NB

$b^x = b^y \quad (b > 0,\ b \neq 1)$

then $x = y$

Exponential Eqns

ex. Solve $2^x = 8$

$x = 3$

ex. solve $2^{x^2-1} = 16$

$2^{x^2-1} = 2^4$

$x^2 - 1 = 4$

$x^2 = 5$

$x = \pm\sqrt{5}$

ex. solve $2^{x^2-1} = 3$

Need logs

LOG REVIEW

$$2^3 = 8 \leftrightarrow \log_2 8 = 2$$

Def^n:

$$b^y = x \longleftrightarrow \log_b x = y$$

Exponential Form	Log Form
$10^3 = 1000$	$\longrightarrow$ $\log_{10} 1000 = 3$
$16^{\frac{1}{2}} = 4$ $\longleftarrow$	$\log_{16} 4 = \frac{1}{2}$
$\left(\frac{1}{16}\right)^{\frac{1}{2}} = \frac{1}{4}$ $\longleftarrow$	$\log_{\frac{1}{16}} \frac{1}{4} = \frac{1}{2}$

ex. Find $\log_2 32$

Let $y = \log_2 32$

$2^y = 32$

$y = 5$

ex. Find $\log_2(-16)$

Let $y = \log_2(-16)$

$2^y = -16$

No sol'n for $y = \log_2(-16)$!!

So $\log(-16)$ UND

ex. $y = \log_2 0$

Let $y = \log_2 0$

$2^y = 0$

No sol'n

So $\log_2 0$ is UND

Fact! $\log_b N$ is UND if $N \leq 0$ Can't have a negative log

RULES OF LOGS

Rules: Assume $b > 0$, $b \neq 1$ and $M, N > 0$

1. $\log_b b = 1$ $(b = b)$
2. $\log_b 1 = 0$ $(b^0 = 1)$
3. $\log_b (MN) = \log_b M + \log_b N$ (proof not necessary)

ex. $\log_{10} 100$

Method A:

$y = \log_{10} 100$

$10^y = 100$

$y = 2$

Method B:

$\log_{10} (10 \cdot 10)$

$= \log_{10} 10 + \log_{10} 10$

$= 1 + 1 = 2$

N.B. $\cancel{\log_b (-3)(-4) = \log_b(-3) + \log_b(-4)}$

b_{UND}

Proof of ③ - Not necessary

Let $x = \log_{10} M$ & $y = \log_b N$

$\downarrow$ $\qquad$ $\downarrow$

$b^x = M$ $\qquad$ $b^y = N$

$b^x \cdot b^y = MN$

$b^{x+y} = MN$

$\downarrow$

$\log(MN) = x + y$

$= \log_b M + \log_b N$

④ $\log_b \frac{M}{N} = \log_b M - \log_b N$

ex. $\log_{10} \frac{1}{10} = \log_{10} 1 - \log_{10} 10 = 0$

Euler's Constant "e"

$e \approx 2.71...$ (irrational #)

$\log x = \log_{10} x$

$\ln x = \log_e x$

⑤ $\log_b N^r = r \cdot \log_b N$

ex. $\log_7 (7)^5$

$= 5 \log_7 7 = 5 \cdot 1 = 5$

⑥ Change of base formula

$$\log_b N = \frac{\log_c N}{\log_c b} \quad \text{for any } c > 0, c \neq 1$$

ex. $\log_3 2 = \frac{\log_{10} 2}{\log_{10} 3} \xrightarrow{\text{calculator}} 0.630929754$

