

Section 4 - Areas Volumes and Arc Lengths

Area between curves

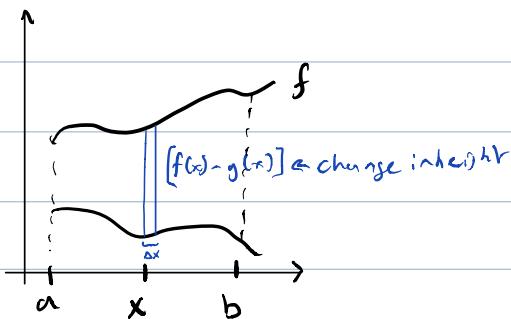
Let $y = f(x)$ and $y = g(x)$ represent two continuous functions over $[a, b]$

Assuming that $f \geq g$ over $[a, b]$

Then the area contained between them is :

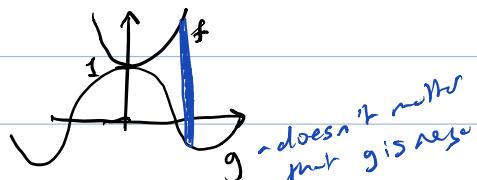
$$\text{Area} = \int_a^b [f(x) - g(x)] dx$$

for 2 rectangles
height of the separation
between two curves



ex. Let $f(x) = x^2 + 1$ and $g(x) = \cos x$

Find the area contained between f and g within the interval $[0, \pi]$



$$\begin{aligned}
 A &= \int_0^{\pi} \left[(x^2 + 1) - (\cos x) \right] dx \\
 &= \left[\frac{x^3}{3} + x - \sin x \right]_0^{\pi} \\
 &= \frac{\pi^3}{3} + \pi
 \end{aligned}$$

A vs:

Consider $f(x) = x^3 - x$ and $g(x) = x^2 - x$

Find the area of the region enclosed between f and g

pts of intersection: $x^3 - x = x^2 - x$

$$x^3 = x^2$$

$$x^3 - x^2 = 0$$

$$x(x^2 - x) = 0$$

$$x(x(x-1)) = 0$$

$$x = 0$$

$$x = 1$$

$$[0, 1] \rightarrow \text{for } x = \frac{1}{2}$$

$$f\left(\frac{1}{2}\right) = -0.375$$

$$g\left(\frac{1}{2}\right) = -0.25$$

so $g \geq f$ over $[0, 1]$

$$A = \int_0^1 [g(x) - f(x)] dx$$

$$= \int_0^1 [(x^2 - x) - (x^3 - x)] dx$$

$$= \int_0^1 (x^2 - x^3) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1$$

$$= \frac{1}{3} - \frac{1}{4}$$

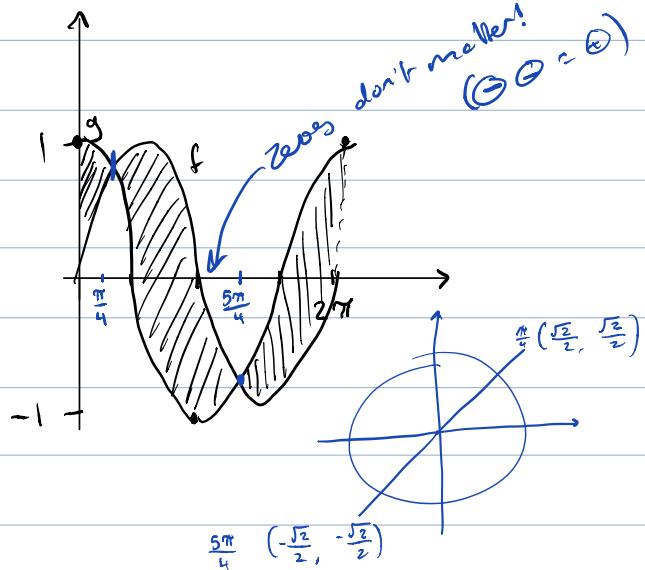
$$= \frac{1}{12}$$

Area

March 20, 2017

$$A = \int_a^b [f(x) - g(x)] dx \quad \text{given that } f(x) \geq g(x) \text{ over } [a, b]$$

Ex. Find the area that is bound between $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[0, 2\pi]$



$$A = \int_0^{\frac{\pi}{4}} [\cos x - \sin x] dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} [\sin x - \cos x] dx + \int_{\frac{5\pi}{4}}^{2\pi} [\cos x - \sin x] dx$$

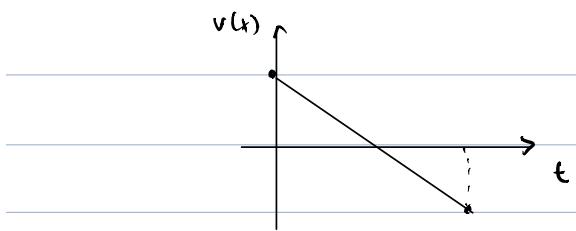
⋮

Special Case: DISPLACEMENT vs DISTANCE, INTEGRATIVE ACTIVITY

displacement = Signed area of the velocity function

$$\text{displacement} = \int_{t_1}^{t_2} v(t) dt$$

distance = area between the velocity function and the t-axis



S.A. = disp

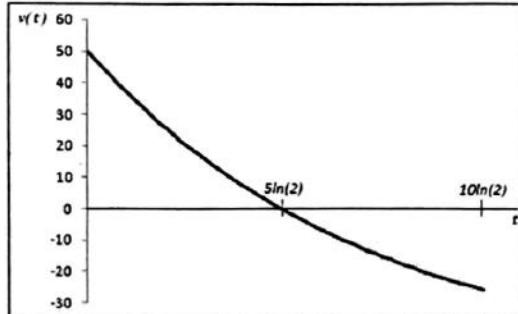
Area = distance

Problem 39

A pilot ejects from his F-14 after it has been severely damaged by the attack of a trig sub (submarine, not substitution). The pilot's vertical velocity (in m/s) at time t is given by

$$v(t) = -50 + 100e^{-0.2t}$$

The graph of $v(t)$ is shown.



