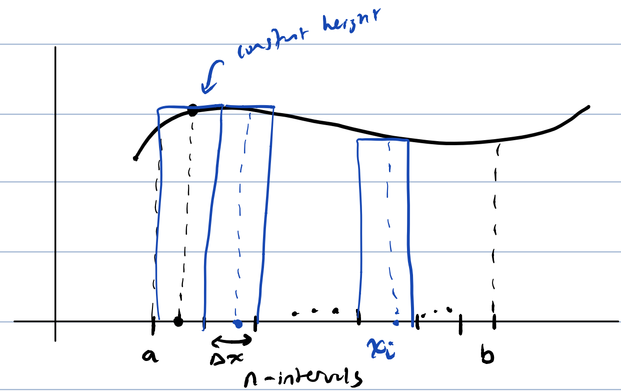
Multiple Integrals

Recall: $\displaystyle\int_a^b f(x)\,dx$ → signed area under curve ① (geometric interpretation $f(x)$: height)

→ displacement (arc length)

→ mass ← important in higher dimensions ②

① 

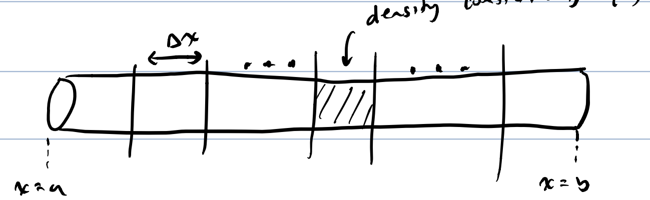
$$\Delta x = \frac{b-a}{n}$$

$$A \approx \sum_{i=1}^{n} bh = \sum_{i=1}^{n} \Delta x\, f(x_i^*)$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \int_a^b f(x)\,dx$$

② Mass! $f(x)$: density at x

Thin rod



chop into n pieces

Assumption: $f(x)$ = density $\approx$ constant on each piece
(per cm)

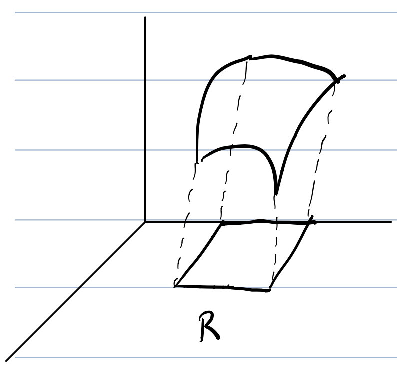
$$\text{Mass} \approx \sum_{i=1}^{n} \text{density} * \text{length}$$

$$= \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

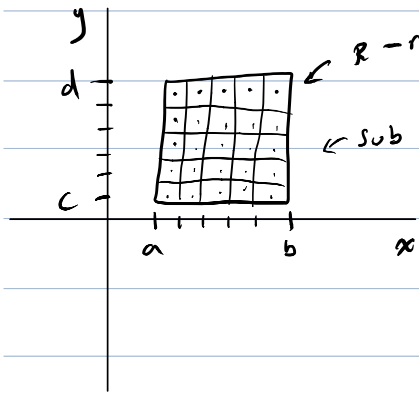
$$\text{Mass} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \int_a^b f(x)\,dx$$

For $z = f(x,y)$ as <u>height</u>



$\iint_R f(x,y)\,dA$ represents volume under the surface

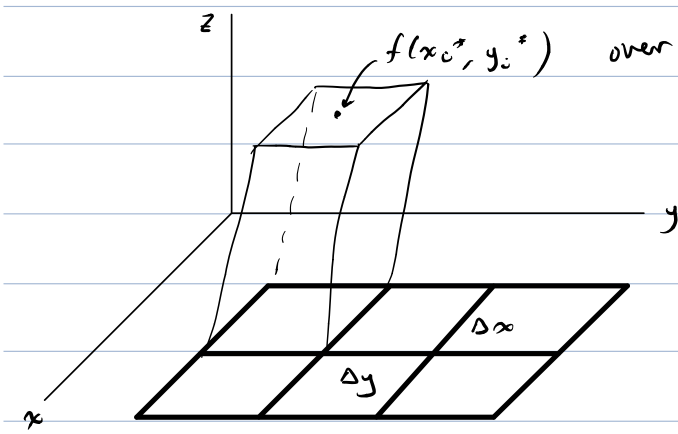


R - rectangle: $[a,b] \times [c,d]$ & cross, but not cross product

sub rectangles

chose an evaluation pt in each sub rectangle (x_i^*, y_i^*)

→ approx - $f(x,y) = f(x_i^*, y_i^*)$
constant on subrectangle



$f(x_i^*, y_i^*)$ over the i^{th} subrectangle is a prism

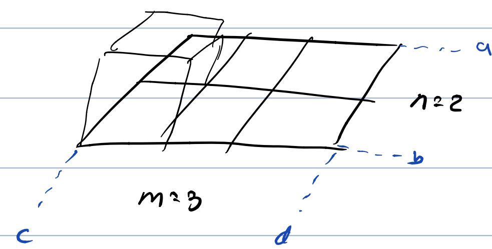
$$V_{prism} = b \cdot w \cdot h$$
$$= \Delta y \, \Delta x \, f(x_i^*, y_i^*)$$
$$\approx \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i^*, y_j^*) \, \Delta x \, \Delta y$$

Then, $\lim\limits_{n,m\to\infty} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i^*, y_j^*)\, \Delta x \Delta y = V = \iint\limits_R f(x,y)\, dA$

April 27

<u>Recall:</u> Double Riemann Sum

Interpreting $z=f(x,y)$ as height



$$\lim_{n,m\to\infty} \sum_{j=1}^{m} \sum_{i=1}^{n} \underbrace{f(x_i^*, y_i^*)\, \Delta x, \Delta y}_{\text{volume of one prism}}$$

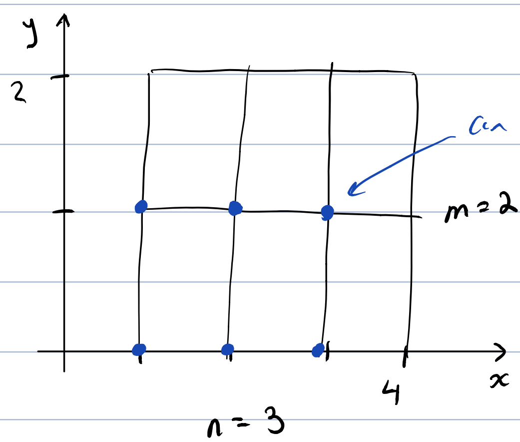
$$\overset{\text{Def}}{=} \iint\limits_R f(x,y)\, dA \quad \leftarrow \text{little bit of area}$$

R: rectangle

$R: [a,b] \times [c,d]$

interval in x — in y

ex. Approximate $\displaystyle\iint_R x^2 y \, dA$ on $[1,4] \times [0,2]$



Can chose to sample anywhere (could even be random)

Choose bottom-left corners

$\Delta x = \frac{4-1}{3} = 1$

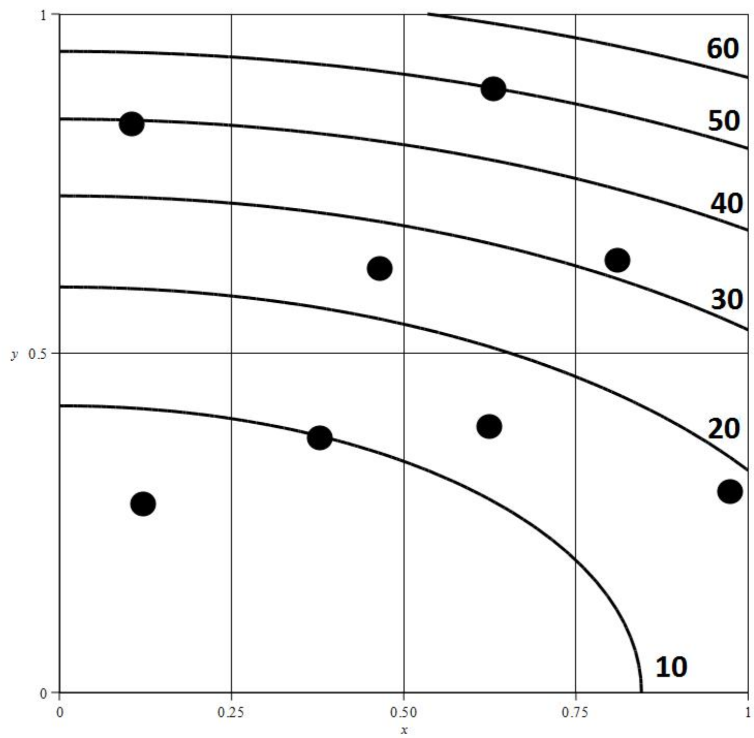
$\Delta y = \frac{2}{2} = 1$

$$V \approx \sum_{j=1}^{2} \sum_{i=1}^{3} f(x_i, y_i)\, \Delta x \, \Delta y$$

$$= \Big(\underbrace{1^2(0) + 2^2(0) + 3^2(0) + 1^2(1) + 2^2(1) + 3^2(1)}_{\text{heights}} \Big) \cdot \underbrace{1 \cdot 1}_{\text{area of base}}$$

$= 14$ units^3

The following is a contour plot of $\rho(x, y)$, a function with gives the density (in g/cm^2) of the (x, y)-plane at any point (x, y).



a) Use the given sample points to estimate the mass of the laminate rectangle $R : [0, 1] \times [0, 1]$.

$\Delta x = \frac{1}{4}$ $\quad m \approx \frac{1}{8}\left(5 + 10 + 15 + 20 + 25 + 30 + 40 + 50\right)$

$\Delta y = \frac{1}{2}$ $\quad = \frac{195}{8} \approx 25\,g$

b) Give an expression for the double integral that corresponds to the mass of this rectangle.

$$\iint_R \rho(x,y)\, dA$$

Answers

a) ≈ 24.3 g

b) $\displaystyle\iint_R \rho(x, y)\, dA$, or as an iterated integral, $\displaystyle\int_0^1\int_0^1 \rho(x, y)\, dx\, dy = \int_0^1\int_0^1 \rho(x, y)\, dy\, dx$

$\int_0^1 \int_0^1 P(x,y)\,dy\,dx$ ← iterated integral on $R: [a,b] \times [c,d]$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$= \int_a^b \boxed{\int_c^d f(x,y)\,dy}\,dx$$

keep x = const,
y then "disappears"
↳ expression in x

Fubini's Theorem (generalization of FTC II)

If $f(x,y)$ is continuous on $R: [a,b] \times [c,d]$,

$$\text{then} \iint f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

ex.
$$\int_1^4 \int_0^2 x^2 y\,dy\,dx = \int_1^4 \left(\int_0^2 x^2 y\,dy \right) dx$$

$$= \int_1^4 \left[\frac{x^2 y^2}{2} \right]_0^2 dx$$

$$= \int_1^4 2x^2\,dx$$

$$= \frac{2}{3} x^3 \Big|_1^4$$

$$= \frac{2}{3} 4^3 - \frac{2}{3}$$

$$= 42$$

<u>ex.</u> For you

$$\int_0^2 \int_1^4 x^2 y \, dx \, dy$$

$$= \int_0^2 \left[\frac{x^3}{3} y \right]_1^4 dy$$

$$= \int_0^2 y \left[\frac{4^3}{3} - \frac{1}{3} \right] dy$$

$$= \left(\frac{4^3}{3} - \frac{1}{3} \right) \frac{y^2}{2} \Big|_0^2$$

$$= \left(\frac{4^3}{3} - \frac{1}{3} \right) \cdot 2$$

$$= 42$$

2. $\displaystyle\iint_R x + y \, dA$, where R is the rectangle $[-1, 3] \times [2, 4]$.