i.e. $3^{0.630929754} = 2$

⑦ $\log_b b^x = x$

ex. $\log_3 3^7 = 7$

⑧ $b^{\log_b x} = x \quad (\rightarrow \log_b x = \log_b x)$

ex. $2^{\log_2 (x^3 - 4)} = x^3 - 4$

Log Rules with e $(M, N > 0)$

① $\ln e = 1$

② $\ln 1 = 0$

③ $\ln (M \cdot N) = \ln M + \ln N$

④ $\ln \left(\frac{M}{N}\right) = \ln M - \ln N$

⑤ $\ln N^r = r \ln N$

⑥ $\log_b N = \frac{\ln N}{\ln b}$

⑦ $\ln e^x = x$

⑧ $e^{\ln x} = x$

ex. $2^{x^2-1} = 3$

$\ln(2^{x^2-1}) = \ln 3$

$(x^2-1)\ln 2 = \ln 3$

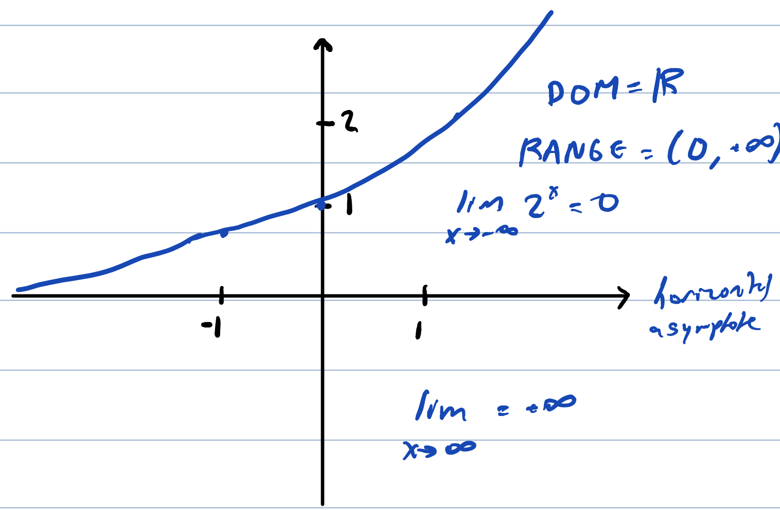
$x^2 - 1 = \frac{\ln 3}{\ln 2}$

$x^2 = \frac{\ln 3}{\ln 2} + 1$

$x = \pm\sqrt{\frac{\ln 3}{\ln 2} + 1}$

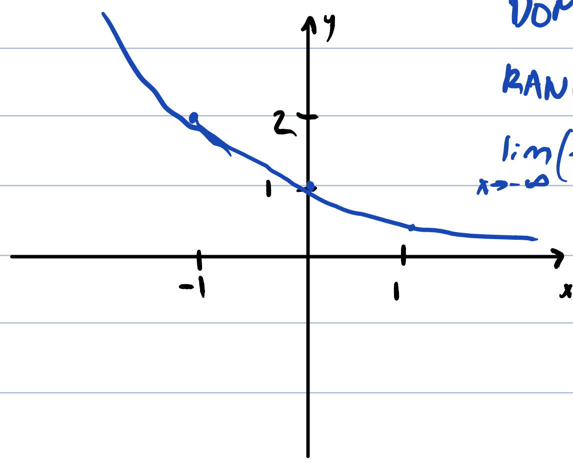
Graphs of Basic Exponential F^{ens}

x	$y = 2^x$
-1	$\frac{1}{2}$
0	1
1	2



$y = b^x \ (b > 1)$

ex. $y = \frac{1}{2}^x$ $(0 < b < 1)$



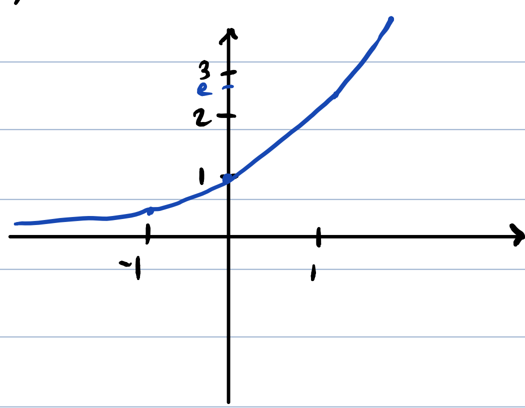
DOM: $\mathbb{R}$

