BASIC CONCEPTS & IDEAS:

solving for real values of x:

$2x+5=0 \rightarrow x=\frac{-5}{2}$

$x^3-2x=0 \rightarrow x(x^2-2)=0 \Rightarrow x=0,\ x=\sqrt{2},\ x=-\sqrt{2}$

$2e^x = e^{2x}+1 \rightarrow (e^x-1)^2=0 \rightarrow e^x=1 \rightarrow x=0$

$(x+2)^2+3=0 \rightarrow$ no real soln!

$\cos x - x = 0 \rightarrow$ good luck buddy (Transcendental eqn)

Find a fcn $y=f(x)$ satisfying the eqn:

① $y'+y=0 \rightarrow y=e^{-x}$ works (can check)

② $y''+y=0 \rightarrow y=\sin x$ works, so does $y=\cos x$, so does any linear combination of sines and cosines works too i.e. $y=A\sin x+B\cos x$

③ $y'+\frac{x^2}{1+x^2}=0 \rightarrow y'=-\frac{x^2}{1+x^2}$ integrate both sides, solve. $y=\int\frac{-x^2}{1+x^2}dx$

Definition

① Ordinary differential equation (ODE)

An equation involving one or more derivatives of an unknown function of one variable. (Goal is to find the fcn that satisfies the equation)

② The "order" of an ODE: The maximum "order" of derivatives shown in the equation.

Examples:

a) $y' - \cos x = 0$ (order 1). Find $y(x) \xrightarrow{\text{Simply an integral}}$ $y(x) = \int \cos x \, dx$

b) $x'' + \omega^2 x = 0$ (order 2). Find $x(t) \to$ SHM

c) $x^2 (y')^4 \sqrt{y'''} + 2e^x y^2 = \pi$ (order 3).

Q: What isn't an "ordinary" diffl eqn

A: "Partial" diffl eqns

Cal III: $z = f(x,t) = x^4 t^3$

$\frac{\partial z}{\partial t} = 3x^4 t^2$ $\frac{\partial z}{\partial x} = 4x^3 t^3$ are the partial derivs of z.

A partial diffl eqn could be:

$\to \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial t^2} = 0$ where $f = f(x,t)$

$\to$ The wave eqn (s/o to Vinet)

Concept of Solution:

A fcn $y=f(x)$ is a soln of a diffl eqn on an interval $x \in I$ (perhaps infinite) when it identically satisfies the diffl eqn on I.

Note: It would not make sense to talk about the soln of a DE at only one specified value of x, because there could be no derivatives → points don't have slopes

E.g. Consider $xy' = 3y - 2x$ for $x \in \mathbb{R}$

Verify that $y(x) = x$ is a soln

LHS $xy' = x \cdot 1 = x$ ✓
RHS $3y - 2x = 3x - 2x = x$ ✓
} Works! RHS = LHS :)

Would $x + C$ work? No! LHS: x

RHS: $x + 3C$

Would Cx work? LHS: $xy' = Cx$

RHS: $3y - 2x = 3Cx - 2x$ only true if $x=1$ (previous answer)

Another try: $y = x + x^3$ LHS: $xy' = x(1 + 3x^2) = x + 3x^3$ ✓

RHS: $3y - 2x = 3(x^3 + x) - 2x = 3x^3 + x$ ✓

WORKS!

