5 - Matrices

October 6, 2017

Definition of matrices

Def'n: An $m \times n$ matrix, A, is a rectangular array of numbers with m rows and n columns.

ex. $\begin{bmatrix} 1 & \pi & e & -2 \\ 1/2 & 0 & 3 & \pi^2 \end{bmatrix}$ is a 2×4 matrix.

Def'n: (i) The ij-entry of matrix A is the number in the i'th row and the j'th column.

(ii) We write $A = (a_{ij})_{m \times n}$ size

formula for the entries

and $a_{ij} = [A]_{ij}$

ex. $A = (i + j)_{2 \times 3}$

$$\begin{array}{c} \\ 2 \end{array} \overset{3}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}}$$

$a_{11} = 1 + 1 = 2$

$a_{12} = 1 + 2 = 3$

$$= \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

Comparing Matrices:

Def'n: Let $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{r \times s}$ be matrices.

Then, we say that A and B are equal if

(i) $m = r$ and $n = s$ (same shape and size)

(ii) for all $i \leq m, j \leq n$, $a_{ij} = b_{ij}$

ex. For which value(s) of x will the following matrices be equal?

(a) $A = \begin{bmatrix} 1 & x \\ 9 & -4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & x \\ x^2 & -4 \end{bmatrix}$

$A = B$ means
$$\begin{aligned} 1 &= 1 \\ x &= x \\ 9 &= x^2 \\ -4 &= -4 \end{aligned}$$

$\rightarrow x^2 = 9$

$\boxed{x = \pm 3}$

(b)

$$A = \begin{bmatrix} 1 \\ x \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

Never equal since A is a 2×1
B is a 2×2

Zero and One Matrix

Def'n: (i) The $m \times n$ zero matrix, $O_{m \times n}$, is the $m \times n$ Matrix with only zero entries.

$$O_{m \times n} = (0)_{m \times n} \quad (\text{where } a_{ij} = 0)$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Note: $(O_{m \times n} A = O_{m \times n})$

(ii) The $n \times n$ identity matrix I_n is

$$I_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note: $(I_n A = A)$

Note: Has to be a square matrix

ex, $I_1 = [1]$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Basic Operations

Scaler Multiplication:

If $A = (a_{ij})_{m \times n}$ and $k \in \mathbb{R}$,

then, $kA = (k a_{ij})_{m \times n}$

Addition or subtraction of Matrices:

If $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$,

then, $A \pm B = (a_{ij} \pm b_{ij})_{m \times n}$

returns matrix of the same shape

ex. Let $A = \begin{bmatrix} 2 & 3 \\ -4 & 12 \end{bmatrix}$, $B = \begin{bmatrix} 7 & -2 \\ 11 & 9 \end{bmatrix}$,

$C = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ -1 & 6 \end{bmatrix}$

(a) $A + 2B = \begin{bmatrix} 2 & 3 \\ -4 & 12 \end{bmatrix} + 2\begin{bmatrix} 7 & -2 \\ 11 & 9 \end{bmatrix}$

$= \begin{bmatrix} 2 & 3 \\ -4 & 12 \end{bmatrix} + \begin{bmatrix} 14 & -4 \\ 22 & 18 \end{bmatrix}$

$= \begin{bmatrix} 16 & -1 \\ 18 & 30 \end{bmatrix}$

(b) $2B - 3C = 2\begin{bmatrix} 7 & -2 \\ 11 & 9 \end{bmatrix} - 3\begin{bmatrix} 1 & 3 \\ 2 & 4 \\ -1 & 6 \end{bmatrix}$

These entries do not correspond to anything in B.

$= \text{DNE}$

Properties of Basic Operations

where A, B, C are $m \times n$ matrices

Addition:

(i) $A + B = B + A$ ~ order of matrices

$O_{m \times n} + A = A$

$A + (B + C) = (A + B) + C$ ~ order of operations

Scaler Multiplication:

(ii) $c(dA) = (cd)A$

$1A = A$

$0A = O_{m \times n}$

Distributive:

(iii) $c(A + B) = cA + cB$

$(c + d)A = cA + dA$

Linear Combinations of Matrices

Def'n: Let $M_1, M_2, \dots, M_k$ be $m \times n$ matrices

We say A is a linear combination of $M_1, M_2, \dots, M_k$

if there are constants $c_1, c_2, \dots, c_k$ so that

$$A = c_1 M_1 + c_2 M_2 + \dots + c_k M_k$$

ex. Write, if possible, $\begin{bmatrix} 5 & 6 \\ -11 & -4 \end{bmatrix}$ as a linear combination

of $\begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix}$ and $\begin{bmatrix} -1 & 3 \\ 4 & 5 \end{bmatrix}$.

$$\begin{bmatrix} 5 & 6 \\ -11 & -4 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 & 3 \\ 4 & 5 \end{bmatrix}$$

So, ① $5 = c_1 - c_2$

② $6 = 4c_1 + 3c_2$

③ $-11 = -c_1 + 4c_2$

④ $-4 = 2c_1 + 5c_2$

Use gaussian elimination:

$$\left[\begin{array}{cc|c} 1 & -1 & 5 \\ 4 & 3 & 6 \\ -1 & 4 & -11 \\ 2 & 5 & -4 \end{array}\right] \xrightarrow[\text{use gaussian elimination}]{\cdots} \begin{array}{cc} c_1 & c_2 \\ \end{array} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So,
$$\begin{bmatrix} 5 & 6 \\ -11 & -4 \end{bmatrix} = 3 \begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix} - 2 \begin{bmatrix} -1 & 3 \\ 4 & 5 \end{bmatrix}$$

Product of Matrices/Matrix Multiplication

→ We want to generalize the dot product to matrices

Def'n: Let $A = \overbrace{\begin{bmatrix} \boxed{\quad v_1 \quad} \\ \boxed{\quad v_2 \quad} \\ \vdots \\ \boxed{\quad v_m \quad} \end{bmatrix}}^{k}$ be an $m \times k$ matrix

and $B = k\left\{\begin{bmatrix} \boxed{w_1} & \boxed{w_2} & \cdots & \boxed{w_n} \end{bmatrix}\right.$ be a $k \times n$ matrix

Note: the number of columns in A is the same as the number of rows in B

Then, AB is the $m \times n$ matrix $\left(\vec{v_i} \cdot \vec{w_j}\right)_{m \times n}$

ex. $A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & 1 \\ 1 & 3 \\ -1 & 1 \end{bmatrix}$