RANGE: $(0, +\infty)$

$\lim_{x \to -\infty} \left(\frac{1}{2}\right)^x = +\infty$

$\lim_{x \to +\infty} \left(\frac{1}{2}\right)^x = 0$

ex. $y = e^x$



Basic Graphs of Log F^ens

$y = \log_2 x$ $(b > 1)$

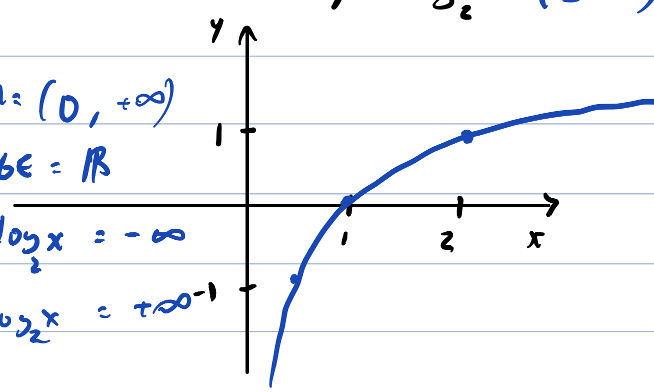
x	$\log_2 x$
$\frac{1}{2}$	-1
1	0
2	1

DOM: $(0, +\infty)$

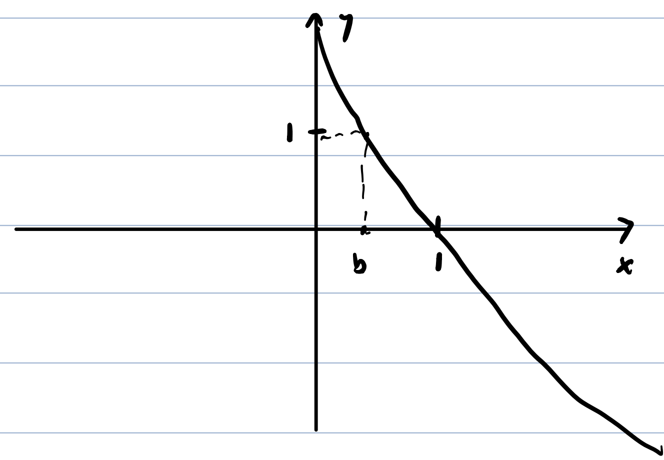
RANGE: $\mathbb{R}$

$\lim_{x \to 0^+} \log_2 x = -\infty$

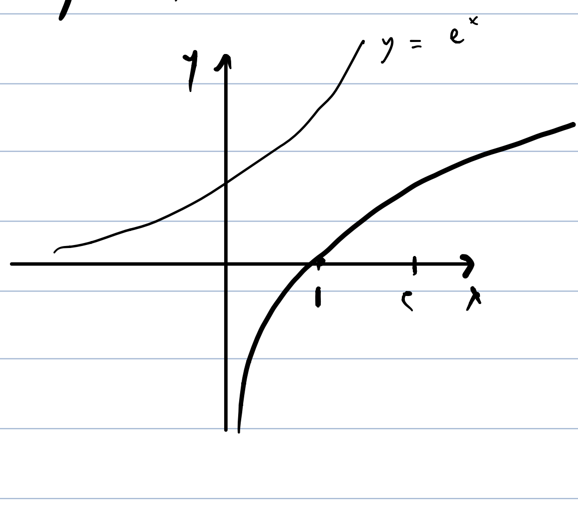
$\lim_{x \to 0^-} \log_2 x = +\infty$



If $0 < b < 1$: $\log_b x$



$y = \ln x$



$f(x) = \ln x$
$f^{-1}(x) = e^x$

$$f^{-1}(f(x)) = e^{f(x)} = e^{\ln x} = x$$

Derivatives

$f(x) = e^x$

$f'(x) = ?$

N.B. Special limit: Don't have to prove

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h}$$

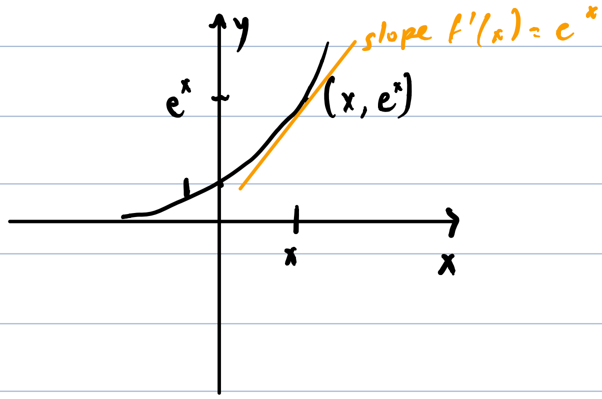
$$= \lim_{h \to 0} e^x \cdot \frac{e^h - 1}{h}$$

important step!

