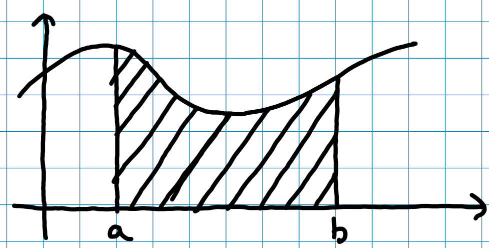
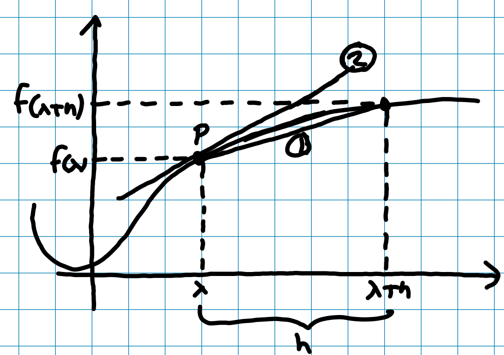
Calculus 2

Areas

→ Finding the area between the curve and the x-axis on the interval $[a, b]$



*Reminder: process to find slope of tangent



① secant: $\dfrac{f(x+h) - f(x)}{x+h - x}$

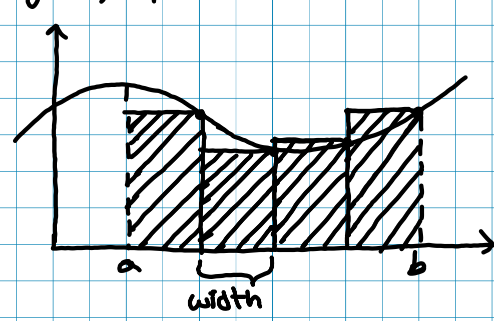
② tangent: $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

Finding area under the curve:

* Seperate into vertical rectangles of same width but different height

→ For height, pick either left or right end point

consider:



- seperate in 4 rectangles
- use right end points

Notation: L_4

Width $= \dfrac{b-a}{4}$

The more rectangles we seperate into, the more accurate the estimation

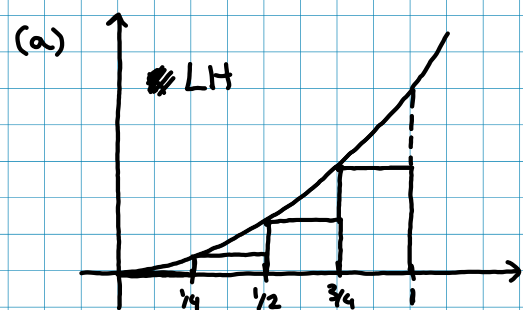
⇒ Using limits, we try to have the number of rectangles approach ∞

→ This will give the most exact estimation

→ No difference between R and L if ∞ rectangles

ex: estimate the area between $y=x^2$ and the x-axis on interval $[0,1]$ using:
(a) 4 rectangles and LH-endpoints (notation: L_4)
(b) 4 rectangles and RH-endpoints (notation. R_4)
(c) 8 rectangles for both R and L endpoints (R_8, L_8)

(a)

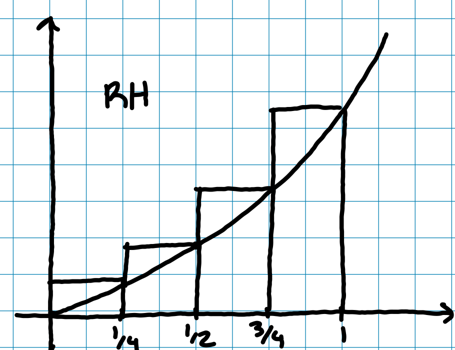


Area(S) ≈ Area(S_1) + Area(S_2) + Area(S_3) + Area(S_4)

* Area = base · height
Area (S_1) = $1/4 \cdot f(0) = 1/4 \cdot 0 = 0$
Area (S_2) = $1/4 \cdot f(1/4) = 1/4 \cdot 1/16 = 1/64$
Area (S_3) = $1/4 \cdot f(1/2) = 1/4 \cdot 1/4 = 1/16$
Area (S_4) = $1/4 \cdot f(3/4) = 1/4 \cdot 9/16 = 9/64$
+ ______
$7/32$

$\therefore L_4 = 7/32 \approx 0.21875$

(b)



Area (S) ≈ Area (S_1) + Area(S_2) + Area(S_3) + Area(S_4)

* Area = base height
$R_4 = 1/4 \cdot f(1/4) + 1/4 (f(1/2)) + 1/4\ f(3/4) + 1/4\ f(1)$
$= 1/64 + 1/16 + 9/64 + 1$
$= 15/32$

$\therefore R_4 = 15/32 \approx 0.46875$

(c)

$R_8 = (1/8)(1/8)^2 + (1/8)(1/4)^2 + (1/8)(3/8)^2 + (1/4)(1/2)^2 + (1/8)(5/8)^2 + (1/8)(3/4)^2 + (1/8)(7/8)^2 + (1/8)(1)^2$

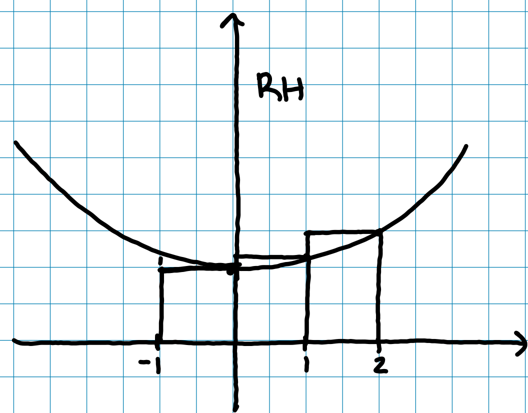
$R_8 = 51/128 \approx 0.398$

$L_8 \approx 0.273$

Note. R_8 and L_8 are closer to eachother than R_4, L_4 b/c we are getting closer to the actual value

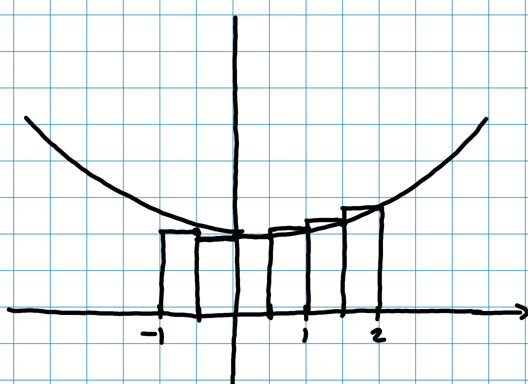
Given: $L_{1000} = 0.3328$, $R_{1000} = 0.3338$ } Guess that area is $1/3$

ex. Estimate the area under $y = x^2 + 1$ on $[-1, 2]$ using: R_3, L_3, R_6, L_6



$$R_3 = 1 \cdot f(0) + 1 \cdot f(1) + 1 \cdot f(2)$$
$$= 1 + 2 + 5$$
$$= 8$$

$$L_3 = 1 \cdot f(-1) + 1 \cdot f(0) + 1 \; f(1)$$
$$= 2 + 1 + 2$$
$$= 5$$



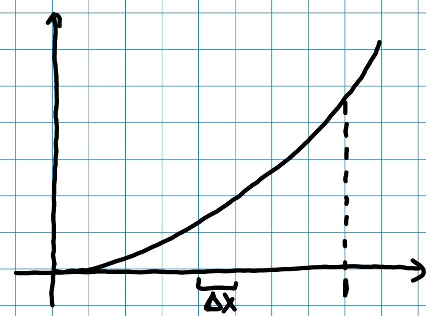
$$R_6 = \tfrac{1}{2} \cdot f(-\tfrac{1}{2}) + \tfrac{1}{2} \cdot f(0) + \tfrac{1}{2} \cdot f(\tfrac{1}{2}) + \tfrac{1}{2} \, f(1) + \tfrac{1}{2} \cdot f(\tfrac{3}{2}) + \tfrac{1}{2} \cdot f(2)$$

$$= 6.875$$

$$L_6 = \tfrac{1}{2} \cdot f(-1) + \tfrac{1}{2} \, f(-\tfrac{1}{2}) + \tfrac{1}{2} \, f(0) + \tfrac{1}{2} \, f(\tfrac{1}{2}) + \tfrac{1}{2} \cdot f(1) + \tfrac{1}{2} \cdot f(\tfrac{3}{2})$$

$$= 5.375$$

ex: consider $y=x^2$ from $[0,1]$. Show that the are is actually $1/3$.



Let n = the number of rectangles

$\therefore$ Base width $= \Delta x = \frac{1}{n}$

$\therefore$ Heights (using RH) are
$f(\frac{1}{n}), f(\frac{2}{n}), f(\frac{3}{n}), \ldots, f(\frac{n}{n})$

$$\therefore R_n = \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)^2 + \left(\frac{1}{n}\right)\left(\frac{3}{n}\right)^2 + \ldots + \left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right)^2 + \left(\frac{1}{n}\right)\left(\frac{n}{n}\right)^2$$

$$\text{Area}(S) = \lim_{n\to\infty} R_n$$

* Property: $1^2+2^2+3^2+\ldots+(n-1)^2+n^2 = \frac{n(n+1)(2n+1)}{6}$

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)\left[\frac{1^2}{n^2}+\frac{2^2}{n^2}+\frac{3^2}{n^2}+\ldots+\frac{(n-1)^2}{n^2}+\frac{n^2}{n^2}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n^3}\left(1^2+2^2+3^2+\ldots+(n-1)^2+n^2\right) \quad \text{* property}$$

$$= \lim_{n\to\infty}\frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} \qquad \frac{\deg 3}{\deg 3} \quad \frac{\infty}{\infty} \quad \therefore \text{use coefficients}$$

$$= \frac{2}{6} = \frac{1}{3}$$

Summation Notation

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n$$

end at $i=n$

$$\sum_{i=1}^{n} a_i$$

start at $i=1$

* "i" is the counter $\Rightarrow$ what is changing

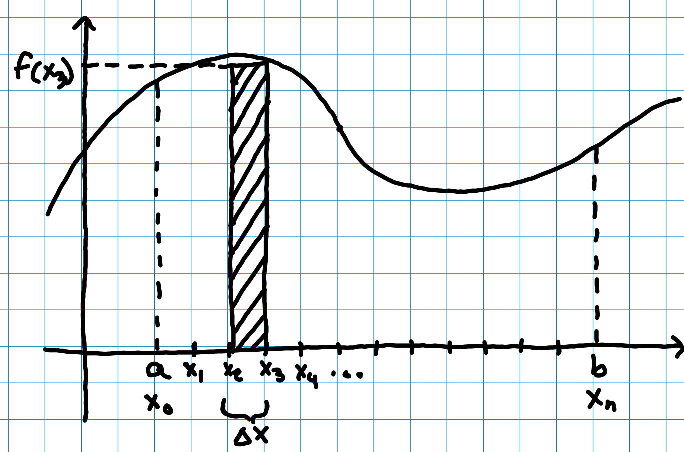
term to be added

ex: $\sum_{i=1}^{n} 2i = \underbrace{2(1)}_{i=1} + \underbrace{2(2)}_{i=2} + 2(3) + \dots + 2(n)$

ex: $\left(\frac{1}{n}\right) f\left(\frac{1}{n}\right) + \left(\frac{1}{n}\right) f\left(\frac{2}{n}\right) + \dots + \left(\frac{1}{n}\right) \cdot f\left(\frac{n}{n}\right)$

$$= \boxed{\sum_{i=1}^{n} \frac{1}{n} \cdot f\left(\frac{1}{n}\right)}$$

In General: What is the area between $y=f(x)$ and the x-axis on $[a,b]$



If we us n rectangles

$\therefore$ $\boxed{\Delta x = \frac{b-a}{n}}$ is base width

and then $x_1, x_2, x_3, \ldots, x_{n-1}, x_n$ are RH endpoints of the corresponding sub-intervals $\therefore$

$$\left.\begin{aligned} x_1 &= a + \Delta x \\ x_2 &= a + 2\Delta x \\ x_3 &= a + 3\Delta x \\ x_n &= a + n\Delta x \end{aligned}\right\} \quad \therefore \quad \boxed{x_i = a + i\Delta x}$$

and the heights are given by $f(x_1), f(x_2), \ldots, f(x_{n-1}), f(x_n)$

$$\therefore\ R_n = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_n)\Delta x$$

$$\therefore\ R_n = \sum_{i=1}^{n} \underbrace{f(x_i)}_{\text{height}} \times \underbrace{\Delta x}_{\text{base}}$$

$$\therefore\ \boxed{\text{Area}(S) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x}$$

Setup limit representing area under $y=f(x)$ on $[a,b]$ given the function

ex: $f(x)=\frac{2x}{x^2+1}$ $\quad 1 \le x \le 3$

$a=1$, $b=3$

① $\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$

② $x_i = a + i\Delta x = 1 + \frac{2i}{n}$

③ $f(x_i) = \frac{2\left(1+\frac{2i}{n}\right)}{\left(1+\frac{2i}{n}\right)^2+1}$

$\therefore$ Area $(S) = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\frac{2\left(1+\frac{2i}{n}\right)}{\left(1+\frac{2i}{n}\right)^2+1}}_{f(x_i)} \underbrace{\left(\frac{2}{n}\right)}_{\Delta x}$

ex: $f(x) = x^2 + \sqrt{1+2x}$ $\quad 4 \le x \le 7$

$a=4$, $b=7$

① $\Delta x = \frac{7-4}{n} = \frac{3}{n}$

② $x_i = a + i\Delta x = 4 + \frac{3i}{n}$

③ $\left(4+\frac{3i}{n}\right)^2 + \sqrt{1+2\left(4+\frac{3i}{n}\right)}$

$\therefore$ Area $(S) = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\left[\left(4+\frac{3i}{n}\right)^2 + \sqrt{1+2\left(4+\frac{3i}{n}\right)}\right]}_{f(x_i)} \underbrace{\left(\frac{3}{n}\right)}_{\Delta x}$

ex: $f(x) = x^3 + \frac{1}{2}\sqrt[3]{x}$ $\quad [-1,4]$

① $\Delta x = \frac{4-(-1)}{n} = \frac{5}{n}$

② $x_i = -1 + \frac{5i}{n}$

③ $f(x_i) = \left(-1+\frac{5i}{n}\right)^3 + \frac{1}{2}\left(-1+\frac{5i}{n}\right)^{1/3}$

$\therefore$ Area $(S) = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(-1+\frac{5i}{n}\right)^3 + \frac{1}{2}\left(-1+\frac{5i}{n}\right)^{1/3}\right]\left(\frac{5}{n}\right)$

Determine a region whose area is the same to the given limit

ex: $\lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\left(\frac{2}{n}\right)}_{\Delta x} \underbrace{\left(5+\frac{2i}{n}\right)^{10}}_{f(x_i)}$

① $\Delta x = \frac{2}{n} = \frac{b-a}{n}$ Let $a=5$ $\therefore b=7$

② $x_i = a + i\Delta x = 5 + \frac{2i}{n}$

If $f(x_i) = \left(5+\frac{2i}{n}\right)^{10}$ $\therefore$ $f(x) = x^{10}$

$\therefore$ $\boxed{f(x) = x^{10} \quad \text{on} \quad [5,7]}$

* Note: could've chosen any a, b

ex chose different a, b for same limit

① $\Delta x = \frac{2}{n} = \frac{b-a}{n}$ Let $a=0$ $\therefore b=2$

② $x_i = a + i\Delta x = \frac{2i}{n}$

If $f(x_i) = \left(5+\frac{2i}{n}\right)^{10}$ $\therefore$ $f(x) = (5+x)^{10}$

$\therefore$ $\boxed{f(x) = (5+x)^{10} \quad \text{on} \quad [0,2]}$

ex: $\lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \frac{1}{1+\frac{3}{n}} + \dots + \frac{1}{1+\frac{n-1}{n}} + \frac{1}{1+\frac{n}{n}}\right]$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{n}\right)\left[\frac{1}{1+\frac{i}{n}}\right]$

① $\Delta x = \frac{1}{n} = \frac{b-a}{n}$ Let $a=1$ $\therefore b=2$

② $x_i = a + i\Delta x = 1 + \frac{i}{1}$

If $f(x_i) = \left(\frac{1}{1+\frac{i}{n}}\right)$ $\therefore$ $f(x) = \frac{1}{x}$

$\therefore$ $\boxed{f(x) = \frac{1}{x} \quad \text{on} \quad [1,2]}$

Summation Rules

Consider if $c \in \mathbb{R}$ is any constant

(a) $\sum_{i=1}^{n} c a_i = c \cdot \sum_{i=1}^{n} a_i$

(b) $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

(c) $\sum_{i=1}^{n} c = c \cdot n$ $\Rightarrow \sum_{i=1}^{n} (1) = 1+1+1+1+\ldots$ (n times)

(d) ① $\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$

$\Rightarrow 1+2+3+\ldots+n-1+n$

$\sum_{i=1}^{10} i = 1+2+3+4+5+6+7+8+9+10$

$10+1 = 11 = (n+1)$
$9+2 = 11 = (n+1)$
$\vdots$
can do 5 times $= \frac{n}{2}$

(e) ② $\boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$

(f) ③ $\boxed{\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2}$

ex: $\sum_{i=1}^{n} i(4i^2-3)$

$= \sum_{i=1}^{n} 4i^3 - \sum_{i=1}^{n} 3i$

$= 4\left(\sum_{i=1}^{n} i^3\right) - 3\left(\sum_{i=1}^{n} i\right)$

$= 4\left(\frac{n(n+1)}{2}\right)^2 - 3\left(\frac{n(n+1)}{2}\right)$

ex: $\sum_{i=1}^{n} (4-3i)^2$

$= \sum_{i=1}^{n} (16-24i+9i^2)$

$= \sum_{i=1}^{n} 16 - 24\sum_{i=1}^{n} i + 9 \sum_{i=1}^{n} i^2$

$= 16n - 24\frac{n(n+1)}{2} + 9\frac{n(n+1)(2n+1)}{6}$

Recall : $\lim_{x\to\infty}$ of type $\frac{\infty}{\infty}$

① Higher power (deg) on top $\Rightarrow \pm\infty$ (sign of highest power)

ex. $\lim_{x\to\infty} \dfrac{x^6+4x^4-3x^2}{3x^5+2x-1} = \infty$

proof: $\lim_{x\to\infty} \dfrac{x^6\left(1+\frac{4}{x^2}-\frac{3}{x^3}\right)}{x^5\left(3+\frac{2}{x^4}-\frac{1}{x^5}\right)} \Rightarrow \dfrac{x^6}{x^5} = x \quad \therefore \quad \infty$

② Higher power (deg) on bottom $\Rightarrow 0$

ex. $\lim_{x\to\infty} \dfrac{x^5+x^4-x^2}{x^6+2x-1} = 0$

proof: $\lim_{x\to\infty} \dfrac{x^5\left(1+\frac{1}{x}-\frac{1}{x^3}\right)}{x^6\left(1+\frac{2}{x^5}-\frac{1}{x^6}\right)} \Rightarrow \dfrac{x^5}{x^6} = \dfrac{1}{x} \quad \therefore \quad 0$

③ Same degree $\Rightarrow$ use coefficients of highest degree of each term that is being multiplied or divided

ex. $\lim_{x\to\infty} \dfrac{x^6+x^4-x^2}{3x^6+2x-1}$ $\dfrac{\text{deg } 6}{\text{deg } 6}$

$= \boxed{\dfrac{1}{3}}$

ex. $\lim_{x\to\infty} \dfrac{(2x-1)^3(3x+2)}{3x^4}$ $\dfrac{\text{deg } 3 \times \text{deg } 1}{\text{deg } 4} = \dfrac{\text{deg } 4}{\text{deg } 4}$

* multiply coefficients of the highest power in each term that is being multiplied

$\dfrac{2^3 \cdot 3}{3} = 8$

Evaluating limits with a summation

2 variables (n and i)

① Simplify the summation (to get rid of i)
② Solve the limit with n

* Note: with respect to the summation, n can be treated as a constant (only i is changing)

ex: $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4}{n}\right)\left(3+\frac{4i}{n}\right)^2$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4}{n}\right)\left(9+\frac{24i}{n}+\frac{16i^2}{n^2}\right)$

- Take Δx portion out of summation (constant)
- Simplify $f(x_i)$ parenthesis with algebra
- Distribute the summation

$= \lim_{n\to\infty} \left(\frac{4}{n}\right)\left(\sum_{i=1}^{n} 9 + \sum_{i=1}^{n} \frac{24i}{n} + \sum_{i=1}^{n} \frac{16i^2}{n^2}\right)$

$= \lim_{n\to\infty} \frac{4}{n}\left[9n + \frac{24}{n}\left(\frac{n(n+1)}{2}\right) + \frac{16}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right]$

$= \lim_{n\to\infty} \frac{36n}{n} + \frac{48\,n(n+1)}{n^2} + \frac{64\,n(n+1)(2n+1)}{6\,n^3}$

* multiply coef. of highest deg. of each term

$\frac{\text{deg } 1}{\text{deg } 1}$ $\frac{\text{deg } 2}{\text{deg } 2}$ $\frac{\text{deg } 3}{\text{deg } 3}$

$36 + 48 \quad \frac{64(2)}{6}$

* Note. For Reiman sums, the deg. is always the same top and bottom

b/c it represents an area, which is neither 0 nor ∞

ex: $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2}{n}\right)\left(\frac{2i}{n}\right)^3 =$

$= \lim_{n\to\infty} \left(\frac{2}{n}\right) \sum_{i=1}^{n} \frac{8i^3}{n^3}$

$= \lim_{n\to\infty} \left(\frac{2}{n}\right) \left[\frac{8}{n^3} \left(\frac{n(n+1)}{2}\right)^2\right]$

$= \lim_{n\to\infty} \frac{16}{n^4} \frac{(n(n+1))^2}{4}$ $\qquad \frac{\deg 4}{\deg 4}$

$= 16/4 = 4$

ex: $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2 =$

$= \lim_{n\to\infty} \left(\frac{1}{n}\right) \sum_{i=1}^{n} \frac{i^2}{n^2}$

$= \lim_{n\to\infty} \frac{1}{n} \left[\frac{1}{n^2} \cdot \frac{n(n+1)(2n+1)}{6}\right]$ $\qquad \frac{\deg 3}{\deg 3}$

$= 2/6 = 1/3$

ex: $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^3 - 2\right] =$

$= \lim_{n\to\infty} \left(\frac{1}{n}\right) \left[\sum_{i=1}^{n} \frac{i^3}{n^3} - \sum_{i=1}^{n} 2\right]$

$= \lim_{n\to\infty} \left(\frac{1}{n}\right) \left[\frac{1}{n^3} \cdot \left(\frac{n(n+1)}{2}\right)^2 - 2n\right]$

$= \lim_{n\to\infty} \frac{[n(n+1)]^2}{n^4 \cdot 2^2} - \frac{2n}{n}$

$= \quad 1/4 \quad - \quad 2$

$= -7/4$

Definite Integral

upper limit of integration

differential

$$\int_a^b \underbrace{f(x)}_{\text{integrand}}\,dx$$

Lower limit of integration

Note:

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x}$$

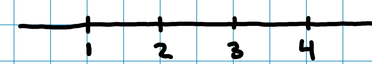
$\therefore$ definite integral also represents the area under $y = f(x)$ on $[a, b]$

ex. Consider the function $f(x) = x^2 + 2x - 5$ and the definite integral $\int_1^4 (x^2+2x-5)\,dx$

(a) Approximate the integral using 3 rectangles

$n = 3$ b/c 3 rectangles

$\Delta x = \frac{4-1}{n} = \frac{4-1}{3} = 1$



$$\int_1^4 (x^2+2x-5)\,dx \approx \text{Area}(S_1) + \text{Area}(S_2) + \text{Area}(S_3)$$
$$= f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1$$
$$= 3 + 10 + 19$$
$$= 32$$

(b) Evaluate exactly the integral $\int_1^4 (x^2+2x-5)\,dx$

① $\Delta x = \frac{3}{n}$
② $x_i = 1 + \frac{3i}{n}$
③ $f(x_i) = \left(1+\frac{3i}{n}\right)^2 + 2\left(1+\frac{3i}{n}\right) - 5$

$$\therefore \int_1^4 (x^2+2x-5)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(1+\frac{3i}{n}\right)^2 + 2\left(1+\frac{3i}{n}\right) - 5\right]\left(\frac{3}{n}\right)$$
$$= \lim_{n\to\infty} \left(\frac{3}{n}\right) \sum_{i=1}^{n} \left[1 + \frac{6i}{n} + \frac{9i^2}{n^2} + 2 + \frac{6i}{n} - 5\right]$$
$$= \lim_{n\to\infty} \left(\frac{3}{n}\right) \sum_{i=1}^{n} \left[-2 + \frac{12i}{n} + \frac{9i^2}{n^2}\right]$$
$$= \lim_{n\to\infty} \left(\frac{3}{n}\right) \left[-2n + \frac{12}{n}\,\frac{n(n+1)}{2} + \frac{9}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right]$$
$$\downarrow$$
$$= 3(-2) + 3(12/2) + 3(9\times 2/6)$$
$$= -6 + 18 + 9$$
$$= 21$$

ex: Express the following limit as a definite integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\left(\frac{4}{n}\right)}_{\Delta x} \underbrace{\underbrace{\left(2+\frac{4i}{n}\right)}_{x_i} \ln\left(1+\underbrace{\left(2+\frac{4i}{n}\right)^2}_{x_i}\right)}_{f(x_i)}$$

① $\Delta x = \frac{4}{n} = \frac{b-a}{n}$ Let $a=2$, $b=6$

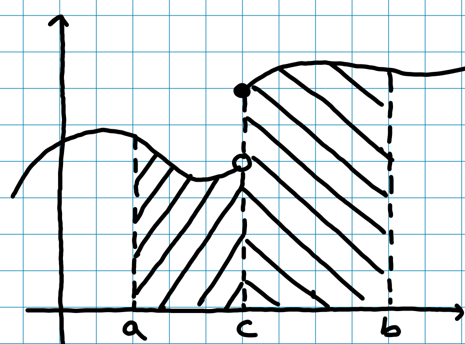
② $x_i = 2+\frac{4i}{n}$

$\therefore = \int_2^6 x \ln(1+x^2)\, dx$

Theorem: If $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ exists

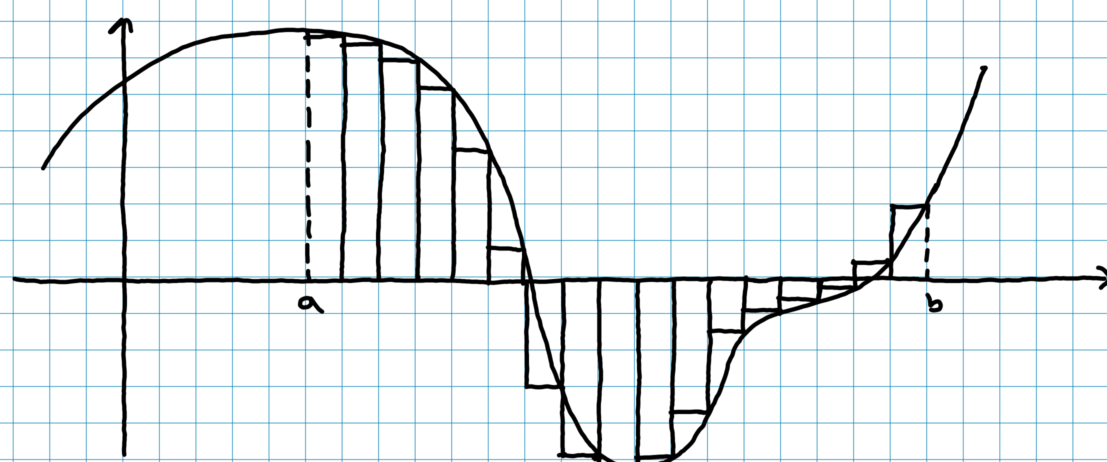
$\therefore$ f is said to be integrable on [a,b]

Note: having a discontinuity does **not** affect integrability



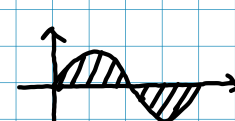
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Consider:



Area ABOVE x-axis $\Rightarrow$ pos. (+)
Area BELLOW x-axis $\Rightarrow$ neg (-)

ex: $\int_0^{2\pi} \sin x \, dx = 0$



Properties of Definite Integral

$$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx} \qquad a > b$$

proof: $\displaystyle \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right)$

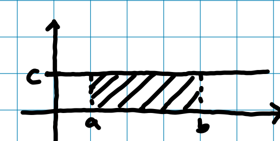
$\displaystyle = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\left[-\left(\frac{a-b}{n}\right)\right]$

$\displaystyle = -\int_b^a f(x)\,dx$

$$\boxed{\int_a^a f(x)\,dx = 0}$$

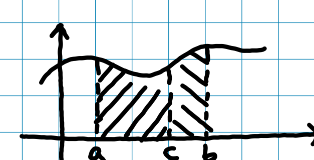
$$\Delta x = \frac{a-a}{n} = 0$$

$$\boxed{\int_a^b c \;\; dx = c(b-a)} \qquad c \in \mathbb{R}$$



$$\boxed{\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx}$$

$$* \quad \boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx}$$



Note. c does not HAVE to be between a and b

$$\boxed{\int_a^b c \cdot f(x)\,dx = c\int_a^b f(x)\,dx}$$

ex: If $\int_0^{10} f(x)\,dx = 17$ and $\int_0^{8} f(x)\,dx = 12$, find $\int_8^{10} f(x)\,dx$

$$\int_0^{10} f(x)\,dx = \int_0^{8} f(x)\,dx + \int_8^{10} f(x)\,dx$$

$$17 = 12 + \int_8^{10} f(x)\,dx$$

$$\therefore \int_8^{10} f(x)\,dx = 5$$

ex: If $\int_0^1 f(x)\,dx = 2$, $\int_1^2 f(x)\,dx = 3$, $\int_0^1 g(x)\,dx = -1$, $\int_0^2 g(x)\,dx = 4$, Find:

(a) $\int_1^2 g(x)\,dx =$

$= \int_0^2 g(x)\,dx - \int_0^1 g(x)\,dx$

$= 4 - (-1) = 5$

$\int_0^2 = \int_0^1 + \int_1^2$

$\int_0^2 - \int_0^1 = \int_1^2$

(b) $\int_0^2 [2f(x) - 3g(x)]\,dx =$

$= 2\int_0^2 f(x)\,dx - 3\int_0^2 g(x)\,dx$

$= 2(2+3) - 3(4) = -2$

$\int_0^2 f = \int_0^1 f + \int_1^2 f$

(c) $\int_1^1 g(x)\,dx =$

$= 0$

(d) $\int_1^2 f(x)\,dx + \int_2^0 g(x)\,dx =$

$= \int_1^2 f(x)\,dx - \int_0^2 g(x)\,dx$

$= 3 - 4 = -1$

(e) $\int_0^2 f(x)\,dx + \int_2^1 g(x)\,dx =$

$= (2+3) - \int_1^2 g(x)\,dx$

$= 5 - (4 - -1) = 0$

$\int_0^2 f = \int_1^2 f + \int_0^1 f$

$\int_0^2 g = \int_1^2 g + \int_0^1 g$

$\int_1^2 g = \int_0^2 g - \int_0^1 g$

ex: $\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$, simplify

$= \int_{-2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$

$= \int_{-2}^{5} f(x)\,dx + \int_{-1}^{-2} f(x)\,dx$

$= \int_{-1}^{5} f(x)\,dx$

Note: c not in between a and b

$a=-1,\ b=5,\ c=-2$

ex: $\int_{0}^{2} 3f(x)\,dx + \int_{1}^{3} 3f(x)\,dx - \int_{0}^{2} 2f(x)\,dx - \int_{1}^{2} 3f(x)\,dx - \int_{2}^{3} 2f(x)\,dx$, simplify

$= \int_{0}^{2} [3f(x) - 2f(x)]\,dx + \int_{1}^{3} 3f(x)\,dx - \int_{1}^{2} 3f(x)\,dx - \int_{2}^{3} 2f(x)\,dx$

$\int_{1}^{3} 3f = \int_{1}^{2} 3f + \int_{2}^{3} 3f$

$\int_{1}^{3} 3f - \int_{1}^{2} 3f = \int_{2}^{3} 3f$

$= \int_{0}^{2} f(x)\,dx + \int_{2}^{3} 3f(x)\,dx - \int_{2}^{3} 2f(x)\,dx$

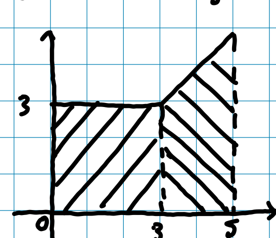
$= \int_{0}^{2} f(x)\,dx + \int_{2}^{3} [3f(x) - 2f(x)]\,dx$

$= \int_{0}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx$

$= \int_{0}^{3} f(x)\,dx$

ex. Find $\int_{0}^{5} f(x)\,dx$ for $f(x) = \begin{cases} 3, & x < 3 \\ x, & x \geq 3 \end{cases}$

$= \int_{0}^{3} f(x)\,dx + \int_{3}^{5} f(x)\,dx$



$\int_{0}^{3} f(x)\,dx = A_1 = 3^2 = 9$

$\int_{3}^{5} f(x)\,dx = A_2 = \frac{(B+b)h}{2} = \frac{(3+5)2}{2} = 8$

$\therefore \int_{0}^{5} f(x)\,dx = 9 + 8 = 17$

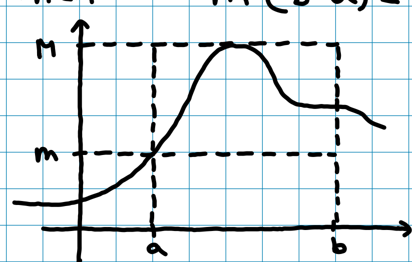
Comparaison Properties of the Definite Integral

(1) If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant 0$

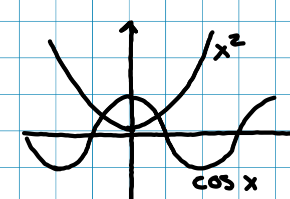
(2) If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$

(3) If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, $m \in \mathbb{R}$, $M \in \mathbb{R}$

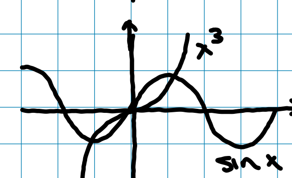
then $m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a)$



<u>Definition</u>: (a) $f(x)$ is "even" if $f(x) = f(-x)$
(ie. symmetric about the y-axis)



(b) $f(x)$ is "odd" if $f(-x) = -f(x)$
(ie. symetry about the y and x axis)



<u>Theorem</u>:

(1) If $f(x)$ is "even" then

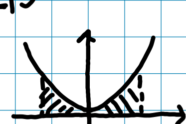
$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

symetric intervals

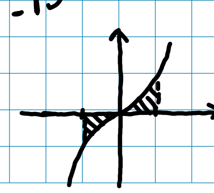
(2) If $f(x)$ is "odd" then

$$\int_{-a}^{a} f(x)\,dx = 0$$

ex: $\int_{-1}^{1} x^2\,dx = 2\int_0^1 x^2\,dx$



ex: $\int_{-1}^{1} x^3\,dx = 0$



Fundamental Theorem of Calculus - I
(FTC-I)

FTC-I : If f is continuous on $[a,b]$,
then F defined by $F(x) = \int_a^x f(t)\,dt$

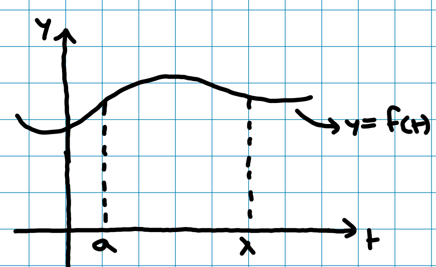
① is continuous on $[a,b]$

② differentiable on (a,b)

③ $F'(x) = f(x)$

consider $F(x) = \int_a^x f(t)\,dt$ ⇒ accumulation fcn

Graph of $f(t)$



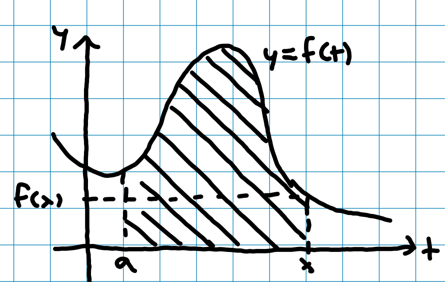
Area accumulates from a to x

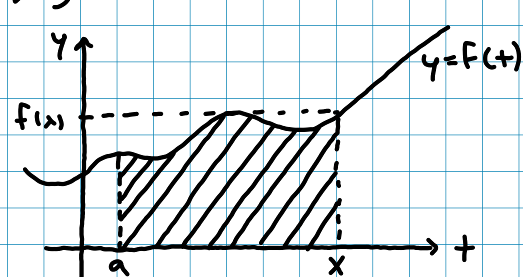
$F(x)$: Function
→ In the graph $f(t)$, it describes the area under f

$F'(x) = f(x)$: derivative of the function (graph above)
→ Describes the rate at which the area changes

Why is $F'(x) = f(x)$?

$F'(x) = \left(\text{how quickly the area is changing}\right)$





The bigger $f(x)$, ⇒ the faster the area changes
∴ The bigger $F'(x)$

Fundamental Theorem of Calculus - I

FTC-I : If f is continuous on $[a,b]$, the F defined by $F(x) = \int_a^x f(t)\,dt$

① is continuous on $[a,b]$

② is differentiable on (a,b)

③ $F'(x) = f(x)$

Proof : Let $f(t)$ be continuous on $[a,b]$

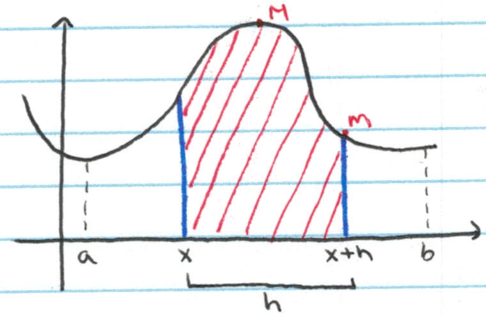
take x and $x+h$ on (a,b), $h \neq 0$

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt = \int_x^{x+h} f(t)\,dt$$

(*) $$\therefore \boxed{\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt}$$

$\because f(t)$ is continuous on $[a,b]$ $\therefore$ continuous on $[x,x+h]$

$\therefore$ By EVT, there exists an absolute max/min for $f(t)$ on $[x,x+h]$



$$m \leq f(t) \leq M$$

$$m \cdot h \leq \int_x^{x+h} f(t)\,dt \leq M \cdot h$$

Note: As $\lim_{h\to 0}$, $m \to f(x)$, $M \to f(x)$

$$\therefore f(x) \cdot h \leq \lim_{h\to 0}\int_x^{x+h} f(t)\,dt \leq f(x) \cdot h$$

Squeeze : $$\lim_{h\to 0}\int_x^{x+h} f(t)\,dt = f(x) \cdot h$$

$$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x)$$

(*) $$\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x) \qquad \therefore F'(x) = f(x)$$

Proof of FTC-I :

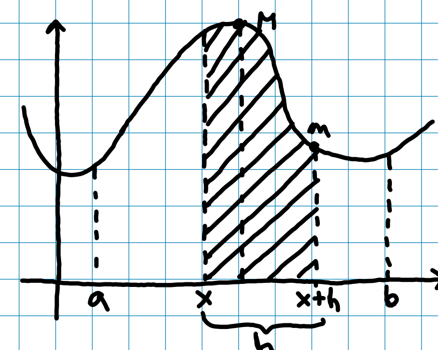
Let $f(t)$ be continuous on $[a,b]$
take x and $x+h$ on (a,b) , $h \neq 0$

Then $F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt = \int_x^{x+h} f(t)\,dt$

$$\therefore \boxed{\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt} \quad (*)$$

For $h>0$, $\because$ $f(t)$ is continuous on $[a,b]$
$\therefore$ $f(t)$ is continuous on $[x, x+h]$

$\therefore$ by Extreme Value Theorem, there exists an absolute max and min for $f(t)$ on $[x, x+h]$



$$\therefore \quad m \leq f(t) \leq M$$

$$\therefore \int_x^{x+h} m\,dt \leq \int_x^{x+h} f(t)\,dt \leq \int_x^{x+h} M\,dt$$

$$m\,h \leq \int_x^{x+h} f(t)\,dt \leq M.h$$

<u>Note</u>: As $\lim_{h\to 0}$, $m \to f(x)$, $M \to f(x)$ $\Rightarrow$ Area $\to$ (base) (height) $h \cdot f(x)$

$$\therefore \quad f(x).h \leq \lim_{h\to 0}\int_x^{x+h} f(t)\,dt \leq f(x).h$$

$\therefore$ By squeeze theorem .

$$\lim_{h\to 0}\int_x^{x+h} f(t)\,dt = f(x).h$$

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x)$$

sub $(*)$

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = f(x)$$

$$\therefore \quad F'(x) = f(x)$$

Applying FTC-I

① If $F(x) = \int_a^x f(t)\,dt$ $\therefore$ $F'(x) = f(x)$

② If $F(x) = \int_a^{g(x)} f(t)\,dt$ $\therefore$ $F'(x) = f(g(x)) \cdot g'(x)$

composite fcn ⇒ chain rule

③ $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x)$

* If asked to find $F'(x^2)$, x^2 is another composite function so we have to proceed by chain rule

ex: $g(x) = \int_0^{x} t^2 \cos t \cdot dt$

$\therefore g'(x) = x^2 \cos x$

ex: $h(x) = \int_1^{x^4} \sec t\,dt$

$h'(x) = \sec(x^4) \cdot 4x^3$

ex: $\ell(x) = \int_x^{x^2} e^{-t^2}\,dt$

$= \int_x^{a} e^{-t^2}\,dt + \int_a^{x^2} e^{-t^2}\,dt$

$= -\int_a^{x} e^{-t^2}\,dt + \int_a^{x^2} e^{-t^2}\,dt$

$\therefore \ell'(x) = -e^{-x^2} + e^{-(x^2)^2} \cdot (2x)$

ex: $\frac{d}{dx}\int_5^{x} (1-\sqrt{\sin t})\,dt = 1-\sqrt{\sin x}$

ex: $\frac{d}{dx}\int_1^{x^3} (1-\sqrt{\sin t})\,dt = (1-\sqrt{\sin x^3})(3x^2)$

ex: $\frac{d}{dt}\int_t^{e^{2t}} \frac{1}{\sqrt{1+x^2}}\,dx = \frac{1}{\sqrt{1+(e^{2t})^2}} \cdot e^{2t} \cdot 2 - \frac{1}{\sqrt{1+t^2}}$

ex: $\frac{d}{dx}\int_{2x-1}^{x^2} \frac{t}{1+t^3}\,dt = \frac{x^2}{1+(x^2)^3}\cdot 2x - \frac{(2x-1)}{1+(2x-1)^3}\cdot 2$

ex: $\frac{d}{d\theta}\int_{\sin\theta}^{\cos\theta} \frac{1}{\sqrt{1-x^2}}\,dx = \frac{1}{\sqrt{1-\cos^2\theta}}\cdot(-\sin\theta) - \frac{1}{\sqrt{1-\sin^2\theta}}\cdot(\cos\theta)$

$= \frac{-\sin\theta}{\sqrt{\sin^2\theta}} - \frac{\cos\theta}{\sqrt{\cos^2\theta}} = -1-1 = -2$

ex: $\frac{d}{dx}\int_{x\sin x}^{x^3} \frac{t}{1-t^2}\,dt = \frac{x^3}{1-(x^3)^2}\cdot 3x^2 - \frac{x\sin x}{1-(x\sin x)^2}\,(x\cos x + \sin x)$

Antiderivatives

→ A fcn F is called "antiderivative" of f on interval I, if $F'(x) = f(x)$ on I

ex: consider $f(x) = x^2$, then

$F(x) = \frac{x^3}{3}$ is an antiderivative of $f(x)$

b/c $\left(\frac{x^3}{3}\right)' = \frac{3x^2}{3} = x^2$

Note: $G(x) = \frac{x^3}{3} + 1$ is also an antiderivative

In general: the C that comes after can be any constant $C \in \mathbb{R}$

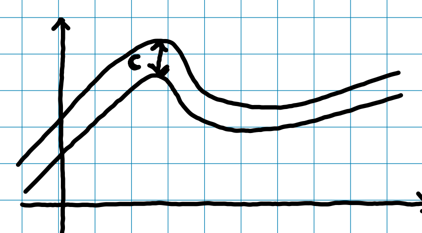
$\because$ the derivative of a constant is 0

$\therefore$ The 'general antiderivative' of $f(x) = x^2$ is $F(x) = \frac{x^3}{3} + C$

<u>Theorem (Cal 1)</u>

If $f'(x) = g'(x)$ on I

then $f(x) - g(x) = C$



$\therefore$ There is an infinite number of antiderivatives for a same fcn

$\therefore$ The difference between all of them is a constant.

Fundamental Theorem of Calculus - II (FTC-II)

$$\boxed{\begin{aligned}&\text{FTC-II: If } f(x) \text{ is continuous on } [a,b] \\ &\qquad \text{and } F'(x) = f(x) \quad (F(x) \text{ is antiderivative}) \\ &\qquad \therefore \int_a^b f(x)\,dx = F(b) - F(a)\end{aligned}}$$

<u>Note</u>: Notation can abbreviate FTC-II

$$F(b) - F(a) = F(x)\Big|_a^b$$

$$\boxed{\int_a^b f(x)\,dx = \underbrace{F(x)}_{\text{general antiderivative}}\Big|_a^b}$$

Fundamental Theorem of Calculus - II

FTC-II : If $f(x)$ is continuous on $[a,b]$
and $F'(x) = f(x)$ ($F(x)$ is the antiderivative)

$$\therefore \int_a^b f(x)\,dx = F(b) - F(a)$$

<u>Proof</u>: Let $f(t)$ be continuous on $[a,b]$

$\circledast$ $\boxed{\text{Let } g(x) = \int_a^x f(t)\,dt}$

FTC-I: $\therefore g'(x) = f(x)$ ($g(x)$ is antiderivative)

Let $F(x)$ be some other antiderivative of $f(x)$

$\therefore F'(x) = g'(x) = f(x)$

$\therefore F'(x) = g'(x)$

$\therefore F(x) = g(x) + C$

$F(b) - F(a) = [g(b) + \cancel{C}] - [g(a) + \cancel{C}]$

$F(b) - F(a) = g(b) - g(a)$

$$\circledast \begin{cases} ① \; g(b) = \int_a^b f(t)\,dt \\ ② \; g(a) = \int_a^a f(t)\,dt = 0 \end{cases}$$

$\therefore F(b) - F(a) = \int_a^b f(t)\,dt$

Proof of FTC-II:

Let $f(t)$ be continuous on $[a,b]$

Let $g(x) = \int_a^x f(t)\,dt$

$\therefore\ g'(x) = f(x)$ (From FTC-I)

$\rightarrow$ $g(x)$ is an antiderivative of $f(x)$

Let $F(x)$ be some other antiderivative of $f(x)$

$\therefore$ Both $F(x)$ and $g(x)$ are antiderivatives of $f(x)$

$F'(x) = f(x) = g'(x)$

$F'(x) = g'(x)$

$\therefore\ F(x) - g(x) = C$

$\therefore\ F(x) = g(x) + C$

Show that: $\int_a^b f(t)\,dt = F(b) - F(a)$

$F(b) - F(a) = [g(b) + C] - [g(a) + C]$

$$\boxed{F(b) - F(a) = g(b) - g(a)}$$

① $g(b) = \int_a^b f(t)\,dt$

② $g(a) = \int_a^a f(t)\,dt = 0$

From: $g(x) = \int_a^x f(t)\,dt$

$\therefore\ F(b) - F(a) = \int_a^b f(t)\,dt - 0$

$$\boxed{F(b) - F(a) = \int_a^b f(t)\,dt}$$

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b$$

$\to F(x)$ is the general antiderivative.

<u>Note</u>: No need to include C b/c it will cancel out

ex: $\int_1^3 e^x\,dx = e^x\Big|_1^3 = e^3 - e^1$

ex: $\int_3^6 \frac{1}{x}\,dx = \ln|x|\Big|_3^6 = \ln 6 - \ln 3 = \ln\frac{6}{3} = \ln 2$

ex: $\int_0^3 (3x^2 + x - 2)\,dx = \left(\frac{3x^3}{3} + \frac{x^2}{2} - 2x\right)\Big|_0^3 =$

$= \left(3^3 + \frac{3^2}{2} - 3 \cdot 2\right) - \left(0^3 + \frac{0^2}{2} - 0\right) = 51/2$

ex: $\int_{-2}^{-1} \frac{x - x^2}{2\sqrt[3]{x}}\,dx =$ * Simplify and then integrate

$= 1/2 \int_{-2}^{-1} (x^{2/3} - x^{5/3})\,dx$

$= 1/2 \left(\frac{x^{5/3}}{5/3} - \frac{x^{8/3}}{8/3}\right)\Big|_{-2}^{-1} = 1/2 \left[\left(\frac{3}{5}(-1)^{5/3} - \frac{3}{8}(-1)^{8/3}\right) - \left(\frac{3}{5}(-2)^{5/3} - \frac{3}{8}(-2)^{8/3}\right)\right]$

ex: $\int_0^3 (t - 5^t)\,dt = \left(\frac{t^2}{2} - \frac{5^t}{\ln 5}\right)\Big|_0^3 = \left(\frac{9}{2} - \frac{5^3}{\ln 5}\right) - \left(\frac{0}{2} - \frac{5^0}{\ln 5}\right)$

ex: $\int_0^1 (1 + u^{1/3} - \frac{2}{5}u^9)\,du = \left(u + \frac{3}{4}u^{4/3} - \frac{2}{5}\frac{u^{10}}{10}\right)\Big|_0^1 = \left(u + \frac{3}{4}u^{4/3} - \frac{u^{10}}{25}\right)\Big|_0^1$

$= \left(1 + \frac{3}{4}(1)^{4/3} - \frac{(1)^{10}}{25}\right) - \left(0 + \frac{3}{4}(0) - \frac{0}{25}\right)$

ex: $\int_e^{2e} \left(\cos x - \frac{1}{x}\right)dx = \left(\sin x - \ln|x|\right)\Big|_e^{2e}$

FTC-II and rational fcn

→ can't integrate numerator and denomenator seperately

⇒ Long division (if $\deg_{num} \geq \deg_{denom}$)

ex: $\displaystyle\int_1^9 \left(\frac{x^3+3x^2-x^{3/2}+3}{\sqrt{x}}\right)dx = \int_1^9 \left(x^{5/2}+3x^{3/2}-x+3x^{-1/2}\right)dx$

$$= \left(\frac{2}{7}x^{7/2}+3\cdot\frac{2}{5}x^{5/2}-\frac{x^2}{2}+3.2x^{1/2}\right)\Big|_1^9$$

$$= \underbrace{\left(\frac{2x^{7/2}}{7}+\frac{6x^{5/2}}{5}-\frac{x^2}{2}+6x^{1/2}\right)}_{F(x)}\Big|_1^9 = F(9)-F(1)$$

ex: $\displaystyle\int_1^3 \frac{6x^6+3x^4+x}{2x^2+1}\,dx$

* Long division.

$$\begin{array}{l|l} 6x^6+3x^4+x & 2x^2+1 \\ \underline{6x^6+3x^4} & 3x^4 \\ \quad 0+x & \end{array}$$

$\therefore \boxed{3x^4+\dfrac{x}{2x^2+1}}$

$$= \int_1^3 \left(3x^4+\frac{x}{2x^2+1}\right)dx$$

$$= \int_1^3 3x^4\,dx + \int_1^3 \frac{x}{2x^2+1}\,dx$$

$$= \left(\frac{3x^5}{5}\right)\Big|_1^3 + \underbrace{\int_1^3 \frac{x}{2x^2+1}\,dx}_{\text{can't integrate this yet}}$$

(see u-substitution)

FTC-II and absolute value fcn

$$|x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$

$\Rightarrow$ split up definite integral into 2

ex. $\displaystyle\int_0^2 |2x-1|\, dx$

$$2x-1 \geq 0$$
$$x \geq 1/2$$

$$\cancel{\int_0^{1/2} (2x-1)\,dx + \int_{1/2}^{2} -(2x-1)\,dx}$$

$$|2x-1| = \begin{cases} 2x-1 & , x \geq 1/2 \\ -(2x-1) & , x < 1/2 \end{cases}$$

$$= \underbrace{\int_0^{1/2} -(2x-1)\,dx}_{x < 1/2} + \underbrace{\int_{1/2}^{2} (2x-1)\,dx}_{x > 1/2}$$

$$= -(x^2-x)\Big|_0^{1/2} + (x^2-x)\Big|_{1/2}^{2}$$

$$= -\left[\left((1/2)^2-(1/2)\right)-\left(0^2-0\right)\right] + \left[\left((2)^2-2\right)-\left((1/2)^2-(1/2)\right)\right]$$

$$= -\left[-1/4\right] + \left[9/4\right]$$

$$= 5/2$$

Common Derivatives and Antiderivatives

$\frac{d}{dx} n = 0$	$\int 0 \, dx = C$
$\frac{d}{dx} x = 1$	$\int 1 \, dx = x + C$
$\frac{d}{dx} x^n = n x^{n-1}$	$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$
$\frac{d}{dx} e^x = e^x$	$\int e^x \, dx = e^x + C$
$\frac{d}{dx} \ln x = \frac{1}{x}$	$\int \frac{1}{x} \, dx = \ln \lvert x \rvert + C$
$\frac{d}{dx} a^x = a^x \ln a$	$\int a^x \, dx = \frac{a^x}{\ln a} + C$

Basic Trigonometric Antiderivatives

$\frac{d}{dx} \sin x = \cos x$	$\int \cos x \, dx = \sin x + C$
$\frac{d}{dx} \cos x = -\sin x$	$\int \sin x \, dx = -\cos x + C$
$\frac{d}{dx} \tan x = \sec^2 x$	$\int \sec^2 x \, dx = \tan x + C$
$\frac{d}{dx} \cot x = -\csc^2 x$	$\int \csc^2 x \, dx = -\cot x + C$
$\frac{d}{dx} \sec x = \sec x \tan x$	$\int \sec x \tan x \, dx = \sec x + C$
$\frac{d}{dx} \csc x = -\csc x \cot x$	$\int \csc x \cot x \, dx = -\csc x + C$
$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$
$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$	$\int -\frac{1}{\sqrt{1-x^2}} \, dx = \arccos x + C$
$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} \, dx = \arctan x + C$
$\frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2}$	$\int -\frac{1}{1+x^2} \, dx = \operatorname{arccot} x + C$
$\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} \, dx = \operatorname{arcsec} x + C$
$\frac{d}{dx} \csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}}$	$\int -\frac{1}{x\sqrt{x^2-1}} \, dx = \operatorname{arccsc} x + C$

Indefinite Integral

Def: Given a fcn $f(x)$,
an antiderivative of $f(x)$ is given by:

$$\int f(x)\,dx$$

and is called an indefinite integral.

Note: $\int f(x)\,dx = F(x) + C$, where $F'(x) = f(x)$

ex: $\int (x^2 + x^{-2})\,dx = \frac{x^3}{3} + \frac{x^{-1}}{-1} + C$

$= \frac{x^3}{3} - \frac{1}{x} + C$

ex: Verify by differentiation that the formula is correct

(a) $\int x\cos x\,dx = x\sin x + \cos x + C$

$\frac{d}{dx} x\sin x + \frac{d}{dx}\cos x + \frac{d}{dx} C =$

$= x\cos x + \sin x + (-\sin x) + 0$

$= x\cos x$

(b) $\int \frac{\sin x}{\cos^2 x}\,dx = \sec x + C$

$\frac{d}{dx}\sec x + \frac{d}{dx} C =$

$= \sec x \tan x + 0$

$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$

ex: If $\frac{dy}{dx} = 2x - 1$, what could $y = f(x)$ be?

$\int (2x-1)\,dx = \frac{2x^2}{2} - x + C = x^2 - x + C$

$\therefore\ y = x^2 - x + C$

If $y(1) = 0$, what is $f(x)$?

$y = x^2 - x + C$

sub $(1, 0)$

$0 = (1)^2 - (1) + C$

$\boxed{C = 0}$

$\therefore\ y = x^2 - x$

U-Substitution

Recall: Differential (dy)

$$\frac{dy}{dx} = f'(x) \Rightarrow \boxed{dy = f'(x)\,dx} \text{ , if } y = f(x)$$

$\therefore$ if $u = f(x)$
$du = f'(x)\,dx$

Rule: If $u = g(x)$ is a differentiable fcn then:

$$\int \underbrace{f(g(x))}_{f(u)}\, \underbrace{g'(x)\,dx}_{du} = \int f(u)\,du$$

$u = g(x)$
$du = g'(x)\,dx$

ex: $\int 2x\sqrt{1+x^2}\ dx =$

$$= \int \underbrace{\sqrt{1+x^2}}_{u} \cdot \underbrace{2x\ dx}_{du}$$

$$= \int \sqrt{u}\ du$$

$$= \frac{u^{3/2}}{3/2} + C$$

$$= \frac{2}{3}(1+x^2)^{3/2}$$

$u = 1+x^2$

$du = 2x\ dx$

ex: $\int x^3 \cos(x^4+2)\,dx =$

$$= \int \cos(u)\ \frac{du}{4}$$

$$= \frac{1}{4}\int \cos(u) \cdot du$$

$$= \frac{1}{4}(\sin u) + C$$

$$= \frac{1}{4}\sin(x^4+2) + C$$

$u = x^4+2$

$du = 4x^3\ dx$

$\frac{du}{4} = x^3\ dx$

ex: $\int \frac{e^{1/x}}{x^2}\ dx =$

$$= \int e^u \cdot (-du)$$

$$= -\int e^u\ du$$

$$= -e^u + C$$

$$= -e^{1/x} + C$$

$u = 1/x = x^{-1}$

$du = -1\,x^{-2}\ dx$

$-du = \frac{1}{x^2}\ dx$

Choose different u

$\int \frac{e^{1/x}}{x^2}\ dx =$

$$= -\int du$$

$$= -u + C$$

$$= -e^{1/x} + C$$

$u = e^{1/x}$

$du = e^{1/x} \cdot (-x^{-2})\ dx$

$-du = \frac{e^{1/x}}{x^2}\ dx$

ex: $\int \frac{x}{2x^2+1}\,dx =$

$u = 2x^2+1$

$du = 4x\,dx$

$\frac{1}{4}\,du = x\,dx$

$= \frac{1}{4}\int \frac{1}{u}\,du$

$= \frac{1}{4}\ln|u| + C$

$= 1/4 \ln|2x^2+1| + C$

* One can <u>NOT</u> have u and x
* Manipulate $u = \ldots$ to transform x into u

ex: $\int \frac{x}{\sqrt{x-4}}\,dx =$

$u = x-4 \Rightarrow x = u+4$

$du = 1\,dx$

$du = dx$

$= \int \frac{x}{\sqrt{u}}\,du$

$= \int \frac{u+4}{\sqrt{u}}\,du$

$= \int (u^{1/2} + 4u^{-1/2})\,du$

$= \frac{u^{3/2}}{3/2} + \frac{4u^{1/2}}{1/2} + C$

$= \frac{2}{3}(x-4)^{3/2} + 8(x-4)^{1/2} + C$

ex: $\int \frac{x^2}{\sqrt{1-x^6}}\,dx =$

* I want the derivative of u to be x^2 (in du) $\therefore$ u has to have x^3

$u = x^3$

$du = 3x^2\,dx$

$\frac{1}{3}\,du = x^2\,dx$

$= \int \underbrace{\frac{1}{\sqrt{1-u^2}}}_{\frac{d}{dx}\arcsin(u)} \frac{du}{3}$

$= \frac{1}{3}\sin^{-1}u + C$

$= 1/3 \sin^{-1}(x^3) + C$

ex: $\int 2x(x^2-1)^4\,dx =$

$= \int u^4\,du$

$= \frac{u^5}{5} + C$

$= \frac{(x^2-1)^5}{5} + C$

$u = x^2-1$

$du = 2x\,dx$

ex: $\int \left(1+\frac{1}{x}\right)^3 \left(\frac{1}{x^2}\right) dx =$

$= -\int u^3\,du$

$= -\frac{u^4}{4} + C$

$= -\frac{1}{4}\left(1+\frac{1}{x}\right)^4 + C$

$u = 1+\frac{1}{x}$

$du = -1 \cdot \frac{1}{x^2}\,dx$

$-du = \frac{1}{x^2}\,dx$

ex: $\int \frac{x^2+3x+7}{\sqrt{x}}\,dx =$

$= \int (x^{3/2}+3x^{1/2}+7x^{-1/2})\,dx$

$= x^{5/2}\cdot\left(\frac{2}{5}\right) + 3x^{3/2}\left(\frac{2}{3}\right) + 7x^{1/2}\cdot\left(\frac{2}{1}\right) + C$

ex: $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx =$

$= \int \cos u \,.\, 2du$

$= 2\int \cos u\,du$

$= 2\sin u + C$

$= 2\sin(\sqrt{x}) + C$

$u = \sqrt{x}$

$du = 1/2\, x^{-1/2}\,dx$

$2\,du = \frac{1}{\sqrt{x}}\,dx$

ex: $\int \frac{4}{x^2}\sin\left(\frac{1}{x}\right) dx =$

$= -4\int \sin u\,du$

$= -4(-\cos u) + C$

$= 4\cos\left(\frac{1}{x}\right) + C$

$u = \frac{1}{x}$

$du = -\frac{1}{x^2}\,dx$

$-du = \frac{1}{x^2}\,dx$

ex: $\int \sin(2x)\cos(2x)\,dx =$

$$\boxed{u = \sin(2x)}$$
$$du = 2\cos(2x)\,dx$$
$$\tfrac{1}{2}\,du = \cos(2x)\,dx$$

$$= \frac{1}{2}\int u\,du$$
$$= \frac{1}{2}\,\frac{u^2}{2} + C$$
$$= \frac{\sin^2(2x)}{4} + C$$

Or using different u

$$\int \sin(2x)\cos(2x)\,dx =$$

$$\boxed{u = \cos(2x)}$$
$$du = 2(-\sin(2x))\,dx$$
$$-\tfrac{1}{2}\,du = \sin(2x)\,dx$$

$$= \int u \cdot -\tfrac{1}{2}\,du$$
$$= -\frac{1}{2}\int u\,du$$
$$= -\frac{1}{2}\,\frac{u^2}{2} + C$$
$$= -\frac{\cos^2(2x)}{4} + C$$

ex: $\int x^5\sqrt{3x^2-2}\,dx =$

$u = 3x^2 - 2 \Rightarrow x^2 = \frac{u-2}{3}$

$du = 6x\,dx$

$\frac{du}{6} = x\,dx$

$= \int x^4\sqrt{3x^2-2}\,x\,dx$

$= \int x^4\sqrt{u}\,\frac{du}{6}$

$= \frac{1}{6}\int\left(\frac{u-2}{3}\right)^2 u^{1/2}\,du$

$= \frac{1}{6}\cdot\frac{1}{9}\int(u^2-4u+4)u^{1/2}\,du$

$= \frac{1}{54}\int(u^{5/2}-4u^{3/2}+4u^{1/2})\,du$

$= \frac{1}{54}\left(\frac{2}{7}\cdot u^{7/2} - 4\cdot\frac{2}{5}u^{5/2} + 4\cdot\frac{2}{3}u^{3/2}\right) + C$

$= \frac{1}{189}\cdot(3x^2-2)^{7/2} - \frac{4}{135}\cdot(3x^2-2)^{5/2} + \frac{4}{81}(3x^2-2)^{3/2} + C$

ex: $\int \frac{(\ln x)^2}{x}\,dx$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$= \int u^2\,du$

$= \frac{u^3}{3} + C$

$= \frac{(\ln x)^3}{3} + C$

ex: $\int \sqrt{\cot x}\,\csc^2 x\,dx =$

$u = \cot x$

$du = -\csc^2 x\,dx$

$-du = \csc^2 x\,dx$

$= -\int\sqrt{u}\,du$

$= -u^{3/2}\cdot\frac{2}{3} + C$

$= -\frac{2}{3}(\cot x)^{3/2} + C$

ex: $\int e^{\tan 2x} \sec^2 2x \, dx$

$u = \tan(2x)$

$du = 2 \cdot \sec^2(2x)\, dx$

$\frac{1}{2} du = \sec^2(2x)\, dx$

$= \frac{1}{2} \int e^u \, du$

$= \frac{1}{2} e^u + C$

$= \frac{e^{\tan 2x}}{2} + C$

ex: $\int 3^{2x} \cot(3^{2x})\, dx$

$u = 3^{2x}$

$du = 3^{2x} \cdot \ln(3) \cdot (2)\, dx$

$\frac{du}{2 \cdot \ln 3} = 3^{2x}\, dx$

$= \frac{1}{2 \ln 3} \cdot \int \underbrace{\cot(u)}_{\cot(u) = \frac{\cos(u)}{\sin(u)}} du$

$= \frac{1}{2 \ln 3} \int \frac{\cos(u)}{\sin(u)} du$

$w = \sin(u)$

$dw = \cos(u)\, du$

* Always use denom.

