Vectors

Geometric vectors

vectors: direction and magnitude

position-free

$\overrightarrow{AB} = \overrightarrow{CD}$

In particular, we put vector $\vec{v}$ in standard position when we make its initial point be the origin.

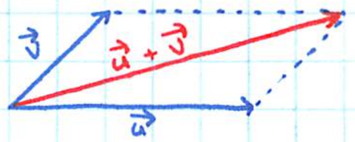
Notation: use lowercase letter corresponding to endpoint to represent vector in standard position

ex: $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$

zero vector: $\vec{0}$ of magnitude zero

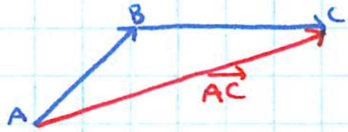
Sum

parallelogram



tip-to-tale

Note: $\boxed{\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}}$ "step over B"

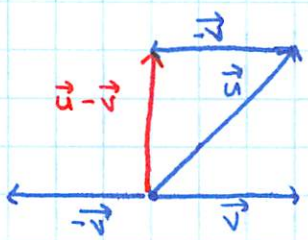


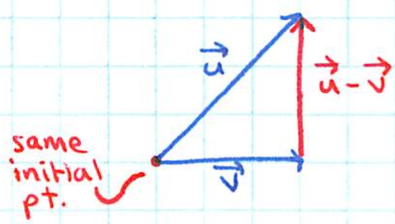
Negative

define $-\vec{v}$ to be the vector with the same magnitude but pointing in opposite direction as $\vec{v}$

Substraction

$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$$





arrow points toward ⊕

Note:

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= \vec{b} - \vec{a} \end{aligned}$$

$$\therefore \quad \boxed{\overrightarrow{AB} = \vec{b} - \vec{a}}$$

Scalar multiplication

$k\vec{v} \Rightarrow$ if $k > 0$: same direction, magnitude $\times k$

$k < 0$: opposite direction, magnitude $\times k$

$k = 0$: $0 . \vec{v} = \vec{0}$

* 2 nonzero vectors $\vec{u}, \vec{v}$ are parallel if one is a scalar multiple of the other.

Properties:

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$

$0 + \vec{v} = \vec{v}$

$\vec{u} + (-\vec{u}) = \vec{0}$

$k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$

$(k+l)\vec{u} = k\vec{u} + l\vec{u}$

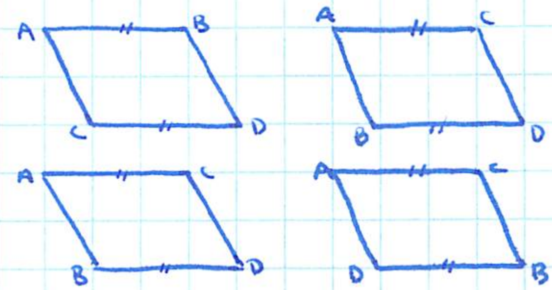
$k(l\vec{u}) = (kl)\vec{u}$

$1\vec{u} = \vec{u}$

Remarks

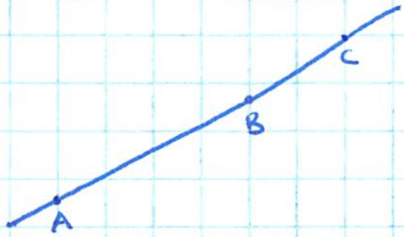
(1) ABCD is a //-gram

if $\boxed{\begin{array}{c}\overrightarrow{AB} = \pm\overrightarrow{CD} \\ \text{or} \\ \overrightarrow{AC} = \pm\overrightarrow{BD}\end{array}}$



if vectors are equal:
- same magnitude
- parallel (scalar multiple)

(2) C is on line AB



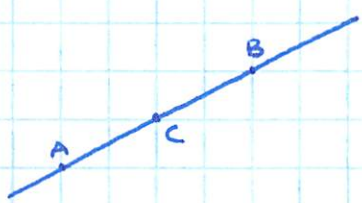
$\vec{AC} = q\vec{AB}$

- it's a scalar multiple
  ↳ they're //

$$\vec{AC} = q\vec{AB}$$
$$(\vec{c} - \vec{a}) = q(\vec{b} - \vec{a})$$
$$\vec{c} = q\vec{b} - q\vec{a} + \vec{a}$$

$$\therefore \vec{c} = q\vec{b} + (1-q)\vec{a}$$

C is on line-segment AB

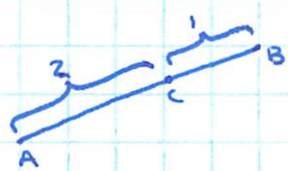


$\vec{AC} = q\vec{AB}$, $0 \le q \le 1$

$\vec{c} = q\vec{b} + (1-q)\vec{a}$

Special case

→ C divides AB in the ratio 2 : 1



Here $q = 2/3$

$\therefore \vec{c} = 2/3\,\vec{b} + 1/3\,\vec{a}$

→ more generally :
C divides AB in ratio n : m

$$\boxed{\vec{c} = \frac{n}{n+m}\vec{b} + \frac{m}{n+m}\vec{a}}$$

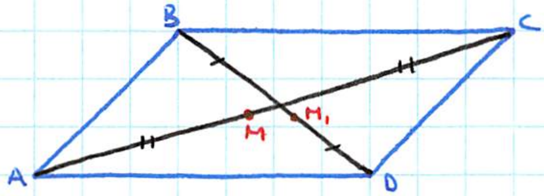
"n" on endpoint

ex: C divides AB 7:3

↳ $\vec{c} = \frac{7}{10}\vec{b} + \frac{3}{10}\vec{a}$

Application: The diagonals of a //-gram intersect at the midpoints

Note: midpoints divide in 1:1 ratio



proof: Let M be midpoint of AC (1:1)
Let M' be midpoint of BD (1:1)

$$\boxed{\begin{array}{l}\text{To show that } M = M' \\ \Rightarrow \text{ show } \vec{m} = \vec{m}'\end{array}}$$

$$\vec{m} = \tfrac{1}{2}\vec{a} + \tfrac{1}{2}\vec{c} = \tfrac{1}{2}(\vec{a}+\vec{c})$$
$$\vec{m}' = \tfrac{1}{2}\vec{b} + \tfrac{1}{2}\vec{d} = \tfrac{1}{2}(\vec{b}+\vec{d})$$

* Also, ABCD is a //-gram

$$\therefore \overrightarrow{AB} = \overrightarrow{DC}$$
$$(\vec{b}-\vec{a}) = (\vec{c}-\vec{d})$$
$$\vec{b}+\vec{d} = \vec{a}+\vec{c}$$

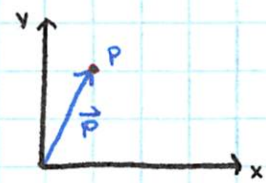
$$\therefore \vec{m} = \tfrac{1}{2}\underbrace{(\vec{a}+\vec{c}) = \tfrac{1}{2}(\vec{b}+\vec{d})}_{=} = \vec{m}'$$

Algebreic Vectors

In standard position, $\vec{p} = \overrightarrow{OP}$ is completely determined by its terminal point P.
So we assign the coordinates of P to be the components of the vector $\vec{p}$.

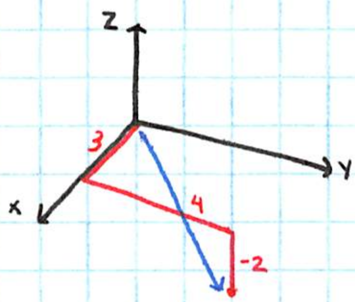
In $\mathbb{R}^2$:

ex: $P(1,3)$ parantheses for coordinates
$\vec{p} = [1,3]$ square-brackets for components

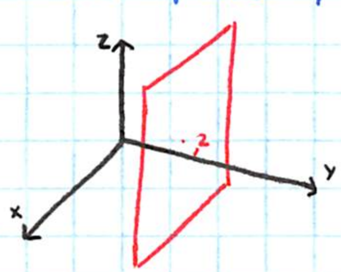


In $\mathbb{R}^3$:

ex: $P(\overset{x}{3}, \overset{y}{4}, \overset{z}{-2})$
$\vec{p} = [3,4,-2]$



ex: the plane $y = 2$



Geometrically, we define $\mathbb{R}^n$ to be the set of all ordered <u>n-tuples</u> of real numbers $(a_1, a_2, \dots, a_n)$: n = # of coordinates/components

ex: $(2, 0, \pi, 3)$ is a 4-tuple in $\mathbb{R}^4$
$\vec{v} = [v_1, v_2, v_3] \in [\mathbb{R}, \mathbb{R}, \mathbb{R}] \equiv \mathbb{R}^3$

$A(a_1, a_2, \dots, a_n)$ in $\mathbb{R}^n$ coordinates
$\vec{a} = [a_1, a_2, \dots, a_n]$ ⇒ square bracket: components

In $\mathbb{R}^n$, the zero vector : $\vec{0} = [0, 0, 0, \dots, 0]$

<u>Note</u>: To compare 2 vectors, they must have the same number of components ($\mathbb{R}^n$)

<u>Sum</u> in agreement with the geometric definition

$$[v_1, v_2, \dots, v_n] + [u_1, u_2, \dots, u_n]$$
$$= [v_1 + u_1, v_2 + u_2, \dots, v_n + u_n]$$

<u>Scalar Multiplication</u> $k[v_1, v_2, \dots, v_n] = [kv_1, kv_2, \dots, kv_n]$

* Suppose that $\vec{v}$ is not in standard position
$$\vec{v} = \overrightarrow{AB} = \vec{b} - \vec{a}$$

$\therefore$ if $\left.\begin{matrix} A(a_1, a_2, \dots, a_n) \\ B(b_1, b_2, \dots, b_n) \end{matrix}\right\}$ $\begin{aligned} \overrightarrow{AB} &= [b_1, b_2, \dots, b_n] - [a_1, a_2, \dots, a_n] \\ &= [b_1 - a_1, b_2 - a_2, \dots, b_n - a_n] \end{aligned}$

ex: Are $\vec{u} = [2, -1, -3]$ and $\vec{v} = [-6, 3, 9]$ // ?