$$\int_{-1}^{3}\int_{2}^{4} (x+y)\, dy\, dx$$

$$= \int_{-1}^{3} \left(xy + \frac{y^2}{2}\right)\Big|_2^4 dx$$

$$= \int_{-1}^{3} \left((4x+8) - (2x+2)\right) dx$$

$$= \int_{-1}^{3} (2x+6)\, dx$$

$$= \left(x^2 + 6x\right)\Big|_{-1}^{3}$$

$$= (9+18) - (1-6)$$

$$= 27 + 5 = 32$$

3. $\displaystyle\iint_R xy\sqrt{x^2+y^2}\, dA$, where R is the rectangle $[0,1]\times[0,1]$.

$$\int_0^1\int_0^1 xy\sqrt{x^2+y^2}\, dy\, dx$$

$$= \int_0^1 x\int_0^1 y\sqrt{x^2+y^2}\, dy\, dx$$

Aside: $\displaystyle\int y\sqrt{x^2+y^2}\, dy$

$$= \frac{1}{2}\int \sqrt{u}\, du$$

$$= \frac{1}{3}u^{3/2} + C$$

$$= \frac{1}{3}(x^2+y^2)^{3/2} + C$$

$$= \int_0^1 x\left[\frac{1}{3}(x^2+y^2)^{3/2}\right]_0^1 dx$$

$$= \int_0^1 \frac{1}{3}x\left[(x^2+1)^{3/2} - x^3\right] dx$$

$$= \underbrace{\frac{1}{3}\int_0^1 x(x^2+1)^{3/2}}_{\text{another substitution}} \quad \underbrace{-\frac{1}{3}\int_0^1 x^4\, dx}_{\text{power rule}}$$

$$\int_0^1\int_0^1 xy\sqrt{x^2+y^2}\, dx\, dy$$

let $u = x^2+y^2$
$du = 2x\, dx$

$$= \int_0^1 \frac{1}{2}\int_{y^2}^{1+y^2} y\sqrt{u}\, du\, dy$$

$$= \int_0^1 \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3} y u^{3/2}\Big|_{y^2}^{1+y^2} dy$$

$$= \int_0^1 \frac{1}{3}y\left[(1+y^2)^{3/2} - |y|^3\right] dy$$

$$= \int_0^1 \left(\frac{1}{3}y(1+y^2)^{3/2} - \frac{1}{3}y^6\right) dy$$

$$= \int_0^1 \frac{1}{3}y(1+y^2)^{3/2} dy - \int_0^1 \frac{1}{3}y^6\, dy$$

let $u = 1+y^2$
$du = 2y$

$$= \frac{1}{6}\int_1^2 u^{3/2}\, du - \frac{1}{3}\frac{y^7}{7}\Big|_0^1$$

$$= \frac{u^{5/2}}{5/2}\cdot\frac{1}{6}\Big|_1^2 - \frac{1}{3}\cdot\frac{1}{7}$$

Observation:

$$\int_a^b \int_c^d \underline{\underline{f(x)}}\, g(y)\, dy\, dx$$

$$= \int_a^b f(x) \underbrace{\left(\int_c^d g(y)\, dy \right)}_{\text{constant}} dx$$

$$= \int_c^d g(y)\, dy \cdot \int_a^b f(x)\, dx$$

April 30

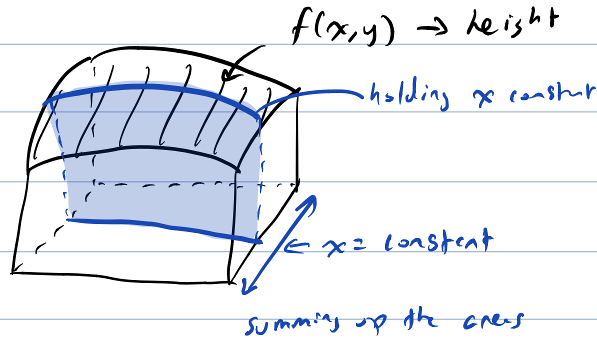
Recall: Double Integrals

$R: [a,b] \times [c,d]$

Fubini's Thm:

$$\iint_R f(x,y)\, dA = \int_a^b \underbrace{\left(\int_c^d f(x,y)\, dy \right)}_{\text{hold } x \text{ constant}} dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

Geometric Motivation



Average value

Cal II: f on $[a, b]$

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

Cal III: $f(x,y)$ on R

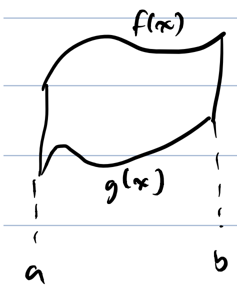
area of region $\rightarrow$ $$\frac{1}{A(R)}\iint_R f(x,y)\,dA$$

Double Integrals over General Regions

→ when R is not necessarily a rectangle

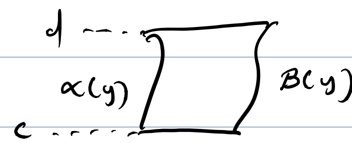
Two types of regions in $\mathbb{R}^2$:

Type I | **Type II**



$a \le x \le b$

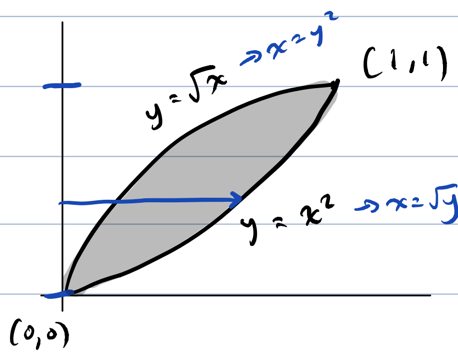
$g(x) \le y \le f(x)$



$c \le y \le d$

$\alpha(y) \le x \le \beta(y)$

Some regions may be both type I & type II



type I: $0 \le x \le 1$

$x^2 \le y \le \sqrt{x}$

type II: $0 \le y \le 1$

$y^2 \le x \le \sqrt{y}$

Result

If R is a type I region

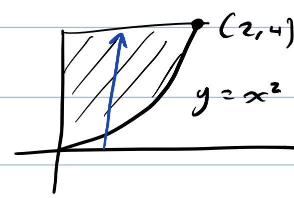
$$\iint_R P(x,y)\,dA = \int_a^b \int_{g(x)}^{f(x)} P(x,y)\,dy\,dx \neq \int_{g(x)}^{f(x)} \int_a^b P(x,y)\,dx\,dy$$

Note: order matters

If R is a type II region

$$\iint_R P(x,y)\,dA = \int_c^d \int_{\alpha(y)}^{\beta(y)} P(x,y)\,dx\,dy$$

ex. Evaluate $\iint_R x y^2\,dA$ where R is



constant bounds in terms of x

Type I: $0 \le x \le 2$

$x^2 \le y \le 4$

$$\int_0^2 \int_{x^2}^4 x y^2\,dy\,dx = \int_0^2 \frac{x y^3}{3}\Big|_{x^2}^{4}\,dx$$

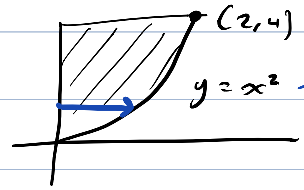
$$= \frac{1}{3}\int_0^2 \left(64x - x^7\right) dx$$

$$= \frac{1}{3}\left(32x^2 - \frac{1}{8}x^8\right)_0^2$$

...

or

Type II:



$$0 \le y \le 4$$
$$0 \le x \le \sqrt{y}$$

$$\int_0^4 \int_0^{\sqrt{y}} x y^2 \, dx \, dy$$

1. Find the average value of $f(x,y) = \dfrac{1+x^2}{1+y^2}$ on the rectangle $[0,1] \times [1,\sqrt{3}]$.

$$\frac{1}{\sqrt{3}-1} \int_1^{\sqrt{3}} \int_0^1 \frac{1+x^2}{1+y^2} \, dx \, dy$$

$$= \frac{1}{\sqrt{3}-1} \int_1^{\sqrt{3}} \frac{1}{1+y^2} \left[1 + \frac{x^3}{3} \right]_0^1 dy$$

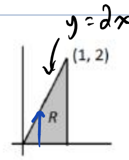
$$= \frac{1}{\sqrt{3}-1} \int_1^{\sqrt{3}} \frac{1}{1+y^2} \cdot \frac{1}{3} \, dy$$

$$= \frac{1}{\sqrt{3}-1} \cdot \frac{1}{3} \arctan y \Big|_{1}^{\sqrt{3}}$$

$$= \frac{1}{3(\sqrt{3}-1)} \left(\frac{\pi}{3} - \frac{\pi}{4} \right)$$

$$= \frac{1}{3(\sqrt{3}-1)} \cdot \frac{\pi}{12}$$

1. Evaluate $\displaystyle\iint_R (2x+2y)\, dA$ where R is given by



<u>Type I</u> :

$$0 \le x \le 1$$

$$0 \le y \le 2x$$

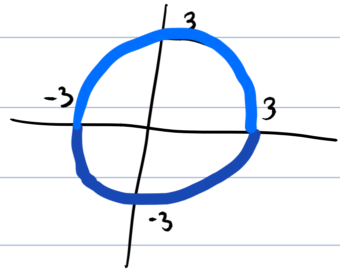
$$\int_0^1 \int_0^{2x} (2x+2y)\, dy\, dx = \int_0^1 (2x + y^2)\Big|_0^{2x}\, dx$$

$$= \int_0^1 (2x + 4x^2)\, dx$$

$$= \left[x^2 + \frac{4}{3} x^3 \right]_0^1$$

$$= 1 + \frac{4}{3} = \frac{7}{3}$$

2. Express as an iterated integral $\iint\limits_R f(x,y)\,dA$, where R is the region $x^2+y^2 \leq 9$.



$-3 \leq x \leq 3$

$-\sqrt{9-x} \leq y \leq \sqrt{9-x}$

Note: not necessarily symmetric

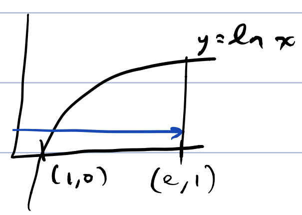
$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x,y)\,dy\,dx \neq 4\int_{0}^{3}\int_{0}^{\sqrt{9-x^2}} f(x,y)\,dy\,dx$$

or algebraically: $x^2+y^2=9$

$y = \pm\sqrt{\boxed{9-x^2}}$

$-3 \leq x \leq 3$

3. Set-up $\iint\limits_R xy\,dA$ where R is the region bounded by $y=\ln x$, $y=0$ and $x=e$ as both a Type I and a Type II integral. Pick one of the integrals to evaluate.



$1 \leq x \leq e$ or $0 \leq y \leq 1$

$0 \leq y \leq \ln x$ $\quad e^y \leq x \leq e$

$$\int_1^e \int_0^{\ln x} xy \, dy \, dx$$

$$\int_0^1 \int_{e^y}^{e} xy \, dx \, dy$$

$$= \frac{1}{2}\int_0^1 yx^2 \Big|_{e^y}^{e} dy = \frac{1}{2}\int_0^1 \left(ye^2 - ye^{2y}\right) dy$$

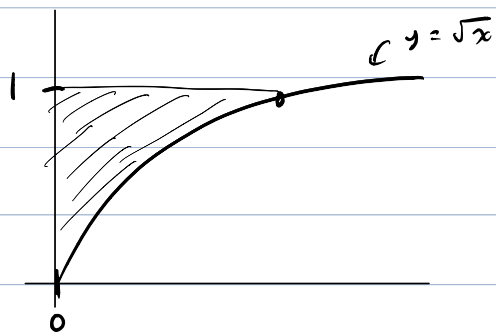
Integration by parts

$$= \frac{1}{2}\left[\frac{1}{2}y^2 e^2\right.$$

4. Consider $\int_0^1 \int_0^{y^2} f(x,y)\, dx\, dy$. Sketch the region of integration and reverse the order of integration.