$= \frac{1}{2 \ln 3} \int \frac{1}{w} dw$

$= \frac{1}{2 \ln 3} \cdot \ln|w| + C$

$= \frac{1}{2 \ln 3} \cdot \ln|\sin(3^{2x})| + C$

ex: $\int \frac{5 - e^x}{e^{2x}} dx$

$= \int \frac{5}{e^{2x}} dx - \int \frac{e^x}{e^{2x}} dx$

$= \int \frac{5}{e^{2x}} dx - \int e^{-x} dx$

$= \int 5e^{-2x} dx - \int e^{-x} dx$

$u = -2x$	$w = -x$
$du = -2dx$	$dw = -dx$
$-\frac{1}{2} du = dx$	$-dw = dx$

$= -5 \cdot \frac{1}{2} \int e^u du - - \int e^w dw$

$= -\frac{5}{2} e^u + e^w + C$

$= e^{-x} - \frac{5}{2} e^{-2x} + C$

Interpretation by Parts (IBP)

$$\boxed{\int f(x) \cdot \frac{d}{dx} g(x) \, dx = f(x) \cdot g(x) - \int g(x) \cdot \frac{d}{dx} f(x) \, dx}$$

Let $u = f(x)$
$v = g(x)$

$$\therefore \boxed{\int u \, dv = uv - \int v \cdot du}$$

<u>Proof</u> (from product rule):

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = \frac{d}{dx} f(x) \cdot g(x) + \frac{d}{dx} g(x) \cdot f(x)$$

$\therefore f(x) \cdot g(x)$ is the antiderivative of:

$$\frac{d\,f(x)}{dx} \cdot g(x) + \frac{d\,g(x)}{dx} \cdot f(x)$$

<u>or</u>

$$\int \left(\frac{df(x)}{dx} \cdot g(x) + \frac{dg(x)}{dx} \cdot f(x) \right) dx = f(x) \cdot g(x)$$

$$\int \frac{df(x)}{dx} \cdot g(x) \, dx + \int \frac{dg(x)}{dx} \cdot f(x) \, dx = f(x) \cdot g(x)$$

$$\therefore \boxed{\int f(x) \cdot \frac{dg(x)}{dx} \cdot dx = f(x) \, g(x) - \int g(x) \cdot \frac{df(x)}{dx} \cdot dx}$$

Let $u = f(x)$ $\qquad$ $v = g(x)$

$du = f'(x) \, dx$ $\qquad$ $dv = g'(x) \, dx$

$= \frac{df(x)}{dx} \cdot dx$ $\qquad$ $= \frac{dg(x)}{dx} \cdot dx$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

Order to chose "u" :

- **L** ogarithmic fcn
- **I** nverse trig fcn
- **A** lgebreic fcn
- **T** rigonometric fcn
- **E** xponential fcn

$\}$ LIATE order of priority

Match Left-hand side of formula:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

ex: $\int x \sin x \, dx$

$$\left[\begin{array}{ll} u = x & dv = \sin x \, dx \\ du = dx & v = -\cos x \end{array}\right]$$

$= -x\cos x - \int -\cos x \, dx$

$= -x\cos x - (-\sin x) + C$

$= -x\cos x + \sin x + C$

ex: $\int \frac{\ln x}{x^2} dx$

$$\begin{array}{ll} u = \ln x & dv = x^{-2} dx \\ du = \frac{1}{x} dx & v = -\frac{1}{x} \end{array}$$

$= -\frac{\ln x}{x} - \int -x^{-2} dx$

$= -\frac{\ln x}{x} + \int x^{-2} dx$

$= -\frac{\ln x}{x} - x^{-1} + C$

ex: $\int (x^2+1)e^{2x} dx$

$$\left[\begin{array}{ll} u = x^2+1 & dv = e^{2x} dx \\ du = 2x\,dx & v = \frac{1}{2} e^{2x} \end{array}\right]$$ IBP 1

$= (x^2+1)\frac{1}{2}e^{2x} - \frac{2}{2}\int x e^{2x} dx$

$$\left[\begin{array}{ll} u = x & dv = e^{2x} dx \\ du = dx & v = \frac{1}{2} e^{2x} \end{array}\right]$$ IBP 2

$= \frac{1}{2}e^{2x}(x^2+1) - \left[\frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x} dx\right]$

$= \frac{1}{2}e^{2x}(x^2+1) - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$

Boomerang!

$\rightarrow$ If $\quad I = f(x) - I \quad \therefore \quad 2I = f(x)$

$\therefore \quad I = f(x)/2$

ex: $\int e^x \sin x \, dx = I$

IBP 1 $\left[\begin{array}{ll} u = \sin x & dv = e^x dx \\ du = \cos x \, dx & v = e^x \end{array} \right.$

$= e^x \sin x - \underbrace{\int e^x \cos x \, dx}$

IBP 2 $\left[\begin{array}{ll} u = \cos x & dv = e^x dx \\ du = -\sin x \, dx & v = e^x \end{array} \right.$

$= e^x \sin x - \left[e^x \cos x - -\int e^x \sin x \, dx \right]$

$= \underbrace{e^x \sin x - e^x \cos x}_{f(x)} - \underbrace{\int e^x \sin x \, dx}_{I}$

$I = f(x) - I$

$2I = f(x)$

$I = \frac{1}{2} f(x)$

$\therefore \int e^x \sin x \, dx = \frac{1}{2} \left(e^x \sin x - e^x \cos x \right) + C$

Proof of $\int \ln x \, dx$ (IBP)

$$\int \ln x \, dx$$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = x$

$$= x \ln x - \int \frac{x}{x} dx$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + C$$

$$= x(\ln x - 1) + C$$

$$\therefore \boxed{\int \ln x \, dx = x(\ln x - 1) + C}$$

Proof of $\int \tan^{-1} x \, dx$ (All inverse trig) (IBP)

$$\int \tan^{-1} x \, dx$$

$\left[u = \tan^{-1} x \qquad dv = dx \right.$

$\left. du = \frac{1}{1+x^2} dx \qquad v = x \right]$

$$= x \tan^{-1} x - \int \underbrace{\frac{x}{1+x^2}}_{\text{sub}} dx$$

$u = 1 + x^2$

$du = 2x \, dx$

$\frac{1}{2} du = x \, dx$

$$= x \tan^{-1} x - \frac{1}{2} \int \frac{1}{u} du$$

$$= x \tan^{-1} x - \frac{1}{2} \ln |u| + C$$

$$= x \tan^{-1} x - \frac{1}{2} \ln |x^2 + 1| + C$$

Definite Integral and u-sub

* Adjust limits of integration with respect to x to limits of integration with respect to u

⇒ Don't need to go back to x

ex: $\int_0^1 x e^{-x^2} dx =$

$u = -x^2$

$du = -2x\,dx$

$-\frac{1}{2} du = x\,dx$

* if $x=0$, $u=0$
$x=1$, $u=-1$

$= -\frac{1}{2}\int_0^{-1} e^u du$

$= -1/2 \cdot e^u \Big|_0^{-1}$

$= -1/2 \left[e^{-1} - e^0\right]$

$= -\frac{1}{2e} + \frac{1}{2}$

* Note: No +C b/c definite integral

Definite Integral and IBP

$$\boxed{\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du}$$

ex: $\int_0^1 x 5^x dx =$

$u = x$ $\quad dv = 5^x dx$

$du = dx$ $\quad v = \frac{5^x}{\ln 5}$

$= \frac{x5^x}{\ln 5}\Big|_0^1 - \int_0^1 \frac{5^x}{\ln 5} dx$

$= \frac{1}{\ln 5} \cdot x5^x\Big|_0^1 - \frac{1}{\ln 5}\int_0^1 5^x dx$

$= \frac{1}{\ln 5} x5^x\Big|_0^1 - \frac{1}{\ln 5}\left(\frac{5^x}{\ln 5}\right)\Big|_0^1$

$= \frac{1}{\ln 5}\left(5^{(1)} - 0\right) - \frac{1}{\ln 5}\left(\frac{5^1}{\ln 5} - \frac{5^0}{\ln 5}\right)$

$= \frac{5}{\ln 5} - \frac{5-1}{(\ln 5)^2}$

$= \frac{5}{\ln 5} - \frac{4}{(\ln 5)^2}$

ex: $\int_0^9 e^{\sqrt{x}}\,dx$

$u = \sqrt{x}$

$du = \frac{1}{2} \cdot x^{-1/2}\,dx$

isolate dx

$2\,du \cdot \sqrt{x} = dx$

sub $\sqrt{x} = u$

$dx = 2u\,du$

If $x = 0$, $u = 0$

$x = 9$, $u = 3$

$= \int_0^3 e^u \cdot 2u\,du$

$= 2\int_0^3 ue^u\,du$

$$\boxed{\begin{array}{ll} w = u & dv = e^u du \\ dw = du & v = e^u \end{array}}$$

$= 2\left(ue^u\Big|_0^3 - \int_0^3 e^u\,du \right)$

$= 2\left(ue^u\Big|_0^3 - e^u\Big|_0^3 \right)$

$= 2\left(3e^3 - 0 - [e^3 - e^0]\right)$

$= 2(3e^3 - e^3 + 1)$

$= 2(2e^3 + 1)$

ex: $\int \frac{\ln\sqrt{x}}{\sqrt{x}}\,dx$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2\,du = \frac{1}{\sqrt{x}}\,dx$

$= 2\int \ln(u)\,du$

$= 2\left[u\ln(u) - u\right] + C$

$= 2\left(\sqrt{x}\ln\sqrt{x} - \sqrt{x}\right) + C$

ex: $\int \frac{\sin 4x}{\tan 4x}\, dx$

$= \int \sin 4x \cdot \frac{\cos 4x}{\sin 4x}\, dx$

$= \int \cos(4x)\, dx$

$u = 4x$
$du = 4\,dx$
$\frac{1}{4} du = dx$

$= \frac{1}{4} \int \cos(u)\, du$

$= 1/4 \cdot \sin(u) + C$

$= 1/4 \ \sin(4x) + C$

ex: $\int \sin(\ln x)\, dx$

IBP 1 $\left[\begin{array}{ll} u = \sin(\ln x) & dv = dx \\ du = \cos(\ln x) \cdot \frac{1}{x}\, dx & v = x \end{array}\right]$

$= x\sin(\ln x) - \int \cos(\ln x) \cdot \frac{x}{x}\, dx$

$= x\sin(\ln x) - \int \cos(\ln x)\, dx$

IBP 2 $\left[\begin{array}{ll} u = \cos(\ln x) & dv = dx \\ du = -\sin(\ln x) \cdot \frac{1}{x}\, dx & v = x \end{array}\right]$

$= x\sin(\ln x) - \left[x\cos(\ln x) - - \int \sin(\ln x) \cdot \frac{x}{x}\, dx \right]$

$= \underbrace{x\sin(\ln x) - x\cos(\ln x)}_{f(x)} - \underbrace{\int \sin(\ln x)\, dx}_{I}$

$I = f(x) - I$
$2I = f(x)$
$I = f(x) \cdot \frac{1}{2}$

$\therefore \int \sin(\ln x)\, dx = \frac{x\sin(\ln x) - x\cos(\ln x)}{2} + C$

IBP: Reduction Formulas

* Repeated IBP

$$\boxed{\int \sin^n x\,dx = -\frac{1}{n}\cos x \cdot \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx}$$

Derivation:

$$\int \sin^n x\,dx = I$$

$$= \int \sin^{n-1} x \cdot \sin x\,dx$$

$$\left[\begin{array}{ll} u = \sin^{n-1} x & dv = \sin x\,dx \\ du = (n-1)\sin^{n-2} x \cdot \cos x\,dx & v = -\cos x \end{array}\right.$$

$$= -\sin^{n-1} x \cdot \cos x - -(n-1)\int \sin^{n-2} x \cdot \underbrace{\cos^2 x}\,dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,(1-\sin^2 x)\,dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\left(\int \sin^{n-2} x\,dx - \int \sin^n x\,dx\right)$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\underbrace{\int \sin^n x\,dx}_{I}$$

$$I = f(x) - (n-1)I$$

$$I + (n-1)I = f(x)$$

$$n \cdot I = f(x)$$

$$I = \frac{1}{n} \cdot f(x)$$

$$\therefore \int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} dx$$

ex: evaluate at $n=4$

$$\int \sin^4 x\,dx = \overset{n=4}{-\frac{1}{4}\cos x \sin^3 x} + \frac{3}{4}\left[\overset{n=2}{-\frac{1}{2}\cos x \sin^1 x} + \frac{1}{2}\int \cancel{\sin^{0}}^{\,1} x\,dx\right]$$

$$= -\frac{1}{4}\cos x \sin^3 x + \frac{3}{4}\left(-\frac{1}{2}\cos x \sin x + \frac{1}{2}(x)\right) + C$$

$$\boxed{\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx}$$

<u>Derivation</u>:

$$\int (\ln x)^n\,dx = I \qquad \left[\begin{array}{ll} u = (\ln x)^n & dv = dx \\ du = n\cdot(\ln x)^{n-1}\cdot\frac{1}{x}\,dx & v = x \end{array}\right.$$

$$= x(\ln x)^n - n\int (\ln x)^{n-1}\cdot\frac{x}{x}\,dx$$

$$= x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$$

ex: evaluate at $n=4$

$$\int (\ln x)^4\,dx =$$

$$= \overset{n=4}{x(\ln x)^4} - 4\Big[\overset{n=3}{x(\ln x)^3} - 3\big[\overset{n=2}{x(\ln x)^2} - 2(\overset{n=1}{x(\ln x)} - 1\int \cancel{(\ln x)^0}^{\,1}\,dx)\big]\Big]$$

$$= x(\ln x)^4 - 4\Big[x(\ln x)^3 - 3\big[x(\ln x)^2 - 2(x(\ln x) - x)\big]\Big] + C$$

$$\boxed{\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx}$$

<u>Derivation</u>:

$$\int x^n e^x\,dx \qquad \left[\begin{array}{ll} u = x^n & dv = e^x\,dx \\ du = n\cdot x^{n-1}\,dx & v = e^x \end{array}\right.$$

$$= x^n e^x - n\int x^{n-1}\cdot e^x\,dx$$

Trigonometric Integral

Recall: Trig identity

$\sin^2 x + \cos^2 x = 1 \Rightarrow$ $\boxed{\begin{aligned}\cos^2 x &= 1 - \sin^2 x \\ \sin^2 x &= 1 - \cos^2 x\end{aligned}}$ Easy case

$\cos 2x = \cos^2 x - \sin^2 x \Rightarrow$ $\boxed{\begin{aligned}\cos^2 x &= \tfrac{1}{2}(1+\cos 2x) \\ \sin^2 x &= \tfrac{1}{2}(1-\cos 2x)\end{aligned}}$ Hard case

$\sin 2x = 2 \sin x \cos x$

Note: ⊛ $\int \cos(n.x)\,dx = \frac{1}{n}\sin(n.x) + C$ (Hard case)

$\int \sin(n.x)\,dx = -\frac{1}{n}\cos(n.x) + C$

Evaluate: $\int \sin^n x \cdot \cos^m x \, dx$ $\quad n = 0, 1, 2, \ldots$ $\quad m = 0, 1, 2, \ldots$

Easy case: m and/or n are odd

$$\boxed{\cos^2 x + \sin^2 x = 1}$$

Hard case: Both m and n are even

$$\boxed{\begin{aligned} \cos^2 x &= 1/2\,(1 + \cos 2x) \\ \sin^2 x &= 1/2\,(1 - \cos 2x) \end{aligned}}$$

ex: $\int \sin^3 x \cos^2 x \, dx$

sin is odd
→ take out one sin
→ work with $u = \cos x$

$= \int \sin^2 x \cdot \cos^2 x \cdot \sin x \, dx$

$= \int (1-\cos^2 x) \cos^2 x \sin x \, dx$

$u = \cos x$
$du = -\sin x \, dx$
$-du = \sin x \, dx$

$= -\int (1-u^2) u^2 \, du$

$= -\int (u^2 - u^4) \, du$

$= -\left(\frac{u^3}{3} - \frac{u^5}{5} \right) + C$

$= -\left(\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5} \right) + C$

ex: $\int_0^{\pi/2} \cos^5 x \, dx$

$= \int_0^{\frac{\pi}{2}} (\cos^2 x)^2 \cdot \cos x \, dx$

$= \int_0^{\frac{\pi}{2}} (1-\sin^2 x)^2 \cos x \, dx$

$u = \sin x$
$du = \cos x \, dx$
if $x = 0$, $u = 0$
$x = \frac{\pi}{2}$, $u = 1$

$= \int_0^1 (1-u^2)^2 \, du$

$= \int_0^1 (1 - 2u^2 + u^4) \, du$

$= \left(u - 2\frac{u^3}{3} + \frac{u^5}{5} \right) \Big|_0^1 = \frac{8}{15}$

ex: $\int_0^{\pi} \underbrace{\sin^2 x}\, dx$ * Hard case

$= \int_0^{\pi} \frac{1}{2}(1-\cos 2x)\, dx$

$= \frac{1}{2}\int_0^{\pi} (1-\cos 2x)\, dx$

$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)\Big|_0^{\pi}$

$= \frac{1}{2}\left(\pi - \frac{1}{2}\cancelto{0}{\sin(2\pi)} - \left(0 - \frac{1}{2}\cancelto{0}{\sin(0)}\right)\right)$

$= \pi/2$

ex: $\int \underbrace{\sin^4 x}\, dx$ $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$= \int \left(\frac{1}{2}(1-\cos 2x)\right)^2 dx$

$= \frac{1}{4}\int \left(1 - 2\cos(2x) + \underbrace{\cos^2 2x}\right) dx$ $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$= \frac{1}{4}\int \left(1 - 2\cos(2x) + \frac{1}{2}(1+\cos 4x)\right) dx$

$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos(2x) + \frac{1}{2}\cos 4x\right) dx$

$= \frac{1}{4}\left[\frac{3}{2}x - 2\cdot\frac{1}{2}\sin(2x) + \frac{1}{2}\cdot\frac{1}{4}\sin(4x)\right] + C$

ex: $\displaystyle\int \frac{\sin^2\sqrt{x}\,\cos^3\sqrt{x}}{\sqrt{x}}\,dx$

$u = \sqrt{x}$
$du = \frac{1}{2\sqrt{x}}\,dx$
$2\,du = \frac{1}{\sqrt{x}}\,dx$

$= 2\int \sin^2 u \cos^3 u \; du$

$= 2\int \sin^2 u \cdot \cos^2 u \cdot \cos u \, du$

$= 2\int \sin^2 u\,(1-\sin^2 u)\cos u\, du$

$w = \sin u$
$dw = \cos u\, du$

$= 2\int w^2\,(1-w^2)\,dw$

$= 2\int (w^2 - w^4)\,dw$

$= 2\left(\frac{w^3}{3} - \frac{w^5}{5}\right) + C$

$= 2\left(\frac{\sin^3\sqrt{x}}{3} - \frac{\sin^5\sqrt{x}}{5}\right) + C$

ex: $\displaystyle\int \sin^2(\pi x)\cos^2(\pi x)\,dx$ *

$u = \pi x$
$du = \pi\,dx$
$\frac{1}{\pi}\,du = dx$

Hard case

$= \frac{1}{\pi}\int \underbrace{\sin^2 u}\cdot\underbrace{\cos^2 u}\,du$

$= \frac{1}{\pi}\int \frac{1}{2}(\underbrace{1-\cos 2u}_{a\,-\,b})\cdot\frac{1}{2}(\underbrace{1+\cos 2u}_{a\,+\,b})\,du$

$= \frac{1}{4\pi}\int (1-\cos^2 2u)\,du$

$= \frac{1}{4\pi}\int \left(1 - \frac{1}{2}(1+\cos 4u)\right)du$

$= \frac{1}{4\pi}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4u\right)du$

$= \frac{1}{4\pi}\left(\frac{1}{2}u - \frac{1}{2}\cdot\frac{1}{4}\sin 4u\right) + C$

$= \frac{1}{8\pi}u - \frac{1}{32\pi}\sin 4u + C$

$= \frac{1}{8\pi}\cdot\pi x - \frac{1}{32\pi}\sin(4\pi x) + C$

Recall: ⊛ $\tan^2 x + 1 = \sec^2 x$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

Evaluate: $\int \sec^n x \tan^m x \, dx$

Easy case: ① n is even ($\sec x$ even)

(i) set aside $\sec^2 x$

(ii) convert $\sec x \to \tan x$ ⊛

(iii) $u = \tan x$

$du = \sec^2 x \, dx$

② m is odd ($\tan x$ odd)

(i) set aside $\sec x \tan x$

(ii) convert $\tan x \to \sec x$ ⊛

(iii) $u = \sec x$

$du = \sec x \tan x \, dx$

ex: $\int \tan^4 x \sec^4 x \, dx$

$= \int \tan^4 x \cdot \sec^2 x \cdot \sec^2 x \, dx$

$= \int \tan^4 x (\tan^2 x + 1) \sec^2 x \, dx$ $\qquad u = \tan x$

$du = \sec^2 x \, dx$

$= \int u^4 (u^2 + 1) \, du$

$= \int (u^6 + u^4) \, du$

$= \frac{u^7}{7} + \frac{u^5}{5} + C$

$= \frac{\tan^7 x}{7} + \frac{\tan^5 x}{5} + C$

ex: $\int \tan^5 x \sec^7 x \, dx$

$= \int (\sec^2 x - 1)^2 \sec^6 x \cdot \sec x \tan x \, dx$ $\qquad u = \sec x$

$du = \sec x \tan x \, dx$

$= \int (u^2 - 1)^2 u^6 \, du$

$= \int (u^4 - 2u^2 + 1) u^6 \, du$

$= \int (u^{10} - 2u^8 + u^6) \, du$

$= \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} + C$

$= \frac{\sec^{11} x}{11} - \frac{2\sec^9 x}{9} + \frac{\sec^7 x}{7} + C$

ex: $\int \tan^3 x \sec^3 x \, dx$

$= \int (\sec^2 x - 1) \sec^2 x \cdot \sec x \tan x \, dx$ $\quad u = \sec x$

$du = \sec x \tan x \, dx$

$= \int (u^2 - 1) u^2 \, du$

$= \int (u^4 - u^2) \, du$

$= \frac{u^5}{5} - \frac{u^3}{3} + C$

$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$

Hard Cases

→ Try to get to an easy case

Use:

① $\boxed{\int \tan x\, dx = \ln|\sec x| + C}$

<u>proof</u>: $\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$

$= \int \frac{\sin x}{\cos x}\, dx$

$u = \cos x$
$du = -\sin x\, dx$
$-du = \sin x\, dx$

$= -\int \frac{1}{u}\, du$

$= -\ln|u| + C$

log property

$= -\ln|\cos x| + C$

$= \ln|\cos^{-1}| + C = \ln|\sec x| + C$

* Any time num. and denom are derivatives of eachother, use u = "denominator"

② $\boxed{\int \sec x\, dx = \ln|\sec x + \tan x| + C}$

<u>proof</u>: $\int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx$

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$

$u = \sec x + \tan x$
$du = (\sec x \tan x + \sec^2 x)\, dx$

$= \int \frac{1}{u}\, du$

$= \ln|\sec x + \tan x| + C$

Odd tan, no sec — not same easy case

ex: $\int \tan^3 x \, dx$

$= \int \tan^2 x \cdot \tan x \, dx$

$= \int (\sec^2 x - 1) \tan x \, dx$

$= \int (\sec^2 x \tan x - \tan x) \, dx$

$= \underbrace{\int \sec^2 x \tan x \, dx}_{EC} - \underbrace{\int \tan x \, dx}_{\ln|\sec x|}$

$u = \tan x$
$du = \sec^2 x \, dx$

$= \int u \, du - \ln|\sec x|$

$= \frac{u^2}{2} - \ln|\sec x| + C$

$= \frac{\tan^2 x}{2} - \ln|\sec x| + C$

Odd sec, no tan

ex: $\int \sec^3 x \, dx = I$

$$\left[\begin{array}{ll} u = \sec x & dv = \sec^2 x \, dx \\ du = \sec x \tan x \, dx & v = \tan x \end{array}\right.$$

$= \sec x \tan x - \int \tan^2 x \sec x \, dx$

$= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$

$= \sec x \tan x - \underbrace{\int \sec^3 x \, dx}_{I} + \underbrace{\int \sec x \, dx}_{\ln|\sec x + \tan x|}$

$I = f(x) - I$

$2I = f(x)$

$I = \frac{1}{2} f(x)$

$\therefore \int \sec^3 x \, dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$

Even tan, no sec

ex: $\int \tan^6 x\,dx$

$$= \int (\sec^2 x - 1)\tan^4 x\,dx$$

$$= \underbrace{\int \sec^2 x \tan^4 x\,dx}_{\text{easy}} - \underbrace{\int \tan^4 x\,dx}_{\text{hard}}$$

$$= \int \tan^4 x \sec^2 x\,dx - \int (\sec^2 x - 1)\tan^2 x\,dx$$

$$= \int \tan^4 x \sec^2 x\,dx - \int \sec^2 x \tan^2 x + \int \tan^2 x\,dx$$

$$= \int \tan^4 x \sec^2 x\,dx - \int \tan^2 x \sec^2 x + \int (\sec^2 x - 1)\,dx$$

$$u = \tan x$$
$$du = \sec^2 x\,dx$$

$$= \int u^4\,du - \int u^2\,du + \cancel{\int (u^2 - 1)\,dx} + \tan x - x$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + \tan x - x + C$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + C$$

Even tan, Odd sec

ex: $\int \tan^4 x \sec^3 x \, dx$

$$= \int (\sec^2 x - 1)^2 \sec^3 x \, dx$$

$$= \int (\sec^4 x - 2\sec^2 x + 1) \sec^3 x \, dx$$

$$= \int (\sec^7 x - 2\sec^5 x + \sec^3 x) \, dx$$

Solve as if odd sec, no tan

Even sec → Always easy

Even sec, odd tan → easy

ex: $\int (\tan^2 x + \tan^4 x)\, dx$

$= \int \tan^2 x \, dx + \int \tan^4 x \, dx$

$= \int (\sec^2 x - 1)\, dx + \int (\sec^2 x - 1) \tan^2 x \, dx$

$= \int (\sec^2 x - 1)\, dx + \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx$

$= \int (\sec^2 x - 1)\, dx + \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1)\, dx$

$u = \tan x$
$du = \sec^2 x$

$= (\cancel{\tan x} - x) + \int u^2 \, du - (\cancel{\tan x} - x)$

$= \frac{u^3}{3} + C$

$= \frac{1}{3} \tan^3 x + C$

OR

$\int (\tan^2 x + \tan^4 x)\, dx = \int \left(\frac{\sin^2 x}{\cos^2 x} + \frac{\sin^4 x}{\cos^4 x} \right) dx$

