8 - Introduction to Complex Numbers

The equation $x^2 = -1$ has no solution in $\mathbb{R}$ b.c. $\sqrt{-1}$ is undefined.

So we can expand the set of real numbers by defining $i = \sqrt{-1}$ $(i^2 = -1)$

We expand $\mathbb{R}$ to the set of complex numbers, denoted by $\mathbb{C}$

ex. a) $3i + 5i = 8i$

real part

imaginary part (just the # in front of i)

b) $-2i^2 + 6i = -2(-1) + 6i = \underset{\text{real part}}{2} + \underbrace{6}_{\text{imaginary part}}i$

General/Rectangular Form of a Complex Number

$$\underset{\text{real part}}{a} + \underset{\text{imaginary part}}{b}\,i$$

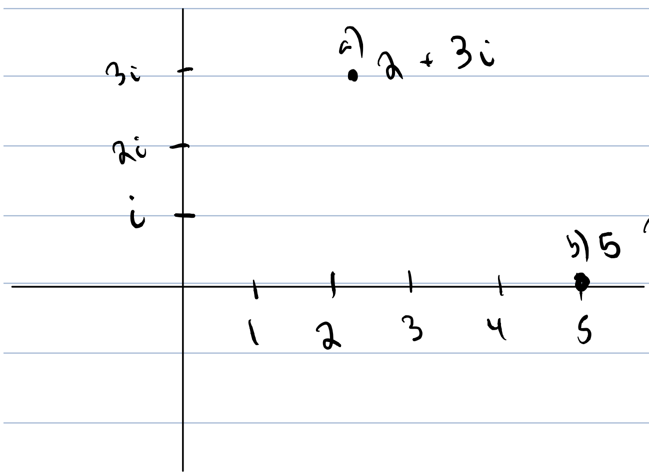
ex. a) $(-i)^2 = (-1)^2 \, i^2 = -1$ b) $0 \cdot i = 0$

Note: Any real number is also a complex number:

If $a \in \mathbb{R}$, then $a \in \mathbb{C}$

Cartesian Plane Representation of Complex Numbers

A useful way of representing complex numbers is by using a Cartesian plane.



$5 = 5 + 0i$

Operations:

ex. Let $z = 2 + 5i$, $w = 3 - 4i$

Find:

a) $w + z$
$= 3 - 4i + 2 + 3i$
$= 5 - i$

b) $3z$
$= 3(2 + 5i)$
$= 6 + 15i$

c) $-3z$
$= -6 - 15i$

d) $z \cdot w$
$= (2 + 5i)(3 - 4i)$

e) $w \cdot z$

$= (3-4i)(2+5i)$

$= 26 + 7i$

$= 6 + 15i - 8i - 20i^2$

$= 6 + 7i + 20$

$= 26 + 7i$

<u>Note</u>: Multiplication is commutative : $zw = wz$

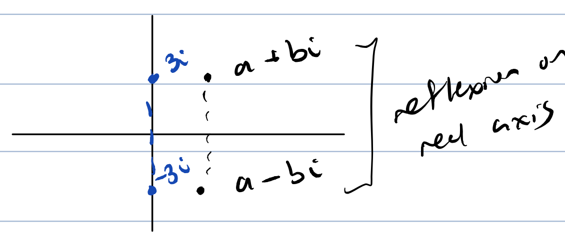
Careful: $(a+bi)(c+di) \neq ac + bdi$

What about division?

<u>Conjugate of a Complex Number</u>:

If $z = a + bi$, we define the complex conjugate of z as

$\bar{z} = a - bi$



<u>ex.</u> a) If $z = 2 + 3i \rightarrow \bar{z} = 2 - 3i$

b) If $z = 3i \rightarrow \bar{z} = -3i$

c) If $z = 5 \rightarrow \bar{z} = 5$

$(5 + 0i) \qquad (5 - 0i)$

d) If $z = 6 - 3i$, find $z \cdot \bar{z}$

$$z \cdot \bar{z} = (6-3i)(6+3i) = 36 - \cancel{18i} + \cancel{18i} - 9i^2$$
$$= 36 + 9$$
$$= 45$$

