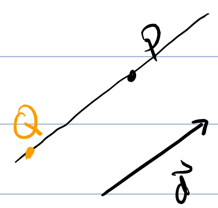
# 2 - Lines

Given a point $P(x_0, y_0, z_0)$ in $\mathbb{R}^3$ and a vector $\vec{d} = \langle a, b, c \rangle$, (direction vector) we want to find a vector expression that is satisfied by the points $Q$ on the line.

How do you represent non-linear equations?



$Q(x, y, z)$ is on the line iff $\overline{PQ} = t\vec{d}$ $t \in \mathbb{R}$

$$\langle x - x_0, y - y_0, z - z_0 \rangle = k \langle a, b, c \rangle$$

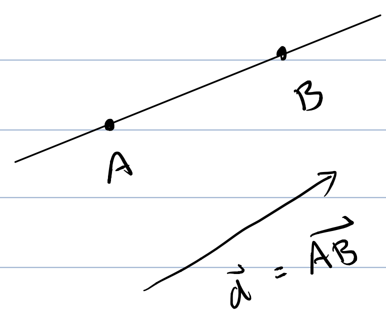
$$\begin{aligned} x - x_0 &= ta \\ y - y_0 &= tb \\ z - z_0 &= tc \end{aligned} \rightarrow \boxed{\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases} \quad t \in \mathbb{R}}$$

ex, a) write parametric equations of the line passing through $A(1, -1, 5)$ parallel to the vector $\vec{d} = \langle 3, 5, 0 \rangle$

$$\begin{cases} x = 1 + 3t \\ y = -1 + 5t \qquad t \in \mathbb{R} \\ z = 5 \end{cases}$$

b) ______ " ________________________ passing through

$A(1,2,3)$ and $B(3,-1,7)$




Take A or B as the point P and the vector $\vec{AB}$ as a direction vector

$\vec{AD} = \langle 2, -3, 4 \rangle$

$$\begin{cases} x = 1 + 2t \\ y = 2 - 3t \qquad t \in \mathbb{R} \\ z = 3 + 4t \end{cases}$$

## Vector Parametric Form

$$l = \begin{cases} x = 1 + 2t \\ y = 2 - 3t \\ z = 3 + 4t \end{cases}$$

We can write the parametric vector form as follows:

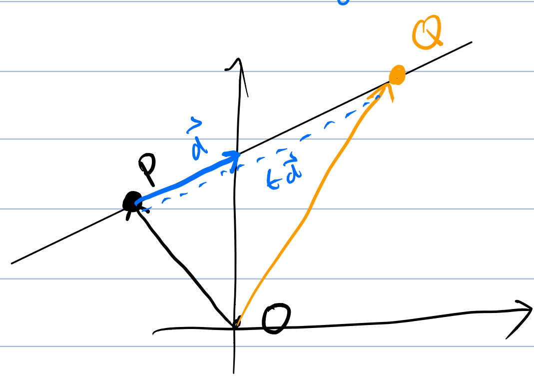
$$\langle x, y, z \rangle = \langle 1 + 2t, 2 - 3t, 3 + 4t \rangle$$

$$\langle x, y, z \rangle = \langle 1, 2, 3 \rangle + t \langle 2, -3, 4 \rangle$$

Can also be written as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \quad t \in \mathbb{R}$$

$$\vec{OQ} = \vec{OP} + t\vec{d}$$




ex. Write a vector parametric equation for the line passing through A(0,0,0) and B(3,2,-5)

$$d = \langle 3, 2, -5 \rangle$$

| Using A as point P | Using B as point P |
|---|---|
| $\langle x, y, z \rangle = t \langle 3, 2, -5 \rangle$ | $\langle x, y, z \rangle = \langle 3, 2, -5 \rangle + t \langle 3, 2, -5 \rangle$ |