$= \int \frac{\sin^2 x \cdot \cos^2 x + \sin^4 x}{\cos^4 x} \, dx$

$= \int \frac{\sin^2 x (\sin^2 x + \cos^2 x)}{\cos^4 x} \, dx$

$= \int \frac{\sin^2 x \cdot (1)}{\cos^4 x} \, dx$

$= \int \tan^2 x \sec^2 x \, dx$ $\quad$ $u = \tan x$, $du = \sec^2 x \, dx$

$= \int u^2 \, du$

$= \frac{u^3}{3} + C$

$= \frac{\tan^3 x}{3} + C$

ex: $\int \cos^2 x \tan^3 x \, dx =$

$$= \int \cos^2 x \cdot \frac{\sin^3 x}{\cos^3 x} \, dx$$

$$= \int \frac{\sin^3 x}{\cos x} \, dx$$

odd sin → put sin aside

$$= \int \frac{(1-\cos^2 x) \cdot \sin x}{\cos x} \, dx$$

$u = \cos x$
$du = -\sin x \, dx$
$-du = \sin x \, dx$

$$= -\int \frac{(1-u^2)}{u} \, du$$

$$= -\int \left(\frac{1}{u} - u\right) du$$

$$= -\left(\ln|u| - \frac{u^2}{2}\right) + C = -\left(\ln|\cos x| - \frac{\cos^2 x}{2}\right) + C$$

ex: $\int \frac{\sin^2 x}{\cos x} \, dx$

odd cos → put cos aside

$$= \int \frac{\sin^2 x}{\cos x} \cdot \frac{\cos x}{\cos x} \, dx$$

$$= \int \frac{\sin^2 x \cdot \cos x}{\cos^2 x} \, dx$$

$u = \sin x$
$du = \cos x \, dx$

$$= \int \frac{\sin^2 x}{(1-\sin^2 x)} \cos x \, dx$$

$$= \int \frac{u^2}{1-u^2} \cdot du$$

* Rational → long division

$$\begin{array}{r|l} u^2 & 1-u^2 \\ \underline{u^2-1} & -1 \\ 1 & \end{array}$$

$$= -1 + \frac{1}{1-u^2}$$

$$= \int \left(-1 + \frac{1}{1-u^2}\right) du$$

$$= -u + \int \frac{1}{1-u^2} \, du$$

$$= -\sin x + \int \frac{1}{1-u^2} \, du$$

see partial fractions

ex: $\int \frac{\tan^3 x}{\cos^4 x}\,dx$

$= \int \frac{\sin^3 x}{\cos^7 x}\,dx$

$= \int \frac{\sin^2 x}{\cos^7 x} \cdot \sin x\,dx$

$= \int \frac{(1-\cos^2 x)}{\cos^7 x} \sin x\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$= -\int \frac{1-u^2}{u^7}\,du$

$= -\int (u^{-7} - u^{-5})\,du$

$= -\left(\frac{u^{-6}}{-6} - \frac{u^{-4}}{-4}\right) + C$

$= -\left(-\frac{1}{6\cos^6 x} + \frac{1}{4\cos^4 x}\right) + C$

ex: $\int \frac{1-\tan^2 x}{\sec^2 x}\,dx$

* split up

$= \int \frac{1}{\sec^2 x}\,dx - \int \frac{\tan^2 x}{\sec^2 x}\,dx$

$= \int \cos^2 x\,dx - \int \frac{\sin^2 x}{\cancel{\cos^2 x}} \cdot \cancel{\cos^2 x}\,dx$

$= \int \frac{1}{2}(1+\cos 2x)\,dx - \int \frac{1}{2}(1-\cos 2x)$

$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) - \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$

$= 1/2 \sin 2x + C$

Trigonometric Sub

Trig Identity to use

Case 1 : $\sqrt{a^2 - x^2}$

$\cos^2 x = 1 - \sin^2 x$

$\boxed{x = a \sin \theta}$

Case 2 : $\sqrt{a^2 + x^2}$

$\sec^2 x = 1 + \tan^2 x$

$\boxed{x = a \tan \theta}$

Case 3 : $\sqrt{x^2 - a^2}$

$\tan^2 x = \sec^2 x - 1$

$\boxed{x = a \sec \theta}$

Case 1

ex: $\int \sqrt{16-x^2}\,dx$

$1-\sin^2 x = \cos^2 x$

$a = 4$

Let $x = a\sin\theta$

$x = 4\sin\theta$

$dx = 4\cos\theta\,d\theta$

$\sqrt{16-x^2} = \sqrt{16-(4\sin\theta)^2}$

$= \sqrt{16-16\sin^2\theta}$

$= \sqrt{16(1-\sin^2\theta)}$

$= \sqrt{16\cos^2\theta}$

$= 4\cdot|\cos\theta|$

* No need Abs. Value

$= \int \overbrace{4\cos\theta}\,\overbrace{(4\cos\theta\,d\theta)}^{dx}$

$= 16\int \cos^2\theta\,d\theta$

$= 16\int \tfrac{1}{2}(1+\cos 2\theta)\,d\theta$

$= 8\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C$

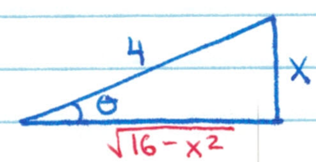
use $\boxed{\sin 2\theta = 2\sin\theta\cos\theta}$

$= 8\left(\theta + \cancel{\tfrac{1}{2}}\cdot\cancel{2}\sin\theta\cos\theta\right) + C$

$\boxed{x = 4\sin\theta}$

① $\sin\theta = \dfrac{x}{4}$

② $\theta = \arcsin\left(\dfrac{x}{4}\right)$



Note: same as original

③ $\cos\theta = \dfrac{\sqrt{16-x^2}}{4}$

$= 8\left(\sin^{-1}\left(\dfrac{x}{4}\right) + \left(\dfrac{x}{4}\right)\cdot\dfrac{\sqrt{16-x^2}}{4}\right) + C$

Case 2

ex: $\displaystyle\int \frac{dx}{(x^2+9)^{3/2}}$

$$= \int \frac{dx}{\left(\sqrt{x^2+9}\right)^3}$$

$$= \int \frac{3\sec^2\theta}{(3\sec\theta)^3}\, d\theta$$

$$= \frac{1}{3}\int \frac{1}{\sec\theta}\, d\theta$$

$$= 1/3 \int \cos\theta \, d\theta$$

$$= 1/3 \sin\theta + C$$

$1 + \tan^2 x = \sec^2 x$

$a = 3$

$x = a\tan\theta$

$x = 3\tan\theta$

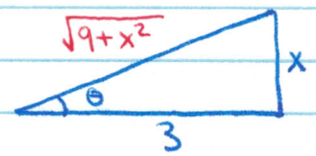
$dx = 3\sec^2\theta \, d\theta$

$$\sqrt{x^2+9} = \sqrt{(3\tan\theta)^2+9}$$
$$= \sqrt{9\tan^2\theta + 9}$$
$$= \sqrt{9(\tan^2\theta+1)}$$
$$= \sqrt{9\sec^2\theta}$$
$$= 3\sec\theta$$

$$\boxed{x = 3\tan\theta}$$

$\tan\theta = \dfrac{x}{3}$

$\theta = \arctan\left(\dfrac{x}{3}\right)$



① $\sin\theta = \dfrac{x}{\sqrt{9+x^2}}$

$$= 1/3\left(\frac{x}{\sqrt{9+x^2}}\right) + C$$

Case 3

ex: $\int \frac{dx}{\sqrt{x^2-16}}$

$\sec^2 x - 1 = \tan^2 x$

$a = 4$

$x = a \sec\theta$

$x = 4 \sec\theta$

$dx = 4 \sec\theta \tan\theta \, d\theta$

$\sqrt{x^2-16} = \sqrt{(4\sec\theta)^2 - 16}$

$= \sqrt{16\sec^2\theta - 16}$

$= \sqrt{16 \tan^2\theta}$

$= 4 \tan\theta$

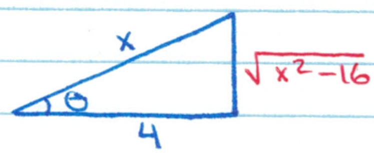
$= \int \frac{4\sec\theta \tan\theta}{4\tan\theta} \, d\theta$

$= \int \sec\theta \, d\theta$

$= \ln|\sec\theta + \tan\theta| + C$

$\boxed{x = 4\sec\theta}$

① $\sec\theta = \frac{x}{4} = \frac{hyp}{adj}$



② $\tan\theta = \frac{\sqrt{x^2-16}}{4}$

$= \ln\left|\left(\frac{x}{4}\right) + \frac{\sqrt{x^2-16}}{4}\right| + C$

ex: $\displaystyle\int \frac{\sqrt{1+x^2}}{x}\,dx$

$1 + \tan^2 x = \sec^2 x$

$a = 1$

$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$\sqrt{1+x^2} = \sqrt{1+(\tan\theta)^2}$

$= \sqrt{\sec^2\theta}$

$= \sec\theta$

$$= \int \frac{\sec\theta \cdot \sec^2\theta\, d\theta}{\tan\theta}$$

$$= \int \frac{\sec^3\theta}{\tan\theta}\, d\theta$$

$$= \int \frac{\sec^2\theta \cdot \sec\theta\tan\theta\, d\theta}{\tan^2\theta}$$

$$= \int \frac{\sec^2\theta}{(\sec^2\theta - 1)} \cdot \sec\theta\tan\theta\, d\theta$$

$u = \sec\theta$

$du = \sec\theta\tan\theta\, d\theta$

$$= \int \frac{u^2}{u^2-1}\, du$$

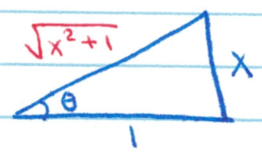
$$u^2 \div (u^2-1): \quad \begin{array}{r} u^2 \\ -(u^2-1) \\ \hline 1 \end{array} \quad \text{quotient } 1$$

$= 1 + \frac{1}{u^2-1}$

$$= \int 1\,du + \underbrace{\int \frac{1}{u^2-1}\,du}_{\text{partial fractions}}$$

$u = \sec\theta$

$\boxed{\tan\theta = x}$



$= \sqrt{x^2+1}$

$$= \sqrt{1+x^2} + \underbrace{\int \frac{1}{u^2-1}\,du}_{\text{partial fraction}}$$

Trinomial in Radical

→ Complete the square

$ax^2 + bx + c \qquad \pm (\tfrac{1}{2} b)^2$

→ make coefficient of x^2 1

ex: $\displaystyle\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$

complete the square:

$-x^2 - 2x + 3 = -(\underbrace{x^2+2x+1}-1) + 3$

$= -[(x+1)^2 - 1] + 3$

$= -(x+1)^2 + 4$

$\displaystyle = \int \frac{x}{\sqrt{4-(x+1)^2}}\,dx$

$u = x+1 \qquad \Rightarrow x = u - 1$

$du = dx$

$\displaystyle = \int \frac{u-1}{\sqrt{4-u^2}}\,du$

$1 - \sin^2\theta = \cos^2\theta$

$a = 2$

$u = 2\sin\theta$

$du = 2\cos\theta\, d\theta$

$\sqrt{4-u^2} = \sqrt{4-(2\sin\theta)^2}$

$= \sqrt{4(1-\sin^2\theta)}$

$= \sqrt{4\cos^2\theta}$

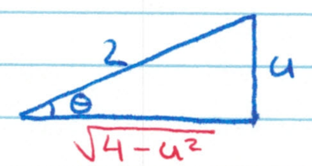
$= 2\cos\theta$

$\displaystyle = \int \frac{(2\sin\theta - 1)(\cancel{2\cos\theta}\, d\theta)}{\cancel{2\cos\theta}}$

$= (-2\cos\theta - \theta) + C$

$\boxed{u = 2\sin\theta}$

$\sin\theta = \dfrac{u}{2}$



① $\theta = \arcsin\left(\dfrac{u}{2}\right)$

② $\cos\theta = \dfrac{\sqrt{4-u^2}}{2}$

$\displaystyle = -2\left(\frac{\sqrt{4-u^2}}{2}\right) - \sin^{-1}\left(\frac{u}{2}\right) + C$

$\displaystyle = -\sqrt{4-(x+1)^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C$

Partial Fraction Decomposition

Good to integrate rational function!

Ratio of 2 polynomials

$$\int \frac{P(x)}{Q(x)}\, dx$$

⇒ Seperate a "hard to integrate" fcn into smaller "easier" partial fractions.

CASE 1

The denominator $Q(x)$ factors into linear terms, none of which repeat.

→ each linear term seperated into fraction

ex: $$\frac{3x^3+x^2-7}{(x+1)(x-3)(x+4)} = \frac{A}{(x+1)} + \frac{B}{(x-3)} + \frac{C}{(x+4)}$$

CASE 2

The denominator $Q(x)$ factors into linear terms, but some repeat.

→ if linear term is repeated (deg), it gets that many fractions (Note: one for each degree)

ex: $$\frac{x^3-x+1}{x^2(x-1)^3} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{(x-1)^3} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)}$$

* Note: Always start with long division if possible:

degree (num) ≥ degree (denom)

Q. How to find A and B.

ex: $\dfrac{8x+29}{\underbrace{(x+7)(x-2)}_{\text{LCD}}} = \dfrac{A}{(x+7)} + \dfrac{B}{(x-2)}$

Multiply both sides by LCD

$$8x+29 = \left[\frac{A}{(x+7)} + \frac{B}{(x-2)}\right](x+7)(x-2)$$

$$8x+29 = A(x-2) + B(x+7)$$

①

$8x+29 = A\underbrace{(x-2)}_{x=2} + B\underbrace{(x+7)}_{x=-7}$

Let:

$\boxed{x=2}$ $8(2)+29 = 0 + B(2+7)$

$45 = 9 \cdot B$

$\boxed{B=5}$

$\boxed{x=-7}$ $8(-7)+29 = A(-7-2) + 0$

$-27 = -9 \cdot A$

$\boxed{A=3}$

② : Expand

$8x+29 = Ax - 2A + Bx + 7B$

$8x+29 = (A+B)x + (-2A+7B)$

coefficient of x must be equal

i $8 = A+B$

ii $29 = -2A+7B$

$A = 8-B$

sub in (ii)

$29 = -2(8-B) + 7B$

$29 = -16 + 2B + 7B$

$45 = 9B$

$\boxed{B=5} \Rightarrow \boxed{A=3}$

$$\therefore \frac{8x+29}{(x+7)(x-2)} = \frac{3}{x+7} + \frac{5}{x-2}$$

ex: $\displaystyle\int \frac{x^4 - 2x^2 + 4x + 1}{(x-1)^2(x+1)}$ $\quad \frac{\text{deg } 4}{\text{deg } 3} \Rightarrow$ long division

$x^3 - x^2 - x + 1$

$$
\begin{array}{rl}
 & x^4 - 2x^2 + 4x + 1 \;\big|\; x^3 - x^2 - x + 1 \\
 - & x^4 - x^3 - x^2 + x \qquad\quad x + 1 \\
\hline
 & x^3 - x^2 + 3x + 1 \\
 - & x^3 - x^2 - x + 1 \\
\hline
 & 4x
\end{array}
$$

$$= \int (x+1)\,dx + \underbrace{\int \frac{4x}{(x-1)^2(x+1)}\,dx}_{\text{partial frac}}$$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{(x+1)}$$

$$4x = A(x+1) + B(x-1)(x+1) + C(x-1)^2$$

$\boxed{x=1}$ C, B out $\quad 4(1) = A(1+1)$ $\quad \boxed{A = 2}$

$\boxed{x=-1}$ A, B out $\quad 4(-1) = C(-1-1)^2$ $\quad \boxed{C = -1}$

Pick any other x-value

$\boxed{x=0}$ $\quad 0 = A(1) + B(-1) + C$

$0 = 2 - B + (-1)$

$\boxed{B = 1}$

$$= \int (x+1)\,dx + \int \frac{2}{(x-1)^2}\,dx + \int \frac{1}{(x-1)}\,dx + \int \frac{-1}{(x+1)}\,dx$$

$$= \frac{x^2}{2} + x + 2\left(-(x-1)^{-1}\right) + \ln|x-1| - \ln|x+1| + C$$

$$= \frac{x^2}{2} + x - \frac{2}{x-1} + \ln|x-1| - \ln|x+1| + C$$

ex: $\displaystyle\int \frac{x^2+2x-1}{x^3-x}\,dx$

$$x^3 - x = x(x^2-1) = x(x-1)(x+1)$$

$$= \int \frac{x^2+2x-1}{x(x-1)(x+1)}\,dx$$

$$\frac{x^2+2x-1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$x^2+2x-1 = A(x-1)(x+1) + B(x)(x+1) + C(x)(x-1)$$

$\boxed{x=0}$ $\quad -1 = A(-1)(1)$

$$\boxed{A = 1}$$

$\boxed{x=1}$ $\quad 1+2-1 = B(1)(2)$

$$\boxed{B = 1}$$

$\boxed{x=-1}$ $\quad 1-2-1 = C(-1)(-2)$

$$\boxed{C = -1}$$

$$= \int \frac{1}{x}\,dx + \int \frac{1}{x-1}\,dx + \int \frac{-1}{x+1}\,dx$$

$$= \ln|x| + \ln|x-1| - \ln|x+1| + C$$

CASE 3

The denominator $Q(x)$, when factored contains an irreducible quadratic, that is <u>not</u> repeated.

ex: $$\frac{x}{(x-2)^3(x^2+4)} = \underbrace{\frac{A}{(x-2)^3} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)}}_{\text{usual}} + \underbrace{\frac{Dx+E}{x^2+4}}_{*}$$

$*$ Q. How to integrate?

$$\int \frac{Dx+E}{x^2+a^2}\,dx$$

$\rightarrow$ split up

$$= \underbrace{D\int \frac{x}{x^2+a^2}\,dx}_{\substack{\text{sub: } u = x^2+a^2 \\ du = 2x\,dx}} + \underbrace{E\int \frac{1}{x^2+a^2}\,dx}_{\substack{\text{Trig sub:} \\ x = a\tan\theta}}$$

$\downarrow$

Note: if $a = 1$

$$\boxed{\int \frac{1}{x^2+1}\,dx = \arctan(x) + C}$$

CASE 4

The denominator $Q(x)$ contains irreducible quadratics, some of which repeat.

ex: $$\frac{x^4}{(x^2-x+1)(x^2+2)^2} = \frac{Ax+B}{(x^2-x+1)} + \frac{Cx+D}{(x^2+2)^2} + \frac{Ex+F}{(x^2+2)}$$

ex: Killer !

$$\int \frac{1}{x^3-1}\,dx$$

Recall: $a^3-b^3=(a-b)(a^2+ab+b^2)$
$a^3+b^3=(a+b)(a^2-ab+b^2)$

$$= \int \frac{1}{(x-1)(x^2+x+1)}\,dx$$

$$\frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

CASE 3

$$1 = A(x^2+x+1) + (Bx+C)(x-1)$$

$\boxed{x=1}$

$$1 = A(3)$$
$$\boxed{A = 1/3}$$

$\boxed{x=0}$

$$1 = A(1) + C(0-1)$$
$$1 - 1/3 = -C$$
$$\boxed{C = -2/3}$$

$\boxed{x=-1}$

any other value

$$1 = A(1) + (-B+C)(-2)$$
$$1 = 1/3 + (-B - 2/3)(-2)$$
$$-1/3 = -B - 2/3$$
$$B = -2/3 + 1/3$$
$$\boxed{B = -1/3}$$

$$\therefore \int \frac{1}{(x-1)(x^2+x+1)}\,dx = \frac{1}{3}\int \frac{1}{x-1}\,dx + \int \frac{-1/3\,x - 2/3}{(x^2+x+1)}\,dx$$

Killer ! (cont'd)

$$\tfrac{1}{3}\int \frac{1}{x-1}\,dx + \int \frac{-\tfrac{1}{3}x - \tfrac{2}{3}}{x^2+x+1}\,dx$$

$$= \tfrac{1}{3}\ln|x-1| + \boxed{\int \frac{-\tfrac{1}{3}x - \tfrac{2}{3}}{x^2+x+1}\,dx} \quad (*)$$

Trig Sub
∴ complete the square (denom)
* Do before "split up"

$$\begin{cases} x^2+x+1 = x^2+x+\tfrac{1}{4}-\tfrac{1}{4}+1 \\ \pm(\tfrac{1}{2}b^2) \quad = (x+\tfrac{1}{2})^2 + \tfrac{3}{4} \end{cases}$$

(*) $$\int \frac{-\tfrac{1}{3}x - \tfrac{2}{3}}{(x+\tfrac{1}{2})^2+\tfrac{3}{4}}\,dx$$

$u = x + \tfrac{1}{2} \Rightarrow x = u - \tfrac{1}{2}$
$du = dx$

$$= \int \frac{-\tfrac{1}{3}(u-\tfrac{1}{2}) - \tfrac{2}{3}}{u^2+\tfrac{3}{4}}\,du$$

$$= \int \frac{-\tfrac{1}{3}u + \tfrac{1}{6} - \tfrac{2}{3}}{u^2+\tfrac{3}{4}}\,du$$

split
$$= -\frac{1}{3}\int \frac{u}{u^2+\tfrac{3}{4}}\,du - \frac{1}{2}\int \frac{1}{u^2+\tfrac{3}{4}}\,du$$

$$\boxed{\begin{array}{l} w = u^2+\tfrac{3}{4} \\ dw = 2u\,du \end{array}}$$

$$\boxed{\begin{array}{c} u^2 + a^2 \\ \therefore a = \tfrac{\sqrt{3}}{2} \end{array}}$$

$u = \frac{\sqrt{3}}{2}\tan\theta$
$du = \sqrt{3}/2\,\sec^2\theta\,d\theta$

$u^2 + 3/4 = (3/4\tan^2\theta) + 3/4$
$= 3/4\sec^2\theta$

$$= -\frac{1}{3}\cdot\frac{1}{2}\int \frac{1}{w}\,dw - \frac{1}{2}\int \frac{\sqrt{3}/2\,\cancel{\sec^2\theta}}{3/4\,\cancel{\sec^2\theta}}\,d\theta$$

$$= -\tfrac{1}{6}\ln|w| - \tfrac{\sqrt{3}}{3}\int d\theta$$

Killer ! (cont'd)

$$-\tfrac{1}{6}\ln|w| - \frac{\sqrt{3}}{3}\int d\theta$$

$$= -\tfrac{1}{6}\ln\left|(x+\tfrac{1}{2})^2 + \tfrac{3}{4}\right| - \frac{\sqrt{3}}{3}\,\theta + C$$

$$u = \frac{\sqrt{3}}{2}\tan\theta$$

$$\theta = \arctan\left(\frac{2u}{\sqrt{3}}\right)$$

$$\theta = \arctan\left(\frac{2(x+\tfrac{1}{2})}{\sqrt{3}}\right)$$

$$= \underbrace{-\tfrac{1}{6}\ln\left|(x+\tfrac{1}{2})^2 + \tfrac{3}{4}\right| - \frac{\sqrt{3}}{3}\arctan\left(\frac{2(x+\tfrac{1}{2})}{\sqrt{3}}\right) + C} = \circledast$$

$$\therefore \int \frac{1}{x^3-1}\,dx = \tfrac{1}{3}\ln|x-1| + \circledast$$

ex: $\displaystyle\int \frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2}\,dx$ CASE 4

$$\frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2} = \frac{Ax+B}{x^2+2x+2} + \frac{Cx+D}{(x^2+2x+2)^2}$$

$$x^3+2x^2+3x-2 = (Ax+B)(x^2+2x+2) + (Cx+D)$$

$$x^3+2x^2+3x-2 = Ax^3+2Ax^2+2Ax+Bx^2+2Bx+2B+Cx+D$$

$$x^3+2x^2+3x-2 = x^3(A)+x^2(2A+B)+x(2A+2B+C)+(2B+D)$$

equate coefficients

① $\boxed{A=1}$

② $2A+B=2 \Rightarrow \boxed{B=0}$

③ $2A+2B+C=3 \Rightarrow 2+C=3 \Rightarrow \boxed{C=1}$

④ $2B+D=-2 \Rightarrow \boxed{D=-2}$

$$= \int \frac{x}{x^2+2x+2}\,dx + \int \frac{x-2}{(x^2+2x+2)^2}\,dx$$

complete the square $\begin{cases} x^2+2x+2 = x^2+2x+1-1+2 \\ \quad = (x+1)^2+1 \end{cases}$

$$= \int \frac{x}{(x+1)^2+1}\,dx + \int \frac{x-2}{[(x+1)^2+1]^2}\,dx$$

(cont'd)

$$= \int \frac{x}{(x+1)^2+1}\,dx + \int \frac{x-2}{[(x+1)^2+1]^2}\,dx$$

$\boxed{\begin{array}{l} u = x+1 \Rightarrow x = u-1 \\ du = dx \end{array}}$

$$= \int \frac{(u-1)}{u^2+1}\,du + \int \frac{u-1-2}{(u^2+1)^2}\,du$$

$$= \int \frac{u}{u^2+1}\,du - \underbrace{\int \frac{1}{u^2+1}\,du}_{\arctan(u)} + \int \frac{u}{(u^2+1)^2}\,du - \underbrace{3\int \frac{1}{(u^2+1)^2}\,du}_{\text{Trig sub}}$$

$\boxed{\begin{array}{l} w = u^2+1 \\ dw = 2u\,du \\ \frac{1}{2}\,dw = u\,du \end{array}}$

$a = 1$

$u = 1 \cdot \tan\theta$

$du = \sec^2\theta\,d\theta$

$u^2+1 = \tan^2\theta + 1$

$= \sec^2\theta$

$$= \frac{1}{2}\int \frac{1}{w}\,dw - \arctan(u) + \frac{1}{2}\int \frac{1}{w^2}\,dw - 3\int \frac{\cancel{\sec^2\theta}\,d\theta}{(\sec^2\theta)^{\cancel{2}}}$$

$$= \frac{1}{2}\ln|w| - \arctan(u) + \frac{1}{2}\left(-w^{-1}\right) - \underbrace{3\int \cos^2\theta\,d\theta}$$

$= \int \cos^2\theta\,d\theta$

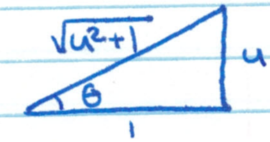
$= \int 1/2\,(1+\cos 2\theta)\,d\theta$

$= 1/2\left(\theta + \frac{1}{2}\sin 2\theta\right)$

$$= \frac{1}{2}\ln|w| - \arctan(u) + \frac{1}{2}\left(-\frac{1}{w}\right) - 3/2\left(\theta + \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right) + C$$

$\tan\theta = u$

$\theta = \arctan(u)$



$\sin\theta = \frac{u}{\sqrt{u^2+1}}$

$\cos\theta = \frac{1}{\sqrt{u^2+1}}$

$$= 1/2\ln\left|(x+1)^2+1\right| - \arctan(x+1) - \frac{1}{2((x+1)^2+1)}$$

$$- \frac{3}{2}\left(\arctan(x+1) + \frac{(x+1)}{\sqrt{(x+1)^2+1}} \cdot \frac{1}{\sqrt{(x+1)^2+1}}\right) + C$$

Turn into rational fcn

ex: $\int \frac{\sqrt{x+4}}{x}\,dx$

$u = \sqrt{x+4} \Rightarrow u^2 - 4 = x$

$du = \frac{1}{2\sqrt{x+4}}\,dx$

$du = \frac{1}{2u}\,dx$

$2u\,du = dx$

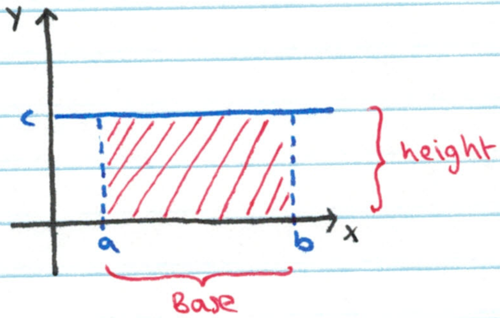
$= \int \frac{u}{u^2-4}(2u\,du)$

$= 2\int \frac{u^2}{u^2-4}\,du$

* use partial fractions

Average Value

consider a constant fcn $f(x) = c$



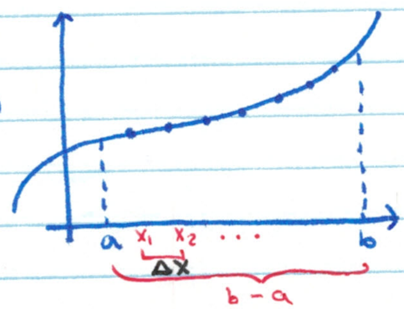
$$\text{Avg}(f) = \begin{pmatrix}\text{Avg}\\ \text{Height}\end{pmatrix} = c$$

Area = (base)(height)

$$\boxed{\text{Height} = \frac{\text{Area}}{\text{Base}}} \Rightarrow \boxed{\begin{pmatrix}\text{Avg}\\ \text{Height}\end{pmatrix} = \frac{\int_a^b f(x)\,dx}{b-a}}$$

<u>Approximate using finite pts</u>

$$f_{avg} \approx \frac{f(x_1)+f(x_2)+f(x_3)+\dots+f(x_n)}{n}$$



$$= \sum_{i=1}^{n} (f(x_i)) \cdot \left(\frac{1}{n}\right)$$

$$= \sum_{i=1}^{n} f(x_i) \cdot \Delta x \cdot \left(\frac{1}{b-a}\right)$$

$$\Delta x = \frac{b-a}{n}$$

$$\boxed{\therefore \frac{1}{n} = \frac{\Delta x}{b-a}}$$

$$= \frac{1}{b-a} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

⊛ To get exact value (not approximation):

$$f_{avg} = \lim_{n\to\infty} \frac{1}{b-a} \cdot \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$\boxed{f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

$$\frac{\text{Area}}{\text{Base}}$$

ex: In a city, the temp. (in F°) t hours after 9 am can be modelled by the fcn:

$$T(t) = 50 + 14\sin\left(\frac{\pi t}{12}\right)$$

Find the average temp. from 9am - 9pm.