- a) Find the pilot's displacement over the time interval $[0, 10\ln(2)]$.

$$\begin{aligned} \text{distance} &= \int_0^{5\ln 2} [(-50 + 100e^{-0.2t}) - 0] dt + \int_{5\ln 2}^{10\ln 2} [0 - (-50 + 100e^{-0.2t})] dt \\ &= \left[-50t - 500e^{-0.2t} \right]_0^{5\ln 2} + \left[50t + 500e^{-0.2t} \right]_{5\ln 2}^{10\ln 2} \\ &\quad 100 e^{-0.2t} \end{aligned}$$

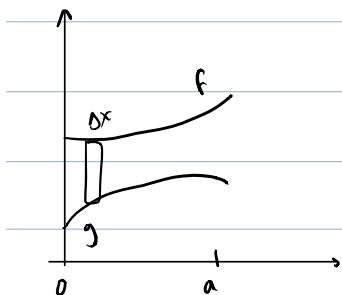
- b) Find the distance the pilot has travelled over the time interval $[0, 10\ln(2)]$.

$$\begin{aligned} \text{displacement} &= \int_0^{10\ln 2} |v(t)| dt \\ &= \int_0^{10\ln 2} [50 + 100e^{-0.2t}] dt \\ &= \left[-50t - 500e^{-0.2t} \right]_0^{10\ln 2} \end{aligned}$$

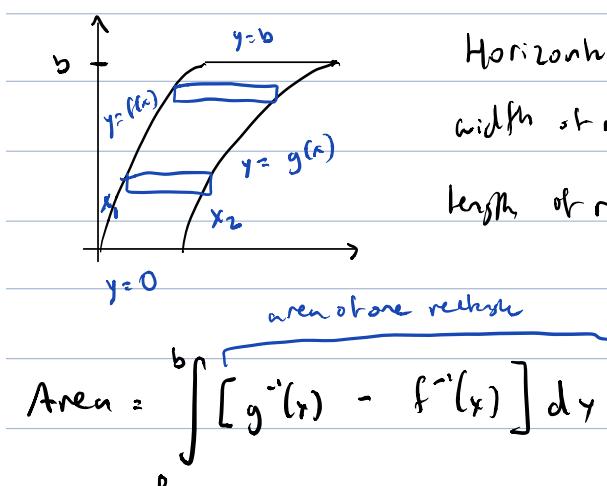
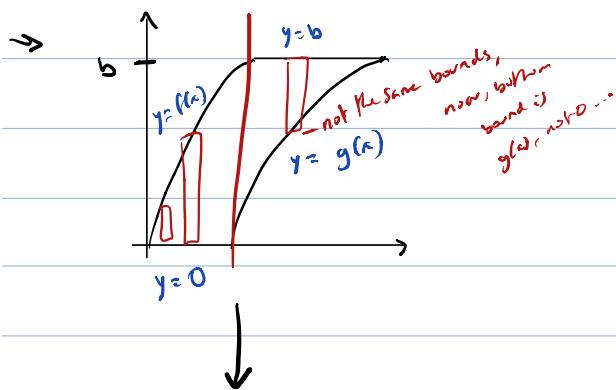
Horizontal rectangle representation

It may be convenient to express (convert) functions of x into functions of y in order to evaluate areas.

→ Classic x -interval

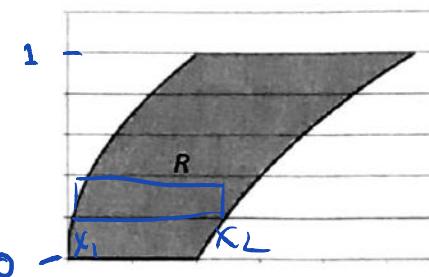


$$A = \int_0^a [f(x) - g(x)] dx$$



Problem 41

Consider the region R (shown below) and its boundary functions:



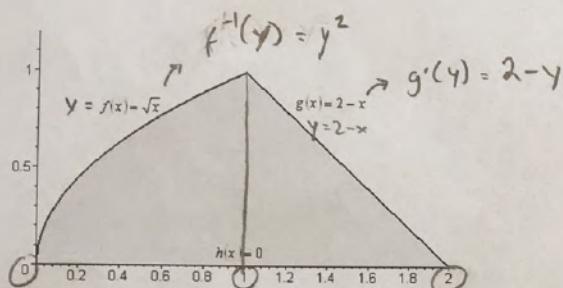
$$\begin{aligned}f(x) &= \sqrt{x} \\g(x) &= \ln(x) \\h(x) &= 1 \\y &= 0\end{aligned}$$

- Set up the x-integral(s) that would be required to compute the area of region R.
- Set up the y-integral(s) that would be required to compute the area of region R.
- Compute the area of region R using either result of b) or c).

$$f: y = \sqrt{x} \rightarrow x_1 = y^2$$

$$g: y = \ln x \rightarrow x_2 = e^y$$

$$\begin{aligned}\text{Area} &= \int_0^1 (e^y - y^2) dy \\&= \left[e^y - \frac{y^3}{3} \right]_0^1 \\&= \left(e - \frac{1}{3} \right) - 1\end{aligned}$$

Problem 44Consider the region that is bounded by the functions $f(x) = \sqrt{x}$, $g(x) = 2 - x$, and the x -axis (which we can think of as $h(x) = 0$).