$0 \leq x \leq y^2$

$0 \leq y \leq 1$



$0 \leq x \leq 1$

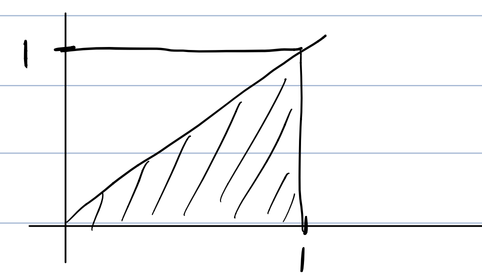
$x^2 \leq y \leq 1$

$$\int_0^1 \int_{x^2}^1 f(x,y)\, dy\, dx$$

5. Evaluate $\int_0^1 \int_y^1 e^{x^2} \, dx \, dy$ (hint: it is not possible in this order).

$y \le x \le 1$ $\qquad$ $0 \le x \le 1$

$0 \le y \le 1$ $\qquad$ $0 \le y \le x$

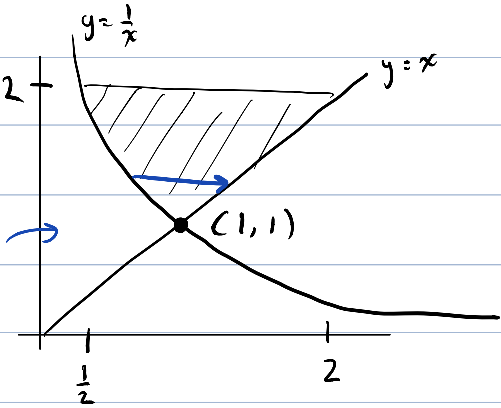


$$\int_0^1 \int_0^x e^{x^2} \, dy \, dx$$

$$= \int_0^1 y e^{x^2} \Big|_0^x dx = \int_0^1 x e^{x^2} dx = \frac{1}{2} e^{x^2} \Big|_0^1$$

$$= \frac{1}{2} e - \frac{1}{2}$$

6. Consider $\iint\limits_R f(x,y)\,dA$, where R is the region bounded by $y = x$, $y = 2$ and $xy = 1$. Express the double integral as iterated integral(s) in each possible order of integration.



also bounded by $x \geq 0$

$xy = 1$

$y = \frac{1}{x}$

$1 \leq y \leq 2$

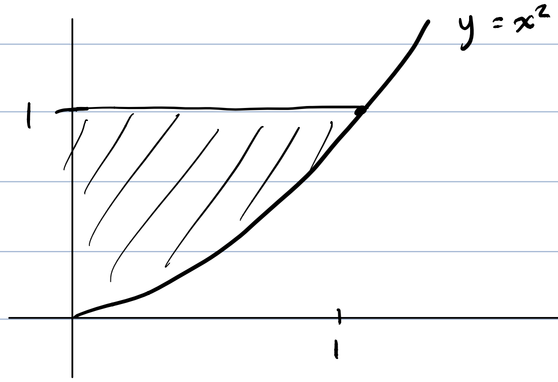
$\frac{1}{y} \leq x \leq y$

$\frac{1}{2} \leq x \leq 1$ & $1 \leq x \leq 2$

$\frac{1}{x} \leq y \leq 2$ & $x \leq y \leq 2$

$$\int_1^2 \int_{1/y}^{y} f(x,y)\,dx\,dy \quad \text{or} \quad \int_{\frac{1}{2}}^{1} \int_{\frac{1}{x}}^{2} f(x,y)\,dy\,dx + \int_1^2 \int_x^2 f(x,y)\,dy\,dx$$

7. Consider $\int_0^1 \int_{x^2}^1 \frac{x^3}{\sqrt{x^4+y^2}}\, dy\, dx$. Sketch the region R of integration, and reverse the order of integration. Then find the average value of $f(x,y) = \frac{x^3}{\sqrt{x^4+y^2}}$ on R (hint: one order of integration will be easier than the other).



$0 \leq y \leq 1$

$0 \leq x \leq \sqrt{y}$

Aside: $\int \frac{x^3}{\sqrt{x^4+y^2}}\, dx$

let $u = x^4 + y^2$

$du = 4x^3\, dx$

$= \frac{1}{4} \int \frac{du}{\sqrt{u}}$

$= \frac{1}{4} \cdot 2\, u^{1/2}$

$= \frac{1}{2} \sqrt{x^4 + y^2}$

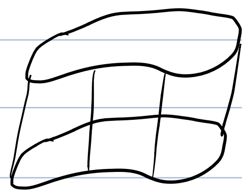
$$\int_0^1 \int_0^{\sqrt{y}} \frac{x^3}{\sqrt{x^4+y^2}}\, dx\, dy$$

$$= \frac{1}{2} \int_0^1 \left[\sqrt{x^4+y^2} \right]_0^{\sqrt{y}} dy$$

$$= \frac{1}{2} \int_0^1 (\sqrt{2} - 1)\, y\, dy$$

$$= \frac{1}{2} (\sqrt{2} - 1) \left. \frac{y^2}{2} \right|_0^1 = \frac{1}{4} (\sqrt{2} - 1)$$

Trick for Area → A(r)



$\leftarrow f(x,y) = \underline{1}$

Volume: $A_b \cdot h = A_b \cdot 1 = A_b = A(R)$

$$\iint_R 1 \, dA$$

$$A(R) = \int_0^1 \int_{x^2}^1 1 \cdot dy\, dx = \int_0^1 y \Big|_{x^2}^1 dx$$

$$= \int_0^1 (1 - x^2)\, dx$$

$$= \left[x - \frac{x^3}{3} \right]_0^1$$

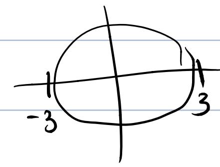
$$= 1 - \frac{1}{3} = \frac{2}{3}$$

Average: $\frac{1}{2/3} \cdot \frac{1}{4} (\sqrt{2} - 1) = \frac{3}{8} (\sqrt{2} - 1)$

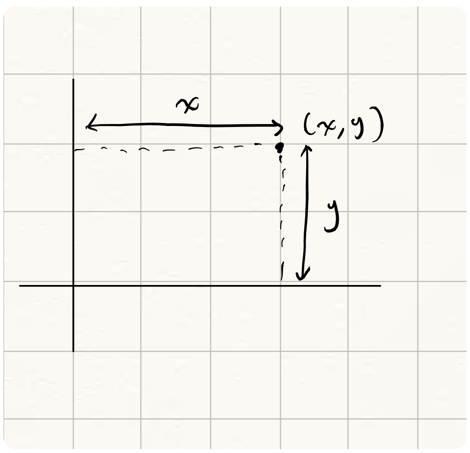
May 2

Polar Coordinates → circle symmetry

Recall $$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x,y)\,dy\,dx$$ ← much nicer in polar

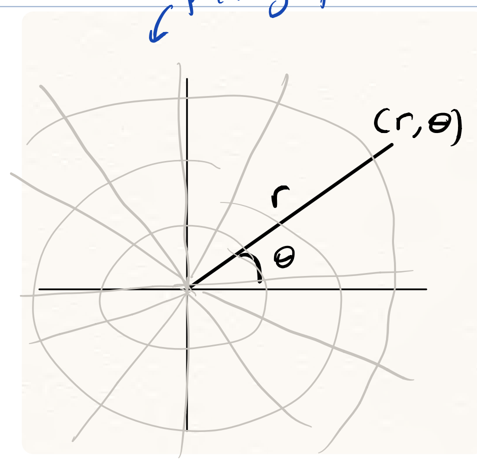


Cartesian - Rectangular

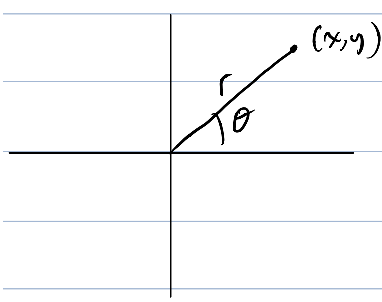


Polar Coordinates

polar graph lines



rectangular ↔ polar



$$\boxed{\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}}$$

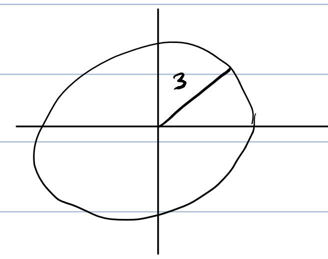
$$\begin{cases} \boxed{r = \sqrt{x^2+y^2}} \\ \dfrac{y}{x} = \dfrac{r\sin\theta}{r\cos\theta} = \tan\theta \\ \theta = \arctan\left(\dfrac{y}{x}\right) \end{cases}$$

ex. (1) circle: $x^2+y^2=9$

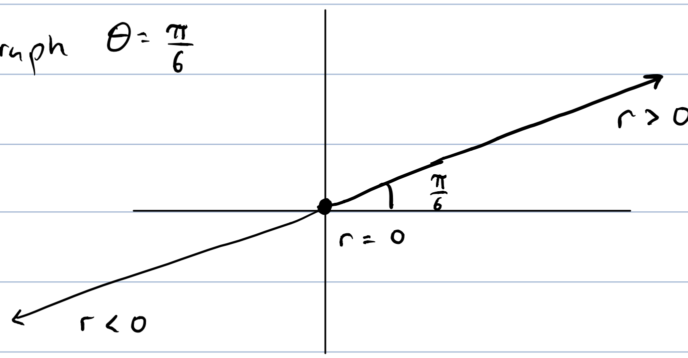
$$\left(\sqrt{x^2+y^2}\right)^2 = 9$$

$$r^2 = 9$$

$$r = 3 \quad (\text{or } r=-3)$$



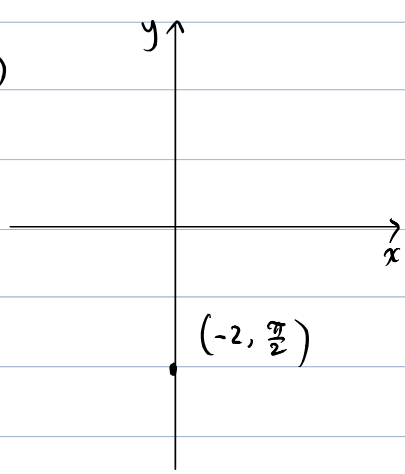
(2) Graph $\theta = \frac{\pi}{6}$



For you (1) a) Graph the pt $(r,\theta) = (-2, \frac{\pi}{2})$

b) Find another representation of $(-2, \frac{\pi}{2})$ with no negative coordinates

(1) a)



b) $(r,\theta) = (2, \frac{3\pi}{2})$

or

$\theta = \frac{3\pi}{2} + 2k\pi \quad k \geq 0$

not unique

② Translate to rectangular coordinates

$r = \cos\theta + 2\sin\theta$

(Hint multiply by r)

