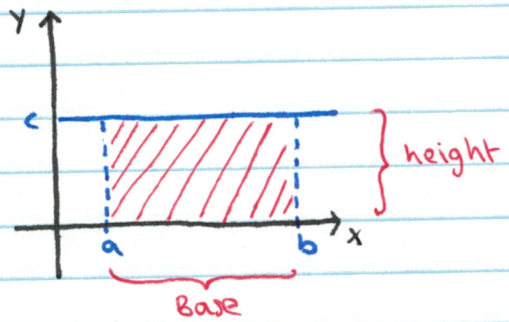
Average Value

consider a constant fcn $f(x) = c$



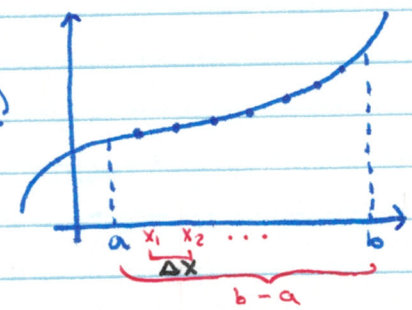
$$\text{Avg}(f) = \begin{pmatrix}\text{Avg}\\ \text{Height}\end{pmatrix} = c$$

$$\text{Area} = (\text{base})(\text{height})$$

$$\boxed{\text{Height} = \frac{\text{Area}}{\text{Base}}} \Rightarrow \boxed{\begin{pmatrix}\text{Avg}\\ \text{Height}\end{pmatrix} = \frac{\int_a^b f(x)\,dx}{b-a}}$$

<u>Approximate using finite pts</u>

$$f_{avg} \approx \frac{f(x_1)+f(x_2)+f(x_3)+\ldots+f(x_n)}{n}$$



$$= \sum_{i=1}^{n} (f(x_i)) \cdot \left(\frac{1}{n}\right)$$

$$= \sum_{i=1}^{n} f(x_i) \cdot \Delta x \cdot \left(\frac{1}{b-a}\right)$$

$$\Delta x = \frac{b-a}{n}$$

$$\boxed{\therefore \frac{1}{n} = \frac{\Delta x}{b-a}}$$

$$= \frac{1}{b-a} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

⊛ To get exact value (not approximation):

$$f_{avg} = \lim_{n\to\infty} \frac{1}{b-a} \cdot \underbrace{\sum_{i=1}^{n} f(x_i)\,\Delta x}$$

$$\boxed{f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

$$\frac{\text{Area}}{\text{Base}}$$

ex: In a city, the temp. (in F°) t hours after 9 am can be modelled by the fcn:

$$T(t) = 50 + 14\sin\left(\frac{\pi t}{12}\right)$$

Find the average temp. from 9am - 9pm.

Let: 9am $\Rightarrow$ $a = 0$
9pm $\Rightarrow$ $b = 12$

$$T_{avg} = \frac{1}{12-0} \cdot \int_a^b T(t)\,dt$$

$$= \frac{1}{12} \int_0^{12} \left(50 + 14\sin\left(\frac{\pi t}{12}\right)\right) dt$$

$u = \frac{\pi t}{12}$

$du = \frac{\pi}{12}\,dt$

$dt = \frac{12}{\pi}\,du$

$t = 0,\ u = 0$

$t = 12,\ u = \pi$

$$= \frac{1}{\pi} \int_0^{\pi} (50 + 14\sin(u))\,du$$

$$= \frac{1}{\pi} \left(50u + 14(-\cos(u))\right)\Big|_0^{\pi}$$

$$= \frac{1}{\pi}\left[50\pi - 14\cos(\pi) - \cancel{50(0)} + 14\cos(0)\right]$$

$$= \frac{1}{\pi}\left[50\pi - 14(-1) + 14(1)\right]$$

$$\approx 59\,°F$$

Mean Value Theorem

Recall: MVT for derivatives:

Let $f(x)$ be

1. continuous on $[a,b]$
2. differentiable on (a,b)

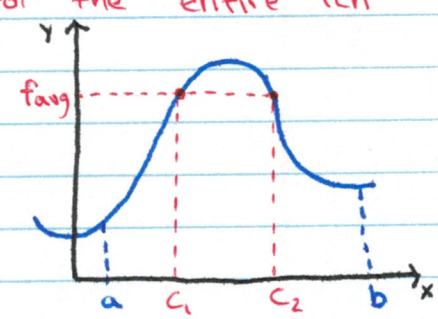
$$\therefore \exists\, c \in (a,b) \text{ s.t } \underbrace{f'(c)}_{\text{slope tangent}} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{slope secant}}$$

