

### 3A) GENERAL VECTOR SPACES

Vectors as magnitude and direction in  $\mathbb{R}^2, \mathbb{R}^3, \dots, \mathbb{R}^n$  are just particular types of vectors... There are other kinds of vectors too!

Defn: A vector is an element of a "vector space"

Well, then what's a vector space?!

Defn: A vector space is a set of elements (called vectors) with 2 operations satisfying a number of properties called "axioms". These 2 operations are called addition and scalar multiplication.

\* Scalars can be  $\mathbb{R}$  or  $\mathbb{C}$ , but they could also be other things

So a vector is just anything that satisfies these axioms...

Axioms 1-5 deal with the addition

Axioms 6-10 involve scalar multiplication.

Looking carefully at these axioms, we can see that functions (as long as they are defined on an interval) are vectors too!

## Definition of VECTOR SPACE

A vector space  $V$  is a non-empty set of elements called vectors with 2 operations, addition and scalar multiplication, satisfying the following axioms.

For all  $\vec{u}, \vec{v}, \vec{w} \in V$ , and all scalars<sup>1</sup>  $r, s$ :

A1)  $\vec{u} + \vec{v} \in V$  (Closure under addition.)

A2)  $\vec{u} + \vec{v} = \vec{v} + \vec{u}$  ← commutativity

A3)  $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

( hence write  $\vec{u} + \vec{v} + \vec{w}$  without the unnecessary parentheses ).

A4) There is a vector  $\vec{t} \in V$  satisfying that for all  $\vec{v} \in V$ :  
 $\vec{v} + \vec{t} = \vec{v}$ .

(  $\vec{t}$  does not depend on  $\vec{v}$ , it is the same for any  $\vec{v} \in V$ .  $\vec{t}$  is the zero-vector:  $\vec{t} = \vec{0}$ . )

A5) For all  $\vec{v} \in V$ , there is an  $\vec{n} \in V$  such that

$$\vec{v} + \vec{n} = \vec{0}.$$

(  $\vec{n}$  depends on  $\vec{v}$ , it is the negative of  $\vec{v}$ :  $\vec{n} = -\vec{v}$ . )

This one implies that the vector space is non empty

S1)  $r\vec{v} \in V$  (Closure under scalar multiplication.)

S2)  $r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}$

S3)  $(r+s)\vec{v} = r\vec{v} + s\vec{v}$

S4)  $(rs)\vec{v} = r(s\vec{v})$

S5)  $1\vec{v} = \vec{v}$

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<sup>1</sup> In Linear Algebra I, we take the word scalar to mean *real numbers*; in Linear Algebra II and in Differential Equations, we take the word scalar to mean either *real numbers* or *complex numbers* depending on the context.

So we saw that there are many types of vectors. As long as something satisfies the (10 axioms) (above) describing a vector space. Examples of vector spaces, as seen in NYC, are  $\mathbb{R}^2$ ,  $\mathbb{R}^3$ ,  $\mathbb{R}^n$  (called euclidean vector spaces). But like we said, other things can be vectors too:

① The set of all  $\frac{\text{fixed size}}{m \times n}$  matrices

Note: The set  $M$  of all matrices is not a vector space.

Counterexample:  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \in M$ ,  $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in M$ .

But  $A+B \notin M$  ( $\because$  undefined)

② The set  $P$  of all polynomials (any degree, including the zero polynomial  $p(x) \equiv 0$ )<sup>degree is undefined or  $-\infty$ .</sup>

Proof that the set  $P_n$  of all polynomials of degree  $\leq n$  (including  $p(x) \equiv 0$ ) is a vector space on handout.

Note: The set of all polynomials of  $P_2$  (degree exactly 2) is NOT a vector space. We can easily find a

Counterexample:  $p(x) = -2x^2 - 7x + 5 \in P_2$

$$q(x) = 2x^2 + 5x - 8 \in P_2$$

$$p(x) + q(x) = -2x - 3 \notin P_2.$$

$\therefore$  Axiom A<sub>1</sub> does not hold  $\Rightarrow$  Not a V.S.

## $\mathbb{P}_2$ is a Vector Space

$$\mathbb{P}_2 = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}^1$$

For any  $p(x) = a + bx + cx^2$ ,

$$p_1(x) = a_1 + b_1x + c_1x^2,$$

$$p_2(x) = a_2 + b_2x + c_2x^2,$$

$$p_3(x) = a_3 + b_3x + c_3x^2,$$

and any  $r, s \in \mathbb{R}$ , we do have:

$$\begin{aligned} A1) \quad p_1(x) + p_2(x) &= (a_1 + b_1x + c_1x^2) + (a_2 + b_2x + c_2x^2) \\ &= a_1 + a_2 + b_1x + b_2x + c_1x^2 + c_2x^2 \\ &= (a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2 \in \mathbb{P}_2 \checkmark \end{aligned}$$

$$\begin{aligned} A2) \quad p_1(x) + p_2(x) &= \dots = (a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2 \\ p_2(x) + p_1(x) &= \dots = (a_2 + a_1) + (b_2 + b_1)x + (c_2 + c_1)x^2 \\ &= (a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2 \checkmark \end{aligned}$$

$$\begin{aligned} A3) \quad p_1(x) + [p_2(x) + p_3(x)] &= (a_1 + b_1x + c_1x^2) + [(a_2 + b_2x + c_2x^2) + (a_3 + b_3x + c_3x^2)] \\ &= a_1 + b_1x + c_1x^2 + [(a_2 + a_3) + (b_2 + b_3)x + (c_2 + c_3)x^2] \\ &= (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)x + (c_1 + c_2 + c_3)x^2 \\ [p_1(x) + p_2(x)] + p_3(x) &= [(a_1 + b_1x + c_1x^2) + (a_2 + b_2x + c_2x^2)] + (a_3 + b_3x + c_3x^2) \\ &= [(a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2] + a_3 + b_3x + c_3x^2 \\ &= (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)x + (c_1 + c_2 + c_3)x^2 \checkmark \end{aligned}$$

A4) The zero polynomial  $t(x) \equiv 0$  for all  $x$ , i.e.,  $t(x) = 0 + 0x + 0x^2 \in \mathbb{P}_2$  does satisfy that

$$p(x) + t(x) = p(x) \quad \text{for all } p(x) \in \mathbb{P}_2.$$

A5) For any  $p(x) = a + bx + cx^2 \in \mathbb{P}_2$ ,  $n(x) = -a - bx - cx^2 \in \mathbb{P}_2$  does satisfy  
 $p(x) + n(x) = 0$  (the zero polynomial).

We write  $n(x) = -p(x)$ .

S1) --> S5) on other side...