Let: 9am $\Rightarrow$ $a = 0$
9pm $\Rightarrow$ $b = 12$

$$T_{avg} = \frac{1}{12-0} \cdot \int_a^b T(t)\,dt$$

$$= \frac{1}{12} \int_0^{12} \left(50 + 14\sin\left(\frac{\pi t}{12}\right)\right)dt$$

$u = \frac{\pi t}{12}$

$du = \frac{\pi}{12}\,dt$

$dt = \frac{12}{\pi}\,du$

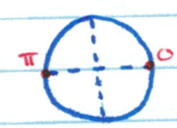
$t = 0,\ u = 0$

$t = 12,\ u = \pi$

$$= \frac{1}{\pi} \int_0^{\pi} (50 + 14\sin(u))\,du$$

$$= \frac{1}{\pi} \left(50u + 14(-\cos(u))\right)\Big|_0^{\pi}$$

$$= \frac{1}{\pi}\left[50\pi - 14\cos(\pi) - \cancel{50(0)} + 14\cos(0)\right]$$



$$= \frac{1}{\pi}\left[50\pi - 14(-1) + 14(1)\right]$$

$$\approx 59\,°F$$

Mean Value Theorem

Recall: MVT for derivatives:

Let $f(x)$ be
1. continuous on $[a,b]$
2. differentiable on (a,b)

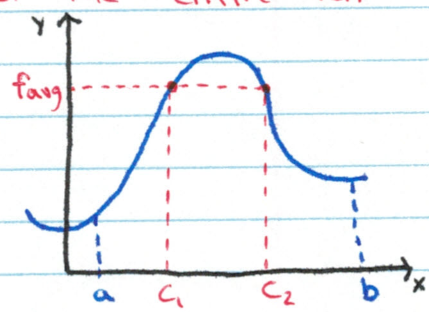
$$\therefore \exists c \in (a,b) \text{ s.t } \underbrace{f'(c)}_{\text{slope tangent}} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{slope secant}}$$

There exist

MVT (for integrals)

$$\boxed{\begin{array}{l}\text{If } f(x) \text{ is continuous on } [a,b] \\ \therefore \exists c \in (a,b) \text{ s.t } f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx\end{array}}$$

ie. There must be a value "c", where the y value at "c" is equal to the average value for the entire fcn



Assume: $f(x)$ is continuous on $[a,b]$

Define: $F(x) = \int_a^x f(t)\,dt$

$\therefore$ FTC-I : ① $F(x)$ continuous on $[a,b]$

* ② $F(x)$ differentiable on (a,b)

③ $F'(x) = f(x)$

Note: Now $F(x)$ satisfies conditions for MVT (derivatives)

$$\therefore \exists c \in (a,b) \text{ s.t } F'(c) = \frac{F(b)-F(a)}{b-a}$$

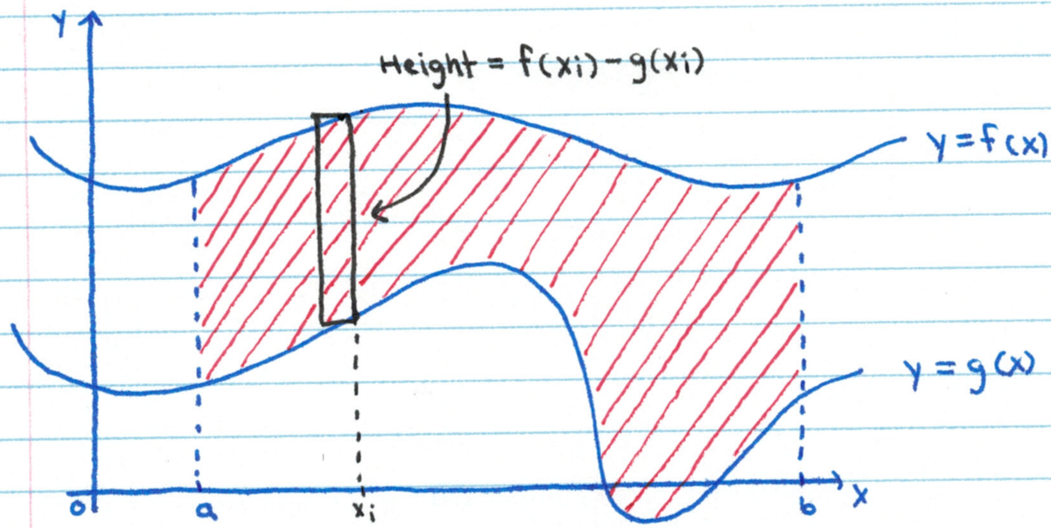
① $F'(c) = f(c)$ by FTC-I

② $F(a) = \int_a^a f(t)\,dt = 0$

③ $F(b) = \int_a^b f(t)\,dt$

$$\therefore f(c) = \frac{1}{b-a} \cdot \int_a^b f(t)\,dt$$

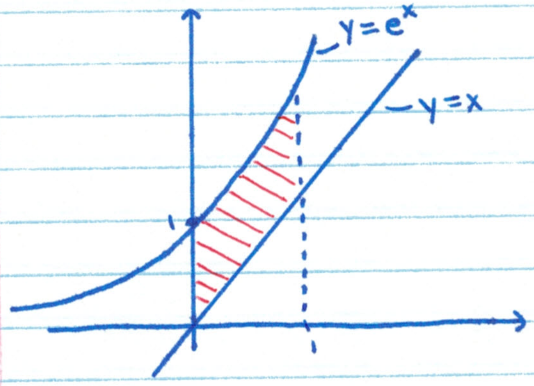
Area Between Curves



Theorem: The area A of the region bounded by the curves $y = f(x)$ and $y = g(x)$, and $x = a$, $x = b$ is:

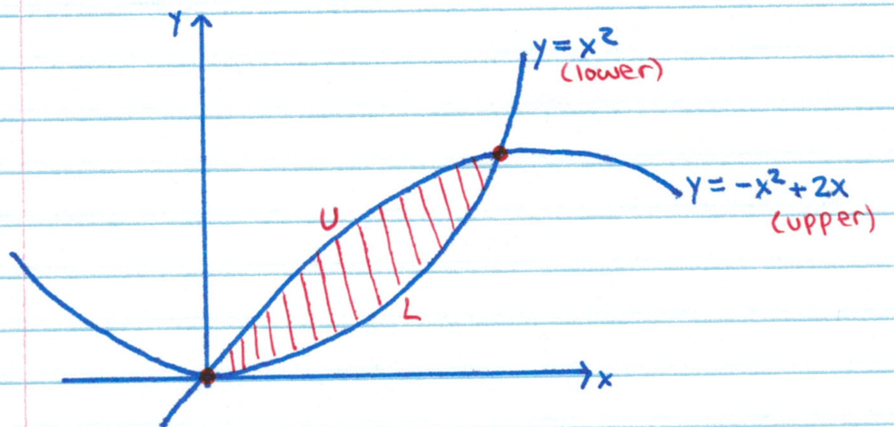
$$\boxed{\text{Area} = \int_a^b (f(x) - g(x))\,dx}$$ (upper - lower)

ex: Find the area of the region bounded by $y = e^x$, $y = x$ and $x=0$, $x=1$



$$\text{Area} = \int_0^1 (e^x - x)\,dx$$
$$= \left(e^x - \frac{x^2}{2}\right)\Big|_0^1$$
$$= (e - 1/2) - (e^0 - 0)$$
$$= e - 3/2$$

ex: ... region enclosed by $y = x^2$ and $y = -x^2 + 2x$



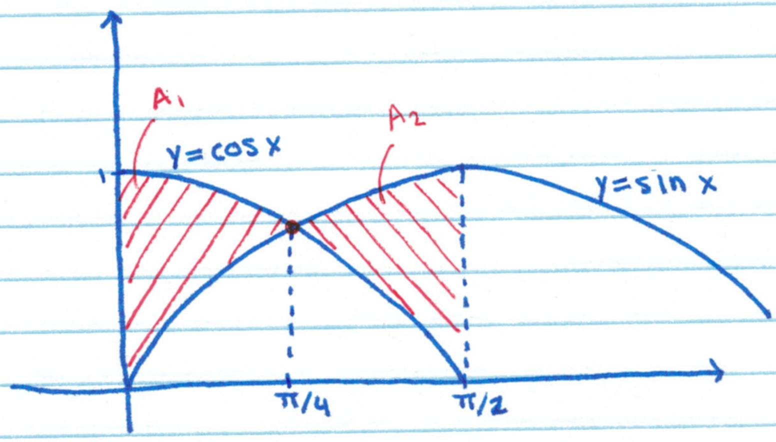
Find bounds of integration (ie. intersection pts)

$$\begin{cases} y = x^2 & ① \\ y = -x^2 + 2x & ② \end{cases}$$

sub ①
$$x^2 = -x^2 + 2x$$
$$0 = 2x - 2x^2$$
$$0 = 2x(1-x)$$
$$x=0 \mid x=1$$
bounds

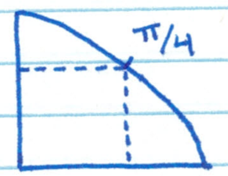
$$\therefore A = \int_0^1 \left((-x^2+2x) - (x^2)\right)dx$$

ex: ... the region bounded by $y = \sin x$, $y = \cos x$, $x = 0$, $x = \pi/2$



Intercept

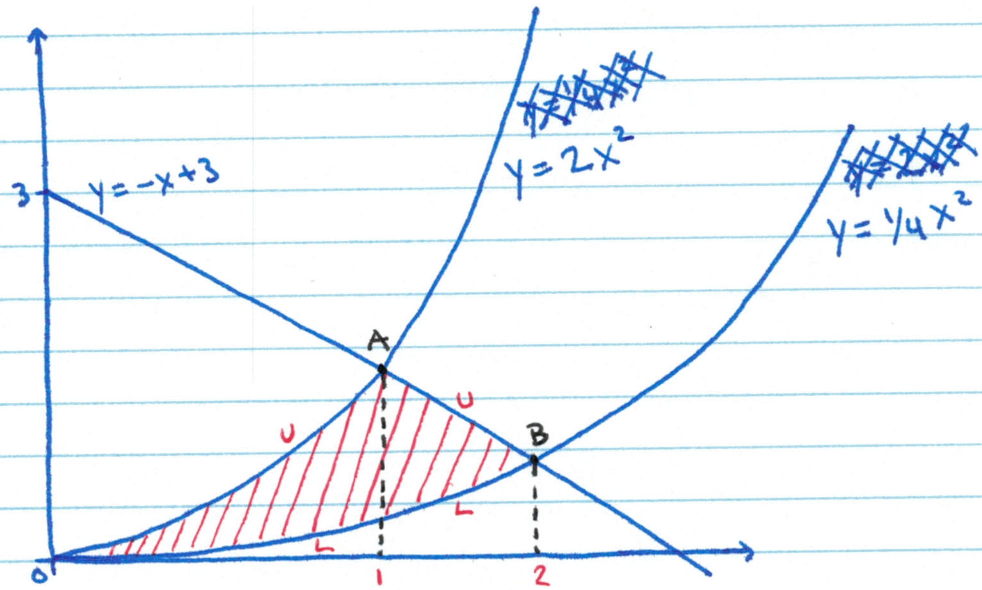
$$\sin x = \cos x$$
$$x = \pi/4$$



Before $\frac{\pi}{4}$: $y = \cos x$ is upper
After $\frac{\pi}{4}$: $y = \sin x$ is upper

$$\therefore \text{Area} = \underbrace{\int_0^{\pi/4} (\cos x - \sin x)\,dx}_{A_1} + \underbrace{\int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx}_{A_2}$$

ex: ... region bounded by $y = 1/4\, x^2$, $y = 2x^2$ and $\underbrace{x + y = 3}_{y = -x + 3}$, $x \geq 0$



intercept A

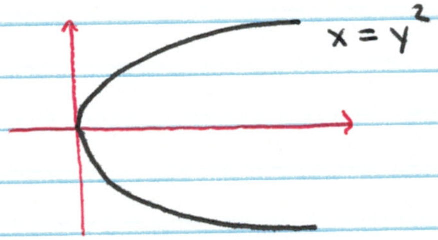
$$\begin{cases} y = -x + 3 \\ y = 2x^2 \end{cases} \qquad \boxed{x = 1}$$

intercept B

$$\begin{cases} y = -x + 3 \\ y = 1/4\, x^2 \end{cases} \qquad \boxed{x = 2}$$

$$\therefore \text{Area} = \int_0^1 \left(2x^2 - \tfrac{1}{4}x^2\right) dx + \int_1^2 \left(-x + 3 - \tfrac{1}{4}x^2\right) dx$$

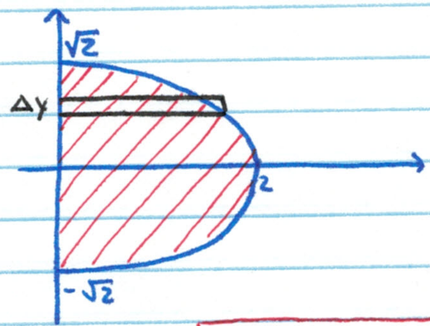
Think of x as a fcn of y
(ie. $x(y) = \dots$)



<u>Note</u>:
- $x = y^2 + c$
  c translation to right (if $c > 0$)
- $x = (y-k)^2$
  k translation up (if $k>0$)

ex: ... region bounded by $\underbrace{y^2 = 2-x}_{x = -y^2+2}$ and the line $x = 0$

→ reflexion
→ +2 right



$$\therefore \text{Area} = \int_{-\sqrt{2}}^{\sqrt{2}} \left(-y^2 + 2 - (0)\right) dy$$

everything with respect to y

$$\boxed{\text{Area} = \int_a^b (\text{right} - \text{left})\, dy \quad \rightarrow a, b \text{ are } y\text{-values}}$$

<u>Compare w.r.t x</u>

$y^2 = 2 - x$

$\boxed{y = \pm\sqrt{2-x}}$ 2 fcns

$$\therefore \ A = \int_0^2 \left(\sqrt{2-x} - -\sqrt{2-x}\right) dx$$

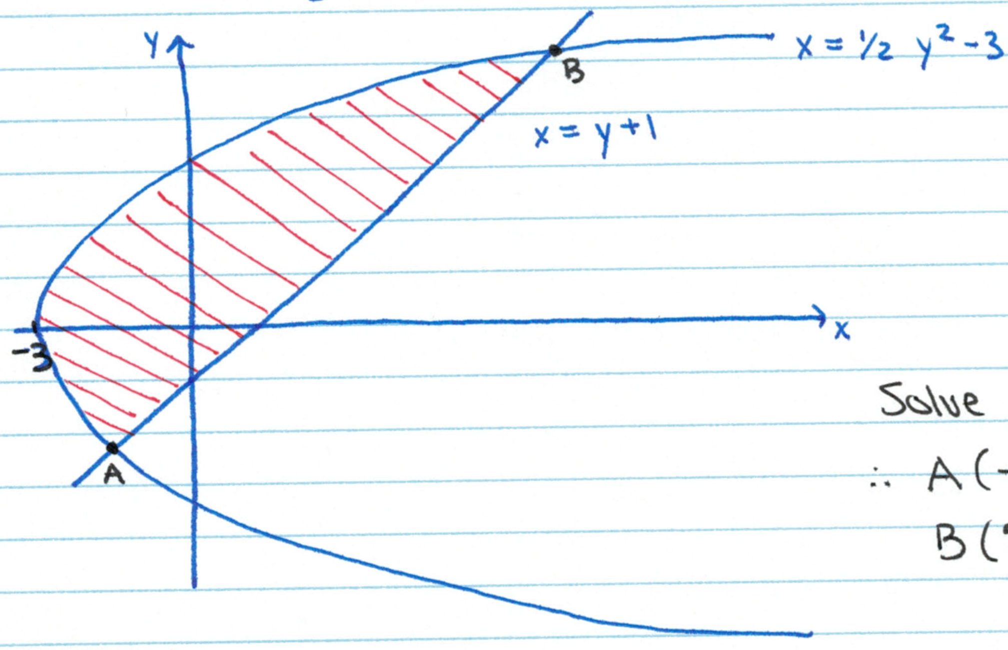
<u>or</u> $A = 2 \cdot \int_0^2 \left(\sqrt{2-x}\right) dx$

ex: ... region enclosed by $y = x - 1$
and parbola $y^2 = 2x + 6$

① Put everything w.r.t y

$x = y + 1$

$x = \frac{1}{2} y^2 - 3$ → translation 3 to the left



Solve A, B

∴ A(-1, -2)

B(5, 4)

∴ limits of integration w.r.t y : $-2 \rightarrow 4$

$$\text{Area} = \int_{-2}^{4} \left(y + 1 - \left(\tfrac{1}{2} y^2 - 3 \right) \right) dy$$

<u>Compare w.r.t x</u>

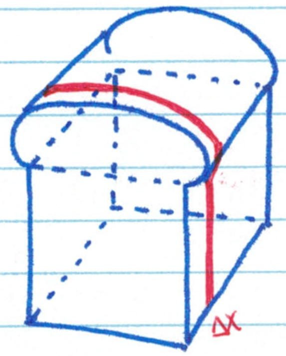
$y = x - 1$ and $y = \pm\sqrt{2x + 6}$

~~Find intercept & ⇒~~ limits of integration ⇒ x

$$\therefore A = \underbrace{\int_{-3}^{-1} \left(\sqrt{2x+6} - -\sqrt{2x+6} \right) dx}_{\text{Before intercept A}} + \underbrace{\int_{-1}^{5} \left(\sqrt{2x+6} - (x-1) \right) dx}_{\text{After intercept A}}$$

Volumes of Revolution

Consider a loaf of bread



(Volume of slice) $= \underbrace{\left(\begin{matrix}\text{surface}\\ \text{area}\end{matrix}\right)}_{A(x)} \Delta X$

$$(\text{Total Volume}) \approx \sum_{i=1}^{n} A(x_i)\,\Delta X$$

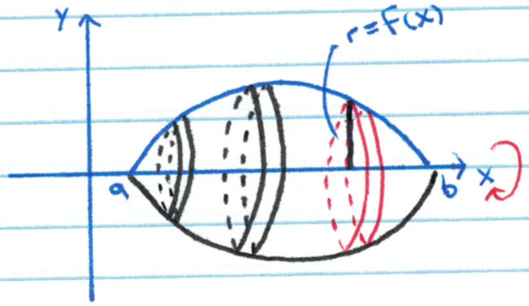
Definition: Let S be a solid that lies between $x=a$ and $x=b$.
If the cross section area of S is $A(x)$, then the volume of S is:

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\,\Delta x = \int_a^b A(x)\,dx$$

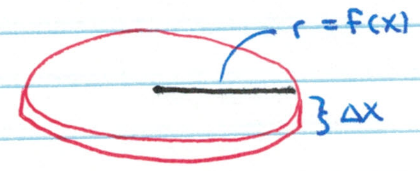
Method 1 (Discs)

Note: cross-sectional cut $\perp$ to the axis of rotation.

consider:



Slice: (disc)

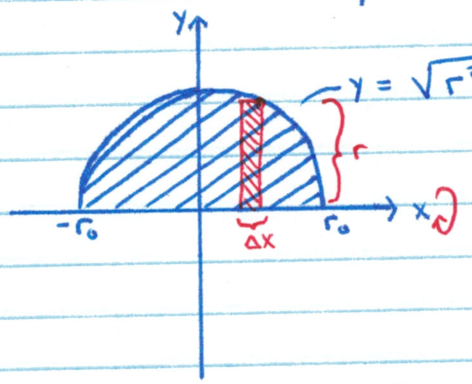


$$A(r) = \pi r^2$$
$$A(x) = \pi (f(x))^2$$
$$\therefore \boxed{V = \pi \int_a^b (f(x))^2 \, dx}$$

ex: Show that the volume of a sphere of radius r_0 is $V = \frac{4}{3}\pi r_0^3$

circle : $x^2 + y^2 = r^2$

$\therefore \; y = \pm\sqrt{r^2 - x^2}$



$\Rightarrow$ rotate about the x-axis

$\rightarrow$ gives you a sphere

$$V = \pi \int_a^b (f(x))^2 \, dx$$

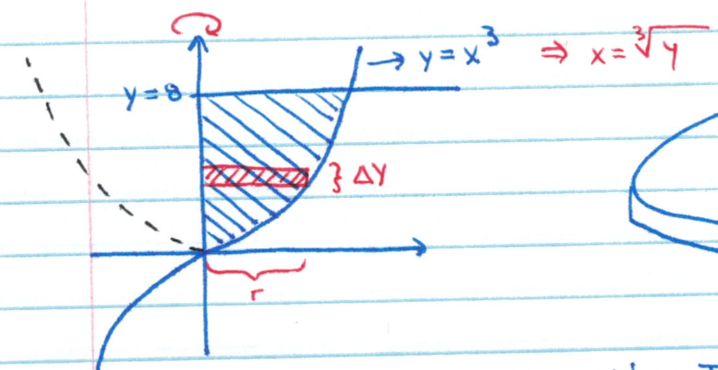
$$= \pi \int_{-r_0}^{r_0} \left(\cancel{\sqrt{r_0^2 - x^2}}\right)^{\cancel{2}} dx$$

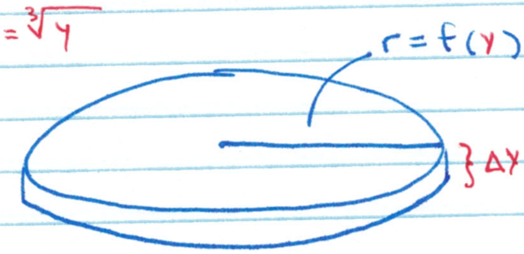
$$= \pi \left(r_0^2 \cdot x - \frac{x^3}{3} \right) \Big|_{-r_0}^{r_0}$$

$$= \pi \left[r_0^3 - \frac{r_0^3}{3} - \left(-r_0^3 + \frac{r_0^3}{3} \right) \right]$$

$$= 4/3 \, \pi r_0^3$$

ex: Find the vol. of a solid object bounded by $y = x^3$, $y = 8$, $x = 0$, rotated about the y-axis





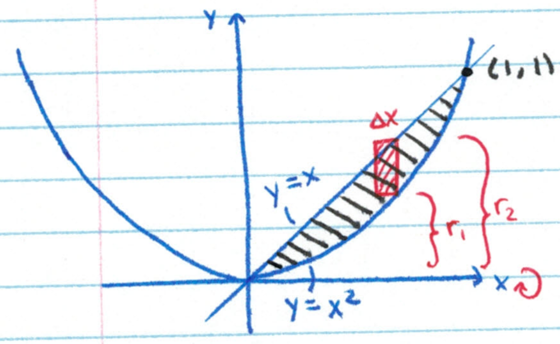
$$\therefore \; V = \pi \int_0^8 (f(y))^2 \, dy$$

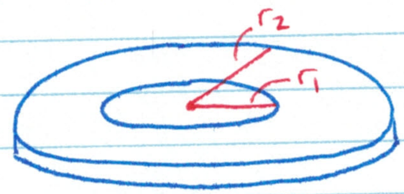
$$= \pi \int_0^8 (\sqrt[3]{y})^2 \, dy$$

* isolate $x \Rightarrow f(y)$
* limits of integration w.r.t y

Method 2 (Washers)

consider: the region enclosed by $y=x$, $y=x^2$ rotated about the x-axis.





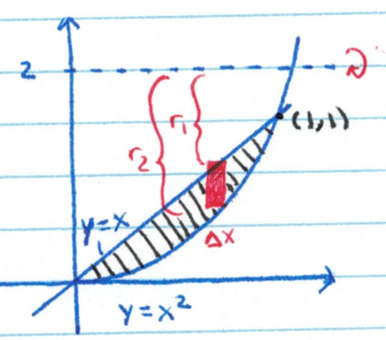
$$\text{Area} = \pi r_2^2 - \pi r_1^2$$
$$= \pi \left(r_2^2 - r_1^2\right)$$

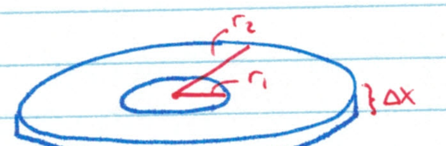
$$\boxed{\begin{aligned} r_1 &= x^2 \\ r_2 &= x \end{aligned}}$$

$$\boxed{V = \int_a^b \pi \left(r_2^2 - r_1^2\right) dx}$$

$$V = \int_a^b A(x)\, dx$$
$$= \pi \int_0^1 \left((x)^2 - (x^2)^2\right) dx$$

ex: same region rotated about $y=2$



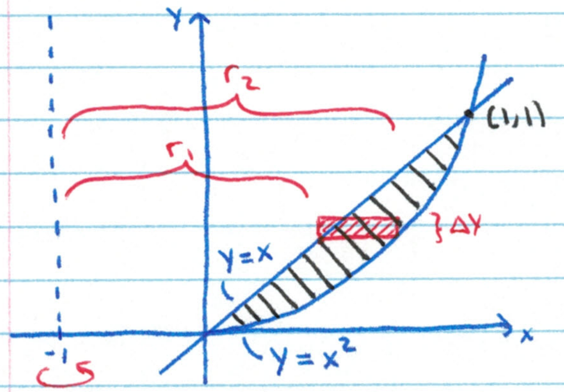


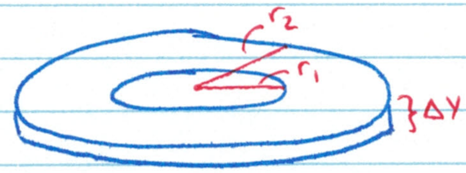
$r = 2 - y_{value}$

$$\boxed{\begin{aligned} r_1 &= 2 - x \\ r_2 &= 2 - x^2 \end{aligned}}$$ $y=x$, $y=x^2$

$$\therefore V = \pi \int_0^1 \left((2-x^2)^2 - (2-x)^2\right) dx$$

ex: ... same region, rotated about $x = -1$





$r = 1 + x_{value}$

$$\boxed{\begin{array}{l} r_1 = 1 + y \\ r_2 = 1 + \sqrt{y} \end{array}}$$

$y = x$

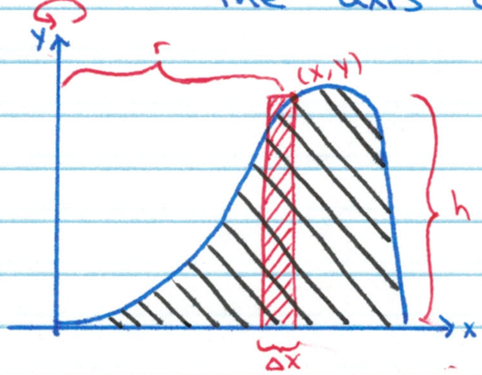
$y = x^2$

limits w.r.t y

$$V = \pi \int_0^1 \left((1+\sqrt{y})^2 - (1+y)^2 \right) dy$$

Method 3 (Shells)

Note: cross section is NOT $\perp$ to the axis of rotation.

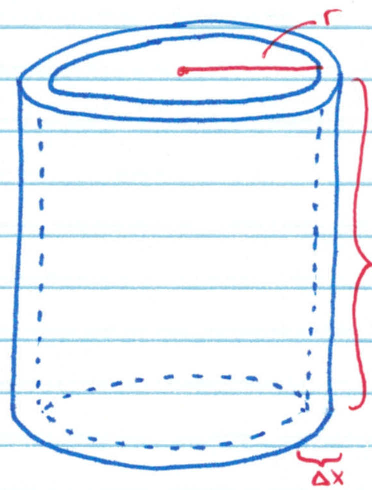


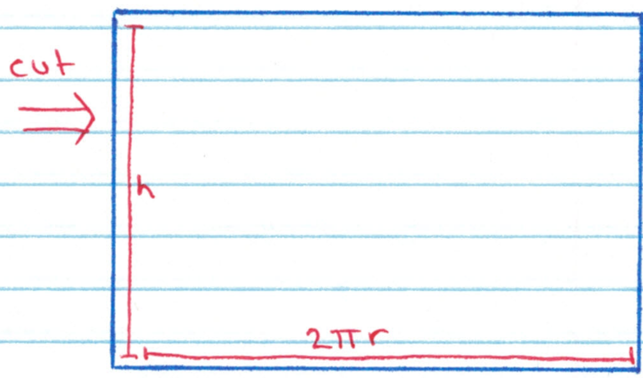
everything w.r.t x

$r = x_{value}$

$h = y_{value}$

$$\boxed{\begin{aligned} r &= x \\ h &= f(x) \end{aligned}}$$

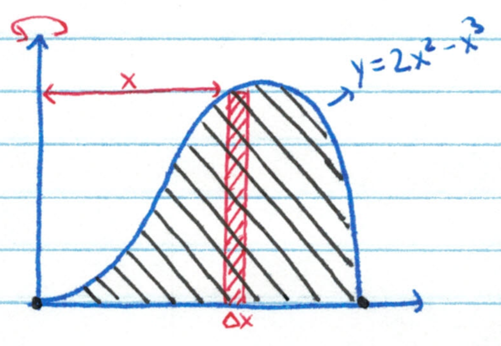




$A(x) = 2\pi r \cdot h$

$$\boxed{V = \int_a^b 2\pi r h \cdot dx}$$

ex: rotate about the y-axis, region bounded by $y = 2x^2 - x^3$ and the x-axis



intercept: $0 = 2x^2 - x^3$

$= x^2(2 - x)$

$x=0 \mid x=2$

$r = x_{value} = x$

$h = y_{value} = 2x^2 - x^3$ $\quad y = 2x^2 - x^3$

$$\therefore V = \int_0^2 2\pi \cdot (x)(2x^2 - x^3)\,dx$$

ex: ... $x = 1 + (y-2)^2$ and $x = 2$

around $y = 1$

Note: values already given as a function of y $\therefore$ probably easy horizontal cut (dy)

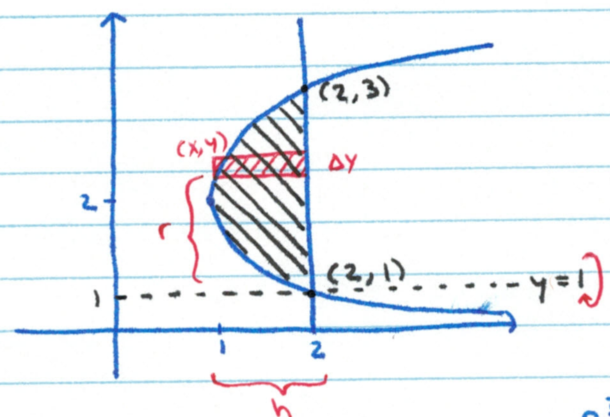
$x = y^2 \Rightarrow$ right 1, up 2 $\Rightarrow x = (y-2)^2 + 1$

Intercepts: limits of integ.