$$\frac{-6}{2} = \frac{3}{-1} = \frac{9}{-3} = -3 \qquad \therefore \vec{v} = -3\vec{u} \quad \text{scalar multiple}$$

$\therefore$ // and opposite

ex: Find the coordinates of P which is 1/3 the distance from A(3,-1,4) to B(2,3,5)

$\xrightarrow{}$ 1:2 ratio

$\vec{a} = [3,-1,4]$ $\vec{b} = [2,3,5]$

P divides $\vec{AB}$ in 1:2 ratio

$\vec{p} = \frac{1}{3}\vec{b} + \frac{2}{3}\vec{a} =$

$= \frac{1}{3}[2,3,5] + \frac{2}{3}[3,-1,4] = [\frac{8}{3}, \frac{1}{3}, \frac{13}{3}]$

$\therefore P(\frac{8}{3}, \frac{1}{3}, \frac{13}{3})$

OR

$\vec{AP} = \frac{1}{3}\vec{AB}$

$\vec{p} - \vec{a} = \frac{1}{3}[2-3, 3--1, 5-4]$

$\vec{p} = \vec{a} + [-\frac{1}{3}, \frac{4}{3}, \frac{1}{3}] = [\frac{8}{3}, \frac{1}{3}, \frac{13}{3}]$

ex: Find the coordinates of the midpoint M between A (a_1, a_2, a_3) and B (b_1, b_2, b_3)

$\xrightarrow{}$ 1:1 ratio

$\vec{m} = \frac{1}{2}\vec{b} + \frac{1}{2}\vec{a}$

$= \frac{1}{2}(\vec{b} + \vec{a})$

$= \left[\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \frac{a_3+b_3}{2}\right]$

Def'n: A linear combination of vectors is a sum of scalar multiples of these vectors. More precisely;
$\vec{v}$ is a linear combination of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ if we can find scalars $k_1, k_2, \dots, k_n$
s.t $\vec{v} = k_1\vec{v}_1 + k_2\vec{v}_2 + \dots + k_n\vec{v}_n$

ex: $\vec{v} = 2\vec{a} - 10\vec{b} + \frac{1}{2}\vec{c}$ is a linear combination of $\vec{a}, \vec{b}, \vec{c}$

ex: $\vec{\omega} = [2,7] = -3\underbrace{[2,-1]}_{``\vec{u}} + 2\underbrace{[4,2]}_{``\vec{v}}$

$\therefore$ $\vec{\omega}$ is a linear combination of $\vec{u}$ and $\vec{v}$

Def'n: → $\hat{\imath} = [1,0]$ and $\hat{\jmath} = [0,1]$ in $\mathbb{R}^2$

since $[a,b] = [a,0] + [0,b] = a\hat{\imath} + b\hat{\jmath}$
$\therefore$ any vector in $\mathbb{R}^2$ can be expressed as a linear combination of $\hat{\imath}$ and $\hat{\jmath}$.

$\hat{\imath}$ and $\hat{\jmath}$ are standard basis vectors of $\mathbb{R}^2$

→ $\hat{\imath} = [1,0,0]$, $\hat{\jmath} = [0,1,0]$, $\hat{k} = [0,0,1]$ in $\mathbb{R}^3$

ex: $[1,-1,3] = \hat{\imath} + -\hat{\jmath} + 3\hat{k}$

→ For $n>3$, we denote the r^{th} standard basis vector by:

$$\hat{e}_r = [0,0,0,\overset{r^{th} \text{ component}}{1},0]$$

$$[a_1, a_2, \dots, a_n] = a_1\hat{e}_1 + a_2\hat{e}_2 + \dots + a_n\hat{e}_n$$

Def'n: Given the set of vectors in $\mathbb{R}^n$
$S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$, the span of S
is the set of all linear combinations of
$\vec{v}_i$'s, which are called generators.

$$\text{Span } S = \text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\} = \{K_1\vec{v}_1 + K_2\vec{v}_2 + \dots + K_r\vec{v}_r\}$$

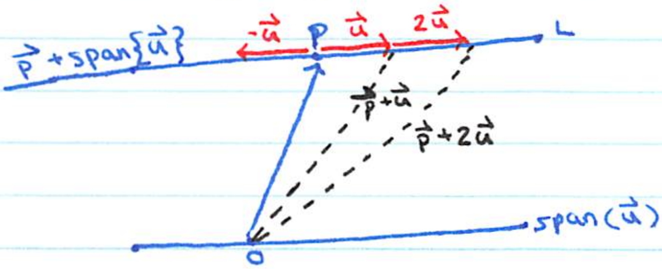
→ To describe a span geometrically, we put all the vectors in the span into standard position and we just look at the terminal points.

* Span = all possible combinations of vectors
Analogy: 2 crayons: red and yellow
can create: red ⟵ ... orange ... ⟶ yellow
· if I add orange crayon: doesn't make difference

ex: (i) Let $\vec{u} \neq \vec{0}$
1 vector: Span$\{\vec{u}\}$ can be interpreted geometrically by looking at all terminal points in st. pos
$K\vec{u}$, K can be any constant
$\therefore$ span$\{\vec{u}\}$ yields a line through the origin
// to $\vec{u}$

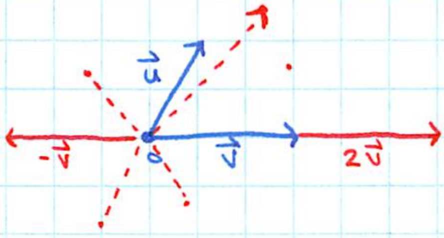
All terminal points make the line

(ii) L is a line parallel to $\vec{u}$ (ie direction vector $\vec{u}$)
through point P ($\therefore$ must add $\vec{p}$)



Vector form of L
$\boxed{\vec{p} + t\vec{u}}$
all terminal points will make the line L

→ Let $\vec{u}$ and $\vec{v}$ be 2 non-zero, non-// (non-collinear) vectors,
Span $\{\vec{u}, \vec{v}\} = \mathbb{R}^2$ is a plane through the origin



By combining scalar multiples of $\vec{u}$ and $\vec{v}$, we can create any point on the plane

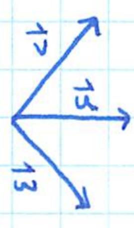
→ If you want the plane parallel to span$\{\vec{u}, \vec{v}\}$ but that does not pass through the origin, it passes through P_0

↳ $\vec{P_0} + \text{span}\{\vec{u}, \vec{v}\}$
$\vec{P_0} + s\vec{u} + t\vec{v}$ $\quad s, t \in \mathbb{R}$

→ Let $\vec{u}, \vec{v}, \vec{w}$ be non-zero, non-coplanar ($\therefore$ non-//)

$\therefore$ span $\{\vec{u}, \vec{v}, \vec{w}\} = \mathbb{R}^3$

⇒ we can create any point in $\mathbb{R}^3$ by combining these vectors in different ways



Note: "n" in $\mathbb{R}^n$ is:
→ not # of vectors
→ is # of components in each vector.

- if span $\{\vec{v_1}, \ldots, \vec{v_r}\} = \mathbb{R}^2$, $r \geq 2$
  → 2 of the vectors may be collinear (line)
  $\therefore$ it requires a third vector that is not parallel

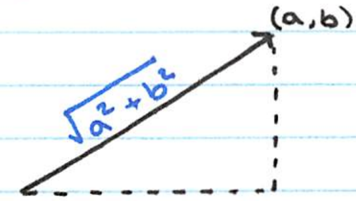
- if span $\{\vec{v_1}, \ldots, \vec{v_r}\} = \mathbb{R}^3$, $r \geq 3$

Norm

Def'n: Let $\vec{v} = [v_1, v_2, \dots, \vec{v}_n]$ in $\mathbb{R}^n$

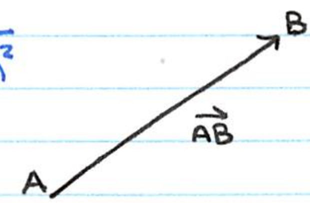
we define the norm of $\vec{v}$ to be:

$$\boxed{\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}}$$



Distance between $A(a_1, a_2, \dots, a_n)$ and $B(b_1, b_2, \dots, b_n)$:

$$\|\vec{AB}\| = \sqrt{(b_1-a_1)^2 + (b_2-a_2)^2 + \dots (b_n-a_n)^2}$$



Theorem:

(i) $\|\vec{v}\| \geq 0$, and if $\|\vec{v}\| = 0$ $\therefore$ $\vec{v} = \vec{0}$

(ii) $\|K\vec{v}\| = |K| \, \|\vec{v}\|$ $\Rightarrow$ proof:

$$\begin{aligned} \|K\vec{v}\| &= \sqrt{(Kv_1)^2 + (Kv_2)^2 + \dots + (Kv_n)^2} \\ &= \sqrt{K^2 (v_1^2 + v_2^2 + \dots + v_n^2)} \\ &= |K| \sqrt{v_1^2 + v_2^2 + \dots + v_n^2} \end{aligned}$$

ex: $\|[-4, 4, -2]\|$

$= \|-2[2, -2, 1]\|$

$= (2) \, \|[2, -2, 1]\|$

$= 2\sqrt{4+4+1} = 6$

Def'n: A unit vector is a vector of norm 1 (denoted by ^ instead of →)

* All standard basis vectors are unit vectors

Theorem: if $\vec{u} \neq \vec{0}$, then

$$\boxed{\hat{u} = \frac{1}{\|\vec{u}\|}\vec{u}}$$

is the unit vector in the same direction as $\vec{u}$.

proof: $\hat{u}$ has the same direction as $\vec{u}$ since it is a positive scalar multiple of $\vec{u}$.

AND

$$\|\hat{u}\| = \left\|\underbrace{\frac{1}{\|\vec{u}\|}}_{K} \cdot \vec{u}\right\| = \frac{1}{\|\vec{u}\|}\|\vec{u}\| = 1$$

ex: Let $\vec{u} = [-1,1,2]$; Find the vectors of length 3 that are // to $\vec{u}$.

