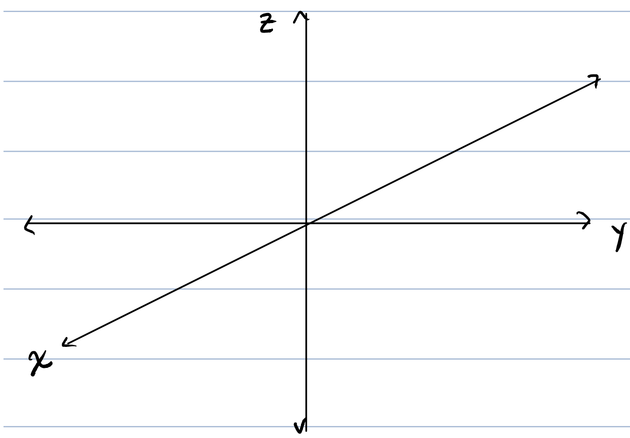
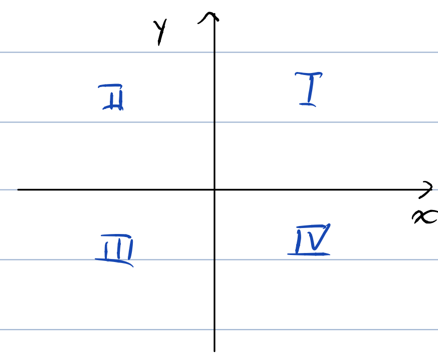
March 14

Types of Graphs in $\mathbb{R}^3$



in $\mathbb{R}^2$

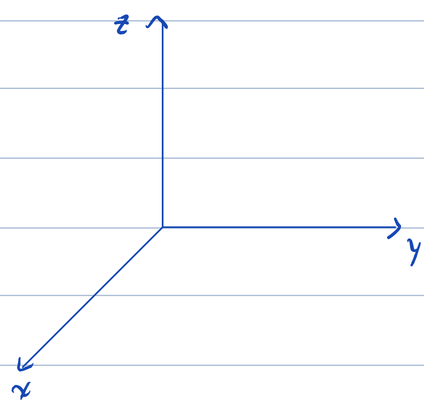


8 - octants

(Right handed system in $\mathbb{R}^3$)

4 - quadrants in $\mathbb{R}^2$

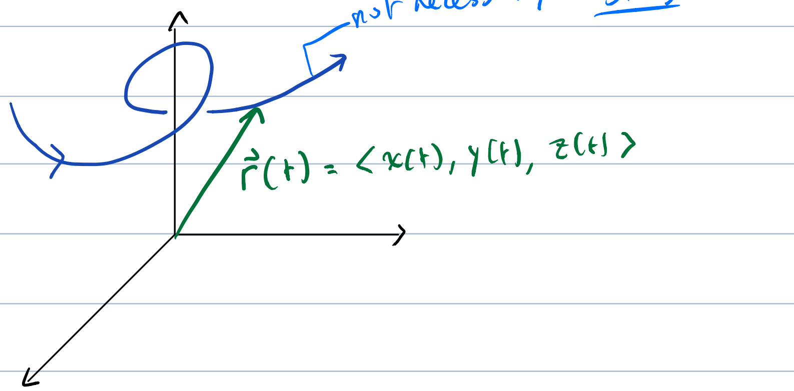
1^{st} octant:



You can have 1D graphs in $\mathbb{R}^3 \rightarrow$ line

2D graphs in $\mathbb{R}^3 \rightarrow$ plane

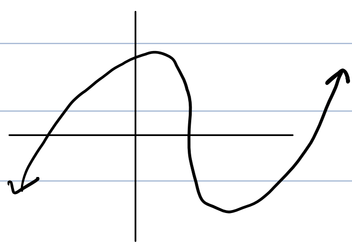
<u>1D graphs</u>: vector functions or space curves



2D graphs:

In $\mathbb{R}^2$: $\underset{\text{dep}}{y} = f(\underset{\text{ind}}{x})$

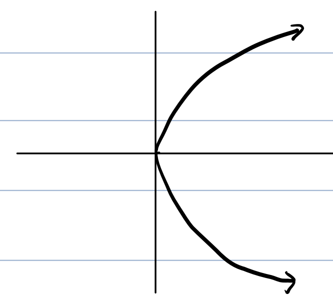
Functions



Vertical line test:

Any vertical intersects the graph at at most 1 point

Non-Functions



$\mathbb{R}^3$:

Functions

$$\underset{\text{dep}}{z} = f(\overbrace{x,y}^{2 \text{ ind}})$$

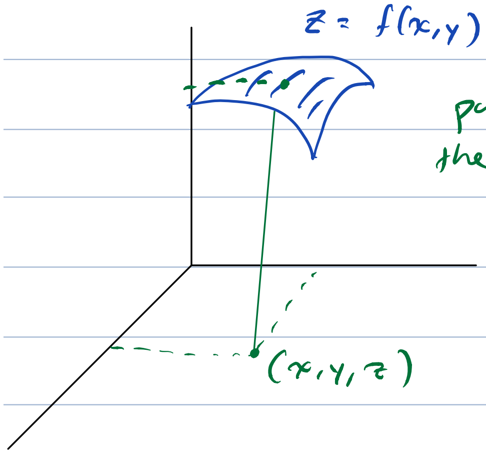
Non-Functions

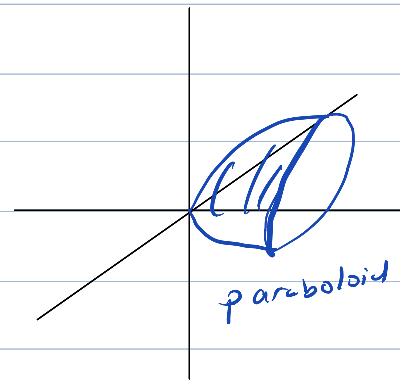
Cylinders

quadratic surfaces

that fail the VLT

may still pass the VLT





Cylinders

A cylinder is the name that is given to a surface that is parallel to one of the axes.

In particular, it may or may not look like a "cylinder" from geometry.

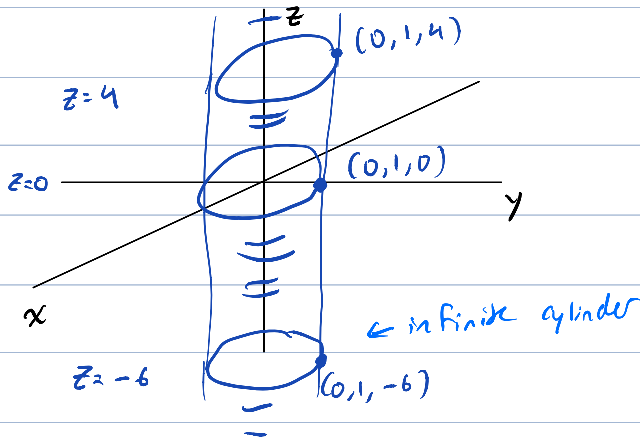
Algebraically: Cylinders occur when one of the coordinate variables (x,y,z) is missing from the equation of the surface.