- Set up the x-integral(s) that would be required to compute the area of region R.
- Set up the y-integral(s) that would be required to compute the area of region R.
- Compute the area of region R using either result of b) or c).

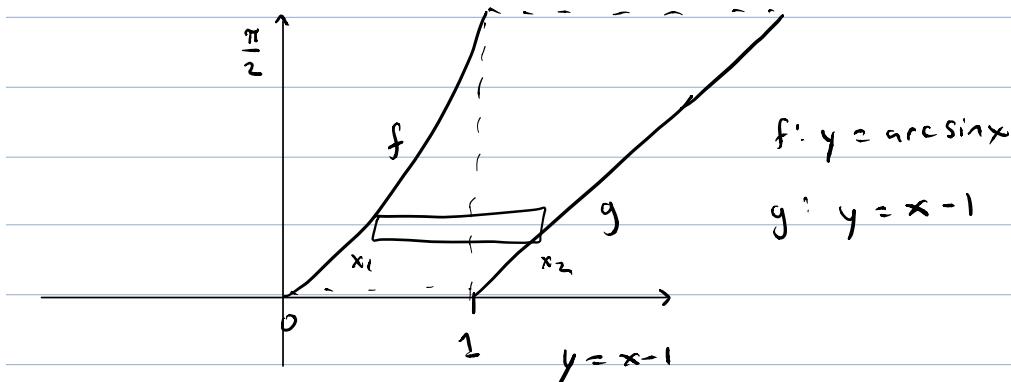
$$a) A_x = \int_0^1 [\sqrt{x} - 0] dx + \int_1^2 [(2-x) - 0] dx$$

$$\begin{aligned}b) A_y &= \int_0^1 [g'(y) - f'(y)] dy \\&= \int_0^1 [(2-y) - y^2] dy\end{aligned}$$

$$c) \begin{aligned}&= \left[2y - \frac{y^2}{2} - \frac{y^3}{3} \right]_0^1 \\&= 2 - \frac{1}{2} - \frac{1}{3}\end{aligned}$$

$$= \frac{7}{6}$$

À vous Set up using both x - and y -integrals the expressions required to compute the area of the following region.



x -integral

$$A_x = \int_0^1 [f(x) - g(x)] dx + \int_1^{\frac{\pi}{2}} [\frac{\pi}{2} - g(x)] dx$$

$$= \int_0^1 \arcsin x dx + \int_1^{\frac{\pi}{2}} [\frac{\pi}{2} - (x-1)] dx$$

y -integral

$$f: y = \arcsin x \rightarrow x = \sin y$$

$$g: y = x - 1 \rightarrow x = y + 1$$

$$A_y = \int_0^{\frac{\pi}{2}} [g^{-1}(y) - f^{-1}(y)] dy$$

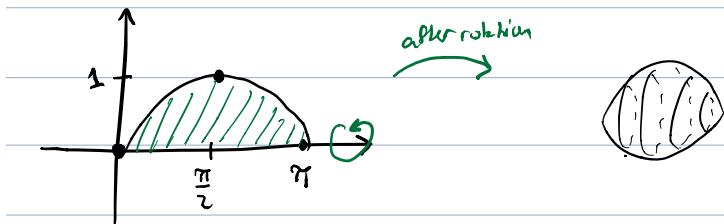
$$= \int_0^{\frac{\pi}{2}} [(y+1) - \sin y] dy$$

Volumes of revolution

March 22, 2017

→ Evaluate volumes that are obtained by rotating regions (enclosed between curves) around an axis of symmetry

e.g. Suppose $f(x) = \sin x$ over $[0, \pi]$ is rotated around the x-axis. What volume would be obtained?

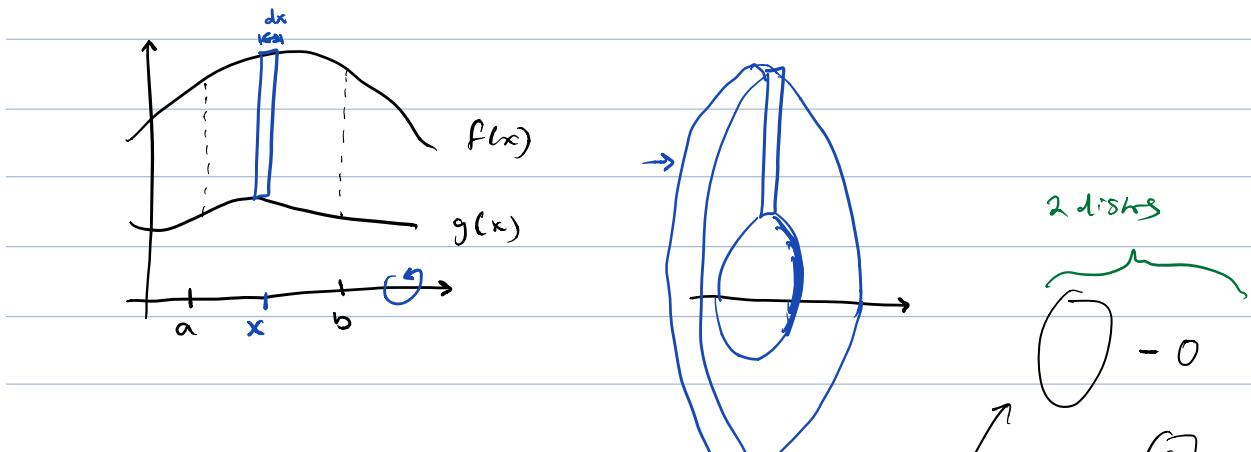


Method 1: Disk/Washer Method

→ Subdivide the area to be rotated into rectangles that are perpendicular to the axis of rotation.