(1) same direction $3\hat{u} = 3 \overbrace{\frac{1}{\|\vec{u}\|}\vec{u}}^{\text{norm } 1}$

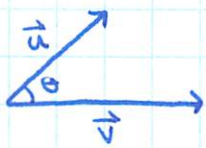
$$= \frac{3}{\sqrt{6}}[-1,1,2]$$

(2) opposite direction $-3\hat{u} = -3 \frac{1}{\|\vec{u}\|}\vec{u}$

$$= -\frac{3}{\sqrt{6}}[-1,1,2]$$

Def'n: Let $\vec{u}, \vec{v} \neq 0$ with a <u>common initial point</u>, they determine a unique angle:

$0 \leq \theta \leq \pi$ called the angle between $\vec{u}$ and $\vec{v}$

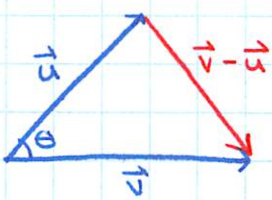


$\theta = 0$ if $\vec{u}, \vec{v}$ have the same direction

$\theta = \pi$ if $\vec{u}, \vec{v}$ have opposite direction

$\theta = \pi/2$ if $\vec{u}, \vec{v}$ are orthogonal ($\perp$)

Law of cos: $c^2 = a^2 + b^2 - 2ab\cos\theta$



$$\|\vec{v}-\vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

$$\|\vec{u}\|\,\|\vec{v}\|\cos\theta = 1/2\left(\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{v}-\vec{u}\|^2\right)$$

$$\|\vec{u}\|\,\|\vec{v}\|\cos\theta = 1/2\left(u_1^2+u_2^2+\dots+u_n^2+v_1^2+\dots+v_n^2 - \underbrace{(v_1-u_1)^2}_{-(v_1^2-2v_1u_1+u_1^2)} - (v_2-u_2)^2 - \dots - (v_n-u_n)^2\right)$$

$$= 1/2\left(--(2v_1u_1 + 2v_2u_2 + \dots + 2v_nu_n\right)$$

$$= v_1u_1 + v_2u_2 + \dots + v_nu_n$$

$\therefore$ <u>Dot product</u> of $\vec{u} = [u_1, u_2, \dots, u_n]$ and $\vec{v} = [v_1, v_2, \dots, v_n]$:

$$\boxed{\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + \dots + u_nv_n}$$ $\Rightarrow$ yields scalar

$$\|\vec{u}\|\,\|\vec{v}\|\cos\theta = \vec{u}\cdot\vec{v}$$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} \xrightarrow[0\leq\theta\leq\pi]{} \theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right)$$

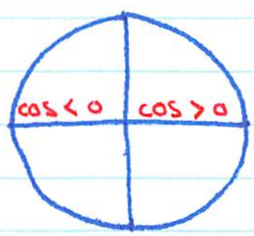
Remarks

(1) $\vec{u} \cdot \vec{v} \in \mathbb{R}$ (dot product of 2 vectors yields a scalar)

(2) $\vec{u} \cdot \vec{v} = 0 \Rightarrow \vec{u} \perp \vec{v}$ $\qquad \cos(\pi/2) = 0$

(3) $\vec{u} \cdot \vec{v} > 0 \Rightarrow \vec{u}, \vec{v}$ form an acute angle $(0 < \theta < \pi/2)$

(4) $\vec{u} \cdot \vec{v} < 0 \Rightarrow \vec{u}, \vec{v}$ form an obtuse angle $(\pi/2 < \theta < \pi)$



$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$

Theorem:

(1) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

(2) $\vec{u} \cdot \vec{0} = 0$

(3) $\boxed{\vec{u} \cdot \vec{u} = \|\vec{u}\|^2}$ $\qquad u_1 u_1 + u_2 u_2 + \dots + u_n u_n$

(4) $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v})$

(5) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ $\qquad u_1(v_1 + w_1) + \dots + u_n(v_n + w_n)$

ex: Find the cosine of the angle between $\vec{u} = [1, 0, -1]$ and $\vec{v} = [2, -1, 1]$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{2+0-1}{\sqrt{2}\,\sqrt{6}} = \frac{1}{2\sqrt{3}}$$

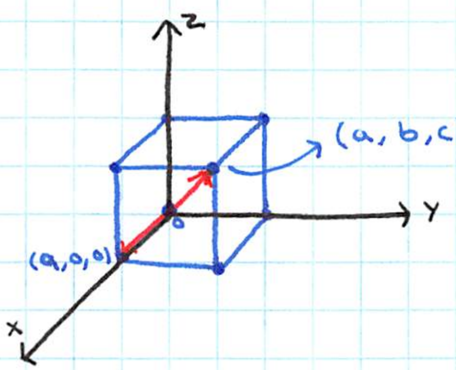
ex: Find "a" s.t $[1, -2a, 1] \perp [0, 3, -5]$

$$0 = \vec{u}\cdot\vec{v} = 0 + (-6a) - 5$$

$$0 = (-6a) + (-5)$$

$$\boxed{a = -5/6}$$

ex: Find the angle between a diagonal of a cube and one of its sides. Cube of length "a"



(a, b, c), but for cube $a = b = c$

$$\left.\begin{aligned}\vec{s} &= [a, 0, 0]\\ \vec{d} &= [a, a, a]\end{aligned}\right\} \cos\theta = \frac{a^2+0+0}{\sqrt{a^2}\,\sqrt{3a^2}} = \frac{a^2}{a^2\sqrt{3}}$$

$$\cos\theta = 1/\sqrt{3} \Rightarrow \theta = \cos^{-1}(1/\sqrt{3})$$

ex: Show that $\underbrace{\|\vec{u}+\vec{v}\|^2}_{\text{RHS}} = \underbrace{\|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\vec{u}\cdot\vec{v}}_{\text{LHS}}$

$$\text{RHS} = \|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$$

$$= \underbrace{\vec{u}\cdot\vec{u}}_{\|\vec{u}\|^2} + 2\vec{u}\cdot\vec{v} + \underbrace{\vec{v}\cdot\vec{v}}_{\|\vec{v}\|^2}$$

Matrix

Def'n: A matrix is a rectangular array of numbers

$$A = \begin{array}{cl} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} & \begin{array}{l} \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \\ \\ \leftarrow \text{row } m \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{column } 1 & \text{column } 2 & \text{column } n \end{array} & \end{array}$$

① count rows (m)

② count columns (n)

$\underset{m \times n}{A}$

a_{ij} entry

i: row #, j: column #

$\Rightarrow$ size $m \times n$

special case: * You can think of a $1 \times n$ matrix as a row vector in $\mathbb{R}^n$

ie. $A = [a_{11}, a_{12}, \ldots, a_{1n}] = \vec{r}_1$

$\therefore$ the rows of an $m \times n$ matrix A are vectors in $\mathbb{R}^n$ and

$$A = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_m \end{bmatrix}$$

* You can think of an $m \times 1$ matrix as a colum vector in $\mathbb{R}^m$

ie. $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} = \vec{c}_1$

$\therefore$ the columns of A are vectors in $\mathbb{R}^m$ and

$$A = [\vec{c}_1 \mid \vec{c}_2 \mid \cdots \mid \vec{c}_n]$$

ex: car dealership sales

$$\begin{array}{c} \\ \text{Smith} \\ \text{Jones} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Subcompact} & \text{Compact} & \text{Luxury} \end{array} \\ \begin{bmatrix} 3 & 2 & 1 \\ 5 & 1 & 0 \end{bmatrix} \end{array}$$

size: $\underbrace{2}_{\#\text{ rows}} \times \underbrace{3}_{\#\text{ columns}}$

ex: $\begin{bmatrix} -2 & 5 \\ 0 & -2 \\ 3 & 6 \end{bmatrix}$

size: 3×2

$a_{21} = 0$

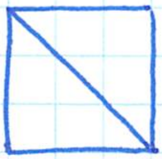
$a_{32} = 6$

$a_{23} = \times \rightarrow$ not there, no 3rd column

$\vec{r_1} = [-2, 5]$ $\vec{c_2} = \begin{bmatrix} 5 \\ -2 \\ 6 \end{bmatrix}$

We will say that A is a <u>square</u> if $\boxed{m=n}$

We may then define the <u>main diagonal</u> of A



$a_{11}, a_{22}, \ldots, a_{nn}$

<u>Note</u>: only for squares

We have $A = B$ only if the 2 matrices have the same size $(m \times n)$ and if their corresponding entries are the same

ex: $\begin{bmatrix} 0 & 2 & 7 \\ 1 & -3 & 5 \end{bmatrix} \neq \begin{bmatrix} 0 & 2 \\ 1 & -3 \end{bmatrix}$

ex: solve $\begin{bmatrix} a & -a \\ c & b \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 6 & 2 \end{bmatrix} \Rightarrow \begin{cases} a = 1 \\ b = 2 \\ c = 6 \end{cases}$

Operations

- **Sum** $A+B$

Note: only exists for matrices of the same size

$$[a_{ij}]+[b_{ij}] = [a_{ij}+b_{ij}]$$

ex: $\begin{bmatrix} 0 & 2 & 7 \\ 1 & -3 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 9 \\ 2 & -2 & 8 \end{bmatrix}$

- **Scalar multiplication**

Let K be any scalar, then

$$K[a_{ij}] = [Ka_{ij}]$$

ex: $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \\ 1 & 0 \end{bmatrix}$ $\quad 2A = \begin{bmatrix} 8 & 4 \\ 2 & 6 \\ 2 & 0 \end{bmatrix}$ $\quad -A = \begin{bmatrix} -4 & -2 \\ -1 & -3 \\ -1 & 0 \end{bmatrix}$