The graph is parallel to the missing variable's axis

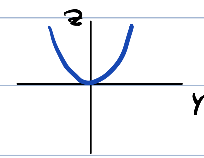
ex. $x^2 + y^2 = 1$ in $\mathbb{R}^3$

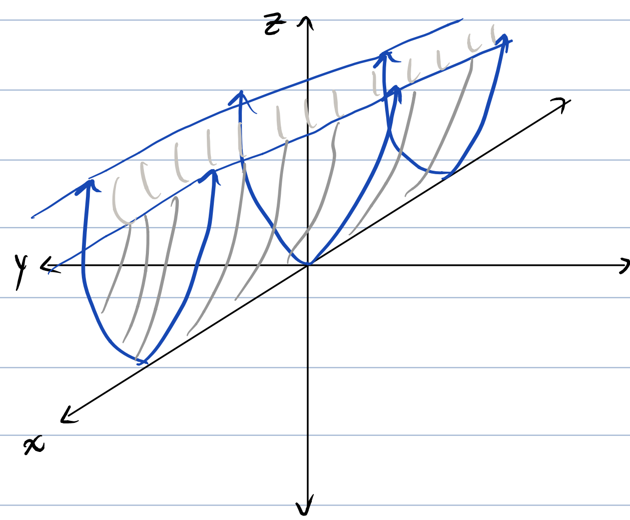
no z

parallel to z-axis



ex. $z = y^2$ in $\mathbb{R}^3$
↑ parabolic cylinder





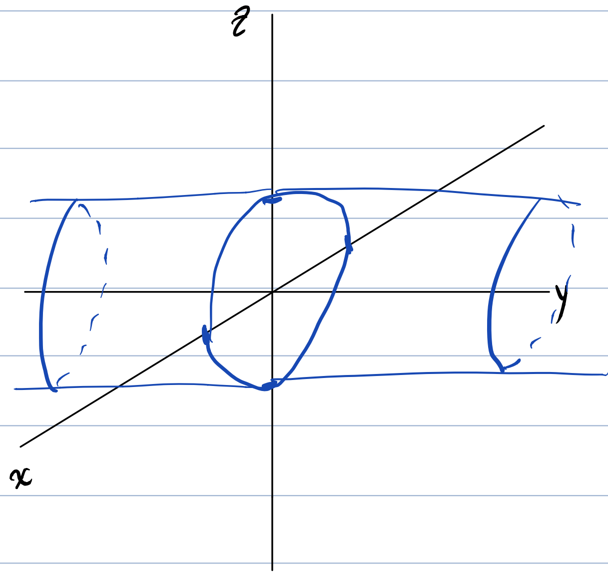
ex. For you: Sketch

① $x^2 + z^2 = 1$

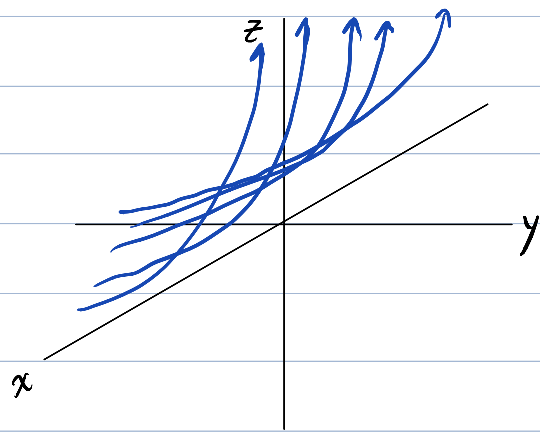
② $z = e^y$

③ $x - y = 0$

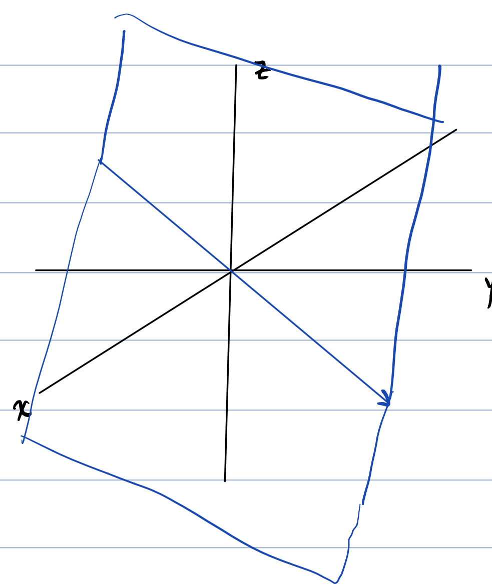
①



②



③



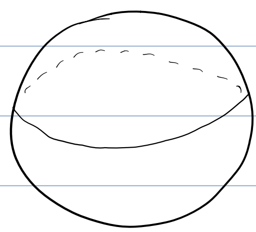
$x - y = 0$

$\vec{n} = \langle -1, 1, 0 \rangle$

Quadratic Surfaces

3D analog to conic sections

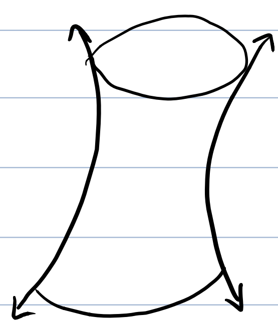
Sphere



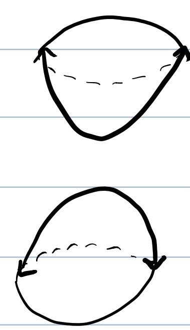
Ellipsoid

Hyperboloids

1 sheet

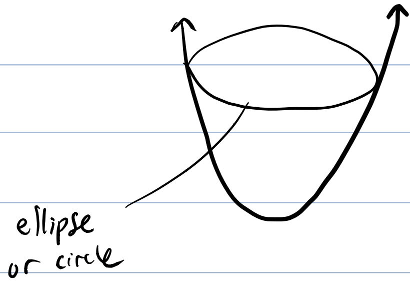


2 sheets

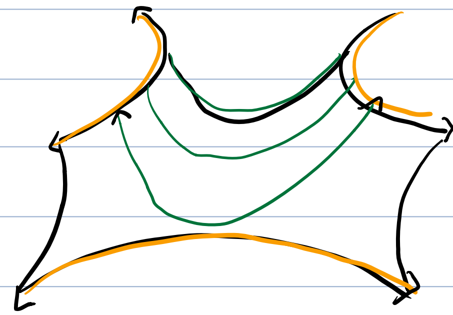


Paraboloids

Elliptic paraboloid ("paraboloid")



Hyperbolic paraboloid



Equations of Quadratic Surfaces

The most general form equation is

$$Ax^2 + By^2 + Cz^2 + Dx + Ey + Fz + \overbrace{\boxed{\cancel{Gxy + Hyz + Ixz}}}^{\text{Rotational Terms}} + J = 0$$

Sphere: centred at $(0,0,0)$ ex. $r=0$: $x^2+y^2+z^2=0$

radius $r>0$ $(0,0,0)$

$x^2+y^2+z^2=r^2$ single point

Ellipsoid: ex. $4x^2 + 4y^2 + z^2 = 12$

=> Investigate by cross section analysis

Examine algebraically cross-sections in the planes:

$x = k$ → for ellipsoids: ellipses

$y = k$ for arbitrary k

$z = k$ → for ellipsoids: circle

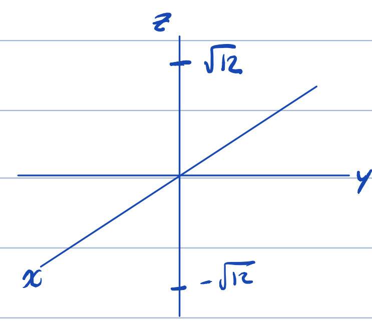
Let $z = k$ for $k \in \mathbb{R}$ ($z = k$ is a plane parallel to xy-plane)

$$4x^2 + 4y^2 = 12 - k^2$$

$$\underbrace{x^2 + y^2}_{\geq 0} = \underbrace{\frac{1}{4}(12 - k^2)}_{r^2} \quad \text{circles}$$

→ $12 - k^2 \geq 0$

$-\sqrt{12} \leq k \leq \sqrt{12}$ → length is $2\sqrt{12}$



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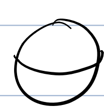
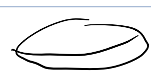
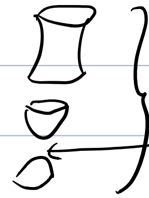
Review Cross sections

sphere  circles

ellipsoid  ellipses & circles

hyperboloid – 1 sheet
– 2 sheet

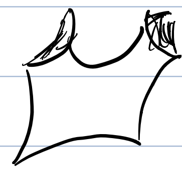


ellipses / circles & hyperbolas

empty cross sections (values of k that are not possible)

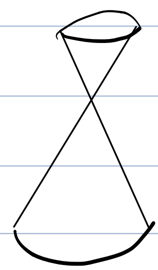
elliptic paraboloid  parabolas & ellipses / circles

hyperbolic paraboloid



hyperbolas & parabolas

Cone



circles / ellipses & hyperbolas
two intersecting lines

ex. Back to previous ex.

Ellipsoid: $4x^2 + 4y^2 + z^2 = 12$

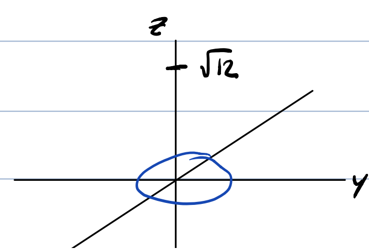
Let $z=k$ $\quad 4x^2 + 4y^2 = 12 - k^2$

$$x^2 + y^2 = \underbrace{3 - \tfrac{1}{2}k^2}_{r^2} \quad \text{circles}$$

$$\geq 0$$

$$\therefore \; k^2 \geq 12$$

$$-\sqrt{12} > k > 12$$



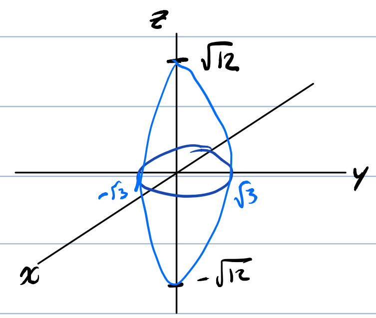
← xy plane $z=k=0$

parallel to xz plane

Let $y = k$: $4x^2 + z^2 = 12 - 4k^2$ ellipses

$$x^2 + \frac{z^2}{4} = 3 - k^2 \geq 0$$

$$-\sqrt{3} \leq k \leq \sqrt{3}$$



parallel to yz plane

Let $x = k$: $y^2 + \frac{z^2}{4} = 3 - k^2$

ellipses

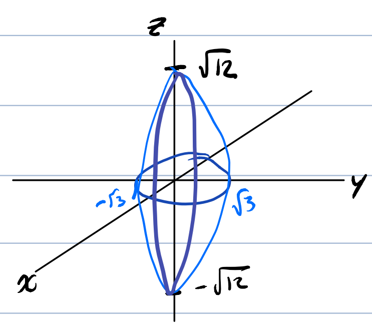
$$-\sqrt{3} \leq k \leq \sqrt{3}$$

$x^2 + y^2 = r^2$ circle

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\quad bx^2 + a^2y^2 = a^2b^2$ ellipse

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$ hyperbola

$x' = ay^2$ parabola



ex. For you

Perform a cross-section analysis in $x=k$, $y=k$, $z=k$ on the following

Name the surface & sketch it.