$r = r\cos\theta + 2r\sin\theta$

$x^2+y^2 = r\cos\theta + 2r\sin\theta$

$x^2+y^2 = x + 2y$

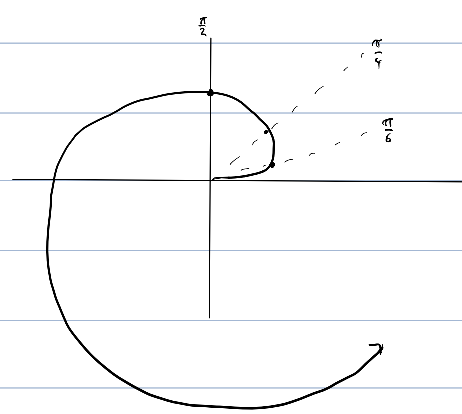
$x^2 - x + y^2 - 2y = 0$

$x^2 - x + \frac{1}{4} + y^2 - 2y + 1 = \frac{5}{4}$

$(x - \frac{1}{2})^2 + (y-1)^2 = \frac{5}{4}$

③ In polar form, graph $r = \theta$

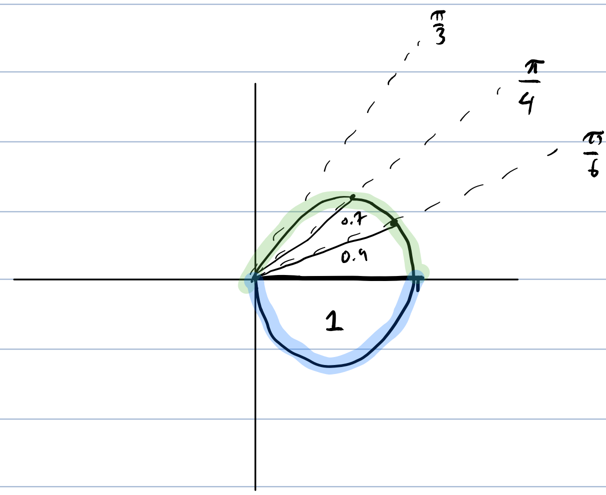
θ	r
0	0
$\frac{\pi}{6}$	$\frac{\pi}{6}$
$\frac{\pi}{2}$	$\frac{\pi}{2}$



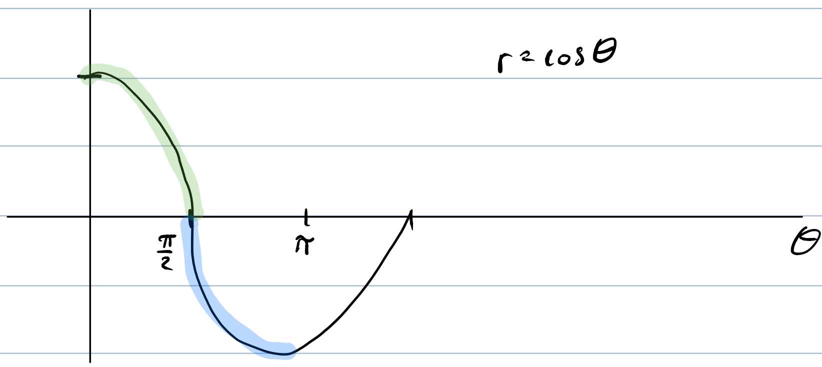
Functions $r = f(\theta)$

ex, Graph $r = \cos\theta$

θ	r
0	1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2} \approx 0.9$
$\frac{\pi}{4}$	$\frac{1}{2} \approx 0.7$
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	$-\frac{1}{2}$



Alternative to table of values

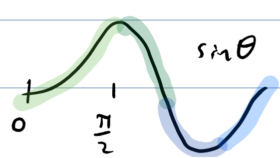


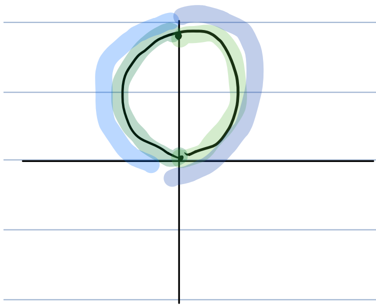
ex, For you

Graph: ① $r = 2\sin\theta$

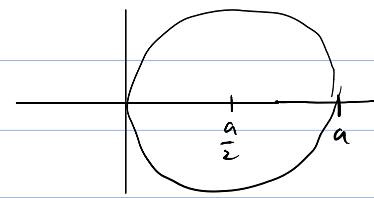
② $r = 3\cos 2\theta$

①

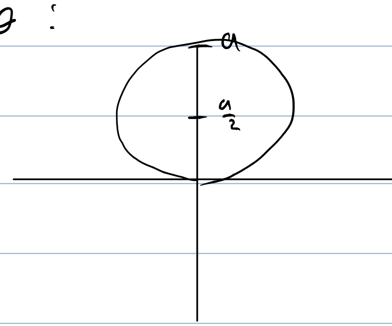




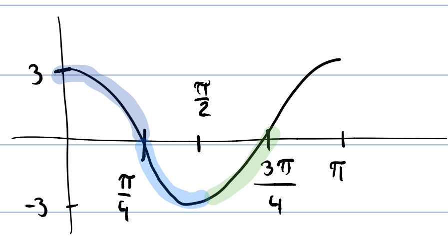
In general: $r = a\cos\theta$ is a circle with center $a/2$

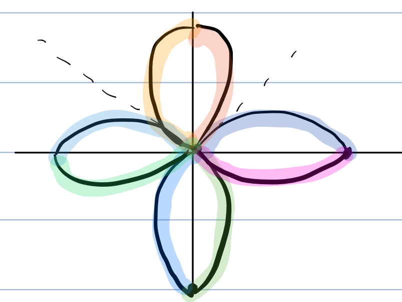


$r = a\sin\theta$:



② $r = 3\cos 2\theta$





Have familiarity with:

circles: $r = k$ constant

$r = a\sin\theta$

$r = a\cos\theta$

lines: $\theta = k$

roses: $r = a\cos(n\theta)$ $n > 1$

$r = a\sin(n\theta)$

Cardioids: $r = a(k + \cos\theta)$

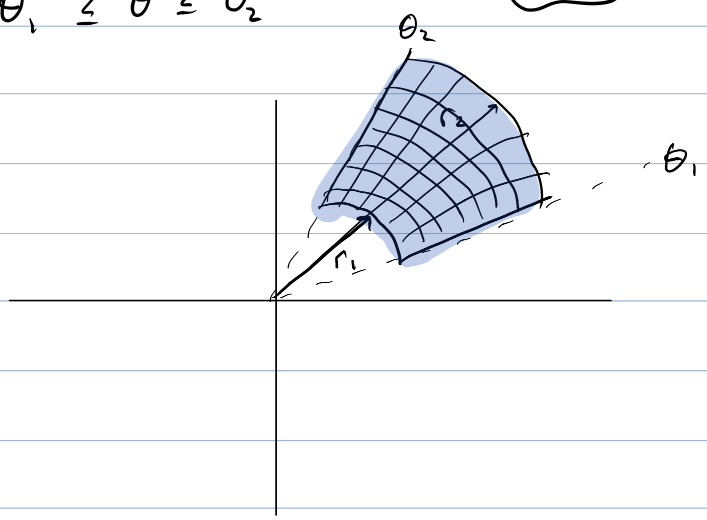
$r = a(k + \sin\theta)$

A polar "rectangle"

$$r_1 \le r \le r_2$$
$$\theta_1 \le \theta \le \theta_2$$

$a \le x \le b$

$c \le y \le d$



We will wish to integrate functions $f(x,y)$ over polar rectangles

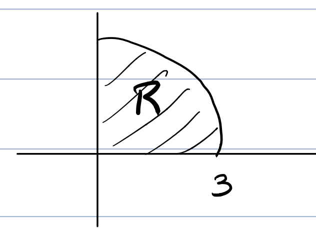
surprise!

$$\iint_R f(x,y)\, dA = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\, r \, dr\, d\theta$$

where R: $r_1 \le r \le r_2$

$\theta_1 \le \theta \le \theta_2$

ex. Integrate $f(x,y) = \sqrt{x^2+y^2}$ on



$0 \le \theta \le \frac{\pi}{2}$

$0 \le r \le 3$

$$\iint_R f(x,y)\, dA = \int_0^{\frac{\pi}{2}}\int_0^3 r \cdot r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^3 r^2\, dr\, d\theta = \int_0^{\frac{\pi}{2}} \left[\tfrac{1}{3} r^3\right]_0^3 d\theta$$

$$= \int_0^{\frac{\pi}{2}} 9\, d\theta = \frac{9\pi}{2}$$

May 4

Polar Coordinates

Review:

$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2$

$= r^2$

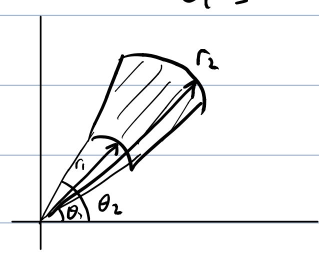
$r = \sqrt{x^2 + y^2}$

functions $r = f(\theta)$

circles, lines through origin, roses, cardioids

polar rectangle $r_1 \le r \le r_2$

$\theta_1 \le \theta \le \theta_2$



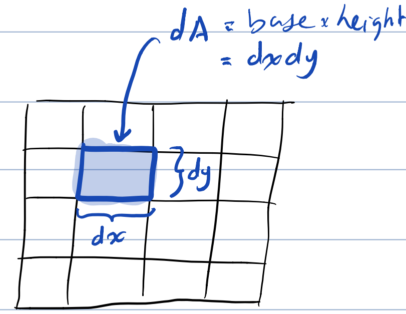
If R is a polar rectangle

!! God's gift

$$\iint_R f(x,y)\, dA = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\, r \, dr\, d\theta$$

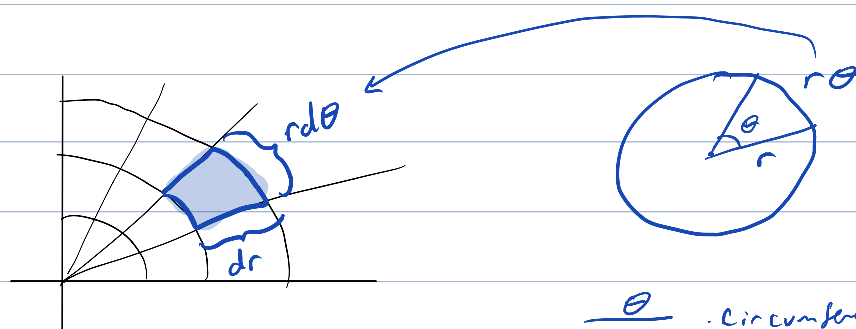
Why the "r"?