<sup>1</sup> This handout actually shows that  $\mathbb{P}_2(\mathbb{R})$  (polynomials with coefficients in  $\mathbb{R}$ ) is a vector space over  $\mathbb{R}$  (using the scalars  $r, s$  in  $\mathbb{R}$ ). We could have generalized to  $\mathbb{P}_2(\mathbb{C})$  is a vector space over  $\mathbb{C}$ .

$$S1) \quad r(p(x)) = r(a + bx + cx^2) = (ra) + (rb)x + (rc)x^2 \in \mathbb{P}_2 \checkmark$$

$$\begin{aligned} S2) \quad r[p_1(x) + p_2(x)] &= r[(a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2] \\ &= r(a_1 + a_2) + r(b_1 + b_2)x + r(c_1 + c_2)x^2 \\ &= (ra_1 + ra_2) + (rb_1 + rb_2)x + (rc_1 + rc_2)x^2 \\ rp_1(x) + rp_2(x) &= [(ra_1) + (rb_1)x + (rc_1)x^2] + [(ra_2) + (rb_2)x + (rc_2)x^2] \\ &= (ra_1 + ra_2) + (rb_1 + rb_2)x + (rc_1 + rc_2)x^2 \checkmark \end{aligned}$$

$$\begin{aligned} S3) \quad (r+s)p(x) &= (r+s)[a + bx + cx^2] \\ &= (r+s)a + (r+s)bx + (r+s)cx^2 \\ &= (ra + sa) + (rb + sb)x + (rc + sc)x^2 \\ rp(x) + sp(x) &= r[a + bx + cx^2] + s[a + bx + cx^2] \\ &= [ra + rbx + rcx^2] + [sa + sbx + scx^2] \\ &= (ra + sa) + (rb + sb)x + (rc + sc)x^2 \checkmark \end{aligned}$$

$$\begin{aligned} S4) \quad (rs)p(x) &= (rs)[a + bx + cx^2] = rs(a) + rs(bx) + rs(cx^2) = rsa + rsbx + rscx^2 \\ r[sp(x)] &= r[s(a + bx + cx^2)] = r[sa + sbx + scx^2] = rsa + rsbx + rscx^2 \checkmark \end{aligned}$$

$$S5) \quad 1(p(x)) = 1(a + bx + cx^2) = 1a + 1bx + 1cx^2 = a + bx + cx^2 = p(x) \checkmark$$

③ The set  $F(D)$  of all real valued function defined on a fixed domain  $D$  forms a V.S.

Remark:  $\mathbb{R}^1 \cong \mathbb{R}$  (but not exactly equal)

1D euclidean vector space

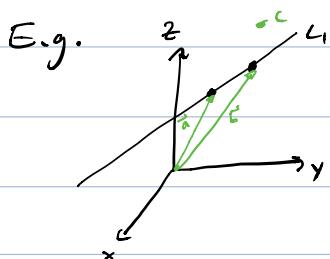
is isomorphic to the real number line

$\mathbb{R}^0$  is then  $\mathbb{R}^0 = \{0\}$ . This is called the trivial vector space.

The set  $E$  of all polynomials with even coefficients is not a vector space (closed under addition  $\Rightarrow A, E$ , but not closed under scalar multiplication  $\Rightarrow S1 \nRightarrow$  e.g. multiply by  $\frac{1}{2}$ ).

Indeed, the notions seen in linear algebra 1 with vectors in  $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \dots, \mathbb{R}^n$  hold for vector spaces in general. The handout below states (or restates) these notions in a general context.

Aside: Lines not through origin are not a subspace.



$L_1$  is not a subspace of  $\mathbb{R}^3$  :  $\vec{a} + \vec{b} = \vec{c}$ , and  
 $\vec{c}$  is not on line. Not closed under addition.

But  $L_1$  is a subset of  $\mathbb{R}^3$ .

# GENERALIZING LINEAR ALGEBRA I TO GENERAL VECTOR SPACES

## Definition of Vector Space

The general definition of a vector space is given on a separate handout.

For a set to be a vector space, it needs to satisfy the 10 axioms<sup>1</sup> A1) – A5) and S1) – S5) there.

In the end, what we realize is that there are many types of vectors apart from just arrows in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ , or even  $\mathbb{R}^n$ . Matrices, polynomials, certain sets of functions, etc. may be considered as *vectors*.

## Definition of Subspace

Start with a known vector space  $V$ , and consider a subset  $W \subseteq V$ .

$W$  is said to be a **subspace** of  $V$  if  $W$  itself happens to be a vector space in its own right.

## Subspace Test

Start with a known vector space  $V$ , and consider a subset  $W \subseteq V$ .

Then  $W$  turns out to be a **subspace** of  $V$  if, and only if,

for any scalar  $r$ , and for any  $\vec{u}, \vec{v} \in W$ :

A4)  $\vec{0} \in W$

A1)  $\vec{u} + \vec{v} \in W$  (closure under vector addition)

S1)  $r\vec{v} \in W$  (closure under scalar multiplication)

ST

\* This means: When we test a completely new set for being a vector space, we need to verify 10 axioms, but when we test a subset of a previously known vector space, we need to check only 3 axioms instead of 10. \*

Remark: It is not necessary either to verify A4) if we know (verify) that  $W$  is non-empty.

## Spans

Recall that the **span** of the set of vectors  $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \subseteq V$  is the set of all linear combinations of those vectors:  $\text{Span}(S) = \left\{ \vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n \mid a_1, a_2, \dots, a_n \in \mathbb{R} \right\}$ .

Remark:  $\mathbb{R}$  could be replaced by another set of scalars.

<sup>1</sup> Some of the 10 axioms of the handout are redundant; ten are given for extra clarity. One could actually define vector spaces using fewer axioms.

It is proven in Linear I that  $\text{Span}(S)$  is always a *subspace* (not only a subset) of  $V$ .

Convention:  $\text{Span}(\emptyset) := \{\vec{0}\}$ .

### Dependence and Independence.

Consider the following vectors of  $V$ :  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ .

They are said to be *linearly dependent* if, and only if, at least one of these can be written as a linear combination of the others. Equivalently, we may say that they are *linearly independent* if none of these can possibly be written as a linear combination of the others.

In practice, we test the linear independence of vectors through the following:

A set  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$  of vectors of  $V$  is linearly independent when

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n = \vec{0} \Rightarrow a_1 = a_2 = \dots = a_n = 0.$$

(That is, all  $a_i$  equal to zero is the *only* possibility.)

In other words, the only way to obtain the zero vector by means of a linear combination of the vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  is through the trivial linear combination.

### Spanning Sets, Bases, and Dimension

A **spanning set** for  $V$ , or a **generating set** of  $V$  is a set  $S$  of vectors of  $V$  that happens to span  $V$ :  
 $\text{Span}(S) = V$ .

A spanning set of  $V$  which is linearly independent is said to be a **basis**  $B$  of  $V$ .

The **dimension** of  $V$  is the cardinality (number of elements in the set) of any one of its possible bases  $B$ :

$$\dim(V) = \# B.$$

**Remark:** It is mentioned without proof in Linear Algebra I, and then proven in Linear Algebra II, that any two bases of  $V$  must necessarily have the same cardinality.