Note: A line with no vector of constants necessarily passes through the origin

If there is no vector of constants, then the line passes through the origin

Symmetric form of the equation of a line

$x = x_0 + at$

$y = y_0 + bt$

$z = z_0 + ct$

$\rightarrow$

$t = \dfrac{x - x_0}{a}$

$t = \dfrac{y - y_0}{b}$

$t = \dfrac{z - z_0}{c}$

$\rightarrow$

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

ex: a) $x = 3 + t$

$y = -4 - t$

$z = t$

$$\frac{x - 3}{1} = \frac{y + 4}{-1} = \frac{z}{1}$$

b) $x = 1 + t$ $\rightarrow$ $x - 1 = 3 - z$ ; $y = 5$

$y = 5$

$z = 3 - t$

c) $x = 2 - t$

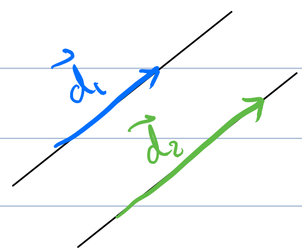
$y = 5$ $\rightarrow$ $\dfrac{x - 2}{-1} = z$ ; $y = 5$

$z = t$

# Relative Positions of Two Lines in $\mathbb{R}^3$

i) Two lines can be coplanar (contained within one plane):

a) Two parallel lines:

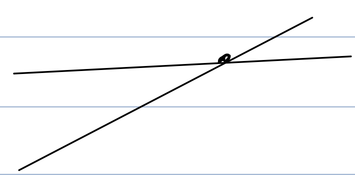



Two lines are parallel iff their direction vectors are parallel

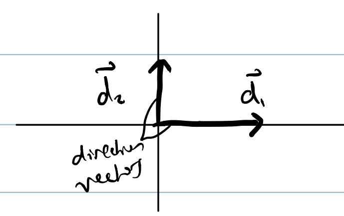
i.e. $\vec{d_1} = k\vec{d_2}$

or $\vec{d_1} \times \vec{d_2} = \vec{0}$

b) Two intersecting lines:



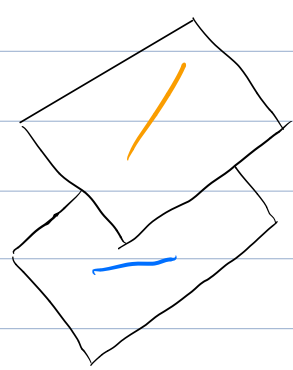
particular case: perpendicular lines




iff $\vec{d_1}$ and $\vec{d_2}$ are $\perp$

iff $\vec{d_1} \cdot \vec{d_2} = 0$

ii) Two skew lines (i.e. not coplanar)



Note: For any two skew lines, there is always a pair of parallel planes each containing one of the lines.

→ skew lines are lines that are not intersecting nor parallel

## Finding intersection of two intersecting lines

ex. $l_1$ :
$$x = -3 + 4t_1$$
$$y = 5 - t_1$$
$$z = -6 + 7t_1$$

$$l_2 : \frac{x-11}{2} = \frac{y-6}{1} = \frac{z+4}{-4}$$

Find the intersection of lines $l_1$ and $l_2$ if possible.

$$l_2 : \frac{x-11}{2} = \frac{y-6}{1} = \frac{z+4}{-4}$$

$$\begin{cases} x = 11 + 2t_2 \\ y = 6 + 2t_2 \\ z = -4 - 4t_2 \end{cases} \quad t \in \mathbb{R}$$

* **Note:** Need to distinguish between 't's on lines

$$\left.\begin{array}{ll} (1) & -3 + 4t_1 = 11 + 2t_2 \\ (2) & 5 - t_1 = 6 + t_2 \end{array}\right\} \text{solve for } t_1 \text{ and } t_2$$

