

Section 3 - Integration Techniques

Feb. 15, 2017

INTEGRATION BY PARTS

When should I use IBP?

→ A u-sub fails

→ Product of • trig × polynomial

• exp × polynomial

• ln, arctan, arcsin × polynomial

• trig × exp/trig

ex. $\int x e^{x^2+1} dx \rightarrow$ by subs.

ex. $\int (x^2+1) e^x dx \rightarrow$ IBP

Recall that (Product Rule)

$$(fg)' = f'g + fg'$$

$$\int (fg)' dx = \int f'g dx + \int fg' dx$$

$$fg = \int f'g dx + \underline{\int fg' dx}$$

$$\boxed{\int fg' dx = fg - \int f'g dx}$$

hope this integral is
easier than $\int f g' dx$

INTEGRATION BY PARTS FORMULA

ex. $\int (\underbrace{3x+1}_{f} \underbrace{\cos x}_{g'}) dx$

$f = 3x+1$ $g' = \cos x$
 $f' = 3$ $g = \sin x$

$$= \int (3x+1) \cos x dx$$

$$= (3x+1) \sin x - \int 3 \sin x dx$$

$$= (3x+1) \sin x - 3 \int \sin x dx$$

$$= (3x+1) \sin x - 3(-\cos x) + C$$

$$= (3x+1) \sin x + 3 \cos x + C$$

ex. $\int (3x+1) \ln x dx$

~~$f = 3x+1$~~ ~~$g' = \ln x$~~

$$\int \ln x ??$$

$$f = \ln x \quad g' = 3x+1$$

$$f' = \frac{1}{x} \quad g = \frac{3}{2}x^2 + x$$

$$I = (\ln x)(\frac{3}{2}x^2 + x) - \int \frac{1}{x} (\frac{3}{2}x^2 + x) dx$$

$$I = (\frac{3}{2}x^2 + x) \ln x - \int (\frac{3}{2}x^2 + x) dx$$

$$I = (\frac{3}{2}x^2 + x) \ln x - \left(\frac{3}{4}x^2 + x \right) + K$$

À vous

$$I = \int (x^2 + 2x + 4) e^x dx$$

$$f = x^2 + 2x + 4$$

$$g' = e^x$$

$$f' = 2x + 2$$

$$g = e^x$$

$$I = e^x (x^2 + 2x + 4) - \underbrace{\int (2x+2) e^x}$$

$$f = 2x + 2 \quad g' = e^x$$

$$f' = 2 \quad g = e^x$$

$$I = e^x (x^2 + 2x + 4) - \left((2x+2)e^x - \int 2e^x dx \right)$$

$$= e^x (x^2 + 2x + 4) - (2x+2)e^x + 2e^x + C$$

PRIORITY LIST for "f"

→ $\ln x, \arcsin x, \operatorname{arccos} x$

→ polynomials

→ exp, trigs

"OUR SHORTCUT METHOD" when $f = \text{polynomial}$

ex. I: $\int (x^2 + 2x + 4) e^x dx$

<u>f</u>	<u>g'</u>
$\oplus x^2 + 2x + 4$	e^x
$\ominus 2x + 2$	$e^x \rightarrow (x^2 + 2x + 4) e^x$
$\oplus 2$	$e^x \rightarrow - (2x + 2) e^x$
$\ominus 0$	$e^x \rightarrow + 2 e^x$
stop at 0	

$$I = (x^2 + 2x + 4)e^x - (2x + 2)e^x + 2e^x + C$$

ex. I: $\int (x^3 + 2x^2 + 4) \sin x dx$

<u>f</u>	<u>g'</u> <small>✓ antiderivatives</small>
$\oplus x^3 + 2x^2 + 4$	$\sin x$
$\ominus 3x^2 + 2$	$-\cos x$
$\oplus 6x + 3$	$-\sin x$
$\ominus 6$	$\cos x$
$\oplus 0$	$\sin x$

$$I = (x^3 + 2x^2 + 4)(-\cos x) - (3x^2 + 2)(-\sin x) + (6x + 2)(\cos x) - 6 \sin x$$

ex. I: $\int \arctan x \, dx$

$$f = \arctan x \quad g' = 1$$

$$f' = \frac{1}{1+x^2} \quad g = x$$

$$\begin{aligned} I &= (\arctan x) x - \int \frac{1}{1+x^2} x \, dx \\ &\quad \text{Let } u = 1+x^2 \end{aligned}$$

$$= x \arctan x - \frac{1}{2} \int \frac{1}{u} du \quad du = 2x \, dx$$

$$= x \arctan x - \frac{1}{2} \ln(1+x^2)$$

ex. $I = \int x \sec^2 x \, dx$

$$\begin{array}{ll} f = x & g' = \sec^2 x \\ f' = 1 & g = \tan x \end{array}$$

$$\begin{array}{ll} \oplus \frac{f}{x} & \frac{g'}{\sec^2 x} \\ \ominus 1 & \tan x \\ 0 & \ln |\sec x| \end{array}$$