(a) $AB = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} = \text{DNE}$

since A is a $2 \times \boxed{3}$

B is a $\boxed{2} \times 2$

(b) $BA = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} \underline{1 \cdot 1 + 3 \cdot 0} & \underline{1 \cdot 2 + 3 \cdot 1} & 1 \cdot (-2) + 3(3) \\ & \cdots & \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 & 7 \\ -2 & 0 & 16 \end{bmatrix}$$

october 10

ex. $\overset{A}{\begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 0 & 5 \end{bmatrix}} \overset{B}{\begin{bmatrix} 1 & 0 & -3 & 6 \\ 0 & 2 & 5 & 1 \end{bmatrix}} = AB$

$3 \times \cancel{2}$ $\qquad$ $\cancel{2} \times 4$

$$= \begin{bmatrix} \underline{2\cdot 1 + 1\cdot 0} & 2\cdot 0 + 1\cdot 2 & & \\ & & & \\ & & & 0\cdot 6 + 5\cdot 1 \end{bmatrix}$$

pos (1,1) pos (1,2) pos (3,4)

3×4

$$= \begin{bmatrix} 2 & 2 & -1 & 13 \\ 1 & 6 & 12 & 9 \\ 0 & 10 & 25 & 5 \end{bmatrix}$$

<u>ex.</u>

$$\underset{2\times \cancel{2}}{\overset{A}{\begin{bmatrix} \boxed{1 \quad 2} \\ 3 \quad 4 \end{bmatrix}}} \underset{\cancel{2}\times 2}{\overset{B}{\begin{bmatrix} 0 & 2 \\ 3 & -1 \end{bmatrix}}}$$

$$AB = \begin{bmatrix} 6 & 0 \\ 12 & 2 \end{bmatrix}$$

$$BA = \begin{bmatrix} \boxed{0 \quad 2} \\ 3 \quad -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 8 \\ 0 & 2 \end{bmatrix}$$

$\therefore AB \neq BA$

ex.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$2 \times \cancel{3}$ $\quad$ $\cancel{3} \times 1$

$$= \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

ex.

$$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 5 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 3 & 0 \end{bmatrix}$$

3×1 $\quad$ 3×3

X

= UNDEFINED

ex.

$$\begin{array}{cc} A & B \\ \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{array} = \begin{array}{c} AB \\ \begin{bmatrix} 3 \end{bmatrix} \end{array}$$

1×1

$$BA = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

3×3

Properties of Matrix Multiplication

Note: Matrix multiplication is not commutative.

i.e. $AB \neq BA$

(i) $(AB)C = A(BC)$ associative

(ii) $A(B+C) = AB + AC$ left distributive

(iii) $(A+B)C = AC + BC$ right distributive

(iv) If $k \in \mathbb{R}$, then $k(B+C) = kB + kC$

(v) If $k \in \mathbb{R}$, then $k(AB) = kAB = A(kB)$

Transpose of Matrix

ex. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

3×2 $\qquad$ 2×3

the transpose of A

or

A transposed

Defn: If $A = (a_{ij})_{m \times n}$, then $A^T = (a_{ji})_{n \times m}$

ex, $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

flip along the main diagonal

Properties of the Transpose

(i) $(A^T)^T = A$

(ii) If $k \in \mathbb{R}$, then $(kA)^T = kA^T$

ex, $\left(2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)^T = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}^T = \begin{bmatrix} 2 & 6 \\ 4 & 8 \end{bmatrix}$

or

$2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = 2\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 4 & 8 \end{bmatrix}$

(iii) $(A+B)^T = A^T + B^T$

(iv) $(AB)^T = B^T \cdot A^T$

ex, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}$

$$AB = \begin{bmatrix} 6 & -3 \\ 12 & -5 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 6 & 12 \\ -3 & -5 \end{bmatrix}$$

$$A^T B^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 4 & -2 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ -3 & -5 \end{bmatrix}$$

$$B^T A^T = (AB)^T$$

Note: $$(\underset{m\times \cancel{n}}{A} \ \underset{\cancel{n}\times p}{B})^T = \underset{p\times \cancel{n}}{B^T} \ \underset{\cancel{n}\times m}{A^T}$$

Reminder:

Zero Matrix

ex. $$\mathbb{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

ex. $$\mathbb{O}_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Properties

(i) $A + \mathbb{0} = A$

(ii) $0A = \mathbb{0}$

scalar zero

Identity Matrix

→ Square matrix

→ in RREF with no row of zeros (In other words, diagonal is 1's)

ex. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

ex.

$$I \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}^{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$\therefore\ I = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

ex, $AI = A$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} I = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$\therefore\ I = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Caution: Not every property of $\mathbb{O}$ and I is the same as those of the number 0 and the number 1.

ex.
$$\underset{3\times 2}{\overset{A}{\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 3 & 6 \end{bmatrix}}} \underset{2\times 2}{\overset{B}{\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}}} = \underset{3\times 2}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}} = \mathbb{O}$$

Note: In $\mathbb{R}$ ~ Real numbers

$ab = 0 \rightarrow a = 0$ or $b = 0$

ex. In $\mathbb{R}$

$\cancel{a}b = \cancel{a}c$ and $a \neq 0$

$\therefore b = c$

For matrices:

$$\overset{A}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \overset{B}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} = \overset{A}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \overset{C}{\begin{bmatrix} 7 & 8 \\ 3 & 4 \end{bmatrix}}$$

$$\begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix}$$

But $B \neq C$

Note:

(i) we don't define matrix division

(ii) In $\mathbb{R}$, if $ax = b$ $(a \neq 0)$, then $x = \frac{b}{a} = a^{-1}b$

(iii) For matrices: if $AX = B$, then $X = A^{-1}B$ if A^{-1} exists

The matrix A^{-1} is called A inverse or the inverse of A.

Defn: Given a square matrix A, if there exists a matrix B such that $AB = I$ and $BA = I$, we say that B is the inverse of A, and we write

$$B = A^{-1}$$

(Note: If $B = A^{-1}$, then $A = B^{-1}$)

October 11

Inverse of a Matrix

Defn: If A is $n \times n$, then, if there exists a matrix B such that $AB = I$ and $BA = I$, then we say that B is <u>the inverse</u> of A and we write $B = A^{-1}$

(if there is no B, then A is singular)

Theorem: Let A be a 2×2 matrix, say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

Then, A is invertible <u>iff</u> $ad - bc \neq 0$.

