

Section 5 - Improper Integrals

An integral is considered improper when its integrand ($f(x)$) is

→ discontinuous at the bounds of integration

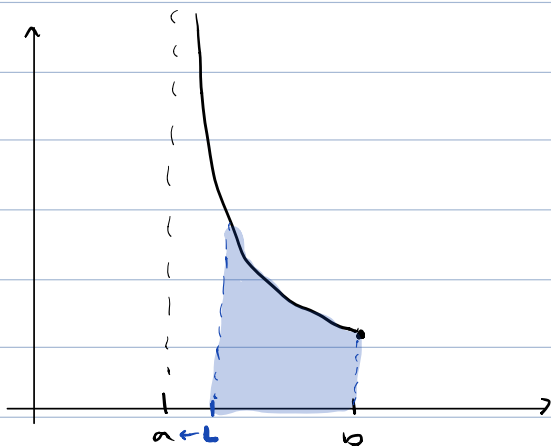
→ discontinuous between the bounds of integration

→ evaluated over infinite bounds

Case 1: Discontinuity at the left bound

Suppose $f(x)$ is discontinuous at $x=a$

$$\text{Then } \int_a^b f(x) dx = \lim_{L \rightarrow a^+} \int_L^b f(x) dx$$



ex. Evaluate the improper integral

$$\int_0^1 \frac{1}{2\sqrt{x}} dx$$

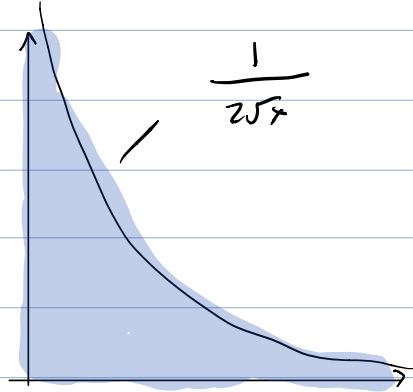
Note $\frac{1}{2\sqrt{x}}$ is discontinuous (undefined)
at $x=0$

$$= \lim_{L \rightarrow 0^+} \int_L^1 \frac{1}{2} x^{-1/2} dx$$

$$= \lim_{L \rightarrow 0^+} \left[\frac{1}{2} \cdot \frac{x^{-1/2}}{-1/2} \right]_L^1$$