$$= e^x \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$= e^x$$

Proof of: If $y = e^x$, $\frac{dy}{dx} = e^x$



ex. $y = \dfrac{x e^x}{x^2+1}$

$$y' = \frac{(x e^x)'(x^2+1) - (x e^x)(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{(x)'e^x + x(e^x)'(x^2+1) - (x e^x)(2x)}{(x^2+1)^2}$$

$$= \frac{(e^x + x e^x)(x^2+1) - 2x^2e^x}{(x^2+1)^2}$$

Chain Rule Version

$$y = e^{(ins)} \rightarrow y' = e^{(ins)} \cdot (ins)'$$

ex. $y = e^{5x} \rightarrow y' = e^{5x}(5x)'$

$$= 5e^{5x}$$

ex. $y = (e^{5x})^3 = e^{15x} \quad \ldots$

ex. $e^y = x$ Find $\frac{dy}{dx}$ Implicit diff^n

$(e^y)' = x'$

$e^y \cdot y' = 1$ → Not! $\cancel{e^{y \cdot y'}}$

$y' = \frac{1}{e^y}$

Proof of $y = \ln x$

ex. $y = \ln x$

$y = \log_e x$

↓

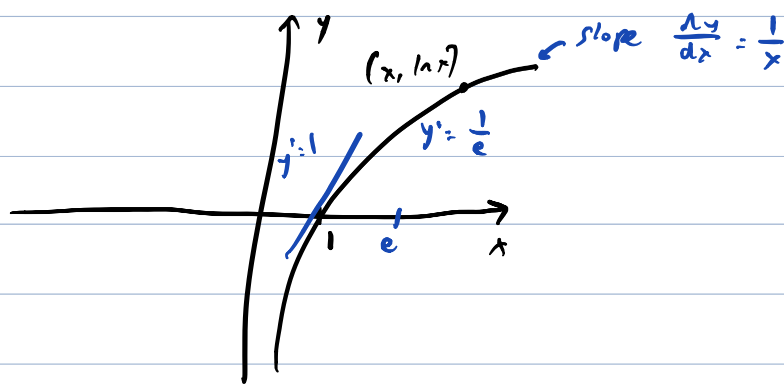
$e^y = x$

⋮

$y' = \frac{1}{e^y}$

$y' = \frac{1}{x}$

So proven that $y = \ln x$, $y' = \frac{1}{x}$



Chain rule version

$y = \ln(ins) \to y' = \frac{1}{ins} \cdot (ins)'$

ex. $y = \ln(x^3)$

Ⓐ $y' = \dfrac{1}{x^3} \cdot 3x^2 = \dfrac{3}{x}$

Ⓑ $y' = 3\ln x = 3[\ln x]'$

$= 3\left(\frac{1}{x}\right) = \frac{3}{x}$

ex. $y = 2^x$ ← Not e^x

$\ln(y) = \ln(2^x)$

$\ln(y) = x\ln 2$

$[\ln(y)]' = [x\ln 2]'$

$\frac{1}{y}\cdot y' = \ln 2\,[x]'$

$y' = y\ln 2$

$y' = 2^x \ln 2$

$y = b^x \rightarrow y' = b^x \ln b$

$y = e^x$

$y = \ln x$

$y = b^x$

Chain Rule Version

$y = b^{ins} \rightarrow y' = b^{ins}\ln b \cdot (ins)'$

ex. $y = 7^{\sin x}$

$\frac{dy}{dx} = 7^{\sin x}\ln 7 \cdot (\sin x)'$

$y' = 7^{\sin x}\ln 7 \cdot \cos x$

$y'' = \ln 7\left[7^{\sin x}\cos x\right]'$

product rule

Given $y = \log_2 x$, $\frac{dy}{dx}$?