In that case,

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

ex.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \longrightarrow A^{-1} = \frac{1}{1\cdot 4 - 2\cdot 3} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

To check: $A \cdot A^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \left(-\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \right)$

$$= -\frac{1}{2} \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \checkmark$$

ex. $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

rows are multiples of each other → parallel lines

$ab - bc = 1\cdot 4 - 2\cdot 2 = 0 \quad \rightarrow \quad ad = bc$ This gives you a row

$\therefore$ Not invertible $\qquad \frac{a}{b} = \frac{c}{d}$ of zeros in RREF

Note! (i) Not invertible is synonymous to singular

(ii) If one row is a multiple of another row, then the matrix is singular ex. Consider the following matrix equation

$$\overset{A}{\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}} X = \overset{B}{\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}}$$

Solve for the matrix X

In $\mathbb{R}$

$3x = 7$

$\frac{1}{3} \cdot 3x = 7 \cdot \frac{1}{3}$

$x = \frac{7}{3}$

$$AX = B$$

If A is invertible, then

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

$$A^{-1} = -\frac{1}{3}\begin{bmatrix} -3 & -2 \\ 0 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix}$$

Notice where A^{-1} is placed (must be consistent on both sides

+ AXA^{-1} is not the same as $A^{-1}AX$

$\therefore$ so A^{-1} must be placed on the left

So,

$$X = \frac{1}{3}\begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 7 & -1 \\ -2 & -1 \end{bmatrix}$$

Powers of Matrices

If A is a $n \times n$ matrix, we define

(i) $A^0 = I$

(ii) $A^n = \underbrace{A A \dots A}_{n \text{ factors}}$ where $n = 1, 2, \dots$

(iii) If A is invertible, we define

$$A^{-n} = (A^{-1})^n$$
$$= \underbrace{A^{-1} A^{-1} \dots A^{-1}}_{n \text{ factors}}$$

<u>ex</u> If $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, find

a) A^2

$$A^2 = AA = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix}$$

b) $A^{-1} = \frac{1}{3} \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}$

c) $A^{-2} = (A^{-1})^2 = \left(\frac{1}{3}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}\right)^2$

$= \frac{1}{9}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}$

$= \frac{1}{9}\begin{bmatrix} 9 & -8 \\ 0 & 1 \end{bmatrix}$

Note: $(A^{-1})^2 = (A^2)^{-1}$

$= \frac{1}{9}\begin{bmatrix} 9 & -8 \\ 0 & 1 \end{bmatrix}$

Properties of Power of Matrices

(i) $A^r A^s = A^{r+s}$

(ii) $(A^r)^s = A^{rs}$

If A is invertible, then:

(iii) $(A^{-1})^{-1} = A$

(iv) $(kA)^{-1} = \frac{1}{k} A^{-1}$

(v) If A^n is invertible and $(A^n)^{-1} = \underbrace{(A^{-1})^n}_{\text{This was from definition}} = A^{-n}$

ICP 5

(i) $A = \begin{bmatrix} 3 & 1 & 4 \\ -2 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 1 & 1 \\ 2 & -4 & 1 \end{bmatrix}$,

$C = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 0 & 4 \end{bmatrix}$

Properties of transpose

Recall For A, any $m \times n$ matrix,

The transpose, A^T of A the $n \times m$ matrix whose columns are the rows of A

ex. $A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & \pi & 5 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 2 & -1 \\ 3 & \pi \\ 1 & 5 \end{bmatrix}$

2×3 $\qquad$ 3×2

Note: If $A = (a_{ij})_{m \times n}$,

then, $A^T = (a_{ji})_{n \times m}$

Recall: (i) $(AB)^T = B^T A^T$

(ii) $(A+B)^T = A^T + B^T$

(iii) $(kA)^T = kA^T$

(iv) $(A^T)^T = A$

(v) $(A^{-1})^T = (A^T)^{-1}$

Symmetric Matrices

Defn: (i) A matrix, A, is symmetric if $A^T = A$

(ii) A matrix, A, is skew-symmetric (antisymmetric) if $A^T = -A$

ex (i) $A = \begin{bmatrix} 1 & 2 & 3 \\ \boxed{2} & 4 & 5 \\ \boxed{3} & 5 & 8 \end{bmatrix}$ is symmetric since $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 8 \end{bmatrix}$

ex. Let A be any matrix and take $B = AA^T$

$n\times m$ $m\times n$

$n\times n$

Show that B is symmetric.

$$B^T = (AA^T)^T = (A^T)^T A^T$$
$$= AA^T$$
$$= B$$

$\therefore B$ is symmetric

Proposition: If A is symmetrical, then A is square.

Proof: Say A is a $m \times n$, A^T is an $n \times m$ matrix.

But, since A is symmetrical,

$$A = A^T$$

m rows n rows

and so, $m = n$.

$\therefore A$ is square.

Note: An $m\times n$ matrix A is square if $m=n$

ex. Let A be a square matrix.

(a) Show that $(A + A^T)$ is symmetric.

$B = (A + A^T)$ we want to show that: $B^T = B$

$$B^T = (A + A^T)^T = A^T + (A^T)^T$$
$$= A^T + A$$
$$= A + A^T = B$$

$\therefore$ $(A + A^T)$ is symmetric

(b) Show that $A - A^T$ is skew-symmetric

we want: $(A - A^T)^T = -(A - A^T)$

$$(A - A^T)^T = A^T - (A^T)^T$$
$$= A^T - A$$
$$= -A + A^T$$
$$= -(A - A^T)$$

$\therefore$ $A - A^T$ is skew-symmetric.

Solving Equations Using Algebra of Matrices

<u>Recall</u>: (i) B is the inverse of A if $\cdot\ BA = I_n$
$\cdot\ AB = I_n$

(ii) Properties of the inverse:

(a) $(AB)^{-1} = B^{-1}A^{-1}$

(b) $(kA)^{-1} = \frac{1}{k}A^{-1}$

(c) $(A^{-1})^{-1} = A$

(d) $(A^n)^{-1} = (A^{-1})^n$ where n is an integer
$= A^{-n}$

<u>ex.</u> Solve for X in the following equations

(a) $X \underbrace{\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}}_{B} = \begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 5 & 1 \end{bmatrix}$

$$\left\{\begin{aligned} XA &= B \\ XAA^{-1} &= BA^{-1} \\ XI_n &= BA^{-1} \\ X &= BA^{-1} \end{aligned}\right.$$

$$X \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} B^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 5 & 1 \end{bmatrix} B^{-1}$$

$$X \cancel{I_n} = \begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 5 & 1 \end{bmatrix} B^{-1}$$