$$= \lim_{L \rightarrow 0^+} [\sqrt{1} - \sqrt{L}]$$

$$= 1 - 0 = 1$$



This improper interval
converges to 1

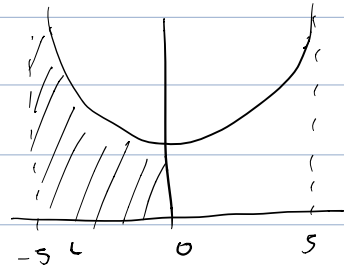
ex. À vous

$$\int_{-5}^0 \frac{1}{\sqrt{25-x^2}} dx$$

$$25-x^2 \geq 0 \text{ if } x \leq 5, x \geq -5$$

$$= \lim_{L \rightarrow -5^+} \int_L^0 \frac{1}{\sqrt{25-x^2}} dx$$

$$= \lim_{L \rightarrow -5^+} \left[\arcsin \frac{x}{5} \right]_L^0$$



$$= \lim_{L \rightarrow -5^+} \left[\arcsin 0 - \arcsin \frac{L}{5} \right]$$

$$= \lim_{L \rightarrow -5^+} \left[-\arcsin \frac{L}{5} \right]$$

$$= -\arcsin \frac{-5}{5}$$

$$= -\left(-\frac{\pi}{2}\right)$$

$$= \frac{\pi}{2}$$

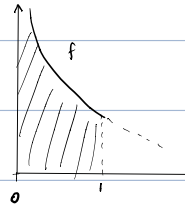
ex. À vous

$$f(x) = \frac{1}{x^{3/2}} \quad \text{dom}(f) =]0, +\infty[$$

a) Find the Area between $x=0$, $x=1$

b) Find the volume in the 1st quadrant between $x=0$, $x=1$

when rotated around the y-axis



$$a) A = \int_0^1 x^{-3/2} dx$$

$$= \lim_{L \rightarrow 0^+} \int_L^1 x^{-3/2} dx$$

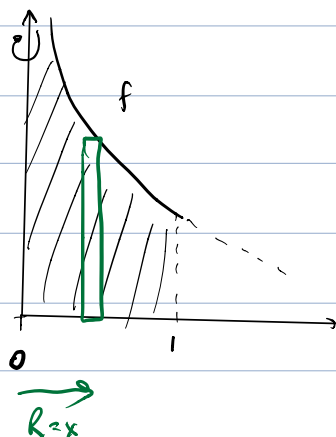
$$= \lim_{L \rightarrow 0^+} \left[\frac{x^{-1/2}}{-1/2} \right]_L^1$$

$$= \lim_{L \rightarrow 0^+} \left[-\frac{2}{\sqrt{x}} \right]_L^1$$

$$= \lim_{L \rightarrow 0^+} \left[-2 + \frac{1}{\sqrt{L}} \right]$$

$$= \infty \text{ (divergent)}$$

b)



$$\begin{aligned} dV &= 2\pi R h w \\ &= 2\pi x \cdot x^{-3/2} dx \\ &= 2\pi x^{-1/2} dx \end{aligned}$$

$$V = \int_0^1 2\pi x^{-1/2} dx$$

$$= 2\pi \int_0^1 \frac{1}{\sqrt{x}} dx$$

$$= 2\pi \lim_{L \rightarrow 0^+} \int_L^1 x^{-1/2} dx$$

$$= 2\pi \lim_{L \rightarrow 0^+} \left[\frac{x^{1/2}}{1/2} \right]_L^1$$

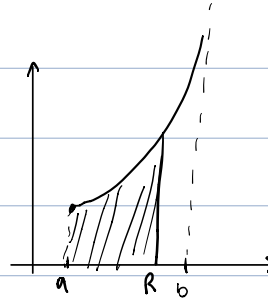
$$= 2\pi \lim_{L \rightarrow 0^+} [2 - 2L]$$

$$= 4\pi$$

Case 2

If $f(x)$ is discontinuous at $x=b$,

$$\text{then } \int_a^b f(x) dx = \lim_{R \rightarrow b^-} \int_a^R f(x) dx$$

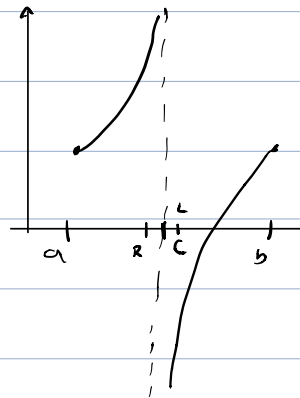


Case 3

If $f(x)$ is discontinuous at $x=c$ where $a < c < b$

$$\text{then } \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$$= \lim_{R \rightarrow c^-} \int_a^R f(x) dx + \lim_{L \rightarrow c^+} \int_L^b f(x) dx$$



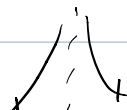
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Improper Integral \rightarrow Case 1, 2, 3

ex. $\int_{-1}^1 \frac{1}{x^2} dx$

$\frac{1}{x^2}$ is undefined at $x=0$

$$= \int_{-1}^0 \frac{1}{x^2} dx + \int_0^1 \frac{1}{x^2} dx$$



$$\begin{aligned}
&= \lim_{R \rightarrow 0^-} \int_{-1}^R \frac{1}{x^2} dx + \lim_{L \rightarrow 0^+} \int_L^1 \frac{1}{x^2} dx \\
&= \lim_{R \rightarrow 0^-} \left[-\frac{1}{x} \right]_{-1}^R + \lim_{L \rightarrow 0^+} \left[-\frac{1}{x} \right]_L^1 \\
&= \lim_{R \rightarrow 0^-} \left[-\frac{1}{R} + \frac{1}{-1} \right] + \lim_{L \rightarrow 0^+} \left[-\frac{1}{1} + \frac{1}{L} \right] \\
&= \underbrace{+\infty - 1}_{\text{diverging}} \quad \underbrace{-1 + \infty}
\end{aligned}$$