Actually, lets try: $y = x + Cx^3$

LHS: $xy' = x(1 + 3Cx^2)$

RHS: $3y - 2x = 3(x^3 + x) - 2x = x^3 + x$

Lets look at the graph of this family of solns

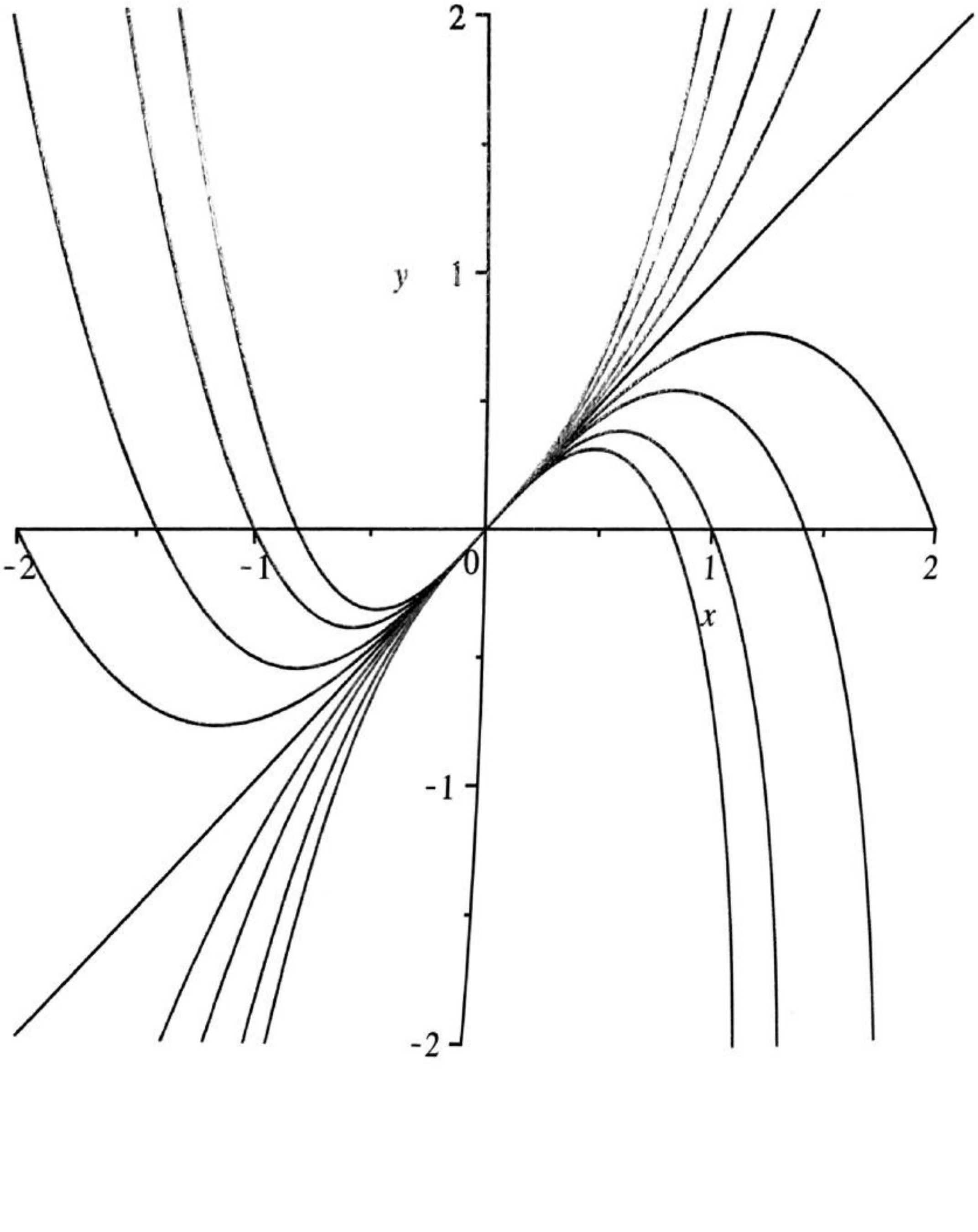
General Solution of

$x\,y' = 3y - 2x$

is

$y = x + C\,x^3$

```
> C0 := 0 : C1 := 0.25 : C2 := 0.5 : C3 := 1 : C4 := 1.5 : C5 := -0.25 : C6 := -0.5 : C7 := -1 :
    C8 := -1.5 :
>
> plot([x + C0·x^3, x + C1·x^3, x + C2·x^3, x + C3·x^3, x + C4·x^3, x + C5·x^3, x + C6·x^3, x + C7·x^3, x
    + C8·x^3], x=-2..2, y=-2..2, scaling = constrained, color = [black, red, red, red, red, blue,
    blue, blue, blue]);
```



A soln of a differential equation involving arbitrary constants "C, D, E..." is called a <u>general solution</u> of the diffl eqn. When we assign particular values to the constants, we have a particular soln.

Now, things are not always that simple and clean w/ diffl eqns...