$$(3) \quad -6 + 7t_1 = -4 - 4t_2$$

$$(1) - 2(2)$$

$$-3 + 4t_1 = 11 + \cancel{2t_2}$$

$$-10 + 2t_1 = -12 - \cancel{2t_2}$$

$-13 + 6t_1 = -1$

$6t_1 = 12$

$\underline{t_1 = 2}$

$5 - (2) = 6 + t_2$

$\underline{t_2 = -3}$

Check in 3

$-6 + 7(2) = -4 - 4(-3)$

$8 = 8$ ✓

Now, subst $t_1 = 2$ into $l_1$

∴ we obtain pt $(5, 3, 8)$

Can do the same for $l_2$ and we find same pt $(5, 3, 8)$

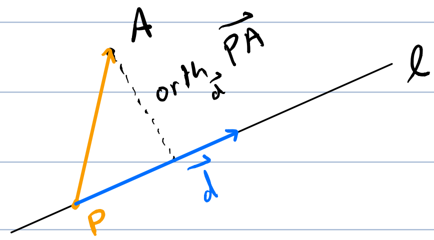
∴ pt $(5, 3, 8)$ is the pt of intersection of $l_1$ and $l_2$

* To determine skew lines, can check if parallel, the check if no intersection. But, there's a better way...

## Distance from a point $A(x_0, y_0, z_0)$ to a line in $\mathbb{R}^3$

Two ways:

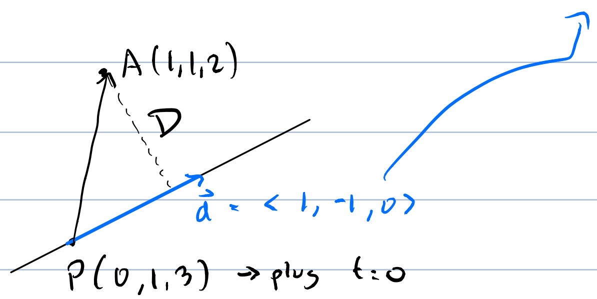
a)




$$D = \|\text{orth}_{\vec{d}}\, \vec{PA}\|$$

ex. Find the distance from the point $A(1,1,2)$ to the line

$\ell: \langle x, y, z \rangle = \langle 0,1,3 \rangle + t\langle 1,-1,0 \rangle$




$$D = \|\text{orth}_{\vec{d}}\, \vec{PA}\|$$

$$= \|\vec{PA} - \text{proj}_{\vec{d}}\, \vec{PA}\|$$

$$= \left\| \overrightarrow{PA} - \frac{\overrightarrow{PA} \cdot \vec{d}}{\|d\|^2} \vec{d} \right\|$$

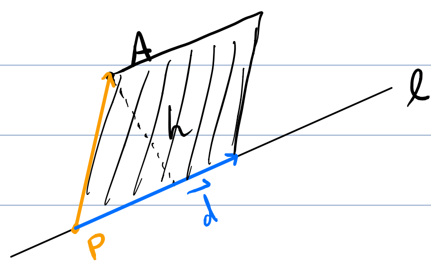
$$= \left\| \overrightarrow{PA} - \frac{\langle 1, 0, -1 \rangle \cdot \langle 1, -1, 0 \rangle}{\| \langle 1, -1, 0 \rangle \|} \langle 1, -1, 0 \rangle \right\|$$

...

$$= \left\| \langle \tfrac{1}{2}, \tfrac{1}{2}, -1 \rangle \right\|$$

$$= \sqrt{\frac{3}{2}}$$

b)




$$h = \frac{\text{Area of Parallelogram}}{\text{length of base}}$$

$$= \frac{\overrightarrow{PA} \times \vec{d}}{\|\vec{d}\|}$$

ex. Previous example with method b

$$h = \frac{\| \langle 1, 0, -1 \rangle \times \langle 1, -1, 0 \rangle \|}{\sqrt{2}}$$

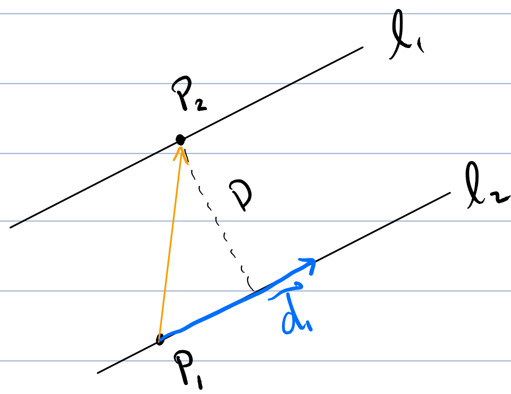
$$\begin{array}{ccc} 1 & 0 & -1 \\ 1 & -1 & 0 \end{array}$$