Rectangular: $[a,b] \times [c,d]$



So, $\iint_R f\,dA = \int_c^d \int_a^b f\,dx\,dy$

Polar:



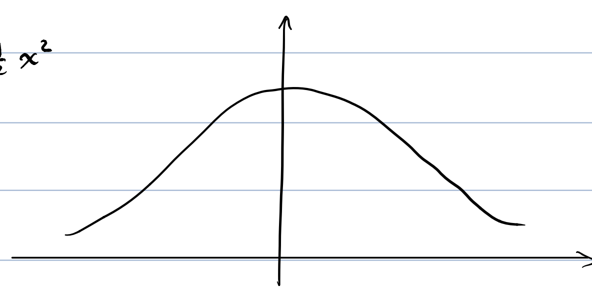
$$dA \approx r\,d\theta\,dr = r\,dr\,d\theta$$

It turns out that this is right

$\frac{\theta}{\text{total}} \cdot \text{circumference}$

$\frac{\theta}{\cancel{2\pi}} \cdot \cancel{2\pi} r$

ex. $f(x) = e^{-\frac{1}{2}x^2}$



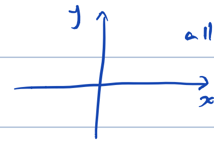
Let $I = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx$ → requires power series

↳ dummy variable

trick: $I^2 = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\,dy$

Recall: $\displaystyle\int_a^b f(x)\,dx \int_c^d g(y)\,dy = \int_a^b\!\!\int_c^d f(x)\,g(y)\,dx\,dy$

$$I^2 = \int_{-\infty}^{\infty}\!\!\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)}\,dx\,dy$$



all $\mathbb{R}^2$

covers all of $\mathbb{R}^2$

$$\overset{\text{to polar}}{=} \int_0^{2\pi}\!\!\int_0^{\infty} e^{-\frac{1}{2}r^2}\, r\,dr\,d\theta$$

$x = r\cos\theta$
$y = r\sin\theta$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} e^{-\frac{1}{2}r^2}\, r\,dr$$

$$= \left[\theta\right]_0^{2\pi} \lim_{t\to\infty} \int_0^{t} e^{-\frac{1}{2}r^2}\, r\,dr$$

let $u = -\frac{1}{2}r^2$
$du = -r\,dr$

$$= 2\pi \lim_{t\to\infty} -\int_0^{-\frac{1}{2}t^2} e^{u}\,du$$

$$= 2\pi \lim_{t\to\infty} \left(-e^{u}\right)\Big|_0^{-\frac{1}{2}t^2}$$

$$= 2\pi \lim_{t\to\infty} \Big(\underbrace{-e^{-\frac{1}{2}t^2}}_{\to 0} + 1\Big)$$

$$= 2\pi$$

$$I^2 = 2\pi$$

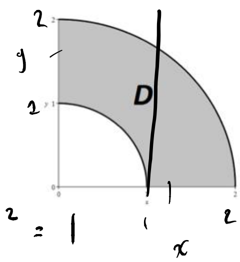
$$I = \sqrt{2\pi}$$ (has to be $\oplus$ since it's above the axes)

1. Consider $\displaystyle\iint_D \sqrt{x^2+y^2}\,dA$ where D is the region given at right.

   a) Set up the integral using rectangular coordinates.

   b) Set up the integral using polar coordinates.

   c) Pick one of the above and evaluate the integral.



$x^2 + y^2 = 1$

a) $0 \le x \le 1$ & $1 \le x \le 2$

$\sqrt{1-x^2} \le y \le \sqrt{4-x^2}$ $\quad 0 \le y \le \sqrt{4-x^2}$

$$\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,dy\,dx + \int_1^2 \int_0^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,dy\,dx$$

b-c) $$\int_0^{\frac{\pi}{2}} \int_1^2 r \cdot r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} d\theta \; \frac{r^3}{3}\Bigg|_1^2 = \left(\frac{8}{3} - \frac{1}{3}\right) \cdot \frac{\pi}{2}$$

$$= \frac{7\pi}{6}$$

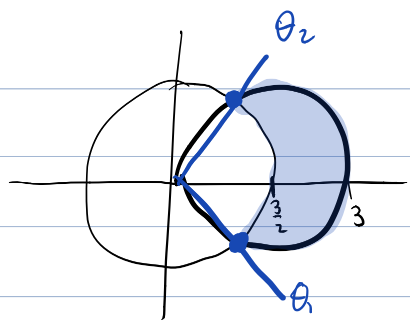
<u>In general:</u> $\theta_1 \le \theta \le \theta_2$

$\alpha(\theta) \le r \le \beta(\theta)$

$$\iint_R f(x,y)\,dA = \int_{\theta_1}^{\theta_2} \int_{\alpha(\theta)}^{\beta(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

2. Set up (but do not evaluate) a double integral in polar coordinates to find the area of eac h of the regions D below. Sketch the region.

a) D is the region outside the circle $r = \frac{3}{2}$ and inside the circle $r = 3\cos\theta$.



$\frac{3}{2} = 3\cos\theta$

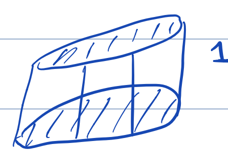
$\frac{1}{2} = \cos\theta$

$\theta = -\frac{\pi}{3}, \frac{\pi}{3}$

$\frac{3}{2} \le r \le 3\cos\theta$

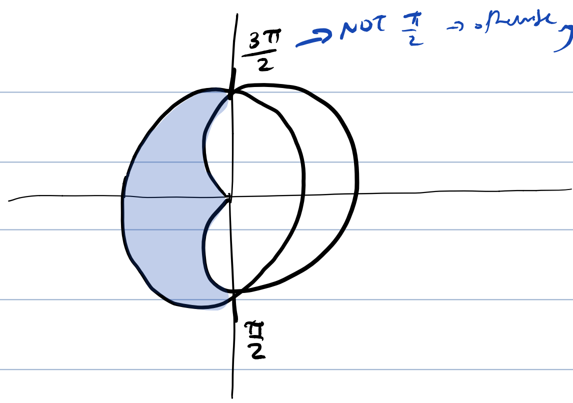
$-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$

$$V = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{\frac{3}{2}}^{3\cos\theta} 1 \cdot r \, dr \, d\theta$$



$V = A_b$

b) D is the region inside the circle $r = 1$ and outside the cardioid $r = 1 + \cos\theta$.



$\frac{3\pi}{2} \Rightarrow$ NOT $\frac{\pi}{2} \rightarrow$ [illegible]

$1 + \cos\theta = 1$

$\cos\theta = 0$ $\quad \theta = \frac{\pi}{2} \rightarrow \frac{3\pi}{2}$

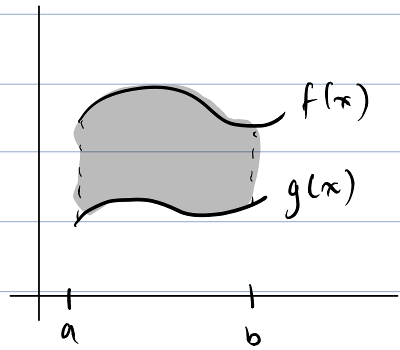
$1 + \cos\theta \le r \le 1$

$\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$

Volumes of Solids in $\mathbb{R}^3$

May 7

Recall Cal II: Area of planar region



$$A = \int_a^b \left(f(x) - g(x)\right)\, dx$$

$\swarrow$ Cal III

or $A = \int_a^b \int_{g(x)}^{f(x)} 1 \, dy\, dx$

Cal III: Volumes

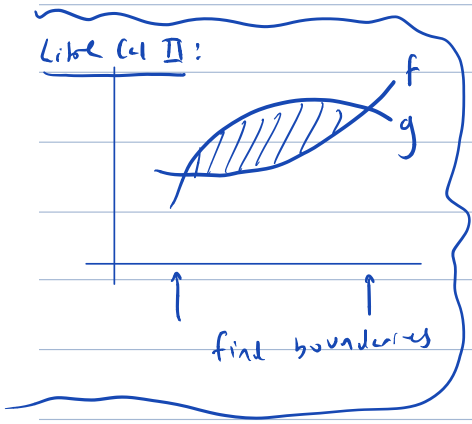
$\nwarrow$ R is the shadow of the solid in the (x, y) plane

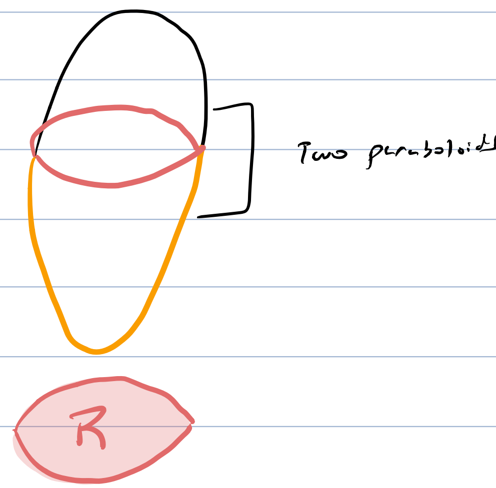
$$\iint_R \left(f(x,y) - g(x,y)\right)\, dA$$

Later: $\iiint_S 1 \, dV$

ex. Find the volume of the solid bounded by

$\underbrace{z = 2x^2 + y^2}_{\text{elliptic paraboloid faces up}}$ & $\underbrace{z = 9 - x^2 - 2y^2}_{\text{elliptic paraboloid faces down } (z = \ominus)}$





The shadow of the boundary curve will give us the region of integration.

$$\begin{cases} z = 2x^2 + y^2 \\ z = 9 - x^2 - 2y^2 \end{cases}$$

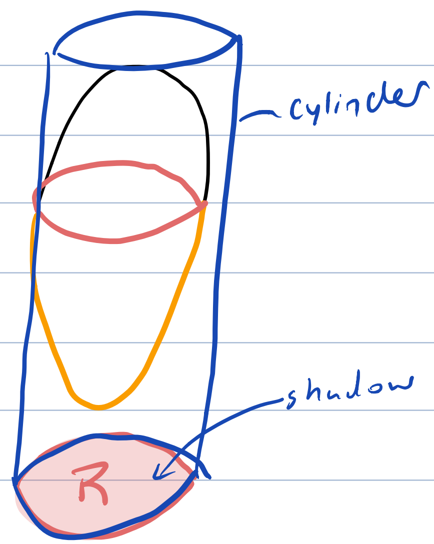
$$2x^2 + y^2 = 9 - x^2 - 2y^2$$

$$3x^2 + 3y^2 = 9$$

$$x^2 + y^2 = 3$$

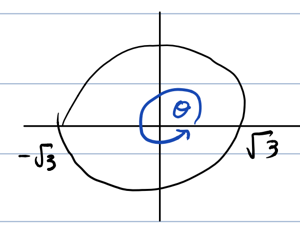
→ in $\mathbb{R}^2$: circle

→ in $\mathbb{R}^3$: cyllinder (parallel to z-axis)



$0 \le r \le \sqrt{3}$

$0 \le \theta \le 2\pi$



$$\text{Vol} = \iint_R \left[9 - x^2 - 2y^2 - (2x^2 - y^2) \right] dA$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} (9 - 3x^2 - 3y^2)\, dy\, dx$$

in polar

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} (9 - 3r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} (9r - 3r^3)\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{3}} (9r - 3r^3)\, dr = \theta \Big|_0^{2\pi} \left(\frac{9}{2} r^2 - \frac{3}{4} r^4 \right)\Big|_0^{\sqrt{3}}$$

$$= 2\pi \left(\frac{27}{2} - \frac{27}{4} \right) = \frac{27\pi}{2} \text{ units}^3$$

ex. Find the volume of the solid bounded by

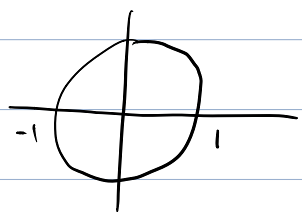
① $z = \overbrace{2 - x^2 - y^2}^{\text{elliptic paraboloid}}$ & $z = 1$

② $z = x^2 + y^2$ & $z = y$

sketch

① $1 = 2 - x^2 - y^2$

$x^2 + y^2 = 1$



$$V = \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[2 - x^2 - y^2 - 1 \right] dy\, dx$$

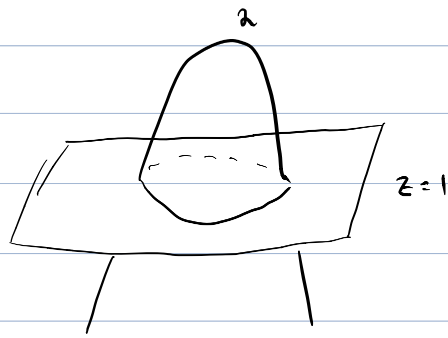
$$\overset{\text{polar}}{=} \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 (r - r^3)\, dr$$