- **Zero matrix**

O_{mn} where each entry is 0

$\therefore$ Clearly $A + O = A$

$A - A = O$

Properties

Let A, B, C be $m \times n$, $K, \ell \in \mathbb{R}$

(i) $A+B = B+A$

(ii) $(A+B)+C = A+(B+C)$

(iii) $A+O = A$

(iv) $A-A = O$

(v) $K(A+B) = KA+KB$

(vi) $(K+\ell)A = KA+\ell A$

(vii) $K(\ell A) = (K\ell)A$

(viii) $1A = A$

ex: $5X - \begin{bmatrix} 7 & 2 & 3 \\ 9 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ 5 & 3 & -5 \end{bmatrix} - X$

$$6X = \begin{bmatrix} 1 & -2 & 4 \\ 5 & 3 & -5 \end{bmatrix} + \begin{bmatrix} 7 & 2 & 3 \\ 9 & -4 & 1 \end{bmatrix}$$

$$X = 1/6 \begin{bmatrix} 8 & 0 & 7 \\ 14 & -1 & -4 \end{bmatrix}$$

- **Transpose**

exchange rows and columns

Let A be $m \times n = [a_{ij}]$

define A^T $n \times m = [a_{ji}]$

ex: $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}$ 1 column → row at a time

$2 \times 3 \longrightarrow 3 \times 2$

Theorem: (i) $(A^T)^T = A$

(ii) $(KA)^T = KA^T$

(iii) $(A+B)^T = A^T + B^T$

Def'n: A is called symetric if $\boxed{A = A^T}$ $\circledast$

→ square $(n \times n)$ and symetric w.r.t the main diagonal

ex: $\begin{bmatrix} 2 & 4 & 5 \\ 4 & 6 & 3 \\ 5 & 3 & -1 \end{bmatrix} = A$ $A^T = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 6 & 3 \\ 5 & 3 & -1 \end{bmatrix}$

ex: Show that if A and B are symetric $n \times n$ matrices, then $2A + B$ is symetric

Note: A, B symetric $\therefore \circledast$ $A = A^T$, $B = B^T$

$$\underbrace{(2A+B)^T = 2A^T + B^T \overset{\circledast}{=} (2A+B)}_{\text{symetric}}$$

Matrix multiplication

Let A be $m \times n$ and B be $n \times p$

equal

Define their product to be the $m \times p$ matrix s.t :

dot product

$$(AB)_{ij} = (i^{th} \text{ row } A) \cdot (j^{th} \text{ column } B)$$

in $\mathbb{R}^n$

$$= [a_{i1}, a_{i2}, \ldots, a_{in}] \cdot \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

$$= a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

ex:

$$\underbrace{\begin{bmatrix} 3 & 2 & 4 & 0 \\ -1 & 0 & 2 & 1 \\ 0 & -1 & 4 & 6 \end{bmatrix}}_{3 \times 4} \begin{array}{cc} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ \multicolumn{2}{c}{\underbrace{\begin{bmatrix} 3 & 2 \\ 5 & 4 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}}_{4 \times 2}} \end{array} = \underbrace{\begin{bmatrix} {}^{1}\ 23 & {}^{2}\ 14 \\ {}^{3}\ -1 & {}^{4}\ 0 \\ {}^{5}\ -1 & {}^{6}\ 8 \end{bmatrix}}_{3 \times 2}$$

Row $\cdot$ Column
A B

CAREFUL: Even when AB and BA both exist and have the same size

$$\left[\quad \underbrace{\underset{m\times n}{A}\ \underset{n\times p}{B}}_{m\times p} \qquad\qquad \underbrace{\underset{n\times p}{B}\ \underset{m\times n}{A}}_{\substack{p=m \\ n\times n}} \quad\right]$$

Need $p = n = m$

(squares of same size)

usually $\boxed{AB \neq BA}$ (even if same size)

* If $AB = BA$, we say that A and B are <u>commute</u>

<u>Note</u>: The order AB or BA is important.

$\Rightarrow$ AB and BA mean different things.

Define the $n \times n$ identity matrix

$$I_n = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \; 1 \end{bmatrix} \qquad a_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{(diagonal)}$$

square

ex: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Theorem:

1. $A(BC) = (AB)C$
2. $A(B+C) = AB + AC$ (keep order: $AB \neq BA$, $AC \neq CA$)
3. $(B+C)A = BA + CA$
4. $K(AB) = (KA)B = A(KB)$ constant can move order, matrices must keep order
5. $IA = A$, $AI = A$
6. $\boxed{(AB)^T = B^T A^T}$ A and B flip

proof: Say $A: m \times n$ $B: n \times p$

- Compare sizes

$\underbrace{(\underset{m\times n}{A}\underset{n\times p}{B})^T}_{m \times p}$ is $p \times m$ | $\underset{p \times n}{(B)^T} \underset{n \times m}{(A)^T}$ is $p \times m$

- Compare entries

$((AB)^T)_{ij} = (AB)_{ji} = (j^{th} \text{ row } A) \cdot (i^{th} \text{ column } B)$

$((B^T)(A^T))_{ij} = (i^{th} \text{ row } B^T) \cdot (j^{th} \text{ column } A^T)$
$= (i^{th} \text{ column } B) \cdot (j^{th} \text{ row } A)$ =

Remarks

1. Say A is $m \times n$ and $\vec{v}$ is a vector in $\mathbb{R}^n$ we can think of $\vec{v}$ as an $n \times 1$ matrix

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad A = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_m \end{bmatrix} = [\vec{c}_1 \mid \vec{c}_2 \mid \cdots \mid \vec{c}_n]$$

we can compute:

$$\boxed{\underbrace{A_{m\times n}\ \vec{v}_{n\times 1}}_{m\times 1} = \underbrace{\begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_m \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\vec{v}} = \begin{bmatrix} \vec{r}_1 \cdot \vec{v} \\ \vec{r}_2 \cdot \vec{v} \\ \vdots \\ \vec{r}_m \cdot \vec{v} \end{bmatrix}}$$

more importantly, if $A = [\vec{c}_1 \mid \vec{c}_2 \mid \cdots \mid \vec{c}_n]$, then

$$\boxed{A\vec{v} = v_1\vec{c}_1 + v_2\vec{c}_2 + \cdots + v_n\vec{c}_n} = [\vec{c}_1 \mid \vec{c}_2 \mid \cdots \mid \vec{c}_n] \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

so the vector $A\vec{v}$ is a linear combination of the column vectors of A, where the coefficients are the components of $\vec{v}$.

2. Given $B = [\vec{c}_1 \mid \vec{c}_2 \mid \cdots \mid \vec{c}_p]$ $\quad n \times p$

then $AB = A[\vec{c}_1 \mid \vec{c}_2 \mid \cdots \mid \vec{c}_p]$

$$\boxed{AB = [A\vec{c}_1 \mid A\vec{c}_2 \mid \cdots \mid A\vec{c}_p]}$$

$$\rightarrow AB = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_m \end{bmatrix} [\vec{c}_1 \mid \vec{c}_2 \mid \cdots \mid \vec{c}_n] = \begin{bmatrix} \vec{r}_1 \cdot \vec{c}_1 & \vec{r}_1 \cdot \vec{c}_2 & \cdots & \vec{r}_1 \cdot \vec{c}_n \\ \vec{r}_2 \cdot \vec{c}_1 & \vdots & & \vdots \\ \vdots & \vdots & & \\ \underbrace{\vec{r}_m \cdot \vec{c}_1}_{A\vec{c}_1} & \cdot & & \vec{r}_m \vec{c}_n \end{bmatrix}$$

Systems of Linear Equations

A) Introduction

Suppose that in $\mathbb{R}^2$, we are asked to write the vector $\begin{bmatrix} 8 \\ 9 \end{bmatrix}$ as a linear combination of the vectors: $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

<u>Note</u>: we know it is possible bc
$\vec{u}$ and $\vec{v}$ are not $//$
$\therefore$ Span $\{\vec{u}, \vec{v}\} = \mathbb{R}^2$

<u>Recall</u>: $//$ if $\vec{u} = k\vec{v}$

$$\begin{bmatrix} 8 \\ 9 \end{bmatrix} = x\vec{u} + y\vec{v}$$

$$= x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2x \\ x \end{bmatrix} + \begin{bmatrix} y \\ 3y \end{bmatrix}$$

$$\begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 2x + y \\ x + 3y \end{bmatrix}$$

component level

$$\left.\begin{cases} 2x + y = 8 \\ x + 3y = 9 \end{cases}\right\}$$ Linear system

variable

constant

coefficient

unique solution $\left\{ \begin{array}{c} x \\ y \end{array} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$

<u>Note</u>: 2×2 - linear system

2 equations

2 variables

Generally:

An $m \times n$ - linear system is:

equations — # variables — All variables are degree 1

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

a_{ij} — i: # equations, j: # variables

- coefficient matrix: $A = \underset{m \times n}{[a_{ij}]}$
- column vector of constants: $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$
- vector of variables: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Def'n: System in matrix form: $\boxed{A\vec{x} = \vec{b}}$

ex: $\underset{2 \times 2}{\overset{A}{\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}}} \overset{\vec{x}}{\begin{bmatrix} x \\ y \end{bmatrix}} = \overset{\vec{b}}{\begin{bmatrix} 8 \\ 9 \end{bmatrix}}$

Def'n: Augmented matrix of system: $\underset{m \times (n+1)}{\boxed{[A \mid \vec{b}]}}$

ex: $\underset{2 \times 3}{\left[\begin{array}{cc|c} 2 & 1 & 8 \\ 1 & 3 & 9 \end{array}\right]}$

$[\vec{c}_1 | \vec{c}_2 | \dots | \vec{c}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{b} \longrightarrow \left[\begin{array}{cccc|c} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n & \vec{b} \\ \downarrow & \downarrow & & \downarrow & \downarrow \end{array}\right]$

$\qquad x_1 \quad x_2 \quad \dots \quad x_n$

- each first entry of each row will be multiplied by x_1,
- second $\rightarrow x_2$

Matrix $\longrightarrow$ Linear System

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b}}$$

component level:

$$\begin{cases} b_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ b_2 = \dots \\ \vdots \\ b_m = a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{cases}$$

we get many equations (linear equations) from $A\vec{x} = \vec{b}$ bc components have to correspond to eachother on both ends of "=".
$\rightarrow$ The set of all these equations forms the linear <u>system</u>.

of solutions

- 2×2 linear system
  → looking at intersection between 2 lines in $\mathbb{R}^2$.
  - 0 sol'n — // lines (never touch)
  - 1 sol'n — 1 intersection
  - ∞ sol'n — superposed lines

- we will show that an $m \times n$ system has:

  0 sol'n — inconsistent

  $\left.\begin{array}{l}1 \text{ sol'n} \\ \infty \text{ sol'n}\end{array}\right\}$ consistent

FIVE STAR

B) Gaus - Jordan and Gaussian Elimination

Basic method to solve a system:
- replace it by a simpler one with the same solution set (= equivalent system)

3 elementary row operations : when performed on an augmented matrix will yield an equivalent system.

