

2 - Integration Methods

objective: To find antiderivatives of "non-basic" functions.

→ We know that $\int \cos x \, dx = \sin x$

$$\int \cos(sx) \, dx = ?$$

$$\int \cos(\ln x) \, dx = ?$$

$$\int \cos(e^x) \, dx = ?$$

$$\int \cos(x^2) \, dx = ?$$

Basic Algebraic Approaches

ex. $\int (4x + 3)^2 \, dx$

→ expand!

$$= \int (16x^2 + 24x + 9) \, dx$$

$$= 16 \int x^2 \, dx + 24 \int x \, dx + 9 \int 1 \, dx$$

$$= 16 \frac{x^3}{3} + 24 \frac{x^2}{2} + 9x + k$$

ex. $\int \frac{3x^2 + 2x + 5}{x} dx$

→ distribute the divisor

$$= \int (3x + 2 + \frac{5}{x}) dx$$

$$= \frac{3}{2}x^2 + 2x + 5 \ln|x| + A$$

Note: not x^1

(needs to be on same dom as $\frac{1}{x}$)

ex. $\int \sin(\frac{x}{2}) \cos(\frac{x}{2}) dx$

→ Try Identities

$$= \int \frac{1}{2} \sin x dx$$

$$= \frac{1}{2} \int \sin x dx$$

$$= -\frac{1}{2} \cos x + C$$

$$\begin{aligned}\text{ex. } & \int \tan^2 x \, dx \\ &= \int (\sec^2 x - 1) \, dx \\ &= \ln|x| - x + C\end{aligned}$$

$$\begin{aligned}\text{ex. } & \int (\sin x + \cos x)^2 \, dx \\ &= \int (\sin^2 x + 2 \sin x \cos x + \cos^2 x) \, dx\end{aligned}$$

$$\begin{aligned}&= \int (2 \sin x \cos x + 1) \, dx \\ &= \int (\sin(2x) + 1) \, dx\end{aligned}$$

SUBSTITUTION

$$\text{ex. } \int \cos x \, dx = \sin x + C \leftarrow \text{basic antiderivative}$$

$$\text{ex. } \int 3x^2 \cos(x^3) \, dx$$

$$\text{Let } u = x^3$$

$$\frac{du}{dx} = u' \rightarrow du = u' dx$$

$$du = 3x^2 dx$$

$$\int 3x^2 \cos(x^3) dx = \int \cos u du$$

$$= \sin u + C$$

$$= \sin(x^3) + C$$

$$\int (2x+5)^{12} dx$$

$$\text{Let } u = 2x+5$$

$$du = 2dx$$

$$dx = \frac{1}{2} du$$

$$\int (2x+5)^{12} dx = \frac{1}{2} \int (2x+5)^{12} 2 du$$

$$= \frac{1}{2} \int u^{12} du$$

$$= \frac{1}{2} \cdot \frac{u^{13}}{13} + C$$

$$= \frac{u^{13}}{26} + C$$

$$= \frac{(2x+5)^{13}}{26} + C$$

$$\text{ex. } \int \frac{e^x}{16 + e^{2x}} dx \quad \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right)$$

$$= \int \frac{1}{16 + (e^x)^2} e^x dx$$

$$\text{Let } u = e^x$$

$$du = e^x dx$$

$$\int \frac{e^x}{16 + e^{2x}} dx = \int \frac{1}{16 + u^2} du$$

$$= \frac{1}{4} \arctan\left(\frac{u}{4}\right) + C$$

$$= \frac{1}{4} \arctan\left(\frac{e^x}{4}\right) + C$$

Survival Guide : Choices of 'u'

→ inner function

→ is there a $u \leftrightarrow u'$ combination

→ when all else fails, let $u = \text{denominator, or switch methods}$

$$\text{ex. } \int \frac{24x^2 + 64x - 16}{4x^3 + 16x^2 - 8x + 14} dx$$

$$\text{Let } v = 4x^3 + 16x^2 - 8x + 14$$

$$dv = (12x^2 + 32x - 8) dx$$

$$\int \frac{24x^2 + 64x - 16}{4x^3 + 16x^2 - 8x + 14} dx = 2 \int \frac{1}{u} du$$

$$= 2 \ln|u| + C$$

$$= 2 \ln |4x^3 + 16x^2 - 8x + 14| + C$$

ex. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

$$\text{Let } u = \sqrt{x}$$

$$du = \frac{1}{2\sqrt{u}} dx$$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 \int \frac{e^{\sqrt{x}}}{2\sqrt{x}} dx$$