There exist

MVT (for integrals)

$$\boxed{\begin{array}{l}\text{If } f(x) \text{ is continuous on } [a,b] \\ \therefore \exists\, c \in (a,b) \text{ s.t } f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx\end{array}}$$

ie. There must be a value "c", where the y value at "c" is equal to the average value for the entire fcn



Assume: $f(x)$ is continuous on $[a,b]$

Define: $F(x) = \int_a^x f(t)\,dt$

$\therefore$ FTC-I : ① $F(x)$ continuous on $[a,b]$

* ② $F(x)$ differentiable on (a,b)

③ $F'(x) = f(x)$

Note: Now $F(x)$ satisfies conditions for MVT (derivatives)

$$\therefore \exists\, c \in (a,b) \text{ s.t } F'(c) = \frac{F(b)-F(a)}{b-a}$$

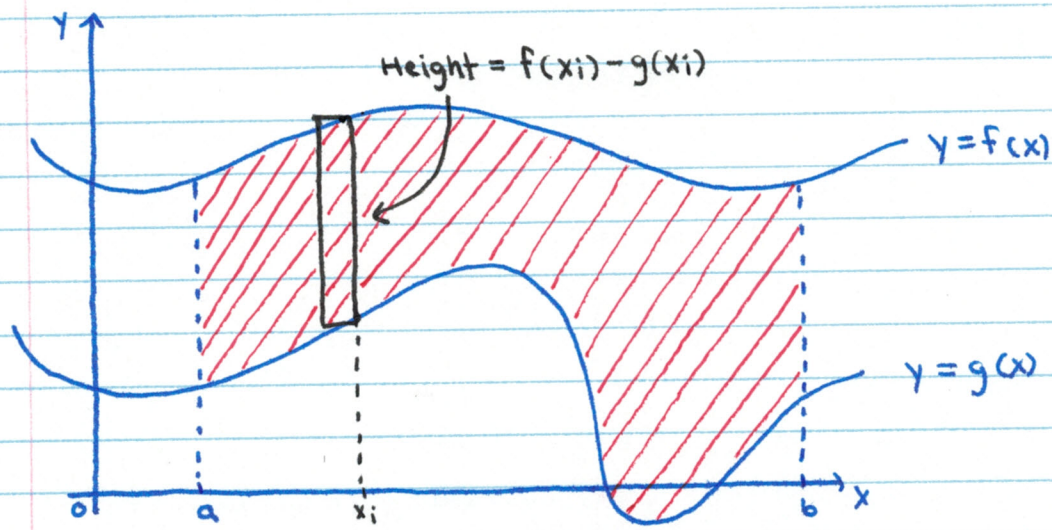
① $F'(c) = f(c)$ by FTC-I

② $F(a) = \int_a^a f(t)\,dt = 0$

③ $F(b) = \int_a^b f(t)\,dt$

$$\therefore f(c) = \frac{1}{b-a} \cdot \int_a^b f(t)\,dt$$

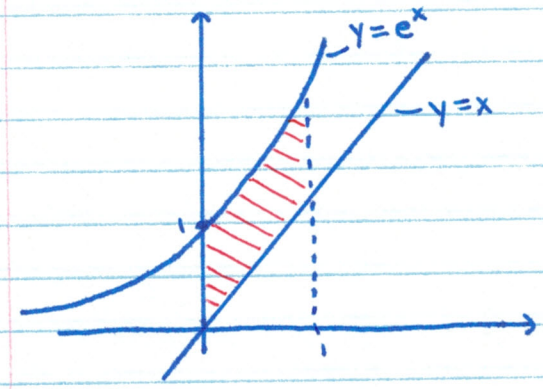
Area Between Curves



<u>Theorem</u>: The area A of the region bounded by the curves $y = f(x)$ and $y = g(x)$, and $x = a$, $x = b$ is:

$$\boxed{\text{Area} = \int_a^b \left(f(x) - g(x)\right) dx}$$ (upper − lower)

ex: Find the area of the region bounded by $y = e^x$, $y = x$ and $x = 0$, $x = 1$