→ Determine the volume generated by that rectangle

Iden: Consider the area between $f(x)$ and $g(x)$ within interval $[a, b]$ rotated around the x-axis.



$$dV \xrightarrow{\text{volume of disk} = \pi r^2 h} / = (0)$$

$$dV = \pi R^2 w - \pi r^2 w$$

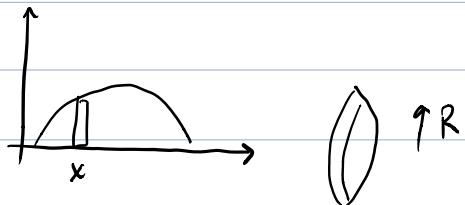
$$= \pi [f(x)]^2 dx - \pi [g(x)]^2 dx$$

$$= \pi ([f(x)]^2 - [g(x)]^2) dx$$

$$V = \int_{\text{domain}} dV$$

$$= \pi \int_a^b ([f(x)]^2 - [g(x)]^2) dx$$

ex. Let $f(x) = \sin x$ over $[0, \pi]$. What volume is obtained if $f(x)$ is rotated around the x -axis?



$$dV = \pi (f(x))^2 dx$$

$$= \pi \sin^2 x dx$$

$$V = \int_0^\pi \pi \sin^2 x dx$$

$$= \pi \int_0^\pi \frac{1}{2} (1 - \cos 2x) dx$$

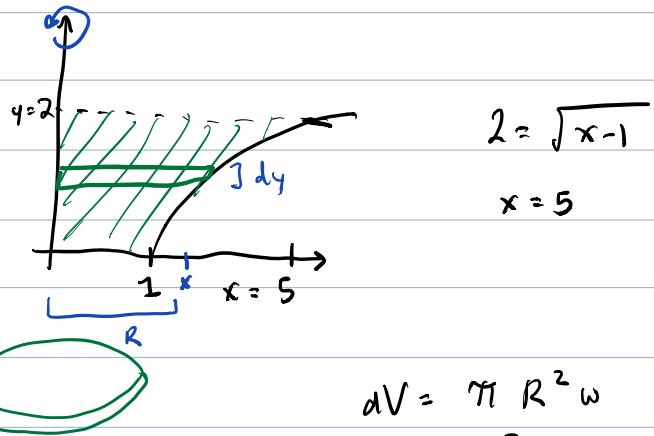
$$= \frac{\pi}{2} \left[\int_0^\pi 1 dx - \frac{1}{2} \int_0^\pi \cos 2x dx \right]$$

$$= \frac{\pi}{2} \left[x \Big|_0^\pi - \frac{1}{2} \sin 2x \Big|_0^\pi \right]$$

$$= \frac{\pi}{2} \left[\pi - \frac{1}{2}(0) \right] = \frac{\pi^2}{2}$$

ex. Let $f(x) = \sqrt{x-1}$

Find the volume generated by $f(x)$ when it is rotated around the y -axis (for values of y between 0 and 2)



$$2 = \sqrt{x-1}$$

$$x = 5$$

$$dV = \pi R^2 w$$

$$= \pi \left[f^{-1}(y) \right]^2 dy \text{ or } = \pi x^2 dy \\ = \pi (y^2 + 1)^2 dy$$

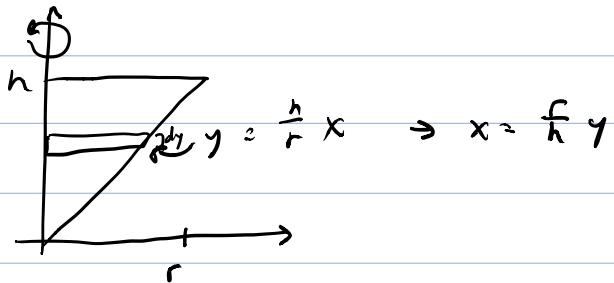
*don't need to convert bounds
b.c. it's w.r.t.
y*

$$V = \int_0^2 \pi (y^2 + 1)^2 dy$$

Note: When using a horizontal rectangle representation:

$$V = \int_c^d \pi [x_2^2 - x_1^2] dy$$

ex.



$$dV = \pi \left(\frac{r}{h} \cdot y\right)^2 dy$$

$$V = \int_0^h \pi \left(\frac{r}{h} \cdot y\right)^2 dy$$

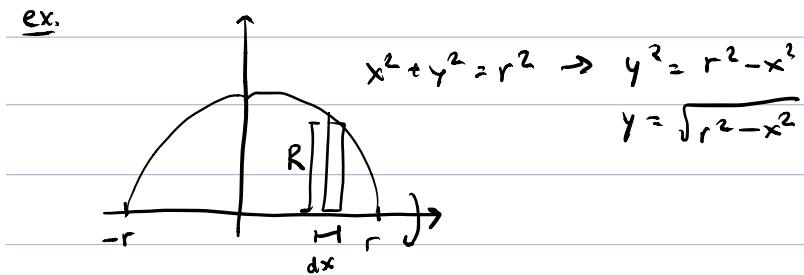
$$= \pi \cdot \frac{r^2}{h^2} \int_0^h y^2 dy$$

$$= \pi \frac{r^2}{h^2} \left[\frac{y^3}{3} \right]_0^h$$

$$= \pi \frac{r^3}{h^2} \cdot \frac{h^3}{3}$$

$$= \frac{\pi r^2 h}{3}$$

ex.