1. Multiply a row by a non-zero constant
   $R_i \leftarrow kR_i$
2. Interchange 2 rows
   $R_i \leftrightarrow R_j$
3. Add a multiple of a row to another row
   $R_i \leftarrow R_i + kR_j$

Def'n: A matrix (not necessarily connected to a system) is said to be in Row-echelon form (REF) if:

1. All rows consisting entirely of zeros are below the non-zero rows

2. The first non-zero entry from the left in each row (called pivot or leading entry) is to the right of the pivots from any row above it. (ie. only zeros under a pivot).

ex: $\begin{bmatrix} 0 & 2 & 3 & 6 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, ex: $\begin{bmatrix} 1 & 6 & 3 \\ 0 & 0 & 5 \end{bmatrix}$

Def'n: The matrix is in reduced row-echelon form (RREF) if in addition:

3. All pivots are equal to 1

4. Each pivot is the only non-zero entry in its column.

ex: $\begin{bmatrix} 1 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, ex: $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

even if these are non-zero, there is no pivot in that column

ex: 0 matrix

FIVE STAR

$$A \underset{\longleftarrow}{\overset{\longrightarrow}{}} B \Rightarrow A \text{ and } B \text{ are row equivalent}$$

sequence of elementary row operations.

<u>Theorem</u>: Every matrix is row-equivalent to a matrix in (R) REF.

<u>Remark</u>: A matrix has infinetly many REFs BUT only <u>one</u> RREF (position and # of pivots are fixed)

Why is RREF nice?

Say the augmented matrix to our system is row equivalent to:

$$\begin{array}{c} \begin{array}{ccccc|c} & s & & & t & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 4 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 2 & 4 \end{array}\right] \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \end{array}$$

RREF

Let us call the variables correspondind to:

→ pivot columns: <u>leading or dependent variables</u> x_1 x_3 x_4

→ and other columns: <u>parameters or free variables</u> x_2 x_5

$$\begin{cases} \underline{x_1} + 4s + 3t = -2 & \Rightarrow x_1 = -2 - 4s - 3t \\ \underline{x_3} - 2t = 0 & \Rightarrow x_3 = 2t \\ \underline{x_4} + 2t = 4 & \Rightarrow x_4 = 4 - 2t \end{cases}$$

<u>General sol'n</u>

$s, t \in \mathbb{R}$

$$\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{bmatrix} -2-4s-3t \\ s \\ 2t \\ 4-2t \\ t \end{bmatrix} = \vec{x}$$

<u>Particular sol'n</u>

ex: $s = t = 0$

$$\begin{bmatrix} -2 \\ 0 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

How to get to (R)REF?

Gaus - Jordan → RREF
Gaussian → REF

<u>step 1</u>: locate leftmost nonzero column and get a non-zero entry (pivot) at its top.
→ if going to RREF make pivot equal to 1.

ex: divide by pivot: 2 6 0 1 $R_1 \leftarrow \frac{1}{2} R_1$ 1 3 0 ½

<u>or</u> to avoid fractions add multiples of rows <u>below</u>:

3 6 0 1 $R_1 \leftarrow R_1 - R_2$ 1 4 -6 -2
2 2 6 3   2 2 6 3

↳ must be: R_1 - "row below"

<u>step 2</u>: Add suitable multiples of the pivot row to the rows below (and above if RREF) to create zeros below the pivot (and above if RREF).

ex: Solve :

$$\begin{cases} x_3 + x_4 + x_5 = 0 \\ x_1 + x_2 - 2x_3 - x_5 = 1 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0 \\ 2x_1 + 2x_2 - x_3 + x_5 = 3 \end{cases}$$

step 1

$$\left[\begin{array}{ccccc|c} 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 1 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 3 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

step 2

$$\left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} \\ \\ R_3 \leftarrow R_3 + R_1 \\ R_4 \leftarrow R_4 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 1 \\ 0 & 0 & 3 & 0 & 3 & 1 \end{array}\right] \begin{array}{l} R_1 \leftarrow R_1 + 2R_2 \\ \\ \\ R_4 \leftarrow R_4 - 3R_2 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 1 \\ 0 & 0 & 0 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3 \leftarrow -\tfrac{1}{3} R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1/3 \\ 0 & 0 & 0 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \leftarrow R_1 - 2R_3 \\ R_2 \leftarrow R_2 - R_3 \\ \\ R_4 \leftarrow R_4 + 3R_3 \end{array}$$

$$\begin{array}{cccccc} & s & & & t & \\ \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 5/3 \\ 0 & 0 & 1 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 1 & 0 & -1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ x_1 & & x_3 & x_4 & & \end{array}$$

Note: 2 rows are multiples of eachother ∴ one of them becomes a zero row.

General sol'n

$$\begin{bmatrix} 5/3 - s - t \\ s \\ 1/3 - t \\ -1/3 \\ t \end{bmatrix} \qquad s, t \in \mathbb{R}$$

ex:

$$\begin{bmatrix} 1 & 3 & 6 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

× need pivot here

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 3 & 6 & 4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

$R_2 \leftarrow R_2 - R_4$

$$\begin{bmatrix} 1 & 3 & 6 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

$R_1 \leftarrow R_1 - 6R_2$

$R_4 \leftarrow R_4 - 2R_2$

$$\begin{bmatrix} 1 & 3 & 0 & 10 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$R_1 \leftarrow R_1 - 10R_3$

$R_2 \leftarrow R_2 + R_3$

$R_4 \leftarrow R_4 - 8R_3$

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Number of solutions in a system $A\vec{x} = \vec{b}$

Let A be any $m \times n$ matrix

Def'n: Rank of A $= rk(A) = r$

$= \#$ of pivots in any REF of A

" $\#$ of nonzero rows

(every nonzero row has pivot)

Nullity of A $= n - r$

$\#$ of columns $= \#$ of unknowns

$rk(A)$

Remark: if A is $m \times n$, then $rk(A) \leq m$ and $rk(A) \leq n$

(ie. at most 1 pivot per row / per column)

Theorem: Given an $m \times n$ system $A\vec{x} = \vec{b}$

1. The system is consistent (ie. has at least 1 sol'n)

   if: $rk([A \mid \vec{b}]) = rk(A)$

   $\Rightarrow$ no pivot in $\vec{b}$ column (to the right of the line)

   $\Rightarrow$ nonzero row has 0 in $\vec{b}$ column

2. Given that the system is consistent, then:

   $n = r \Rightarrow$ one sol'n (pivot in every column)

   $r < n \Rightarrow \infty$ sol'n (Note: $n - r = \#$ parameters $=$ nullity)

Def'n: A homogeneous system is one for which all constant terms are zero:

$$\boxed{A\vec{x} = \vec{0}}$$

ex: $\begin{cases} x_1 - x_2 = 0 \\ x_2 + 3x_3 = 0 \end{cases}$

Note: such a system is always consistent since it has $\vec{x} = \vec{0}$ as a sol'n. ($\therefore$ at least one sol'n)

In fact, $\vec{0}$ is called the trivial sol'n.

Theorem: An $m \times n$ $A\vec{x} = \vec{0}$ has only the trivial sol'n if $rk(A) = n$

(Recall: when $r = n$: only 1 sol'n)

Theorem: A homogeneous system with more variables than equations has ∞'ly many solutions.

proof: Say $m \times n$ with $n > m \geq r$

$\therefore n - r > 0 \Rightarrow$ # of paramerers

$\rightarrow$ if there are parameters

$\therefore \infty$'ly many sol'n.

Non-homogeneous system

$\boxed{A\vec{x} = \vec{b}}$ $\longrightarrow$ $\boxed{A\vec{x} = \vec{0}}$

its associated homogeneous system

ex:
$$\begin{cases} x_1 - 2x_2 - x_3 + 3x_4 = 0 \\ -2x_1 + 4x_2 + 5x_3 = 3 \\ x_1 - 2x_2 - 2x_3 + x_4 = -1 \end{cases}$$

RREF of

$A\vec{x} = \vec{b}$ vs $A\vec{x} = \vec{0}$

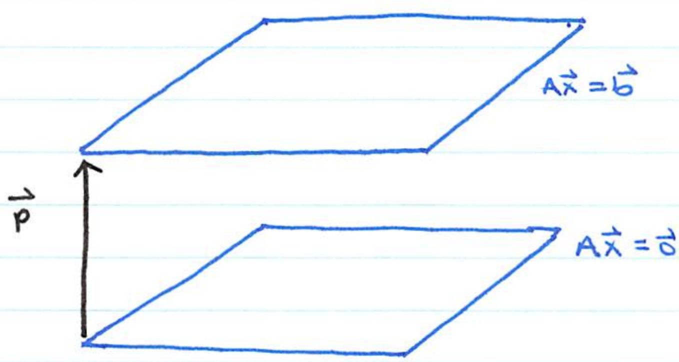
$$\begin{array}{c} \begin{array}{cccccc} & s & & t & & \end{array} \\ \left[\begin{array}{cccc|c} 1 & -2 & 0 & 5 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \qquad \begin{array}{c} \begin{array}{cccccc} & s & & t & & \end{array} \\ \left[\begin{array}{cccc|c} 1 & -2 & 0 & 5 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

General sol'n to $A\vec{x}=\vec{b}$ — Particular sol'n $A\vec{x}=\vec{b}$ — General sol'n $A\vec{x}=\vec{0}$

$$\begin{bmatrix} 1+2s-5t \\ s \\ 1-2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2s-5t \\ s \\ -2t \\ t \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \vec{p}$$

$s = t = 0$



c) Linear Systems as Vector Equations

Recall: Given the $m \times n$ - linear system
$A\vec{x} = \vec{b}$ where:

$$A = [\vec{c}_1 \mid \vec{c}_2 \mid \ldots \mid \vec{c}_n] \quad \text{and}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

linear combination

we have: $A\vec{x} = x_1\vec{c}_1 + x_2\vec{c}_2 + \ldots + x_n\vec{c}_n$

Consequently, the system is equivalent to the vector equation

$$\boxed{x_1\vec{c}_1 + x_2\vec{c}_2 + \ldots + x_n\vec{c}_n = \vec{b}} \quad \text{in } \mathbb{R}^m$$

which is another way to write $[\vec{c}_1 \mid \vec{c}_2 \mid \ldots \mid \vec{c}_n]\vec{x} = \vec{b}$

Theorem: The system $A\vec{x} = \vec{b}$ is consistent if $\vec{b}$ is a linear combination of the column vectors of A.
(ie. if it is in their span).