$$= \frac{\| \langle -1, -1, -1 \rangle \|}{\sqrt{2}}$$

$$= \sqrt{\frac{3}{2}}$$

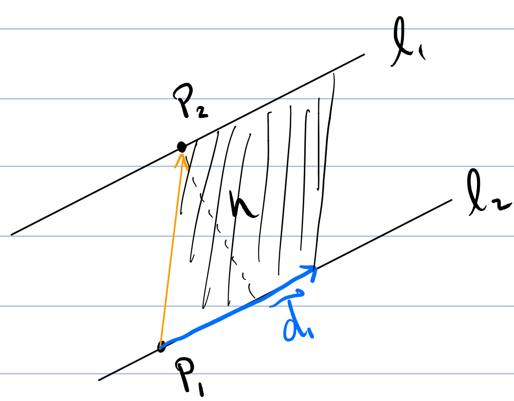
## Distance between two parallel lines

a)




$$D = \| \operatorname{orth}_{\vec{d}_1} \overrightarrow{P_1 P_2} \|$$

b)

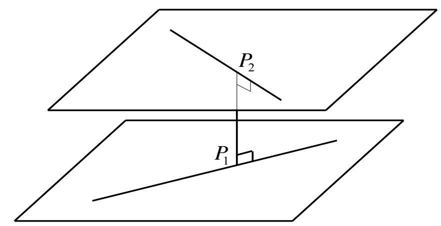



$$h = \frac{\text{Area of Parallelogram}}{\text{norm of base}} = \frac{\| \overrightarrow{P_1 P_2} \times \vec{d} \|}{\| \vec{d} \|}$$

**201-105** **ICP 3**

**1.** Consider the lines $l_1: \begin{array}{l} x = 3 - t \\ y = t \\ z = -2 \end{array}$ and $l_2: \begin{array}{l} x = 6 + 3t \\ y = -5 - t \\ z = 4 + t \end{array}$

a) Show that $L_1$ and $L_2$ are ***skew lines*** .




b) Find equations of two parallel planes, each containing one of the lines above.

c) Find the distance between the two lines.

**2.** a) Show that the lines $l_1 : [x, y, z] = [-2, 3, 1] + t[-4, -2, 2]$ and $l_2 : [x, y, z] = [1, 1, 2] + t[2, 1, -1]$ are parallel.

b) Consider the point $A(-2,3,1)$ on line $l_1$ in (a). Find the point on the line $l_2$ closest to the point $A$.

c) Compute the distance between the two lines in two ways.

September 14, 2017

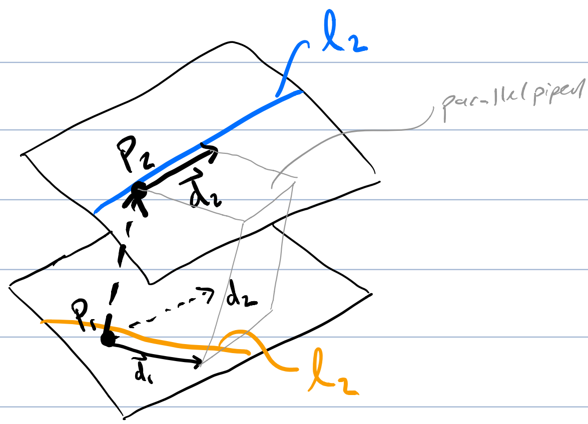
# Skew Lines

- skew lines are lines that are **not coplanar.**

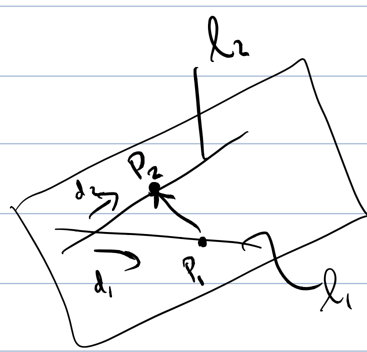
Q: How to identify skew lines?