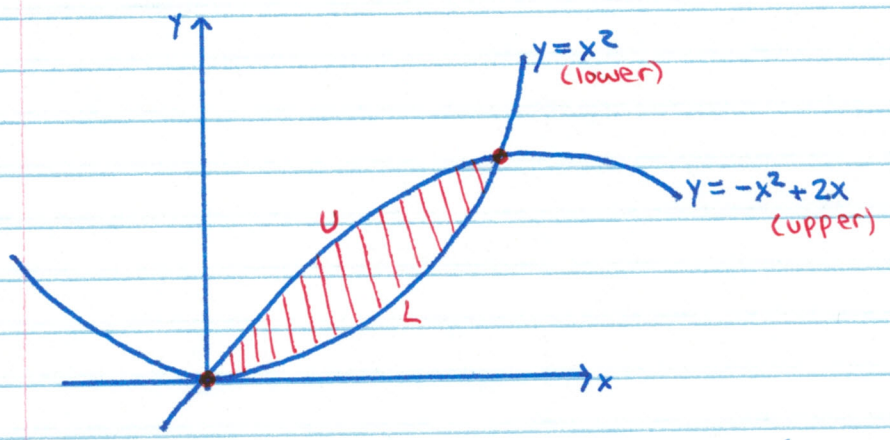
$$\text{Area} = \int_0^1 (e^x - x)\,dx$$

$$= \left(e^x - \frac{x^2}{2}\right)\Big|_0^1$$

$$= \left(e - \tfrac{1}{2}\right) - \left(e^0 - 0\right)$$

$$= e - 3/2$$

ex: ... region enclosed by $y = x^2$ and $y = -x^2 + 2x$



Find bounds of integration (ie. intersection pts)

$$\begin{cases} y = x^2 & ① \\ y = -x^2 + 2x & ② \end{cases}$$

sub ①

$$x^2 = -x^2 + 2x$$

$$0 = 2x - 2x^2$$

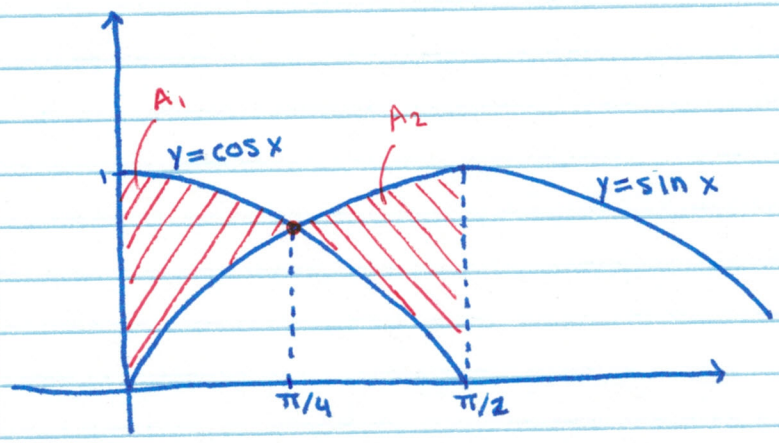
$$0 = 2x(1 - x)$$

$$x = 0 \mid x = 1$$

bounds

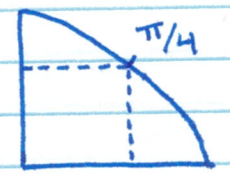
$$\therefore A = \int_0^1 \left((-x^2 + 2x) - (x^2)\right) dx$$