1BP

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ex. Warm Up

$$\int 2x^3 \cos(x^2) dx$$

Let $u = x^2$

$$= \int x^2 \cos(u) 2x dx \quad du = 2x dx$$

$$= \int u \cos u du$$

$$f = u \quad g' = \cos u$$
$$f' = 0 \quad g = \sin u$$

$$= u \sin u + \cos u + C$$

$$= x^2 \sin x^2 + \cos x^2 + C$$

ex. $\int (\ln x)^m dx$

$$= x(\ln x)^m - \int m(\ln x)^{m-1} \cdot \frac{1}{x} dx$$

$$f = (\ln x)^m \quad g' = 1$$
$$f' = m(\ln x)^{m-1} \cdot \frac{1}{x} \quad g = x$$

$$\boxed{\int (\ln x)^m dx = x(\ln x)^m - m \int (\ln x)^{m-1} dx}$$

chain rule

This pattern is called a **REDUCTION FORMULA**

$$\text{If } m=1 \quad \int \ln x dx = x \ln x - \int 1 dx$$
$$= x \ln x - x + C$$

$$\text{If } m=2 \quad \int (\ln x)^2 dx = x(\ln x)^2 - 2 \int \ln x dx$$

$$= x(\ln x)^2 - 2(x\ln x - x) + C$$

$$\underline{\text{ex.}} \quad \int_1^e (\ln x)^3 dx$$

$$= x(\ln x)^3 \Big|_1^e - 3 \int_1^e (\ln x)^2 dx$$

$$= e - 3 \left[x(\ln x)^2 \Big|_1^e - 2 \int_1^e \ln x dx \right]$$

$$= e - 3 \left[e - 2 \left(x \ln x \Big|_1^e - \int_1^e (\ln x)^0 dx \right) \right]$$

$$= e - 3e + 6(e - x \Big|_1^e)$$

$$= -2e + 6(e - (e-1))$$

$$= -2e + 6$$

À vous²

Find a reduction formula for

$$\int x^m \cos x dx$$

$$f = x^m$$

$$g' = \cos x$$

$$f' = mx^{m-1}$$

$$g = \sin x$$

$$= x^m \sin x - \int mx^{m-1} \sin x dx$$

$$= x^m \sin x - \left[-mx^{m-1} \cos x + \int (m-1)x^{m-2} \cos x dx \right]$$

$$f = x^{m-1}$$

$$g' = \sin x$$

$$f' = (m-1)x^{m-2}$$

$$g = -\cos x$$

$$= x^m \sin x + m x^{m-1} \cos x - m(m-1) \int x^{m-2} \cos x dx \quad \text{Need form to stay}$$

Bonne chance...

$$\int_0^\pi x^4 \cos x dx$$

$$= \left[x^4 \sin x + 4x^3 \cos x \right]_0^\pi - (4)(3) \int x^2 \cos x dx$$

$$= -4\pi^3 - 12 \left[\left[x^2 \sin x + 2x \cos x \right]_0^\pi - (2)(1) \int x \cos x dx \right]$$

$$= -4\pi^3 - 12(-2\pi - 2 \sin x \Big|_0^\pi)$$

$$= -4\pi^3 + 24\pi$$

Your favourite nightmare...

$$\text{Solve } I = \int e^x \sin x dx$$

$$f = \sin x \quad g' = e^x$$

$$f' = \cos x \quad g = e^x$$

$$I = e^x \sin x - \int e^x \cos x dx$$

Need to be consistent here! otherwise

$$= e^x \sin x - \left[e^x \cos x + \int e^x \sin x dx \right] \quad \text{you will just undo what you}$$

$$f = \cos x \quad g' = e^x \\ f' = -\sin x \quad g = e^x$$

$$I = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

$$\text{But... } I = \int e^x \sin x \, dx$$

so

$$I = e^x \sin x - e^x \cos x - I$$

$$2I = e^x \sin x - e^x \cos x$$

$$I = \frac{1}{2} (e^x \sin x - e^x \cos x)$$

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$$\text{ex, Solve } I = \int e^{3x} \cos 2x \, dx$$

$$f: e^{3x} \quad g' = \cos 2x$$

$$f' = 3e^{3x} \quad g = \int \cos 2x \, dx \quad \text{Let } u = 2x \\ = \frac{1}{2} \sin(2x)$$

$$I = \frac{1}{2} e^{3x} \sin(2x) - \frac{3}{2} \int e^{3x} \sin(2x) \, dx$$

$$f: e^{3x} \quad g' = \sin(2x)$$

$$f' = 3e^{3x} \quad g = \int \sin(2x) \, dx \\ = -\frac{1}{2} \cos 2x$$

$$I = \frac{1}{2} e^{3x} \sin(2x) - \frac{3}{2} \left[-\frac{1}{2} e^{3x} \cos 2x + \frac{3}{2} \int e^{3x} \cos 2x \, dx \right]$$

