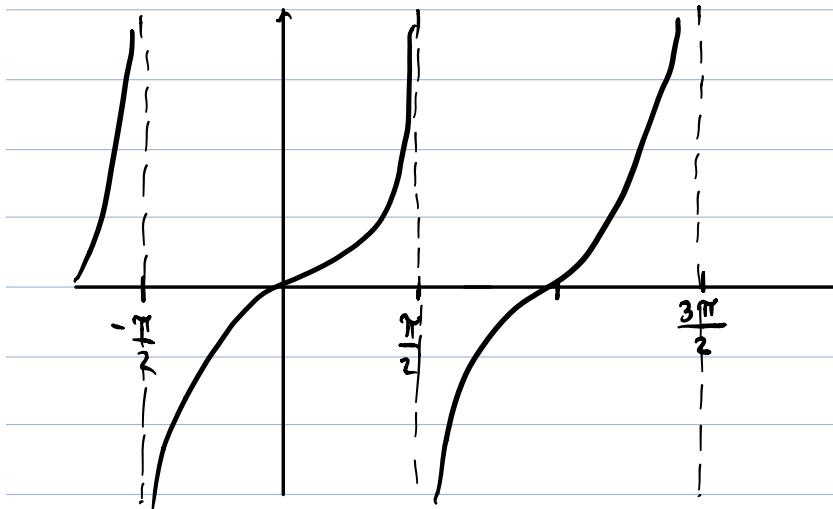
Graph of $y = \sin x$ - know how to draw graph of sin, cos, tan



Graph of $y = \cos x$



$$y = \tan x$$



$$\text{DOM} = \left\{ x \mid x \neq \frac{\pi}{2} + k\pi \quad k = \pm 0, \pm 1, \pm 2, \dots \right\}$$

$$\text{Range} = \mathbb{R}$$

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \tan x = \infty$$

$$\lim_{x \rightarrow \frac{\pi}{2}^+} \tan x = -\infty$$

Because $y = \sin(x)$ and $y = \cos(x)$ are both continuous everywhere, we can "cheat" to evaluate limits for sin and cos

ex. $\lim_{x \rightarrow \frac{\pi}{3}} \frac{x \sin x}{\cos x + 2} \stackrel{ch}{=} \frac{\frac{\pi}{3} \sin \frac{\pi}{3}}{\cos(\frac{\pi}{3}) + 2} \stackrel{use\ calculator}{\approx}$

ex. $\lim_{x \rightarrow \frac{\pi}{2}} \frac{x \sin x}{\cos(\pi) + 2} \stackrel{ch}{=} \frac{\frac{\pi}{2} \sin \frac{\pi}{2}}{\cos(\frac{\pi}{2}) + 2} = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4} \stackrel{use\ unit\ circle}{=}$

ex. $\lim_{x \rightarrow 0^+} x^2 \sin\left(\frac{1}{x}\right) \stackrel{ch}{=} 0^+ \sin\left(\frac{1}{0^+}\right) \leftarrow \text{Indeterminate Form}$

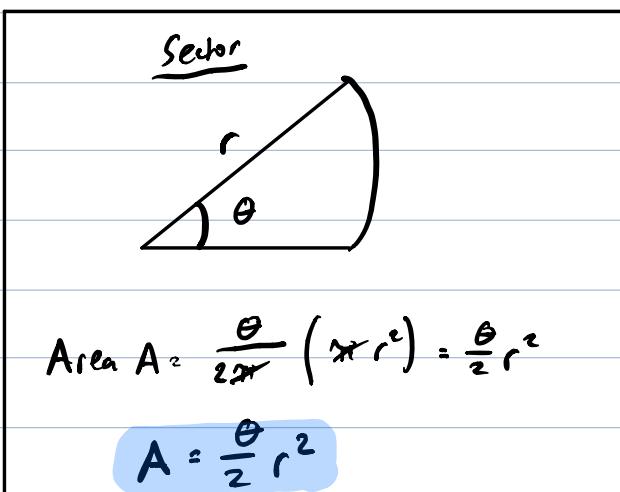
Use Squeeze Theorem

ex. $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = \frac{0}{0} \leftarrow \text{INDET}$

THIS IS AN IMPORTANT LIMIT! PROVE THAT IT'S = 1

can be
used in
test

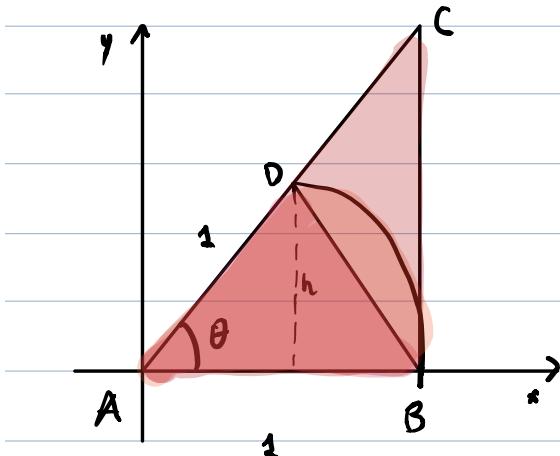
Proof:



Theorem: $\lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} = 1$

Proof:

Area

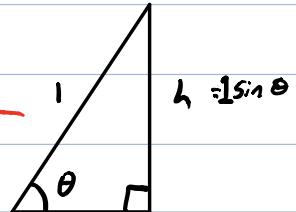


$$A_1 = \Delta ABD$$

$$A_2 = \text{sector } ABO$$

$$A_3 = \Delta ABC$$

$$A_1 = \frac{1}{2} \text{ base} \times \text{height} = \frac{1}{2}(1)h = \frac{1}{2} \sin \theta$$



$$A_2 = \frac{\theta}{2} r^2 = \frac{\theta}{2} (1)^2 = \frac{1}{2} \theta$$

$$\begin{aligned} A_3 &= \frac{1}{2} \text{ base} \times \text{height} = \frac{1}{2}(1)\overline{BC} = \frac{1}{2} \tan \theta \\ &= \frac{1}{2} \frac{\sin \theta}{\cos \theta} \end{aligned}$$

$$\tan \theta = \frac{\text{opp}}{\text{adj}}$$

$$\text{adj} \tan \theta = \text{opp}$$

$$\tan \theta = \frac{\text{opp}}{\text{adj}}$$

N.B.

$$A_1 \leq A_2 \leq A_3$$

$$\frac{1}{2} \sin \theta \leq \frac{1}{2} \theta \leq \frac{1}{2} \frac{\sin \theta}{\cos \theta}$$

and

$$\frac{1}{2} \sin \theta \leq \frac{1}{2} \theta \quad \left| \begin{array}{l} \frac{1}{2} \theta \leq \frac{1}{2} \frac{\sin \theta}{\cos \theta} \\ \cos \theta \leq \frac{\sin \theta}{\theta} \end{array} \right.$$
$$\frac{\sin \theta}{\theta} \leq 1$$

$$\text{So } \cos \theta \leq \frac{\sin \theta}{\theta} \leq 1$$

But...

$$\lim_{\theta \rightarrow 0^+} \cos \theta = \cos(0) = 1$$

4

$$\lim_{\theta \rightarrow 0^+} 1 = 1$$

By squeeze thm:

$$\lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} = 1$$

Also... $\lim_{\theta \rightarrow 0^-} \frac{\sin \theta}{\theta} = 1$ (don't need to prove it)

① $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$

② $\lim_{\theta \rightarrow 0} \frac{\cos(\theta) - 1}{\theta} = 0$

$$\text{Proof: } \lim_{\theta \rightarrow 0} \frac{\cos(\theta) - 1}{\theta} = \lim_{\theta \rightarrow 0} \frac{(\cos(\theta) - 1)(\cos(\theta) + 1)}{\theta (\cos(\theta) + 1)}$$

$$= \lim_{\theta \rightarrow 0} \frac{\cos^2 \theta - 1}{\theta (1 + \cos \theta)}$$

$$= \lim_{\theta \rightarrow 0} \frac{-\sin^2 \theta}{\theta (1 + \cos \theta)} \quad \begin{matrix} \sin^2 \theta + \cos^2 \theta = 1 \\ \cancel{\sin^2 \theta} \end{matrix}$$

$$= \lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \cdot \frac{-\sin \theta}{1 + \cos \theta} \right)$$

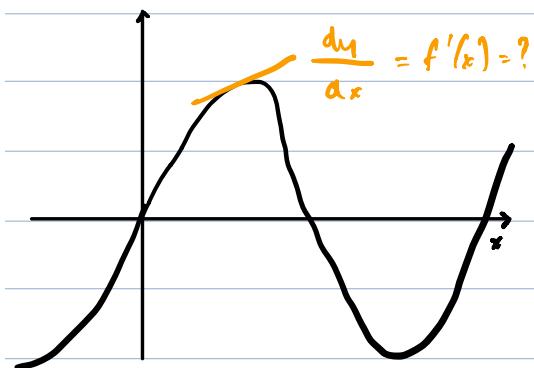
$$= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \rightarrow 0} \frac{-\sin \theta}{1 + \cos \theta}$$

$$= 1 \cdot \frac{\sin(0)}{1 + \cos(0)}$$

$$= 1 \cdot \frac{0}{2}$$

$$= 0$$

Now, consider $y = f(x) = \sin(x)$



Important proof

$$\begin{aligned}
 ① f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin x \cosh h - \sin x + \sin h \cos x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin x (\cosh h - 1) + \sin h \cos x}{h} \\
 &= \lim_{h \rightarrow 0} \left[\sin x \cdot \frac{\cosh h - 1}{h} + \lim_{h \rightarrow 0} \left[\frac{\sin h}{h} \cdot \cos x \right] \right] \\
 &= \sin x \cdot \lim_{h \rightarrow 0} \frac{\cosh h - 1}{h} + \cos x \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} \\
 &= \sin x \cdot 0 + \cos x \cdot 1 \\
 &\boxed{f'(x) = \cos x}
 \end{aligned}$$