For example, consider the 1st order ODE

$(y')^2 - xy' + y = 0$. Lets verify that $y = Cx - C^2$ is a general soln

$\cancel{C^2} - \cancel{x}C + \cancel{C}x - \cancel{C^2} = 0$ ☑

Thus, $C=0$: $y=0$

$C=1$: $y = x-1$ $C=-1$: $y = -x-1$

$C=2$: $y = 2x-4$ $C=-2$: $y = -2x-4$

$C=3$: $y = 3x-9$ $C=-3$: $y = -3x-9$

Now look:

Try $y = \frac{x^2}{4}$ as a possible solution.

Then, $(y')^2 - xy' + y = 0$

$\left(\frac{2x}{4}\right)^2 - x\left(\frac{2x}{4}\right) + \left(\frac{x^2}{4}\right) = 0$

$\frac{x}{4} - \frac{x^2}{2} + \frac{x^2}{4} = 0$

$\frac{x^2}{2} - \frac{x^2}{2} = 0$ ☑ . Thus $y = \frac{x^2}{4}$ is also a solution even though it is not part of the "general soln" $y = Cx - C^2$

Graphically represented on next page

$$(y')^2 - xy' + y = 0$$

has a

Singular Solution

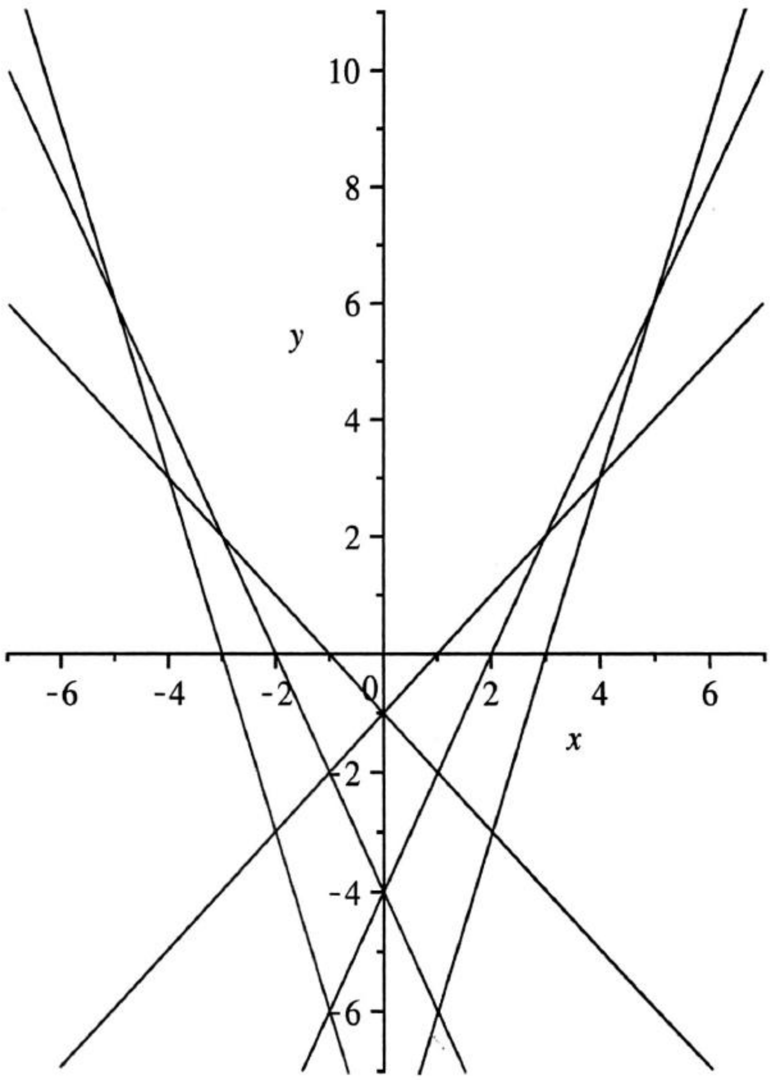
→ A soln

Family Forming the General Solution

$$y = C\,x - C^2$$

for C = 0, 1, −1, 2, −2, 3, −3

```
> plot([0,x-1,-x-1,2*x-4,-2*x-4,3*x-9,-3*x-9],x=-7..7,y=-7..11,
  color=red,scaling=constrained);
```