$y = \log_2 x \overset{\text{change of base}}{=} \frac{\ln x}{\ln 2} = \frac{1}{\ln 2} \cdot \ln x$

$y' = \frac{1}{\ln 2} [\ln x]'$

$= \frac{1}{x \ln 2}$

$$\boxed{y = \log_b x \rightarrow y' = \frac{1}{x \ln b}}$$

ex. $y = \log_7 x \rightarrow \frac{dy}{dx} = \frac{1}{x \ln 7}$

Chain Rule Version

$y = \log_b (\text{ins}) \rightarrow y' = \frac{1}{(\text{ins}) \ln b} \cdot (\text{ins})'$

① $y = e^x \rightarrow y' = e^x$

② $y = b^x \rightarrow y' = b^x \ln b$

③ $y = \ln x \rightarrow y' = \frac{1}{x}$

④ $y = \log_b x \rightarrow y' = \frac{1}{x \ln b}$

ex. $y = x^e \rightarrow y' = e x^{e-1}$ ← Note (just power rule)

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ex. $e^{xy} + \ln y = x^3$

$$\left[e^{xy} + \ln y\right]' = \left[x^3\right]'$$

$$\left[e^{xy}\right]' + \left[\ln y\right]' = \left[x^3\right]'$$

$$e^{xy}(xy)' + \frac{1}{y}y' = 3x^2$$

$$e^{xy}(x'y + xy') + \frac{y'}{y} = 3x^2$$

$$e^{xy}(y + xy') + \frac{y'}{y} = 3x^2$$

$$ye^{xy} + xy'e^{xy} + \frac{y'}{y} = 3x^2$$

$$y^2e^{xy} + xyy'e^{xy} + y' = 3x^2y$$

$$y'(xye^{xy} + 1) = 3x^2y - 3y^2e^{xy}$$

$$y' = \frac{3x^2y - y^2e^{xy}}{xye^{xy} + 1}$$

ex. $y = \dfrac{\sqrt[3]{x}\,(x^3+1)^4}{x-1}$

Can use quotient rule then...

Alternative method:

$$\ln y = \ln \frac{\sqrt[3]{x}\,(x^3+1)^4}{x-1}$$

Can skip to this step

$$= \ln \sqrt[3]{x}\,(x^3+1)^4 - \ln(x-1)$$

$$= \ln \sqrt[3]{x} + \ln(x^3+1)^4 - \ln(x-1)$$

$$\ln y = \frac{1}{3}\ln x + 4\ln(x^3+1) - \ln(x-1)$$

$$[\ln y]' = \left[\frac{1}{3}\ln x + 4\ln(x^3+1) - \ln(x-1)\right]'$$

$$\frac{1}{y}y' = \frac{1}{3}\cdot\frac{1}{x} + 4\cdot\frac{1}{x^3}(x^3+1)' - \frac{1}{x-1}(x-1)'$$

$$y' = y\left[\frac{1}{3x} + \frac{12}{x^3} - \frac{1}{x-1}\right]$$

Need to present in terms of 'x'

$$y' = \frac{\sqrt[3]{x}\,(x^3+1)^4}{x-1}\left[\frac{1}{3x} + \frac{12}{x^3} - \frac{1}{x-1}\right]$$

ex. $y = x^4 \rightarrow \frac{dy}{dx} = 4x^3$ ← must be a constant

must be a constant

ex. $y = 4^x \rightarrow \frac{dy}{dx} = 4^x \ln 4$

ex. $y = x^x \rightarrow \frac{dy}{dx} = \cancel{x\, x^{x-1}}$

$= \cancel{x^x \ln x}$

Use Logarithmic Differentiation:

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

$$[\ln y]' = [x \ln x]'$$

$$\frac{1}{y}\cdot y' = x' \ln x + x(\ln x)'$$

$$= (1)(\ln x) + \cancel{x}\cdot\frac{1}{\cancel{x}}$$

$$\frac{1}{y}\cdot y' = \ln x + 1$$

$$y' = y(\ln x + 1)$$

$$y' = x^x(\ln x + 1)$$

In general, if:

$$y = f(x)^{g(x)} \quad (f(x) \text{ \& } g(x) \text{ are not constant } f^{\text{cns}})$$