$$= 2 \int e^u du$$

$$= 2 e^u + C$$

$$= 2 e^{\sqrt{x}} + C$$

ex. $\int \frac{\ln(\ln t)}{t \ln t} dt$

$$\text{Let } u = \ln t \quad \text{for } t \geq 0$$

$$du = \frac{1}{t} dt$$

$$\int \frac{\ln(\ln t)}{t \ln t} dt = \int \frac{\ln(\ln t)}{\ln t} \cdot \frac{1}{t} dt$$

$$= \int \frac{\ln u}{u} du$$

$$\text{Let } z = \ln u$$

$$dz = \frac{1}{u} du$$

$$\int \frac{\ln u}{u} du = \int z dz$$

$$= \frac{z^2}{2} + C$$

$$= \frac{\ln^2(u)}{2} + C$$

$$\text{or... } u = \ln(\ln t)$$

$$du = \frac{1}{\ln t} \cdot \frac{1}{t} dt$$

$$= \frac{\ln^2(\ln t)}{2} + C$$

$$\text{Ex} \quad \int (2y+1) \sqrt{y-3} dy$$

$$\text{Let } u = y-3$$

$$du = dy$$

$$\int (2(u+3) + 1) \sqrt{u} du$$

$$= \int (2u + 7) \sqrt{u} du$$

$$= \int (2u^{3/2} + 7u^{1/2}) du$$

$$= 2 \int u^{3/2} du + 7 \int u^{1/2} du$$

$$= 2 \frac{u^{5/2}}{5/2} + 7 \frac{u^{3/2}}{3/2} + C$$

$$= \frac{4}{5} u^{5/2} + \frac{14}{3} u^{3/2} + C$$

$$= \frac{4}{5} (y-3)^{5/2} + \frac{14}{3} (y-3)^{3/2} + C$$

Problem 27

Use a substitution to solve each of the following (definite or indefinite) integrals

$$1) \int \frac{2x+1}{4x^2 + 4x + 3} dx$$

$$2) \int \frac{\cos(\sqrt{x})\sqrt{1+\sin(\sqrt{x})}}{\sqrt{x}} dx$$

$$3) \int \frac{x dx}{\sqrt{16-x^4}}$$

$$4) \int \frac{x^5}{\sqrt{x^3+4}} dx$$

$$1) \int \frac{2x+1}{4x^2 + 4x + 3} dx$$

$$\text{Let } u = 4x^2 + 4x + 3$$

$$du = (8x + 4) dx$$

$$\begin{aligned} \int \frac{2x+1}{4x^2 + 4x + 3} dx &= \frac{1}{4} \int \frac{1}{4x^2 + 4x + 3} \cdot 4(2x+1) dx \\ &= \frac{1}{4} \int \frac{1}{u} du \\ &= \frac{1}{4} \ln|u| + C \\ &= \frac{1}{4} \ln|4x^2 + 4x + 3| + C \end{aligned}$$

$$2) \int \frac{\cos(\sqrt{x}) \sqrt{1+\sin(\sqrt{x})}}{\sqrt{x}} dx$$

Let $u = 1 + \sin(\sqrt{x})$

$$du = \left(\cos(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \right) dx$$

$$\int \frac{\cos(\sqrt{x}) \sqrt{1 + \sin(\sqrt{x})}}{\sqrt{x}} dx = 2 \int \sqrt{1 + \sin(\sqrt{x})} \cos(\sqrt{x}) \frac{1}{2\sqrt{x}} dx$$

$$= 2 \int u^{1/2} du$$

$$= 2 \cdot \frac{u^{3/2}}{3/2} + C$$

$$= \frac{4}{3} u^{3/2} + C$$

$$= \frac{4}{3} (1 + \sin \sqrt{x})^{3/2} + C$$

$$*3) \int \frac{x}{\sqrt{16 - x^4}} dx = \frac{1}{2} \int \frac{1}{\sqrt{16 - (x^2)^2}} 2x dx$$

$$\text{Let } u = x^2 \quad = \frac{1}{2} \int \frac{1}{\sqrt{16 - u^2}} du$$

$$du = 2x dx$$

$$= \frac{1}{2} \arcsin\left(\frac{u^2}{4}\right) + C$$

$$*4) \int \frac{x^5}{\sqrt{x^3 + 4}} dx = \frac{1}{3} \int \frac{x^3}{\sqrt{x^3 + 4}} 3x^2 dx$$