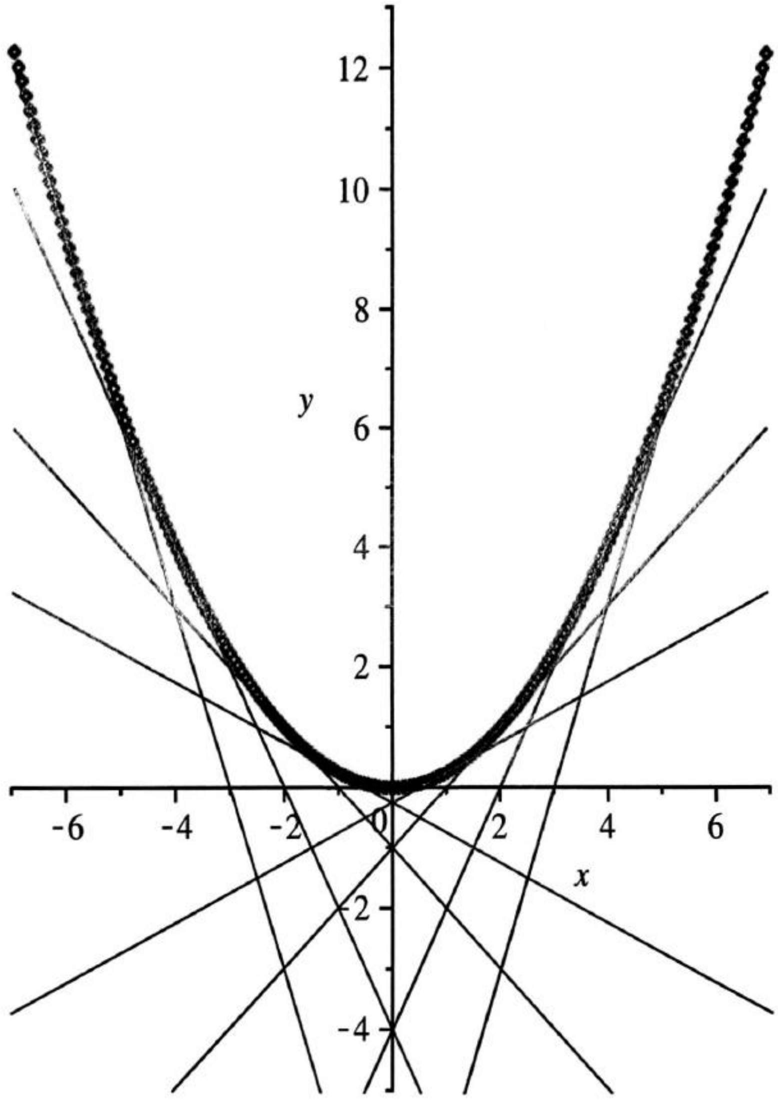


```
>
>
>
```

Singular Solution

$$y = x^2/4$$

```
>
>
> plot([x^2/4,0,x/2-1/4,-x/2-1/4,x-1,-x-1,2*x-4,-2*x-4,3*x-9,-3*
  x-9],x=-7..7,y=-5..13,color=[blue,red,red,red,red,red,red,red,
  red,red],style=[point,line,line,line,line,line,line,line,line,
  line],scaling=constrained);
```



Such a soln is called a singular soln of the diffl eqn.

We will do our best to try to find out when a diffl eqn does or does not have singular solns. Singular soln ≡ a soln to DE not encompassed by general soln.

Lets try to guess some solns to DE^s to build our intuition

① $y' = 2xy$. Could be $y = e^{x^2}$

$(e^{x^2})' = 2xe^{x^2} = 2xy$ ☑

Actually, $y = Ce^{x^2}$ is the gen'l soln

② $y'' + 9y = 0$ Looks like SHM...

$y = \cos(3x)$ is a soln? $y'' = -9\cos(3x)$

$-9\cos 3x + 9y = 0$

$-9\cos(3x) + 9\cos(3x) = 0$

Note: $\sin 3x$ also works

Actually $y = A\sin(3x) + B\cos(3x)$ is the general soln

③ Guess the soln to $y' = -\frac{x}{y}$?