If $y = \sin x$, then $\frac{dy}{dx} = \cos x$

i.e. $(\sin x)' = \cos x$

ex. If $f(x) = \sin x$, what is the slope of tang. line at

a) $\frac{\pi}{2}$

$$f'\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

refer to unit circle

b) π

$$f'(\pi) = \cos \pi = -1$$

c) 0

$$f'(0) = \cos 0 = 1$$

$$\text{ex. } y = x \sin x$$

$$\begin{aligned} \frac{dy}{dx} &= (x)' \sin x + x (\sin x)' \\ &= 1 \sin x + x \cos x \\ &= \sin x + x \cos x \end{aligned}$$

$$\text{ex. } y = \sin^2 x$$

$$\frac{dy}{dx} = (\sin x)^2 \leftarrow (\sin x)^r$$

$$\begin{aligned} \frac{dy}{dx} &= 2(\sin x)^{1-1} \cdot (\sin x)' \\ &= 2 \sin x \cdot \cos x \quad \checkmark \\ &= \sin(2x) \quad \leftarrow \text{can simplify to this, but not necessary} \end{aligned}$$

⑤ Now, consider $y = \cos x$

$$\frac{dy}{dx} = -\sin x$$

$$\text{i.e. } (\cos x)' = -\sin x$$

Proof Let $f(x) = \cos x$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h}$$

$$\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$$

$$= \lim_{h \rightarrow 0} \frac{\cos x \cdot \cos h - \sin x \cdot \sin h - \cos x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cos x \cdot \cos h - \cos x}{h} - \frac{\sin x \cdot \sin h}{h}$$

$$= \cos x \cdot \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} - \frac{\sin h}{h} \cdot \sin x$$

October 14, 2016

③ $y = \tan x \rightarrow \frac{dy}{dx} = \sec^2 x$

Proof: $y = \tan x = \frac{\sin x}{\cos x}$

$$\frac{dy}{dx} = \frac{(\sin x)' \cdot \cos x - \sin x \cdot (\cos x)'}{(\cos x)^2}$$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

$$(4) \quad y = \cot x \rightarrow \frac{dy}{dx} = -\csc^2 x$$

* memory trick:
 $y = \cot x \rightarrow \frac{dy}{dx} = -\csc^2 x$
 (Same for rules 5&6)

$$(5) \quad y = \sec x \rightarrow \frac{dy}{dx} = \sec x \tan x$$

Proof $y = \sec x = \frac{1}{\cos x}$

$$\frac{dy}{dx} = \frac{(1)' \cos x - 1 (\cos x)'}{(\cos x)^2} = \frac{0(\cos x) - (-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}$$

$$= \tan x \cdot \sec x$$

* Need to know Proofs

$$(6) \quad y = \csc x \rightarrow \frac{dy}{dx} = -\csc x \cot x$$

ex. $y = \frac{\tan^4 x}{x^2 + 1}$

$$\frac{dy}{dx} = \frac{(\tan^4 x)'(x^2 + 1) - (\tan^4 x)(x^2 + 1)'}{(x^2 + 1)^2}$$

IN S chain rule
 \downarrow
 $[(\tan x)^4]'$
 $= 4 \tan^3 x \cdot (\tan x)'$

$$= \frac{4 \tan^3 x (\tan x)' (x^2 + 1) - (\tan^4 x)(2x)}{(x^2 + 1)^2}$$

$$= \frac{4\tan^3 x (\sec^2 x)(x^2+1) - 2x(\tan^4 x)}{(x^2+1)^2}$$

ex. $y = \tan(x^4)$, $\frac{dy}{dx} = ?$

This is a composition of 2 functions: $y = f(x) = \tan(x)$
 $y = g(x) = x^4$

Chain rule: If $y = f(g(x)) = f(u)$ where $u = g(x)$

then: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

For $y = \tan(x^4)$

Let $u = x^4 \rightarrow \frac{du}{dx} = 4x^3$

$y = \tan(u) \rightarrow \frac{dy}{du} = \sec^2(u)$

$\therefore \frac{dy}{du} = \sec^2(u) \cdot 4x^3 = \sec^2(x^4) \cdot 4x^3$

Summary

$y = \tan x \rightarrow \frac{dy}{dx} = \sec^2(x) \cdot (x^4)'$

Chain Rule Structure: $y = \tan(\text{ins}) \rightarrow \frac{dy}{dx} = \sec^2(\text{ins}) \cdot (\text{ins})'$