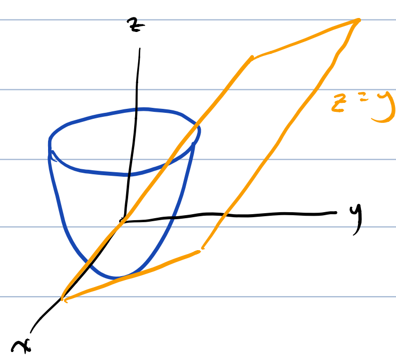
$$= \left[\theta \right]_0^{2\pi} \cdot \left[\tfrac{1}{2} r^2 - \tfrac{1}{4} r^4 \right]_0^1$$

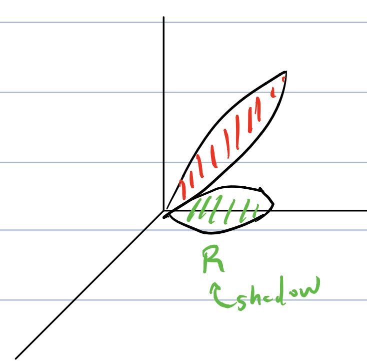
$$= 2\pi \left(\frac{1}{2} - \frac{1}{4} \right)$$

$$= \frac{\pi}{2}$$



(2) $z = x^2 + y^2$ & $z = y$

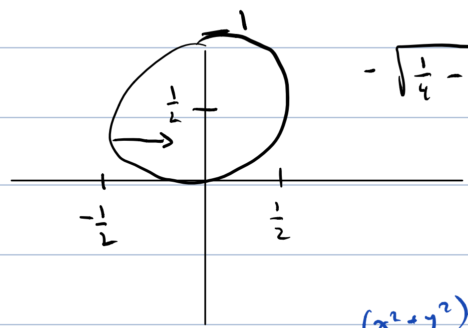




$y = x^2 + y^2$

$0 = x^2 + y^2 - y + \frac{1}{4} - \frac{1}{4}$

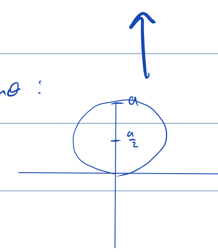
$\frac{1}{4} = x^2 + \left(y - \frac{1}{2}\right)^2$



$-\sqrt{\frac{1}{4} - \left(y - \frac{1}{2}\right)^2} \le x \le \sqrt{\frac{1}{4} - \left(y - \frac{1}{2}\right)^2}$

$0 \le y \le 1$

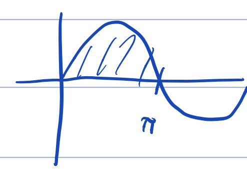
$r = a\sin\theta$:



$y - (x^2 + y^2)$

$$\int_0^{\pi}\int_0^{\sin\theta} (r\sin\theta - r^2)\, r\, dr\, d\theta$$

$r = \sin\theta$



$$= \int_0^{\pi} \left(\frac{1}{3} r^3 \sin\theta - \frac{1}{4} r^4\right)\Big|_0^{\sin\theta} d\theta$$

$$= \int_0^{\pi} \left(\frac{1}{3}\sin^4\theta - \frac{1}{4}\sin^4\theta\right) d\theta$$

$$= \frac{1}{12}\int_0^{\pi} \sin^4\theta\, d\theta = \frac{1}{2}\int_0^{\pi} (\sin^2\theta)^2\, d\theta$$

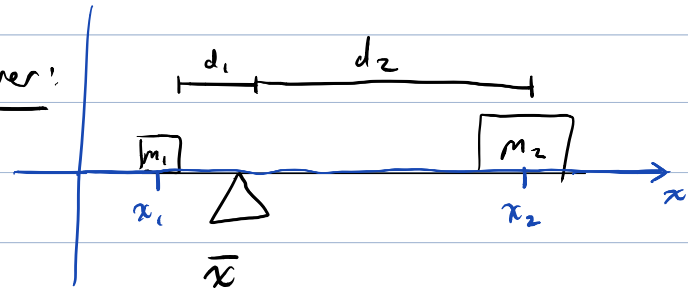
half angle : $\sin^2\theta = \frac{1 - \cos 2\theta}{2}$;

$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$

...

Application: Center of Mass

Simple lever:



balance pt:

$$\boxed{m_1 d_1 = m_2 d_2}$$

(from torque)

If m_1, m_2, x_1 & x_2 are known, calculate $\bar{x}$

$$d_2 = x_2 - \bar{x}$$

$$d_1 = \bar{x} - x_1$$

$$m_1 d_1 = m_2 d_2$$

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

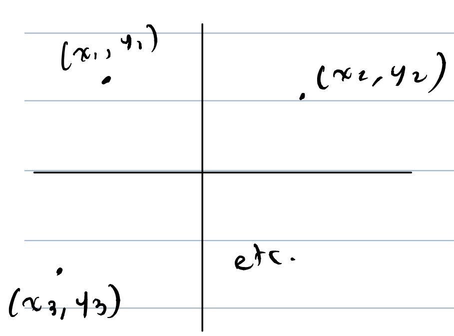
$$m_1\bar{x} - m_1 x_1 = m_2 x_2 - m_2\bar{x}$$

$$\bar{x} = \frac{m_1 x_1 + m_2 x_1}{m_1 + m_2}$$

← total moment about the centre $\sum x_i m_i$

← total mass $\sum m_i$

In the plane:



centre of mass: $(\bar{x}, \bar{y})$

notation

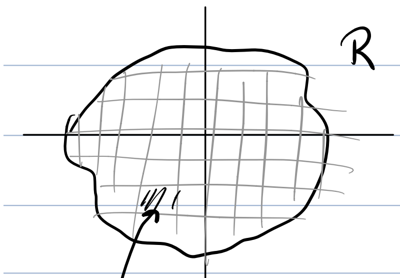
where $\bar{x} = \dfrac{\sum x_i m_i}{\sum m_i} = \dfrac{M_y}{M}$ ← moment about y-axis

$\bar{y} = \dfrac{\sum y_i m_i}{\sum m_i} = \dfrac{M_x}{M}$ ← moment about x-axis

distances from x-axis

Centre of Mass of a Laminate Region

infinitessimally thin layer



Total mass of R, given that the density at any point (x,y) is $\rho(x,y)$ (g/cm^2)

Assume that $\rho(x,y)$ is constant over one rectangle

$$\lim_{n,m\to\infty} \sum_{j=1}^{m} \sum_{i=1}^{n} \underbrace{\underbrace{\rho(x_i, y_i)}_{\text{density}} \underbrace{\Delta x \Delta y}_{\text{area}}}_{\text{mass of one subrectangle}}$$

$$= \iint_R \rho(x,y)\, dA = M$$

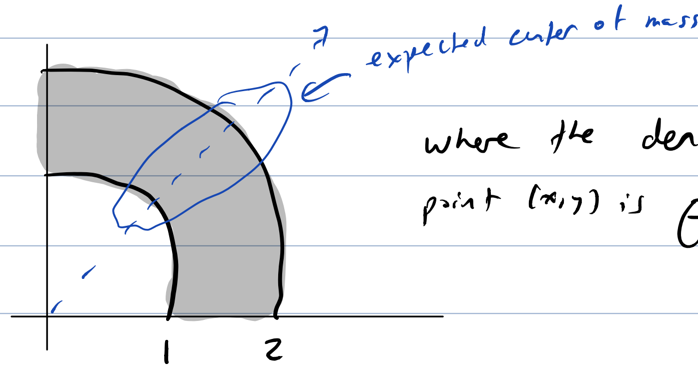
Similarly, the moments can be calculated

$$M_y = \iint_R x\rho(x,y)\, dA \qquad M_x = \iint_R y\rho(x,y)\, dA$$

$(\bar{x}, \bar{y})$ = centre of mass

$$\bar{x} = \frac{M_y}{M} \; ; \; \bar{y} = \frac{M_x}{M}$$

<u>ex</u> Calculate the coordinates of the centre of mass of



where the density at any point (x,y) is $\rho(x,y) = x + y$

$$M = \text{Total Mass} = \iint_R \rho(x,y)\, dA$$

$0 \le \theta \le \frac{\pi}{2}$

$1 \le r \le 2$

$$= \int_0^{\frac{\pi}{2}} \int_1^2 (r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta)\, d\theta \int_1^2 r^2\, dr$$

$$\ldots = \frac{14}{3} \quad \Leftarrow \text{maple}$$

M_y = Moment around y-axis

$$= \iint_R x \rho(x,y)\, dA$$

$$= \int_0^{\frac{\pi}{2}} \int_1^2 r\cos\theta \, (r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$$

maple

$$= \int_0^{\frac{\pi}{2}} (\cos^2\theta + \sin\theta\cos\theta)\, d\theta \int_1^2 r^3\, dr \quad \dots \quad = \frac{30+15\pi}{6}$$

M_x = Moment around x-axis

$$= \iint_R y \rho(x,y)\, dA$$

$$= \int_0^{\frac{\pi}{2}} \int_1^2 r\, y\, \theta\, (r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$$

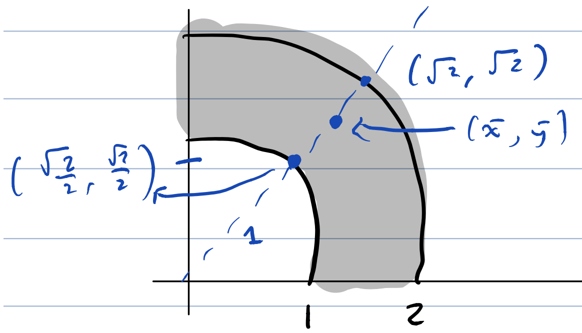
$$= \int_0^{\frac{\pi}{2}} (\sin^2\theta + \sin\theta\cos\theta)\, d\theta \int_1^2 r^3\, dr$$

maple

$$= \frac{30+15\pi}{6} = M_y$$

$$\bar{x} = \frac{M_y}{M} = \frac{30+15\pi}{18} \cdot \frac{3}{14} \approx 1.033$$

$\bar{y}$ = same



May 9

<u>Recall:</u> R – region of plane: "laminate"

$\rho(x,y)$ in g/cm^3 density at each point (x,y) in R

M = total mass = $\iint_R \rho(x,y)\,dA$

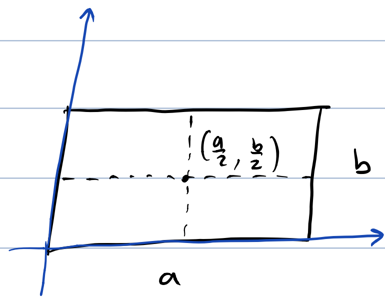
M_y = moment about y-axis = $\iint_R x\rho(x,y)\,dA$

M_x = moment about x-axis = $\iint_R x\rho(x,y)\,dA$

$(\bar{x}, \bar{y})$ = centre of mass = $\left(\frac{M_y}{M}, \frac{M_x}{M}\right)$

New: "Centroid" - centre of mass when ρ = constant

ex. Consider a rectangle of length a, width b & constant density ρ. Find the centroid of the rectangle.