ex: ... the region bounded by $y = \sin x$, $y = \cos x$, $x = 0$, $x = \pi/2$



<u>Intercept</u>

$$\sin x = \cos x$$
$$x = \pi/4$$

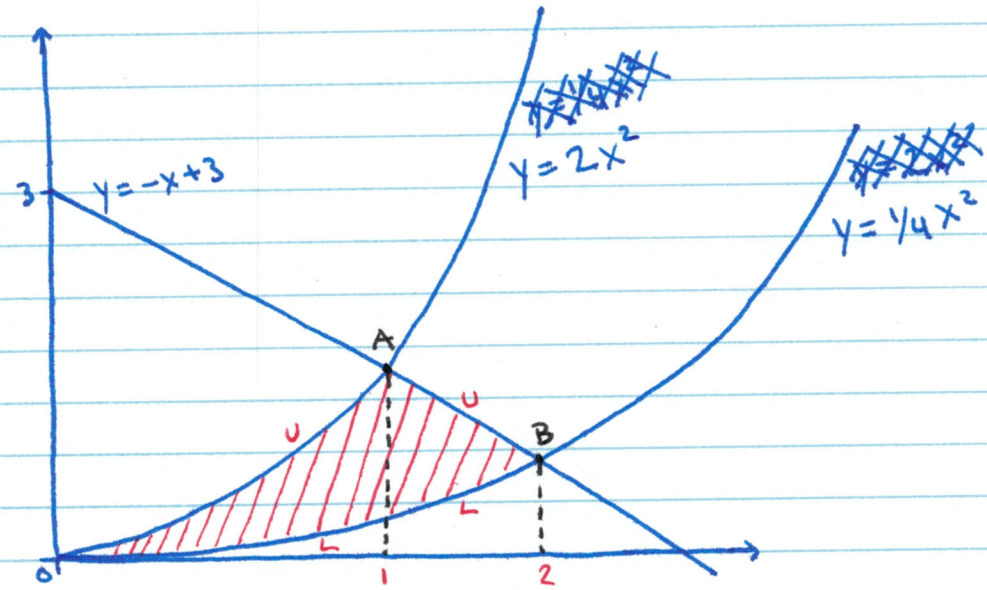


Before $\frac{\pi}{4}$: $y = \cos x$ is upper

After $\frac{\pi}{4}$: $y = \sin x$ is upper

$$\therefore \text{Area} = \underbrace{\int_0^{\pi/4} (\cos x - \sin x)\,dx}_{A_1} + \underbrace{\int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx}_{A_2}$$

ex: ... region bounded by $y = \frac{1}{4}x^2$, $y = 2x^2$ and $\underbrace{x + y = 3}_{y = -x+3}$, $x \geq 0$



intercept A

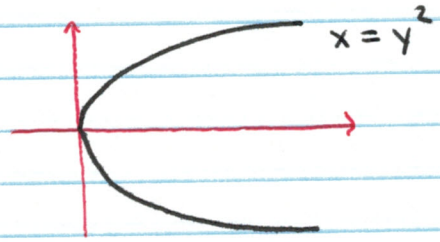
$$\begin{cases} y = -x+3 \\ y = 2x^2 \end{cases} \qquad \boxed{x = 1}$$

intercept B

$$\begin{cases} y = -x+3 \\ y = \frac{1}{4}x^2 \end{cases} \qquad \boxed{x = 2}$$

$$\therefore \text{Area} = \int_0^1 \left(2x^2 - \tfrac{1}{4}x^2\right) dx + \int_1^2 \left(-x+3 - \tfrac{1}{4}x^2\right) dx$$

Think of x as a fcn of y
(ie. $x(y) = \ldots$)

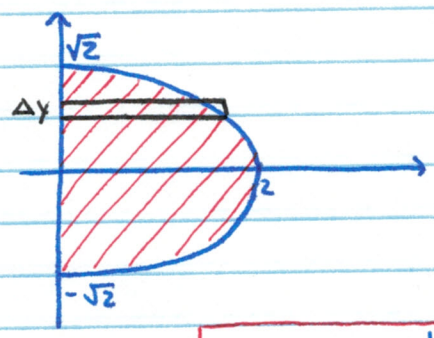


Note: • $x = y^2 + c$
c translation to right (if $c > 0$)

• $x = (y - k)^2$
k translation up (if $k > 0$)

ex: ... region bounded by $\underbrace{y^2 = 2 - x}_{x = -y^2 + 2}$ and the line $x = 0$

→ reflexion
→ +2 right



$$\therefore \text{Area} = \int_{-\sqrt{2}}^{\sqrt{2}} \left(-y^2 + 2 - (0)\right) dy$$