$$\text{ex. } y = \tan(\sin x)$$

$$\begin{aligned}\frac{dy}{dx} &= \sec^2(\sin x) \cdot (\sin x)' \\ &= \sec^2(\sin x) \cdot (\cos x)\end{aligned}$$

$$\text{ex. } y = \sin(5x)$$

$$\frac{dy}{dx} = \cos(5x) \cdot (5x)'$$

$$= 5 \cos(5x)$$

Deriv Formulas Norman

$$\textcircled{1} \quad y = x^r \rightarrow \frac{dy}{dx} = r x^{r-1}$$

$$\textcircled{2} \quad y = \sin x \rightarrow \frac{dy}{dx} = \cos x$$

$$\textcircled{3} \quad y = \cos x \rightarrow \frac{dy}{dx} = -\sin x$$

$$\textcircled{4} \quad " = \tan x \rightarrow " = \sec^2 x$$

$$\textcircled{5} \quad " = \cot x \rightarrow " = -\csc^2 x$$

Chain Rule Version

$$y = (\sin x)^r \rightarrow \frac{dy}{dx} = r (\sin x)^{r-1} \cdot (\sin x)'$$

$$y = \sin(\cos x) \rightarrow \frac{dy}{dx} = \cos(\cos x) \cdot (-\sin x)'$$

$$y = \cos(\sin x) \rightarrow " = -\sin(\sin x) \cdot (\cos x)'$$

$$= \tan(\cos x) \rightarrow " = \sec^2(\cos x) \cdot (-\sin x)'$$

:

$$\begin{array}{l|l} \textcircled{6} \quad " = \sec x \rightarrow " = \sec x \tan x & " = \sec(\ln s) \rightarrow " = \sec() \cdot \tan() \cdot ()' \\ \textcircled{7} \quad " = \csc x \rightarrow " = -\csc x \cot x & " = \csc() \rightarrow " = -\csc() \cot() \cdot ()' \end{array}$$

ex. $y = 5 \sec(x^3 - 2x)$

$$\frac{dy}{dx} = 5 [\sec(x^3 - 2x)]'$$

$$= 5 [\sec(x^3 - 2x) \tan(x^3 - 2x) \cdot (x^3 - 2x)']$$

$$= 5 \sec(x^3 - 2x) \tan(x^3 - 2x) (3x^2 - 2)$$

ex. $y = \sin(x)$

$$\begin{aligned} y' &= \cos(x) \cdot (x)' \\ &= \cos(x) \end{aligned}$$

shyrid way
to do it, but
works

ex. $y = \sin(2)$ constant

$$\frac{dy}{dx} = 0$$

ex. $y = \sin^5(t)$

$$\frac{dy}{dx} = 0$$

$$\frac{dy}{dt} = ?$$

$$\begin{aligned}y &= (\sin(t^4))^5 \\&= 5(\sin(t^4))^4 (\sin(t^4))' \\&= 5 \sin^4(t^4) \cdot \cos t^4 \cdot (t^4)' \\&= 5 \sin^4(t^4) \cos t^4 \cdot 4t^3 \\&= 20t^3 \sin^4 t^4 \cos t^4 \leftarrow \text{Should simplify to this step}\end{aligned}$$