$\vdots$

$$X = \begin{bmatrix} 8 & 3 \\ -9 & -4 \\ 20 & 7 \end{bmatrix}$$

(b) $(ABXC)^T A = C^T A$

$$(ABXC)^T A A^T = C^T A A^T$$

$$\left((ABXC)^T\right)^T = \left(C^T\right)^T$$

$$ABXC = C$$

$$\cancel{(AB)^{-1}}\,\cancel{AB}XC\,\cancel{C^{-1}} = (AB)^{-1}$$

$$X = (AB)^{-1}$$

$$= B^{-1}A^{-1}$$

(c) $$(5X^T)^{-1} - \begin{array}{c} A \\ \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix} \end{array} = 0$$

$$(5X^T)^{-1} = A$$

$$\left((5X^T)^{-1}\right)^{-1} = A^{-1}$$

$$5X^T = A^{-1}$$

$$X^T = \frac{1}{5}A^{-1}$$

$$(X^T)^T = \left(\frac{1}{5}A^{-1}\right)^T$$

$$X = \frac{1}{5}(A^{-1})^T$$

$$A^{-1} = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}^{-1} = \frac{1}{\underbrace{(-6)}_{ad} - \underbrace{(-5)}_{bc}} \begin{bmatrix} 2 & 1 \\ -5 & -3 \end{bmatrix}$$

$$= - \begin{bmatrix} 2 & 1 \\ -5 & -3 \end{bmatrix}$$

So, $X = \frac{1}{5} \left(- \begin{bmatrix} 2 & 1 \\ -5 & -3 \end{bmatrix} \right)^T$

$$= -\frac{1}{5} \begin{bmatrix} 2 & -5 \\ 1 & -3 \end{bmatrix}$$

201-NYC **ICP5**

1. Solve for the matrix X in the following expressions:

a)

$$(3X)^{-1} = \begin{bmatrix} -2 & 1 \\ -2 & 3 \end{bmatrix}$$

b) $X\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

c) $AX - 3X = B$ *where* $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$

d) $XA - 3X = C$ *where* $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$

e) $AX - BX = C$ *where* $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$

f) $ABX^T + 3X^T = I$ where $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}$,

2. Find the inverse of matrix $A = \begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix}$ and use it to solve the linear system $\begin{aligned} -3x + y &= 5 \\ 2x + 2y &= 0 \end{aligned}$

October 17

Some trivial proofs...

Recall: If A is invertible, then A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Proof: Show that $A^T (A^{-1})^T = I$

$$A^T (A^{-1})^T = (A A^{-1})^T = I^T = I \checkmark$$

Theorem: If A, B are invertible, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof:
$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(B(B^{-1}A^{-1})) \\ &= A((BB^{-1})A^{-1}) \\ &= A(IA^{-1}) \\ &= AA^{-1} \\ &= \checkmark\end{aligned}$$

ex. $ABX^T + 3X^T = I$ Have to factor on the right!

$$(AB + 3I)X^T = I$$

$$X^T = (AB + 3I)^{-1} I$$

$$X = ((AB + 3I)^{-1})^T$$

$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 4 & 0 \end{bmatrix}$

2×3 $\quad$ 3×2

$$= \left(\left(\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right)^{-1} \right)^{T}$$

$$= \left(\begin{bmatrix} 5 & 3 \\ 4 & 4 \end{bmatrix}^{-1} \right)^{T}$$

$$= \frac{1}{8} \begin{bmatrix} 4 & -3 \\ -4 & 5 \end{bmatrix}^{T}$$

$$\therefore \; X = \frac{1}{8} \begin{bmatrix} 4 & -4 \\ -3 & 5 \end{bmatrix}$$

Note: $XA + BX = I$

$\therefore$ Can't factor out X

ex. $\underset{A}{\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}} X - X \underset{B}{\begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}} = \underset{C}{\begin{bmatrix} 3 & -8 \\ 5 & -11 \end{bmatrix}}$

$$\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -8 \\ 5 & -11 \end{bmatrix}$$

$$\begin{cases} ① \; 3a + c - a - 2b = 3 \\ ② \; 3b + d - 4a = -8 \\ ③ \; -a + 2c - c - 2d = 5 \\ ④ \; -b + 2d - 4c = -11 \end{cases}$$

Solve the system...

Method for finding A^{-1} for n x n matrices

STEPS:

① Given a $n \times n$ matrix A, we can write the matrix $[A|I]$ by writing the $n \times n$ identity matrix I next to it.

ex. $A = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 4 & -1 \\ 2 & 5 & -4 \end{bmatrix}$, $[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 3 & 4 & -1 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$

② Reduce the matrix $[A|I]$ to its RREF.

$$\begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 3 & 4 & -1 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_2-3R_1} \begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 4 & -10 & -3 & 1 & 0 \\ 0 & 5 & -10 & -2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 5 & -10 & -2 & 0 & 1 \\ 0 & 4 & -10 & -3 & 1 & 0 \end{bmatrix} \xrightarrow{R_2-R_3} \begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 4 & -10 & -3 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3-4R_2} \begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & -10 & -7 & 5 & -4 \end{bmatrix} \xrightarrow{-\frac{1}{10}R_3} \begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & \frac{7}{10} & -\frac{1}{2} & \frac{2}{5} \end{bmatrix}$$

$$\xrightarrow{R_1-3R_3} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -\frac{11}{10} & \frac{3}{2} & -\frac{6}{5} \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & \frac{7}{10} & -\frac{1}{2} & \frac{2}{5} \end{array}\right]$$

(3) If A turns into I, then I turns into A^{-1}

i.e. $[A \mid I] \rightsquigarrow \ldots \rightsquigarrow [I \mid A^{-1}]$

If not (i.e. we get a row of zeros in A), then A is singular (not invertible)

$$\therefore A^{-1} = \begin{bmatrix} -\frac{11}{10} & \frac{3}{2} & -\frac{6}{5} \\ 1 & -1 & 1 \\ \frac{7}{10} & -\frac{1}{2} & \frac{2}{5} \end{bmatrix} = \frac{1}{10}\begin{bmatrix} & & \\ & \cdots & \\ & & \end{bmatrix}$$

Check: $AA^{-1} = I$

$\vdots$

$$= \frac{1}{10}\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} = I \checkmark$$

Question: Ok.... but why does this work?

Elementary Matrices

ex. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{I}{\overset{R_1 \leftrightarrow R_2}{\rightsquigarrow}}$ ex. $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{I}{\overset{R_2 + 3R_1}{\rightsquigarrow}}$ ex. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 17 \end{bmatrix} \overset{\frac{1}{17} R_3}{\rightsquigarrow} I$

Defn: A $n \times n$ matrix is an elementary matrix if it was obtained from I using one single elementary row operation (ERO).