everything with respect to y

$$\boxed{\text{Area} = \int_a^b (\text{right} - \text{left})\, dy}$$

→ a, b are y-values

<u>Compare w.r.t x</u>

$y^2 = 2 - x$

$\boxed{y = \pm\sqrt{2 - x}}$ 2 fcns

$$\therefore A = \int_0^2 \left(\sqrt{2-x} - -\sqrt{2-x}\right) dx$$

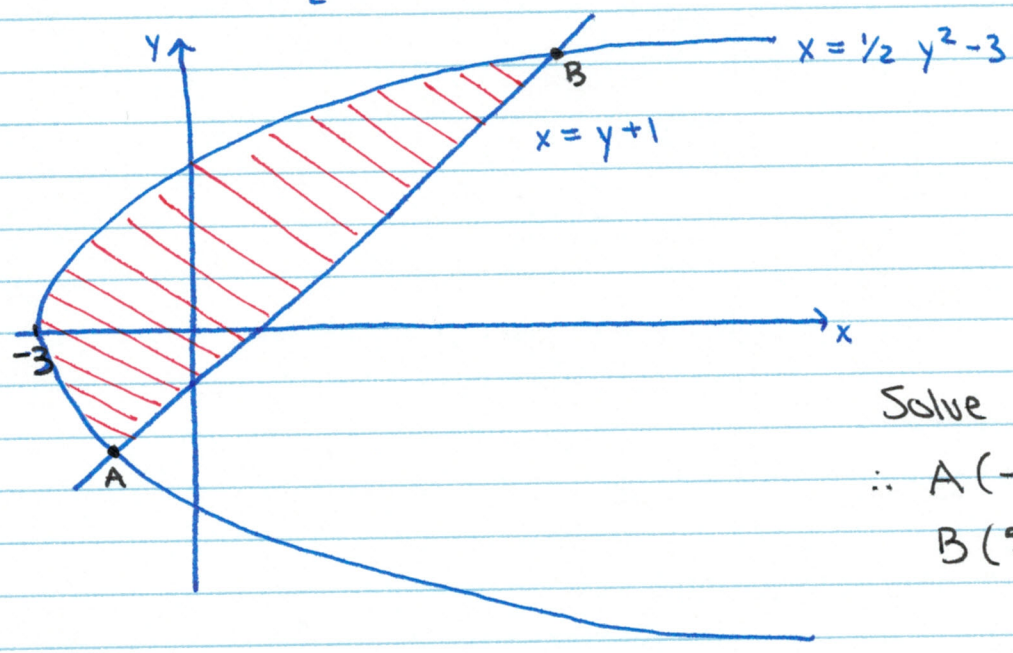
or $$A = 2 \cdot \int_0^2 \left(\sqrt{2-x}\right) dx$$

ex: ... region enclosed by $y = x - 1$
and parbola $y^2 = 2x + 6$

① Put everything w.r.t y

$x = y + 1$

$x = \frac{1}{2} y^2 - 3$ → translation 3 to the left



Solve A, B

∴ $A(-1, -2)$

$B(5, 4)$

∴ limits of integration w.r.t y : $-2 \to 4$

$$\text{Area} = \int_{-2}^{4} \left(y + 1 - \left(\tfrac{1}{2} y^2 - 3 \right) \right) dy$$

<u>Compare w.r.t x</u>

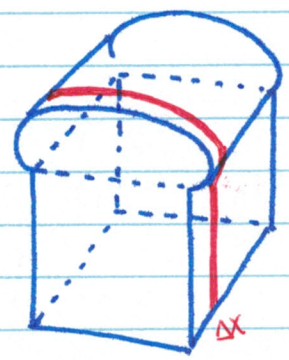
$y = x - 1$ and $y = \pm\sqrt{2x+6}$

~~Find intercept~~ limits of integration ⇒ x

$$\therefore A = \underbrace{\int_{-3}^{-1} \left(\sqrt{2x+6} - -\sqrt{2x+6} \right) dx}_{\text{Before intercept A}} + \underbrace{\int_{-1}^{5} \left(\sqrt{2x+6} - (x-1) \right) dx}_{\text{After intercept A}}$$

Volumes of Revolution

Consider a loaf of bread



$$(\text{Volume of slice}) = \underbrace{(\text{surface area})}_{A(x)} \Delta X$$

$$(\text{Total Volume}) \approx \sum_{i=1}^{n} A(x_i)\,\Delta X$$

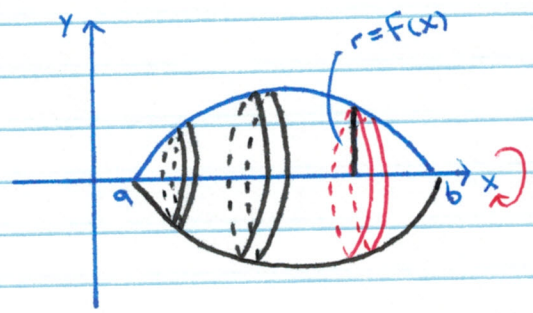
<u>Definition</u>: Let S be a solid that lies between $x=a$ and $x=b$. If the <u>cross section area</u> of S is $A(x)$, then the volume of S is:

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\,\Delta x = \int_a^b A(x)\,dx$$

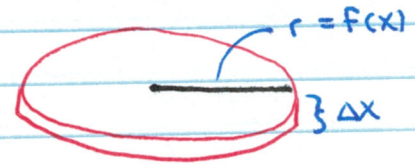
Method 1 (Discs)

Note: cross-sectional cut $\perp$ to the axis of rotation.

consider:



Slice: (disc)



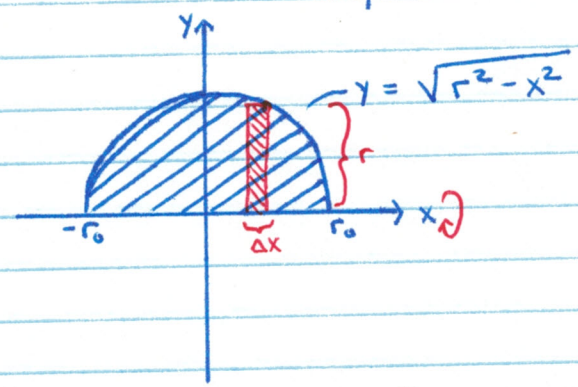
$$A(r) = \pi r^2$$
$$A(x) = \pi (f(x))^2$$

$$\therefore \boxed{V = \pi \int_a^b (f(x))^2 \, dx}$$

ex: Show that the volume of a sphere of radius r_0 is $V = \frac{4}{3}\pi r_0^3$

circle : $x^2 + y^2 = r^2$

$\therefore \; y = \pm\sqrt{r^2 - x^2}$



$\Rightarrow$ rotate about the x-axis

$\rightarrow$ gives you a sphere

$$V = \pi \int_a^b (f(x))^2 \, dx$$

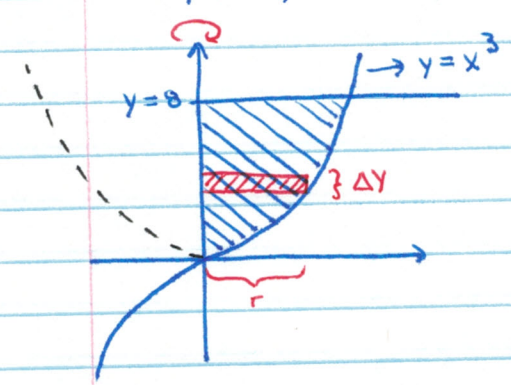
$$= \pi \int_{-r_0}^{r_0} \left(\sqrt{r_0^2 - x^2}\right)^2 dx$$

$$= \pi \left(r_0^2 \cdot x - \frac{x^3}{3}\right)\Big|_{-r_0}^{r_0}$$

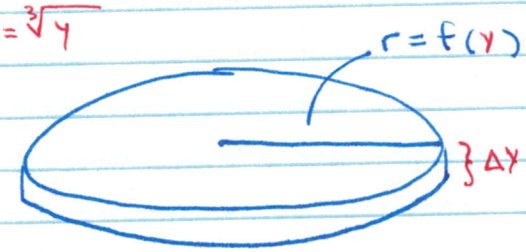
$$= \pi\left[r_0^3 - \frac{r_0^3}{3} - \left(-r_0^3 + \frac{r_0^3}{3}\right)\right]$$

$$= 4/3 \, \pi r_0^3$$

ex: Find the vol. of a solid object bounded by $y = x^3$, $y = 8$, $x = 0$, rotated about the y-axis



$\Rightarrow x = \sqrt[3]{y}$



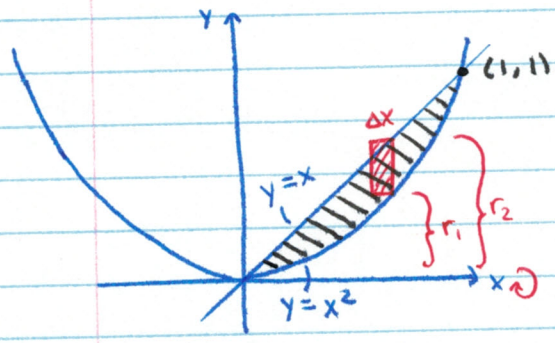
$$\therefore \; V = \pi \int_0^8 (f(y))^2 \, dy$$

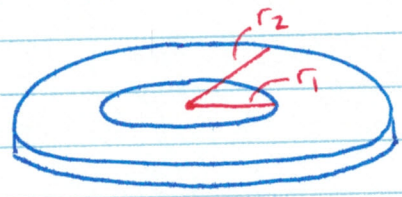
$$= \pi \int_0^8 \left(\sqrt[3]{y}\right)^2 dy$$

* isolate $x \Rightarrow f(y)$
* limits of integration w.r.t y

Method 2 (Washers)

consider: the region enclosed by $y=x$, $y=x^2$ rotated about the x-axis.