$$\text{ex. } \sin(xy) = x^3 + y^4, \quad \frac{dy}{dx} = ?$$

must use implicit diff

$$[\sin(xy)]' = [x^3 + y^4]'$$

$$\cos(xy) \cdot (xy)' = (x^3)' + (y^4)'$$

$$\cos(xy) \cdot [x'y + xy'] = 3x^2 + (4y^3 \cdot y')$$

$$\cos(xy) \cdot [y + xy'] = 3x^2 + 4y^3 y'$$

$$y \cos(xy) + xy' \cdot \cos(xy) = 3x^2 + 4y^3 y'$$

$$xy' \cos(xy) - 4y^3 y' = 3x^2 - y \cos(xy)$$

$$y'(x \cos(xy) - 4y^3) = 3x^2 - y \cos(xy)$$

$$y' = \frac{3x^2 - y \cos(xy)}{x \cos(xy) - 4y^3}$$

Inverse Functions - Trig Twist

$$f^{-1}(f(f(x))) = x \quad (x \text{ in dom } f)$$

$$f(f^{-1}(x)) = x \quad (x \text{ in dom } f^{-1})$$

P-205 ex 5
Not on Assignment

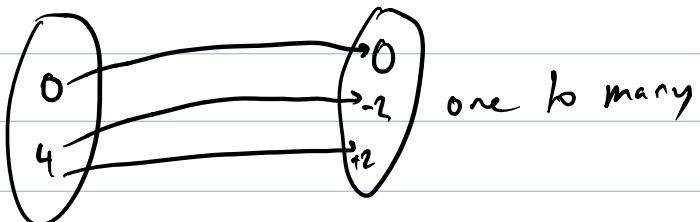
Consider $g(x) = x^2$

Dom : I R

Range : $[0, +\infty]$



If we "reverse" arrows



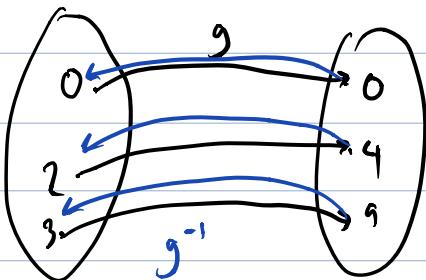
So $g(x) = x^2$ has no inverse f^{-1} , but there is an inverse relationship

We can still talk about an inverse f^{-1} related to $g(x) = x^2$ by restricting its domain:

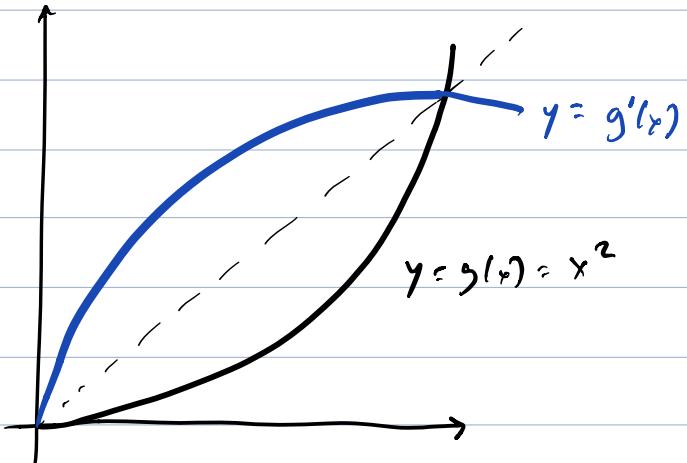
$$g(x) = x^2 \quad (x \geq 0)$$

$$\text{DOM} = [0, +\infty)$$

$$\text{RANGE} = [0, +\infty)$$

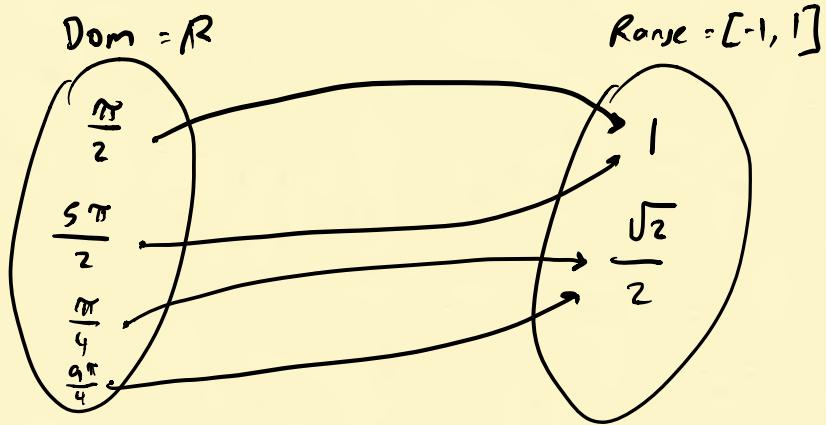


Graph:



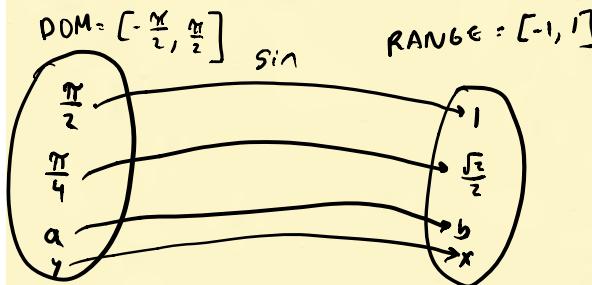
INVERSE OF TRIG FCNS

ex. $y = f(x) = \sin x$



Many to one, so inverse is not a function

In order to talk about an inverse function for $f(x) = \sin x$, we restrict the domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

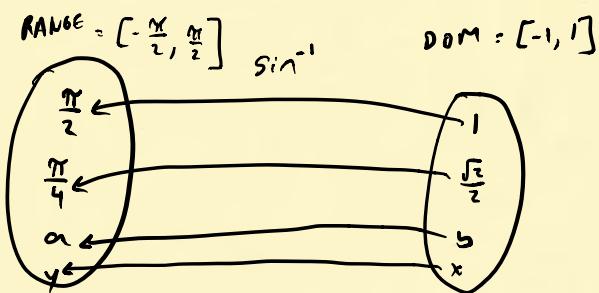


$$\sin \frac{\pi}{2} = 1$$

$$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\sin a = b$$

$$\sin y = x$$



$$\sin^{-1}(1) = \frac{\pi}{2}$$

$$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

$$\sin^{-1}(b) = a$$

$$\sin^{-1}(x) = y$$

Defn

$$y = \sin^{-1}(x) \leftrightarrow \sin(y) = x$$

$$\text{where } -1 \leq x \leq 1 \quad \text{and} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