$$dV = \pi (\sqrt{r^2 - x^2})^2 dx$$

$$= \pi (r^2 - x^2) dx$$

$$V = \int_{-r}^r \pi (r^2 - x^2) dx$$

$$= \pi \int_{-r}^r (r^2 - x^2) dx$$

$$= \pi \cdot \left[r^2 x - \frac{x^3}{3} \right]_{-r}^r$$

$$= \pi \left[\left(r^3 - \frac{r^3}{3} \right) - \left(-r^3 + \frac{r^3}{3} \right) \right]$$

$$= \pi \left[2r^3 - \frac{2r^3}{3} \right]$$

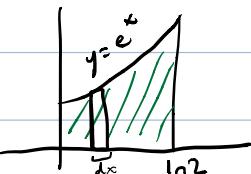
$$= \frac{\pi}{3} (6r^3 - 2r^3)$$

$$= \frac{4}{3} \pi r^3$$

X

ex. Mr. Nasty

Consider the region:



Set up (but do not compute) the integral(s) needed to find the volume obtained when the region is rotated:

1) around the x-axis

2) around the y-axis

3) around $y=3$

4) around $y=-1$

① $R = y = e^x$

$$V = \int_0^{\ln 2} \pi (e^x)^2 dx$$

$$= \int_0^{\ln 2} \pi e^{2x} dx$$

Let $u = 2x$

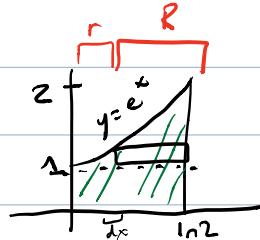
$$= \frac{1}{2} \int_0^{\ln 4} \pi e^u du$$

$$du = 2dx$$

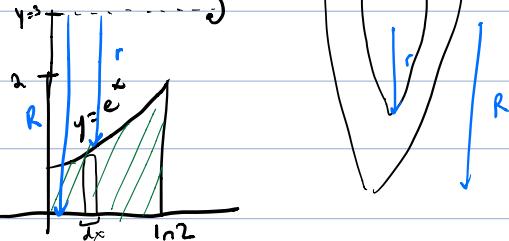
:

②

$$V = \int_0^1 \pi (\ln 2)^2 dy + \int_1^2 \pi [(\ln 2)^2 - (\ln y)^2] dy$$



③



March 23, 2017

$$R = 3$$

$$r = 3 - e^x$$

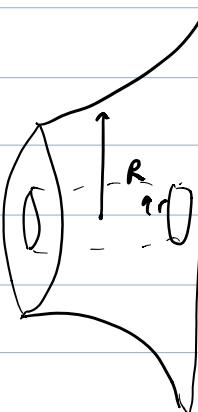
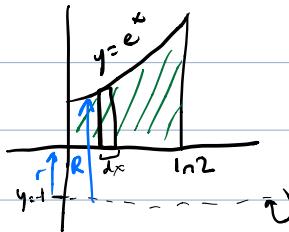
$$dV = \pi R^2 w - \pi r^2 w$$

$$= \pi (R^2 - r^2) w$$

$$= \pi (q - (3-e^x)^2) dx$$

$$V = \int_0^{\ln 2} \pi [q - (3-e^x)^2] dx$$

(4)



$$R = e^x - (-1) = e^x + 1$$

$$r = 0 - (-1) = 1$$

$$dV = \pi (R^2 - r^2) w$$

$$= \pi ((e^x+1)^2 - 1) dx$$

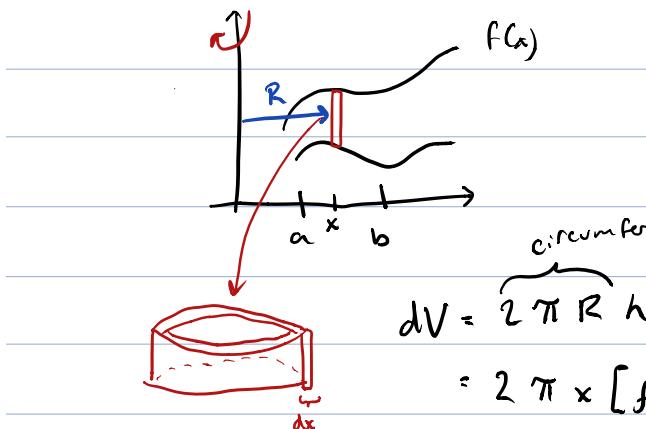
$$V = \int_0^{\ln 2} \pi [(e^x+1)^2 - 1] dx$$

Shell Method

This method is used when rectangles are drawn parallel to the axis of rotation.

Visualization!