$$\text{Let } u = x^3 + 4 \quad = \frac{1}{3} \int \frac{u^{-4}}{\sqrt{u}} du \quad \text{where } x^3 = u - 4$$

$$du = 3x^2 dx$$

$$= \frac{1}{3} \int \left(u^{-1/2} - \frac{4}{\sqrt{u}}\right) du$$

$$= \frac{1}{3} \left[\int v^{1/2} - 4 \int v^{-1/2} \right] dv$$

$$= \frac{1}{3} \left[\frac{v^{3/2}}{3/2} - 4 \frac{v^{1/2}}{1/2} \right]$$

$$= \frac{2}{9} v^{3/2} - \frac{8}{3} v^{1/2} + C$$

$$= \frac{2}{9} (x^3 + 4)^{3/2} - \frac{8}{3} (x^3 + 4)^{1/2} + C$$

still to come...

→ Definite integrals involving a subs?

→ Initial condition problems

u-sub involving definite integrals

$$\underset{\text{bounds of } x}{\left\{ \begin{array}{l} ⑤ \\ ⑥ \end{array} \right\}} \int f(x) dx \rightarrow \underset{\text{bounds of } u}{\left\{ \begin{array}{l} ⑦ \\ ⑧ \end{array} \right\}} \int g(u) du$$

Ex. $\int_1^e \frac{(1 + \ln x)^3}{x} dx$

ʃ f(x) cont'd over specified interval

f(x) defined over specified interval

Bounds

$$x = e \rightarrow u = 1 + \ln e$$

$$= 2$$

$$\text{Let } u = 1 + \ln x$$

$$x = 1 \rightarrow u = 1 + \ln 1$$

$$du = \frac{1}{x} dx$$

$$= 1$$

$$\int_1^e \frac{(1 + \ln x)^3}{x} dx = \int_1^e (1 + \ln x)^3 \cdot \frac{1}{x} dx$$

$$= \int_1^e u^3 du$$

$$= \frac{u^4}{4} \Big|_1^2$$

No need to go back to x
if bonds are in u

$$= 4 - \frac{1}{4}$$

$$= \frac{15}{4}$$

N.B. $\frac{(u^4 - k)}{4} \Big|_1^2$

$$= \left(\frac{16 - 1}{4} \right) - \left(\frac{1 + 16}{4} \right)$$

So no need for k !

$$\text{Ex. } \int_{-2}^2 \cos(t^2) t^3 dt$$

$$\text{Let } f(t) = \cos(t^2) t^3 dt$$

$$f(-t) = \cos((-t)^2) (-t)^3 dt = -\cos(t^2) t^3 dt$$

$$-f(t) = -\cos(t^2) t^3 dt$$

Recall even if $f(x) = f(-x)$

$$\text{so } f(-t) = -f(t)$$

odd if $f(-x) = -f(x)$

$$\text{So } f(t) \text{ is odd}$$

Also note $\cos(-x) = \cos(x)$

$$\text{So } \int_{-2}^2 \cos(t^2) t^3 dt = 0$$

... would have been useful
to prove $\cos(t^3)$ is even.

$$\text{or let } u = t^2$$

Bonds

$$du = 2t dx$$

$$t=2 \rightarrow u=4$$

$$t=-2 \rightarrow u=4$$

$$\int_{-2}^2 \cos(t^2) t^3 dt = \frac{1}{2} \int_{-2}^2 \cos(t^2) t^2 2t dt$$

$$= \frac{1}{2} \int_4^4 \cos(u) u du$$

$$= 0$$

$$\text{ex. } \int_0^{\pi/2} \sin x \cos x dx$$

$$\text{Let } u = \sin x$$

Bounds

$$du = \cos x dx$$

$$x = \frac{\pi}{6} \Rightarrow u = \frac{1}{2}$$

$$x = 0 \Rightarrow u = 0$$

$$\int_0^{1/2} u du$$

$$= \frac{u^2}{2} \Big|_0^{1/2}$$

$$= \frac{1}{2} \cdot \frac{1}{4}$$

$$= \frac{1}{8}$$

$$\text{ex. } \int_0^{\pi/4} \tan x dx = \int_0^{\pi/4} \frac{\sin x}{\cos x} dx$$