Recall that if  $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$  is a basis of  $V$ , then any vector  $\vec{w} \in V$  can be written *uniquely* as a linear combination of the basis vectors; that is, writing  $\vec{w}$  as  $\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n$  can be made with only one possible set of values for  $a_1, a_2, \dots, a_n$ .

In a sense, a basis of  $V$  turns out to be a generating set for  $V$  that has the least number of elements possible while still spanning  $V$ .

$P$  is a vector space.  $P_2$  is a subset of  $P$  and a subspace in  $P$ .

$P_{\neq}$  is a subset of  $P$ , but it is not a subspace (as shown above).

Remark: • Any vector space  $V$  is a subset and a subspace of itself.

• Whatever  $V$  is, it must have a zero vector (blk of  $A^4$ ).

Then  $W = \{\vec{0}\} \subseteq V$  is a subspace of  $V$  called the trivial subspace or the zero subspace.

E.g. Let  $D_3$  be the set of all diagonal  $3 \times 3$  matrices.

Is this a subspace

(No need to check all 10 axioms)

$\hookrightarrow D_3 \subseteq M_3$ , a known V.S., so we don't need to check all 10 axioms, it is sufficient to check closure under linear combinations and presence of a zero element

1)  $\vec{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in D_3 \quad \checkmark \quad \therefore 0 \in D_3$

2) Let  $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \in D_3$ , let  $B = \begin{bmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{bmatrix} \in D_3$ .  $A+B = \begin{bmatrix} a+d & 0 & 0 \\ 0 & b+e & 0 \\ 0 & 0 & c+f \end{bmatrix} \in D_3 \quad \checkmark$   
∴ closed under addition

3) Let  $r \in \mathbb{R}$  and  $A \in D_3$ .  $rA = r \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} ra & 0 & 0 \\ 0 & rb & 0 \\ 0 & 0 & rc \end{bmatrix} \in D_3 \quad \checkmark$  ∴ closed under scalar mult.

$\therefore D_3$  is a subspace of  $M_3$ , i.e. it is indeed a vector space

E.g. Let  $W = \{a+bx+cx^2+dx^3 \in P_3 \mid a+3d=0\}$

1) Zero element?

$$\underset{a}{a} + \underset{b}{0}x + \underset{c}{0}x^2 + \underset{d}{0}x^3 \quad a+3d = 0+3 \cdot 0 = 0 \quad \checkmark$$

$$\therefore 0 \in W \quad \checkmark$$

2) Closure under addition:

$$p_1(x) = a_1 + b_1x + c_1x^2 + d_1x^3 \mid a_1 + 3d_1 = 0$$

$$p_2(x) = a_2 + b_2x + c_2x^2 + d_2x^3 \mid a_2 + 3d_2 = 0$$

$$p_1(x) \in W \text{ and } p_2(x) \in W$$

$$\begin{aligned} p_1(x) + p_2(x) &= a_1 + b_1x + c_1x^2 + d_1x^3 + a_2 + b_2x + c_2x^2 + d_2x^3 \\ &= (\underbrace{a_1 + a_2}_{\text{new } a}) + (\underbrace{b_1 + b_2}_{\text{new } b})x + (\underbrace{c_1 + c_2}_{\text{new } c})x^2 + (\underbrace{d_1 + d_2}_{\text{new } d})x^3 \end{aligned}$$

$$a_1 + a_2 + 3(d_1 + d_2) = a_1 + a_2 + 3d_1 + 3d_2 = (a_1 + 3d_1) + (a_2 + 3d_2) = 0 \quad \checkmark$$

$\therefore p_1(x) + p_2(x) \in W$ . Closed under addition

3) Let  $r$  be a scalar and let  $p(x) \in W$

$$r p(x) = r(a+bx+cx^2+dx^3)$$

$$= (\underbrace{ra}_{\text{new } a}) + r bx + r cx^2 + (\underbrace{rd}_{\text{new } d})x^3$$

$$ra + 3rd = r(a+3d) = r(0) = 0 \quad \checkmark$$

$\therefore$  closed under scalar multiplication.

Let  $C^k(I)$  denote the set of func whose  $k^{\text{th}}$  derivative exists

and is cont' on a given interval  $I$ . We write  $C(I)$  to mean

$C^0(I)$ , i.e. the set of all cont' func on  $I$  ( $0^{\text{th}}$  derivative is cont')

E.g. Is  $C(I)$  a vector space?

First, we note that  $C(I) \subseteq F(I)$ , the known vector space of all funcs defined on an interval (see notes above).

$\therefore$  We may use the subspace test.

1)  $f(x) = 0$  is cont<sup>s</sup>  $\forall x \in I \therefore 0 \in S$

2) Let  $f(x) \& g(x) \in C(I)$ ,

then  $f(x) + g(x)$  is cont<sup>s</sup> (proved in cal I  $\rightarrow$  using limit laws)

$\therefore$  closed under addition

3) Let  $r$  be a scalar &  $f(x) \in C(I)$ ,

then  $f(x)$  is cont<sup>s</sup> (proved in cal I)

$\therefore$  closed under scalar multiplication

Thus  $C(I)$  is a subspace of  $F(I)$  and is itself a vector space.

E.g. Is  $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$  a linear combination of

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}.$$

Does  $M = aA + bB + cC$ .

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} c & 2c \\ c & 2c \end{bmatrix}$$

$$a+c=1$$

$$b+2c=2$$

$$3=b+c$$

$$4=a+2c$$

system is  
inconsistent  
(can check)

$\therefore M$  not in  $\text{span } \{A, B, C\}$

Does  $f(x) = 2 \sin(x + \frac{\pi}{6})$  belong to  $\text{span}\{\sin x, \cos x\}$ ?

$$f(x) = 2 \left[ \sin x \cos \frac{\pi}{6} + \sin \frac{\pi}{6} \cos x \right]$$

$$f(x) = 2 \left[ \sin x \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cos x \right]$$

$$f(x) = \underbrace{\frac{\sqrt{3}}{2}}_a \sin x + \underbrace{\frac{1}{2}}_b \cos x$$

$$\therefore a \sin x + b \cos x = 2 \sin(x + \frac{\pi}{6}) \Rightarrow$$

E.g. Describe the span of  $S = \{p_1(x), p_2(x), p_3(x)\} \subseteq \mathbb{P}$

$$\text{where } p_1(x) = x^2 - x \quad p_2(x) = x^3 + x + 2 \quad p_3(x) = 2x^3 + 4$$

i.e. Find conditions describing when a vector of  $\mathbb{P}$  is in span or not.