FIVE STAR

* Determine if $\vec{v}$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$ (ie. if $\vec{v}$ is in their span)

ex: determine whether $\vec{v} = [3, -12, -4]$ is a linear combination of $\vec{v}_1 = [1, -2, 0]$ and $\vec{v}_2 = [2, -1, 2]$

(here $\vec{v}_1 \not\parallel \vec{v}_2$ $\therefore$ we are asking if $\vec{v}$ is in their span $\Rightarrow$ the plane they generate in $\mathbb{R}^3$).

Solve: $x_1 \vec{v}_1 + x_2 \vec{v}_2 = \vec{v}$

$$x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -12 \\ -4 \end{bmatrix}$$

Augmented $\Leftrightarrow$

$$\begin{array}{c} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ -2 & -1 & -12 \\ 0 & 2 & -4 \end{array}\right] \\ \begin{array}{cc} x_1 & x_2 \end{array} \end{array} \qquad R_2 \leftarrow R_2 + 2R_1$$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 3 & -6 \\ 0 & 2 & -4 \end{array}\right] \qquad R_2 \leftarrow R_2 - R_3$$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{array}\right] \qquad \begin{array}{l} R_1 \leftarrow R_1 - 2R_2 \\ R_3 \leftarrow R_3 - 2R_2 \end{array}$$

$$\begin{array}{c} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cc} x_1 & x_2 \end{array} \end{array}$$

unique sol'n: $x_1 = 7$
$x_2 = -2$

$\therefore$ $\vec{v} = 7\vec{v}_1 + (-2)\vec{v}_2$

$\therefore$ yes

ex: determine if $\vec{v} = [3,-4,-6]$ is in the span of $\vec{v}_1 = [1,2,3]$ $\vec{v}_2 = [-1,-1,-2]$ and $\vec{v}_3 = [1,4,5]$

Solve $\vec{v} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 2 & -1 & 4 & -4 \\ 3 & -2 & 5 & -6 \end{array}\right] \quad \begin{array}{l} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & 2 & -10 \\ 0 & 1 & 2 & -15 \end{array}\right] \quad R_3 \leftarrow R_3 - R_1$$

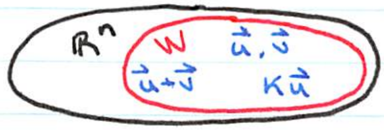
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & 2 & -10 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

pivot on right $\therefore$ 0 sol'ns

$\therefore$ $\vec{v}$ is not a linear comb. of $\vec{v}_1, \vec{v}_2, \vec{v}_3$
(there are no x_1, x_2, x_4 that would make $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{v}$)

(Note: we conclude that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are coplanar since they do not span $\mathbb{R}^3$)

Def'ns:



1. A subset W in $\mathbb{R}^n$ is closed under vector addition if for all $\vec{u}, \vec{v}$ in W $(\in W)$, $\vec{u} + \vec{v}$ is also in W.

2. A subset W of $\mathbb{R}^n$ is closed under scalar vector multiplication if for all $\vec{u}$ in W and all K in $\mathbb{R}$, $K\vec{u}$ is also in W.

3. A non-empty subset W of $\mathbb{R}^n$ is called subspace in $\mathbb{R}^n$ if it is closed under vector addition and scalar multiplication.

Remark: Since $0\vec{u} = \vec{0}$, a subspace always contains the zero vector.

To verify that a subset is a subspace:

- is $\vec{0}$ in it?
- closed under vector $\oplus$?
- closed under scalar $\otimes$?

Note: (1) $\{\vec{0}\}$ is the zero subspace of $\mathbb{R}^n$

(2) $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$

ex: check if the following subsets are subspaces of $\mathbb{R}^3$

(1) $\{[a,0,b]\}$

↳ all triples with 2^{nd} component $=0$

- $\vec{0}$ in it : ✓

- vector + :

$[a_1,0,b_1]$, $[a_2,0,b_2] \in W$

$[a_1,0,b_1]+[a_2,0,b_2]$

$= [a_1+a_2, 0, b_1+b_2]$ ✓

- scalar × :

$K[a,0,b] = [Ka, 0, Kb]$ ✓

∴ it is a subspace

(2) $\{[x,y,1]\}$

- $\vec{0}$ in it ? <u>No</u>

∴ not a subspace

(3) $\{[x,y,z] \text{ s.t } z \geq 0\}$

* inequality ⇒ scalar ×

$[0,0,1] \in W$

$-1[0,0,1]=[0,0,-1]$

↳ not in W

∴ not a subspace

(4) $\{[x,y,z] \text{ s.t } |x|=|y|\}$

* Abs. value ⇒ vector +

$[1,1,0] \in W$

$[2,-2,0] \in W$

<u>But</u> : $[1,1,0]+[2,-2,0] = [3,-1,0]$

$|x| \neq |y|$

↳ not in W

∴ not a subspace

(5) $\{[x,y,z] \text{ s.t } xy = xz\}$

Note: true if $y = z$

but, if $x = 0$ $\therefore$ y not necessarily equal to z

$[1, 1, 1] \in W$ $(1)1 = (1)1 = 1$

and $[0, 2, 1] \in W$ $(0)2 = (0)1 = 0$

but: $[1,1,1] + [0,2,1] = [1,3,2]$

$\hookrightarrow$ not in W

$\therefore$ not a subspace.

Theorem: Let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ in $\mathbb{R}^n$

Then $W = \text{span } S$ is a subspace of $\mathbb{R}^n$

Recall: span S is all the possible linear combinations of $\vec{v}_i$'s

proof:

- $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \dots + 0\vec{v}_r$

- closed under vector $+$:

$$(a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_r\vec{v}_r) + (b_1\vec{v}_1 + b_2\vec{v}_2 + \dots + b_r\vec{v}_r)$$

$$= \underbrace{(a_1+b_1)\vec{v}_1 + (a_2+b_2)\vec{v}_2 + \dots + (a_r+b_r)\vec{v}_r}_{\text{is a linear combination} \; \therefore \text{ in } W}$$

- closed under scalar $\times$:

$$K(a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_r\vec{v}_r)$$

$$= \underbrace{Ka_1\vec{v}_1 + Ka_2\vec{v}_2 + \dots + Ka_r\vec{v}_r}_{\text{is a linear combination} \; \therefore \text{ in } W}$$

Def'n: • $W = \text{span } S$ is called the linear span of S

• we say that the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ span W or that they are generators for W or that they form a spanning set for W.

Note: it can be shown that all subspaces of $\mathbb{R}^n$ can be linear spans.

Def'n: Let A be $m \times n$ with $A = [\vec{c}_1 \mid \vec{c}_2 \mid \dots \mid \vec{c}_n]$

Then the subspace: $\text{span}\{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\}$ is called the column space of A (in $\mathbb{R}^m$)

column space of A = $\text{span}\{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\}$

Consequently, $\vec{b}$ is in the column space of A

if values $x_1, x_2, \dots x_n$ exist s.t

$$\vec{b} = x_1\vec{c}_1 + x_2\vec{c}_2 + \dots + x_n\vec{c}_n$$

(ie. if $A\vec{x} = \vec{b}$ is consistent)

(*) Do the vectors $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r}\}$ in $\mathbb{R}^n$ span $\mathbb{R}^n$?

Take any vector $[x_1, x_2, \ldots, x_n]$ in $\mathbb{R}^n$.

Is $[x_1, x_2, \ldots, x_n]$ a linear comb. of $\vec{v_1}, \vec{v_2}, \ldots$?

ie. is $\left[\begin{array}{c|c|c|c|c} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_r} & \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} \end{array}\right]$ consistent? Always?

→ if there is a pivot in every row
($rk([\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r}]) = n$)
∴ span $\mathbb{R}^n$ ⇒ always consistent
→ can yield any $[x_1, x_2, \ldots, x_n]$ in $\mathbb{R}^n$

→ if I have a zero row
∴ there is a condition for consistency
$[\,0 \quad 0 \quad 0 \mid \underbrace{\cdots}_{=0}\,]$

This condition of form $ax + by + cz = 0$ (in $\mathbb{R}^3$)
is the condition for the vectors that
can be spanned (they form a plane)
→ condition = plane equation

<u>Note</u>: if I want to check if a vector
is in a plane (have plane equation),
simply plug it into the plane equation
and check if it works.