USE LOGARITHMIC DIFFERENTIATION

ex. $y = 2^{\sin x}$

$\frac{dy}{dx} = 2^{\sin x} \cdot \ln 2 \cdot (\sin x)'$

$= 2^{\sin x} \cdot \cos x$ ← N.B. log diff^n would have worked, but it's longer

ex. $y = (x^3+1)^{\sin x}$

Must use log diff^n

$\ln y = \ln\left[(x^3+1)^{\sin x}\right]$

$\ln y = \sin x \cdot \ln(x^3+1)$

$[\ln y]' = [\sin x \cdot \ln(x^3+1)]'$

$\frac{1}{y} \cdot y' = (\sin x)'(\ln(x^3+1)) + \sin x\,(\ln(x^3+1))$

$= \cos x \cdot \ln(x^3+1) + \sin x \cdot \frac{1}{x^3+1} \cdot 3x^2$

$y' = (x^3+1)^{\sin x}\left(\cos x \cdot \ln(x^3+1) + \frac{3x^2}{x^3+1}\right)$

L'HÔPITAL'S RULE (L'H)

INTRODUCTION

ex.

$$\lim_{x \to 2} \frac{x^2-4}{x-2} \overset{ch}{=} \frac{0}{0}$$ ← indet.

$$\lim_{x \to 2} \frac{(x-2)(x+2)}{x-2} = 4$$

ex.

$$\lim_{x \to \infty} \frac{x^3-2x+1}{2x^3+4} = \frac{\infty}{\infty}$$ ← INDET

$$= \lim_{x \to \infty} \frac{\cancel{x^3}\left(1-\frac{2}{x^2}+\frac{1}{x^3}\right)}{\cancel{x^3}\left(2+\frac{4}{x^3}\right)}$$

$$= \frac{1}{2}$$

ex. $$\lim_{x \to 0} \frac{e^{3x}-1}{x} = \frac{0}{0}$$

What algebra ??

Need another method called L'HOPITAL'S RULE

L'Hôpital's Rule:

If $\lim \frac{f(x)}{g(x)}$ leads to an indet. form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$

then $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$

(any form of lim)

ex. $\lim_{x\to 2} \frac{x^2-4}{x-2} = \frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to 2} \frac{2x}{1}$

$= 4$

ex. $\lim_{x\to\infty} \frac{x^3-2x+1}{2x^3+4} = \frac{\infty}{\infty}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{3x^2-2}{6x^2}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{6\cancel{x}}{12\cancel{x}} = \frac{1}{2}$

ex. $\lim_{x\to 0} \frac{e^{3x}-1}{x} = \frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to 0} \frac{3e^{3x}}{1}$

$= 3e^0$

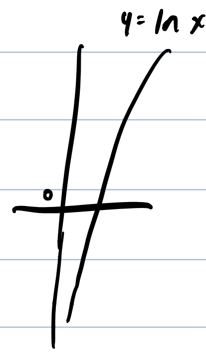
$= 3$

ex. $\lim_{x\to 1} \dfrac{3x^2-1}{x^2+1}$ Need to check to see if result is $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$= \frac{2}{2} = 1$ $\boxed{\text{CAUTION}}$

$\overset{L'H}{\cancel{=}}$ $\cancel{\frac{6x}{2x} = 3}$

ex. $\lim_{x\to 0^+} x \ln x$ $\xrightarrow{\text{recall}}$



$= (0)\cdot(-\infty)$

$\uparrow$

INDET. FORM

Cannot apply L'H rule yet

$\lim_{x\to 0^+} x \cdot \ln x$

$= \lim_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}} = \dfrac{-\infty}{+\infty}$

$\overset{L'H}{=} \lim_{x\to 0^+} \dfrac{\frac{1}{x}}{-x^{-2}}$

$= \lim_{x\to 0} (-x)$

$= 0$

N.B. for $0 \cdot (-\infty)$, ans will not always be 0.