Note: (1) Given a $m \times n$ matrix A and a $m \times m$ elementary matrix E.

Left-multiplying A by E : EA is the same as applying the corresponding ERO directly to A.

ex. $\underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \xrightarrow{R_2 - 3R_1} \underset{E}{\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}}$

$\rightarrow$ EA $\qquad$ B

$$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \overset{B}{\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 3 \end{bmatrix}}$$

Notes ② An elementary matrix E is invertible and its inverse E^{-1} can be obtained by peforming the inverse (opposite) ERO.

ex. $E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

$\downarrow$ applying formula for 2×2

$$E^{-1} = \frac{1}{1} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

To create E, we used the ERO $R_2 - 3R_1$

$$\overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \xrightarrow{R_2 - 3R_1} \overset{E}{\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}}$$

inverse ERO

$$\overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \xrightarrow{R_2 + 3R_1} \overset{E^{-1}}{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}$$

Defn: Given two $m \times n$ matrices A, B, we say that A is <u>row equivalent</u> to B (notation: $A \sim B$) if there exists a sequence $\theta_1, \theta_2, \ldots, \theta_k$ of ERO's such that

$$A \xrightarrow{\theta_1} A_1 \xrightarrow{\theta_2} A_2 \rightarrow \cdots \xrightarrow{\theta_k} B$$

ex.

(a) A matrix A and its RREF are row-equivalent

i.e. $A \sim R$

(b) If E is elementary, it's now row equivalent to I.

i.e. $E \sim I$

ex. Show that $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 5 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 10 & -4 \\ 2 & 6 & -1 \end{bmatrix}$

are row equivalent.

$$\overset{A}{\begin{bmatrix} 2 & 1 & 1 \\ 0 & 5 & -2 \end{bmatrix}} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 5 & -2 \\ 2 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 + R_1} \begin{bmatrix} 0 & 5 & -2 \\ 2 & 6 & -1 \end{bmatrix} \xrightarrow{2R_1} \begin{bmatrix} 0 & 10 & -4 \\ 2 & 6 & -1 \end{bmatrix}$$

October 18

ex. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, the corresponding ERO is $\xrightarrow{SR_2}$

Proposition: If $A \sim B$, then there exists a sequence of elementary matrices $E_1, E_2, \dots, E_k$

such that $E_k \dots E_2 E_1 A = B$

Proof: $A_1 = E_1 A$

$A_2 = E_2 A_1 = E_2 E_1 A$

$\vdots$

$B = A_k = E_k A_{k-1}$

$\left\{ A \xrightarrow{\theta_1} A_1 \xrightarrow{\theta_2} A_2 \right.$

ex. (from last class)

$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 5 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 10 & -4 \\ 2 & 6 & -1 \end{bmatrix}$

(a) Find three elementary matrices E, F, G such that

$GFE\,A = B$

$$\overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \xrightarrow{R_1+R_2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = E$$

$$\overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \xrightarrow{2R_2} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = F$$

$$\overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = G$$

So, $GFEA = B$

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{flips the rows}} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 5 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 5 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 5 & -2 \end{bmatrix}$$

$$= \underset{B}{\begin{bmatrix} 0 & 10 & -4 \\ 2 & 6 & -1 \end{bmatrix}}$$

(b) Find an invertible matrix P

$$B = PA$$

$$P = GFE = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

(c) Find three elementary matrices H, J, L such that

$$L\,J\,H\,B = A$$

$$B \xrightarrow{R_1 \leftrightarrow R_2} A_2 \xrightarrow{\frac{1}{2}R_2} A_1 \xrightarrow{R_1 - R_2} A$$

$$\overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = H$$

$$\vdots \quad \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = J$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = L$$

So $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} B = A$

Note: $L = E^{-1}$

$J = F^{-1}$

$H = G^{-1}$

i.e. $B = GFEA$

$A = E^{-1}F^{-1}G^{-1}B$

In general, if $E_k \ldots E_2 E_1 A = B$

then $E_1^{-1} E_2^{-1} E_3^{-1} B = A$

Proposition 2: If $A \sim I$, then A is invertible

Proof: If $A \sim I$, then there exists a sequence of elementary matrices $E_1, E_2, \ldots, E_k$

such that $\underbrace{E_k \ldots E_2 E_1}_{A^{-1}} A = I$

So $A^{-1} = E_k \ldots E_2 E_1$ and A is invertible

Note: $A^{-1} = E_k \ldots E_2 E_1 I$

So the same sequence $\xrightarrow{\theta_1} \xrightarrow{\theta_2} \cdots \xrightarrow{\theta_k}$ that turns A into I will turn I into A^{-1}

So $[A \mid I] \overset{e_1}{\rightsquigarrow} \dots [I \mid A^{-1}]$

Proposition 3: If $A \sim I$, then A can be written as a product of elementary matrices

Proof: If $E_k \dots E_1 A = I$,

$E_1^{-1} \dots E_k^{-1} I = A$

$E_1^{-1} \dots E_k^{-1} = A$

But the inverse of an elementary matrix is also elementary. So A is a product of elem matrices.

ex Let $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Write A as a product of elem. matrices.

$$\overset{A}{\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 + 2R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}}_{E1}$$

$$I \xrightarrow{R_1 + 2R_3} \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E2}$$

So $E_2 E_1 A = I$

$E_1^{-1} E_2^{-1} I = A$

So $E_1^{-1} E_2^{-1} = A$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Invertibility Theorem

If A is a $n \times n$ matrix, then the following statements are equivalent:

(i) A is invertible

(ii) $A \sim I$ (i.e. A is row equivalent to I)

(iii) A can be written as a product of elementary matrices

(iv) The rank of A is n

Solving a Linear System using A^{-1}

<u>Note:</u> Any linear system can be seen as a matrix equation

<u>ex.</u> $x - y + 2z = 5$

$2x + y - z = 0$

We can write the system as an equality of matrices

$$\overset{A}{\begin{bmatrix}1 & -1 & 2\\ 2 & 1 & -1\end{bmatrix}}\overset{X}{\begin{bmatrix}x\\ y\\ z\end{bmatrix}} = \overset{B}{\begin{bmatrix}5\\ 0\end{bmatrix}}$$

$$AX = B$$

Note! In this example, A is 2×3, so A is not invertible

Note! In cases where A is $n \times n$ (square), we can solve the system by solving the matrix eqn as follows.

$$AX = B$$
$$X = A^{-1}B$$

ex Use A^{-1} to solve the system

$$x - 2y = 6$$
$$2x + 3y = -1$$

$$\overset{A}{\begin{bmatrix}1 & -2\\ 2 & 3\end{bmatrix}}\overset{X}{\begin{bmatrix}x\\ y\end{bmatrix}} = \overset{B}{\begin{bmatrix}6\\ -1\end{bmatrix}}$$