* What 2 vectors span the plane $ax+by+cz=0$?

ie. what vectors $[x,y,z]$?
solve for $[x,y,z]$

$$\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

↳ Augmented : $\left[\begin{array}{ccc|c} a & b & c & 0 \end{array}\right]$

→ factor out parameters from sol'n
⇒ obtain 2 lin. indep vectors
↳ these 2 will span the plane

ex: Find 2 vectors spanning the plane $x+4y-5z=0$

$$\begin{array}{cccc} & s & t & \\ \left[1 \right. & 4 & -5 & \left| \; 0 \right] \end{array}$$ RREF

Gen. Sol'n $\begin{array}{c} x \\ y \\ z \end{array}\begin{bmatrix} -4s+5t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ vectors $\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$ span the plane.

ex: The basis vectors : $\hat{e}_1, \hat{e}_2, \hat{e}_3, \ldots, \hat{e}_n$ span $\mathbb{R}^n$ because :
any vector in $\mathbb{R}^n$ $[a_1, a_2, \ldots, a_n]$ can be written as $[a_1, a_2, \ldots, a_n] = a_1\hat{e}_1 + a_2\hat{e}_2 + \ldots + a_n\hat{e}_n$

ex: Determine if $\vec{v}_1 = [2,-1,3]$, $\vec{v}_2 = [4,1,2]$ $\vec{v}_3 = [8,-1,8]$ span $\mathbb{R}^3$

Take any vector $[x,y,z]$
is $[x,y,z]$ a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$?

$$\left[\begin{array}{ccc|c} 2 & 4 & 8 & x \\ -1 & 1 & -1 & y \\ 3 & 2 & 8 & z \end{array}\right] \quad R_1 \leftarrow R_1 + R_2$$

is the system consistent

$$\left[\begin{array}{ccc|c} 1 & 5 & 7 & x+y \\ -1 & 1 & -1 & y \\ 3 & 2 & 8 & z \end{array}\right] \quad \begin{array}{l} R_2 \leftarrow R_2 + R_1 \\ R_3 \leftarrow R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 7 & x+y \\ 0 & 6 & 6 & x+2y \\ 0 & -13 & -13 & z-3x-3y \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 7 & x+y \\ 0 & -13 & -13 & z-3x-3y \\ 0 & 6 & 6 & x+2y \end{array}\right] \quad R_2 \leftarrow R_2 + 2R_3$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 7 & x+y \\ 0 & -1 & -1 & z-x+y \\ 0 & 6 & 6 & x+2y \end{array}\right] \quad R_3 \leftarrow R_3 + 6R_2$$

zero column

$$\left[\begin{array}{ccc|c} 1 & 5 & 7 & x+y \\ 0 & -1 & -1 & z-x+y \\ 0 & 0 & 0 & \underline{-5x+8y+6z} \end{array}\right]$$

∴ No : not $\underline{\text{always}}$ consistent ∴ doesn't span ALL of $\mathbb{R}^3$
(Note: vectors are coplanar)

These vectors span the plane :
$\boxed{-5x+8y+6z=0}$ (the condition for $[x,y,z]$ to be in span)

$\vec{v}_i$'s failed to span entire space bc REF $A = [\vec{v}_1 | \cdots | \vec{v}_r]$ had a zero row.

Theorem: Let A be $m \times n$, then TFAE (the following statements are equivalent)

1. For all $\vec{b}$ in $\mathbb{R}^m$, the system $A\vec{x} = \vec{b}$ is consistent

2. The column space of A $(= \text{span}\{\vec{c}_1, \dots, \vec{c}_n\})$ is $\mathbb{R}^m$

3. $rk(A) = m$ (one pivot in every row. (no zero row))

Recap:

- $\vec{v}$ = lin. comb. of $\vec{v}_1, \dots, \vec{v}_r$?

  solve $x_1\vec{v}_1 + \dots + x_r\vec{v}_r = \vec{v}$

- Do $\vec{v}_1, \dots, \vec{v}_r$ span $\mathbb{R}^n$?

  $rk(\underbrace{[\vec{v}_1 | \dots | \vec{v}_r]}_{A''}) = n$ (pivot in every row)

  $A\vec{x} = \vec{b}$ always consistent

Note: $\mathbb{R}^n$

n : # of components in vector

column vector $\vec{c}_1 = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

n = number of rows

$\therefore$ $rk(A) = n \Rightarrow$ pivot in every row.

D) Homogeneous Systems

Theorem: The solution set to a homogeneous system $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$ called the:
solution space to the system
OR nullspace of A

Recall: sol'n set is the "$\vec{x}$"

proof: • $\vec{0}$ in it? $\Rightarrow$ $A\vec{0} = \vec{0}$ ✓ (trivial sol'n in it)

• closed under vector + :
Let $\vec{v}_1 \in W \Rightarrow A\vec{v}_1 = \vec{0}$
$\vec{v}_2 \in W \Rightarrow A\vec{v}_2 = \vec{0}$
is $(\vec{v}_1 + \vec{v}_2)$ also in W ?
$A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2 = \vec{0} + \vec{0} = \vec{0}$ ✓

• closed under scalar x :
Let $\vec{v} \in W \Rightarrow A\vec{v} = 0$

$A(k\vec{v}) = k(A\vec{v}) = k\vec{0} = \vec{0}$ ✓

One can switch back and forth between the representation of a subspace as a linear span and as a solution set.

Linear Span
(generators: $\vec{v}_i$'s)

vectors in subspace are lin. comb

Sol'n Space
(homogeneous)

vectors in subspace are sol'ns to eqcns

Solve
factor out parameters

ex: $\begin{bmatrix} 2s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

generators

put in REF $\left[\begin{array}{c|c|c|c} \vec{v}_1 & \cdots & \vec{v}_r & \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} \end{array}\right]$

ask for consistency
→ conditions = homogeneous sys

ex: $\left[\begin{array}{cc|c} 1 & 5 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & x_2 + x_1 + 3x_3 \\ 0 & 0 & x_4 - x_1 + 5x_3 \end{array}\right]$

hom. system. $\begin{cases} x_2 + x_1 + 3x_3 = 0 \\ x_4 - x_1 + 5x_3 = 0 \end{cases}$

ex: Find generators for the sol'n space

$$\begin{cases} x_1 + x_3 + x_4 = 0 \\ 3x_1 + 2x_2 + 5x_3 + x_4 = 0 \\ 4x_2 + 4x_3 - 4x_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 3 & 2 & 5 & 1 & 0 \\ 0 & 4 & 4 & -4 & 0 \end{array}\right] \qquad R_2 \leftarrow R_2 - 3R_1$$

rows multiples of eachother

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & -2 & 0 \\ 0 & 4 & 4 & -4 & 0 \end{array}\right] \qquad R_3 \leftarrow R_3 - 2R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad R_2 \leftarrow \tfrac{1}{2} R_2$$

$$\begin{array}{c} \begin{array}{ccccc} & & s & t & \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

General Sol'n

$$\begin{bmatrix} -s-t \\ -s+t \\ s \\ t \end{bmatrix} \quad s,t \in \mathbb{R} \quad \overset{*}{=} \quad \underbrace{s \overset{s=1,\ t=0}{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}} + t \overset{s=0,\ t=1}{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}}_{\text{lin. comb.}}$$

$\therefore \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ span the sol'n space.

generators

ex: express:

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\} \quad \text{in} \quad \mathbb{R}^4$$

as a sol'n space.

A vector $\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is in W if

we can find scalars k_1, k_2, k_3 s.t

$$k_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

constants

variables

$\Leftrightarrow$ Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & x_1 \\ 0 & 1 & 1 & x_2 \\ -1 & 0 & 1 & x_3 \\ 1 & 0 & -1 & x_4 \end{array}\right] \quad \begin{array}{l} R_3 \leftarrow R_3 + R_1 \\ R_4 \leftarrow R_4 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & x_1 \\ 0 & 1 & 1 & x_2 \\ 0 & 2 & 2 & x_1 + x_3 \\ 0 & -2 & -2 & -x_1 + x_4 \end{array}\right] \quad \begin{array}{l} R_3 \leftarrow R_3 - 2R_2 \\ R_4 \leftarrow R_4 + 2R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & x_1 \\ 0 & 1 & 1 & x_2 \\ 0 & 0 & 0 & x_1 - 2x_2 + x_3 \\ 0 & 0 & 0 & -x_1 + 2x_2 + x_4 \end{array}\right]$$

conditions

$\therefore \vec{v}$ will be in span (ie system is consistent) if:

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ -x_1 + 2x_2 + x_4 = 0 \end{cases}$$

homogeneous system

E) Non-Homogeneous System

Theorem: if $\vec{v}$ is a particular sol'n to $A\vec{x} = \vec{b}$, then for any sol'n $\vec{w}$ of its associated homogeneous system $(A\vec{x} = \vec{0})$:

1. $\vec{v}_0 + \vec{w}$ is a sol'n of $A\vec{x} = \vec{b}$
2. and all its sol'ns arise this way

proof: (1) $A(\vec{v}_0 + \vec{w}) = \underbrace{A\vec{v}_0}_{\vec{b}} + \underbrace{A\vec{w}}_{\vec{0}} = \vec{b} + \vec{0} = \vec{b}$

(2) Let $\vec{v}$ be any sol'n to $A\vec{x} = \vec{b}$
(show $\vec{v} = \vec{v}_0 + \vec{w}$ for some $\vec{w}$)

$$\vec{v} = v_0 + \overbrace{(\vec{v} - \vec{v}_0)}^{\vec{w}}$$

$\hookrightarrow$ $A(\vec{v} - \vec{v}_0) = A\vec{v} - A\vec{v}_0 = \vec{b} - \vec{b} = \vec{0}$

ex:
$$\begin{cases} 2x_1 + x_2 - x_3 - 4 = -1 \\ 3x_1 + x_2 + x_3 - 2x_4 = -2 \\ -x_1 - x_2 + 2x_3 + x_4 = 2 \\ -2x_1 - x_2 \quad + 2x_4 = 3 \end{cases}$$

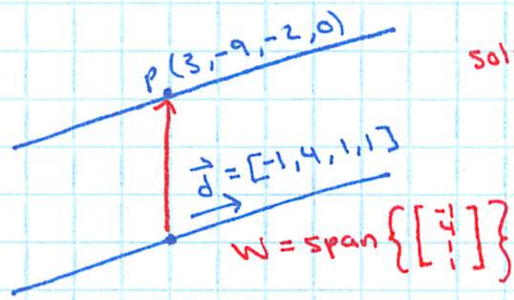
has General sol'n:

particular sol'n $s=0$

General sol'n to associated hom. sys

$$\begin{bmatrix} 3-s \\ -9+4s \\ -2+s \\ s \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ -9 \\ -2 \\ 0 \end{bmatrix}}_{\vec{v}_0} + \underbrace{s \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}}_{\vec{w}}$$

And:
$$\begin{bmatrix} 3 \\ -9 \\ -2 \\ 0 \end{bmatrix} + \underbrace{s \overbrace{\begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}}^{\vec{d}\,:\ \text{direction vector}}}_{W = \text{span} \Rightarrow \text{line in } \mathbb{R}^4}$$



sol'n set $= \vec{p} + W$

Linear Independence

A set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ in $\mathbb{R}^n$ is said to be linearly independent if the equation $x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_r\vec{v}_r = \vec{0}$ has only the trivial sol'n.