Suppose the area between f and g within $[a, b]$ is rotated along the y -axis.



$$dV = \underbrace{2\pi R h w}_{\text{circumference}} \\ = 2\pi \times [f(x) - g(x)] dx$$

$$V = \int_a^b 2\pi \times [f(x) - g(x)] dx$$

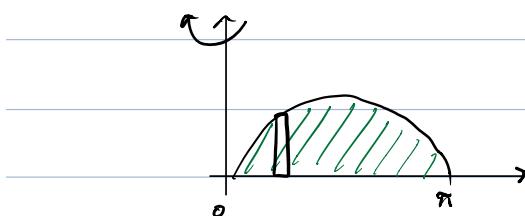
or

$$\int_c^d 2\pi y [y_2 - y_1] dy \quad \begin{pmatrix} \text{if rotation is around} \\ x\text{-axis} \end{pmatrix}$$

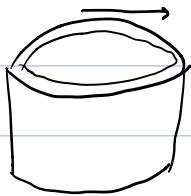
\uparrow or $f^{-1}(y)$ \uparrow or $y^{-1}(y)$

Ex. Let $f(x) = \sin x$ over $[0, \pi]$

What volume is generated if the region below $f(x)$ and above the x -axis is rotated around the y -axis?



rectangles parallel to the axis \rightarrow Shell method



$$dV = 2\pi R h w$$

$$= 2\pi \times [\sin x - 0] dx$$

$$= 2\pi \times \sin x dx$$

$$V = \int_0^{\pi} 2\pi \times \sin x dx$$

$$= 2\pi \int_0^{\pi} x \sin x dx$$

$$\begin{array}{rcl} f & & g' \\ \oplus x & & \sin x \end{array}$$

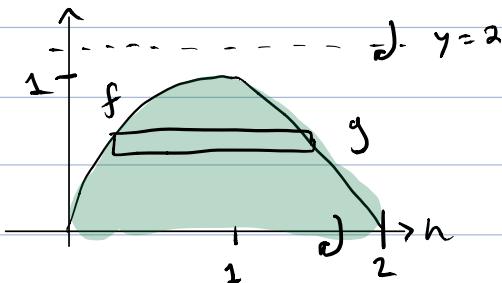
$$= 2\pi \left[-x \cos x + \sin x \right]_0^{\pi}$$

$$\begin{array}{rcl} \ominus 1 & & -\cos x \\ \ominus 0 & & -\sin x \end{array}$$

:

$$\begin{array}{rcl} \oplus 0 & & -\sin x \end{array}$$

Ex. Consider the following region:



$$f: y = \sqrt[3]{x}$$

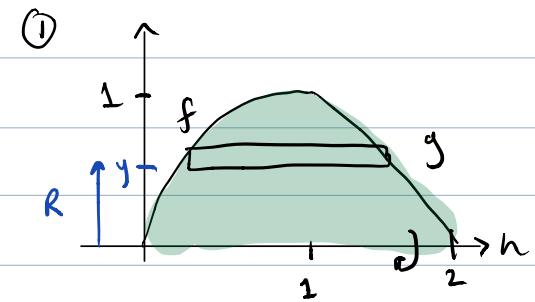
$$g: y = 2 - x$$

$$h: y = 0$$

Set up (using the Shell Method) the integrals required to find the volume obtained when the region is rotated:

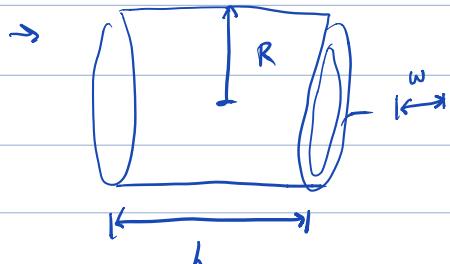
(1) around the x-axis

(2) around $y = 2$



$$f: y = \sqrt[3]{x} \rightarrow y^3 = x$$

$$g: y = 2 - x \rightarrow x = 2 - y$$

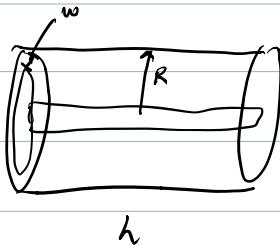
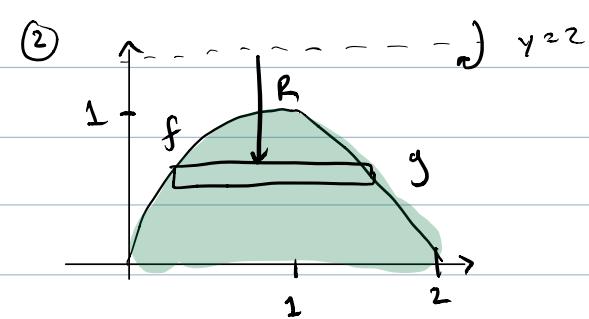


$$dV = 2\pi R h w$$

$$= 2\pi y [g^{-1}(y) - f^{-1}(y)] dy$$

$$= 2\pi y [(2-y) - y^3] dy$$

$$V = \int_0^1 2\pi y (2-y-y^3) dy$$



$$R = 2-y$$

$$h = x_2 - x_1 = 2-y - y^3$$

$$w = dy$$

$$0 \leq y \leq 1$$

$$dV = 2\pi R h w$$

$$= 2\pi (2-y) (2-y-y^3) dy$$

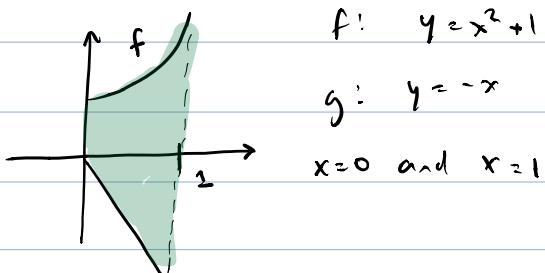
$$V = \int_0^1 2\pi (2-y) (2-y-y^3) dy$$