Let  $p(x) = ax^3 + bx^2 + cx + d$ . Let  $r, s, t$  be scalars

$$p(x) = r p_1(x) + s p_2(x) + t p_3(x).$$

$$ax^3 + bx^2 + cx + d = rx^2 - rx + sx^3 + sx + 2sx^3 + 4t$$

$$ax^3 + bx^2 + cx + d = (2t+s)x^3 + (r+s)x^2 + (s-r)x + (2s+4t)$$

↑ What is this equality? "Equality of vectors"

∴ left must be same polynomial as right.

$$a = 2t + s$$

$$b = r$$

$$c = s - r$$

$$d = 2s + 4t$$

$$\left[ \begin{array}{ccc|c} 0 & 1 & 2 & a \\ 0 & 1 & 0 & b \\ -1 & 1 & 0 & c \\ 0 & 2 & 4 & d \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 0 & 0 & b \\ 0 & 1 & 0 & b+c \\ 0 & 0 & 1 & \frac{a-b-c}{2} \\ 0 & 0 & 0 & d-2a \end{array} \right]$$

consistent if  
 $d-2a=0$ . i.e.,  
can be found if  
 $2a=d$

i.e. A vector  $p(x) = ax^3 + bx^2 + cx + d$  is in  $\text{span}(\{p_1, p_2, p_3\})$

when  $d=2a$

E.g. Consider a homogeneous linear DE

$$\textcircled{*} \quad y^{(n)} + p_{n-1} y^{(n-1)} + \dots + p_2 y'' + p_1 y' + p_0 y = 0$$

The set of all its solutions form a vector space. We then call  
the "solution space" of the DE " $S$ " for set of all solutions

Note: Doesn't really mean anything to say that differential equations themselves  
form vector spaces...

Proof: We have that  $S$  is a subset of the known vector space

$C^{(n)}(I)$  (cont's  $n$  times differentiable on an interval  $I$ ).

We prove using the subspace test.

1) 0 element.  $f(x) \equiv 0$  satisfies  $y^{(n)} + p_{n-1} y^{(n-1)} + \dots + p_2 y'' + p_1 y' + p_0 y = 0$   
 $\therefore 0 \in S$   $\square$

2) Closure under addition?

Let  $y = f(x)$  and  $y = g(x)$  be solutions to  $y^{(n)} + p_{n-1} y^{(n-1)} + \dots + p_1 y' + p_0 y = 0$

meaning  $f^{(n)} + \sum_{i=0}^{n-1} f^{(i)} p_i(x) = 0$  and  $g^{(n)} + \sum_{i=0}^{n-1} g^{(i)} p_i(x) = 0$

$$\begin{aligned} & f^{(n)} + g^{(n)} + \sum_{i=0}^{n-1} (f^{(i)} + g^{(i)}) p_i(x) \\ &= f^{(n)} + g^{(n)} + \sum_{i=0}^{n-1} f^{(i)} p_i(x) + g^{(i)} p_i(x) \\ &= f^{(n)} + g^{(n)} + \sum_{i=0}^{n-1} f^{(i)} p_i(x) + \sum_{i=0}^{n-1} g^{(i)} p_i(x) \end{aligned}$$

$$= (g^{(n)} + \sum_{i=0}^{n-1} g^{(i)} p_i(x)) + (f^{(n)} + \sum_{i=0}^{n-1} f^{(i)} p_i(x))$$

$$= 0 + 0 = 0 \quad \therefore \text{closed under addition.}$$

3) Closed under scalar multiplication?

Let  $f(x)$  be a soln to  $\star$ . i.e.  $f^{(n)} + \sum_{i=0}^{n-1} f^{(i)} p_i = 0$ . Let  $r$  be a scalar.

Then  $rf(x)$  is homogeneous qn given

$$rf^{(n)} + \sum_{i=0}^{n-1} (rf^{(i)}) p_i$$

$$= rf^{(n)} + r \sum_{i=0}^{n-1} f^{(i)} p_i = r(f^{(n)} + \sum_{i=0}^{n-1} f^{(i)} p_i) = r \cdot (0) = 0$$

$\therefore$  Closed under scalar multiplication

So, solns to linear homogeneous ODEs are a vector spaces.

Corollary: The Superposition Principle

If  $f_1(x), f_2(x), \dots, f_k(x)$  are known solns of a homogeneous linear ODE,

then all linear combinations of these solns

$$f = a_1 f_1 + a_2 f_2 + \dots + a_n f_n.$$

In sexy words,  $\text{span}\{f_1, f_2, \dots, f_n\} \subseteq$  of soln space of the DE.

Note that linearity and homogeneity are truly required for this.

Example:  $y'' + y = 1$  has solns  $y_1 = 1 + \cos x$  and  $y_2 = 1 + \sin x$

$$(1 + \cos x)'' + 1 + \cos x = -\cos x + 1 + \cos x = 1 \quad \square$$

$$\text{AND } (1 + \sin x)'' + 1 + \sin x = -\sin x + 1 + \sin x = 1 \quad \square$$

$$\text{BUT } y = y_1 + y_2 = (1 + \cos x + 1 + \sin x)'' + (\sin x + 1 + \cos x + 1)$$

$$= -\cos x - \sin x + \sin x + \cos x + 2 = 2 \neq 1$$

Example:  $y'' - y = 0$  has  $y_1 = 1$  and  $y_2 = x^2$  as solns.

$$y_1 = 1: (1)'' - 1(1)' = 0 - 0 = 0$$

$$y_2: (x^2)''(x^2)' - x(x^2)' = 2x^2 - 2x^2 = 0$$

However,  $y_1 + y_2$  is NOT a soln.

$$(x^2+1)''(x^2+1) - x(x^2+1)'$$

$$2x^2 + 2 - 2x^2 = 2 \neq 0 \therefore \text{NOT a soln}$$

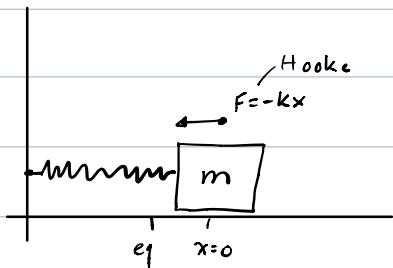
**Corollary 2:**

A linear ODE has no singular solns, i.e. its general soln contains all its possible solutions.

Idea of proof:

If  $f_1$  were claimed to be a singular solution, then  $f_1$  together with the other known solns  $f_2, f_3, \dots, f_k$  would satisfy that  $a_1 f_1 + a_2 f_2 + \dots + a_k f_k$  are also solutions. If that is the case  $a_1=1$ , and  $a_2=a_3=\dots=a_k=0$ . Therefore, the soln  $f_1$  is already contained in the general soln.

**Application: Simple Harmonic Motion**



$$\sum F_x = ma_x$$

$$-kx = ma$$

$$-\frac{k}{m}x(t) = x''(t) \quad \text{let } \omega = \sqrt{\frac{k}{m}}$$

$$x'' + \omega^2 x = 0 \quad \text{linear homogeneous ODE}$$

Can easily guess two solns:  $y_1 = \cos(\omega t)$ ,  $y_2 = \sin(\omega t)$ .