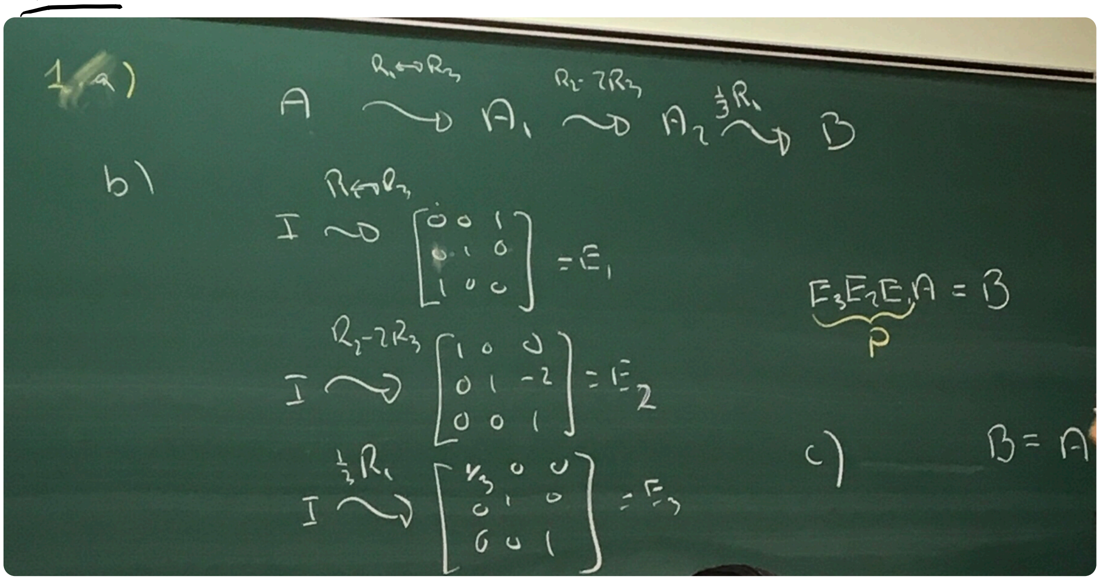
$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 16 \\ -13 \end{bmatrix}$$

$$\therefore \begin{cases} x = \frac{16}{7} \\ y = -\frac{13}{7} \end{cases}$$

October 24

ICP 6



2

$$\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} \xrightarrow{R_1 + 2R_2} \begin{bmatrix} 1 & 5 \\ 2 & 2 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 5 \\ 0 & -8 \end{bmatrix} \xrightarrow{-\frac{1}{8}R_2} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 - 5R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I \xrightarrow{R_1 + 2R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = E_1$$

$$I \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = E_2 \qquad I \xrightarrow{-\frac{1}{8}R_2} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{8} \end{bmatrix} = E_3$$

$$I \xrightarrow{R_1 - 5R_2} \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix} = E_4$$

$$E_4 E_3 E_2 E_1 A = I$$

and $E_1 E_2 E_3 E_4 = A$

<u>Remember:</u> If A is a $n \times n$ matrix, then we define the determinant of A as

$$\det(A) = a_{i1} C_{i1} + a_{i2} C_{i2} + \dots + a_{in} C_{in}$$

for any i integer in $[1, n]$

or

$$\det(A) = a_{1i} C_{1i} + a_{2i} C_{2i} + \dots a_{ni} C_{ni}$$

for any i integer in $[1, n]$

<u>ex.</u> $A = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & -1 & 3 \\ 2 & 0 & 3 & 0 \\ 0 & 1 & 0 & 6 \end{bmatrix}$

On row 3: $\det(A) = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} + a_{34}C_{34}$

Cofactor expansion:

$$= 2(-1)^4 \begin{vmatrix} 2 & 0 & 5 \\ 1 & -1 & 3 \\ 1 & 0 & 6 \end{vmatrix} + 0 + 3(-1)^6 \begin{vmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 1 & 6 \end{vmatrix} + 0$$

$$= 2\left(0 + (-1)(-1)^4 \begin{vmatrix} 2 & 5 \\ 1 & 6 \end{vmatrix} + 0\right) + 3\left(1\cdot(-1)^2 \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} + 0 + 0\right)$$

using col 2   using col 1

$= -14 + 9$

$= -5$

Or: Shortcut: (Recall scalar triple product)

Let $A = \begin{bmatrix} 2 & 0 & 5 \\ 1 & -1 & 3 \\ 1 & 0 & 6 \end{bmatrix}$

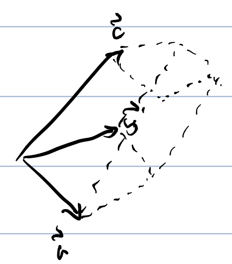
$$\begin{array}{ccccc} 2 & 0 & 5 & 2 & 0 \\ 1 & -1 & 3 & 1 & -1 \\ 1 & 0 & 6 & 1 & 0 \end{array} \qquad -12$$

Note: Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

$\vec{c} = \langle c_1, c_2, c_3 \rangle$

Then, the volume of parallelpiped is



$$\text{vol} = \left| \vec{a} \cdot (\vec{b} \times \vec{c}) \right| = \det\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \right)$$

Properties of a Determinant

(1) $\det(A^T) = \det(A)$

ex. $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 5 \\ 0 & -1 & 2 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 2 & 5 & 2 \end{bmatrix}$

using col 2 : $\det(A) = 1 \cdot \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} - (-1) \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix}$

$= 1$

using row 2 : $\det(A^T) = 1 \cdot \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = 1$

(2) Let A, B be $n \times n$ matrices

$\det(AB) = \det(A)\det(B)$

ex: $A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$

$AB = \begin{bmatrix} 3 & 6 \\ 5 & 12 \end{bmatrix}$

$|AB| = \begin{vmatrix} 3 & 6 \\ 5 & 12 \end{vmatrix} = 6$ $\quad |A| = -2 \quad |B| = -3$

Note: $\det(AB) = \det(A)\det(B)$
$= \det(B)\det(A)$
$= \det(BA)$

(3) $\det(A^{-1}) = \dfrac{1}{\det(A)}$ $\quad$ $\det(A^2) = \det(A)\det(A)$
$= (\det(A))^2$