N.B. $\sin^{-1} \neq \frac{1}{\sin}$

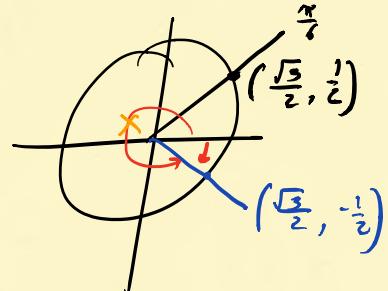
Ex. Find $\sin^{-1}\left(-\frac{1}{2}\right)$

Let $y = \sin^{-1}\left(-\frac{1}{2}\right)$

$$\sin y = -\frac{1}{2} \quad \left(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\right)$$

From unit circle: $y = -\frac{\pi}{6}$

i.e. $\sin^{-1}\left(\frac{1}{2}\right) = -\frac{\pi}{6}$



Ex. Find $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

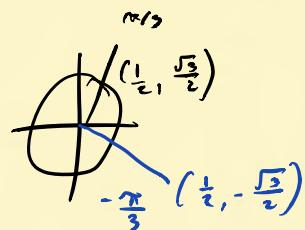
Let $y = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

↓ defn

$$\sin y = -\frac{\sqrt{3}}{2} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$y = -\frac{\pi}{3}$$

i.e. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$



$$f^{-1}(f(x)) = x \quad (x \text{ in dom } f)$$

$$f(f^{-1}(x)) = x \quad (x \text{ in dom } f^{-1})$$



Fact ① $\sin^{-1}(\sin x) = x$ true for all x in $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\textcircled{2} \quad \sin(\sin^{-1}x) = x \quad " \quad [-1, 1]$$

N.B.

$\sin^{-1}(3) \leftarrow \text{UND}$. not in $\text{dom} = [-1, 1]$

$$\text{ex. } \sin(\sin^{-1}\frac{1}{3}) = \frac{1}{3}$$

$$\sin(\sin^{-1}3) = \text{UND}$$

$$\text{ex. } \sin(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)) = -\frac{\sqrt{3}}{2}$$

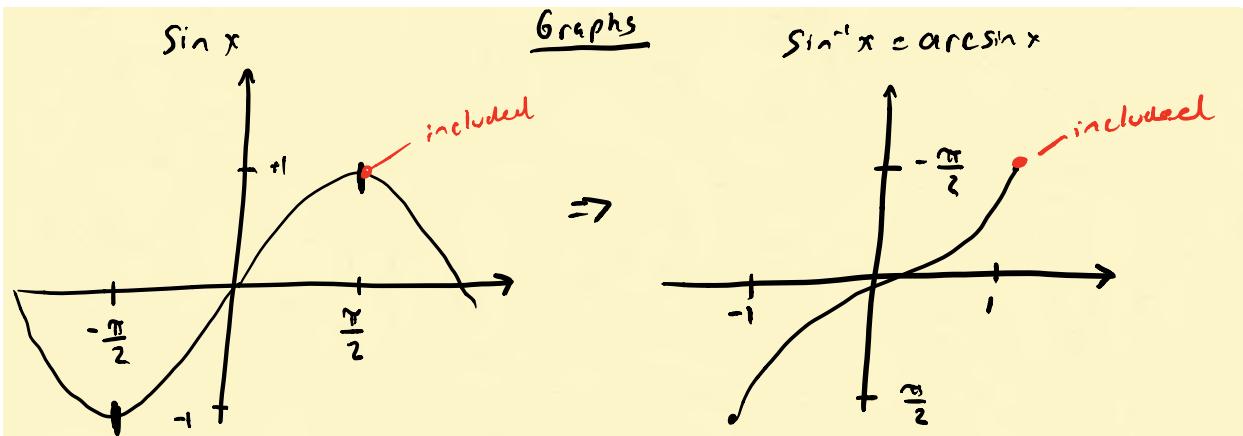
$$\text{ex. } \sin^{-1}(\sin \frac{\pi}{7}) = \frac{\pi}{7}$$

$$\text{ex. } \sin^{-1}(\sin \frac{5\pi}{2}) \leftarrow \begin{array}{l} \text{Rule ① cannot} \\ \text{be used, but still} \\ \text{defined} \end{array}$$