One way: - Check that they don't intersect
- check that they are not parallel

Other way:




Note: If $l_1$ and $l_2$ intersect, then $\overrightarrow{P_1P_2} \cdot (\vec{d}_1 \times \vec{d}_2) = 0$
(scalar triple product)



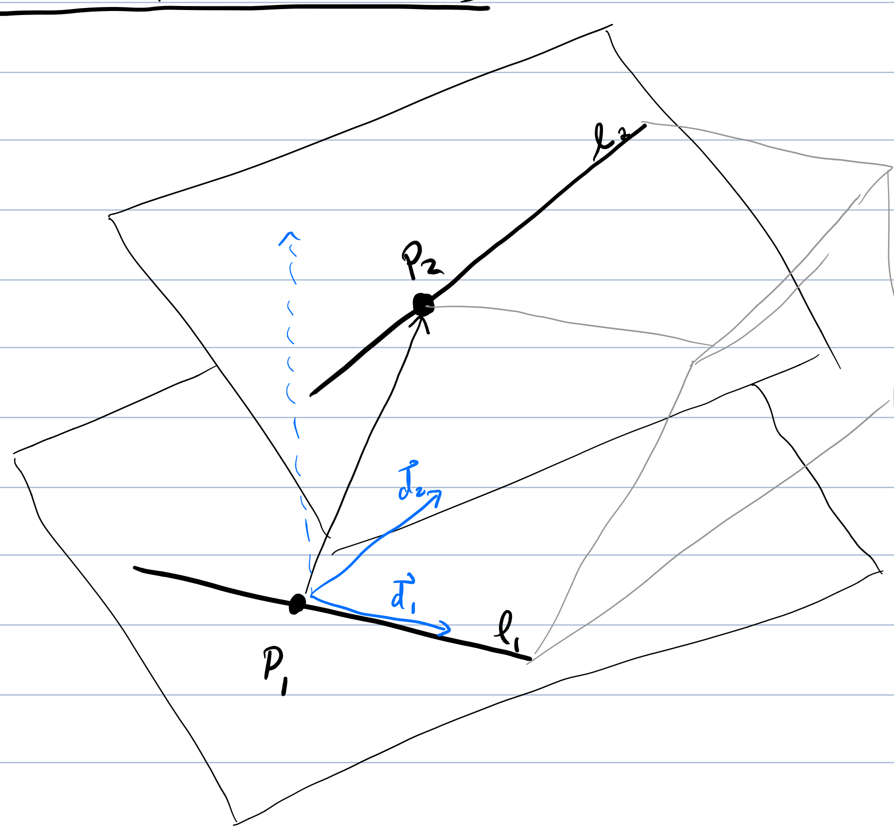
Note: If $\ell_1$ and $\ell_2$ are parallel, then $\overrightarrow{P_1P_2} \cdot \underbrace{(\vec{d}_1 \times \vec{d}_2)}_{\vec{0}} = 0$

So,

Two lines $\ell_1$ and $\ell_2$ in $\mathbb{R}^3$ are skew (not coplanar) if and only if $\overrightarrow{P_1P_2} \cdot (\vec{d}_1 \times \vec{d}_2) \neq 0$

where $P_1$ and $P_2$ are random points on $\ell_1$ and $\ell_2$ respectively.

## Distance between skew lines



$\vec{n} = \vec{d_1} \times \vec{d_2}$

$$D = \left\| \operatorname{proj}_{\vec{n}} \overrightarrow{P_1 P_2} \right\|$$

$$\boxed{D = \frac{\left| \overrightarrow{P_1 P_2} \cdot (\vec{d_1} \times \vec{d_2}) \right|}{\| \vec{d_1} \times \vec{d_2} \|}}$$