In general: $\det(A^n) = (\det(A))^n$

Proof $AA^{-1} = I$
$\det(AA^{-1}) = \det(I)$
$\det(A)\det(A^{-1}) = 1$
$\det(A^{-1}) = \dfrac{1}{\det(A)}$

ex. Suppose A is such that $A^2 = I$, then what can you say about $\det(A)$?

$$A^2 = I$$
$$\det(A^2) = \det(I)$$
$$(\det(A))^2 = 1$$
So $\det(A) = \pm 1$

ex. Suppose A is such that $A = A^T$, then what can we say about $\det(A)$?

$$A = A^T$$
$$\det(A) = \det(A^T)$$
$$\det(A) = \det(A)$$
You can say nothing!

like $x = x$

ex. Suppose A is skew-symmetric, i.e. $A^T = -A$, what are the possible values of $\det(A)$

$$A^T = -A$$
$$\det(A^T) = \det(-A)$$
$$\det(A) = (-1)^n \det(A)$$
$$\det(A) = (-1)^n \det(A)$$

If n is even:

$\det(A) = \det(A)$

$\therefore$ So $\det(A) \in \mathbb{R}$

If n is odd:

$\det(A) = -\det(A)$

$\therefore \det(A) = 0$

like $x = -x$

$2x = 0$

$x = 0$

Oct. 25

Back to properties...

(4) If E is an elem. matrix, then $\det(E) \neq 0$

ex. $\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} \xrightarrow{R_1 \leftrightarrow R_2} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{E}$ $\det(E) = -1$

$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} \xrightarrow{3R_2} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}_{E}$ $\det(E) = 3$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

Triangular Matrices

ex. $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 5 & 6 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix}$,

$C = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Def'n: A $n \times n$ matrix is <u>upper triangular</u> if $a_{ij} = 0$ when $i > j$

<u>lower triangular</u> if $a_{ij} = 0$ when $j > i$

not too important

<u>diaganol</u> if $a_{ij} = 0$ when $i \neq j$

(<u>i.e.</u> both upper and lower triangular)

<u>ex.</u> Find $\det(A)$ if $A = \begin{bmatrix} 2 & 0 & 3 & 4 & 7 & -1 \\ 0 & 3 & 4 & -2 & 5 & 6 \\ 0 & 0 & -1 & 7 & 2 & 756 \\ 0 & 0 & 0 & 2 & 1075 & -36 \\ 0 & 0 & 0 & 0 & 3/17 & \pi \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

along last row

$$\det(A) = 1(-1)^{12} \begin{vmatrix} 2 & 0 & 3 & 4 & 7 \\ 0 & 3 & 4 & -2 & 5 \\ 0 & 0 & -1 & 7 & 2 \\ 0 & 0 & 0 & 2 & 1075 \\ 0 & 0 & 0 & 0 & 3/17 \end{vmatrix} = 1\left(\frac{3}{17}(-1)^{10} \begin{vmatrix} 2 & 0 & 3 & 4 \\ 0 & 3 & 4 & -2 \\ 0 & 0 & -1 & 7 \\ 0 & 0 & 0 & 2 \end{vmatrix}\right)$$

$$= 1 \cdot \frac{3}{17} \cdot \left(2(-1)^{8} \begin{vmatrix} 2 & 0 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & -1 \end{vmatrix}\right) = 1 \cdot \frac{3}{17} \cdot 2\left((-1)(-1)^{6} \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}\right) = 1 \cdot \frac{3}{17} \cdot 2 \cdot (-1) \cdot 2 \cdot 3$$

$$= -\frac{36}{17}$$

Just the product of the elements on the diagonal

(5) If A is a $n \times n$ triangular matrix,

then $\det(A) = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn} = \prod_{i=1}^{n} a_{ii}$

(i.e. the product of the entries on the main diagonal)

The following three properties tell us now how the determinant of a matrix A is affected by an ERO.

(6) If matrix B is obtained from matrix A by interchanging two rows,

then $\det(B) = -\det(A)$

eg. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = - \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$

$R_1 \leftrightarrow R_2$

(7) If matrix B is obtained from matrix A by multiplying one row by a constant $k \in \mathbb{R}$,

then, $\det(A) = k \det(B)$

ex.

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \xrightarrow{5R_2} \underset{B}{\begin{bmatrix} 1 & 2 \\ 15 & 20 \end{bmatrix}}$$

$\det(A) = -2 \qquad \det(B) = -10$

ex. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \frac{1}{5}\begin{vmatrix} 1 & 2 \\ 15 & 20 \end{vmatrix}$

Note: (a) If $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \frac{1}{5}\begin{vmatrix} 1 & 2 \\ 15 & 20 \end{vmatrix}$, then $\begin{vmatrix} 1 & 2 \\ 15 & 20 \end{vmatrix} = 5\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

(b) If A is a $n \times n$ matrix, and $k \in \mathbb{R}$, then $\det(kA) = k^n \det(A)$

ex. $\det(5A) = 25(-2)$

$= -50$

(8) If matrix B is obtained from matrix A by adding a multiple of one row to another row,

then, $\det(B) = \det(A)$

ex. $\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$

$\det(A) = -2 \qquad \det(B) = -2$

Finding the determinant (the easy way)

Using properties 6, 7, 8, we can find the determinant of a large matrix by using row reduction to turn it into a triangular matrix.

ex. Let $A = \begin{bmatrix} 0 & 2 & -4 & 5 \\ 3 & 0 & -3 & 6 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{bmatrix}$

$$\begin{vmatrix} 0 & 2 & -4 & 5 \\ 3 & 0 & -3 & 6 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{vmatrix} \underset{(R_1 \leftrightarrow R_2)}{=} - \begin{vmatrix} 3 & 0 & -3 & 6 \\ 0 & 2 & -4 & 5 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{vmatrix}$$

$$\underset{(\frac{1}{3} R_1)}{=} -3 \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & -4 & 5 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{vmatrix} \underset{\begin{pmatrix} R_3 - 2R_1 \\ R_4 - 5R_1 \end{pmatrix}}{=} -3 \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & -4 & 5 \\ 0 & 4 & 7 & 3 \\ 0 & -1 & 2 & -4 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -4 \\ 0 & 4 & 7 & 3 \\ 0 & 2 & -4 & 5 \end{vmatrix} = \underline{3} \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -4 \\ 0 & 0 & 15 & -33 \\ 0 & 0 & 0 & -13 \end{vmatrix}$$

$(R_2 \leftrightarrow R_1)$

$\begin{pmatrix} R_3 + 4R_2 \\ R_4 + 2R_2 \end{pmatrix}$

Now we have a triangular matrix!