$2 = (y-2)^2 + 1$

$1 = (y-2)^2$

$y = 2 \pm 1$ $\boxed{y = 1 \\ y = 3}$



(Shells)

w.r.t y

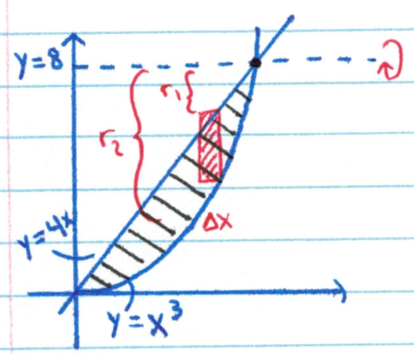
$r = y_{value} - 1$ $\qquad r = y - 1$

$h = 2 - x_{value}$ $\qquad h = 2 - [(y-2)^2 + 1]$

$h = 1 - (y-2)^2$

$$\therefore V = \int_1^3 2\pi (y-1)(1-(y-2)^2)\, dy$$

ex: ... $y = x^3$, $x \geq 0$, $y = 4x$ around $y = 8$

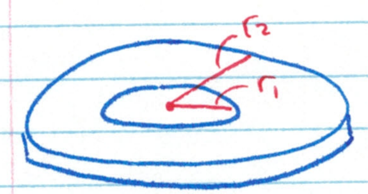


Intercept: $x^3 = 4x$

$\boxed{x = 2} \Rightarrow \boxed{y = 8}$

(washers)

$\boxed{r_1 = 8 - 4x \\ r_2 = 8 - x^3}$ $\qquad r = 8 - y_{value}$



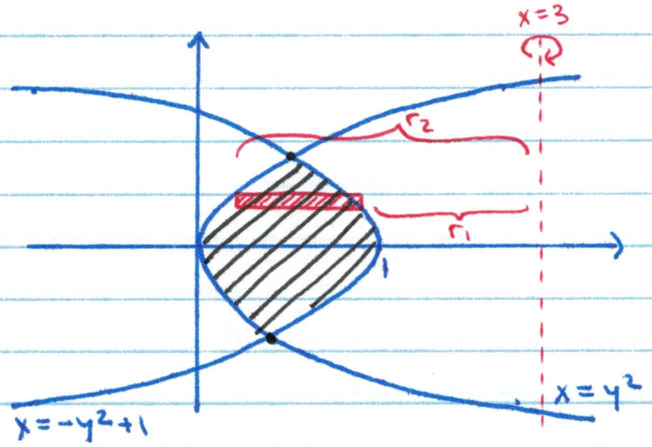
$$V = \pi \int_0^2 \left((8-x^3)^2 - (8-4x)^2 \right) dx$$

ex: ... $x = y^2$, $x = -y^2 + 1$ easy (dy)

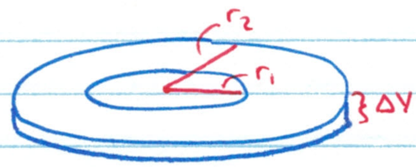
rotated about $x = 3$

$x = -y^2 + 1$

- reflexion
- 1 right



(washer)



Intercept: $y^2 = -y^2 + 1$

$y^2 = 1/2$

$\boxed{y = \pm\sqrt{1/2}}$

$r = 3 - X_{value}$

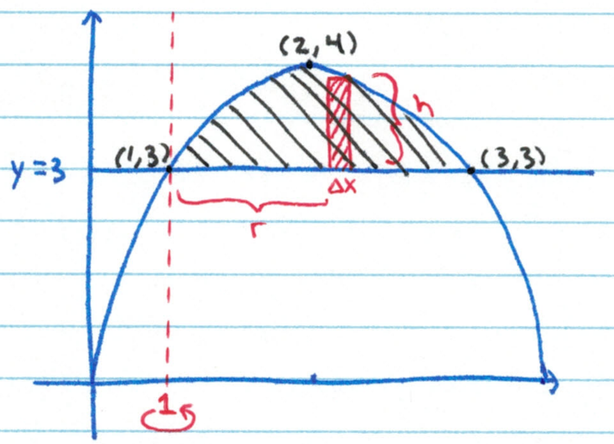
$\boxed{\begin{array}{l} r_1 = 3 - (-y^2 + 1) \\ r_2 = 3 - (y^2) \end{array}}$

$$\therefore \quad V = \pi \int_{-\sqrt{1/2}}^{\sqrt{1/2}} \left((3 - y^2)^2 - (2 + y^2)^2 \right) dy$$

ex: ... $y = -x^2 + 4x$, $y = 3$
rotate about $x = 1$

$$\begin{aligned} y &= -x^2 + 4x \\ &= -(\underline{x^2 - 4x + 4} - 4) \\ &= -(x-2)^2 + 4 \end{aligned} \qquad \therefore V(2,4)$$

Intercept: $3 = -x^2 + 4x$
$x^2 - 4x + 3 = 0$ $\qquad \left.\begin{matrix} S = 4 \\ P = 3 \end{matrix}\right\rangle 3, 1$



Note: Washers would be too difficult (dy)
→ switch to w.r.t y
→ have 2 fcns ±

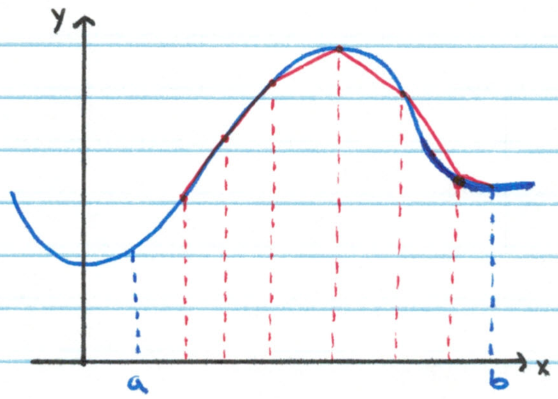
Note: Given w.r.t x
∴ use Δx (dx)

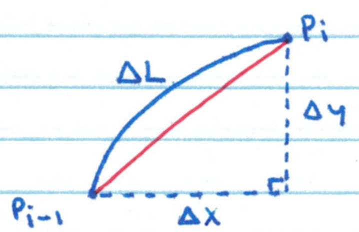
(Shells)

w.r.t x $\begin{cases} r = X_{value} - 1 \\ h = Y_{value} - 3 \end{cases}$ $\qquad \boxed{\begin{aligned} r &= x - 1 \\ h &= (-x^2 + 4x) - 3 \end{aligned}}$

$$V = 2\pi \int_1^3 (x-1)(-x^2 + 4x - 3)\, dx$$

Arc Length





Let L be actual distance

$$\Delta L \approx |P_i - P_{i-1}|$$

$$L \approx \sum_{i=1}^{n} |P_i - P_{i-1}|$$

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_i - P_{i-1}|$$

Note: $|P_i - P_{i-1}| = \sqrt{\Delta x^2 + \Delta y^2}$

factor out Δx to resemble Riemann

$$= \sqrt{(\Delta x)^2 \left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right.}$$

$$= \Delta x \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

Note: As $n \to \infty$

$$\frac{\Delta y}{\Delta x} \to \frac{dy}{dx}$$

$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad \Delta x$$

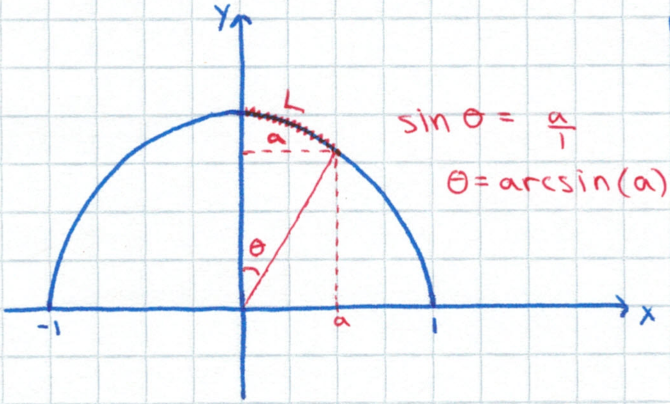
$$\therefore L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \ \Delta x$$

$$\boxed{L = \int_a^b \sqrt{1 + (f'(x))^2} \ dx}$$

condition:
if $f'(x)$ is continous on $[a,b]$

ex: Find arclength of $y = \sqrt{1-x^2}$ from $x=0$, $x=a$

$\boxed{r=1}$



$\sin\theta = \frac{a}{1}$

$\theta = \arcsin(a)$

$P = 2\pi r$

$L = \theta \cdot r$ $\quad \boxed{r=1}$

$\boxed{L = \theta} \Rightarrow L = \arcsin(a)$

Use arclength formula: $L = \int_a^b \sqrt{1+(f'(x))^2}\, dx$

$f(x) = \sqrt{1-x^2}$

① $f'(x) = (-\cancel{2}x) \cdot \frac{1}{\cancel{2}\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$

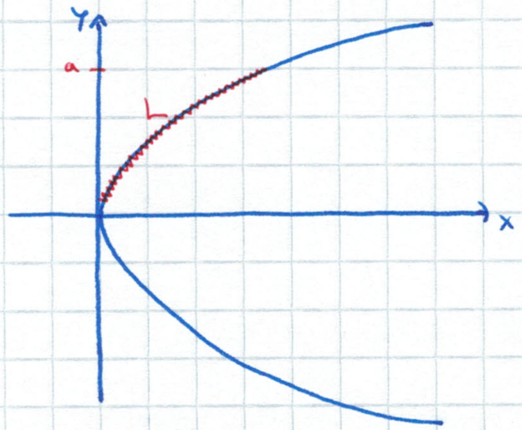
② $1+(f'(x))^2 = 1 + \left(\frac{-x}{\sqrt{1-x^2}}\right)^2 = 1 + \frac{x^2}{1-x^2} = \frac{1-\cancel{x^2}+\cancel{x^2}}{1-x^2}$

③ $L = \int_0^a \sqrt{\frac{1}{1-x^2}}\, dx$

$= \int_0^a \underbrace{\frac{1}{\sqrt{1-x^2}}}_{\text{arcsin}}\, dx$

$= \sin^{-1}(x)\Big|_0^a = \sin^{-1}(a) - \cancelto{0}{\sin^{-1}(0)} = \boxed{\sin^{-1}(a)}$

ex: arclength of $x = y^2$ from $y=0$ to $y=a$



$$L = \int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$f(y) = y^2$$

$$f'(y) = 2y$$

$$\therefore L = \int_0^a \sqrt{1+(2y)^2}\, dy$$

$$L = \int_0^a \sqrt{1+(2y)^2}\, dy$$

$$= \int_0^{\tan^{-1}(a)} \sqrt{1+\tan^2\theta} \cdot \frac{\sec^2\theta}{2}\, d\theta$$

$$= \frac{1}{2}\int_0^{\tan^{-1}(a)} \sec^3\theta \; d\theta$$

<u>Trig Sub</u>

You can do this

$$\left[\begin{array}{l} 2y = (1)\tan\theta \\ 2\,dy = \sec^2\theta\, d\theta \end{array}\right.$$

$$dy = \frac{\sec^2\theta}{2}\, d\theta$$

if $y = 0$, $\theta = \tan^{-1}(0) = 0$

$y = a$, $\theta = \tan^{-1}(a)$ //

$\theta = \arctan(2y)$

know how to solve (Boomerang)

$$\left[\begin{array}{ll} u = \sec\theta & dv = \sec^2\theta\, d\theta \\ du = \sec\theta\tan\theta\, d\theta & v = \tan\theta \end{array}\right]$$

$$= \frac{1}{2}\left[\sec\theta\tan\theta\Big|_0^{\tan^{-1}(a)} - \int_0^{\tan^{-1}(a)} \tan^2\theta\sec\theta\, d\theta\right]$$

$$I = \tfrac{1}{2}\left[\sec\theta\tan\theta\Big|_0^{\tan^{-1}(a)} - \left(\underbrace{\int_0^{\tan^{-1}(a)} \sec^3\theta\, d\theta}_{I} - \int_0^{\tan^{-1}(a)} \sec\theta\, d\theta\right)\right.$$

$$I = \tfrac{1}{2}\left(f(x) - I\right)$$

$$3I = f(x)$$

$$\frac{f(x)}{3} = I$$

$$\therefore I = \tfrac{1}{3}\left[\sec\theta\tan\theta\Big|_0^{\tan^{-1}(a)} + \int_0^{\tan^{-1}(a)} \sec\theta\, d\theta\right]$$

ex: arclength of $y = 5x^{3/2}$, $0 \leq x \leq 5$

$$L = \int_0^5 \sqrt{1+(f'(x))^2}\, dx$$

$$= \int_0^5 \sqrt{1 + \frac{225}{4}x}\, dx$$

$$= \int_1^{282.25} \frac{4}{225}\sqrt{u}\, du$$

$$= \frac{4}{225}\int_1^{282.25} \sqrt{u}\, du$$

① $f'(x) = 5 \cdot 3/2 \cdot x^{1/2}$

$= \frac{15}{2}\sqrt{x}$

② $1+(f'(x))^2 = 1 + \left(\frac{15}{2}\sqrt{x}\right)^2$

$= 1 + \frac{225}{4}x$

$u = 1 + \frac{225}{4}x$

$du = dx \cdot \frac{225}{4}$

$\frac{4}{225}du = dx$

---

if $x = 0$, $u = 1$

$x = 5$, $u = 282.25$

ex: $y = \sqrt{x - x^2} + \sin^{-1}(\sqrt{x})$; $0 \leq x \leq \pi/2$

① $f'(x) = (1-2x) \cdot \frac{1}{2\sqrt{x-x^2}} + \frac{1}{2\sqrt{x}} \cdot \frac{1}{\sqrt{1-(\sqrt{x})^2}}$

$= \frac{1-2x}{2\sqrt{x-x^2}} + \frac{1}{2\sqrt{x(1-x)}}$

$= \frac{1-2x+1}{2\sqrt{x-x^2}} = \frac{2-2x}{2\sqrt{x-x^2}} = \frac{\cancel{2}(1-x)}{\cancel{2}\sqrt{x-x^2}}$

② $1+(f'(x))^2 = 1 + \left(\frac{1-x}{\sqrt{x-x^2}}\right)^2$

$= \frac{x-x^2}{x-x^2} + \frac{(1-x)^2}{x-x^2}$

$= \frac{x-\cancel{x^2}+1-2x+\cancel{x^2}}{x-x^2}$

$= \frac{(1-x)}{x(1-x)} = 1/x$

$$\therefore L = \int_0^{\pi/2} \sqrt{1/x}\, dx$$

Improper Integral

→ when an integral does not satisfy the condition of FTC-II :
f must be continuous on [a,b]

<u>Type I</u> (Infinite interval)

$$\int_a^{\infty} f(x)\,dx = \lim_{t \to \infty} \int_a^{t} f(x)\,dx$$

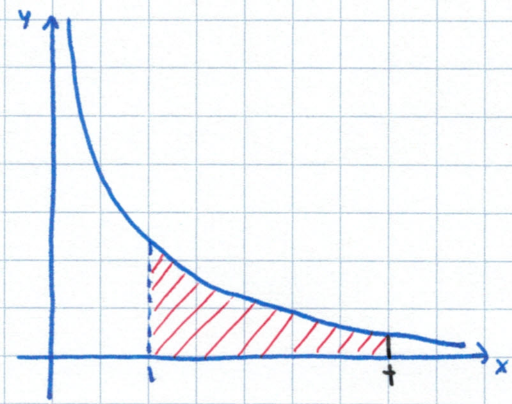
The integral → "<u>converges</u>" if limit exists
→ "<u>diverges</u>" if limit DNE

<u>Note</u>: $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{+\infty} f(x)\,dx$$

$$= \lim_{t \to -\infty} \int_t^{a} f(x)\,dx + \lim_{s \to \infty} \int_a^{s} f(x)\,dx$$

* DO NOT : $$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t \to \infty} \int_{-t}^{t} f(x)\,dx$$

ex: consider the area under $y = \frac{1}{x^2}$ above the x-axis from $x = 1 \Rightarrow x = +\infty$



$$\int_1^\infty \frac{1}{x^2}\,dx =$$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$$

$$= \lim_{t\to\infty} \left(-\frac{1}{x}\right)\Bigg|_1^t$$

$$= \lim_{t\to\infty} \left(-\cancel{\frac{1}{t}}^{\,0} - -\frac{1}{1}\right) = \boxed{1}$$

ex: $$\int_0^\infty e^{-kx}\,dx$$

$k > 0$

$u = -kx$

$du = -k\,dx$

$-\frac{1}{k}\,du = dx$

$x = 0,\ u = 0$

$x = t,\ u = -kt$

$$= \lim_{t\to\infty} \int_0^t e^{-kx}\,dx$$

$$= \lim_{t\to\infty} \int_0^{-kt} -\frac{1}{k} e^u\,du$$

$$= \lim_{t\to\infty} -\frac{1}{k}\left(e^u\right)\Bigg|_0^{-kt}$$

$$= \lim_{t\to\infty} -\frac{1}{k}\left(e^{-kt} - e^0\right)$$

$$= \lim_{t\to\infty} -\frac{1}{k}\left(e^{-\infty} - 1\right)$$

$$= -\frac{1}{k}(0 - 1)$$

$$= \boxed{\frac{1}{k}}$$

Type II (discontinuity interval)

(i) If f is discontinuous at $x = b$, then:

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

(ii) If f is discontinuous at $x = a$, then:

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

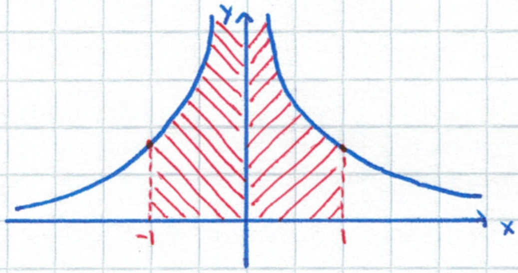
Consider:

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \left(-\frac{1}{x}\right)\Big|_{-1}^{1} \qquad \text{FTC-II}$$

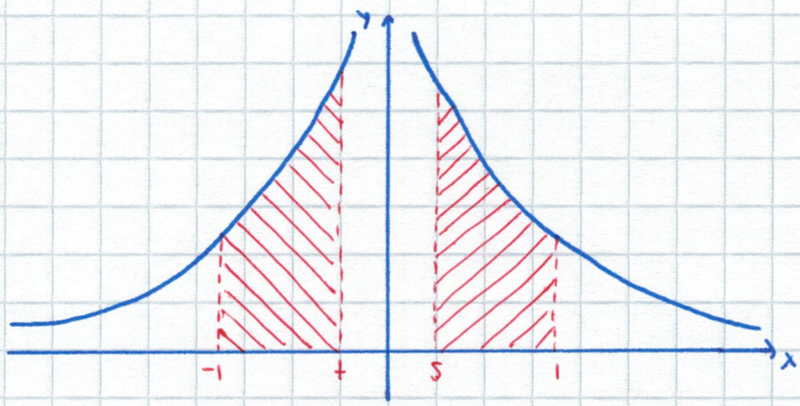
$$= -\frac{1}{1} - \left(-\frac{1}{-1}\right)$$

$$= -2 \quad \otimes \Rightarrow \text{FTC-II cannot be applied bc there is a discontinuity}$$

$\frac{1}{x^2}$ Restriction: $x^2 \neq 0$, $x \neq 0$



* Problem: vertical asymptote at $x=0$
→ discontinuous



$$\therefore \int_{-1}^{1} \frac{1}{x^2}\,dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{x^2}\,dx + \lim_{s\to 0^+} \int_{s}^{1} \frac{1}{x^2}\,dx$$

$$= \lim_{t\to 0^-} \left(-\frac{1}{x}\right)\Big|_{-1}^{t} + \lim_{s\to 0^+} \left(-\frac{1}{x}\right)\Big|_{s}^{1}$$

$$= \lim_{t\to 0^-} \Big(-\underbrace{\frac{1}{t}}_{-\infty} - -\frac{1}{-1}\Big) + \lim_{s\to 0^+} \Big(-\frac{1}{1} - -\underbrace{\frac{1}{s}}_{+\infty}\Big)$$

$$= -(-\infty) + (+\infty) = \infty \quad \text{DNE} \therefore \text{divergent}$$

ex: $\displaystyle\int_0^2 x^2 \ln x \, dx$

Restrictions:
$\ln x \Rightarrow x > 0$

$$= \lim_{t \to 0^+} \int_t^2 x^2 \ln x \, dx$$

$$\left[\begin{array}{ll} u = \ln x & dv = x^2\,dx \\ du = \frac{1}{x}\,dx & v = \frac{x^3}{3} \end{array}\right.$$

$$= \lim_{t \to 0^+} \left[\frac{x^3 \ln x}{3} \Big|_t^2 - \int_t^2 \frac{x^2}{3} \, dx \right]$$

$$= \lim_{t \to 0^+} \left[\tfrac{1}{3}\left(8 \ln 2 - t^3 \ln(t)\right) - \tfrac{1}{3}\left(\frac{x^3}{3}\right)\Big|_t^2 \right]$$

$$= \lim_{t \to 0^+} \left[\tfrac{1}{3}\left(8 \ln 2 - \underbrace{t^3}_{0} \cdot \underbrace{\ln(t)}_{-\infty}\right) - \tfrac{1}{9}\left(8 - \underbrace{t^3}_{0}\right) \right]$$

$\circledast$

$$\left\{ \begin{aligned} \circledast \lim_{t \to 0^+} t^3 \ln(t) &= \lim_{t \to 0^+} \frac{\ln(t)}{t^{-3}} \; \frac{\infty}{\infty} \overset{H}{=} \lim_{t \to 0^+} \frac{\frac{1}{t}}{-3t^{-4}} \\ &= \lim_{t \to 0^+} \frac{t^3}{-3} \\ &= \boxed{0} \end{aligned} \right.$$

$$= \tfrac{1}{3}(8 \ln 2 - 0) - \tfrac{1}{9}(8)$$

$\therefore$ convergent.

Seperable Differential Equations

Def'n: A "differential equation" is an equation involving a fcn and some of its derivatives.

ex: $y' = 2x-1$

$\frac{dy}{dx} = 2x-1$

$\int dy = \int (2x-1)\,dx$

$\boxed{y = x^2 - x + C}$

$P(1,0)$

$0 = 1^2 - 1 + C$

$\boxed{C = 0}$

$\therefore\ y = x^2 - x$

ex: show that $y = ae^{-x} + be^{-x}$ satisfies the DE: $y'' = -2y' - y$

$y'' + 2y' + y = 0$

$y' = -a.e^{-x} - be^{-x}$

$y'' = ae^{-x} + be^{-x}$

$\therefore\ (ae^{-x} + be^{-x}) + 2(-ae^{-x} - be^{-x}) + (ae^{-x} + be^{-x}) = 0$

<u>Def'n</u>: A "seperable differential equation" is a DE in which: $\frac{dy}{dx}$ can be factored as a function of x times a function of y.

$\left(\text{ie. } \frac{dy}{dx} = f(x) \cdot g(y) \right)$

if $$\frac{dy}{dx} = (f(x)) \cdot (g(y))$$

$$\frac{dy}{g(y)} = f(x)\,dx$$

integrate → $$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

* The problem of solving DE that satisfy the initial condition:

$$\underset{\text{initial time}}{y(t_0)} = \underset{\text{initial value}}{y_0}$$

(ie. the curve $y(t)$ passes through the point $P(t_0, y_0)$)

is called an "initial value problem".

<u>Note</u>: → Find C
→ Is fcn $+$ or $-$?
(you have a y value so you know if it is $+$ or $-$)

ex: (a) Solve the DE

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

$$\frac{dy}{dx} = (x^2)\cdot\left(\frac{1}{y^2}\right)$$

$$\int y^2\,dy = \int x^2\,dx$$

$$\frac{y^3}{3} = \frac{x^3}{3} + C$$

This C keeps its value

$$y^3 = x^3 + 3C$$

$$y = \sqrt[3]{x^3 + 3C}$$

(b) Find the sol'n that satisfies the initial condition: $y(0) = 2$ $\therefore$ $P(0,2)$

use: $y^3 = x^3 + 3C$

sub P

$$(2)^3 = (0)^3 + 3C$$

$$\boxed{8 = 3C}$$

$$\boxed{\therefore\ y = \sqrt[3]{x^3 + 8}}$$

ex: Solve DE

$$\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}$$

$$\int (2y + \cos y)\,dy = \int 6x^2\,dx$$

$$y^2 + \sin y = 6\frac{x^3}{3} + C$$

$$\boxed{y^2 + \sin y = 2x^3 + C}$$

implicit equation

ex: Solve the DE:

(i) $y' = x^2 y$

$$\frac{dy}{dx} = x^2 y$$

$$\int \frac{1}{y} \cdot dy = \int x^2 \, dx$$

$$\ln|y| = \frac{x^3}{3} + C$$

isolate y

$$e^{(x^3/3 + C)} = |y|$$

$$\boxed{y = \pm e^{(x^3/3 + C)}}$$

(ii) $\frac{dL}{dt} = K L^2 \ln(t)$ $\qquad L(1) = -1$

$\therefore P(1, -1)$

$$\int \frac{1}{L^2} \, dL = \int K \ln(t) \, dt$$

$$-L^{-1} = K \cdot (t\ln(t) - t) + C$$

$$\therefore L = -\frac{1}{K(t\ln(t) - t) + C}$$

use $-\frac{1}{L} = K \cdot t(\ln(t) - 1) + C$

sub $P(1, -1)$

$$\frac{-1}{-1} = K \cdot (1)(\cancelto{0}{\ln(1)} - 1) + C$$

$$\boxed{1 + K = C} \qquad \therefore L = -\frac{1}{K(t\ln(t) - t) + (1+K)}$$

ex: Solve the DE

(i) $(y^2 + xy^2)\, y' = 1$

$$y^2(1+x)\,\frac{dy}{dx} = 1$$

$$\int y^2\, dy = \int \frac{1}{1+x}\, dx$$

$$\frac{y^3}{3} = \ln|1+x| + C$$

$$\therefore \quad y = \sqrt[3]{3\ln|1+x| + 3C}$$

(ii) $\dfrac{dp}{dt} = t^2 p - p + t^2 - 1$

$$\frac{dp}{dt} = p(t^2 - 1) + (t^2 - 1)$$

$$\frac{dp}{dt} = (t^2 - 1)(p + 1)$$

$$\int \frac{1}{p+1}\, dp = \int (t^2 - 1)\, dt$$

$$\ln|p+1| = \frac{t^3}{3} - t + C$$

$$|p+1| = e^{(t^3/3 - t + C)}$$

$$\therefore \quad p = \pm\, e^{(t^3/3 - t + C)} - 1$$

(iii) $\frac{dy}{dx} + e^{y+x} = 0$

$$\frac{dy}{dx} = -e^{y+x}$$

$$\frac{dy}{dx} = -e^{y} \cdot e^{x}$$

$$\int e^{-y}\,dy = -\int e^{x}\,dx$$

$$-e^{-y} = -e^{x} + C$$

$$e^{-y} = e^{x} - C$$

$$-y = \ln(e^{x} - C)$$

$$y = -\ln(e^{x} - C)$$

(iv) $\frac{dy}{dx} = \frac{\ln x}{xy}$ $\quad$ $y(1) = 2 \Rightarrow P(1,2)$

$$\int y\,dy = \int \frac{\ln x}{x}\,dx$$

$u = \ln x$
$du = \frac{1}{x}\,dx$

$$\frac{y^2}{2} = \int u\,du$$

$$\frac{y^2}{2} = \frac{u^2}{2} + C$$

$$y^2 = (\ln x)^2 + 2C$$

$$y = \pm\sqrt{(\ln x)^2 + 2C}$$

C-value
use: $y^2 = (\ln x)^2 + 2C$

$$(2)^2 = (\cancel{\ln(1)})^2 + 2C$$

$$\boxed{4 = 2C}$$

$\therefore\ y = \pm\sqrt{(\ln(x))^2 + 2C}$

pick pos. value bc $P(1,2)$ $\therefore$ "y" positive

$\therefore\ y = \sqrt{(\ln x)^2 + 4}$

ex. Solve the DE:

(i) $\dfrac{du}{dt} = \dfrac{2t + \sec^2 t}{2u}$ $\quad u(0) = -5$

$\int 2u\,du = \int (2t + \sec^2 t)\,dt$

$u^2 = t^2 + \tan(t) + C$

$u = \pm\sqrt{t^2 + \tan(t) + C}$ $\Rightarrow$ pick neg. value bc initial condition

use: $u^2 = t^2 + \tan(t) + C$

$(-5)^2 = (0)^2 + \tan(0) + C$

$25 = 0 + 0 + C$

$\boxed{25 = C}$

$\therefore u = -\sqrt{t^2 + \tan(t) + 25}$

(ii) $\dfrac{dy}{dx} = \sqrt{xy}$ $\quad y(1) = 2$

$\int y^{-1/2}\,dy = \int x^{1/2}\,dx$

$y^{1/2} \cdot \left(\frac{2}{1}\right) = x^{3/2} \cdot \left(\frac{2}{3}\right) + C$

$\sqrt{y} = \frac{1}{3}x^{3/2} + \frac{C}{2}$

$y = \left(\frac{1}{3}x^{3/2} + \frac{C}{2}\right)^2$

C-value:

use: $\sqrt{y} = \frac{1}{3}x^{3/2} + \frac{C}{2}$

$\sqrt{2} = \frac{1}{3} \cdot (1)^{3/2} + \frac{C}{2}$

$\boxed{\sqrt{2} - \frac{1}{3} = \frac{C}{2}}$

$\therefore y = \left(\frac{1}{3}x^{3/2} + \sqrt{2} - \frac{1}{3}\right)^2$

ex. Solve the DE:

$$y' \tan x = a + y \qquad y\left(\frac{\pi}{3}\right) = a$$

$$\frac{dy}{dx} \tan x = a + y \qquad x \in (0, \pi/2)$$

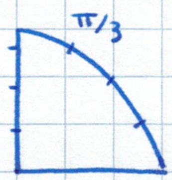
$$\int \frac{1}{a+y}\, dy = \int \frac{1}{\tan x}\, dx$$

$$\ln|a+y| = \int \frac{\cos x}{\sin x}\, dx \qquad \begin{aligned} u &= \sin x \\ du &= \cos x\, dx \end{aligned}$$

$$\ln|a+y| = \int \frac{1}{u}\, du$$

$$\ln|a+y| = \ln|\sin x| + C$$

<u>C-value</u> : sub $P\left(\frac{\pi}{3}, a\right)$



$$\ln|a+a| = \ln|\sin \pi/3| + C$$

$$\ln|2a| = \ln|\sqrt{3}/2| + C$$

$$\boxed{\ln|2a| - \ln|\sqrt{3}/2| = C}$$

$$\therefore \ln|a+y| = \ln|\sin x| + C$$

$$|a+y| = e^{\ln|\sin x| + C}$$

$$a + y = \pm e^{\ln|\sin x| + C}$$

$$\therefore y = \pm e^{\ln|\sin x| + \ln|2a| - \ln|\sqrt{3}/2|} - a$$

Sequences and Series

Sequences

$$\{a_n\} = \{a_1, a_2, a_3, a_4, \ldots, a_n\} = \{a_i\}_{i=1}^{n}$$

General form $\Rightarrow$ a_n

ex: Find the general form:

(i) $\{\overset{n=1}{\tfrac{1}{2}}, \overset{2}{\tfrac{2}{3}}, \overset{3}{\tfrac{3}{4}}, \overset{4}{\tfrac{4}{5}}, \ldots\} \Rightarrow a_n = \frac{n}{n+1}$

(ii) $\{\overset{1}{\tfrac{3}{5}}, -\overset{2}{\tfrac{4}{25}}, \overset{3}{\tfrac{5}{125}}, -\overset{4}{\tfrac{6}{625}}, \ldots\} \Rightarrow a_n = \left(\frac{n+2}{5^n}\right) \cdot \underbrace{(-1)^{n+1}}_{\text{to account for alternating } +/-}$

skip 2 ∴ 2n

(iii) $\{1, \tfrac{1}{3}, \tfrac{1}{5}, \tfrac{1}{7}, \tfrac{1}{9}, \ldots\} \Rightarrow a_n = \frac{1}{2n-1}$

(iv) $\{\overset{n=1}{3}, \overset{2}{2}, \overset{3}{\tfrac{4}{3}}, \overset{4}{\tfrac{8}{9}}, \overset{5}{\tfrac{16}{27}}, \ldots\} \Rightarrow a_n = \frac{2^{(n-1)}}{3^{(n-2)}}$

Defn: A sequence $\{a_n\}$ is said to:

(i) "converge" if $\lim_{n\to\infty} a_n$ exists

(ii) "diverge" if $\lim_{n\to\infty} a_n$ DNE

Note: A sequence has a restricted domain:

$n =$ counted numbers : $\{1, 2, 3, 4, 5, \ldots\}$

ex: (a) $a_n = \ln(1+n^2) - \frac{1}{2}\ln(n^3+4n)$

$$\lim_{n\to\infty} \left[\overset{\infty}{\ln(1+n^2)} \overset{-}{-} \overset{\infty}{\tfrac{1}{2}\ln(n^3+4n)} \right]$$

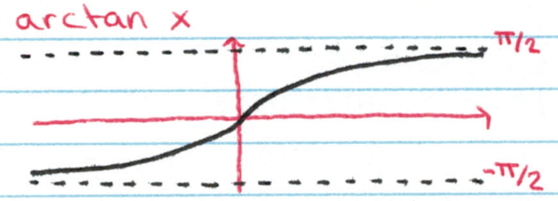
$$= \lim_{n\to\infty} \ln\left(\frac{1+n^2}{\sqrt{n^3+4n}} \right)$$

$$\frac{\deg(2)}{\deg(3/2)} \Rightarrow +\infty$$

$$= \ln(+\infty) = +\infty \qquad \therefore \text{ divergent}$$

(b) $a_n = e^{2n/(n+2)} \cdot \tan^{-1}\left(\frac{n}{2}\right)$

Recall: $y = \arctan x$



$$\lim_{n\to\infty} e^{2n/(n+2)} \cdot \underbrace{\tan^{-1}\left(\tfrac{n}{2}\right)}_{\pi/2}$$

$$⊛ \lim_{n\to\infty} \frac{2n}{n+2} = \frac{2}{1} \quad \frac{\deg(1)}{\deg(1)}$$

$$= e^2 \cdot \pi/2 \qquad \therefore \text{ convergent}$$

(c) $a_n = \dfrac{(2n-1)!}{(2n+1)!}$

Factorials: $n! = (n)(n-1)(n-2) \ldots (3)(2)(1)$
$= (n)(n-1)(n-2)!$

$$\lim_{n\to\infty} \frac{\cancel{(2n-1)!}}{(2n+1)\cdot(2n)\cdot\cancel{(2n-1)!}}$$

$$= \lim_{n\to\infty} \frac{1}{2n(2n+1)} = \frac{1}{\infty} = 0 \qquad \therefore \text{ convergent}$$

(d) $a_n = \dfrac{3\sin(2n)}{2+\sqrt{n}}$ (squeeze)

$$-1\left(\frac{3}{2+\sqrt{n}}\right) \leq \frac{3\sin(2n)}{2+\sqrt{n}} \leq 1\left(\frac{3}{2+\sqrt{n}}\right)$$

$$\left.\begin{array}{l} \lim\limits_{n\to\infty} -\dfrac{3}{2+\sqrt{n}} = -\dfrac{1}{\infty} = 0 \\ \lim\limits_{n\to\infty} \dfrac{3}{2+\sqrt{n}} = \dfrac{1}{\infty} = 0 \end{array}\right\} \therefore \lim_{n\to\infty} \frac{3\sin(2n)}{2+\sqrt{n}} = 0$$

$\therefore$ convergent.