Bounds

$$\text{Let } u = \cos x$$

$$x = \frac{\pi}{4} \Rightarrow u = \frac{\sqrt{3}}{2}$$

$$du = -\sin x dx$$

$$x = 0 \Rightarrow u = 1$$

$$= - \int_0^{\pi/6} \frac{1}{\cos x} (-\sin x) dx$$

$$= - \int_1^{\sqrt{3}/2} \frac{1}{u} du$$

$$= - \ln|u| \Big|_1^{\sqrt{3}/2}$$

$$= - \ln \frac{\sqrt{3}}{2} = \ln \frac{2}{\sqrt{3}} \quad \text{properties of } \ln$$

$$\ln \left(\frac{\sqrt{3}}{2} \right)^{-1} = - \ln \frac{\sqrt{3}}{2}$$

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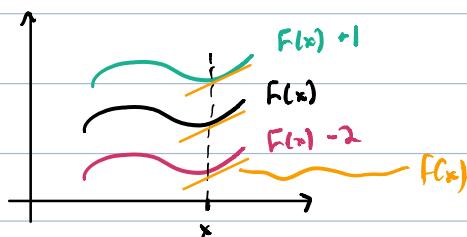
Antiderivative families

Let $f(x)$ represent a continuous function over $[a, b]$ and assume that
 $F(x)$ is an antiderivative of $f(x)$

Then $F(x) + k$ is also an antiderivative of $f(x)$

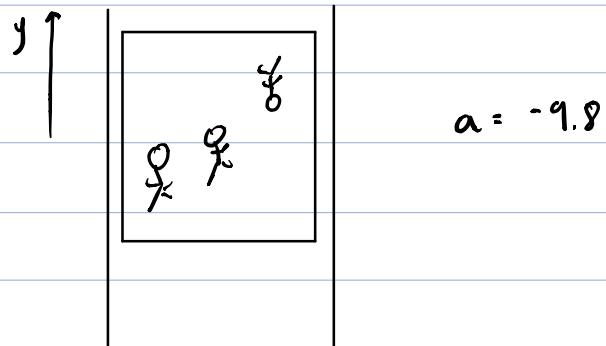
$$F(x) \xrightarrow{\text{derivative}} F'(x) = f(x)$$

$$F(x) + k \xrightarrow{\text{derivative}} F'(x) = f(x)$$



Initial Condition Problems

ex. The terrible trio are in an elevator whose cables (yes all of them) happened to rupture



$$a = -9.8$$

Find $y(t)$ if the elevator was 150 m above the ground and heading up at a speed of 5 m/s when the cables broke ($t=0$)

$$a(t) = -9.8$$

$$v'(t) = -9.8$$

$$\begin{aligned} v(t) &= \int v'(t) dt \\ &= \int -9.8 dt \end{aligned}$$

$$v(t) = -9.8t + C$$

$$\text{If } t=0, v=5$$

$$5 = -9.8(0) + C$$

$$C = 5$$

$$\text{so } v(t) = -9.8t + 5 = y'(t)$$

$$\begin{aligned}y(t) &= \int y'(t) dt \\&= \int (-9.8t + 5) dt\end{aligned}$$

$$= \frac{9.8t^2}{2} + 5t + k$$

$$y(t) = -4.9t^2 + 5t + k$$

$$y = 150 \text{ when } t=0$$

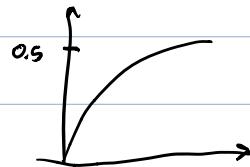
$$150 = -4.9(0)^2 + 5(0) + k \Rightarrow k = 150$$

$$y(t) = -4.9t^2 + 5t + 150$$

ex. The rate of taxation in Sweden is represented by the function:

$$T'(x) = 0.5 - 0.5e^{-0.2x}$$

where x is the annual income
in thousands of euros



A person who earns $x = 100^{(\text{th})}$ pays $35^{(\text{th})}$ of euros in tax. Find $T(x)$.

$$T'(x) = 0.5 - 0.5e^{-0.2x}$$

$$T(x) = \int T'(x) dx$$

$$= \int (0.5 - 0.5e^{-0.2x}) dx$$

$$= 0.5x - 0.5 \int e^{-0.2x} dx$$

Let $u = -0.2x$

$$du = -0.2 dx$$

$$= 0.5x - 0.5 \left(\frac{1}{-0.2}\right) \int e^{-0.2x} (-0.2 dx)$$

$$= 0.5x + 2.5 \int e^u du$$

$$= 0.5x + 2.5 e^u + C$$

$$T(x) = 0.5x + 2.5 e^{-0.2x} + C$$

when $x = 100$, $T = 35$

$$35 = 0.5(100) + 2.5 e^{-0.2(100)} + C$$

$$C = -15$$

$$\text{So } T(x) = 0.5x - e^{-0.2x} - 15$$

Alternative (algebraic) strategies:

If substitution fails, the following strategies can be considered.