Advice

When evaluating areas or volumes of revolution

- ① Choose your rectangle representation
- ② Adapt the method to the rectangles

ex. Set up



$$f: y = x^2 + 1$$

$$g: y = -x$$

$$x=0 \text{ and } x=1$$

① Area

② Volume around y-axis

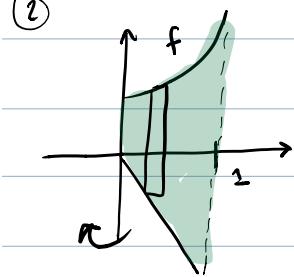
③ Volume around $x=1$

① Vertical rectangles

$$\int_0^1 (y_2 - y_1) dx$$

$$= \int_0^1 [(x^2 + 1) - (-x)] dx = \int_0^1 (x^2 + x + 1) dx$$

(2)



$$dV = 2\pi Rhw$$

$$= 2\pi \times ((x^2 + 1) - (-x)) dx$$

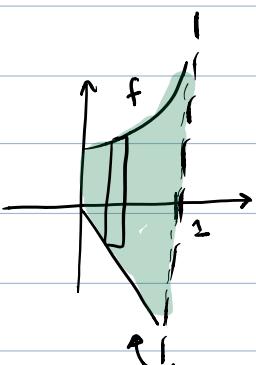
$$= 2\pi \times (x^2 + x + 1) dx$$



$$V = \int_0^1 2\pi \times (x^2 + x + 1) dx$$

$$\xrightarrow{R}$$

(3)



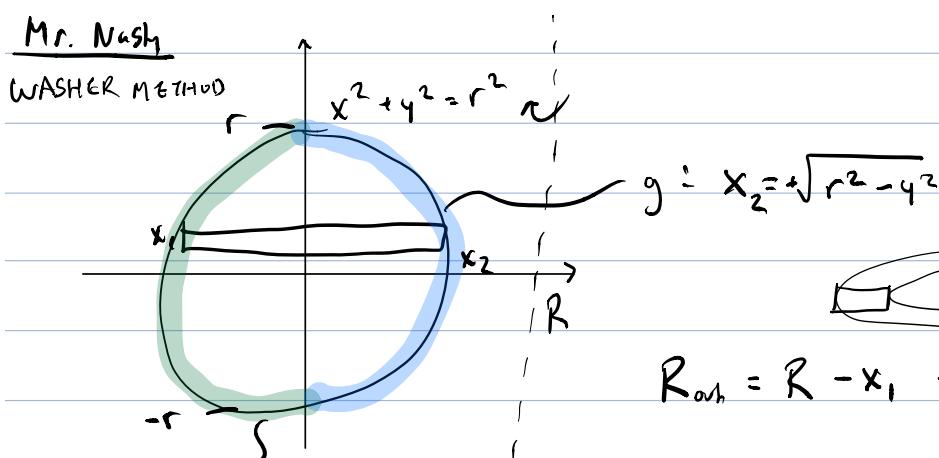
$$dV = 2\pi Rhw$$

$$= 2\pi (1-x)(x^2 + x + 1) dx$$

$$V = \int_0^1 2\pi (1-x)(x^2 + x + 1) dx$$

Mr. Nash

WASHER METHOD



$$R_{out} = R - x_1 = R - (-\sqrt{r^2 - y^2}) \\ = R + \sqrt{r^2 - y^2}$$

$$f: x_1 = -\sqrt{r^2 - y^2}$$

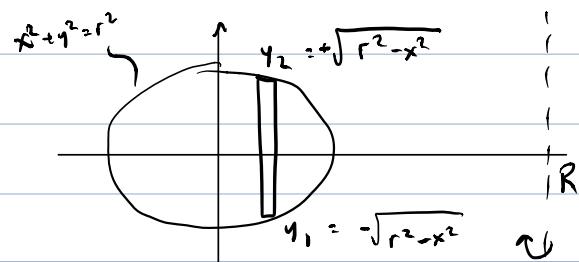
$$R_{in} = R - x_2 = R - \sqrt{r^2 - y^2}$$

$$dV = \pi R^2 w$$

$$= \pi \left[(R + \sqrt{r^2 - y^2})^2 - (R - \sqrt{r^2 - y^2})^2 \right] dy$$

$$V = \int_{y=-r}^{y=r} \pi \left[(R + \sqrt{r^2 - y^2})^2 - (R - \sqrt{r^2 - y^2})^2 \right] dy$$