$\frac{5\pi}{2}$ & $\frac{\pi}{2}$ are coterminal since $\frac{5\pi}{2} = \frac{\pi}{2} + 2\pi$

$$\text{So } \sin\left(\frac{5\pi}{2}\right) = \sin \frac{\pi}{2}$$

$$\text{So } \sin^{-1}(\sin \frac{5\pi}{2}) = \sin^{-1}(\frac{\pi}{2}) = \frac{\pi}{2}$$



ex. $\cos(\sin^{-1} \frac{1}{5})$

$$\downarrow$$

$$y = \sin^{-1} \frac{1}{5} \quad \left(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \right)$$

↓ defn

$$\sin y = \frac{1}{5}$$

$$\text{adj}^2 + 1^2 = s^2$$

$$\text{adj} = \sqrt{s^2 - 1^2}$$

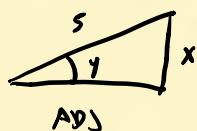
$$\text{so } \cos(\sin^{-1} \frac{1}{5}) = \cos(y) = \frac{\text{adj}}{s} = \frac{\sqrt{s^2 - 1^2}}{s}$$

ex. Rewrite $\cos(\sin^{-1} \frac{x}{5})$ in "algebraic form" \hookrightarrow no trig

$$\text{Let } y = \sin^{-1} \frac{x}{5}$$

$$\downarrow$$

$$\sin y = \frac{x}{5}$$



$$\text{adj}^2 + x^2 = s^2$$

$$\text{adj} = \sqrt{s^2 - x^2}$$

$$\text{so } \cos(\sin^{-1} \frac{x}{5}) = \cos(y) = \frac{\text{adj}}{s} = \frac{\sqrt{s^2 - x^2}}{s}$$

Restrictions: $-1 \leq \frac{x}{5} \leq 1$
 ↑
 Don't need to include
 it's implied

Question: If $y = \sin^{-1} x$, $\frac{dy}{dx} = ?$ ← PROOF (know it!)

Soln: $y = \sin^{-1} x$ ↓ defⁿ
 $\sin(y) = x$ ↓
use implicit diffⁿ

$$[\sin(x)]' = [x]'$$

↓

$$\cos(y) \cdot y' = 1$$

$$y' = \frac{1}{\cos(y)}$$

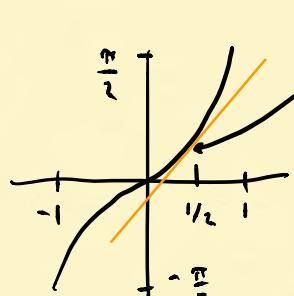
Back to $\sin(y) = x$



$$\text{ADJ} = \sqrt{1-x^2} \leftarrow \text{Pythag}$$

$$\cos(y) = \frac{\Delta}{H} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

$$\text{So } y' = \frac{1}{\sqrt{1-x^2}}$$



$$\left(\frac{1}{2}, \sin^{-1}\frac{1}{2}\right) = \left(\frac{1}{2}, \frac{\pi}{6}\right)$$

$$\text{slope} = \frac{1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-\left(\frac{1}{2}\right)^2}} = \frac{2}{\sqrt{3}}$$

Chain Rule Version:

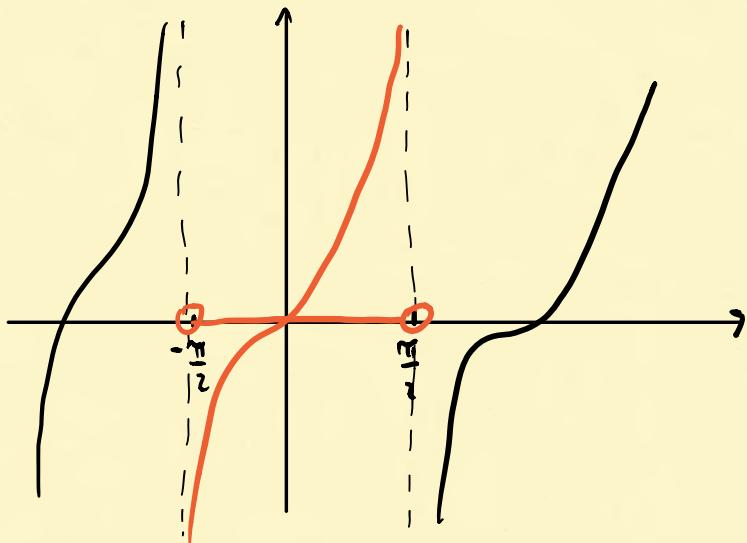
$$\text{If } y = \sin^{-1}(u) \text{ then } \frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} (u')'$$

Ex. $y = \sin^{-1}(3x)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sqrt{1-(3x)^2}} (3x)' \\ &= \frac{3}{\sqrt{1-9x^2}}\end{aligned}$$

* Only responsible for $\sin^{-1}(x)$ & $\tan^{-1}(x)$ [not $\cos^{-1}(x)$]