$y = \pm\sqrt{c^2 - x^2}$ or $x^2 + y^2 = c^2$ (a circle)

Take ⊕ve:

$$y' = -\frac{\cancel{2}x}{\cancel{2}\sqrt{c^2-x^2}} = -\frac{x}{\sqrt{c^2-x^2}}$$

A first application:

Quantum physics tells that radioactive substances decay at a rate proportional to the amount present

Q: Starting with 2g of $^{226}_{88}Ra$ at $t=0$, describe the amount remaining at later times t.

A: Let $y(t)$ be the amount present at time $t \geq 0$
↳ in grams ↳ in seconds

Information: $y \propto y' \Rightarrow y'(t) = k\,y(t)$. Um... $y = e^{kt}$ works

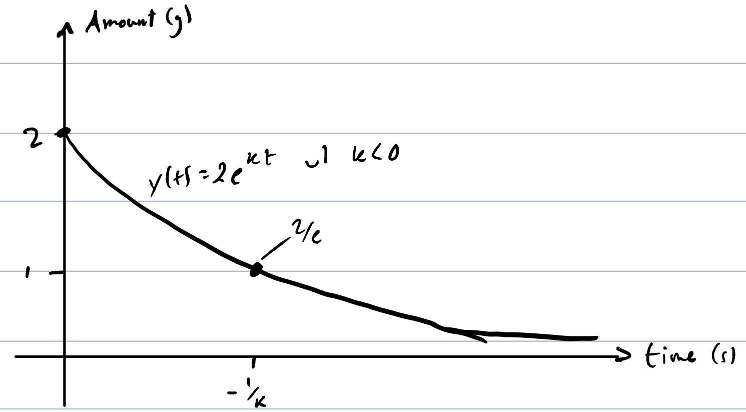
General soln: $y = Ce^{kt}$

For $^{226}_{88}Ra$ $k = -1.4 \cdot 10^{-11}\ s^{-1}$

$y(0) = 2 \Rightarrow Ce^{k(0)} = 2 \Rightarrow C = 2$

$\therefore y(t) = 2e^{kt}$ is a particular soln

Graphically:



In Linear, gen'l soln had __all__ possible soln, but in DE, the general soln where # free variables corresponds to the order of the DE may not encompass every possible solution (i.e. $\exists$ singular solns).
The simplest case where we can settle the question of singular solns or not. Very Simple DE: $y' - f(x) = 0$.
Indeed, this is $y' = f(x) \Rightarrow y(x) = \int f(x)\,dx = F(x) + C$ From cal II, we proved that this contains __all__ possible solns to $y = \int f(x)\,dx$

RECALL ON SOME ANTIDIFFERENTIATION THEORY

(Why the "+C" Does Give <u>ALL</u> the Possible Antiderivatives)

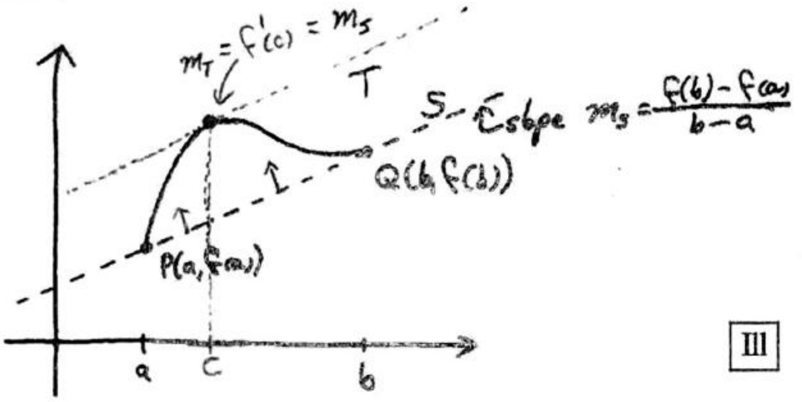
It all starts with a theorem from Calculus I:

<u>(Lagrange's) Mean Value Theorem: MVT</u>

Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b).