(e) $a_n = \sqrt[n]{2^{1+3n}}$

$$\lim_{n\to\infty} 2^{\left(\frac{1+3n}{n}\right)\circledast}$$

$$= 2^3 = 8$$

$\circledast \lim\limits_{n\to\infty} \dfrac{3n+1}{n} = 3$

$\therefore$ convergent

(f) $a_n = \dfrac{(\ln(n))^2}{n}$

$$\lim_{n\to\infty} \frac{(\ln(n))^2}{n} \frac{\infty}{\infty} \overset{H}{=} \lim_{n\to\infty} \frac{2\cdot\ln(n)\cdot{}^1/n}{1}$$

$$= \lim_{n\to\infty} \frac{2\ln(n)}{n}$$

$$\overset{H}{=} \lim_{n\to\infty} \frac{2\cdot(^1/n)}{1}$$

$$= \frac{2}{\infty} = 0$$

$\therefore$ convergent

For what values of r does $\{r^n\}$ converge

(ie. when does $\lim_{n\to\infty} r^n$ exist)

if...

$\boxed{r>1}$ $\lim_{n\to\infty}(r^n) = +\infty$ DNE $\therefore$ div.

• $\boxed{r=1}$ $\lim_{n\to\infty}(1)^n = 1$ $\therefore$ cgt.

• $\boxed{-1<r<1}$ $\lim_{n\to\infty}(r^n) = 0$ $\therefore$ cgt.

ex: $\lim_{n\to\infty}\left(\frac{1}{4}\right)^n = 0$

$\boxed{r=-1}$ $\lim_{n\to\infty}(-1)^n$ DNE $\therefore$ div.

$\{-1, 1, -1, 1, -1, 1, \ldots\}$

$\boxed{r<-1}$ $\lim_{n\to\infty}(r^n)$ DNE $\therefore$ div.

$\therefore a_n = r^n$ is:

cgt. if $\boxed{-1<r\leq 1}$

and div. elsewhere.

Ex: determine cgt. or div. of the sequences:

(a) $a_n = n^2 e^{-n}$

$$\lim_{n\to\infty} \underset{\infty}{n^2} \cdot \underset{0}{e^{-n}} = \lim_{n\to\infty} \frac{n^2}{e^n} \quad \frac{\infty}{\infty}$$

$$\overset{(H)}{=} \lim_{n\to\infty} \frac{2n}{e^n}$$

$$\overset{(H)}{=} \lim_{n\to\infty} \frac{2}{e^n} = \frac{1}{\infty} = 0 \qquad \therefore \text{cgt}$$

(b) $a_n = 1 + 2\left(\frac{4}{5}\right)^n$

$$\lim_{n\to\infty} 1 + 2\underbrace{\left(\frac{4}{5}\right)}_{r<1}^n = 1 \qquad \therefore \text{cgt.}$$

$r^n = 0$

(c) $a_n = \dfrac{2\sin(n^2)}{n + \ln(n)}$

$$\lim_{n\to\infty} \frac{2\sin(n^2)}{n+\ln(n)} \qquad \text{(squeeze)}$$

$$-\frac{2}{n+\ln(n)} \le \frac{2\sin(n^2)}{n+\ln(2)} \le \frac{2}{n+\ln(n)}$$

$$\left.\begin{aligned} &\lim_{n\to\infty} -\frac{2}{n+\ln(n)} = -\frac{2}{\infty} = 0 \\ &\lim_{n\to\infty} \frac{2}{n+\ln(n)} = \frac{2}{\infty} = 0 \end{aligned}\right\} \therefore \lim_{n\to\infty} \frac{2\sin(n^2)}{n+\ln(n)} = 0$$

$\therefore$ cgt.

(d) $a_n = \left(1 + \frac{2}{n}\right)^n$

$$\lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^n \quad 1^\infty$$

$$= \lim_{n\to\infty} e^{\ln(1+2/n)^n}$$

$$= \lim_{n\to\infty} e^{n \cdot \ln(1+2/n)} \circledast$$

$$= e^2$$

$\therefore$ cgt.

$$\circledast \lim_{n\to\infty} \underbrace{n}_{\infty} \cdot \underbrace{\ln(1+2/n)}_{0}$$

$$= \lim_{n\to\infty} \frac{\ln(1+2/n)}{n^{-1}}$$

$$\stackrel{H}{=} \lim_{n\to\infty} \frac{-2 \cdot \cancel{n^{-2}} \cdot \frac{1}{1+2/n}}{-\cancel{n^{-2}}}$$

$$= \lim_{n\to\infty} \frac{2}{1 + \underbrace{2/n}_{0}} = 2$$

Note: Limit definition of e

$$\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a$$

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

Infinite Series

Consider an infinite sum:

$$\boxed{\sum_{n=1}^{\infty} a_n} = \lim_{m\to\infty} \sum_{n=1}^{m} a_n$$

This is called an infinite series

$$\sum_{n=1}^{\infty} a_n \approx \sum_{n=1}^{m} a_n = S_m$$

(partial) sum

increase the accuracy by using larger number of terms (m)

Def'n: The series $\sum_{n=1}^{\infty} a_n$ if $\sum_{n=1}^{m} a_n = S_m$

(i) "converges" if $\lim_{m\to\infty} S_m$ exists

(ii) "diverges" if $\lim_{m\to\infty} S_m$ DNE

Consider:

$$\{a_n\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right\} = \left\{\frac{1}{2^n}\right\}$$

↳ sequence

→ What happens if we add up all values?

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 1$$

↳ series

travel: - half of length (1)
- half of what is left
- half of what is left

...

As $n \to \infty$: approach 1
But never pass 1 $\therefore$ cgt.

consider: $\sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right) \approx \sum_{n=1}^{m} a_n = S_m$

$$\left.\begin{array}{llll} m=1 & S_1 = \frac{1}{2} & & = \frac{1}{2} \\ m=2 & S_2 = \frac{1}{2} + \frac{1}{4} & & = \frac{3}{4} \\ m=3 & S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} & & = \frac{7}{8} \\ m=4 & S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} & & = \frac{15}{16} \end{array}\right\} \text{Pattern } S_m = \frac{2^m - 1}{2^m}$$

$$\therefore \boxed{S_m = \frac{2^m - 1}{2^m}}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{m\to\infty} S_m = \lim_{m\to\infty} \frac{2^m - 1}{2^m}$$

$$\lim_{m\to\infty} \frac{2^m - 1}{2^m} \overset{H}{=} \lim_{m\to\infty} \frac{2^m \ln(2)}{2^m \ln(2)} = \boxed{1}$$

Geometric Series

The geometric series (for $a \neq 0$):

$$\boxed{\sum_{n=1}^{\infty} ar^{n-1}} \quad \text{is} \quad \begin{cases} \text{cgt.} & \text{if} \quad |r| < 1 \\ \text{div.} & \text{if} \quad |r| \geq 1 \end{cases}$$

Also, if cgt. then:

$$\boxed{\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}} = S = \lim_{m \to \infty} S_m$$

<u>Note</u>: convert into this specific form

consider:
$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots$$

r is common ratio as we go from one term to the next. (ie. divide one number by the previous one)

a is the first term being added

* Change counter to $n=1$
→ if decrease counter increase inside
→ if increase counter decrease inside

Q. For what values of r will geometric series converge ?

if: $\boxed{r=1}$ $\sum_{n=1}^{\infty} a(1)^{n-1} = a+a+a+a+\ldots$ div.

$$S_m = m \cdot a$$

$\boxed{r=-1}$ $\sum_{n=1}^{\infty} a(-1)^{n-1} = a-a+a-a+a\ldots$

$$\left.\begin{aligned} S_1 &= a \\ S_2 &= 0 \\ S_3 &= a \\ S_4 &= 0 \end{aligned}\right\} \lim_{m\to\infty} S_m \text{ DNE}$$ div.

$\boxed{r>1}$
$\boxed{r<-1}$ $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots = \pm\infty$

div.

$\boxed{0<r<1}$ $\sum_{n=1}^{\infty} ar^{n-1}$

$$S_m = a + ar + ar^2 + \ldots + ar^{m-2} + ar^{m-1}$$
$$r \cdot S_m = ar + ar^2 + ar^3 + \ldots + ar^{m-1} + ar^m$$

consider: $S_m - rS_m = a - ar^m$

$$S_m(1-r) = a - ar^m$$

$$\boxed{S_m = \frac{a-ar^m}{1-r}}$$

evaluate: $\lim_{m\to\infty} S_m$

$$= \lim_{m\to\infty} \frac{a(1-\overset{\to 0}{r^m})}{1-r}$$

Recall:
$|r|<1 \therefore \lim_{m\to\infty} r^m = 0$
$|r|>1 \therefore \lim_{m\to\infty} r^m$ DNE

$$\therefore \boxed{\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}}$$ cgt.

ex: convergent or divergent ?

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n}$

put in form $\sum_{n=1}^{\infty} a r^{n-1}$

$= \sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{1}{2^{n-1}}$ $\quad r = 1/2 \Rightarrow |r| < 1 \therefore$ cgt $\quad$ cgt.

$a = 1/2$

$= \frac{a}{1-r} = \frac{1/2}{-1/2+1} = \boxed{1}$

(b) $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$

$= \sum_{n=1}^{\infty} \frac{3}{3^n} \cdot 2^{2n}$

$= \sum_{n=1}^{\infty} 3 \cdot \left(\frac{4}{3}\right)^n$

$= \sum_{n=1}^{\infty} 3 \cdot \frac{4}{3} \cdot \left(\frac{4}{3}\right)^{n-1}$ $\quad |r| = 4/3 > 1 \quad \therefore$ div.

(c) $\sum_{n=1}^{\infty} 6(0.9)^{n-1}$ $\quad r = 0.9 \Rightarrow |r| < 1 \quad \therefore$ cgt.

$a = 6$

$S = \frac{a}{1-r} = \frac{6}{1-0.9} = 60$

(d) $\sum_{n=3}^{\infty} \frac{10^n}{9^{n-1}}$

change counter

$= \sum_{n=1}^{\infty} \frac{10^{n+2}}{9^{n+1}} = \sum_{n=1}^{\infty} \frac{10^3}{9^2} \left(\frac{10}{9}\right)^{n-1}$ $\quad |r| = \frac{10}{9} > 1 \quad \therefore$ div

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\sqrt{2})^n}$

$$= \sum_{n=1}^{\infty} \frac{1}{\sqrt{2}} \cdot \left(\frac{1}{\sqrt{2}}\right)^{n-1} \qquad |r| = 1/\sqrt{2} < 1 \quad \therefore \quad \text{cgt.}$$

$$a = 1/\sqrt{2}$$

$$S = \frac{a}{1-r} = \frac{1/\sqrt{2}}{1 - 1/\sqrt{2}}$$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{\pi^n}{3^{n+1}}$

$$= \sum_{n=1}^{\infty} \frac{\pi}{3^2}\left(\frac{\pi}{3}\right)^{n-1} \qquad |r| = \pi/3 > 1 \quad \therefore \quad \text{div.}$$

ex: express the number $1.53\overline{42}$ as a ratio of 2 integers

$$1.53\overline{42} = 1.53 + 0.0042 + 0.000042 + \ldots$$

$$= \frac{153}{100} + \underbrace{\left(\frac{42}{10^4} + \frac{42}{10^6} + \frac{42}{10^8} + \ldots\right)}$$

$r = \frac{1}{100}$ ← divide any term by previous one

$a = \frac{42}{10^4}$ ← first term

$$S = \frac{a}{1-r} =$$

$$= \frac{42 \cdot 10^{-4}}{1 - 10^{-2}}$$

$$= \frac{7}{1650}$$

$$\therefore 1.53\overline{42} = \frac{153}{100} + \frac{7}{1650}$$

$$= \frac{15189}{9900}$$

$\boxed{\text{Telescoping Series}}$

Form: $\sum_{n=1}^{\infty} (a_n - a_{n+1})$

ex: Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is cgt

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \Rightarrow 1 = A(n+1) + B(n)$$

$\boxed{n=0} \Rightarrow A = 1$

$\boxed{n=-1} \Rightarrow B = -1$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$S_m = \underbrace{\left(1 - \cancel{\tfrac{1}{2}}\right)}_{n=1} + \underbrace{\left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right)}_{n=2} + \underbrace{\left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right)}_{n=3} + \dots + \underbrace{\left(\cancel{\tfrac{1}{m-1}} + \cancel{\tfrac{1}{m}}\right)}_{n=m-1} + \underbrace{\left(\cancel{\tfrac{1}{m}} - \tfrac{1}{m+1}\right)}_{n=m}$$

$$\therefore \boxed{S_m = 1 - \frac{1}{m+1}}$$

$$\therefore \lim_{m\to\infty} S_m = \lim_{m\to\infty} \left(1 - \frac{1}{m+1}\right) = 1$$

$\therefore \sum a_n$ cgt.

$\sum a_n = 1$

ex: $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+4}\right)$

$$S_m = \left(1 - \cancel{\tfrac{1}{5}}\right) + \left(\tfrac{1}{2} - \cancel{\tfrac{1}{6}}\right) + \left(\tfrac{1}{3} - \cancel{\tfrac{1}{7}}\right) + \left(\tfrac{1}{4} - \cancel{\tfrac{1}{8}}\right) + \left(\cancel{\tfrac{1}{5}} - \cancel{\tfrac{1}{9}}\right) +$$
$$\dots + \left(\cancel{\tfrac{1}{m-3}} - \tfrac{1}{m+1}\right) + \left(\cancel{\tfrac{1}{m-2}} - \tfrac{1}{m+2}\right) + \left(\cancel{\tfrac{1}{m-1}} - \tfrac{1}{m+3}\right) + \left(\cancel{\tfrac{1}{m}} - \tfrac{1}{m+4}\right)$$

$$\therefore S_m = 1 + 1/2 + 1/3 + 1/4$$
$$- \frac{1}{m+1} - \frac{1}{m+2} - \frac{1}{m+3} - \frac{1}{m+4}$$

ex: $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} \left[\ln(n) - \ln(n+1)\right]$

$$S_m = \underbrace{(\ln(1) - \cancel{\ln(2)})}_{n=1} + \underbrace{(\cancel{\ln(2)} - \cancel{\ln(3)})}_{n=2} + \dots + \underbrace{(\cancel{\ln(m)} - \ln(m+1))}_{n=m}$$

$$S_m = \ln(1) - \ln(m+1)$$

$$\therefore \lim_{m\to\infty} S_m = \lim_{m\to\infty} \left(\cancelto{0}{\ln(1)} - \ln(\underset{\infty}{m}+1)\right) = -\infty$$

$\therefore$ div.

n^{th} term test

Theorem: if $\sum_{n=1}^{\infty} a_n$ is cgt. then

$$\lim_{n\to\infty} a_n = 0$$

$\therefore$ Test for divergence

If $\lim_{n\to\infty} a_n \neq 0$ or DNE

then $\sum a_n$ is divergent

Note: the test can only confirm divergence

if $\lim_{n\to\infty} a_n = 0$ $\therefore$ test inconclusive

$\rightarrow$ can be div./cgt.

Note: Rates of growth: $x^x > x! > a^x > x^a$

ex: does $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ converge or diverge

$\lim_{n\to\infty} \frac{n^2}{5n^2+4}$ $\frac{\deg(2)}{\deg(2)}$ $= \boxed{\frac{1}{5}} \neq 0$ $\therefore \sum a_n$ cgt. by n^{th}

The Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$

Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ is div

even though $\lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$

Show that $\lim_{m\to\infty} S_m = +\infty$ DNE

consider only S_{2^m} (subsequence $\Rightarrow$ every term is included in S_m)

$m=1$ $S_2 = 1 + \frac{1}{2}$

$m=2$ $S_4 = 1 + \frac{1}{2} + \overbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}^{>1/2} > 1 + 1/2 + 1/2$

$m=3$ $S_8 = 1 + \frac{1}{2} + \overbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}^{>1/2} + \overbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}^{>1/2} > 1 + 1/2 + 1/2 + 1/2$

$m=4$ $S_{16} = 1 + \frac{1}{2} + \overbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}^{>1/2} + \overbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}^{>1/2}$
$+ \overbrace{\left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right)}^{>1/2} > 1 + 1/2 + 1/2 + 1/2 + 1/2$

$\therefore S_{2^m} > 1 + m\left(\frac{1}{2}\right)$ it keeps growing by $\frac{1}{2}$

$\therefore \lim_{m\to\infty} S_{2^m} > \lim_{m\to\infty} 1 + m(1/2) = +\infty$

$\therefore \lim_{m\to\infty} S_{2^m} = +\infty$

$\therefore \lim_{m\to\infty} S_m = +\infty$ $\therefore \sum \frac{1}{n}$ is div.

Integral Test

Suppose $f(x)$ is: ① continuous
② positive
③ decreasing on $[1, \infty)$

Let $a_n = f(n)$, then

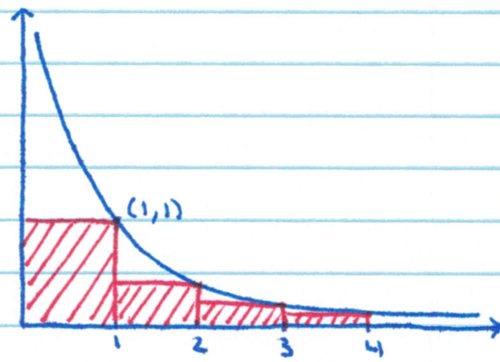
$$\boxed{\sum a_n \approx \int_1^\infty f(x)\,dx}$$

→ behave in similar way
→ converge together.

consider: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$

consider: $f(x) = \frac{1}{x^2}$

and $\int_1^\infty \frac{1}{x^2}\,dx = 1$



$$\begin{pmatrix}\text{sum of}\\ \text{rectangles}\end{pmatrix} = \overset{\Delta x \,\cdot\, f(x)}{(1)(1)} + (1)(1/4) + (1)(1/9) + (1)(1/16) + \dots$$

$$= 1 + \underbrace{1/4 + 1/9 + 1/16 + \dots}$$

remove bc $\int_1^\infty f(x)dx$

$$1/4 + 1/9 + 1/16 + \dots < \int_1^\infty \frac{1}{x^2}\,dx = 1$$

$\therefore\ 1/4 + 1/9 + 1/16 + \dots < 1$

$\therefore\ 1 + 1/4 + 1/9 + 1/16 + \dots < 2$ $\quad \therefore \sum a_n$ cgt.

ex: Test for convergence vs divergence

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$f(x) = \frac{1}{x^2+1}$

① continuous
② pos
③ decreasing $[1,\infty)$

$$\int_1^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty}\left(\tan^{-1}(x)\right)\Big|_1^t$$

$$= \lim_{t\to\infty}\left[\tan^{-1}(t) - \tan^{-1}(1)\right]$$

$$= \pi/2 - \pi/4 = \pi/4 \quad \therefore \text{cgt}$$

$\therefore \sum a_n$ cgt.

(b) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

$f(x) = \frac{n^2}{e^{n^3}}$

$\left\{\begin{array}{l}\text{continuous}\\ \text{positive}\\ \text{decreasing}\end{array}\right.$

$$\int_1^\infty \frac{x^2}{e^{x^3}}\,dx = \lim_{t\to\infty}\int_1^t \frac{x^2}{e^{x^3}}\,dx$$

$u = x^3$
$du = 3x^2\,dx$
$\frac{1}{3}\,du = x^2\,dx$

$x = 1,\ u = 1$
$x = t,\ u = t^3$

$$= \lim_{t\to\infty}\frac{1}{3}\int_1^{t^3}\frac{1}{e^u}\,du$$

$$= \lim_{t\to\infty}\frac{1}{3}\left(-\frac{1}{e^u}\right)\Big|_1^{t^3}$$

$$= \lim_{t\to\infty} -\frac{1}{3}\left(\cancelto{0}{\frac{1}{e^{t^3}}} - \frac{1}{e}\right) = \frac{1}{3e} \quad \therefore \text{cgt}$$

$=$

$\therefore \sum a_n$ cgt.

$\boxed{\text{P - Series}}$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p}} \quad \text{is} \quad \begin{cases} \text{cgt} & \text{if} \quad p>1 \\ \text{div.} & \text{if} \quad p \le 1 \end{cases}$$

The value of p affects how fast the fcn goes to zero:
- very big P ⇒ fast ⇒ cgt
- very small P ⇒ slow ⇒ div.

Q. For what values of P is the series cgt?

$\boxed{P<0}$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^p}$ $\quad$ n^{th} term $\lim_{n\to\infty} \frac{1}{n^p} = +\infty$ $\quad \therefore$ div

$\boxed{P=1}$ $\quad \sum_{n=1}^{\infty} \frac{1}{n}$ $\quad$ harmonic series $\quad \therefore$ div

$\boxed{P>0}$ $\quad f(x) = \frac{1}{x^p}$: continuous, decreasing, positive

$$\int_1^{\infty} x^{-p}\,dx = \lim_{t\to\infty} \left(\frac{x^{-p+1}}{1-p}\right)\Bigg|_1^t$$

$$= \lim_{t\to\infty} \left(t^{1-p}\right)\left(\frac{1}{1-p}\right) - \left(\frac{1}{1-p}\right)$$

(*)

if $0<P<1$ $\quad 1-p>0 \quad \therefore \lim_{t\to\infty} t^{1-p} = +\infty$

(*) DNE $\therefore$ div.

$p>1$ $\quad 1-p<0 \quad \therefore \lim_{t\to\infty} t^{1-p} = 0$

(*) exists $\therefore$ cgt

ex: cgt or div ?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$ $\quad p = \sqrt{2} > 1 \quad \therefore$ cgt.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^{e}}$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{e-1}} \qquad e-1 > 1 \qquad \therefore \text{ cgt.}$$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\sqrt{n} + 4}{n^2} \right)$

$$= \underbrace{\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2}}_{p = 3/2} + \underbrace{\sum_{n=1}^{\infty} \frac{4}{n^2}}_{p = 2} \qquad p > 1 \quad \therefore \text{ cgt}$$

Note: can only split up a series when all series are convergent.

Comparaison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms:

(i) if $\sum b_n$ is cgt. and $a_n \leq b_n$

$\therefore \sum a_n$ is cgt

(ii) if $\sum b_n$ is div. and $a_n \geq b_n$

$\therefore \sum a_n$ is div

Note: to find b_n, use dominant term from numerator and denominator

Limit Comparaison Test

if $\lim_{n\to\infty} \frac{a_n}{b_n} \frac{0}{0} = C$ Note: $C > 0$, C exists

then: $\sum a_n \approx \sum b_n$

→ they behave in the same way

→ converge together

ex:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

$\sum b_n = \sum \frac{1}{2^n}$

Geometric series: $|r| = 1/2 < 1$ $\therefore$ cgt

$\dfrac{1}{2^n+1} < \dfrac{1}{2^n}$

larger denom

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1} < \sum_{n=1}^{\infty} \frac{1}{2^n}$

$\therefore \sum \frac{1}{2^n+1}$ also cgt.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2-1}}$

$b_n = \frac{1}{\sqrt{n^2}} = \frac{1}{n}$

$\sum \frac{1}{n}$ is div (Harmonic)

$\dfrac{1}{\sqrt{n^2-1}} > \dfrac{1}{n}$

smaller denom

$\therefore \sum \frac{1}{\sqrt{n^2-1}}$ is also div.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{\underbrace{n^5}_{\text{dominant}}+n^2}}$

$b_n = \frac{1}{\sqrt{n^5}}$

P-series: $P = 5/2 > 1$ $\therefore$ cgt

$\dfrac{1}{\sqrt{n^5+n^2}} < \dfrac{1}{\sqrt{n^5}}$

$\therefore \sum \frac{1}{\sqrt{n^5+n^2}}$ is also cgt.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

$b_n = \frac{1}{2^n}$

Geometric: $|r| = 1/2 < 1$ $\therefore$ cgt

$\dfrac{1}{2^n-1} > \dfrac{1}{2^n}$

smaller denom

$\therefore$ Test inconclusive
(cannot use comparaison test)

ex: Limit comparaison

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

$b_n = \frac{1}{2^n}$

Geometric series: $|r| = 1/2 < 1 \therefore$ cgt

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} \frac{\infty}{\infty} \overset{H}{=} \lim_{n\to\infty} \frac{2^n \ln(2)}{2^n \cdot \ln(2)} = \boxed{1}$$

$\therefore \sum a_n \approx \sum b_n$

$\therefore \sum a_n$ is also cgt.

(b) $\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{n^5 + 5}}$

$b_n = \frac{n^2}{\sqrt{n^5}} = \frac{1}{n^{1/2}}$

P-series: $p = 1/2 < 1 \therefore$ div.

$$\lim_{n\to\infty} \frac{(2n^2 + 3n)}{\sqrt{n^5 + 5}} \cdot \frac{\sqrt{n^5}}{n^2} \quad \frac{\deg(5/2)}{\deg(5/2)} = \frac{2}{1} = \boxed{2}$$

$\therefore \sum a_n \approx \sum b_n$

$\therefore \sum a_n$ also div.

(c) $\sum_{n=1}^{\infty} \frac{(1 + \frac{1}{n})^2}{e^n}$

$b_n = \frac{1}{e^n}$

Geometric: $|r| = 1/e < 1 \quad \therefore$ cgt

$$\lim_{n\to\infty} \frac{(1 + \frac{1}{n})^2}{\cancel{e^n}} \cdot \left(\frac{\cancel{e^n}}{1}\right) = \boxed{1}$$

$\therefore \sum a_n \approx \sum b_n$

$\therefore \sum a_n$ also cgt.

(d) $\sum_{n=1}^{\infty} \left(\frac{n + 4^n}{n^2 + 6^n}\right)$

$b_n = \frac{4^n}{6^n} = \left(\frac{4}{6}\right)^n$

Geometric: $|r| = 2/3 < 1 \quad \therefore$ cgt

$$\lim_{n\to\infty} \left(\frac{4^n + n}{6^n + n^2}\right) \cdot \frac{6^n}{4^n} = \lim_{n\to\infty} \frac{\cancel{4^n}(1 + n/4^n)}{\cancel{6^n}(1 + n^2/6^n)} \cdot \frac{\cancel{6^n}}{\cancel{4^n}} = \boxed{1}$$

$\therefore \sum a_n \approx \sum b_n$

$\therefore \sum a_n$ is also cgt.

Ratio Test

comparing one term to the term that comes after

Let $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

(i) if $L < 1$ $\therefore$ $\sum a_n$ cgt.

(ii) if $L > 1$ $\therefore$ $\sum a_n$ div.