① $4x^2 - 2y^2 - z^2 = 4$

Let $x = k$ $\quad 4k^2 - 4 = 2y^2 + z^2 \geq 0$

$\quad\quad\quad \geq 0 \quad\quad\quad \geq 0$

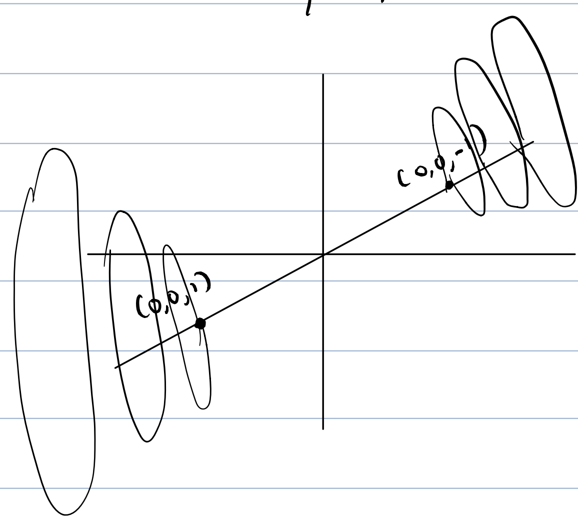
$k^2 \geq 1 \quad -1 \leq k \geq 1$ $\quad$ gap → hyperboloid of 2 sheets

→ looking for ellipses & hyperbolas in cross-sections

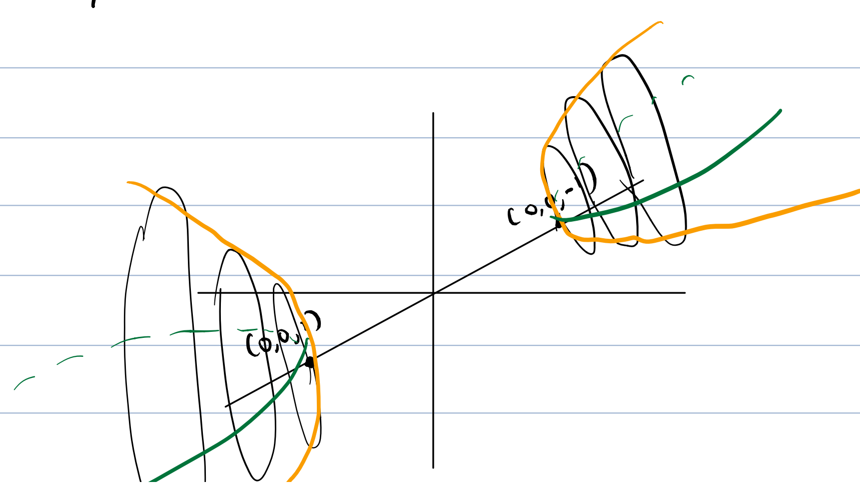
at $k = 1$: $2y^2 + z^2 = 0$

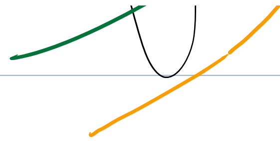
$y = 0,\ z = 0$

at $k = -1$: $y = 0,\ z = 0$



Let $y = k$: $\quad 4x^2 - z^2 = 4 + 2k^2$ $\quad$ hyperbola $\quad$ no restrictions $z \in \mathbb{R}$





Let $z=k$ ... $z \in \mathbb{R}$ no restriction hyperbola

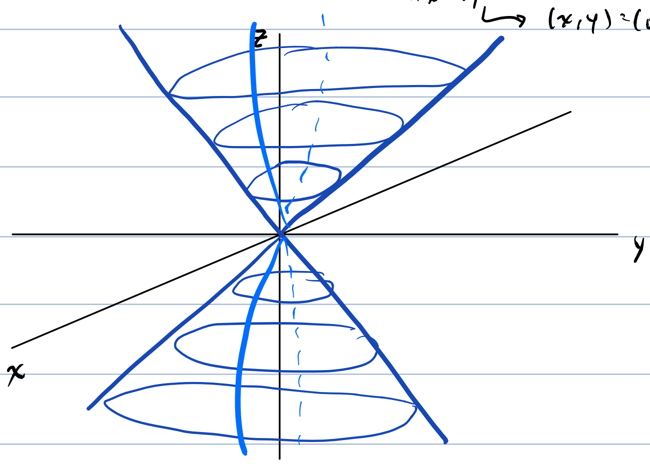
Tip: Find ellipses first

② $2x^2+y^2-z^2=0$

Let $x=k$: $y^2-z^2=-2k^2$ hyperbola for $k\in\mathbb{R}$
↳ when $k=0$ $y=\pm z$ ∴ straight line

Let $y=k$: $2x^2-z^2=-k^2$ hyperbola for $k\in\mathbb{R}$
↳ $(0,0,0)$

Let $z=k$: $2x^2+y^2=k^2$ ellipse for $k\in\mathbb{R}$
↳ when $k=0$: $2x^2+y^2=0$
↳ $(x,y)=(0,0)$



double cone

Vector Functions

A vector function is a function with one real input and whose output is a vector.

for us

$\vec{r}(t) = \langle x(t), y(t) \rangle \in \mathbb{R}^2$

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle \in \mathbb{R}^3$

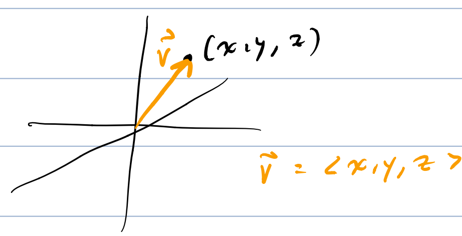
ex. You know some from linear

line: $\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle v_1, v_2, v_3 \rangle$

$$\vec{r}(t) = \langle \underbrace{x_0 + tv_1}_{x(t)}, \underbrace{y_0 + tv_2}_{y(t)}, \underbrace{z_0 + t v_3}_{z(t)} \rangle$$

$$= x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

$\hat{r}(t)$ is giving the position vector at t that changes (through time)



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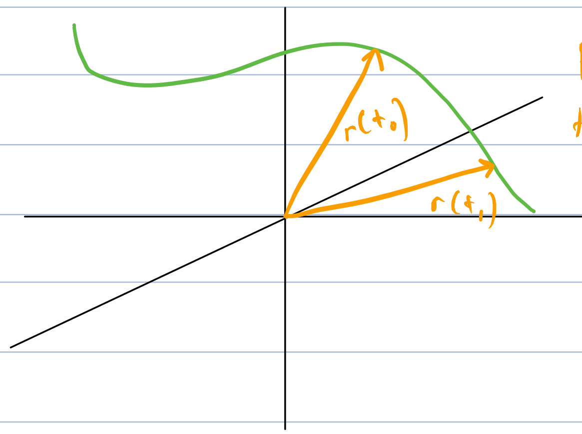
Vector Functions (in $\mathbb{R}^3 \to$ space curves)

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ $\quad t \to$ time

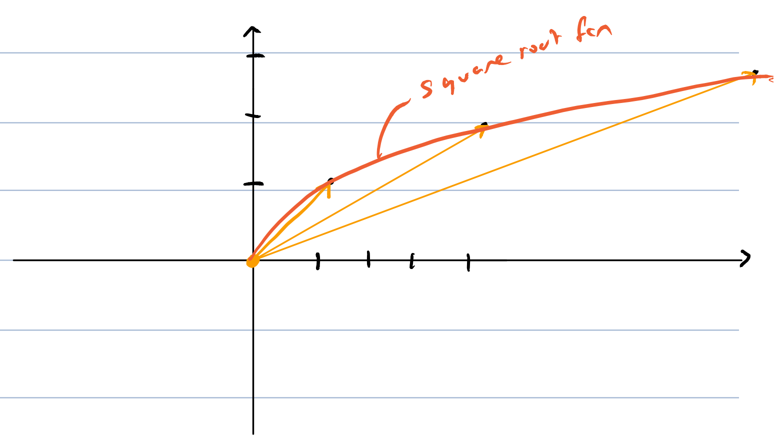


position vector moving through time

ex. (in $\mathbb{R}^2$)

① $\vec{r}(t) = \langle t, \sqrt{t} \rangle \quad t \in [0, \infty)$

t	$\vec{r}$
0	$\langle 0,0 \rangle = \vec{0}$
1	$\langle 1,1 \rangle$
4	$\langle 4,2 \rangle$
9	$\langle 9,3 \rangle$

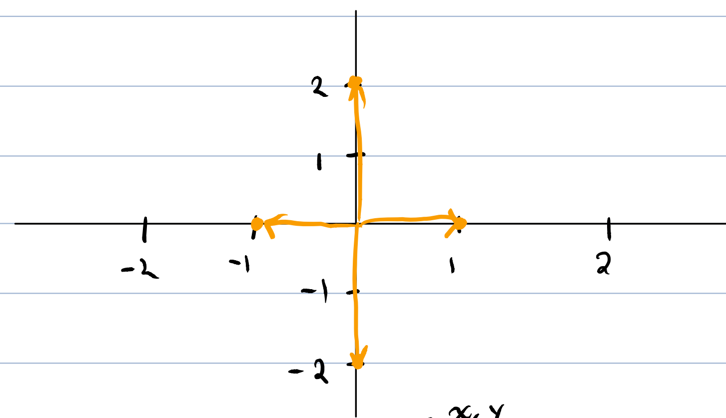


In general in $\mathbb{R}^2$, $y = f(x)$ can be parameterized as
$\vec{r}(t) = \langle t, f(t) \rangle$ $\quad \hookrightarrow t$ is the parameter

(2)

a) Make a table of values and sketch $\vec{r}(t) = \langle \cos t, 2\sin t \rangle$

t	$r(t)$
0	$\langle 1, 0 \rangle$
$\frac{\pi}{2}$	$\langle 0, 2 \rangle$
π	$\langle -1, 0 \rangle$
$\frac{3\pi}{2}$	$\langle 0, -2 \rangle$



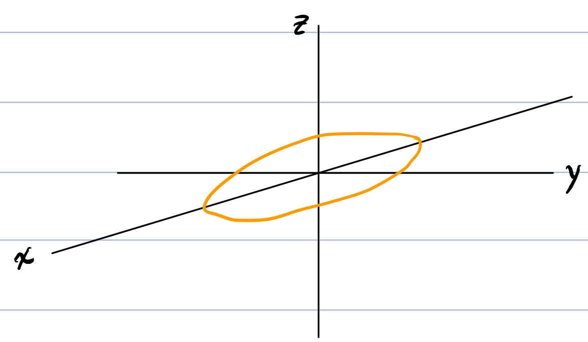
x, y

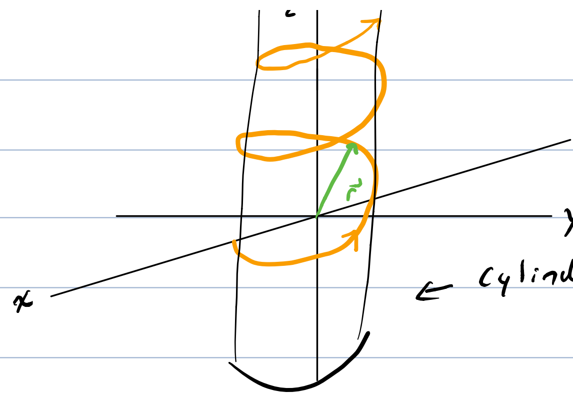
b) Identify the curve & give its cartesian equation

An ellipse $\quad x^2 + \frac{y^2}{4} = 1$

(3) sketch the space curve $\vec{r}(t) = \langle \cos(t), 2\sin(t), t \rangle$

If $z = 0$ hypothetically: $\langle \cos t, 2\sin t, 0 \rangle$





← cylinder parallel to z-axis

⑤ Show that $\vec{r}(t) = \langle \sin t \cos t, \sin^2 t, \cos t \rangle$ lies on the unit sphere.