$$I = \frac{1}{2} e^{3x} \sin 2x + \frac{3}{4} e^{3x} \cos 2x - \frac{9}{4} I$$

$$I = \frac{4}{13} \left[\frac{1}{2} e^{3x} \sin 2x + \frac{3}{4} e^{3x} \cos 2x \right]$$

$$I = \frac{2}{13} e^{3x} \sin 2x + \frac{3}{13} e^{3x} \cos 2x + C$$

$$\underline{\text{ex.}} \int 2^{2x} e^{2x} dx$$

$$= \frac{1}{2} \int 2^v e^v du$$

$$= \frac{1}{2} \int (2e)^v dv$$

⋮

$$\underline{\text{ex. I:}} \int \sin^n x dx$$

$$I = \int \underbrace{\sin^{n-1} x}_f \underbrace{\sin x dx}_g$$

$$f = \sin^{n-1} x$$

$$g' = \sin x$$

$$f' = (n-1) (\sin x)^{n-2} \cdot \cos x \quad \text{chain rule} \quad g = -\cos x$$

$$I = -(\sin x)^{n-1} \cos x + \int (n-1) \sin^{n-2} x \cos^2 x dx$$

$$I = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) dx$$

$$I = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx - (n-1) \int \sin^n x dx$$

$$I = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx - (n-1) I$$

$$n I = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx$$

$$I = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x dx$$

ex. solve $I = \int \sin^3 x \, dx$

$$I = -\frac{1}{3} \sin^2 x \cos x - \frac{2}{3} \cos x + C$$

ex. $I = \int \sec^n x \, dx$

Hint: $\tan^2 x = \sec^2 x - 1$

$$I = \int \sec^{n-2} x \sec^2 x \, dx$$

$$f = (\sec x)^{n-2}$$

$$g' = \sec^2 x \, dx$$

$$f' = (n-2) \sec^{n-3} x \cdot \sec x \tan x$$

$$g = \int \sec^2 x \, dx$$

$$g = \tan x$$

$$I = \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \left[\int \sec^n x \, dx - \int \sec^{n-2} x \, dx \right]$$

$$I = \sec^{n-2} x \tan x - (n-2) I + (n-2) \int \sec^{n-2} x \, dx$$

$$(n-1) I = \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x \, dx$$

$$I = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$$

ex. solve $I = \int \sec^4 x \, dx$

$$I = \frac{1}{3} \sec^3 x \tan x + \frac{2}{3} \int \sec^2 x \, dx$$

$$= \frac{1}{3} \sec^2 x \tan x + \frac{2}{3} \tan x + C$$

----- END OF MATERIAL FOR TEST 1 ----- Feb. 22

- IBP ✓ } when to use these?
Trigonometric Powers } → when subst fails
→ sin/cos combos
→ sec/tan combos

Case 1 Odd Powers of "cos"

ex: $\int \sin^3 x \cos^5 x \, dx$

Step 1: Take one $\cos x$ aside with dx

$$= \int \sin^2 x \cos^4 x \underbrace{\cos x \, dx}_{\text{"I want to be a du one day..."}}$$

Step 2: Replace all remaining $\cos^2 x$ by $(1 - \sin^2 x)$
since we start with odd powers

$$= \int \sin^2 x (1 - \sin^2 x) (1 - \sin^2 x) \cos x \, dx$$

Step 3: Let $u = \sin x \rightarrow du = \cos x \, dx$ yay!

$$= \int u^2 (1 - u^2) (1 - u^2) du$$

Step 4: expand / distribute and antiderivative

$$= \int (v^2 - 2v^4 + v^6) dv$$

$$= \frac{1}{3}v^3 - \frac{2}{5}v^5 + \frac{1}{7}v^7 + C$$

$$= \frac{1}{3}\sin^2 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C$$

À vous $\int \sin^4(\ln x) \cos^3(\ln x) dx$

$$\text{Let } u = \ln x$$

$$= \int \sin^4 u \cos^3 du$$

$$du = \frac{1}{x} dx$$

$$= \int \sin^4 u \cos^2 \cos du$$

$$= \int \sin^4 u (1 - \sin^2 u) \cos du$$

$$\text{Let } w = \sin u \rightarrow dw = \cos u du$$

$$= \int w^4 (1 - w^2) dw$$

$$= \int (w^4 - w^6) dw$$

$$= \frac{1}{5}w^5 - \frac{1}{7}w^7 + \boxed{C}$$

Case 2 Odd powers of 'sin'