→ expansion

→ trig identities

→ split the numerator (in fraction)

→ completion of squares

→ long division

ex $\int \frac{2x+5}{x^2+4} dx$ } split the numerator

$$= \int \frac{2x}{x^2+4} dx + \int \frac{5}{x^2+4} dx$$

$$= \int \frac{1}{x^2+4} 2x dx + 5 \int \frac{1}{x^2+4} dx$$

\downarrow Don't need
to write dx

$$= \ln(x^2+4) + 5 \cdot \frac{1}{2} \arctan \frac{x}{2} + C$$

abs. val can be
removed since it is
always positive

ex $\int \frac{1}{x^2+8x+25} dx$

Completing the square? $\rightarrow x^2 + 8x + (\frac{1}{2}8)^2 - (\frac{1}{2}8)^2 + 25$

$$= x^2 + 8x + 16 - 16 + 25$$

$$= (x+4)^2 + 9$$

\downarrow
let $u = x+4$
 $du = dx$

$$a^2 + 2ab + b^2$$

$$= \int \frac{1}{u^2+3^2} du$$

$$= \frac{1}{3} \arctan \left(\frac{x+4}{3} \right) + C$$

ex. $\int \frac{1}{\sqrt{20x-x^2}} dx$

$$= \int \frac{1}{-(x^2 - 20x + 100) + 100} dx$$

Completing the square

$$= \int \frac{1}{100 - (x-10)^2} dx$$

$$= \int \frac{1}{100-u^2} du \quad \text{Let } u = x-10$$

$du = dx$

$$= \arcsin\left(\frac{u}{10}\right)$$

$$= \arcsin\left(\frac{x-10}{10}\right) + C$$

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À vous

$$\int \frac{dx+1}{x^2+6x+10} dx$$

$$= \int \frac{2x+1}{x^2+6x+9-9+10} dx$$

$$= \int \frac{2x+1}{(x+3)^2+1} dx$$

$$= \int \frac{2(u-3)+1}{u^2+1} du \quad \text{Let } u = x+3 \\ du = dx$$

$$= \int \frac{2u-5}{u^2+1} du$$

$$= \int \frac{2u}{u^2+1} du - 5 \int \frac{1}{u^2+1} du$$

↓
Let $w = u^2+1$
 $dw = 2u du$

$$= \int \frac{1}{w} dw - 5 \operatorname{arctan}(u)$$

$$= \ln |(x+3)^2+1| - 5 \operatorname{arctan}(x+3) + C$$

Ex. What would you do?

$$\int \frac{6x^3 + 5x^2 + 3x + 1}{x^2 + 6x + 10} dx$$

higher or equal power
on top

improper rational expression

$$\begin{array}{r} 6x - 31 \\ \hline x^2 + 6x + 10 \left[\begin{array}{r} 6x^3 + 5x^2 + 3x + 1 \\ -(6x^3 + 36x^2 + 60x) \\ \hline -31x^2 - 57x + 1 \end{array} \right] \\ \hline -(-31x^2 - 186x - 310) \\ \hline 129x + 311 \end{array}$$

$$\text{So } \int \frac{6x^3 + 5x^2 + 3x + 1}{x^2 + 6x + 10} dx = \int \underbrace{6x - 31}_{\text{direct}} + \frac{129x + 311}{x^2 + 6x + 10}$$

$\textcircled{1}, \text{ not } \textcircled{2}$

Completing the square + substitution
+ split numerator...

À vous

$$\text{Ex. } \int \frac{1}{1 - \sin x} dx$$

$$= \int \frac{1}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x} dx$$

$$= \int \frac{1 + \sin x}{1 - \sin^2 x} dx$$

$$= \int \frac{1 + \sin x}{\cos^2 x} dx$$

$$= \int \frac{1}{\cos^2 x} + \int \frac{\sin x}{\cos^2 x} dx$$

$$= \int \sec^2 x dx + \int \sec x \tan x dx$$

$$= \tan x + \sec x + C$$

ex. $\int \frac{1}{e^x + e^{-x}} dx$

$$= \int \frac{e^x + 1}{e^x(e^x + e^{-x})} dx$$

$$= \frac{e^x}{(e^x)^2 + 1}$$

:

ex. $\int \sec x dx$

$$= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} dx$$

$$= \int \frac{\sec^2 x + \tan x}{\sec x + \tan x} dx \quad u - u' \text{ combo}$$