(show that $x(t)$, $y(t)$, $z(t)$ satisfy the equation of the surface)

⑥ Find the intersection of

$\vec{r} = \langle t, t+1, t^2 - 2t + 1 \rangle$

and the paraboloid $z = x^2 + y^2$

⑤ Unit Sphere: $x^2 + y^2 + z^2 = 1$

$x(t) = \sin t \cos t \qquad y(t) = \sin^2 t \qquad z(t) = \cos t$

$$(\sin t \cos t)^2 + (\sin^2 t)^2 + \cos^2 t \stackrel{?}{=} 1$$

$$\sin^2 t \cos^2 t + \sin^4 t + \cos^2 t \qquad 1$$

$$\sin^2 t (1 - \sin^2 t) + \sin^4 t + (1 - \sin^2 t) \qquad 1$$

$$\cancel{\sin^2 t} - \cancel{\sin^4 t} + \cancel{\sin^4 t} + 1 - \cancel{\sin^2 t} \qquad 1$$

$$1 \qquad 1$$

⑥ $$(t^2-2t+1) = t^2 + (t+1)^2$$

$$\cancel{t^2} - 2t \cancel{+1} = \cancel{t^2} + t^2 + 2t \cancel{+1}$$

$$0 = t^2 + 4t$$

$$0 = t(t+4)$$

$$t = 0, \quad t = -4$$

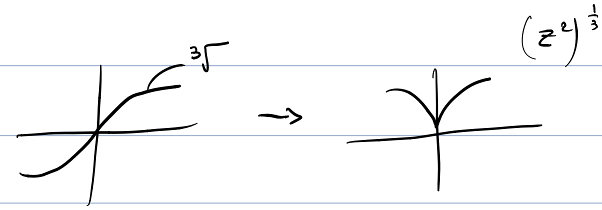
$$\begin{cases} \vec{r}(0) = \langle 0, 1, 1 \rangle \\ r(-4) = \langle -4, -3, 25 \rangle \end{cases}$$

<u>A note on projections:</u>

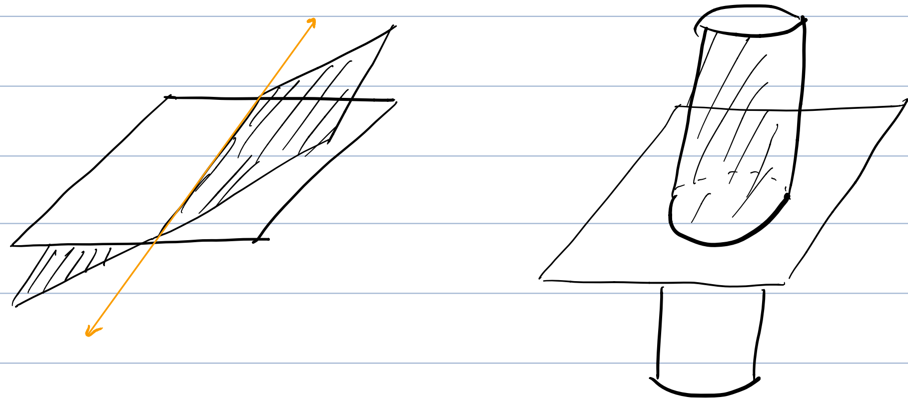
The twisted cubic : $\vec{r}(t) = \langle t, t^2, t^3 \rangle$

The technique of projections

	looking down z-axis $\downarrow$ <u>xy</u>	<u>yz</u>	<u>xz</u>
$x = t$			
$y = t^2$	$y = x^2$	$z = y^{3/2}$	$z = x^3$
$z = t^3$		$\downarrow$ $y = z^{2/3}$	



Intersection of surfaces



Two surfaces intersect in a curve

ex. Find a vector function that represents the curve of intersection of

$$\begin{cases} x^2+y^2=1 & \leftarrow \text{circular cylinder parallel to } z \\ y+z=2 & \leftarrow \text{plane parallel to } x \end{cases}$$

$\hookrightarrow z = 2-y$

In $\mathbb{R}^2$, we know that the circle can be parameterized as:

$\vec{r}(t) = \langle \cos t, \sin t \rangle$

$\vec{r}_2(t) = \langle \cos t, \sin t, 2-\sin t \rangle$

Calculus of Vector Functions

Limits: $\lim_{t\to a} \vec{r}(t) \equiv \langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t) \rangle$

is defined as

if these limits exist.

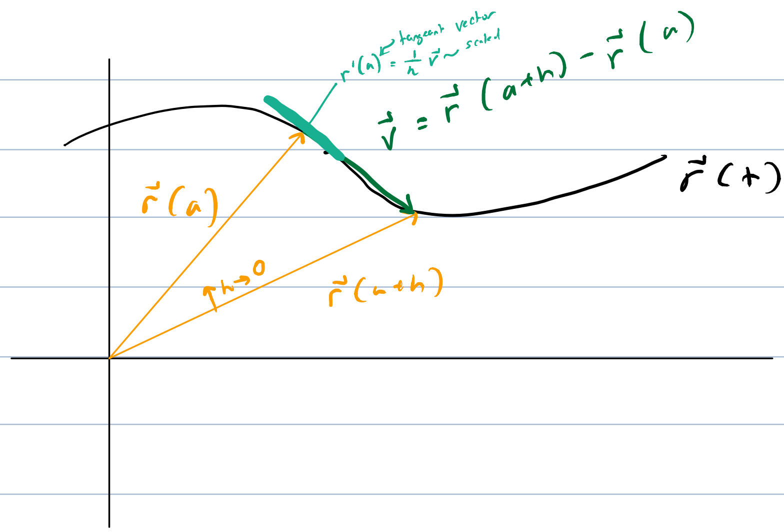
Continuity: A vector function $\vec{r}(t)$ is continuous at $t=a$ if each of its component functions is continuous at $t=a$.

The derivative of a vector function → The tangent vector

Recall: From Cal I: $y=f(x)$ derivative at $x=a$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

By analogy: $\vec{r}\,'(a) = \lim_{h\to 0} \frac{\vec{r}(a+h) - \vec{r}(a)}{h}$ ← vector substraction



$\vec{r}'(a)$ is the tangent vector to $\vec{r}(t)$ at $t = a$

$$\vec{r}'(a) = \left\langle \lim_{h \to a} \frac{x(a+h) - x(a)}{h}, \lim_{h \to a} \frac{y(a+h) - y(a)}{h}, \lim_{h \to a} \frac{z(a+h) - z(a)}{h} \right\rangle$$

$= \langle x'(a), y'(a), z'(a) \rangle$ if these limits exist.

Unit tangent vector: $T(a) = \dfrac{\vec{r}'(a)}{\|\vec{r}'(a)\|}$ where $\|\vec{r}'(a)\| \neq 0$

ex. If $\vec{r}(t) = \langle t, t^2, t^3 \rangle$

Find $T(1)$

$\vec{r}'(t) = \langle 1, 2t, 3t^2 \rangle$

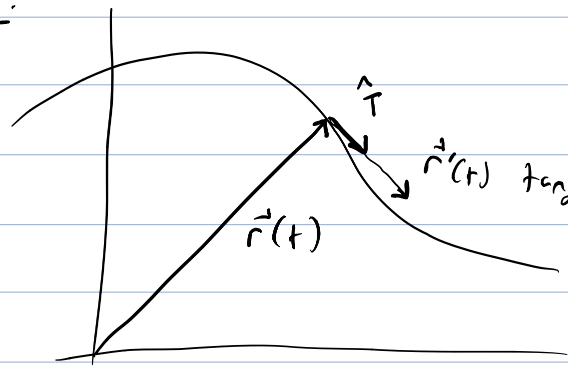
$T(1) = \dfrac{\langle 1, 2, 3 \rangle}{\sqrt{14}}$ → unit tangent

March 21

A5 due Monday

Recall:

$$\hat{T}(t) = \frac{1}{\|\vec{r}\,'(t)\|}\,\vec{r}\,'(t)$$



Note: $\|r'(t)\| \to$ speed

ex. For you

a) Show that $P(-1,2,-3)$ lies on the curve

$$\vec{r}(t) = t\,\hat{\imath} + (t^2+1)\,\hat{\jmath} + (t^3+2t)\,\hat{k}$$

b) Find the angle between $\hat{r}$ and $\hat{r}'$ at P.

c) Find a vector equation for the tangent line to $\hat{r}$ at P.

$$\begin{cases} t = -1 \\ t^2+1 = 2 \\ t^3+2t = -3 \end{cases}$$ candidate, must check that other eqns are satisfied

a) $-1 = t$

$2 = t^2+1$

$2 \overset{?}{=} (-1)^2+1 \quad t=-1$

$= 2$ ✓

$-3 = t^3+2t$

$-3 \overset{?}{=} (-1)^3+2(-1) \quad t=-1$

$= -1-2 = -3$ ✓

b) $\vec{r}\,'(t) = \hat{\imath} + 2t\,\hat{\jmath} + (3t^2+2)\,\hat{k}$

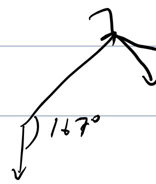
$\vec{r}(-1) = -\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$