ex: $\int \sin^5 x \, dx$

Step 1: Place a $\sin x$ aside with dx

Step 2: Replace all remaining $\sin^2 x$ by $(1 - \cos^2 x)$

Step 3: Let $u = \cos x \Rightarrow du = -\sin x \, dx$

Step 4: Expand and integrate

$$= \int \sin^4 x \sin x \, dx$$

$$= \int (1 - \cos^2 x)^2 \sin x \, dx$$

$$= - \int (1 - u^2)^2 \, du$$

$$\text{Let } u = \cos x$$

$$du = -\sin x \, dx$$

$$= - \int (1 - 2u^2 + u^4) \, du$$

$$= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + \textcircled{j}$$

À vous

$$\int_{\sqrt{x}}^{\sqrt{5x}} \frac{\sin^3(\sqrt{x})}{\cos^4(\sqrt{x})} \, dx$$

$$= \int_4^9 \frac{\sin^2(\sqrt{x})}{\cos^4(\sqrt{x})} \cdot \sin(\sqrt{x}) \frac{1}{\sqrt{x}} dx$$

$$= \int_4^9 \frac{1 - \cos^2(\sqrt{x})}{\cos^4(\sqrt{x})} \cdot \sin(\sqrt{x}) \frac{1}{\sqrt{x}} dx$$

Let $u = \cos(\sqrt{x})$

$$= -2 \int_{\cos(2)}^{\cos(3)} \frac{1 - u^2}{u^4} du$$

$$du = -\sin(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} dx$$

Bounds

$$x = 9 \Rightarrow u = \cos(3)$$

$$x = 4 \Rightarrow u = \cos(2)$$

$$= -2 \int_{\cos(2)}^{\cos(3)} (u^{-4} - u^2) du$$

$$= -2 \left[\left(-\frac{1}{3u^3} + \frac{1}{u} \right) \right]_{\cos(2)}^{\cos(3)}$$

$$= -2 \left[\left(-\frac{1}{3\cos^3 3} + \frac{1}{\cos 3} \right) - \left(-\frac{1}{3\cos^3 2} + \frac{1}{\cos 2} \right) \right]$$

Case 3 Even cos and even sin

$$\text{ex. } \int \sin^2 x dx \quad \text{or} \quad \int \cos^2 x dx$$

Replace all $\sin^2 x$ by $\frac{1}{2}(1 - \cos 2x)$

and all $\cos^2 x$ by $\frac{1}{2}(1 + \cos 2x)$

$$\underline{\text{ex.}} \quad \int \sin^2 x \, dx$$

$$= \frac{1}{2} \int (1 - \cos 2x) \, dx \quad \text{Let } u = 2x \\ du = 2 \, dx$$

$$= \frac{1}{4} \int (1 - \cos u) \, dx$$

$$= \frac{1}{4} u - \frac{1}{4} \sin u + C$$

$$= \frac{1}{2} x - \frac{1}{4} \sin 2x + C$$

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$$\underline{\text{ex.}} \quad \int \sin^2\left(\frac{1}{4}x\right) \cos^2\left(\frac{1}{4}x\right) \, dx$$

$$= \frac{1}{4} \int (1 - \cos \frac{1}{2}x)(1 + \cos \frac{1}{2}x) \, dx$$

$$= \frac{1}{4} \int \left(1 - \cos^2 \frac{1}{2}x\right) \, dx$$

$$= \frac{1}{4} \int \left(1 - \left(\frac{1}{2}(1 + \cos x)\right)\right) \, dx \stackrel{\text{OR}}{=} \frac{1}{4} \int \sin^2\left(\frac{1}{2}x\right) \, dx$$

$$= \frac{1}{8} \int (1 - \cos x) \, dx$$

$$= \frac{1}{8} \int (1 - \cos x) \, dx$$

$$= \frac{1}{8} x - \frac{1}{8} \sin x + C$$

$$= \frac{1}{8} x - \frac{1}{8} \sin x + C$$

Application

$$\text{N.B. period } T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2$$

Let $v(t) = 8 \cos(\pi t)$ represent the velocity of a mass of 1kg oscillating at the end of a spring.

a) Find \bar{v} over $[0, 2]$

$$\bar{v} = \frac{1}{2} \int_0^2 8 \cos \pi t \, dt$$

$$= 4 \int_0^2 \cos \pi t \, dt$$

$$= \frac{4}{\pi} \int_0^{2\pi} \cos u \, du$$

$$= \frac{4}{\pi} \sin u \Big|_0^{2\pi}$$

$$= 0$$

$$\text{Let } u = \pi t$$

$$du = \pi dt$$

Boundary

$$t=2 \rightarrow u = 2\pi$$

$$t=0 \rightarrow u = 0$$

b) What is the average kinetic energy of this mass?