SHELL METHOD



$$dV = 2\pi(\text{radius}) hw$$

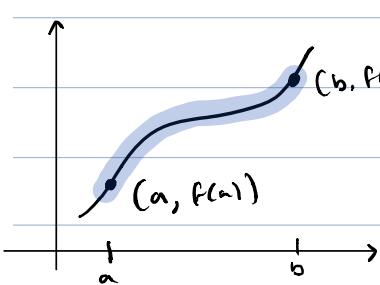
$$= 2\pi (R-x) (2\sqrt{r^2-x^2}) dx$$

$$\text{radius} = R-x$$

$$h = (\sqrt{r^2 - x^2}) - (-\sqrt{r^2 - x^2}) \\ = 2\sqrt{r^2 - x^2}$$

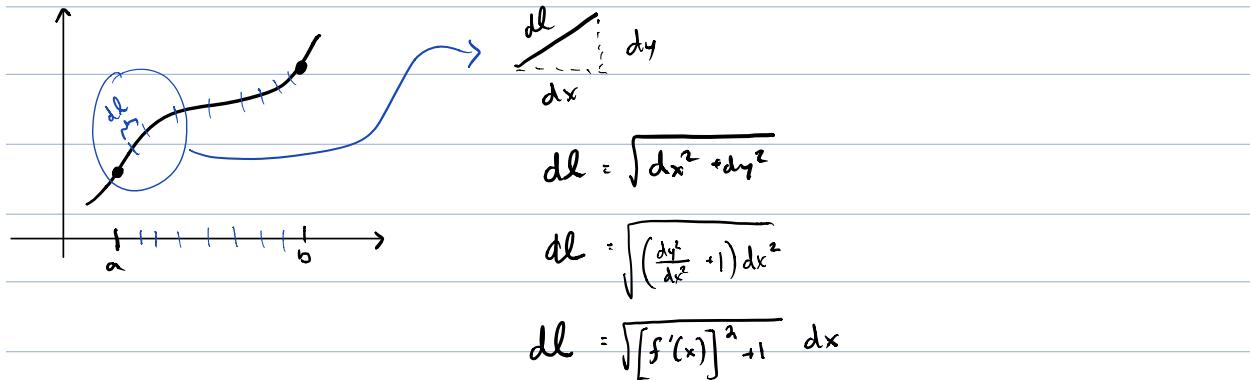
$$V = \int_{-r}^r [2\pi(R-x)(2\sqrt{r^2-x^2}) dx]$$

Arc Length of a Curve



Arc length = distance required to travel from $(a, f(a))$ to $(b, f(b))$ while remaining on the curve.

How do we do this?



Therefore,

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx \quad \text{or} \quad L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

ex. Let $f(x) = \ln(\sec x)$ $0 \leq x \leq \frac{\pi}{4}$

Find the arc length generated by $f(x)$ over the given interval.

$$f'(x) = [\ln(\sec x)]' = \frac{1}{\sec x} \cdot \cancel{\sec x \tan x}$$

$$= \tan x$$

$$1 + [f'(x)]^2 = 1 + \tan^2 x \\ = \sec^2 x$$

$$\sqrt{1 + [f'(x)]^2} = \sec x \quad (\text{sign is taken care of in the given interval})$$

$$\begin{aligned} L &= \int_0^{\frac{\pi}{4}} \sqrt{1 + [f'(x)]^2} dx \\ &= \int_0^{\frac{\pi}{4}} \sec x dx \\ &= \left[\ln |\sec x + \tan x| \right]_0^{\frac{\pi}{4}} \end{aligned}$$

$$= \ln(\sqrt{2} + 1)$$

ex À vous

Let $f(x) = \frac{1}{3}x^{3/2} - x^{1/2}$ over $[1, 4]$. Find L

$$f'(x) = \frac{1}{2}x^{1/2} - \frac{1}{2}x^{-1/2}$$

$$= \frac{1}{2}\sqrt{x} - \frac{1}{2\sqrt{x}}$$

$$1 + (f'(x))^2 = 1 + \left(\frac{1}{2}\sqrt{x} - \frac{1}{2\sqrt{x}} \right)^2$$

$$= 1 + \left[\frac{1}{4}x - \frac{1}{2} + \frac{1}{4x} \right]$$

$$= \frac{1}{2} + \frac{1}{4}x + \frac{1}{4x} = \left(\frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2} \right)^2$$

$$\sqrt{1 + [f(x)]^2} = \sqrt{\frac{1}{2} + \frac{1}{4}x + \frac{1}{4x}}$$

OR

$$L = \int_1^4 \sqrt{\frac{1}{2} + \frac{1}{4}x + \frac{1}{4x}} dx$$

$$= \int_1^4 \sqrt{\frac{1}{4}(2 + x + \frac{1}{x})} dx$$

$$= \int_1^4 \sqrt{\frac{x^2 + 2x + 1}{x}} dx$$

$$= \int_1^4 \sqrt{\frac{(x+1)^2}{x}} dx$$

$$= \int_1^4 (x+1) x^{-1/2} dx$$

$$= \int_1^4 \left[\frac{x^{3/2}}{2 \cdot 3/2} + \frac{x^{1/2}}{2 \cdot 1/2} \right] dx$$

$$= \left[\frac{1}{3} x^{3/2} + \frac{1}{2} x^{1/2} \right]_1^4$$

...

$$= \int_1^4 \left[x^{1/2} + x^{-1/2} \right] dx$$

Ex. $y = \frac{2}{3} x^{3/2}$ over $[1, 4]$

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Arc length of a curve

Ex. Let $f(x) = \frac{1}{2} \ln(\sin x) + \frac{1}{2} \ln(\cos x)$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx \quad \text{or} \quad L = \int_c^d \sqrt{1 + [g'(y)]^2} dy \quad \text{where } g' = f^{-1}$$

Find the length of curve f within $[\frac{\pi}{6}, \frac{\pi}{3}]$

$$f'(x) = \frac{1}{2} \cdot \frac{1}{\sin x} \cdot \cos x - \frac{1}{2} \cdot \frac{1}{\cos x} \cdot \sin x \\ = \frac{1}{2} \cot x - \frac{1}{2} \tan x$$

$$[f'(x)]^2 = \frac{1}{4} \cot^2 x - 2 \cdot \frac{1}{2} \cot x \cdot \frac{1}{2} \tan x + \frac{1}{4} \tan^2 x$$

$$= \frac{1}{4} \cot^2 x - \frac{1}{2} + \frac{1}{4} \tan^2 x$$

$$1 + [f'(x)]^2 = \frac{1}{4} \cot^2 x - \frac{1}{2} + \frac{1}{4} \tan^2 x$$

$$= \left(\frac{1}{2} \cot x + \frac{1}{2} \tan x \right)^2$$

$$\sqrt{1 + [f'(x)]^2} = \frac{1}{2} \cot x + \frac{1}{2} \tan x$$

$$L = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left[\frac{1}{2} \cot x + \frac{1}{2} \tan x \right] dx \\ = - \left[\frac{1}{2} \ln(\sin x) - \frac{1}{2} \ln(\cos x) \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$