$\vec{r}\,'(-1) = \hat{\imath} - 2\hat{\jmath} + 5\hat{k}$

$\vec{r}(-1)\cdot\vec{r}\,'(-1) = -1 - 4 - 15$

$= -20 = \|\vec{r}(-1)\|\,\|\vec{r}\,'(-1)\|\cos\theta$

$\cos\theta = \dfrac{-20}{\sqrt{14}\,\sqrt{30}} = 167.4^\circ$

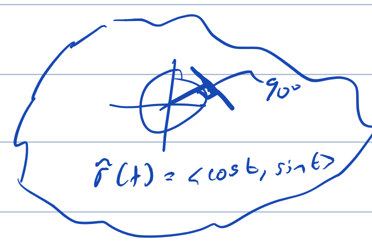


$\phi = 180^\circ - 167.4 = 12.6^\circ$

Note : not $= 90^\circ$

c)

$$\langle x, y, z\rangle = \underbrace{\langle -1, 2, -3\rangle}_{P} + t\,\underbrace{\langle 1, -2, 5\rangle}_{\vec{r}\,'(-1)}$$

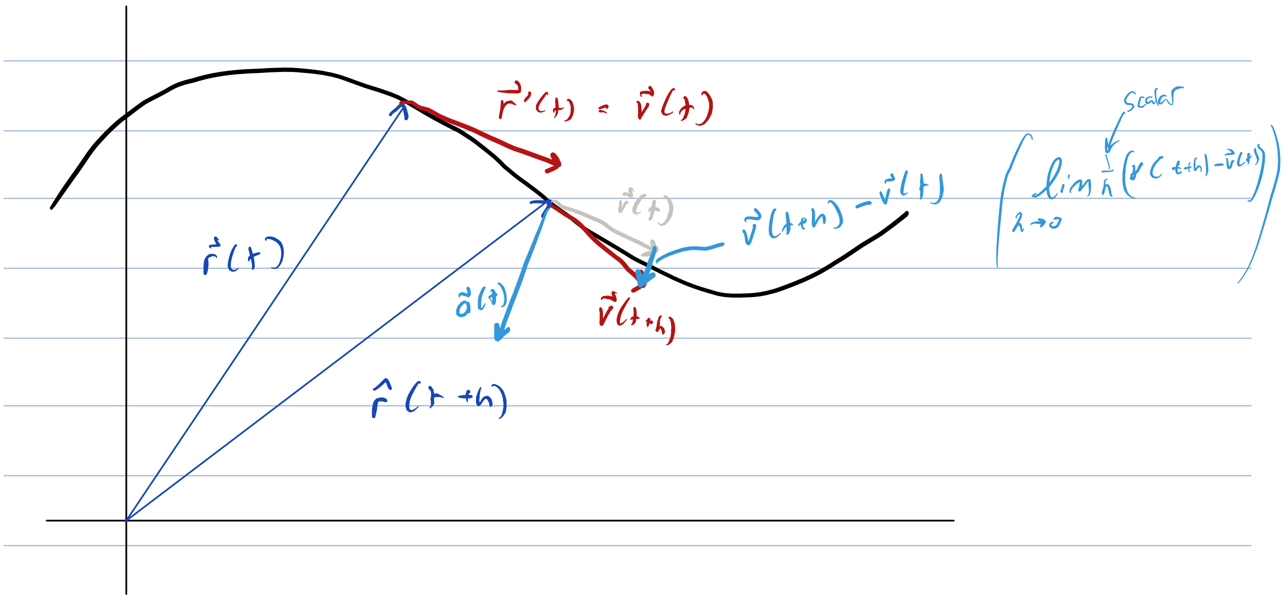


Motion in 3D - Space

If the position of a particle (travelling through space) at time t is given by $\vec{r}(t)$.

Then, the velocity of the particle is given by $\vec{v}(t) = r'(t)$

and the acceleration $\vec{a}(t) = v'(t) = r''(t)$



acceleration points in the direction of curvature

ex. ① A moving particle starts at position $\vec{r}_0 = \hat{i}$, with $\vec{v}_0 = \hat{i} - \hat{j} + \hat{k}$ and acceleration at time t given by $\vec{a}(t) = 4t\hat{i} + 6t\hat{j} + \hat{k}$.

Find the velocity and the position at any time t.

$$\vec{a}(t) = \langle a_1(t), a_2(t), a_3(t) \rangle$$

$$\int \vec{a}(t)\,dt = \langle \int a_1(t), \int a_2(t), \int a_3(t) \rangle$$

$$= \langle A_1(t) + C_1, A_2(t) + C_2, A_3(t) + C_3 \rangle$$

$$= \langle A_1(t), A_2(t), A_3(t) \rangle + \langle C_1, C_2, C_3 \rangle$$

$$= \vec{A}(t) + \vec{C}$$

$\vec{v}(t) = \int \vec{a}(t)\,dt$

$= \int \langle 4t, 6t, 1 \rangle\, dt$

$= \langle 2t^2, 3t^2, t \rangle + \vec{C}$

$\vec{v}(0) = \langle 1, -1, 1 \rangle = 2(0)^2, 3(0)^2, 0 \rangle + \vec{C}$

$\vec{C} = \langle 1, -1, 1 \rangle$

$\therefore \vec{v}(t) = \langle 2t^2, 3t^2, t \rangle + \langle 1, -1, 1 \rangle = \langle 2t^2+1, 3t^2-1, t+1 \rangle$

$\vec{x}(t) = \int \langle 2t^2+1, 3t^2-1, t+1 \rangle\, dt$

$= \langle \frac{2}{3}t^3 + t, t^3 - t, \frac{1}{2}t^2 + t \rangle + \vec{D}$

$\vec{x}(0) = \langle 1, 0, 0 \rangle = \langle 0, 0, 0 \rangle + \vec{D}$

$\vec{x}(t) = \langle \frac{2}{3}t^3 + t, t^3 - t, \frac{1}{2}t^2 + t \rangle + \langle 1, 0, 0 \rangle$

$= \langle \frac{2}{3}t^3 + t + 1, t^3 - t, \frac{1}{2}t^2 + t \rangle$

(2) A particle moves with acceleration $\vec{a}(t) = \langle t, e^t, e^{-t} \rangle$ and $\vec{v}_0 = \hat{k}$.

Find its speed at $t = 1$.

$\vec{v}(t) = \int \langle t, e^t, e^{-t} \rangle\, dt$

$\vec{v}(t) = \langle \frac{1}{2}t^2, e^t, -e^{-t} \rangle + \vec{c}$

$\vec{v}(0) = \langle 0,0,1 \rangle = \langle 0, 1, -1 \rangle + \vec{c}$

$\vec{c} = \langle 0, -1, -2 \rangle$

$\therefore \vec{v}(t) = \langle \frac{1}{2}t^2, e^t, -e^{-t} \rangle + \langle 0, -1, 2 \rangle$

$\vec{v}(1) = \langle \frac{1}{2}, e-1, -\frac{1}{e} + 2 \rangle$

$\|\vec{v}(1)\| = 2.42$ time u/dist n

Derivative Rules

① $\frac{d}{dt}\vec{c} = \vec{0}$

② $\frac{d}{dt}\left(\vec{u}(t) \pm \vec{v}(t)\right) = \vec{u}'(t) \pm \vec{v}'(t)$

③ i) $\frac{d}{dt}\left(c\,\vec{u}(t)\right)$

ii) $\frac{d}{dt}\left(\alpha(t)\,\vec{u}(t)\right)$

$\alpha(t)$ → Scalar fcn

ex. $\langle t^2, t^4 - t^2, 1 \rangle = t^2 \langle 1, t^2 - 1, \frac{1}{t^2} \rangle$

④ $\frac{d}{dt}\left(\vec{u}(t) \cdot \vec{v}(t)\right) = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t)\,\vec{v}'(t)$

⑤ $\frac{d}{dt}\left(\vec{u}(t) \times \vec{v}(t)\right) = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$

⑥ Chain rule $\frac{d}{dt}\left(\vec{u}(\alpha(t))\right) = \vec{u}'(\alpha(t))\,\alpha'(t)$

March 23

Derivative Rule

$$\left[\vec{u}(t)\cdot\vec{v}(t)\right]' = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$$

Proof: (in $\mathbb{R}^3$) Let $\vec{u}(t) = \langle u_1(t), u_2(t), u_3(t)\rangle$

$\vec{v}(t) = \langle v_1(t), v_2(t), v_3(t)\rangle$

$$\left[\vec{u}(t)\cdot\vec{v}(t)\right]' = \left(u_1(t)v_1(t) + u_2(t)v_2(t) + u_3(t)v_3(t)\right)'$$

$$= \left(u_1(t)v_1(t)\right)' + \left(u_2(t)v_2(t)\right) + \left(u_3(t)v_3(t)\right)'$$

$$= u_1'(t)\,v_1(t) + u_1(t)v_1'(t) + u_2'(t)\,v_2(t) + u_2(t)\,v_2'(t)$$
$$u_3'(t)\,v_3(t) + u_3(t)v_3'(t)$$

$$= u_1'(t)v_1(t) + u_2'(t)\,v_2(t) + u_3'(t)v_3(t)$$
$$+ u_1(t)v_1'(t) + u_2(t)\,v_2'(t) + u_3(t)\,v_3'(t)$$

$$= \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$$

Result #1

If $\vec{r}(t)$ is a curve with constant magnitude of $\vec{r}$, then $\vec{r}(t)$ and $\vec{r}'(t)$ are perpendicular.