$$K(t) = \frac{1}{2} m [v(t)]^2$$

$$= \frac{1}{2} \cdot 1 \left(8 \cos(\pi t) \right)^2$$

$$= 32 \cos^2 \pi t$$

$$\bar{K}_{[0,2]} = \frac{1}{2} \int_0^2 32 \cos^2 \pi t$$

$$= 16 \int_0^2 \cos^2 \pi t dt$$

$$= 16 \cdot \frac{1}{2} \int_0^2 (1 + \cos 2\pi t) dt$$

$$= \frac{8}{2\pi} \int_0^2 (1 + \cos 2\pi t) 2\pi dt \quad \text{Let } u = 2\pi t$$

$$= \frac{8}{2\pi} \int_0^{4\pi} (1 + \cos u) du$$

$$= \frac{8}{2\pi} \left[u + \sin u \right]_0^{4\pi}$$

$$du = 2\pi dt$$

boundaries

$$t=2 \rightarrow u=4\pi$$

$$t=0 \rightarrow u=0$$

$$= \frac{8}{2\pi} [(4\pi + 0) - (0 + 0)]$$

$$= \frac{8}{2\pi} \cdot 4\pi$$

$$= 16$$

Case 4 Odd powers of 'tan'

$$\text{ex. } \int \sec^3 x \tan^3 x dx$$

Step 1: Move 1 sec x tan x aside with dx

$$= \int \sec^2 x \tan^2 x \sec x \tan x \, dx$$

Step 2: Change all remaining $\tan^2 x$ into $\sec^2 x - 1$

$$= \int \sec^2 x (\sec^2 - 1) \sec x \tan x \, dx$$

Step 3: Let $u = \sec x$, $du = \sec x \tan x \, dx$

$$= \int u^2 (u^2 - 1) \, du$$

Step 4: Expand and integrate

$$= \int (u^4 - u^2) \, du$$

$$= \frac{1}{5} u^5 - \frac{1}{3} u^3 + C$$

$$= \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C$$

Case 5 Even powers of 'sec'

ex. $\int \sec^6 x \, dx$

Step 1: Move 1 sec²x aside with dx

$$= \int \sec^4 x \sec^2 x \, dx$$

Step 2: Transform all remaining sec²x into tan²x + 1

$$= \int (\tan^2 x + 1)^2 \sec^2 x \, dx$$

Step 3: Let $u = \tan x \rightarrow du = \sec^2 x$

Step 4: Expand, distribute and integrate.

$$= \int (u^2 + 1)^2 \, du$$

$$= \int (u^4 + 2u^2 + 1) \, du$$

$$= \frac{1}{5} \tan^5 x + \frac{2}{3} \tan^3 x + \tan x + C$$

À vous

$$\int \sec^4 x \tan^3 x \, dx$$

As an even secant

$$= \int \sec^2 x \tan^3 x \sec^2 x \, dx$$

$$= \int (1 + \tan^2 x) \tan^3 x \sec^4 x dx$$

Let $v = \tan x$

$$dv = \sec^2 x dx$$

$$= \int (1 + v^2) v^3 dv$$

$$= \int (v^3 + v^5) dv$$

$$= \frac{1}{4} \tan^4 x + \frac{1}{6} \tan^6 x + C$$

As an odd tangent

$$= \frac{1}{6} \sec^6 x - \frac{1}{4} \sec^4 x + K$$

If you use this identities, final answer will only differ by a constant

Case 6 OMG IDK WTS

ex. $\int \sec x dx$

$$= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$$u = \sec x + \tan x$$

$$dv = (\sec x \tan x + \sec^2 x) dx$$

$$= \int \frac{1}{u} du$$

$$= \ln |u| + C$$

$$= \ln |\sec x + \tan x| + C$$

$$\text{ex. } I = \int \sec^3 x dx$$

$$f = \sec x \quad g' = \sec^2 x$$

$$f' = \sec x \tan x \quad g = \tan x$$

$$I = \sec x \tan x - \int \sec x \tan^2 x dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$$

$$I = \sec x \tan x - \underbrace{\int \sec^3 x dx}_{I_1} + \int \sec x dx$$

$$I = \frac{1}{2} (\sec x \tan x + \ln |\sec x + \tan x|) + C$$

Trig Substitutions

March 1

When should I use trig subs?