All linear combinations are solutions:  $y = A \sin(\omega t) + B \cos(\omega t)$

and there are no other solutions ( $\therefore$  no singular solutions)

E.g. Consider the set  $A_3$  of  $M_3$  ( $3 \times 3$  matrices) consisting of all the anti-symmetric (aka skew-symmetric) matrices in  $M_3$  (i.e.  $M^T = -M$ ).

Recall that matrices in  $A_3$  are of the form

$$A_3 = \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$$

The 3 <sup>anti-symmetric</sup> matrices

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

These 3 matrices span  $A_3$ : Any matrix  $\begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$  in  $A_3$  is  $aA + bB + cC$ .

Check independence:  $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = 0$  has only trivial soln

$$a \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -b \\ 0 & 0 & 0 \\ b & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -c \\ 0 & c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Only works w/  $a=b=c=0$ . i.e.  $\exists!$  trivial soln  $\Rightarrow$  independent!

$\therefore A, B, C$  are independent and the set  $\{A, B, C\}$  spans  $A_3$

$\Rightarrow \{A, B, C\}$  is a basis for  $A_3$  and  $\dim(A_3) = 3$ .

E.g. What is the dimension of  $P_3$  (all polynomials of the form  $ax^3 + bx^2 + cx + d$  |  $a, b, c, d$  are scalars).

(Clearly, the 4 polynomials  $p_1(x) = 1$ ,  $p_2(x) = x$ ,  $p_3(x) = x^2$ ,  $p_4(x) = x^3$  are 4 polynomials in  $P_3$  that clearly span  $P_3$ .  $\therefore$  Any  $ax^3 + bx^2 + cx + d = a_1p_1(x) + a_2p_2(x) + a_3p_3(x) + a_4p_4(x)$ )

Moreover: Put  $a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) + a_4 p_4(x) = 0$   
 $\therefore \exists!$  soln (trivial soln).  $a_1 = a_2 = a_3 = a_4 = 0$       Equality of vectors  
must be satisfied by x

$\therefore$  Linearly independent

$\Rightarrow \text{Basis}(P_3) = \{p_1(x), p_2(x), p_3(x), p_4(x)\}$  and  $\text{Dim}(P_3) = 4$

Aside:  $\text{Dim}(P_2) = 3$

$\text{Dim}(P_{10}) = 11$

$\text{Dim}(P) = \infty$  !!! (countably infinite)

E.g. Consider the vector space  $F(I)$  of all cont' functions

defined on a fixed interval  $I$ . Are  $f(x) = \sin x, g(x) = \cos x, h(x) = \tan x$   
 on  $I = (-\frac{\pi}{2}, \frac{\pi}{2})$  linearly dependant or independant

Try: Put  $a f(x) + b g(x) + c h(x) = 0$

$$a \sin x + b \cos x + c \tan x = 0$$

↑ equality of vectors,  
no true if x.

$$\text{If } x=0 \quad a \sin(0) + b \cos(0) + c \tan(0) = 0 \Rightarrow b = 0$$

$$\text{If } x=\frac{\pi}{4} \quad a \sin\left(\frac{\pi}{4}\right) + b \cos\left(\frac{\pi}{4}\right) + c \tan\left(\frac{\pi}{4}\right) = 0 \Rightarrow \frac{1}{\sqrt{2}}a + \frac{1}{\sqrt{2}}b + c = 0$$

$$\text{If } x=\frac{\pi}{6} \quad a \sin\left(\frac{\pi}{6}\right) + b \cos\left(\frac{\pi}{6}\right) + c \tan\left(\frac{\pi}{6}\right) = 0 \Rightarrow \frac{1}{2}a + \frac{\sqrt{3}}{2}b + \frac{1}{\sqrt{3}}c = 0$$

$$\frac{1}{\sqrt{2}}a + c = 0, \quad b = 0, \quad \frac{1}{2}a + \frac{1}{\sqrt{3}}c = 0$$

$$-\sqrt{2}c = a \quad ; \quad \frac{1}{2}a + \frac{1}{\sqrt{3}}c = 0$$

$$-\frac{\sqrt{2}}{\sqrt{2}}c + \frac{1}{\sqrt{3}}c = 0$$

$$c\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}\right) = 0 \Rightarrow c = 0 \Rightarrow a = 0.$$

$\therefore \exists!$  soln (trivial soln).  $\Rightarrow \{\sin(x), \cos(x), \tan(x)\}$  is independant.

### 3B) Complex Numbers

By defn,  $i$  is a soln to  $x^2 + 1 = 0$ . I.e.  $i^2 = -1$

$$\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}\}.$$

Note that the vector space of dimension 2 over the scalar  $\mathbb{R}$ .  $\vec{v}_1 = 1$ ,  $\vec{v}_2 = i$ . Any  $\vec{v} \in \mathbb{C} = a\vec{v}_1 + b\vec{v}_2$ . (If scalar is  $\mathbb{C}$ , one dimension vector space)

$$\text{Let } z = a+bi \quad R_{\mathbb{C}}(z) = a$$

$$\underline{\text{Im}(z) = b} \rightarrow \text{NOT } \underline{\underline{b}}$$

If  $a=0$ ,  $z$  is <sup>purely</sup> "imaginary". If  $b=0$ ,  $z$  is "real"

Multiplication requires distribution:

$$z_1 = a+bi, \quad z_2 = c+di$$

$$\begin{aligned} z_1 \cdot z_2 &= (a+bi)(c+di) = ac + adi + bci + bdi^2 \\ &= \underbrace{(ac - bd)}_{\text{real}} + \underbrace{(ad + bc)i}_{\text{imaginary}}. \end{aligned}$$

$$\text{Given } z = a+bi$$

Its complex conjugate is  $\bar{z} = a-bi$ . The norm/modulus/magnitude of  $z$  is  $\|z\| = \sqrt{a^2 + b^2} \in \mathbb{R}$ .

$$\text{Note that } z\bar{z} = (a+bi)(a-bi) = a^2 - bi^2 = a^2 + b^2 = \|z\|^2 \quad (z\bar{z} = \|z\|^2).$$

$$\text{We use the conjugate for division. } \frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{\bar{z}_2}{\bar{z}_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \frac{z_1 \bar{z}_2}{\|z_2\|^2}$$
$$\left( \frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{\|z_2\|^2} \right).$$

## Interlude on Roots and Factors

$$\text{Consider } p(x) = x^3 + 4x^2 - 11x - 30$$

Could  $x=4$  be a root? No! Doesn't divide 30 into integer

Could  $x=2$  be a root? Maybe... Does divide 30 into an integer

Could  $x=-6$  be a root? Maybe... Does divide 30 into an integer

Could  $x=-\frac{1}{2}$  be a root? Nope

Ok this is weird... What is this rational roots then anyways?

### Rational Roots Theorem:

Let  $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  be a polynomial of degree  $n$  ( $a_n \neq 0$ ) defined over  $\mathbb{Z}$  (i.e.  $a_0, a_1, \dots, a_{n-1}, a_n$  are all integers).