In general: If $z = a + bi$

$$z \cdot \bar{z} = a^2 + b^2$$

Absolute Value: If $z = a + bi$

$$|z| = \sqrt{z \cdot \bar{z}} = \sqrt{a^2 + b^2}$$

ex. a) If $z = 2 + 3i$

$$|z| = \sqrt{2^2 + 3^2}$$
$$= \sqrt{13}$$

b) If $z = 5$

$$|z| = \sqrt{5^2 + (0i)^2}$$
$$= \sqrt{25}$$
$$= 5$$

c) If $z = -5$

$$|z| = \sqrt{(-5)^2} = 5$$

Division

Let z and w be two complex numbers with $w \neq 0$,

then, $\frac{z}{w} = \frac{z}{w} \cdot \frac{\bar{w}}{\bar{w}}$ (multiply numberator and denominator by the conjugate of the denominator)

$= \frac{z\,\bar{w}}{|w|}$

ex. Let $z = 3+4i$, $w = 1-2i$, $x = 5i$

a) $\frac{z}{w} = \frac{3+4i}{1-2i}$

$= \frac{(3+4i)(1+2i)}{5}$

$= \frac{3+4i+6i+4\cdot 2i^2}{5}$

$= \frac{-5+10i}{5} = -1+2i$

b) $\frac{1}{z} = \frac{1}{3+4i}$

$= \frac{3-4i}{3^2+4^2}$

$= \frac{3-4i}{25}$

$= \frac{3}{25} - \frac{4}{25}i$

c) $\frac{x}{w}$

$= \frac{5i}{1-2i}$

$= \frac{5i(1+2i)}{1^2+2^2}$

$= \frac{5i+10i^2}{5}$

$= \frac{-10+5i}{5} = -2+i$

d) $\frac{z}{x}$

$= \frac{3+4i}{5i} \frac{(-5i)}{(-5i)}$

$= \frac{-15i-20i^2}{25} = \frac{4}{5} - \frac{3}{5}i$

Note: $\sqrt{-5} = \sqrt{(-1)(5)} \quad \sqrt{-1}\,\sqrt{5} = \sqrt{5}\,i$

Note: $i^0 = 1$

$i = i$

$i^2 = -1$

$i^3 = -i \quad (i^2 \cdot i)$

$i^4 = 1 \quad ((i^2)^2)$

$i^5 = i^4 i$

$= (i^2)^2 i$

$= i$

$i^6 = (i^2)^3 = (-1)^3 = -1$

$i^7 = i^6 \cdot i = (i^2)^3 \cdot i = i -$

ex. a) $i^{57} = i^{56}\, i$

$= (i^2)^{28}\, i$

$= (+1)\, i$

$= i$

b) $\sqrt{-12} + \sqrt{-27}$

$= i\sqrt{12} + i\sqrt{27}$

$= i\left(\sqrt{12} + \sqrt{27}\right)$

$= \left(\sqrt{12} + \sqrt{27}\right) i$

$= \left(2\sqrt{3} + 3\sqrt{3}\right) i$

$= 5\sqrt{3}\, i$

c) $\sqrt{-9}\,\sqrt{-4}$

$= \sqrt{9}\,i\,\sqrt{4}\,i$

$= \sqrt{36}\, i^2$

$= -6$

Careful: $\sqrt{-9} \cdot \sqrt{-9} \neq \sqrt{(-9)(-9)}$

$\sqrt{a}\sqrt{b} = \sqrt{ab}$ doesn't work if $a<0$ & $b<0$

both negative

Solving Equations in $\mathbb{C}$

Solve $x^2 - 4x + 6 = 0$ in $\mathbb{C}$

Note: In $\mathbb{R}$, $b^2 - 4ac = (-4)^2 - 4 \cdot 1 \cdot 6 = 16 - 24 = -8 < 0$

So the eqn has no sol'n

In $\mathbb{C}$:

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(6)}}{2} = 2 \pm \frac{\sqrt{16 - 24}}{2}$$

$$= 2 \pm \frac{\sqrt{-8}}{2}$$

$$= 2 \pm \frac{\sqrt{8}}{2} i$$

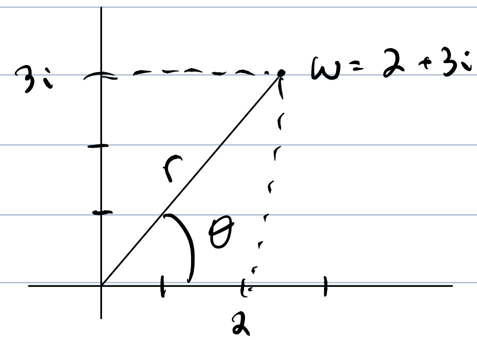
$$= 2 \pm \frac{\cancel{2}\sqrt{2}}{\cancel{2}} i$$

$$= 2 + \sqrt{2}\, i \text{ and } 2 - \sqrt{2}\, i$$

Sol'n: $x \in \{2 + \sqrt{2}\, i, 2 + \sqrt{2}\, i\}$

Polar Form (trigonometric form)

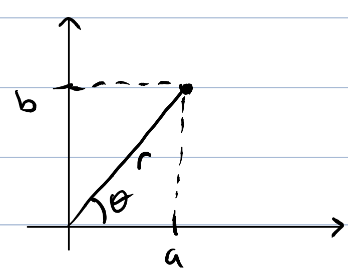
ex. $w = 2 + 3i$ can be seen as the point $(2, 3)$ in the Cartesian plane



$$r = \sqrt{2^2 + 3^2} = \sqrt{13}$$

$$\theta = \arctan\left(\frac{3}{2}\right) \approx 56.3^\circ$$

In general:



The point (a, b) can be represented by polar coordinates (r, θ)

$$r = \sqrt{a^2 + b^2} = |z|$$

abs. val

and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$\cos\theta = \frac{a}{r} \rightarrow a = r\cos\theta$

$\sin\theta = \frac{b}{r} \rightarrow b = r\sin\theta$

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We'll represent the number $z = a + bi$

as

$$z = r\cos\theta + r\sin\theta\, i$$

or

$z = r \operatorname{cis}\theta$ where $\operatorname{cis}\theta = \cos\theta + \sin\theta\, i$

ex.

a) Express the number $z = 3\operatorname{cis}\left(\frac{\pi}{3}\right)$ in rectangular form

$$\begin{aligned} z &= 3\operatorname{cis}\left(\tfrac{\pi}{3}\right) \\ &= 3\cos\left(\tfrac{\pi}{3}\right) + 3\sin\left(\tfrac{\pi}{3}\right) i \\ &= 3\cdot\tfrac{1}{2} + 3\,\tfrac{\sqrt{3}}{2}\, i \\ &= \tfrac{3}{2} + \tfrac{3\sqrt{3}}{2}\, i \end{aligned}$$

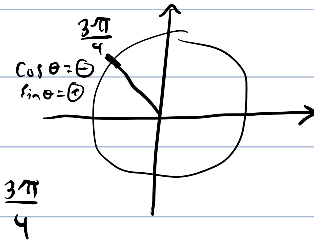
b) Find the polar form for $z = -\sqrt{2} + \sqrt{2}\, i$

$$r = |z| = \sqrt{(-\sqrt{2})^2 + (\sqrt{2})^2} = 2$$

$$\cos\theta = \frac{a}{r} = \frac{-\sqrt{2}}{2}$$

$$\sin\theta = \frac{b}{r} = \frac{\sqrt{2}}{2}$$

So, $\theta = \frac{3\pi}{4}$

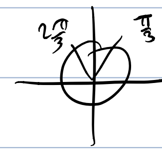


So $z = 2\,\text{cis}\left(\frac{3\pi}{4}\right)$

c) Convert to polar form: $z = -1 + \sqrt{3}\,i$

$$r = |z| = \sqrt{1+3} = 2$$

$$\left.\begin{array}{l}\cos\theta = -\frac{1}{2} \\ \sin\theta = \frac{\sqrt{3}}{2}\end{array}\right] \theta = \frac{2\pi}{3}$$



$$\theta = \frac{2\pi}{3}$$

$$z = 2\,\text{cis}\left(\frac{2\pi}{3}\right)$$

d) Convert to polar form: $z = 4i$

$$r = \sqrt{4^2}$$
$$= 4$$

$$\left.\begin{array}{l}\cos\theta = \frac{0}{4} = 0 \\ \sin\theta = \frac{4}{4} = 1\end{array}\right] \theta = \frac{\pi}{2}$$

$$z = 4\,\text{cis}\left(\frac{\pi}{2}\right)$$

e) Convert to polar form: $z = -5$

$r = 5$

$\cos\theta = -1$
$\sin\theta = 0$

$z = 5\,\text{cis}(\pi)$

Multiplying and Dividing in Polar Form

Multiplication:

Let $z = r_1(\cos\alpha + i\sin\alpha)$, $w = r_2(\cos\beta + i\sin\beta)$

$$zw = r_1 \cdot r_2\left[\cos(\alpha+\beta) + i\sin(\alpha+\beta)\right]$$

Proof: $zw = (r_1\cos\alpha + r_1\sin\alpha\, i)(r_2\cos\beta + r_2\sin\beta\, i)$

$= r_1r_2\cos\alpha\cos\beta + r_1r_2\cos\alpha\sin\beta\, i + r_1r_2\sin\alpha\cos\beta\, i$
$\quad + r_1r_2\sin\alpha\sin\beta\, i^2$

$= r_1r_2\left[\underbrace{(\cos\alpha\cos\beta - \sin\alpha\sin\beta)}_{\text{real part}} + (\underbrace{\cos\alpha\sin\beta + \sin\alpha\cos\beta}_{\text{imaginary part}})i\right]$

$= r_1r_2\left[\cos(\alpha+\beta) + \sin(\alpha+\beta)\, i\right]$

ex. Let $z = 8 \text{ cis}\left(\frac{5\pi}{3}\right)$, $w = 4 \text{ cis}\left(\frac{2\pi}{3}\right)$

Find zw.

$zw = 32 \text{ cis}\left(\frac{7\pi}{3}\right)$

$= 32 \text{ cis}\left(\frac{\pi}{3}\right)$

Division!

$$\frac{z}{w} = \frac{r_1}{r_2}\left[\cos(\alpha - \beta) + i \sin(\alpha - \beta)\right] = \frac{r_1}{r_2} \cos(\alpha - \beta)$$

ex. a) Let $z = 8 \text{ cis}\left(\frac{5\pi}{3}\right)$, $w = 4 \text{ cis}\left(\frac{2\pi}{3}\right)$

$\frac{z}{w} = 2 \text{ cis}\left(\frac{\cancel{3}\pi}{\cancel{3}}\right) = 2 \text{ cis}(\pi)$

b) Convert to rect. form

$\frac{z}{w} = -2$

Note: Let $z = r(\cos\theta + i \sin\theta)$

a) $z^2 = r^2(\cos 2\theta + i \sin 2\theta)$
$= r^2 \text{ cis } 2\theta$

b) $z^3 = z\, z^2 = z\, r^2 \text{cis } 2\theta$
$= r^3 \text{ cis } 3\theta$

So, in general, we have De Moivre's Theorem:

$$z^n = r^n \left(\cos(n\theta) + i \sin(n\theta)\right)$$

ex. Find $\left(-\sqrt{2} + i\sqrt{2}\right)^5$

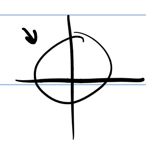
→ convert to polar form

$z = -\sqrt{2} + i\sqrt{2}$

$r = \sqrt{2+2} = 2$

$\cos\theta = \dfrac{-\sqrt{2}}{2}$

$\sin\theta = \dfrac{\sqrt{2}}{2}$



$z = 2 \operatorname{cis}\left(\dfrac{3\pi}{4}\right)$

$\therefore z^5 = 2^5 \operatorname{cis}\left(\dfrac{15\pi}{4}\right) = 32 \operatorname{cis}\left(\dfrac{7\pi}{4}\right)$

ex. Find cube roots of $8i$,

i.e. solve $z^3 = 8i$

→ Find $8i$ in polar form:

$z^3 = 8 \operatorname{cis} \frac{\pi}{2}$

$z^3 = r^3 \operatorname{cis} 3\theta = 8 \operatorname{cis} \frac{\pi}{2}$

$\therefore r^3 = 8 \qquad \therefore 3\theta = \frac{\pi}{2} + 2\pi K$

$r = 2$

$K=0: \quad 3\theta = \frac{\pi}{2} \rightarrow \theta = \frac{\pi}{6}$

$K=1: \quad 3\theta = \frac{\pi}{2} + 2\pi = \frac{5\pi}{2} \rightarrow \theta = \frac{5\pi}{6}$