$\hookrightarrow$ Show $\vec{r} \cdot \vec{r}\,' = 0$

<u>Proof:</u> $\|\vec{r}(t)\| = K$

$\|\vec{r}(t)\|^2 = k^2$

$\left(\vec{r}(t) \cdot \vec{r}(t)\right)' = \left(k^2\right)'$

$\vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t) = 0$

$2\vec{r}\,'(t) \cdot \vec{r}(t) = 0$

$r'(t) \cdot \vec{r}(t) = 0$

$\therefore \vec{r}(t) \perp \vec{r}\,'(t)$

<u>Result #2</u>

$\neq \|r'(t)\|$

If $\vec{r}(t) \neq \vec{0}$, then $\|\vec{r}(t)\|' = \frac{1}{\|\vec{r}(t)\|} \vec{r}(t) \cdot \vec{r}\,'(t)$

Let $\|\vec{r}(t)\|$

$$\left(\|\vec{r}(t)\|^2\right)' = \left(\vec{r}(t) \cdot \vec{r}(t)\right)' = \left(k^2\right)'$$

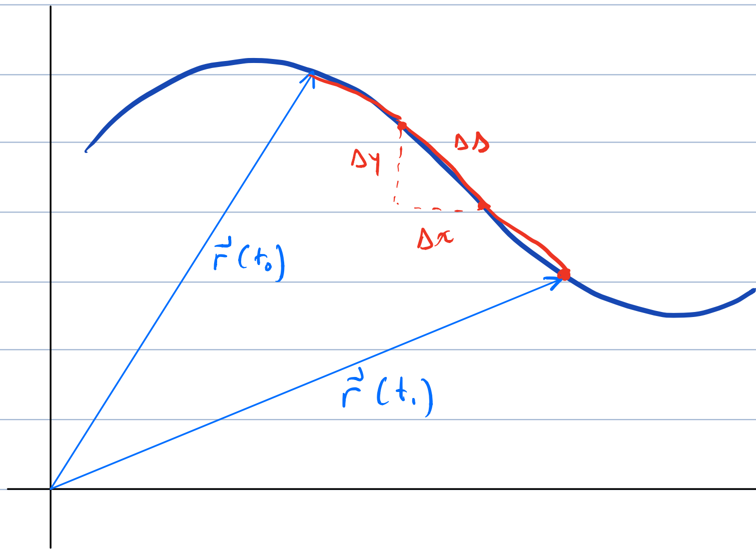
$$= \vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t)$$

$$\cancel{2}\|\vec{r}(t)\| \left(\|\vec{r}(t)\|\right)' = \cancel{2}\vec{r}\,'(t) \cdot \vec{r}(t)$$

$$\|\vec{r}(t)\|' = \frac{1}{\|\vec{r}(t)} \vec{r}\,'(t) \cdot \vec{r}(t)$$

Arc Length

Idea: Approximate a curve by line segments



spacium

$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2} \cdot \frac{\Delta t}{\Delta t}$$

$$= \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \; \Delta t$$

Let n be # segments

as $n \to \infty$

$\Delta t \to dt$ $\quad \Delta y \to dy$

$\Delta x \to dx$ $\quad \Delta s \to ds$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; dt$$

$$= \sqrt{x'(t)^2 + y'(t)^2} \; dt$$

$$= \|\vec{r}\,'(t)\| \, dt$$

$\vec{r}(t) = \langle x(t), y(t) \rangle$

$\vec{r}\,'(t) = \langle x'(t), y'(t) \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{x'(t)^2 + y'(t)^2}$

$$\boxed{s = \int_{t_0}^{t_1} \|\vec{r}\,'(t)\|\, dt}$$

Alternate way of thinking: motion

$\vec{r}(t)$ - position

$\|\vec{r}\,'(t)\|$ - speed

$\int \|\vec{r}\,'(t)\|\, dt$ - distance along path

ex. (1) Let $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ → helix

Find the length of the helix from $t=0$ to $t=\pi$

(A particle is travelling along the helix. Find the total distance it travels from $t=0$ to $t=\pi$)

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$

$$S = \int_0^{\pi} \sqrt{2}\, dt = \sqrt{2}\Big|_0^{\pi} = \sqrt{2}\,\pi$$

② $\vec{r}(t) = \frac{1}{2}t^2\,\hat{i} + \frac{2\sqrt{2}}{3}\,t^{3/2}\,\hat{j} + t\,\hat{k}$

Find the length on $[1,4]$

$$r'(t) = \langle t, \sqrt{2}\sqrt{t}, 1 \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{t^2 + 2t + 1}$$

$$= \sqrt{(t+1)^2}$$

$$= t + 1$$

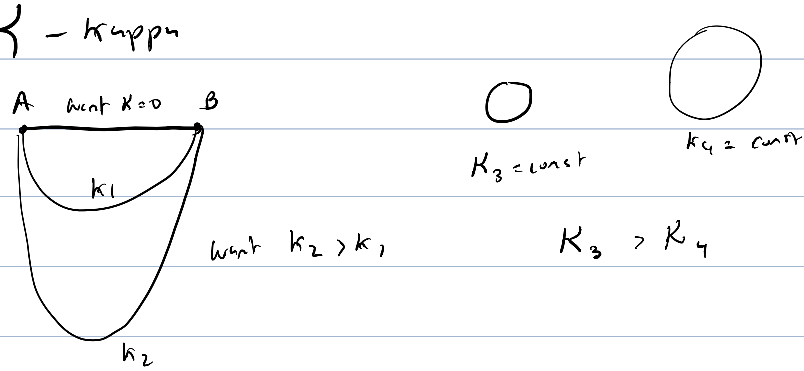
$$S = \int_1^4 (t+1)\, dt$$

$$= \left(\frac{t^2}{2} + t\right)\Big|_1^4 = 8 + 4 - \frac{1}{2} - 1 = \frac{21}{2}$$

Curvature

→ how tightly a curve changes direction over a unit distance

K – kappa



The Arclength Function

$$S(t) = \int_0^t \|\vec{r}\,'(u)\|\, du$$

dummy variable

A function which measures the arclength from $t = 0$ to any input value along a given curve $\vec{r}(t)$

ex. a) Find $S(t)$ for $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

$\vec{r}(u) = \langle \cos u, \sin u, u \rangle$

$\vec{r}\,'(u) = \langle -\sin u, \cos u, 1 \rangle$

$\|\vec{r}\,'(u)\| = \sqrt{2}$

$$S(t) = \int_0^t \sqrt{2}\, du$$

$$= \sqrt{2}\, u \Big|_0^t$$

$$s = \sqrt{2}\, t$$

b) Reparameterize the helix in terms of arclength

Change from t to s

$$s = \sqrt{2}\, t \Rightarrow t = \frac{1}{\sqrt{2}} s = t(s)$$

$$\vec{r}_2(s) = \vec{r}(t(s))$$

$$= \left\langle \cos\left(\tfrac{1}{\sqrt{2}} s\right), \sin\left(\tfrac{1}{\sqrt{2}} s\right), \tfrac{1}{\sqrt{2}} s \right\rangle$$

the helix with distance instead of time as a parameter

ex. Let $\vec{r} = \langle t, 1+2t, 3-2t \rangle$. Reparameterize $\vec{r}$ in terms of arclength

$$\vec{r}\,' = \langle 1, 2, -2 \rangle$$
$$\|\vec{r}\,'\| = \sqrt{9} = 3$$

$$s(t) = \int_0^t 3\, du$$

$$s = 3t$$

$$\therefore t = \frac{s}{3}$$

$$r(s) = \left< \frac{s}{3},\ 1 + \frac{2}{3}s,\ 3 - \frac{2}{3}s \right>$$

March 28

Recall: $S(t) = \int_0^t \|\vec{r}\,'(u)\|\, du$ → arclength fcn

$$\frac{ds}{dt} = \|r'(u)\| \quad \text{(FTC)}$$

Curvature

Def'n: The curvature K of a vector curve is defined as

$$K = \left\| \frac{d\hat{T}}{ds} \right\| \sim \text{scaler}$$

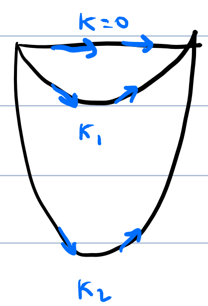
Unpacking

$\hat{T}$ – unit tangent vector

↳ magnitude = 1 = constant

$\hat{T}(s)$ parametized in terms $\frac{d\hat{T}}{ds}$ – the instantaneous change in $\hat{T}$ with respect to s.

Since the magnitude of $\hat{T}$ is constant, any change in $\hat{T}$ comes from a change in direction.



K_2 has a bigger change in the vector components than K_1

Unfortunately, calculating $\left\|\frac{d\hat{T}}{ds}\right\|$ directly is painful.

Given $\vec{r}(t)$, steps:

often very difficult

1. Reparameterize $s(t) = \int_0^t \|\vec{r}\,'(u)\|\,du \Rightarrow \vec{r}(s)$
2. Find $\hat{T}(s) = \frac{1}{\|\vec{r}\,'(s)\|}\vec{r}\,'(s)$
3. $\frac{d\hat{T}}{ds}$
4. $\left\|\frac{d\hat{T}}{ds}\right\|$

To avoid this heartache, we observe:

chain rule

$$\kappa = \left\| \frac{d\hat{T}}{ds} \right\| = \left\| \frac{d\hat{T}}{dt} \cdot \frac{dt}{ds} \right\|$$

related to the 2^{nd} derivative

$$= \left\| \frac{\frac{d\hat{T}}{dt}}{\frac{ds}{dt}} \right\| = \frac{\| \hat{T}'(t) \|}{\| \vec{r}'(t) \|}$$

<u>ex.</u> Find the curvature κ of a straight line ($\kappa = 0$)

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle = \langle x_0 + t v_1, y_0 + t v_2, z_0 + t v_3 \rangle$$

$$\kappa = \frac{\| \hat{T}'(t) \|}{\| \vec{r}'(t) \|}$$

$$\vec{r}'(t) = \langle v_1, v_2, v_3 \rangle$$

$$\hat{T}(t) = \frac{1}{\| \vec{r}'(t) \|} \vec{r}'(t) = \frac{1}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \langle v_1, v_2, v_3 \rangle$$

$$\hat{T}'(t) = \left(\frac{1}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \langle v_1, v_2, v_3 \rangle \right)'$$

$$= \langle 0, 0, 0 \rangle$$

$$K = \frac{\| \langle 0,0,0 \rangle \|}{\sqrt{v_1^2 + v_2^2 + v_3^2}} = \frac{0}{\sqrt{\dots}} = 0 \quad \checkmark$$

ex. Show that the curvature of a circle with radius 'a' is constant.