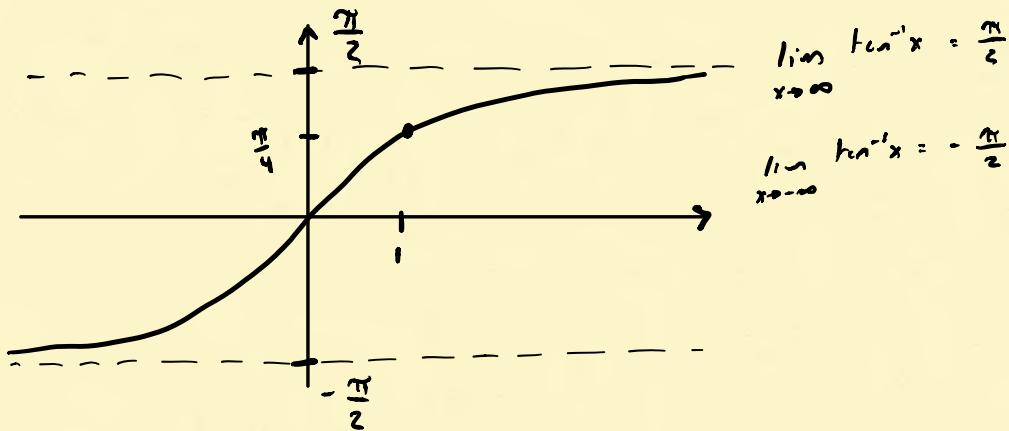
$$y = \tan^{-1} x = \text{Arctan } x$$



Working Defn

$$\begin{aligned}y = \tan^{-1} x &\leftrightarrow \tan y = x \\ -\frac{\pi}{2} < y < \frac{\pi}{2}, \quad -\infty < x < \infty\end{aligned}$$

Graph of Inverse $\tan^{-1}(x)$



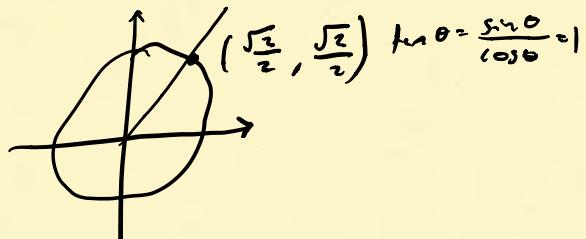
ex. $\tan^{-1}(1)$

$$\text{Let } y = \tan^{-1}(1)$$

$$\tan y = 1 \quad \left(-\frac{\pi}{2} < y < \frac{\pi}{2}\right)$$

$$y = \frac{\pi}{4}$$

$$\text{So } \tan^{-1}(1) = \frac{\pi}{4}$$



$$\textcircled{1} \quad \tan^{-1}(\tan x) = x \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$$

$$\textcircled{2} \quad \tan(\tan^{-1}x) = x \quad (-\infty < x < \infty)$$

$$\text{ex. } \tan(\tan^{-1}3) = 3$$

$$\text{ex. } \tan^{-1}(\tan \frac{\pi}{4}) = \frac{\pi}{4}$$

$$\text{ex. } \tan^{-1}(\tan \frac{9\pi}{4}) = \cancel{\frac{9\pi}{4}} = \underbrace{\tan^{-1}(\tan \frac{\pi}{4})}_{\text{co-terminal}} = \frac{\pi}{4}$$

$$\text{so } \tan \frac{9\pi}{4} = \tan \frac{\pi}{4}$$

Derivatives of $\tan^{-1} x$

If $y = \tan^{-1} x$, then $\frac{dy}{dx} = \frac{1}{1+x^2}$

Proof

$$y = \tan^{-1} x$$



$$\tan y = x$$

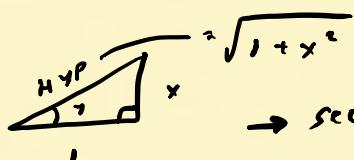
$$[\tan y]' = [x]'$$

$$\sec^2(y) \cdot y' = 1$$

$$y' = \frac{1}{\sec^2 y}$$

$$y' = \frac{1}{1+x^2}$$

$$\tan y = \frac{x}{1}$$



$$\sec y = \frac{1}{\cos y} = \sqrt{1+x^2}$$

Chain Rule Version:

$$y = \tan^{-1} (\ln s) \rightarrow \frac{dy}{ds} = \frac{1}{1 + (\ln s)^2} (\ln s)'$$

$$\text{ex. } q(t) = 3t \operatorname{Arctan} st$$

$$\begin{aligned} q'(t) &= [3t]' \operatorname{Arctan} st + 3t [\operatorname{Arctan} st]' \\ &= 3 \operatorname{Arctan} st + 3t \cdot \frac{1}{1 + (st)^2} \cdot (st)' \end{aligned}$$

$$= 3 \operatorname{Arctan} st + \frac{st}{1+2st^2}$$