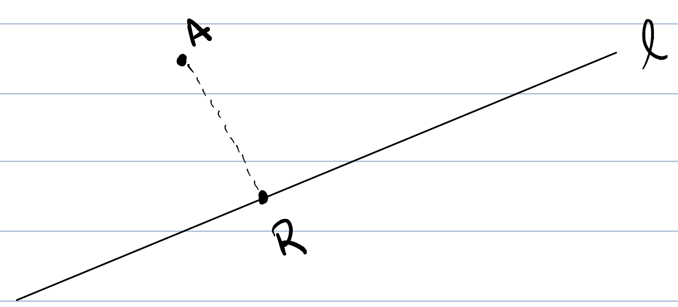
← volume of parallelpiped

← base of parallelpi

... makes sence because

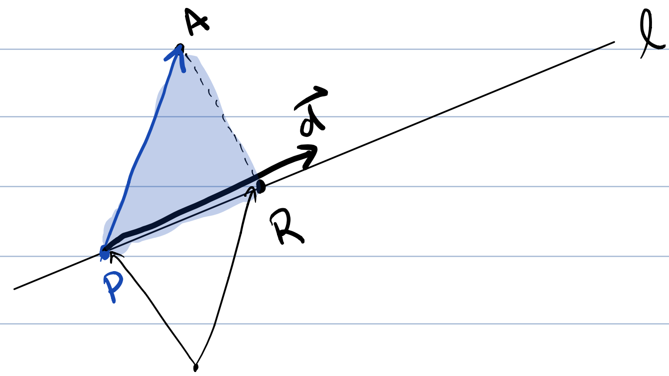
$$\underbrace{h}_{D = \text{height of parallelpiped}} = \frac{v}{b}$$

## Finding the closest point (point to line)





Given a line $\ell$ in $\mathbb{R}^3$ and a point $A$, not on the line, find the pt $R$ on the line that is closest to $A$.




① Chose a random pt P on the line

② Find $\text{proj}_{\vec{d}}\overrightarrow{PA}$.

$$\boxed{\text{So } \overrightarrow{OR} = \overrightarrow{OP} + \text{proj}_{\vec{d}}\overrightarrow{PA}}$$

pt R

ICP $l_1 : [x, y, z] = [-2, 3, 1] + t[-4, -2, 2]$ and $l_2 : [x, y, z] = [1, 1, 2] + t[2, 1, -1]$

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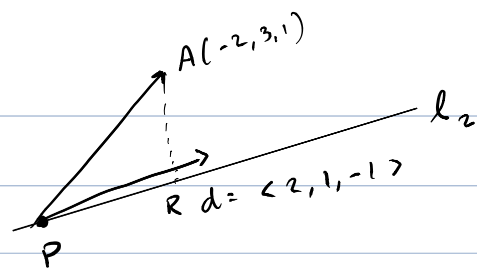


So $\overrightarrow{OR} = \overrightarrow{OP} + \text{proj}_{\vec{d}}\overrightarrow{PA}$

$$= \langle 1, 1, 2 \rangle + \frac{-6 + 2 + 1}{2^2 + 1^2 + (-1)^2}\langle 2, 1, -1 \rangle$$

$= \langle 1,1,2 \rangle - \frac{1}{2} \langle 2,1,-1 \rangle$

$\vec{OR} = \langle 0, \frac{1}{2}, \frac{5}{2} \rangle$

$\therefore$ Pt $R\left(0, \frac{1}{2}, \frac{5}{2}\right)$

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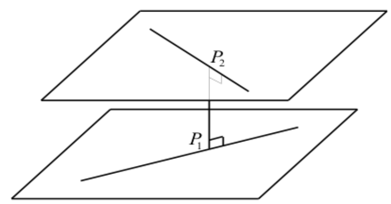


$\vec{P_1P_2} = \langle 3, -5, 6 \rangle$

$d_1 = \langle -1, 1, 0 \rangle$

$d_2 = \langle 3, -1, 1 \rangle$

$\vec{P_1P_2} \cdot (\vec{d_1} \times \vec{d_2}) = -14 \neq 0$

$\therefore$ lines are skew

c) Compute the distance between the two lines in two ways.

$$D = \frac{\left| \overrightarrow{P_1P_2} \cdot (\vec{d_1} \times \vec{d_2}) \right|}{\|\vec{d_1} \times \vec{d_2}\|}$$

$$= \frac{|-14|}{\|\langle 1, 1, -2 \rangle\|}$$

$$= \frac{14}{\sqrt{6}}$$