→ When a u-sub fails

→ ① $x^2 - a^2$ form

② $a^2 - x^2$ form

③ $a^2 + x^2$ form

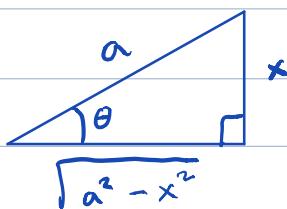
Case 1: $a^2 - x^2$ form

ex. $\int \frac{1}{(9-x^2)^{1/2}} dx$

Survival Guide:

Step 1: Let $x = a \sin \theta \rightarrow \sin \theta = \frac{x}{a} = \frac{\text{opp}}{\text{hyp}}$

$dx = a \cos \theta d\theta$



Let $x = 3 \sin \theta$

$dx = 3 \cos \theta d\theta$

A right triangle is shown with the vertical leg labeled x , the horizontal leg labeled $\sqrt{3^2 - x^2}$, and the hypotenuse labeled 3 . The angle at the bottom-left vertex is labeled θ .

$$= \sqrt{3^2 - x^2}$$

$$\int \frac{1}{(9-x^2)^{3/2}} dx$$

$$= \int \frac{1}{(9 - 9\sin^2\theta)^{3/2}} \cdot 3\cos\theta d\theta$$

Step 2: Replace all $1-\sin^2\theta$ by $\cos^2\theta$ and simplify

$$= \int \frac{3\cos\theta}{(9\cos^2\theta)^{3/2}} d\theta$$

$$= \int \frac{3\cos\theta}{27\cos^3\theta} d\theta$$

$$= \int \frac{1}{9\cos^2\theta} d\theta$$

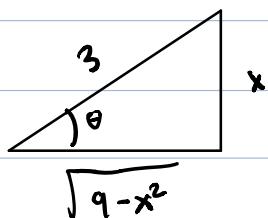
Step 3: Good luck! Trig powers ...

$$= \int \frac{1}{9} \sec^2\theta d\theta$$

$$= \frac{1}{9} \ln|\sec\theta + \tan\theta| + C$$

Step 4: Return to x 's (using your \triangle)

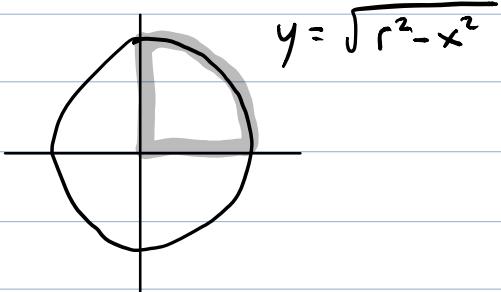
$$= \frac{1}{9} \frac{x}{\sqrt{9-x^2}} + C$$



$$= \frac{x}{\sqrt{r^2 - x^2}} + k$$

Ex. Show that the area of a circle is πr^2 .

$$x^2 + y^2 = r^2$$



Area of a circle = 4 × area of a quarter circle

$$\text{Area} = 4 \int_0^r \sqrt{r^2 - x^2} dx$$

$$\text{Let } x = r \sin \theta$$

Bounds

$$dx = r \cos \theta d\theta$$

$$x=0 \rightarrow 0 = \sin \theta$$

$\theta = 0$ and π
not in domain



$$x = r \Rightarrow 1 = \sin \theta$$

$$\theta = \frac{\pi}{2}$$

$$A = 4 \int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2 \sin^2 \theta} \cdot r \cos \theta d\theta$$

$$= 4 \int_0^{\frac{\pi}{2}} \sqrt{r^2(1-\sin^2\theta)} r \cos\theta d\theta$$

$$= 4 \int_0^{\frac{\pi}{2}} r \cos\theta r \cos\theta d\theta \quad \text{Doesn't have to be } |r| \text{ since } r \geq 0 \text{ in domain we're integrating}$$

$$= 4r^2 \int_0^{\frac{\pi}{2}} \cos^2\theta d\theta \quad (\text{even } \cos\theta)$$

$$= 4r^2 \int_0^{\frac{\pi}{2}} \frac{1}{2} (1 + \cos 2\theta) d\theta$$

$$= \frac{2r^2}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2\theta) 2d\theta \quad u = 2\theta \quad \theta = 0 \rightarrow u = 0 \\ du = 2d\theta \quad \theta = \frac{\pi}{2} \rightarrow u = \pi$$

$$= r^2 \int_0^{\pi} (1 + \cos u) du$$

$$= r^2 \left[u + \sin u \right]_0^\pi$$

$$= r^2 (\pi)$$

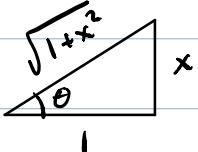
$$= \pi r^2$$

À vous

$$\int \frac{1}{\sqrt{x^2+1}} dx$$

Let $x = \tan \theta$
 $dx = \sec^2 \theta d\theta$

$$= \int \frac{1}{\sqrt{\tan^2 \theta + 1}} \sec^2 \theta d\theta$$



$$= \int \frac{1}{\sec \theta} \sec^2 \theta d\theta \quad \text{assume } \sec \theta \geq 0$$

$$= \int \sec \theta d\theta$$

↓ New formula (case 6)

$$= \ln |\sec \theta + \tan \theta| + k$$

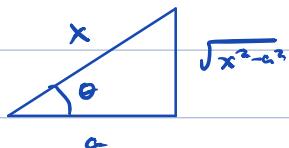
$$= \ln \left| \frac{\sqrt{1+x^2}}{1} + \frac{x}{1} \right| + k \quad \text{subs from triangle}$$

$$= \ln |\sqrt{1+x^2} + x| + k$$

Recall $\int \frac{1}{\sqrt{x^2-1}} dx = \arccos x$

Case 3 $x^2 - a^2$ forms

Steps: ① Let $x = a \sec \theta \rightarrow \sec \theta = \frac{x}{a} \rightarrow$
 $dx = a \sec \theta \tan \theta d\theta$