$$M = \iint_R \rho \, dA$$

$\leftarrow$ fubini's theorem

$$= \rho \underbrace{\int_0^a \int_0^b 1 \, dy \, dx}_{\text{Area (height = 1)}}$$

$$= \rho ab$$

$$M_y = \iint_R x \rho \, dA$$

$$= \rho \int_0^a \int_0^b x \, dy \, dx$$

$$= \rho \int_0^a \left[xy \right]_0^b dx$$

$$= P \int_0^a b\, x\, dx$$

$$= P b \left. \frac{x^2}{2} \right|_0^a$$

$$= \frac{P a^2 b}{2}$$

$$\bar{x} = \frac{M_y}{M} = \frac{\frac{\cancel{P} a^{\cancel{2}} \cancel{b}}{2}}{\cancel{P} \cancel{a} \cancel{b}} = \frac{a}{2}$$

$$M_x = \iint_R y e\, dA = \dots = \frac{P a b^2}{2}$$

$$\bar{y} = \frac{M_x}{M} = \frac{\frac{P a b^3}{2}}{P a b} = \frac{b}{2}$$

centroid $= \left(\frac{a}{2}, \frac{b}{2} \right)$

Triple Integrals

$$\iiint_S f(\underbrace{x, y, z}_{\text{in } \mathbb{R}^3})\, dV$$

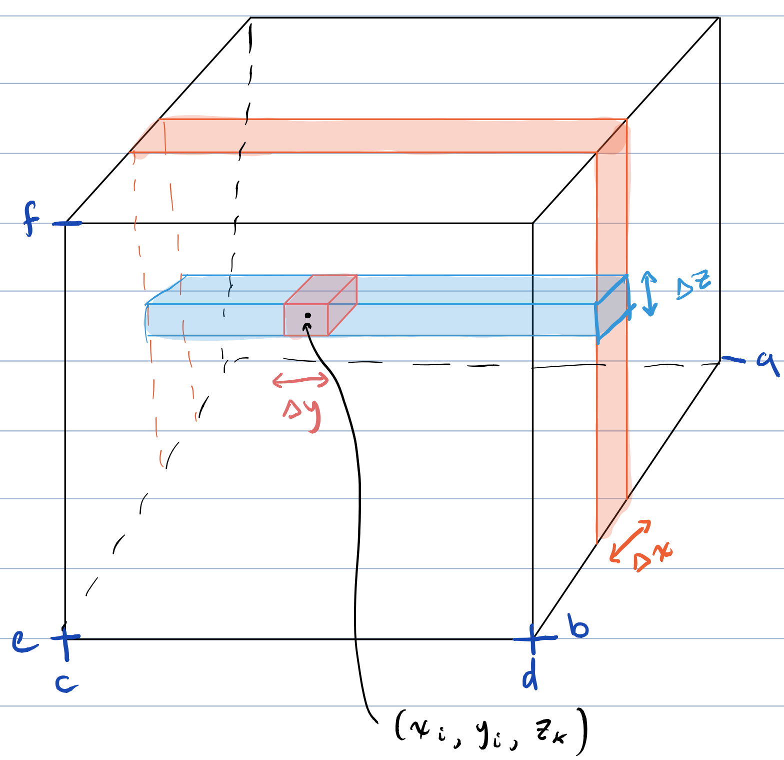
differential of volume

solid in $\mathbb{R}^3$

Easy case: S = Box

$$B : [a,b] \times [c,d] \times [e,f]$$

$$x \in [a,b],\quad y \in [c,d],\quad z \in [e,f]$$



if $f(x,y,z)$ is density (g/cm^3)

$$\underbrace{\iiint_B}_{\text{mass}} f(x,y,z)\,dV = \lim_{n,m,p\to\infty} \sum_{k=1}^{p}\sum_{j=1}^{m}\sum_{i=1}^{n} \underbrace{f(x_i, y_j, z_k)}_{\text{density}} \underbrace{\Delta x\,\Delta y\,\Delta z}_{\text{volume}}$$

density $\times$ volume

Approximation: $f(x_i, y_i, z_k)$ is constant in the ijk^{th} sub box.

Fubini on a Box

If $g(x,y,z)$ is continuous on box $B: [a,b]\times[c,d]\times[e,f]$, then

$$\iiint_B f(x,y,z)\,dV = \int_a^b\int_c^d\int_e^f f(x,y,z)\,dz\,dy\,dx$$

1. Find the total mass of the box $[0,1] \times [0,2] \times [0,3]$ if the density at each point is given by $\rho(x,y,z) = xy + yz + xz$. Use the following orders of integration: a) $dx\,dy\,dz$ and b) $dz\,dx\,dy$

a) $$M = \int_0^3\int_0^2\int_0^1 (xy + yz + xz)\,dx\,dy\,dz$$

$$= \int_0^3\int_0^2 \left(\frac{1}{2}x^2y + xyz + \frac{1}{2}x^2z\right)\Big|_0^1\,dy\,dz$$

$$= \int_0^3\int_0^2 \left(\frac{1}{2}y + yz + \frac{1}{2}z\right)dy\,dz$$

$$= \int_0^3 \left(\frac{1}{4} y^2 + \frac{1}{2} y^2 z + \frac{1}{2} y z \right)_0^2 dz$$

$$= \int_0^3 \left(1 + 2z + z \right) dz$$

$$= \left(z + \frac{3}{2} z^2 \right)_0^3$$

$$= 3 + \frac{27}{2}$$

$$= \frac{33}{2}$$

b) $$\int_0^2 \int_0^1 \int_0^3 (xy + yz + xz) \, dz \, dx \, dy$$

$$= \int_0^2 \int_0^1 \left(zxy + \frac{1}{2} y z^2 + \frac{1}{2} x z^2 \right)_0^3 dx \, dy$$

$$= \int_0^2 \int_0^1 \left(3xy + \frac{9}{2} y + \frac{9}{2} x \right) dx \, dy$$

$$= \int_0^2 \left(\frac{3}{2} x^2 y + \frac{9}{2} xy + \frac{9}{4} x^2 \right)_0^1 dy$$

$$= \int_0^2 \left(\frac{3}{2} y + \frac{9}{2} y + \frac{9}{4} \right) dy$$

$$= \left(\frac{3}{4} y^2 + \frac{9}{4} y^2 + \frac{9}{4} y \right)\Big|_0^2$$

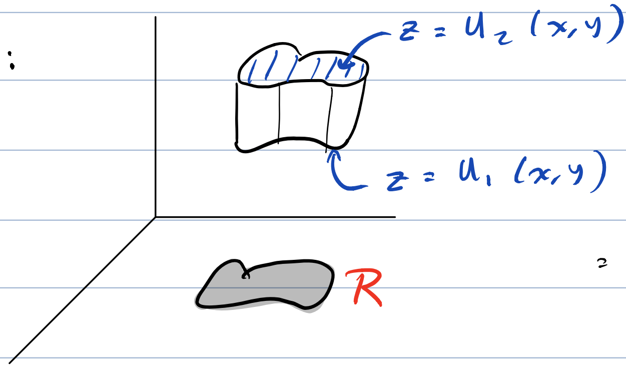
$$= 3 \cdot 4 + \frac{9}{2}$$

$$= 12 + \frac{9}{2}$$

$$= \frac{33}{2}$$

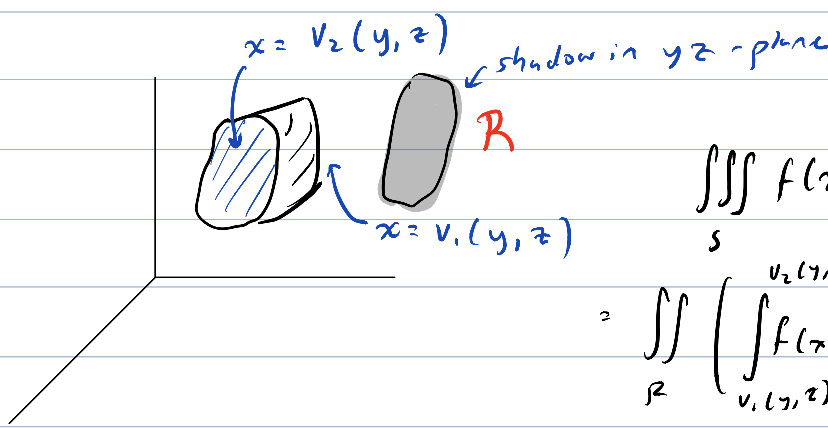
When the solid is not a box: Integrals over General Solids

Type 1:



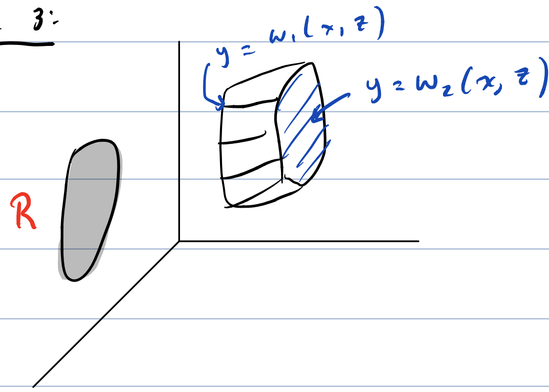
$$\iiint_S f(x,y,z)\,dV = \iint_R \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA$$

Type 2:



$$\iiint_S f(x,y,z)\,dV = \iint_R \left(\int_{v_1(y,z)}^{v_2(y,z)} f(x,y,z)\,dx \right) dA$$

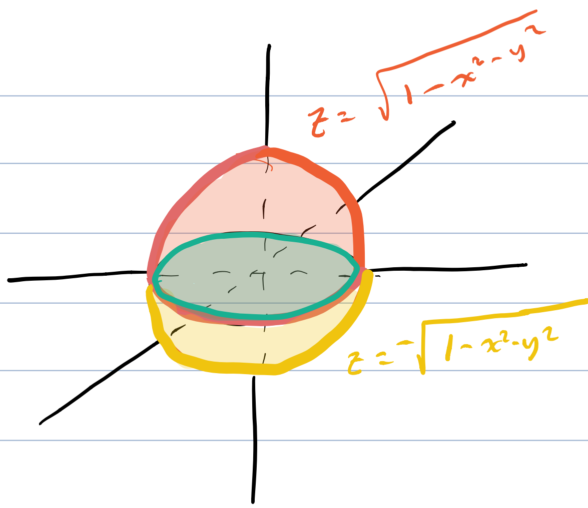
Type 3:



$$\iiint_S f(x,y,z)\,dV = \iint_R \left(\int_{w_1(x,z)}^{w_2(x,z)} f(x,y,z)\,dy \right) dA$$

2. Set up the triple integral of $f(x, y, z)$ over the sphere $x^2 + y^2 + z^2 \leq 1$.

Sphere

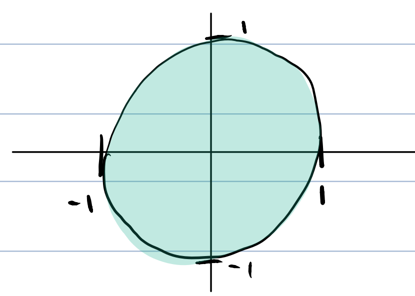


Boundary : $x^2+y^2+z^2=1$

$$\iiint_S f(x,y,z)\,dV$$

$$= \iint_R \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dA$$

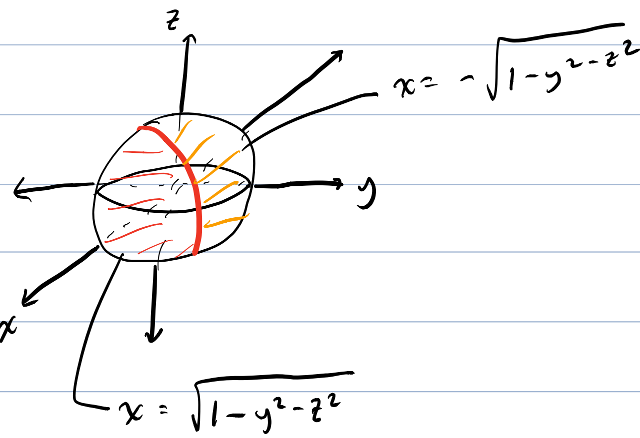
$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$$



$$-1 \leq x \leq 1$$

$$-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$$

Try with $dx\,dz\,dy$:



$$\iiint_S f(x,y,z)\,dV = \iint_R \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} f(x,y,z)\,dx\,dA$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} f(x,y,z)\,dx\,dz\,dy$$