If  $p(x)$  has a rational root  $\pm \frac{r}{s}$  in lowest terms, then necessarily

$r$  must be a divisor of  $a_0$ .

and  $s$  must be a divisor of  $a_n$ .

Remark:  $p(x)$  could have other roots... but these would be irrational.

Reconsidering  $p_1(x) = x^3 + 4x^2 - 11x - 30$ . Possible rational roots are:  $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30$ .  $-2, 3, -5$  are roots of this polynomial.

E.g.  $q(x) = 2x^2 - x - 6$ . Possible rational roots are  $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$ .

(Recall that numerator must divide integer coefficient at lowest power and denominator must divide the integer coefficient at highest power).

Factor Theorem:

If  $r$  is a root of a polynomial  $p(x)$ , then  $(x-r)$  is a factor.

Note: There may be other factors and/or an extra constant as well.

For example,  $p(x) = 2x^3 - x - 6$  has  $p(-2) = -8 + 16 + 22 - 30 = 0$ . This means  $(x+2)$  is a factor of  $p(x)$ .

Indeed,  $p(x) = x^3 + 4x^2 - 11x - 30 = (x+2)(x+5)(x-3)$ .

As we can see, factors and roots are very much related.

Back to Complex Numbers

$$z = a+bi, \bar{z} = a-bi, \|z\| = \sqrt{a^2+b^2}, \text{ Saw } z\bar{z} = \dots = \|z\|^2$$

Properties of Complex Conjugation

- |  |   |
|--|---|
| ① $\bar{\bar{z}} = z$                                | ④ $\overline{z^n} = (\bar{z})^n$  |
| ② $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$ | ⑤ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ |
| ③ $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$   |   |

If  $\bar{z} = z$ , then  $z$  is Real

If  $\bar{z} = -z$ , then  $z$  is imaginary

Think  $z = a+bi$ :

$$\operatorname{Re}(z) = \frac{1}{2}(z + \bar{z}) = a$$

$$\operatorname{Im}(z) = \frac{1}{2i}(z - \bar{z}) = b$$

Suppose  $z \in \mathbb{C}$ ,

Then consider the polynomial  $(x-z)(x-\bar{z})$

It expands as  $x^2 - x\bar{z} - z\bar{x} + z\bar{z}$

$$x^2 - x(z + \bar{z}) + \|z\|^2$$

$$x^2 - 2\operatorname{Re}(z)x + \|z\|^2 \quad \text{This whole thing is real}$$

### Roots of complex numbers

- If I know  $z^n$ , then what is/are the values of  $z$ ?

Suppose  $z^n = r \operatorname{cis} \theta = r \operatorname{cis}(\theta + 2\pi k)$ ;  $k \in \mathbb{N}$  satisfies...

Recall that raising a complex number to  $n^{\text{th}}$  power raises  $r$  to the  $n^{\text{th}}$  power and multiplies  $\theta$  by  $n$ .

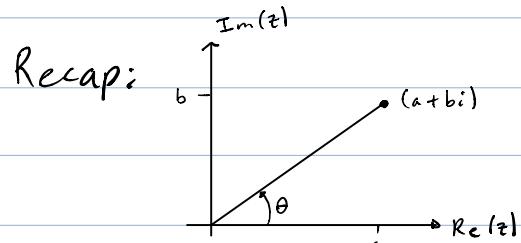
Going backwards means we take  $n^{\text{th}}$  (positive) root of  $r$  and divide angle by  $n$ .

Formally:

If  $z = r \operatorname{cis} \theta = r \operatorname{cis}(\theta + 2\pi k)$  (for  $k \in \mathbb{N}$ )

then,  $z^{1/n} = \sqrt[n]{r} \cdot \operatorname{cis}\left(\frac{\theta + 2\pi k}{n}\right)$   $k$  takes values from 0 to  $n-1$

\*The above formula always gives exactly  $n$  distinct roots



$$z = a + bi$$

$$z = r[\cos \theta + i \sin \theta]$$

$$\text{DeMoivre: } z^n = r^n [\cos(n\theta) + i \sin(n\theta)]$$

Roots of complex numbers: Given  $z^n$ ,  $z = r^{1/n} \operatorname{cis}\left(\frac{\theta + 2\pi k}{n}\right)$  where  $k$  takes on values from 0 to  $n-1$

DeMoivre showed that any non zero number (real or complex) has  $n$  distinct  $n^{\text{th}}$  roots.

E.g. Find all 3<sup>rd</sup> roots of unity i.e. all solns to

$$z^3=1 \text{ where } z \in \mathbb{C}.$$

$$\Rightarrow z = \sqrt[3]{1} \Rightarrow |z| = \sqrt{1^2} \Rightarrow |z| = \boxed{|r|=1} \rightsquigarrow \text{when } z=1, \theta=0 \text{ in complex plane}$$

$$\therefore z = 1^{1/3} \operatorname{cis}\left(\frac{0+2\pi k}{3}\right)$$

$$k=0 \Rightarrow z_0 = 1 \operatorname{cis}\left(\frac{0+2\pi(0)}{3}\right) = \operatorname{cis}(0) = 1 \Rightarrow z_0 = 1$$

$$k=1 \Rightarrow z_1 = 1 \operatorname{cis}\left(\frac{0+2\pi(1)}{3}\right) \Rightarrow z_1 = \operatorname{cis}\left(\frac{2\pi}{3}\right)$$

$$k=2 \Rightarrow z_2 = 1 \operatorname{cis}\left(\frac{0+2\pi(2)}{3}\right) \Rightarrow z_2 = \operatorname{cis}\left(\frac{4\pi}{3}\right)$$

Thus, the 3 cube roots of unity are:

$$z_0 = 1, z_1 = \operatorname{cis}\left(\frac{2\pi}{3}\right), z_2 = \operatorname{cis}\left(\frac{4\pi}{3}\right)$$

$$z^n = w \Rightarrow w^n = z$$

Suppose  $w$ , an  $n^{\text{th}}$  root of a complex number  $z$ , is known.

i.e.  $z^n = w$ . Let  $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}$  be the  $n^{\text{th}}$  roots of unity.

We claim that all  $n^{\text{th}}$  roots of  $z$  are:

$$w_0 = w \varepsilon_0$$

$$w_1 = w \varepsilon_1$$

$$\vdots$$

$$w_{n-1} = w \varepsilon_{n-1}$$

Proof: Let  $w_i : 0 \leq i \leq n-1$  be a root of  $z$  and let  $\{\varepsilon_i\}_{i=0}^{n-1}$

be the  $n$  roots of unity of  $z$ . If  $w$  is a known root of

$z$ , then  $w_i = w \varepsilon_i$  is also a root of  $z$ .

Because  $(w_i)^n = (w \cdot \varepsilon_i)^n = w^n \varepsilon_i^n = z \cdot 1 = z$ .

$$(w_i)^n = z \Rightarrow z^n = w_i.$$

$\therefore w_i$  is also an  $n^{\text{th}}$  root of  $z$ . ■

Are these  $n$  roots always distinct?