$\mathbb{R}^2$: $\vec{r}(t) = \langle a\cos t, a\sin t \rangle$

$\vec{r}\,'(t) = \langle -a\sin t, a\cos t \rangle$

$$\hat{T}(t) = \frac{1}{a\sqrt{\sin^2 t + \cos^2 t}} \langle -a\sin t, a\cos t \rangle$$

$$= \langle -\sin t, \cos t \rangle$$

$\hat{T}'(t) = \langle -\cos t, -\sin t \rangle$

$$K = \frac{\sqrt{\cos^2 t + \sin^2 t}}{a\sqrt{\sin^2 t + \cos^2 t}} = \frac{1}{a} = c \quad \checkmark$$

where c is constant

Def'n: For any curve $\vec{r}(t)$, we define the radius of curvature

$$r_K = \frac{1}{K}$$

Also, we can show that:

$$K = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$$

ex. a) Apply this formula to find $K(t)$ for

$\vec{r}(t) = \langle t, t^2, t^3 \rangle$

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$

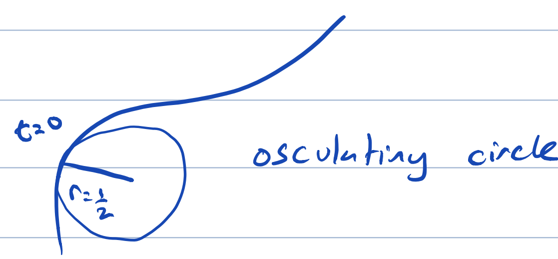
$\vec{r}\,''(t) = \langle 0, 2, 6t \rangle$

$\vec{r}\,'(t) \times \vec{r}\,''(t) = \langle 6t^2, -6t, 2 \rangle = 2\langle 3t^2, -3t, 1 \rangle$

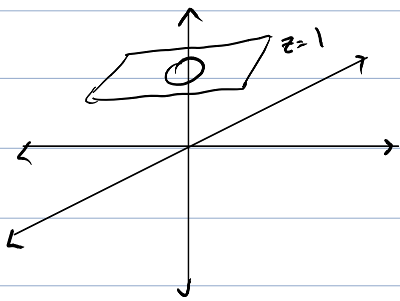
$$K(t) = \frac{2\sqrt{9t^4 + 9t^2 + 1}}{(9t^4 + 4t^2 + 1)^{3/2}}$$

b) Find the radius of curvature at $t=0$

$r_K = \frac{1}{K(0)} = \frac{1}{2}$



ex. a) Find $K(0)$ for $\vec{r}(t) = \langle 3\cos 2t, 3\sin 2t, 1\rangle$



circle in a plane

$\vec{r}\,'(t) = \langle -6\sin 2t, 6\cos 2t, 0\rangle$

$\vec{r}\,''(t) = \langle -12\cos 2t, -12\sin 2t, 0\rangle$

don't need the whole thing for single pt

$\vec{r}\,'(0) \times \vec{r}\,''(0) = \langle 0, 6, 0\rangle \times \langle -12, 0, 0\rangle$

$= \langle 0, 0, 72\rangle$

$K(0) = \dfrac{72}{6^3} = \dfrac{1}{3}$

b) Find $K(t)$ for any t for $\vec{r}(t) = e^t\hat{i} + \sqrt{2}\,t\hat{j} + e^{-t}\hat{k}$

$\vec{r}(t) = \langle e^t, \sqrt{2}\,t, e^{-t}\rangle$

$\vec{r}\,'(t) = \langle e^t, \sqrt{2}, -e^{-t}\rangle$

$\vec{r}\,''(t) = \langle e^t, 0, e^{-t}\rangle$

$\vec{r}\,'(t) \times \vec{r}\,''(t) = \langle \sqrt{2}e^{-t}, -2, -\sqrt{2}e^t\rangle$

$\|\vec{r}\,'(t) \times \vec{r}\,''(t)\| = \sqrt{2e^{-2t} + 4 + 2e^{2t}} = \sqrt{2}\sqrt{e^{2t} + 2 + e^{-2t}}$

$= \sqrt{2}\left(e^t + e^{-t}\right)$

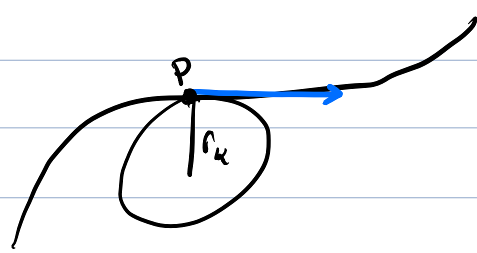
$$\kappa(t) = \frac{\sqrt{2}\,(e^t + e^{-t})}{(e^t+e^{-t})^3} = \frac{\sqrt{2}}{(e^t+e^{-t})^2}$$

Note: In $\mathbb{R}^2$, you will show that

$$\kappa = \frac{|y''|}{(1+(y')^2)^{3/2}}$$

Osculating Circle

Def'n: The osculating circle for a curve at a point P is the circle passing through P which shares the tangent vector (has a parallel tangent vector) to the curve at P and which has radius $r_\kappa = \frac{1}{\kappa}$, where κ is the curvature of the curve at P.



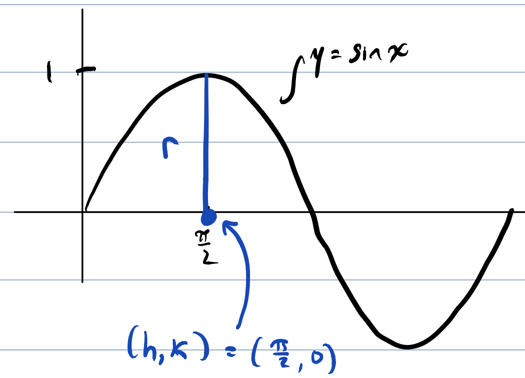
ex. Find the osculating circle to $y = \sin x$ at $x = \frac{\pi}{2}$. Sketch.

$y' = \cos x \rightarrow y'(\frac{\pi}{2}) = 0$

$y'' = -\sin x \rightarrow y''(\frac{\pi}{2}) = -1$

$$K = \frac{|y''|}{(1+(y')^2)^{3/2}} = \frac{|-1|}{1} = 1$$

$$r_K = = \frac{1}{K} = 1$$



$$(x-h)^2 + (y-k)^2 = r^2$$

$$\therefore \left(x - \frac{\pi}{2}\right)^2 + y^2 = 1$$

ex. Find the eqn of the osculating circle at each pt. Sketch.

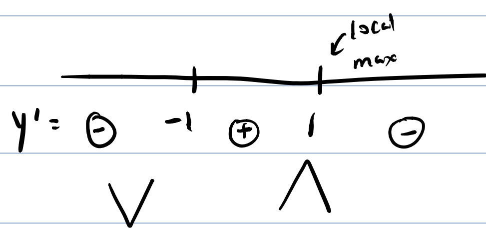
a) $y = 3x - x^3$ at local max

b) $y = x^2$ at $x = 1$

a) $y' = 3 - 3x^2$

$y' = 0 = 3 - 3x^2$

$x = 1, \ x = -1$



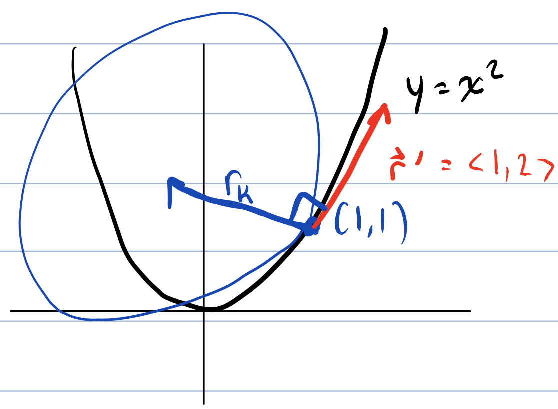
$$K = \frac{|y''|}{(1-(y')^2)^{3/2}} = \frac{6x}{(1-(3-3x^2)^2)^{3/2}}$$

Recall: $\kappa > 0$ $\quad \underbrace{\kappa = \left\| \frac{d\hat{T}}{ds} \right\|}_{\text{def'n}} = \frac{\|\hat{T}'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$ March 28

In $\mathbb{R}^2$: $y = f(x)$ $\quad \kappa = \frac{|y''|}{(1 + (y')^2)^{3/2}}$

From last day

Find the eqn of the osculating (kissing) circle to $y = x^2$ at $x = 1$. Sketch.



$y' = 2x \rightarrow y'(1) = 2$

$y'' = 2$

$$\kappa(1) = \frac{|2|}{(1 + 2^2)^{3/2}} = \frac{2}{5^{3/2}}$$

$r_K = \frac{1}{K} = \frac{5^{3/2}}{2}$

radius of curvature

(i.e. radius of circle)

$\vec{r}(t) = \langle t, f(t) \rangle$

$= \langle t, t^2 \rangle \rightarrow$ at $t=1 \leadsto (1,1)$

$r'(t) = \langle 1, 2t \rangle$

$r'(1) = \langle 1, 2 \rangle$

Centre (h,k)

$(x-h)^2 + (y-k)^2 = r^2$

perpendicular to $\vec{r}\,'$

$\vec{n} = \langle 2, -1 \rangle$? ... check $\vec{n}\,' \cdot \vec{r} = 0$ ✓

But need left & up

$\vec{n} = \langle -2, 1 \rangle$

$\vec{n}_1 = \frac{r_K}{\|\vec{n}\|} \vec{n}$ scale so it's the right length

$= \frac{5^{3/2}/2}{5^{1/2}} \langle -2, 1 \rangle$

$= \frac{5}{2} \langle -2, 1 \rangle$

$= \langle -5, \frac{5}{2} \rangle$

$\langle h, k \rangle = \langle 1, 1 \rangle + \langle -5, \frac{5}{2} \rangle = \langle -4, \frac{7}{2} \rangle$