May 11

Recall: $\underbrace{\iiint_S \rho(x,y,z)\,dV}_{\text{Total mass}}$

density (g/cm^3)

$\underbrace{\iiint_S 1\,dV}$ Volume of S

3 types: $\displaystyle \underbrace{\iint_R}_{} \int_{z_1(\underbrace{x,y}_{})}^{z_2(x,y)} p(x,y,z)\,dz\,dA$

R is in the xy-plane

$$\iint_R \int_{y_1(x,z)}^{y_2(x,z)} p(x,y,z)\,dy\,dA$$

R is in the xz-plane

$$\iint_R \int_{x_1(y,z)}^{x_2(y,z)} p(x,y,z)\,dx\,dA$$

yz-plane

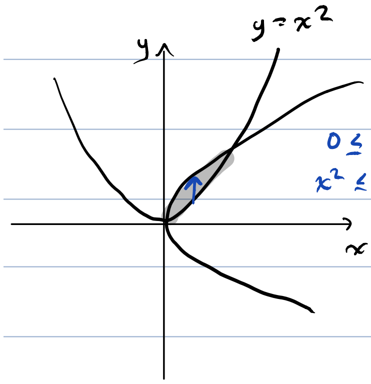
ex. Find the total mass of the solid S bounded by

$\underbrace{y = x^2,\quad x = y^2}_{\text{parabolic cylinder}},\quad \underbrace{z = x+y}_{\text{plane}}$

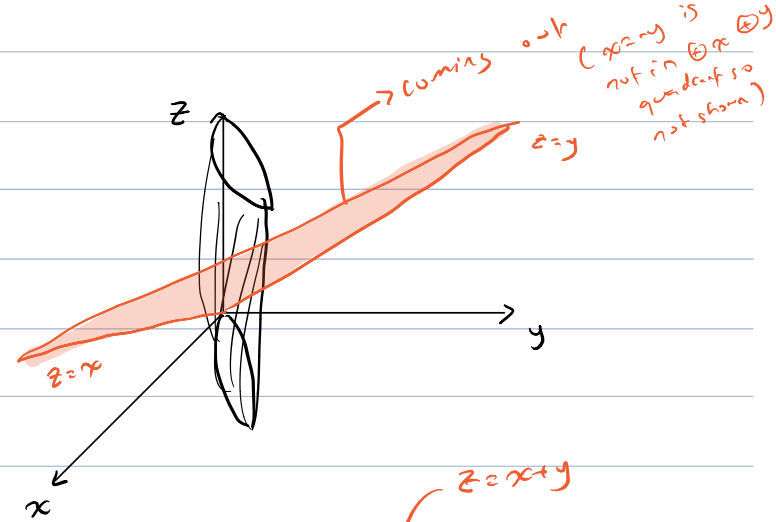
if the density of the solid at each pt (x,y,z) is

$p(x,y,z) = xy$

$$M = \iiint_S p(x,y,z)\,dV$$



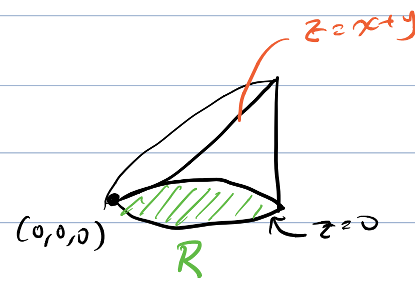
$0 \le x \le 1$
$x^2 \le y \le \sqrt{x}$



$z = x + y$

$y = 0: \quad z = x$ | $z = 0: \quad x = -y$

$x = 0: \quad z = y$



$$\iint_R \int_0^{x+y} xy \, dz \, dA$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} xy \, dz \, dy \, dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} \Big[xyz \Big]_0^{x+y} dy \, dx$$

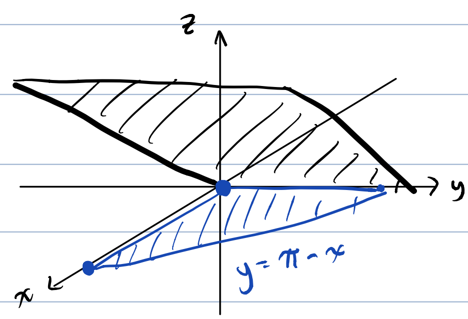
$$= \int_0^1 \int_{x^2}^{\sqrt{x}} \big(xy(x+y) - 0 \big) \, dy \, dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} \left(x^2 y + x y^2\right) dy\, dx$$

$$= \int_0^1 \left(\frac{1}{2} x^2 y^2 + \frac{1}{3} x y^3 \right]_{x^2}^{\sqrt{x}} dx$$

...

3. Evaluate $\iiint_S \sin(y)\, dV$, where S is the portion of 3-space that lies below the plane $z = x$ and above the triangular region with vertices $(0, 0, 0)$, $(\pi, 0, 0)$, and $(0, \pi, 0)$.



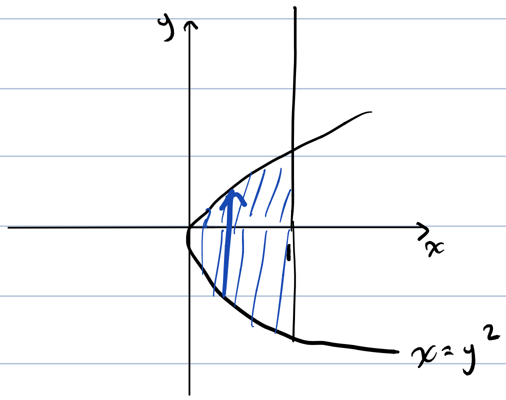
$0 \le x \le \pi$

$0 \le y \le \pi - x$

$$\iint_R \int_0^x \sin y \, dz\, dA$$

$$= \int_0^\pi \int_0^{\pi - x} \int_0^x \sin y \, dz\, dy\, dx$$

4. Find the total mass of the solid S whose density at each point (x, y, z) is given by $\rho(x, y, z) = 6xy$ if S is bounded above by the plane $z = 1+x+y$ and below by the region in the xy–plane bounded by the curves $x = y^2$ and $x = 1$.

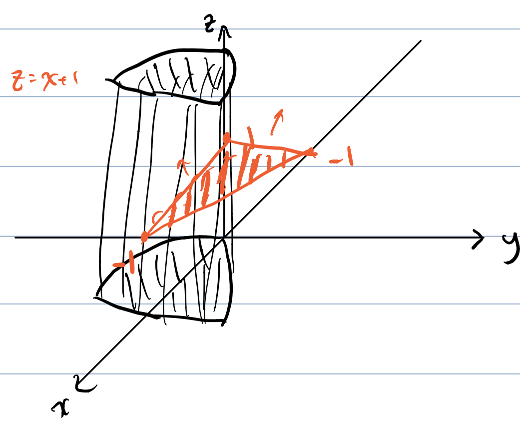


for $-1 \le y \le 1$
$y^2 \le x \le 1$

easier to integrate y^2 than $\sqrt{x}$

$0 \le x \le 1$

$-\sqrt{x} \le y \le \sqrt{x}$



$z = 1 + x + y$

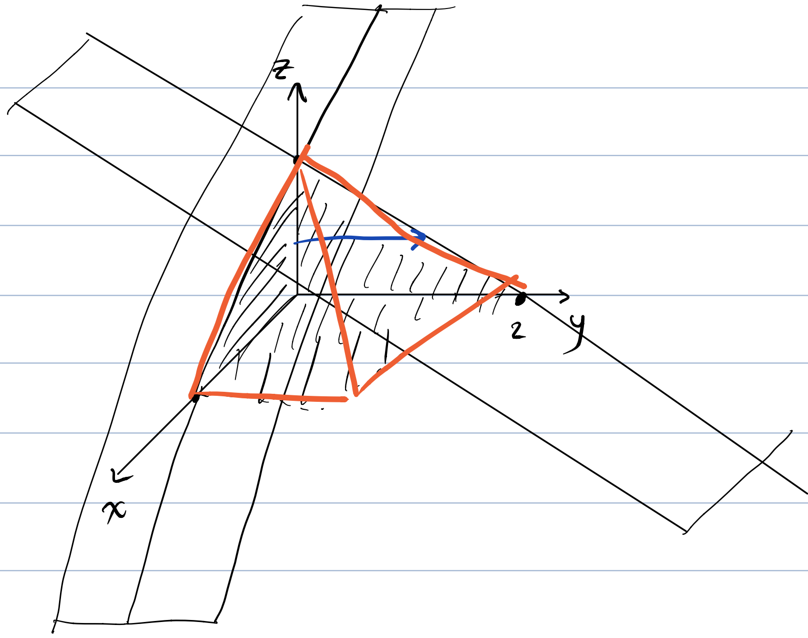
$x = 0$: $z = y + 1$

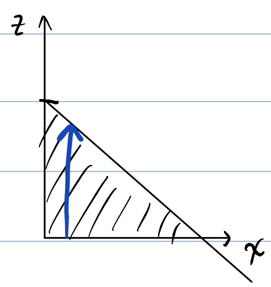
$y = 0$: $z = x + 1$

$0 \le z \le 1 + x + y$

$$\int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{1+x+y} 6xy \, dz \, dx \, dy$$

5. Find the volume in the first octant of the solid bounded by $x + z = 1$ and $y + 2z = 2$.





$0 \leq x \leq 1$

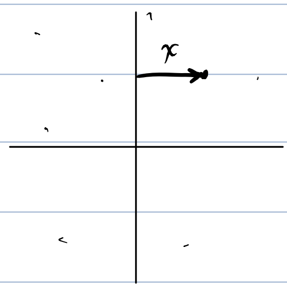
$0 \leq z \leq 1 - x$

$$V = \int_0^1 \int_0^{1-x} \int_0^{2-2z} 1 \, dy \, dz \, dx$$

6. Find the centre of mass of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$, where $\rho(x, y, z) = y$.

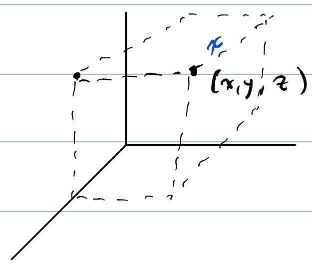
<u>Note:</u> Centre of Mass

$\mathbb{R}^2$:



$$\bar{x} = \frac{M_y}{M} = \frac{\iint x\rho(x,y)\,dA}{M}$$

$\mathbb{R}^3$:



x is the distance from yz-plane

M_{yz} = 1st moment about yz-plane

$$= \iiint_S x\rho(x,y,z)\,dV$$

$$M_{xz} = \iiint_S y\rho(x,y,z)\,dV$$

$$M_{xy} = \iiint_S z\rho(x,y,z)\,dV$$

$$M = \iiint_S \rho(x,y,z)\,dV$$

Centre of mass $(\bar{x}, \bar{y}, \bar{z})$

$$\bar{x} = \frac{M_{yz}}{M} \; ; \; y = \frac{M_{xz}}{M} \; ; \; z = \frac{M_{xy}}{M}$$

Note: The notes from the last 1-2 classes have not been added.