Ex: Find a cube root of  $z = -\frac{27}{\sqrt{2}} + \frac{27}{\sqrt{2}}i$ . Use the n roots

of unity to find all cube roots of  $z$ .

$$z = -\frac{27}{\sqrt{2}} + \frac{27}{\sqrt{2}}i \Rightarrow |z| = \sqrt{\left(-\frac{27}{\sqrt{2}}\right)^2 + \left(\frac{27}{\sqrt{2}}\right)^2}$$

$$|z| = \sqrt{\frac{27^2}{2} + \frac{27^2}{2}} = \sqrt{27^2}$$

$$|z| = 27$$

$$z = 27 \left[ -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right]$$

$$\cos \theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{3\pi}{4} \text{ or } -\frac{3\pi}{4} \quad \therefore \theta = \frac{3\pi}{4}$$

$$\sin \theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

$$\Rightarrow z = 27 \operatorname{cis}\left(\frac{3\pi}{4}\right)$$

$$z^{1/3} = 27^{1/3} \operatorname{cis}\left(\frac{3\pi}{3 \cdot 4}\right) \Rightarrow z^{1/3} = 3 \operatorname{cis}\left(\frac{\pi}{4}\right) = \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}i$$

$\therefore \omega = \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}i$  is a cube root of  $z$ .

Cube roots of unity?  $z^3 = 1 = \cos(0) + i\sin(0)$

$$\Rightarrow z^{1/3} = \operatorname{cis}\left(\frac{0+2\pi k}{3}\right)$$

If  $k=0$

$$\xi_0 = \operatorname{cis}(0)$$

$$\xi_0 = 1$$

If  $k=1$

$$\xi_1 = \operatorname{cis}\left(\frac{2\pi}{3}\right)$$

$$\xi_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

If  $k=2$

$$\xi_2 = \operatorname{cis}\left(\frac{4\pi}{3}\right)$$

$$\xi_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\omega_0 = \omega \cdot \xi_0 = \left(\frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}i\right)(1) \Rightarrow \omega_0 = \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}i$$

$$\omega_1 = \omega \cdot \xi_1 = \left(\frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}i\right) \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$\omega_1 = \frac{-3}{2\sqrt{2}} + \frac{3\sqrt{3}}{2\sqrt{2}}i - \frac{3}{2\sqrt{2}}i + \frac{3\sqrt{3}}{2\sqrt{2}}i^2 = \frac{-3}{2\sqrt{2}} \left((\sqrt{3}+1) + (1-\sqrt{3})i\right) \Rightarrow \omega_1 = \frac{-3}{2\sqrt{2}} \left((\sqrt{3}+1) + (1-\sqrt{3})i\right)$$

$$\omega_2 = \omega \cdot \xi_2 = \left(\frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}i\right) \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$\omega_2 = \frac{-3}{2\sqrt{2}} - \frac{3\sqrt{3}}{2\sqrt{2}}i - \frac{3}{2\sqrt{2}}i - \frac{3\sqrt{3}}{2\sqrt{2}}i^2 = \frac{-3}{2\sqrt{2}} (\sqrt{3}+1)i \Rightarrow \omega_2 = \frac{-3}{2\sqrt{2}} (\sqrt{3}+1)i$$

We could also have found all the cube roots of  $z = \frac{-27}{\sqrt{2}} + \frac{27}{\sqrt{2}}i$  using polar coordinates.

Recall that  $z = 27 \text{ cis}(\frac{3\pi}{4})$ .

$$\Rightarrow z^{\frac{1}{3}} = 27^{\frac{1}{3}} \text{ cis} \left( \frac{\frac{3\pi}{4} + 2\pi k}{3} \right)$$

$$\Rightarrow z^{\frac{1}{3}} = 3 \text{ cis} \left( \frac{\pi}{4} + \frac{2\pi k}{3} \right)$$

If  $k=0$

$$w_0 = 3 \text{ cis} \left( \frac{\pi}{4} \right)$$

If  $k=1$

$$w_1 = 3 \text{ cis} \left( \frac{11\pi}{12} \right)$$

If  $k=2$

$$w_2 = 3 \text{ cis} \left( \frac{19\pi}{12} \right)$$

$$w_0 = \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}i \quad * \text{As obtained on previous page.}$$

These have messed up  $\theta$ 's, but plugging the numbers into the calculator shows that  $w_0$ ,  $w_1$ , and  $w_2$  are the same whether we obtain them by multiplying a known  $n^{\text{th}}$  root by each  $n^{\text{th}}$  root of unity or whether we use the DeMoivre method in polar coordinates.

Interlude:  $c^2 = a^2 + b^2 \Rightarrow c^2 = 2 \Rightarrow c = \sqrt{2}$ .

Pythagoras' cult believed that the only numbers that really existed were the rational numbers ( $\mathbb{Q} = \frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{Z}$ ).

However, a mathematician of the pythagorean cult showed that the hypotenuse of the triangle with unit edge lengths ( $h = \sqrt{2}$ ) was NOT a ratio of integers. This was a problem, because how could the hypotenuse of this triangle not be a number? They needed to expand their notion of what a number was...

E.g.  $x^2 + 3x + 2 = 0$  has 2 real roots (-1 & 2)

$x^2 + 4x + 4$  has 1 real root (-2)

$x^2 + 1$  has no real roots...

↳ " $i \equiv \sqrt{-1}$ " → Back to the heart

## The Fundamental Theorem of Algebra:

Any non constant polynomial over  $\mathbb{C}$  has a complex root.

Let  $p(x)$  be a polynomial of degree  $n$  over  $\mathbb{C}$  ( $n \neq 0$ ).

If  $p(x)$  has a root, call it  $r_1 \Rightarrow (x-r_1)$  is a factor of  $p(x)$ .

So  $p(x) = (x-r_1) \underbrace{q(x)}_{\text{degree } n-1}$ .  $q(x)$  will also have a root, call it  $r_2$ , and  $(x-r_2)$  would then be a factor.  $p(x) = (x-r_1)(x-r_2) \underbrace{A(x)}_{\text{degree } n-2}$

We can repeat this process until  $p(x)$  is factored into  $n$  linear terms.

$$\text{So, } p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = K \underbrace{(x-r_1)(x-r_2)\dots(x-r_n)}_{\text{constant}}$$

where  $r_1, r_2, \dots, r_n$  are the  $n$  complex roots of the polynomial.

Note: These roots do not need to be complex.

The number of times the same root appears in the complex linear factorization of  $p(x)$  is called the multiplicity of  $p(x)$ .

Recap: Any polynomial of degree  $n$  has  $n$  roots\* in the complex plane and therefore can be factored into  $n$  linear terms.

$$\text{i.e. } p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = (x-r_1)(x-r_2)\dots(x-r_n)$$

where  $r_1, r_2, \dots, r_n$  are the roots of  $p(x)$ . These roots are not necessarily distinct. (\*If we count complex roots and multiplicities)

There is a very important corollary of the FTA to the real polynomials.