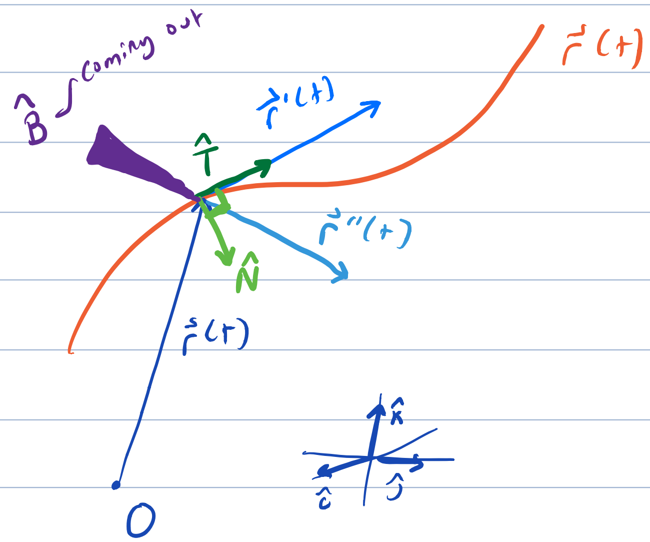
$$(x+4)^2 + \left(y - \frac{7}{2}\right)^2 = \frac{125}{4}$$

TNB - Frame

$\hat{T}$ - unit tangent

$\hat{N}$ - unit normal

$\hat{B}$ - unit binormal



$$\hat{T}(t) = \frac{1}{\|\vec{r}\,'(t)\|} r'(t)$$

$$\hat{N}(t) = \frac{1}{\|\hat{T}'(t)\|} \hat{T}'(t)$$

$\hat{B}(t) = T'(t) \times \hat{N}(t) \rightarrow \|\hat{B}\| = 1$ recall $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\,\|\vec{v}\| \sin\theta$

$$\|\hat{B}(t)\| = \|\hat{T}(t) \times \hat{N}(t)\|$$

$$= \|\hat{T}(t)\|\,\|\hat{N}(t)\| \cancel{\sin 90}^{1}$$

$$= 1 \cdot 1 \quad \checkmark$$

3 related planes

osculating plane

contains $\hat{T}$, $\hat{N}$

normal plane

contains $\hat{N}$, $\hat{B}$

rectifying plane

contains $\hat{B}$, $\hat{T}$

ex. $\vec{r}(t) = \langle t, \frac{1}{2}t^2, t^2 \rangle$

Find $\hat{T}(0)$, $\hat{N}(0)$, $\hat{B}(0)$ 4 the three planes

$\vec{r}'(t) = \langle 1, t, 2t \rangle$

$\|\vec{r}'(t)\| = \sqrt{1 + 5t^2}$

$$\hat{T}(t) = \frac{1}{\sqrt{1 + 5t^2}} \langle 1, t, 2t \rangle$$

$$\hat{T}'(t) = \left(\frac{1}{\sqrt{1+5t^2}}\right)' \langle 1, t, 2t \rangle + \frac{1}{\sqrt{1+5t^2}} \left(\langle 1, t, 2t \rangle\right)'$$

$$= -\frac{1}{2} \left(1+5t^2\right)^{-3/2} \cdot 10t \langle 1, t, 2t \rangle + \left(1+5t^2\right)^{-1/2} \langle 0, 1, 2 \rangle$$

No more derivatives $\rightarrow$ plug in $t=0$

$$\hat{T}(0) = \frac{1}{\sqrt{1+0}} \langle 1, 0, 0 \rangle = \langle 1, 0, 0 \rangle$$

$$\hat{T}'(0) = \vec{0} + \langle 0, 1, 2 \rangle$$

$$\hat{N}(0) = \frac{1}{\sqrt{5}} \langle 0, 1, 2 \rangle$$

$$\hat{B}(0) = \langle 1, 0, 0 \rangle \times \frac{1}{\sqrt{5}} \langle 0, 1, 2 \rangle$$
$$= \frac{1}{\sqrt{5}} \langle 0, -2, 1 \rangle$$

<u>planes</u>

From linear: $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

point-normal form: (x_0, y_0, z_0) point on plane

$\langle a, b, c \rangle = \vec{n}$ normal of plane

point $\vec{r}(0) = \langle 0, 0, 0 \rangle$ $\quad (x_0, y_0, z_0) = (0, 0, 0)$

osculating	normal	rectifying
$\vec{n} \parallel \hat{B}$	$\vec{n} \parallel \hat{T}$	$\vec{n} \parallel \hat{N}$
$\vec{n} = \langle 0,-2,1\rangle$	$\vec{n} = \langle 1,0,0\rangle$	$\vec{n} = \langle 0,1,2\rangle$
$\therefore -2y + z = 0$	$\therefore x = 0$	$\therefore y + 2z = 0$

For each of the situations below:

a) Find the vectors $\hat{T}$, $\hat{N}$ and $\hat{B}$ (the unit tangent, unit normal and unit binormal vectors) to $\vec{r}$ at P.

b) Find the general equations of the osculating, normal and rectifying planes to $\vec{r}$ at P.

c) Decompose the acceleration vector $\vec{a}(t) = \vec{r}''(t)$ into its tangential and normal components

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1. $\vec{r}(t) = \langle t^2, 3, -1+2t\rangle$ at the point $P(1,3,-3)$

a) $\vec{r}'(t) = \langle 2t, 0, 2\rangle$

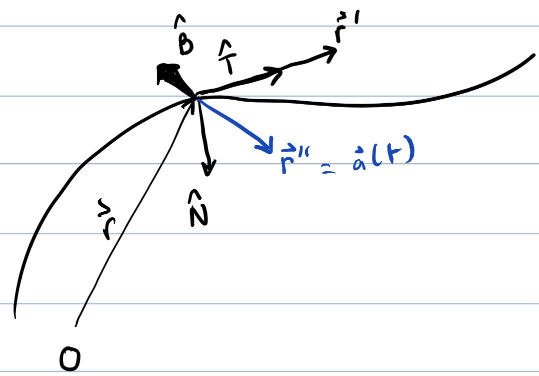
$$\hat{T}(t) = \frac{1}{\sqrt{4t^2+4}}\langle 2t,0,2\rangle$$

$$\hat{T}'(t) = \left(\frac{1}{\sqrt{4t^2+4}}\right)'\langle 2t,0,2\rangle + \frac{1}{\sqrt{4t^2+4}}\langle 2,0,0\rangle$$

$$= (4t^2+4)^{-3/2}\cdot 8t\cdot\langle 2t,0,2\rangle + (4t^2+4)^{-1/2}\langle 2,0,0\rangle$$

$$\begin{cases} t^2 = 1 \\ 3 = 3 \\ -1 + 2t = -3 \end{cases} \rightsquigarrow t = -1$$

$$\hat{T}(-1) = \frac{1}{\sqrt{8}} \langle -2, 0, 2 \rangle$$



April 3

<u>Goal:</u> ① Decompose $\vec{a}(t)$ into tangential component $a_T(t)$
and normal component $a_N(t)$

Express $\vec{a}(t) = a_T(t)\,\hat{T}(t) + a_N(t)\,\hat{N}(t)$

can be done using projections (but we're not going to...)

② Prove $\kappa = \dfrac{\|\vec{r}\,'(t) \times r''(t)\|}{\|\vec{r}\,'(t)\|^3}$

① Examine

$$\hat{T}(t) = \frac{1}{\|\vec{r}\,'(t)\|}\,\vec{r}\,'(t) \qquad \left(\text{also } \hat{N}(t) = \frac{1}{\|\hat{T}'(t)\|}\,\hat{T}'(t) \right)$$

$$\Rightarrow r'(t) = \|\vec{r}\,'(t)\|\,\hat{T}(t)$$

$$\vec{a}(t) = r''(t) = \left(\|\vec{r}\,'(t)\|\,\hat{T}(t) \right)'$$

product rule

$$= \|\vec{r}\,'(t)\|'\,\hat{T}(t) + \|\vec{r}\,'(t)\|\,\underbrace{\hat{T}'(t)\cdot\frac{1}{\|\hat{T}'(t)\|}}_{\hat{N}}\cdot\underbrace{\frac{\|\hat{T}'(t)\|}{\|\vec{r}\,'(t)\|}}_{K}\cdot\|\vec{r}\,'(t)\|$$

Recall: $K = \dfrac{\|\hat{T}'(t)\|}{\|\vec{r}\,'(t)\|}$

Let $\quad \nu(t) = \text{speed} = \|\vec{r}\,'(t)\|$

$$\vec{a}(t) = \underbrace{\nu'(t)\,\hat{T}(t)}_{a_T(t) = \nu(t)} + \underbrace{\nu^2 K\,\hat{N}(t)}_{a_N(t) = \nu(t)^2 K}$$

② κ formula

$$\vec{r}''(t) = \|\vec{r}'(t)\|'\,\hat{T}(t) + \|\vec{r}'(t)\|^2 \kappa \hat{N}(t)$$

Examine $$\vec{r}'(t) \times \vec{r}''(t) = \cancel{\|\vec{r}'(t)\|'\Big(\underbrace{\vec{r}'(t) \times \hat{T}(t)}_{\vec{r}'(t) \parallel \hat{T}(t)}\Big)} \to \vec{0}$$

$$+ \|\vec{r}'(t)\|^2 \kappa \left(\vec{r}'(t) \times \hat{N}(t)\right)$$

$$\|\vec{r}'(t) \times \vec{r}''(t)\| = \|\vec{r}'(t)\|^2 \kappa \|\vec{r}'(t) \times \hat{N}(t)\|$$

$$= \|\vec{r}'(t)\|^2 \kappa \|\vec{r}'(t)\| \, \underbrace{\cancel{\|\hat{N}(t)\|}}_{1 \text{ (unit vector)}} \, \underbrace{\cancel{\sin\theta}}_{1} \quad \hookrightarrow 90^\circ$$

$$\kappa = \frac{\|\vec{r}''(t) \times \vec{r}'(t)\|}{\|\vec{r}'(t)\|^3}$$

$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin\theta$