② Replace all $\sec^2 \theta - 1$ by $\tan^2 \theta$ and simplify

③ Solve the trig power that results from ②

④ Go back to u's

ex. $\int \frac{2e^x}{\sqrt{4e^{2x}-25}} dx$

$$= \int \frac{2e^x}{\sqrt{(2e^x)^2 - 25}} dx$$

$$= \int \frac{1}{\sqrt{u^2 - 25}} du \quad \begin{array}{l} \text{Let } u = 2e^x \\ du = 2e^x dx \end{array}$$

$$= \int \frac{1}{\sqrt{25\sec^2\theta - 25}} 5 \quad \begin{array}{l} \text{Let } u = 5\sec\theta \\ du = 5\sec\theta\tan\theta d\theta \end{array}$$

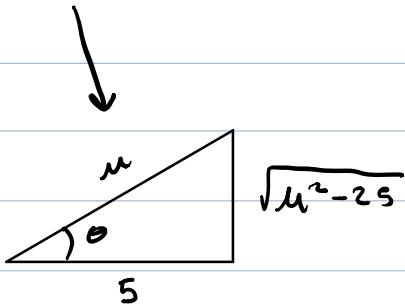
$$= \frac{1}{5} \int \frac{1}{\sqrt{\sec^2\theta + 1}} 5 \sec\theta \tan\theta d\theta$$

$$= \int \frac{1}{5} \sec\theta \tan\theta d\theta$$

$$= \int \sec\theta d\theta$$

$$= \ln |\sec\theta + \tan\theta| + K$$

$$= \ln \left| \frac{u}{5} + \frac{\sqrt{u^2 - 25}}{5} \right| + K$$



$$= \ln \left| \frac{2e^x + \sqrt{4e^{2x}-2s}}{s} \right| + K$$

N.B. also = $\ln |2e^x + \sqrt{4e^{2x}-2s}|$

March 13, 2017

Partial Fraction Decomposition

When to use?

→ not basic integral

→ u-sub fails

→ proper rational func (power on top is less), where the denominator has more than one factor

ex.
$$\int \frac{2x^2+x-1}{x^3-3x^2+3x-9} dx$$

$$\underbrace{x^3-3x^2+3x-9}_{x^2(x-3)+3(x-3)}$$

$(x^2+3)(x-3)$ ← denominator has 2 factors (linear + quadratic)

The decomposition of a proper fraction depends on the nature of its denominator's factors

→ linear factor ($l(x)$) $\longrightarrow \frac{A}{l(x)}$

→ repeated linear factor ($(l(x))^k$) $\longrightarrow \frac{A_1}{l(x)} + \frac{A_2}{(l(x))^2}$

$+ \dots + \frac{A_k}{(l(x))^k}$

\rightarrow quadratic factors ($q(x)$) $\rightarrow \frac{Ax + B}{q(x)}$

\rightarrow repeated quadratic factors ($(q(x))^n$)

$$\longrightarrow \frac{A_1x + B_1}{q(x)} + \frac{A_2x + B_2}{(q(x))^2} + \dots + \frac{A_nx + B_n}{(q(x))^n}$$

ex. what form would the decomposed version of

$$\frac{3x^2 + 2x - 1}{x^3(x^2 + 1)}$$

repeated linear quadratic

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx + E}{x^2 + 1}$$

A vous

Decompose

$$\frac{4x^5 + 3x^2 - 7x + 1}{x^2(x-1)(x-2)^2(x^2+4)^2}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x-2} + \frac{E}{(x-2)^2} + \frac{Fx + G}{x^2+4} + \frac{Hx + I}{(x^2+4)^2}$$

Now the mechanics

ex, find the partial fraction decomposition for

$$\frac{2x+3}{x^2-9x-10} = \frac{2x+3}{(x-10)(x+1)} = \frac{A}{x-10} + \frac{B}{x+1}$$

What A and B satisfy this identity?