(iii) if $L = 1$ $\therefore$ inconclusive

ex: $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

pair off terms

$$\lim_{n\to\infty} \left| \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} \right| = \lim_{n\to\infty} \left| \underbrace{\frac{(n+1)^3}{n^3}}_{1} \cdot \frac{\cancel{3^n}}{3^{\cancel{n}+1}} \right|$$

$$= 1 \cdot \frac{1}{3} = \boxed{\frac{1}{3}} \quad \therefore \text{ cgt.}$$

ex:

(a) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\lim_{n\to\infty} \frac{(n+1)^{(n+1)}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^{\cancel{(n+1)}}}{\cancel{(n+1)} \cdot \cancel{n!}} \cdot \frac{\cancel{n!}}{n^n}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \qquad 1^\infty$$

$$\lim_{n\to\infty} \left(1+\frac{a}{n}\right)^n = e^a$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \boxed{e}$$

$e > 1 \therefore \sum a_n$ div

(b) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

$$\lim_{n\to\infty} \frac{((n+1)!)^2}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2} = \lim_{n\to\infty} \frac{(2n)!}{(2n+2)!} \cdot \frac{((n+1)\cdot n!)^2}{(n!)^2}$$

$$= \lim_{n\to\infty} \frac{\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}} \cdot \frac{(n+1)^2 \cdot \cancel{(n!)^2}}{\cancel{(n!)^2}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} \qquad \frac{\deg(2)}{\deg(2)}$$

$$= \frac{1}{2\cdot 2} = \boxed{1/4}$$

$1/4 < 1 \therefore \sum a_n$ cgt.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{100^n}$

$$\lim_{n\to\infty} \frac{(n+1)!}{\cancel{100^{n+1}}} \cdot \frac{\cancel{100^n}}{n!} = \lim_{n\to\infty} \frac{(n+1)\,\cancel{n!}}{\cancel{n!}} \cdot \frac{1}{100} = \lim_{n\to\infty} \frac{n+1}{100} = +\infty$$

$\therefore \sum a_n$ div.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n \cdot n!}{5.8.11\cdots(3n+2)}$

$$\lim_{n\to\infty} \frac{2^{\cancel{n+1}} \cdot (n+1)!}{\cancel{5}.\cancel{8}.\cancel{11}\cdots(3(n+1)+2)} \cdot \frac{\cancel{5}.\cancel{8}.\cancel{11}\ldots(3n+2)}{\cancel{2^n} \cdot n!}$$

$$= \lim_{n\to\infty} \frac{2 \cdot (n+1) \cdot \cancel{n!}}{\cancel{n!}} \cdot \frac{\cancel{(3n+2)}}{\cancel{(3n+2)} \cdot (3(n+1)+2)}$$

$$= \lim_{n\to\infty} \frac{2\cdot(n+1)}{3n+5} \quad \frac{\deg(1)}{\deg(1)} \quad = \boxed{\frac{2}{3}} < 1 \quad \therefore \sum a_n \text{ cgt}$$

ex: For which values of $K \in \mathbb{Z}^+$ is the series cgt:

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(Kn)!} \qquad \text{ie. when is } \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$$

$$\lim_{n\to\infty} \frac{((n+1)!)^2}{(K(n+1))!} \cdot \frac{(Kn)!}{(n!)^2}$$

$$= \lim_{n\to\infty} \frac{((n+1)\cdot n!)^2}{(n!)^2} \cdot \frac{(Kn)!}{(Kn+K)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 \cdot \cancel{(n!)^2}}{\cancel{(n!)^2}} \cdot \frac{\cancel{(Kn)!}}{(Kn+K)(Kn+K-1)(Kn+K-2)\cdots(Kn+1)\cdot\cancel{(Kn)!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(Kn+K)(Kn+K-1)\cdots(Kn+1)} \qquad \frac{\deg(2)}{\deg(K)}$$

if $K < 2 \Rightarrow \infty \therefore$ div

if $K > 2 \Rightarrow 0 \therefore$ cgt

if $K = 2 \Rightarrow \displaystyle\lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \boxed{\frac{1}{4}} < 1$

$\therefore$ cgt when $K \geq 2$

$\boxed{\text{Root Test}}$

Let $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$

(i) if $L < 1$ $\therefore$ $\sum a_n$ cgt.

(ii) if $L > 1$ $\therefore$ $\sum a_n$ div.

(iii) if $L = 1$ $\therefore$ inconclusive

ex: $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$

$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \boxed{\frac{2}{3}} < 1$ $\therefore \sum a_n$ is cgt.

ex: $\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^{5n}$

$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n}{n+1}\right)^{5n}} = \lim_{n\to\infty} \left(\frac{2n}{n+1}\right)^5 = \frac{2^5}{1^5} = \boxed{2^5} > 1$ $\therefore$ div.

ex: $\sum_{n=1}^{\infty} \left(1 + \frac{3}{n}\right)^{n^2}$

$\lim_{n\to\infty} \sqrt[n]{\left(1+\frac{3}{n}\right)^{n^2}} = \lim_{n\to\infty} \left(1+\frac{3}{n}\right)^n = \boxed{e^3} > 1$ $\therefore$ div.

HARD CASES

(i) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n!}{n^n}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{n!}{n^n}\right)^n} = \lim_{n\to\infty}\frac{n!}{n^n} = * = \lim_{n\to\infty}\frac{\cancel{n}\cdot\overbrace{(n-1)(n-2)(n-3)\dots(3)(2)\cancel{(1)}}^{n-2\text{ terms}}}{\cancel{n}\cdot\underbrace{(n)(n)(n)\dots(n)(n)(n)}_{n-1\text{ terms}}}$$

Statement

$$\boxed{\overset{<n}{(n-1)}\overset{<n}{(n-2)}\overset{<n}{(n-3)}\dots\overset{<n}{(3)}\overset{<n}{(2)} < n^{n-2}}$$

$$\therefore\ * < \lim_{n\to\infty}\left(\frac{n^{n-2}}{n^{n-1}}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$$

$$\therefore\ * < 0$$

$$\underbrace{0 \leq}_{\text{forsure}} \lim_{n\to\infty}\sqrt[n]{\left(\frac{n!}{n^n}\right)^n} < 0$$

$\therefore \displaystyle\lim_{n\to\infty}\sqrt[n]{\left(\frac{n!}{n^n}\right)^n} = 0$ by squeeze theorem

$\therefore\ 0<1\ \therefore$ cgt.

(ii) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{6^n}{n!}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{6^n}{n!}\right)^n} = \lim_{n\to\infty}\frac{6^n}{n!} \overset{*}{=} \lim_{n\to\infty}\left(\frac{6}{n}\right)\underbrace{\overset{<1}{\left(\frac{6}{n-1}\right)}\overset{<1}{\left(\frac{6}{n-2}\right)}\cdots\overset{<1}{\left(\frac{6}{7}\right)}}_{<1}\underbrace{\left(\frac{6}{6}\right)\left(\frac{6}{5}\right)\left(\frac{6}{4}\right)\left(\frac{6}{3}\right)\left(\frac{6}{2}\right)\left(\frac{6}{1}\right)}_{C}$$

$$* < \lim_{n\to\infty}\frac{6}{n}(1)\cdot C = 0$$

$\underline{0 \leq}\ *\ < 0$ $\quad\therefore \displaystyle\lim_{n\to\infty}\frac{6^n}{n!} = 0$ by squeeze

$\therefore \sum a_n$ cgt.

(iii) $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^n}{n^{4n}}$

$$\lim_{n\to\infty}\sqrt[n]{\frac{(n!)^n}{n^{4n}}} = \lim_{n\to\infty}\frac{n!}{n^4} = \lim_{n\to\infty}\frac{n(n-1)(n-2)(n-3)(n-4)!}{n\cdot n\cdot n\cdot n}$$

$$= \lim_{n\to\infty}\overset{=1}{\left(\frac{n}{n}\right)}\overset{=1}{\left(\frac{n-1}{n}\right)}\overset{=1}{\left(\frac{n-2}{n}\right)}\overset{=1}{\left(\frac{n-3}{n}\right)}\cdot(n-4)!$$

$= +\infty > 1 \quad \therefore \sum a_n$ div ex: cgt or div. ?

(a) $\sum_{n=1}^{\infty} \frac{1}{n+3^n}$

$b_n = \frac{1}{3^n}$

Geometric: $|r| = 1/3 < 1$ $\therefore$ cgt.

$\frac{1}{n+3^n} < \frac{1}{3^n}$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n+3^n}$ is cgt (comparaison)

(b) $\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$

$\lim_{n\to\infty} \sqrt[n]{\frac{(2n+1)^n}{n^{2n}}} = \lim_{n\to\infty} \frac{2n+1}{n^2}$ $\frac{\deg(1)}{\deg(2)} = \boxed{0} < 1$

$\therefore \sum a_n$ cgt (root test)

(c) $\sum_{n=1}^{\infty} \frac{n}{n+2}$

$\lim_{n\to\infty} \frac{n}{n+2}$ $\frac{\deg(1)}{\deg(1)} = \boxed{1} \neq 0$ $\therefore \sum a_n$ div (n^{th} term)

(d) $\sum_{n=1}^{\infty} n^2 e^{-n}$

$\lim_{n\to\infty} \left| \frac{(n+1)^2}{e^{\cancel{n+1}}} \cdot \frac{\cancel{e^n}}{n^2} \right| = \lim_{n\to\infty} \underbrace{\frac{(n+1)^2}{n^2}}_{1} \cdot \frac{1}{e} = \boxed{\frac{1}{e}} < 1$

$\therefore \sum a_n$ cgt (ratio)

(e) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$

$b_n = \frac{1}{n \cdot \sqrt{n^2}} = \frac{1}{n^2}$

P-series: $p = 2 > 1$ $\therefore$ cgt

$\frac{1}{n\sqrt{n^2+1}} < \frac{1}{n^2}$

larger denom

$\therefore \sum a_n$ cgt (comparaison)

(f) $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n^2}\right)$

$\lim_{n\to\infty} \cos\left(\frac{1}{n^2}\right) = \cos(0) = \boxed{1} \neq 0 \quad \therefore \sum a_n$ div (n^{th} term)

(g) $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

$$\lim_{n\to\infty} \frac{(n+1)!}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n!} = \lim_{n\to\infty} \frac{(n+1)\cdot\cancel{n!}}{\cancel{n!}} \cdot \frac{\cancel{e^{n^2}}}{e^{\cancel{n^2}+2n+1}}$$

$$= \lim_{n\to\infty} \frac{(n+1)}{e^{2n+1}} \quad \frac{\infty}{\infty}$$

$$\overset{H}{=} \lim_{n\to\infty} \frac{1}{2\cdot e^{2n+1}} = \boxed{0} < 1$$

$\therefore \sum a_n$ cgt (ratio)

(h) $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+2n^2)^n}$

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^{2n}}{(1+2n^2)^n}} = \lim_{n\to\infty} \frac{n^2}{1+2n^2} \quad \frac{\deg(2)}{\deg(2)} = \boxed{\frac{1}{2}} < 1$$

$\therefore \sum a_n$ cgt (root)

(i) $\sum_{n=1}^{\infty} \frac{n\cdot\ln(n)}{(n+1)^3}$

Power Series

Def'n: A series of form

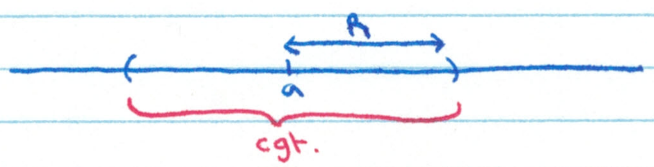
$$\sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + \dots$$

is called a "power series" that is centered at $x=a$.

Note: For some values of x $\sum_{n=1}^{\infty} C_n (x-a)^n$ is cgt. and for others it is div.

3 possibilities:

1. The series converges for all $x \in \mathbb{R}$
2. There is some positive value R s.t the series converges for $|x-a| < R$, and diverges for $|x-a| > R$.
3. The series converges only for $x=a$
   $R=0$ (no radius, only one value)



For the values of x where $\sum_{n=1}^{\infty} c_n x^n$ is cgt. it is equal to $f(x)$.

ie. $f(x) = \sum_{n=1}^{\infty} C_n (x-a)^n$ only within "Radius of Convergence" ex: What is the radius of convergence for the given series?

→ Always use Ratio Test

→ Treat x like constant until getting rid of $\lim_{n\to\infty}$

Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ ⇒ for convergence

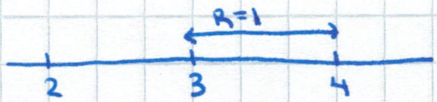
(a) $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n+1}$

$$\lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{(n+2)}\cdot\frac{(n+1)}{(x-3)^n}\right| < 1$$

$$= \lim_{n\to\infty}\left|\underbrace{\frac{\cancel{(n+1)}}{\cancel{(n+2)}}}_{1}\cdot\frac{(x-3)^{\cancel{n}+1}}{\cancel{(x-3)^n}}\right|$$

$$= |x-3|$$

⇒ Need $|x-3| < 1$ ∴ $R=1$ centered at $a=3$



(b) $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$

$$\lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+1}}\cdot x^{2(n+1)}}{(2(n+1))!}\cdot\frac{(2n)!}{\cancel{(-1)^n}\cdot x^{2n}}\right|$$

Cancel out (-1) bc absolute value.

$$= \lim_{n\to\infty}\left|\frac{x^{\cancel{2n}+2}}{\cancel{x^{2n}}}\cdot\frac{\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(2n+1)}\right| = \left|\frac{x^2}{\infty}\right| = 0 \quad \forall x\in\mathbb{R}$$

∴ No matter what value of x, it always equals 0

∴ Need $0 < 1$ ⇒ always true

∴ Radius of convergence is $R=\infty$
(ie. Σa_n cgt. $\forall x\in\mathbb{R}$)

(c) $\sum_{n=1}^{\infty} n!\, x^n$

$$\lim_{n\to\infty}\left|\frac{(n+1)!\cdot x^{\cancel{n+1}}}{n!\cdot \cancel{x^n}}\right|$$

$$\lim_{n\to\infty}\left|(n+1)\cdot x\right| \qquad \text{Need } < 1 \text{ for cgt.}$$

Note: if $\boxed{x=0}$ $\therefore$ $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$ (cgt)

for all other values, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = +\infty$ (div)

$\therefore$ Radius of convergence $R=0$, cgt. only at $x=0$

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{4^n}(x+3)^n$

$$\lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+1}}\cdot(n+1)\cdot(x+3)^{\cancel{n+1}}}{4^{\cancel{n+1}}}\cdot\frac{\cancel{4^n}}{\cancel{(-1)^n}\cdot n\cdot\cancel{(x+3)^n}}\right| < 1$$

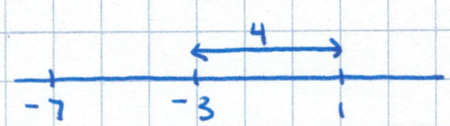
$$= \lim_{n\to\infty}\left|\underbrace{\frac{(n+1)}{n}}_{1}\cdot\frac{(x+3)}{4}\right|$$

$$= \left|\frac{x+3}{4}\right|$$

$\Rightarrow$ Need $\left|\frac{1}{4}(x+3)\right| < 1$

$$\boxed{|x+3| < 4}$$

$\therefore$ Radius of cgt $R=4$, centered at $a=-3$



Taylor Series / Maclaurin Series

$f(x) \approx T_n(x)$

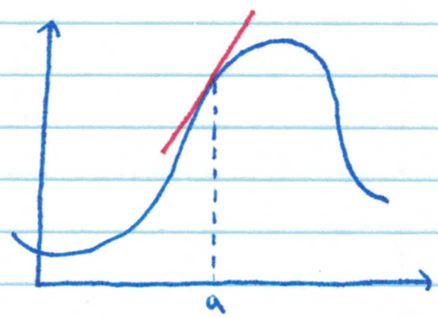
$f(x) = \lim_{n \to \infty} T_n(x)$

<u>Recall</u>: The N^{th} degree Taylor Polynomial is:

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!} \cdot (x-a) + \frac{f''(a)}{2!} \cdot (x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \ldots$$

consider:



$T_0(x) = f(a)$

$T_1(x) = f(a) + f'(a)(x-a)$

Require: ① $T_1(a) = f(a)$
② $T_1'(a) = f'(a)$

$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2} \cdot (x-a)^2$

Now require: ① $T_2(a) = f(a)$ → share point
② $T_2'(a) = f'(a)$ → share slope of tan.
③ $T_2''(a) = f''(a)$ → share concavity

consider:

constant

$$T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$T_3'(x) = 0 + f'(a) + \frac{f''(a)}{\cancel{2}} \cdot \cancel{2}(x-a) + \frac{f'''(a)}{\cancel{3!}\,2} \cdot \cancel{3}(x-a)^2$$

$$\therefore \boxed{T_3'(a) = f'(a)}$$

$$T_3''(x) = 0 + 0 + f''(a) + \frac{f'''(a)}{\cancel{2}} \cdot \cancel{2}(x-a)$$

$$\therefore \boxed{T''(a) = f''(a)}$$

$$T_3'''(x) = 0 + 0 + 0 + f'''(a)$$

$$\therefore \boxed{T'''(a) = f'''(a)}$$

Def'n The "**Taylor Series**" of $y = f(x)$ at $x = a$ is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Note: if $a=0$, then we call it the "**MacLaurin Series**"

$$f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

ex: Find the MacLaurin Series of $f(x) = e^x$

$\underset{a=0}{\text{MacLaurin}}$

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$$

$$\left.\begin{array}{l|l} f(x) = e^x & f(0) = e^0 = 1 \\ f'(x) = e^x & f'(0) = e^0 = 1 \\ f''(x) = e^x & f''(0) = e^0 = 1 \\ f'''(x) = e^x & f'''(0) = e^0 = 1 \end{array}\right\} f^n(0) = 1$$

$$\therefore\ e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Note: only equal to eachother within radius of convergence (where cgt.)

ex: Find the Taylor Series of $f(x) = e^x$ centered at $a = 3$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!} (x-3)^n$$

$$\left.\begin{array}{l} f(3) = e^3 \\ f'(3) = e^3 \\ f''(3) = e^3 \\ f'''(3) = e^3 \end{array}\right\} f^n(3) = e^3$$

$$\therefore\ e^x = \sum_{n=0}^{\infty} \frac{e^3}{n!} (x-3)^n$$

Note: Series alternating +/-

$$(-1)^n \Rightarrow \begin{array}{l} n \text{ "even" } (+) \\ n \text{ "odd" } (-) \end{array}$$

$$(-1)^{n+1} \Rightarrow \begin{array}{l} n \text{ "even" } (-) \\ n \text{ "odd" } (+) \end{array}$$

ex: Find the Taylor series for ln(x) centered at a=2

$$\sum_{n=0}^{\infty} \frac{f^n(2)}{n!}(x-2)^n$$

$f(x) = \ln(x)$	$f(2) = \ln(2)$	$n=0$	Pattern does not apply to $n=0$
$f'(x) = 1/x$	$f'(2) = +(2)^{-1}$	$n=1$	$\therefore$
$f''(x) = -x^{-2}$	$f''(2) = -(2)^{-2}$	$n=2$	Start pattern at $n=1$
$f'''(x) = 2x^{-3}$	$f'''(2) = 2(2)^{-3}$	$n=3$	
$f''''(x) = -3.2x^{-4}$	$f^4(2) = -3.2(2)^{-4}$	$n=4$	
$f^5(x) = 4.3.2x^{-5}$	$f^5(2) = 4.3.2(2)^{-5}$	$n=5$	

$$f^{(n)} = (-1)^{n+1}(n-1)! \cdot (2)^{-n}$$

$$\therefore \ln(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(2)}{n!}(x-2)^n = \underbrace{\ln(2)}_{n=0} + \underbrace{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot(n-1)!}{(2)^n}\cdot\frac{(x-2)^n}{n!}}_{n=1\to\infty}$$

$$\frac{(n-1)!}{n.(n-1)!} = \frac{1}{n}$$

$$\therefore \boxed{\ln(x) = \ln(2) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-2)^n}{(2)^n \cdot n}}$$

→ Find radius of convergence

$$\lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+2}}\cdot(x-2)^{\cancel{n+1}}}{(2)^{\cancel{n+1}}\cdot(n+1)}\cdot\frac{\cancel{(2)^n}\cdot n}{\cancel{(-1)^{n+1}}\cdot\cancel{(x-2)^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-2)}{2}\cdot\underbrace{\frac{n}{(n+1)}}_{1}\right|$$

$$= \left|\frac{1}{2}(x-2)\right|$$

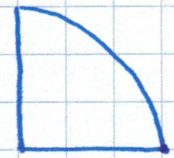
$\Rightarrow$ Need $|1/2(x-2)| < 1$

$|x-2| < 2$ $\quad\therefore R = 2$, centered at $a=2$

ex: Find the MacLaurin Series for $f(x) = \sin(x)$

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$$

n=0	$f(x) = \sin(x)$	$f(0) = 0$
n=1	$f'(x) = +\cos(x)$	$f'(0) = 1$
n=2	$f''(x) = -\sin(x)$	$f''(0) = 0$
n=3	$f'''(x) = -\cos(x)$	$f'''(0) = -1$
n=4	$f^4(x) = +\sin(x)$	$f^4(0) = 0$



$$\therefore \sin(x) = \overset{n=0}{0} + \overset{n=1}{1.(x)^1} + \overset{n=2}{0} + \overset{n=3}{\frac{(-1)}{3!}(x)^3} + \overset{n=4}{0} + \overset{n=5}{\frac{(1)}{5!}(x)^5} + \overset{n=6}{0} + \overset{n=7}{\frac{(-1)}{7!}(x)^7} \ldots$$

Drop all zeros, change counter

$$= \overset{n=0}{(x)^1} + \overset{n=1}{\frac{(-1)(x)^3}{3!}} + \overset{n=2}{\frac{(1)(x)^5}{5!}} + \overset{n=3}{\frac{(-1)(x)^7}{7!}} + \overset{n=4}{\frac{(1)(x)^9}{9!}} \ldots$$

$$= \boxed{\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x)^{2n+1}}$$

ex: Find the MacLaurin Series for $\frac{1}{(x-2)^2}$

$f(x) = (x-2)^{-2}$	$f(0) = \frac{1}{(-2)^2}$ $n=0$
$f'(x) = -2(x-2)^{-3}$	$f'(0) = -\frac{2}{(-2)^3}$ $n=1$
$f''(x) = +3.2(x-2)^{-4}$	$f''(0) = \frac{3.2}{(-2)^4}$ $n=2$
$f'''(x) = -4.3.2(x-2)^{-5}$	$f'''(0) = -\frac{4.3.2}{(-2)^5}$ $n=3$

$$f^n(0) = \frac{(n+1)!}{2^{n+2}}$$

<u>or</u>

$$f^n(0) = \frac{(-1)^n \cdot (n+1)!}{(-2)^{n+2}} = \frac{\cancel{(-1)^n} \cdot (n+1)!}{\cancel{(-1)^{n+2}} \cdot (2)^{n+2}}$$

$$\therefore \frac{1}{(x-2)^2} = \sum_{n=0}^{\infty} \frac{(n+1)!}{(2)^{n+2}} \cdot \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{(n+1)}{2^{n+2}} \cdot x^n$$

→ Find R

$$\lim_{n\to\infty} \left| \frac{(n+2) \cdot x^{\cancel{n}+1}}{2^{\cancel{n}+3}} \cdot \frac{2^{\cancel{n}+2}}{(n+1) \cdot \cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} \left| \underbrace{\frac{(n+2)}{(n+1)}}_{1} \cdot \frac{1}{2} \cdot x \right|$$

$$= \left| \tfrac{1}{2} x \right|$$

$\Rightarrow$ Need $\left| \tfrac{1}{2} x \right| < 1$

$|x| < 2$ $\therefore R = 2$, centered at $a = 0$

ex: Find the MacLaurin Series for $\sin(\pi x)$

$f(x) = \sin(\pi x)$	$f(0) = 0$	$n=0$
$f'(x) = \pi \cdot \cos(\pi x)$	$f'(0) = \pi$	$n=1$
$f''(x) = -\pi^2 \sin(\pi x)$	$f''(0) = 0$	$n=2$
$f'''(x) = -\pi^3 \cos(\pi x)$	$f'''(0) = -\pi^3$	$n=3$
$f^4(x) = \pi^4 \sin(\pi x)$	$f^4(0) = 0$	$n=4$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$$

$$\therefore \sin(\pi x) = \overset{n=0}{0} + \overset{1}{\pi x} + \overset{2}{0} + \overset{3}{\frac{(-\pi^3)}{3!}} x^3 + \overset{4}{0} + \overset{5}{\frac{(\pi^5)}{5!}} x^5 + \cdots$$

change counter

$$= \overset{n=0}{\pi x} + \overset{n=1}{\frac{(-\pi^3)}{3!}} x^3 + \overset{n=2}{\frac{\pi^5}{5!}} x^5 + \overset{n=3}{\frac{(-\pi^7)}{7!}} x^7 + \cdots$$

$$\therefore \boxed{\sin(\pi x) = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{(2n+1)}}{(2n+1)!} \cdot x^{(2n+1)}}$$

Integrative Activity - Rectilinear Motion

Def'n: Motion along a straight line is classified as "**Rectilinear Motion**"

CAL 2 CAL 1

Position fcn : $s(t) = \int v(t)\,dt$ $\qquad v(t) = s'(t) = \frac{ds}{dt}$

Velocity fcn : $v(t) = \int a(t)\,dt$

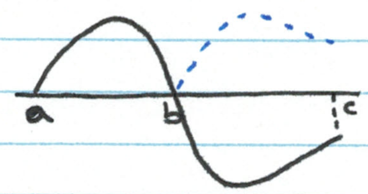
Acceleration fcn: $a(t) = \frac{dv}{dt}$

Speed: $|v(t)|$

FTC-II

Displacement $(s(t_2) - s(t_1))$: from t_1 to t_2 $\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$

Total distance travelled: From t_1 to t_2: $\int_{t_1}^{t_2} |v(t)|\,dt$



$$|v(t)| = \begin{cases} v(t) & \text{if } v(t) \geq 0 \Rightarrow [a,b] \\ -v(t) & \text{if } v(t) < 0 \Rightarrow (b,c] \end{cases}$$

$$\therefore \int_a^c |v(t)|\,dt = \int_a^b v(t)\,dt - \int_b^c v(t)\,dt$$

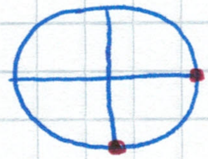
substract when neg.

ex: Let $v(t) = \cos\left(\frac{\pi}{2} t\right)$ [cm/s]

(a) Find displacement over interval $[0, 3]$

$$\text{displacement} = \int_0^3 \cos\left(\frac{\pi}{2} t\right) dt = \frac{2}{\pi} \sin\left(\frac{\pi}{2} t\right)\Big|_0^3$$

$$= \frac{2}{\pi}\Big(\underbrace{\sin\left(\tfrac{3\pi}{2}\right)}_{-1} - \underbrace{\sin(0)}_{0}\Big)$$



$$= \boxed{-\frac{2}{\pi}}$$

(b) Find total distance travelled on $[0, 3]$

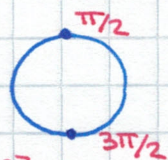
$$\int_0^3 \left|\cos\left(\frac{\pi}{2} t\right)\right| dt$$

when is $\cos\left(\frac{\pi}{2} t\right)$ pos or neg? Value must be between $[0, 3]$

$$\cos\left(\frac{\pi}{2} t\right) = 0$$

$$t = 1, 3, 5, \ldots$$

← not between $[0, 3]$



	0	1	3
$\cos\left(\frac{\pi}{2} t\right)$	⊕	⊖	

$$\therefore \left|\cos\left(\frac{\pi}{2} t\right)\right| = \begin{cases} \cos\left(\frac{\pi}{2} t\right) & \text{on } [0, 1] \\ \cos\left(\frac{\pi}{2} t\right) & \text{on } (1, 3] \end{cases}$$

$$\therefore \int_0^3 \left|\cos\left(\frac{\pi}{2} t\right)\right| dt = \int_0^1 \cos\left(\frac{\pi}{2} t\right) dt - \int_1^3 \cos\left(\frac{\pi}{2} t\right) dt$$

$$= \frac{2}{\pi} \sin\left(\frac{\pi}{2} t\right)\Big|_0^1 - \frac{2}{\pi} \sin\left(\frac{\pi}{2} t\right)\Big|_1^3$$

$$= \frac{2}{\pi}\Big(\underbrace{\sin\left(\tfrac{\pi}{2}\right)}_{1} - \underbrace{\sin(0)}_{0}\Big) - \frac{2}{\pi}\Big(\underbrace{\sin\left(\tfrac{3\pi}{2}\right)}_{-1} - \underbrace{\sin\left(\tfrac{\pi}{2}\right)}_{1}\Big)$$

$$= \frac{2}{\pi}(1) - \frac{2}{\pi}(-2)$$

$$= \boxed{6/\pi}$$

Integrative Activity - Exponential Growth/Decay

Theorem: If $y(t)$ is a quantity at time t and the rate of change of y is proportional to $y(t)$, then:

$$\boxed{\frac{dy}{dt} = K \cdot y(t)}$$

- $\boxed{\frac{dy}{dt}}$ is always changing
  → it's the rate of change at instant t
- $\boxed{K}$ is a constant
  → growth factor
  → percentage of population (if pop. increase ∴ X% of bigger number)

consider solving this DE:

$$\int \frac{1}{y}\, dy = K \int dt$$

$$\ln|y| = K \cdot (t) + C$$

$$|y| = e^{Kt + C}$$

$$y = \pm e^{Kt + C}$$

$$y = \pm e^{C} \cdot e^{Kt}$$

Let $e^{C} = A$

$$y = \pm A \cdot e^{Kt}$$

↳ for this application, only consider positive

$$\therefore \boxed{y = Ae^{Kt}}$$

consider $y(0) = A \cdot e^{K(0)} = A$

$\therefore A = y_0$ ⇒ initial quantity.

$$\therefore \boxed{y = y_0 \cdot e^{Kt}}$$

ex: Suppose the initial population of 10000 bacteria grows exponentially at a rate of 1% per hour and that $y(t) = y$ is the number of bacteria present after t hours.

$\to K$ (growth factor)

$K = 0.01$

$y_0 = 10\,000$

(a) What is the differential equation that models population growth of bacteria.

$$\frac{dy}{dt} = K \cdot y(t)$$

$$\frac{dy}{dt} = (0.01) \cdot y(t)$$

(b) Find the formula for $y(t)$

$$y = y_0 \cdot e^{kt}$$

$$y(t) = 10\,000 \cdot e^{(0.01)t}$$

(c) How long does it take for population to double?

$$y(t) = 2y_0$$

$$\therefore \; 2\cancel{y_0} = \cancel{y_0}\, e^{(0.01)t}$$

$$2 = e^{(0.01)t}$$

$$\ln(2) = (0.01)t$$

$$\boxed{t = \frac{\ln(2)}{0.01}} \quad [\text{hours}]$$

(d) How long does it take for population to reach 45000 bacteria?

$$45000 = 10000 \cdot e^{(0.01) \cdot t}$$

$$4.5 = e^{(0.01)t}$$

$$\ln(4.5) = (0.01)t$$

$$\boxed{t = \frac{\ln(4.5)}{0.01}}$$

ex: An E-coli cell divides into 2 cells every 20 min
The initial population is 60 cells

(a) Find the expression for the number of cells after t hours:

$K \Rightarrow$ Solve for

$y(t) = y_0 e^{Kt}$

$y(1/3) = 2 \cdot 60 = 120$ [1/3 hours = 20 min]

$120 = 60 \cdot e^{K \cdot 1/3}$

$\ln(2) = K \cdot 1/3$

$K = 3 \cdot \ln(2)$

$\therefore \quad y(t) = 60 \cdot e^{3 \cdot \ln(2) t}$

(b) Find the number of cells after 8 hours

$y(8) = 60 \cdot e^{3\ln(2) \cdot (8)}$

$y(8) = 1.01 \times 10^9$

(c) when will the population be 20 000 cells

$20000 = 60 \cdot e^{3\ln(2) t}$

$\frac{1000}{3} = e^{3\ln(2) \cdot t}$

$\ln\left(\frac{1000}{3}\right) = 3\ln(2) \cdot t$

$\frac{\ln\left(\frac{1000}{3}\right)}{3\ln(2)} = t$

$t = 2.79$ hours