Consider: $\lim_{x \to \infty} \left[\frac{4}{x} \cdot (-x)\right] = 0 \cdot (-\infty)$

$= \lim_{x \to \infty} \frac{-4\cancel{x}}{\cancel{x}} = -4$

N.B. Checking with 0s and ∞s

Determinate vs. Indeterminate

Consider: $\lim_{x \to 0^+} x = \underset{\substack{\uparrow \\ \text{Determinate} \\ \text{Form}}}{\ln 0^+} = -\infty$

ex. $\lim_{x \to +\infty} e^x = \underset{\substack{\uparrow \\ \text{Determinate} \\ \text{form}}}{e^{\infty}} = +\infty$

$\infty + \infty = \overset{\text{determinate}}{\infty}$

ex. $\lim_{x \to \infty} (x + e^x) = \infty + e^{\infty} = \infty$

ex. $\infty - \infty$ ← INDET

ex. $\lim_{x \to \infty} \left[(x+4) - x\right] = \underset{\substack{\uparrow \\ \text{INDET}}}{\infty - \infty}$

$= 4$

October 27, 2016

INDETERMINATE FORMS	DETERMINATE FORMS
$\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ - L'HOPITAL'S	$\infty + \infty = \infty$
$\infty - \infty$	$\ln(0^+) = -\infty$
	$\ln(\infty) = \infty$
$0 \cdot \infty$	$e^{\infty} = \infty$
$\infty^0, 0^{\infty}, 1^{\infty}$	$e^{-\infty} = 0$
	$\frac{1}{0^-} = -\infty$; $\frac{1}{0^+} = +\infty$
	$\arctan(\infty) = \frac{\pi}{2}$
	$\arctan(-\infty) = -\frac{\pi}{2}$
	$\infty \cdot \infty = \infty$
	$\sqrt{\infty} = \infty$
	$\infty^3 = \infty$
	$\frac{1}{\infty} = 0$

ex. $\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$

$= \frac{1}{0^+} - \frac{1}{0^+}$

$= \infty - \infty$ ← INDET

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$$

$$= \lim_{x\to 1^+} \frac{x-1 + \ln x}{(\ln x)(x-1)} = \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x\to 1^+} \frac{1 - \frac{1}{x}}{\frac{1}{x}(x-1) + \ln x}$$

$$= \lim_{x\to 1^+} \frac{\left(1 - \frac{1}{x}\right)x}{\left(\frac{1}{x}(x-1) + \ln x\right)x}$$

take out fraction

$$= \lim_{x\to 1^+} \frac{x-1}{x-1+x\ln x} = \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x\to 1^+} \frac{1}{1+\ln x + x\cdot\frac{1}{x}}$$

$$= \lim_{x\to 1^+} \frac{1}{2+\ln x} = \frac{1}{2}$$

Don't say ∞/∞ not safe

__ex.__ $$\lim_{x\to\infty}\left[\frac{1}{x}\cdot x\right] = \frac{1}{\infty}\cdot\infty = 0\cdot\infty$$

$$= \lim_{x\to\infty} 1 = 1$$

ex. $\lim_{x\to 0^+} x\ln x = 0^+ \ln(0^+)$

$= 0 \cdot -\infty$

$= \lim_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \dfrac{\ln(0^+)}{\frac{1}{0^+}} = \dfrac{-\infty}{+\infty}$

$\overset{L'H}{=} \lim_{x\to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} (-x) = 0$

ex. $\lim_{x\to\infty} (x^3 + e^x) = \infty^3 + e^\infty$

$= \infty + \infty$

$= \infty$

ex. $\lim_{x\to\infty} xe^{-x} = \infty \cdot e^{-\infty} = \infty \cdot 0$

$= \lim_{x\to\infty} \dfrac{x}{e^x} = \dfrac{\infty}{\infty}$

$\overset{L'H}{=} \dfrac{1}{e^x} = \dfrac{1}{\infty} = 0$

ex. $\lim_{x\to\infty} \dfrac{-x^4 - 2x^3 + 4x - 1}{5x^4 + 7x^2 - 7x + 2}$

$= \dfrac{\lim_{x\to\infty} -x^4 - 2x^3 + 4x + 1}{\lim_{x\to\infty} 5x^4 + 7x^2 - 7x + 2}$