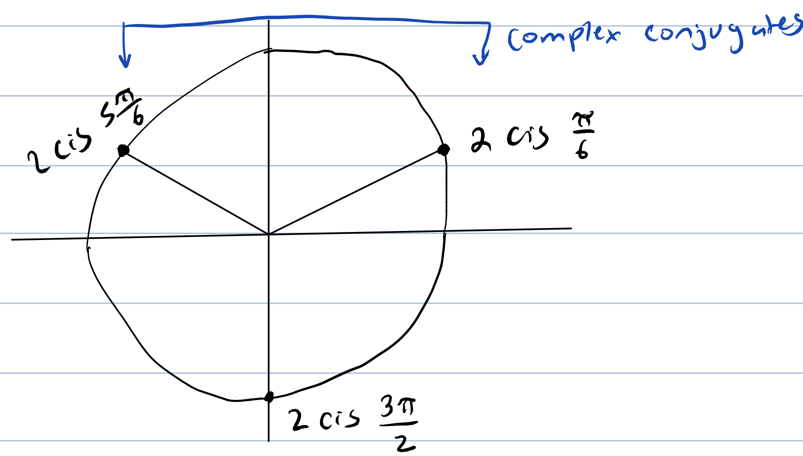
$K=2: \quad 3\theta = \frac{\pi}{2} + 4\pi = \frac{9\pi}{2} \rightarrow \theta = \frac{9\pi}{6} = \frac{3\pi}{2}$

$K=3: \quad 3\theta = \frac{\pi}{2} + 6\pi = \frac{13\pi}{2} \rightarrow \theta = \frac{13\pi}{6} = \frac{\pi}{6}$

$\therefore$ So $8i$ has three cube roots.

$z_1 = 2 \operatorname{cis} \frac{\pi}{6}, \quad z_2 = 2 \operatorname{cis} \frac{5\pi}{6}, \quad z_3 = 2 \operatorname{cis} \frac{3\pi}{2}$

<u>Note:</u>



<u>ex.</u> a) Solve for z in $\mathbb{C}$

$6z^2 + 12z + 7 = 0$

$$z = \frac{-12 \pm \sqrt{12^2 - 4(6)(7)}}{12}$$

$$= -1 \pm \frac{\sqrt{144 - 168}}{12}$$

$$= -1 \pm \frac{\sqrt{-24}}{12}$$

$$\sqrt{24} = \sqrt{6 \cdot 4} = 2\sqrt{6}$$

$$= -1 \pm \frac{\sqrt{24}\, i}{12}$$

$$= -1 \pm \frac{\sqrt{6}}{6} i$$

$$z \in \left\{ -1 + \frac{\sqrt{6}}{6} i,\ -1 - \frac{\sqrt{6}}{6} i \right\}$$

b) $(3i)^5$

$z = 3i$

$z = 3 \operatorname{cis}\left(\frac{\pi}{2}\right)$

$z^5 = 243 \operatorname{cis}\left(\frac{5\pi}{2}\right)$

$= 243 \operatorname{cis}\left(\frac{\pi}{2}\right) = 343\, i$

c) $$\frac{(1+2i)(3-i)}{2+i}$$

$$= \frac{(1+2i)(3-i)}{(2+i)(2-i)} (2-i)$$

$$= \frac{(3 + 2 + 6i - i)(2 - i)}{4 + 1}$$

$$= \frac{\cancel{5}(1 + i)(2 - i)}{\cancel{5}}$$

$$= 2 + 1 + 2i - i$$

$$= 3 + i$$

d) $(1 + i)^7$

$z = 1 + i$

$r = \sqrt{2}$

$$\left.\begin{array}{l} \cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \sin\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \end{array}\right] \theta = \frac{\pi}{4}$$

$r = \sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right)$

$$(\sqrt{2})^7 = \left(\left(\sqrt{2}\right)^2\right)^3 \sqrt{2}$$
$$= 8\sqrt{2}$$

$$z^7 = (\sqrt{2})^7 \operatorname{cis}\left(\frac{7\pi}{4}\right)$$

$$= 8\sqrt{2} \operatorname{cis}\left(\frac{7\pi}{4}\right)$$