If not, they are linearly dependent and any sol'n gives a dependency relationship.
↳ Non-zero sol'n

$\boxed{x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_r\vec{v}_r = \vec{0}}$
- only sol'n: $\vec{x} = \vec{0}$ — lin. indept.
- ∞'ly many sol'n — lin. dept.

Note: Always at least one sol'n bc homogeneous system always has $\vec{x} = \vec{0}$ as 1 sol'n.

→ Linearly independent means only one way of writing $\vec{0}$ as a linear combination of the vectors

Let $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_r$ be r vectors in $\mathbb{R}^n$

$\therefore$ $x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_r\vec{v}_r = \vec{0}$ corresponds to $n \times r$ homogeneous system $A\vec{x} = \vec{0}$, $A = [\vec{v}_1 | \ldots | \vec{v}_r]$

if: $rk(A) = \#$ vectors (pivot in every column) $\rightarrow$ no parameters: 1 soln — Lin. Indept.

$rk(A) < \#$ vectors (some parameters) — Lin Dept

To be Linearly indept, we need:

$\#$ vectors $\leq n$

(ie. # columns) — # of components (ie. # rows)

bc if more columns than rows, some columns will not have pivots $\rightarrow$ $\therefore$ more than one sol'n

<u>Theorem</u>: if you have ℓ vectors in $\mathbb{R}^n$

and $\ell > n$, then

$\rightarrow$ the vectors are linearly dependent.

ex: 9 vectors in $\mathbb{R}^7$
$\rightarrow$ always linearly dependent.

<u>Remarks</u>:

1. If any of the $\vec{v}_i$'s is $\vec{0}$ then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ is dependent.

2. If S is lin. dept and $S \subseteq S'$, then S' is also lin. dept.

3. S is lin. dept. iff any vectors in their span can be <u>uniquely</u> written as a linear combination of the vectors in S.

Theorem: Non-zero vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ form a lin. dependent set in $\mathbb{R}^n$ iff one of them is a linear combination of the others.

Both directions

proof: Say we have dependency relationship:

$$K_1\vec{v}_1 + K_2\vec{v}_2 + \ldots + K_r\vec{v}_r = \vec{0} \qquad \text{with } K \neq 0$$

$$\Rightarrow \vec{v}_1 = \underbrace{-\frac{K_2}{K_1}\vec{v}_2 - \frac{K_3}{K_1}\vec{v}_3 - \ldots - \frac{K_r}{K_1}\vec{v}_r}_{\text{lin combination}}$$

proof: say $\vec{v}_1 = K_2\vec{v}_2 + K_3\vec{v}_3 + \ldots + K_r\vec{v}_r$

$$\Rightarrow 1\vec{v}_1 - K_2\vec{v}_2 - K_3\vec{v}_3 - \ldots - K_r\vec{v}_r = \vec{0}$$

$1 \neq 0$ $\therefore$ dept. relationship

(1) $\{\vec{u}\}$ lin. indep $\Rightarrow \vec{u} \neq \vec{0}$ bc if $x_1\vec{u} = \vec{0}$ and $\vec{u} = \vec{0}$ $\therefore x_1 \neq 0$ is possible

(2) $\{\vec{u}, \vec{v}\}$ lin. indep $\Rightarrow$ not // not colinear

dep $\Rightarrow$ if $\vec{u}$ = lin comb $\vec{v}$
$\vec{u} = K\vec{v}$
$\hookrightarrow \vec{u} \parallel \vec{v}$

(3) $\{\vec{u}, \vec{v}, \vec{w}\}$ lin. indep $\Rightarrow$ not coplanar

From matrix point of view

Theorem: Let A be $m \times n$, TFAE

1. The column vectors of A are linearly independent.

2. In REF, A has a pivot in every column.

3. Each consistent system $A\vec{x} = \vec{b}$ has a unique sol'n.

4. The RREF of A is $\begin{bmatrix} I_n \\ O \end{bmatrix}$ } $m-n$ rows of zeros

5. Nullity of A = $n - rk(A) = 0$

Remark: Any non-zero vector in the nullspace of A (ie sol'n space to $A\vec{x} = \vec{0}$) gives the coefficients of a dependency relationship between the column vectors of A.

$$A\vec{x} = \vec{0}$$

$$\Rightarrow \underbrace{x_1\vec{c}_1 + x_2\vec{c}_2 + \dots + x_n\vec{c}_n = \vec{0}}_{\text{if } \vec{x} \neq \vec{0},\ \therefore \text{ dependency relationship.}}$$

ex: $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & -1 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \vec{0}$

$\nearrow$ $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ in nullspace of A

$\searrow$ $2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-1)\begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \vec{0}$

$2\,\vec{c}_1 - 1\,\vec{c}_2 + 1\,\vec{c}_3 = \vec{0}$

Basis and Dimension

Say $W = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$

if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ are lin. dependent, then one of these vectors, say $\vec{v}_1$, is a lin. combination of the others.

$\therefore \quad W = \text{span}\{\vec{v}_2, \vec{v}_3, \ldots, \vec{v}_r\}$

we can repeat this step until we get a lin. independent spanning set.

Def'n: A basis of a subspace W is a lin. independent spanning set.

Theorem: Any 2 basis for W have the same number of vectors. This number is called the dimension of W.

In $\mathbb{R}^n$, we have the "standard" basis

$$\{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\} \Rightarrow \dim(\mathbb{R}^n) = n$$

Subspaces of $\mathbb{R}^n$

dim 0 $\{\vec{0}\}$

dim 1 line

dim 2 plane

$\vdots$

$n > 3 \begin{cases} \text{dim } n-1 & \text{hyperplane} \\ \text{dim } n & \mathbb{R}^n \end{cases}$

ex: Find a subset of $\vec{v}_1 = [1,-1,5,2]$, $\vec{v}_2 = [-2,3,1,0]$, $\vec{v}_3 = [4,-5,9,4]$, $\vec{v}_4 = [-7,10,-2,-2]$

which is a basis for their span.

Find dependency relationship between these vectors:

Solve: $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 = \vec{0}$

$$\left[\begin{array}{cccc|c} 1 & -2 & 4 & -7 & 0 \\ -1 & 3 & -5 & 10 & 0 \\ 5 & 1 & 9 & -2 & 0 \\ 2 & 0 & 4 & -2 & 0 \end{array}\right] \begin{array}{l} \\ R_2 \leftarrow R_2 + R_1 \\ R_3 \leftarrow R_3 - 5R_1 \\ R_4 \leftarrow R_4 - 2R_1 \end{array} \quad \left[\begin{array}{cccc|c} 1 & -2 & 4 & -7 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 11 & -11 & 33 & 0 \\ 0 & 4 & -4 & 12 & 0 \end{array}\right] \begin{array}{l} R_1 \leftarrow R_1 + 2R_2 \\ \\ R_3 \leftarrow R_3 - 11R_2 \\ R_4 \leftarrow R_4 - 4R_2 \end{array}$$

$$\begin{array}{ccccc} & & s & t & \\ \hline \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
$$\begin{array}{cccc} x_1 & x_2 & & \end{array}$$

Both are non-zero vectors in nullspace of A

$\therefore$ gives dependency relationship

General Sol'n: $\begin{bmatrix} -2s+t \\ s-3t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{cases} -2\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0} \\ \vec{v}_1 - 3\vec{v}_2 + \vec{v}_4 = \vec{0} \end{cases} \Rightarrow \begin{array}{l} \vec{v}_3 = 2\vec{v}_1 - \vec{v}_2 \\ \vec{v}_4 = -\vec{v}_1 + 3\vec{v}_2 \end{array}$$

vectors correspond to parameters column.

vectors correspond to pivot column.

> **Note**: vectors that we "keep" are not final vectors in RREF.
> → originally written $\vec{v}_i$'s

$\vec{v}_3, \vec{v}_4$ are in span of $\vec{v}_1, \vec{v}_2$

$\therefore$ can eliminate

→ keep only $\vec{v}_1, \vec{v}_2$

$\therefore \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} = \text{span}\{\underline{\vec{v}_1, \vec{v}_2}\}$

vectors correspond to pivots

$\therefore$ lin. indep. $\therefore$ BASIS

2 vectors in basis

$\therefore \dim = 2$ $\quad (= \text{rk}(\vec{v}_1 | \vec{v}_2 | \vec{v}_3 | \vec{v}_4))$

It can be shown that the set of generators we get for a sol'n space by solving it and factoring out the parameters of the general sol'n are always lin. indept.
→ They form a basis for the sol'n space

∴ if by factoring out parameters, the vectors obtained form a basis
→ # of parameters = # vectors in basis
" dimension

ex: Sol'n space to:

$$\{x_1 - x_2 + 2x_3 - 10x_4 + x_5 = 0 \quad \text{in } \mathbb{R}^5$$

4 parameters

$$\begin{array}{cccccc} & s & t & q & r & \\ \left[1 \right. & -1 & 2 & -10 & 1 & \left| \; 0 \right] \\ x_1 & & & & & \end{array}$$

General sol'n

$$\begin{bmatrix} s - 2t + 10q - r \\ s \\ t \\ q \\ r \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + q\begin{bmatrix} 10 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

↑ ↑ ↑ ↑

These 4 vectors form basis
∴ dim = 4

Note: dim = 4 = n − 1
= 5 − 1

⇒ ∴ hyperplane
(= sol'n space to ONE equation in $\mathbb{R}^n$ with n>3)

Recap: Dimension = # of vectors in basis

Given ... → Linear span dim = rk

→ Sol'n space dim = # parameters
= nullity

W

Linear span of $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$

$A = [\vec{v}_1 | \vec{v}_2 | \cdots | \vec{v}_r]$

W

column space of A
(ie span of $\vec{v}_i$'s)

To find basis, put A in REF

basis = columns corresponding to pivot columns

Dim = rank $[\vec{v}_1 | \cdots | \vec{v}_r]$

Sol'n space to hom. system

$A\vec{x} = \vec{0}$

W

nullspace of A
(ie. sol'n to $A\vec{x} = \vec{0}$)

To find basis, solve system : RREF.
Factor out parameters in general sol'n
→ vectors of basis

Dim = # parameters
= nullity