→ Multiply both sides by $(x-10)(x+1)$

$$2x+3 = A(x+1) + B(x-10)$$

if $x = 10$:

$$23 = 11A$$

$$A = \frac{23}{11}$$

if $x = -1$:

$$1 = -11B$$

$$B = -\frac{1}{11}$$

Therefore

$$\frac{2x+3}{(x-10)(x+1)} = \frac{\frac{23}{11}}{x-10} + \frac{-\frac{1}{11}}{x+1}$$

À nous

Solve $\int \frac{2x+3}{x^2-9x-10} dx$

$$= \int \frac{2x+3}{x^2-9x-10} dx = \int \frac{11}{x+1} dx$$

$$= \frac{23}{11} \ln|x-10| - \frac{1}{11} \ln|x+1| + C$$

ex. Decompose and integrate

$$\frac{4x^2 + 7x + 9}{x^3(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1}$$

$$4x^2 + 7x + 9 = Ax^2(x+1) + Bx(x+1) + C(x+1) + Dx^3$$

$$x = -1 :$$

$$4 - 7 + 9 = -D$$

$$D = -6$$

$$x = 0 :$$

$$9 = C$$

$$C = 9$$

→ Let's compare powers

$$4x^2 + 7x + 9 = Ax^3 + Ax^2 + Bx^2 + Bx + Cx + C + Dx^3$$

$$x^3: \quad 0 = A + 0$$

$$0 = A - 6$$

$$A = 6$$

$$x^2: \quad 4 = A + B$$

$$4 = 6 + B$$

$$B = -2$$

$$x: \quad 7 = B + C$$

$$7 = 7 \quad \checkmark$$

$$x^0: \quad 9 = C$$

$$C = 9 \quad \checkmark$$

March 16, 2017

Mr. Memy

$$I = \int \frac{3x^4 + 2x^3 + 6x^2 + 3x + 3}{x(x^2+1)^2} dx$$

$$I = \int \left(\frac{A}{x} + \frac{Bx+C}{x+1} + \frac{Dx+E}{(x^2+1)^2} \right) dx$$

$$3x^4 + 2x^3 + 6x^2 + 3x + 3 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$$

if $x=0$

$$\boxed{A=3}$$

Expand and compare powers

$$3x^4 + 2x^3 + 6x^2 + 3x + 3 =$$

$$Ax^4 + 2Ax^2 + A + Bx^4 + Cx^3 + Bx^2 + Cx + Dx^2 + Ex$$

x^4 :

$$3 = A + B \rightarrow \boxed{B=0}$$

x^3 :

$$2 = C \rightarrow \boxed{C=2}$$

x^2 :

$$6 = 2A + B + D$$

$$6 = 2(3) + 0 + D$$

$$6 = 6 + D \rightarrow \boxed{D=0}$$

x° :

$$3 = C + E$$

$$3 = 2 + E \rightarrow E = 1$$

x° :

$$3 = A$$

$$I = \underbrace{\int \frac{3}{x} dx}_{I_1} + \underbrace{\int \frac{2}{x^2+1} dx}_{I_2} - \underbrace{\int \frac{1}{(x^2+1)^2} dx}_{I_3}$$

$$I_1 = 3 \ln|x|$$

$$I_2 = 2 \arctan x$$

$$I_3 = \int \frac{1}{(x^2+1)^2} dx$$

Trig sub: Let $\frac{x}{1} = \tan \theta$

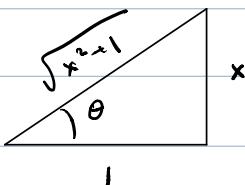
$$dx = \sec^2 \theta d\theta$$

$$= \int \frac{1}{(\tan^2 \theta + 1)^2} \sec^2 \theta d\theta$$

$$= \int \frac{1}{\sec^2 \theta} \sec^2 \theta d\theta$$

$$= \int \frac{1}{\sec^2 \theta} d\theta$$

$$= \int \cos^2 \theta d\theta \quad \text{even cos}$$

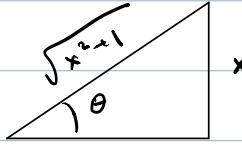


$$= \frac{1}{2} \int (1 + \cos 2\theta) d\theta$$

$$= \frac{1}{2} \int 1 d\theta + \frac{1}{2} \int \cos 2\theta d\theta$$

$$= \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta$$

$$= \frac{1}{2} \arctan x + \frac{1}{4} \cdot 2 \sin \theta \cos \theta$$



$$= \frac{1}{2} \arctan x + \frac{1}{2} \cdot \frac{x}{\sqrt{x^2+1}} \cdot \frac{1}{\sqrt{x^2+1}}$$

$$\sin \theta = \frac{x}{\sqrt{x^2+1}}$$

$$\cos \theta = \frac{1}{\sqrt{x^2+1}}$$

$$\text{So } I = 3 \ln |x| + 2 \arctan x + \frac{1}{2} \arctan x + \frac{1}{2} \cdot \frac{x}{x^2+1} + C$$

$$= 3 \ln |x| + \frac{5}{2} \arctan x + \frac{1}{2} \cdot \frac{x}{x^2+1} + C$$