For polynomials, look at the first term for $x \to \infty$

$= \dfrac{-\infty}{\infty}$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{-4x^3 - 6x + 4}{20x^3 + 14x - 7} = \frac{-\infty}{\infty}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{-12x^2 - 6}{60x^2 + 14} = \frac{-\infty}{\infty}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{-24\cancel{x}}{120\cancel{x}} = \frac{-24}{120} = -\frac{1}{5}$$

in other words, better to factor

$$= \lim_{x\to\infty} \frac{\cancel{x^4}\left(-1 - \frac{2}{x} - \frac{4}{x^3} + \frac{1}{x^4}\right)}{\cancel{x^4}\left(5 + \frac{7}{x^2} - \frac{7}{x^3} + \frac{2}{x^4}\right)} = \frac{-1}{5}$$

ex. $\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = \frac{\infty}{\infty}$

ex. $\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3x^{2/3}}} = \lim_{x\to\infty} \frac{3x^{2/3}}{x} = \lim_{x\to\infty} \frac{3}{x^{1/3}} = \frac{3}{\infty} = 0$

ex. $\lim_{x\to\infty} (2^x)^{\frac{1}{x}} = (2^\infty)^{\frac{1}{\infty}} = \infty^0$

$= \lim_{x\to\infty} 2^1 = 2$

Indet Forms of Type $0^0, 1^\infty, 0$

ex. $\lim_{x \to \infty} \left(\frac{1}{2^x}\right)^{\frac{1}{x}} = \left(\frac{1}{2^\infty}\right)^{\frac{1}{\infty}} = \frac{1}{\infty} = 0^0$

$= \lim_{x \to \infty} \frac{1^{\frac{1}{x}}}{(2^x)^{\frac{1}{x}}} = \frac{1}{2}$

If we change '2' to '3' above we still get 0^0, but answer is $\frac{1}{3}$. So 0^0 is INDET.

To deal with forms $0^0, 1^\infty, \infty^0$ we need to recall a few things:

① $\ln N^r = r \ln N$

② $e^{\ln N} = N$

③ $\lim_{x \to 0^+} \ln x = \ln(0^+) = -\infty$

④ $\lim e^{f(x)} = e^{\lim f(x)}$

ex. $\lim_{x \to 0^+} x^x = 0^0$ No algebra to come up with answer

$\lim_{x \to 0^+} e^{\ln(x^x)}$ by ②

$$= e^{\lim_{x\to 0^+} \ln(x^x)} \quad \text{by } ④$$

$$\text{So } \lim_{x\to 0^+} \ln(x^x) \overset{①}{=} \lim_{x\to 0^+} x\ln x \overset{③}{=} 0^+\cdot \ln(0^+) = 0\cdot\infty$$

$$= \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}} = \frac{\ln 0^+}{\frac{1}{0^+}} = \frac{-\infty}{\infty}$$

$$\overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -x = 0$$

$$\text{So } e^{\lim_{x\to 0^+} \ln(x^x)} = e^0 = 1$$

ex. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^5 = 1^5 = 1$

ex. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = 1^\infty$

$$\lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^x} = e^{\lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x}$$

$$\lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x$$

$$= \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \infty\cdot 0$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(\cancel{-\frac{1}{x^2}}\right)}{\cancel{-\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1$$

$y = \frac{1}{x}$

$y' = x^{-1}$

$= -x^{-2}$

$= -\frac{1}{x^2}$

$$\text{So } e^{\lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x} = e^1 = e$$

ex. $\lim_{x\to\infty} x^{\frac{1}{x}} = \infty^0$

$$= \lim_{x\to\infty} e^{\ln\left(x^{\frac{1}{x}}\right)}$$

$$= e^{\lim_{x\to\infty} \ln\left(x^{\frac{1}{x}}\right)} \overset{\text{see below}}{=} e^0 = 1$$

instead of this

$$\text{So} \lim_{x\to\infty} \ln x^{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{1}{x} \cdot \ln x = 0^+ \cdot \infty$$

$$= \lim_{x\to\infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$ — Don't need to write 0^+

$$\text{So } e^{\lim_{x\to\infty} \ln\left(x^{\frac{1}{x}}\right)} = e^0 = 1$$