Theorem: Any non constant polynomial over  $\mathbb{R}$  factors completely as a product of linear (degree 1) and quadratic (degree 2) factors that are all defined over  $\mathbb{R}$ .

Proof: Let  $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  be a polynomial of degree  $n$ . Let  $r_1, r_2, \dots, r_R$  be the real roots of  $p(x)$ . Let  $s_1, s_2, s_3, \dots, s_C$  be the complex roots of  $p(x)$ . By the conjugate roots theorem, if  $s_i \in \mathbb{C}$  is a root  $\bar{s}_i \in \mathbb{C}$  is also a root.  $\therefore$  The complex roots of  $p(x)$  are

$s_1, \bar{s}_1, s_2, \bar{s}_2, \dots, s_{\frac{C}{2}}, \bar{s}_{\frac{C}{2}}$ . The factor theorem tells us that

$$p(x) = (x - r_1)(x - r_2) \cdots (x - r_R)(x - s_1)(x - \bar{s}_1)(x - s_2)(x - \bar{s}_2) \cdots (x - s_{\frac{C}{2}})(x - \bar{s}_{\frac{C}{2}})$$

$$p(x) = (x - r_1)(x - r_2) \cdots (x - r_R) \cdot [x^2 - x\bar{s}_1 - x s_1 + s_1 \bar{s}_1] \cdots [x^2 - x\bar{s}_{\frac{C}{2}} - x s_{\frac{C}{2}} + s_{\frac{C}{2}} \bar{s}_{\frac{C}{2}}]$$

$$\text{Recall: } \operatorname{Re}(z) = \frac{1}{2}(z + \bar{z}) \Rightarrow -2\operatorname{Re}(z) = -(z - \bar{z})$$

$$\text{Recall: } z \cdot \bar{z} = \|z\|^2 \text{ and } \|z\|^2 \in \mathbb{R}.$$

$$\Rightarrow p(x) = \underbrace{(x - r_1)(x - r_2) \cdots (x - r_R)}_{\text{Linear terms}} \cdot \underbrace{[x^2 - 2\operatorname{Re}(s_1)x + \|s_1\|^2] \cdots [x^2 - 2\operatorname{Re}(s_{\frac{C}{2}})x + \|s_{\frac{C}{2}}\|^2]}_{\text{Quadratic terms (all have real coefficients)}}$$

$$\text{where } R + C = n \text{ (by FTA).}$$

This theorem is what we used in partial fraction decomposition.

Ex: Factor  $x^4 + 16$  completely over  $\mathbb{R}$ .

$$\text{Solve } x^4 + 16 = 0 \quad \rightarrow z^4 = 16(\cos(\pi) + i\sin(\pi))$$

$$x^4 = -16 \quad \rightarrow z = 2(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}))$$

$$\Rightarrow w = \sqrt{2} + \sqrt{2}i \text{ is a fourth root of } -16$$

Now let's find the fourth roots of unity.

$$\begin{aligned}
 z^4 &= 1 \\
 z^4 &= \text{cis}(0) \\
 z &= \text{cis}\left(\frac{0+2\pi k}{4}\right) \\
 z &= \text{cis}\left(\frac{k\pi}{2}\right)
 \end{aligned}
 \quad \left. \begin{array}{l}
 \text{if } k=0 \quad \text{if } k=1 \quad \text{if } k=2 \quad \text{if } k=3 \\
 \varepsilon_0 = \text{cis}(0) \quad \varepsilon_1 = \text{cis}\left(\frac{\pi}{2}\right) \quad \varepsilon_2 = \text{cis}(\pi) \quad \varepsilon_3 = \text{cis}\left(\frac{3\pi}{2}\right) \\
 \varepsilon_0 = 1 \quad \varepsilon_1 = i \quad \varepsilon_2 = -1 \quad \varepsilon_3 = -i
 \end{array} \right\}$$

$$\begin{aligned}
 \omega_0 &= \omega \varepsilon_0 = (\sqrt{2} + \sqrt{2}i)(1) \Rightarrow \omega_0 = \sqrt{2} + \sqrt{2}i \\
 \omega_1 &= \omega \varepsilon_1 = (\sqrt{2} + \sqrt{2}i)(i) \Rightarrow \omega_1 = -\sqrt{2} + \sqrt{2}i \\
 \omega_2 &= \omega \varepsilon_2 = (\sqrt{2} + \sqrt{2}i)(-1) \Rightarrow \omega_2 = -\sqrt{2} - \sqrt{2}i \\
 \omega_3 &= \omega \varepsilon_3 = (\sqrt{2} + \sqrt{2}i)(-i) \Rightarrow \omega_3 = \sqrt{2} - \sqrt{2}i
 \end{aligned}
 \quad \left. \begin{array}{l}
 \omega_0, \omega_1, \omega_2, \omega_3 \text{ are the four roots of } p(x) = x^4 + 16. \\
 \text{Note that } \overline{\omega_0} = \omega_3 \text{ and that } \overline{\omega_1} = \omega_2. \text{ (tkn conjugate roots thm...)}
 \end{array} \right\}$$

$$\Rightarrow x^4 + 16 = (x - \omega_0)(x - \overline{\omega_0})(x - \omega_1)(x - \overline{\omega_1})$$

$$x^4 + 16 = (x^2 - 2 \operatorname{Re}(\omega_0)x + |\omega_0|^2)(x^2 - 2 \operatorname{Re}(\omega_1)x + |\omega_1|^2)$$

$$x^4 + 16 = (x^2 - 2\sqrt{2}x + 4)(x^2 + 2\sqrt{2}x + 4)$$

Lo and behold, this polynomial factors completely as a product of terms of degree  $n \leq 2$  over  $\mathbb{R}$ . This was predicted by above theorem.

Ex: Suppose  $f(x) = x^2 + x$ . What is  $f(1+i)$ ?

$$f(1+i) = (1+i)^2 + (1+i)$$

$$f(1+i) = (1+2i+i^2) + (1+i)$$

$$f(1+i) = 2i + 1 + i$$

$$f(1+i) = 1 + 3i$$

Now, what if  $f(x)$  was more complicated? Like if we had  $f(x) = \operatorname{arccot}(x)$ . What would  $\operatorname{arccot}(1+i)$  be?

We need to mix complex numbers with... Power Series!!

FACT: The power series expansions of many important funcs like  $e^x, \sin x, \cos x$  can be used when  $x \in \mathbb{C}$ .

Important power series are:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots$$

We can have  $e^x$  with  $x = i\theta$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} \dots$$

$$e^{i\theta} = \underbrace{\left[ 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \right]}_{\cos \theta} + i \underbrace{\left[ \frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \right]}_{\sin \theta}$$

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \text{Euler's Formula!}$$

\* If  $\theta = \pi$   $e^{i\pi} = \cos \pi + i \sin \pi = -1 + 0$

$$\therefore e^{i\pi} + 1 = 0$$