Case 4 Infinite Bounds

Definite integrals with an infinite bound are considered improper:

$$\int_a^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_a^R f(x) dx \quad \text{don't need to write } R \rightarrow \infty^-$$

$$\int_{-\infty}^b f(x) dx = \lim_{L \rightarrow -\infty} \int_L^b f(x) dx$$

OR

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x) dx &= \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx \\
&= \lim_{L \rightarrow -\infty} \int_L^c f(x) dx + \lim_{R \rightarrow \infty} \int_c^R f(x) dx
\end{aligned}$$

ex. $\int_1^{\infty} \frac{1}{1+x^2} dx$

$$= \lim_{R \rightarrow \infty} \int_1^R \frac{1}{1+x^2} dx$$

$$= \lim_{R \rightarrow \infty} \left[\arctan x \right]_1^R$$

$$= \lim_{R \rightarrow \infty} [\arctan R - \arctan 1]$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

ex. $\int_1^{\infty} \frac{1}{x^p} dx$

for $p \neq 1$

$$= \lim_{R \rightarrow \infty} \int_1^R \frac{1}{x^p} dx$$

$$= \lim_{R \rightarrow \infty} \left[\ln|x| \right]_1^R$$

$$= \lim_{R \rightarrow \infty} [\ln|R| - 0]$$

$$= \infty \text{ Diverges}$$

for $p > 1$

$$= \lim_{R \rightarrow \infty} \int_1^R \frac{1}{x^p} dx$$

$$= \lim_{R \rightarrow \infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^R$$

$$= \lim_{R \rightarrow \infty} \left[\frac{1}{(1-p)x^{p-1}} \right]_1^R \quad \text{recall: } p > 1$$

$$= \lim_{R \rightarrow \infty} \left[\frac{1}{(1-p)R^{p-1}} - \frac{1}{1-p} \right]$$

$$= 0 - \frac{1}{1-p}$$

$$= \frac{1}{p-1}$$

for $0 < p < 1$

:

$$= \lim_{R \rightarrow \infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^R$$

where $0 < p < 1$

$$= \lim_{R \rightarrow \infty} \left[\frac{R^{-p+1}}{-p+1} - \frac{1}{-p+1} \right]$$

$= \infty$ diverges

Conclusion

The "p-integral" $\int_1^{\infty} \frac{1}{x^p} dx$

converges to $\frac{1}{p-1}$ if $p > 1$, but

diverges for all cases where $0 < p \leq 1$

ex. $\int_1^{\infty} \frac{1}{x^3} dx = \frac{1}{3-1} = \frac{1}{2}$

ex. $\int_1^{\infty} \frac{1}{\sqrt{x}} dx$ diverges

ex.

a) Find an antiderivative for $f(x) = x^2 e^{-x}$

$F(x) = \int x^2 e^{-x} dx$	\oplus x^2	\xrightarrow{f}	$\xrightarrow{g'}$ e^{-x}	$\int e^{-x} dx$
	\ominus $2x$	\xrightarrow{f}	$-e^{-x}$	$= - \int e^u du$
	\oplus 2	\xrightarrow{f}	e^{-x}	$= -e^{-x}$
	\ominus 0	\xrightarrow{f}	$-e^{-x}$	

So

$$\begin{aligned} F(x) &= -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + K \\ &= -e^{-x}(x^2 + 2x + 2) + K \end{aligned}$$

b) Solve $\int_0^{\infty} x^2 e^{-x} dx$

$$= \lim_{R \rightarrow \infty} \int_0^R x^2 e^{-x} dx$$

$$= \lim_{R \rightarrow \infty} \left[-e^{-x} (x^2 + 2x + 2) \right]_0^R$$

$$= \lim_{R \rightarrow \infty} \left[-e^{-R} (R^2 + 2R + 2) + 1(2) \right] = -\frac{\infty}{\infty} + 2$$

$$= -\lim_{R \rightarrow \infty} \left(\frac{R^2 + 2R + 2}{e^R} \right) + 2$$

$$\stackrel{L'H}{=} -\lim_{R \rightarrow \infty} \frac{2R + 2}{e^R} + 2$$

$$= -\lim_{R \rightarrow \infty} \frac{2}{e^R} + 2$$

$$= 0 + 2$$

$$= 2$$