$= 585$

ex. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, and suppose $\det(A) = 7$

Let B be a 3×3 matrix such that $\det(B) = 3$

Find

a) $\det\left((2A)^{-1}\right)$

$$= \frac{1}{\det(2A)} \quad \text{or} = \frac{1}{2}\det(A^{-1})$$

$\vdots$

$$= \frac{1}{2^3 \det(A)}$$

$$= \frac{1}{8(7)} = \frac{1}{56}$$

b) $\det(3A^2 (B^T)^{-1})$

$= 3^3 \det(A^2 (B^{-1})^T)$

$= 3^3 \det(A^2) \det(B^{-1})$

$= \dfrac{27 (\det(A))^2}{\det(B)}$

$= \dfrac{27 (7)^2}{3} = 441$

c)
$$\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix} \underset{(\ )^T}{=} \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = - \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -7$$

$(R_2 \Leftrightarrow R_3)$

d)
$$\begin{vmatrix} 3a & 3b & 3c \\ 4g+2d & 4h+2e & 4i+2f \\ -d & -e & -f \end{vmatrix} = -3 \begin{vmatrix} a & b & c \\ 4g+2d & 4h+2e & 4i+2f \\ d & e & f \end{vmatrix}$$

$\left(-\frac{1}{3} R_1, -R_3\right)$

$$= 3 \begin{vmatrix} a & b & c \\ d & e & f \\ 4g+2d & 4h+2e & 4i+2f \end{vmatrix} = 3 \begin{vmatrix} a & b & c \\ d & e & f \\ 4g & 4h & 4i \end{vmatrix} = 12 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$(R_2 \Leftrightarrow R_3)$ $(R_3 - 2R_2)$ $\left(\frac{1}{4} R_3\right)$

$= 12\,(7)$

$= 84$

Theorem: Let A be a $n \times n$ matrix. A is invertible iff $\det(A) \neq 0$

(i.e. A is singular iff $\det(A) = 0$)

A "if and only if" B

$A \Leftrightarrow B$

Note: A "if" B

$A \Leftarrow B$

Note: A "only if" B ex "I'll go to the cinema with you only if you pay"

$A \Rightarrow B$

Proof:

pt1: ($\Rightarrow$) Show that if A is invertible, the $\det(A) \neq 0$

If A is invertible, then A can be written as a product of elem. matrices

i.e. there exists $E_1, E_2 \ldots E_k$ such that

$$A = E_1 E_2 \ldots E_k$$

So $\det(A) = \det(E_1 \ldots E_k)$

$= \det(E_1)\det(E_2)\ldots\det(E_k)$ det (elem matrix) $\neq 0$

$\quad\quad \neq 0 \quad\quad \neq 0 \quad\quad\quad \neq 0$

$\therefore \det(A) \neq 0$

pt 2: ($\Leftarrow$) Show that, if $\det(A) \neq 0$, then A is invertible

Let R = RREF of A, i.e. $A \sim R$

So there exists a sequence of elem. matrices such that

$E_k \ldots E_2 E_1 A = R$

we want to show that $R = I$ (b.c. $E_k \ldots E_2 E_1$ would be equal to A^{-1})

$\det(E_k \ldots E_1 A) = \det(R)$

$\det(E_k) \cdot \ldots \cdot \det(E_1)\det(A) = \det(R)$

$\neq 0 \quad\quad\quad \neq 0 \quad \neq 0$ (part of premise in theorem)

So $\det(R) \neq 0$

Note: R cannot have a row of zeros

So $R = I$

So $A \sim I$

$\therefore$ A is invertible

Note: If E is elementary, then $\det(E) \neq 0$

a) $\overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \xrightarrow{R_1 \leftrightarrow R_2} \overset{E}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$

$\det(I) = 1$ $\quad$ $\det(E) = -1$

b) $\overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \xrightarrow{kR_2} \overset{E}{\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}}$

$\det(I) = 1$ $\quad$ $\det(E) = k$

c) $\overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \xrightarrow{R_2 - 3R_1} \overset{E}{\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}}$

$\det(I) = 1$ $\quad$ $\det(E) = 1$

ex. For what value(s) of k is the matrix

$A = \begin{bmatrix} k-3 & -2 \\ -2 & k-2 \end{bmatrix}$ singular?

$\det(A) = (k-3)(k-2) - (-2)(-2)$

A is singular iff $\det(A) = 0$

$0 = (k-3)(k-2) - 4$

$0 = k^2 - 3k - 2k + 6 - 4$

$0 = k^2 - 5k + 2$

$k = \dfrac{5 \pm \sqrt{25 - 4(2)}}{2} = \dfrac{5 \pm \sqrt{17}}{2}$

$$k_1 = \frac{5+\sqrt{17}}{2}, \quad k_2 = \frac{5-\sqrt{17}}{2}$$

ex. $$\begin{aligned} k^2x + ky + 2z &= 0 \\ 2x + y + z &= 0 \\ -5z &= 0 \end{aligned}$$

For what value(s) of k does the system have only the trivial solution? = when matrix of system is invertible (invertibility theorem)

i.e. $\det(A) \neq 0$

$$A = \begin{bmatrix} k^2 & k & 2 \\ 2 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix}$$

Can use cofactor expansion

$$\det(A) \neq 0$$

$$-5 \begin{vmatrix} k^2 & k \\ 2 & 1 \end{vmatrix} \neq 0$$

$$-5(k^2 - 2k) \neq 0$$

$$k(k-2) \neq 0$$

$$k \neq 0, \quad k \neq 2$$

$$\therefore k \in \mathbb{R} \setminus \{0, 2\}$$

Cramer's Rule

Let $\underset{n\times n}{A}\,\underset{n\times 1}{X} = \underset{n\times 1}{B}$ be a linear system.

If A is invertible, then the unique solution of the system is given by:

$$x_1 = \frac{\det(A_1)}{\det(A)}$$

$$x_2 = \frac{\det(A_2)}{\det(A)}$$

$$\vdots$$

$$x_n = \frac{\det(A_n)}{\det(A)}$$

where $\underline{A_j}$ is the matrix A where column j is replaced by B

ex. $2x + 3y = 1$
$x - y = 2$

$$\longrightarrow \overset{A}{\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Check that A is invertible, i.e. $\det(A) \neq 0$

$$\det(A) = \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = -5 \neq 0$$

$$x = \frac{\det(A_1)}{\det(A)} \qquad y = \frac{\det(A_2)}{\det(A)}$$

$$= \frac{\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix}}{-5} \qquad = \frac{\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}}{-5}$$

$$= \frac{7}{5} \qquad = -\frac{3}{5}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7/5 \\ -3/5 \end{bmatrix}$$