Then $\exists c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$. $\boxed{\text{IVT}}$

<u>Illustration:</u>



$\boxed{\text{III}}$

If $f(x)$ is conts and diffble on $[a,b]$, then $\exists$ c on (a,b) | $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

The MVT says that the average rate of change between the two points is obtained instantaneously at one point, or, that there is a place c where the slope there equals the *average slope* (*<u>Mean Value</u>* "of the slope") on the interval.

The question that the MVT will resolve is the following...

Suppose we are told that a function has derivative zero on an interval. Then the function could be a constant, for sure. But could there be other possibilities? Are the only functions differentiating to zero constant functions? The answer is yes, and the proof relies on the MVT:

<u>Proposition: Functions with Vanishing Derivatives</u>

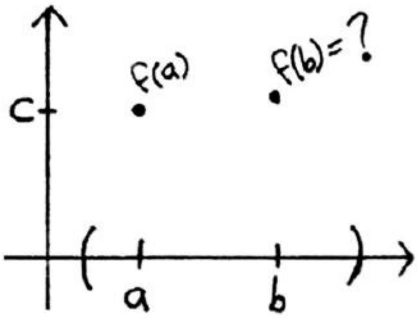
If $F(x)$ satisfies $F'(x) \equiv 0$ on an interval I,

then necessarily $F(x)$ must be a constant function:

$$F(x) = C, \quad \forall x \in I.$$

Proof:

Let $a \in I$, let $C = f(a)$.
We show that for any other $b \in I$, $f(b) = C$ also.



The MVT applies for f on $[a,b]$:

$$\exists c \in (a,b) \text{ s.t. } f'(c) = \frac{f(b) - f(a)}{b - a}$$

Now $f'(c)$ is necessarily zero by assumption, so that we must have:

$$f(b) = f(a) = C \text{ as well.}$$

$\boxed{\text{FwVD}}$

Corollary: Functions with Equal Derivatives

Two functions having identical derivatives on an interval I can only differ by a constant.

Proof:

Say $F(x)$ and $G(x)$ satisfy $F'(x) = G'(x)$, $\forall x \in I$.
Consider the difference $H(x) = G(x) - F(x)$.
The previous proposition applies to H since

$$\begin{aligned} H'(x) &= [G(x) - F(x)]' \\ &= G'(x) - F'(x) \quad \text{by the } \textit{Difference Rule} \text{ for differentiation.} \\ &= 0 \quad \text{on } I. \end{aligned}$$

Thus $H(x) = G(x) - F(x) \equiv C$ on I.

$\boxed{\text{FwED}}$

Conclusion 1:

If: $F(x)$ is *an* antiderivative for $f(x)$ on an interval I,
then: all antiderivatives of $f(x)$ on I must be of the form $F(x) + C$ where C is a constant.

Conclusion 2:

All solutions of the differential equation $y' = f(x)$
are the functions of the form $F(x) + C$,
where $F(x)$ is an antiderivative of $f(x)$ and C is an arbitrary constant.

So, our first "result" about singular solns (or their absence...) is:

For DEs of form $y' - f(x) = 0$, there are <u>no</u> singular solns.

Often, solns of DEs are in implicit form.

For example, consider the DE $y^2(y'-2x) = x^2(1+2yy')$.

Is the implicit fcn $y^3 - x^3 = 3x^2y^2$ is a soln (in implicit form)

This means that the slope of this fcn can be obtained (by imp. diffn)

$$\frac{d}{dx}\left[y^3 - x^3\right] = \frac{d}{dx}\left[3x^2y^2\right]$$

$$3y^2y' - 3x^2 = 6xy^2 + 6x^2yy'$$

$$y'(3y^2 + 6x^2y) = 6xy^2 + 3x^2$$

$$y' = \frac{6xy^2 + 3x^2}{3y^2 + 6x^2y}$$

subbing this back into the DE:

$$y^2\left(\frac{6xy^2 + 3x^2}{3y^2 + 6x^2y} - 2x\right) = x^2\left(1 + 2y\left(\frac{6xy^2 + 3x^2}{3y^2 + 6x^2y}\right)\right)$$

$\vdots$

etc.