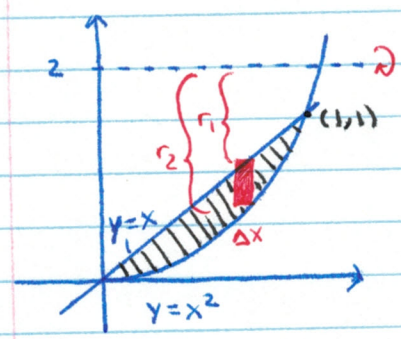
$$\text{Area} = \pi r_2^2 - \pi r_1^2$$
$$= \pi\left(r_2^2 - r_1^2\right)$$

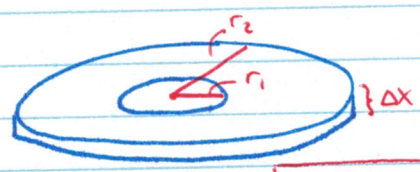
$$\boxed{\begin{aligned} r_1 &= x^2 \\ r_2 &= x \end{aligned}}$$

$$\boxed{V = \int_a^b \pi\left(r_2^2 - r_1^2\right) dx}$$

$$V = \int_a^b A(x)\, dx$$
$$= \pi \int_0^1 \left((x)^2 - (x^2)^2\right) dx$$

ex: same region rotated about $y=2$





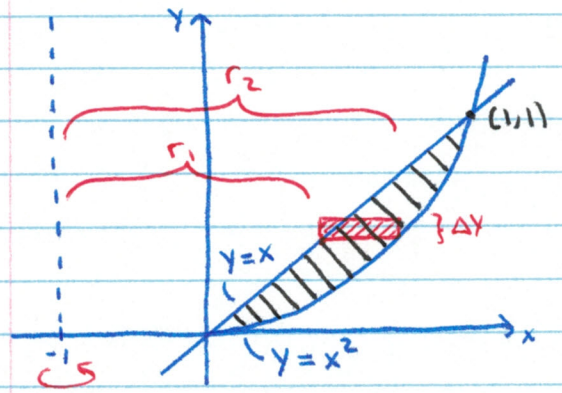
$r = 2 - y_{value}$

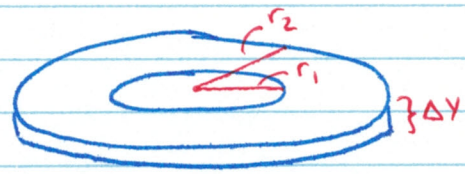
$$\boxed{\begin{aligned} r_1 &= 2 - x \\ r_2 &= 2 - x^2 \end{aligned}}$$

$y = x$
$y = x^2$

$$\therefore V = \pi \int_0^1 \left((2-x^2)^2 - (2-x)^2\right) dx$$

ex: ... same region, rotated about $x = -1$





$r = 1 + x_{value}$

$$\boxed{\begin{aligned} r_1 &= 1 + y \\ r_2 &= 1 + \sqrt{y} \end{aligned}}$$

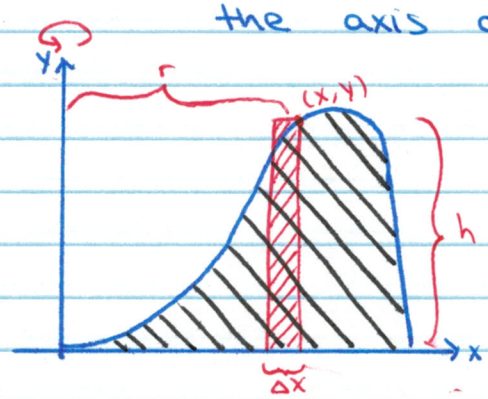
$y = x$
$y = x^2$

limits w.r.t y

$$V = \pi \int_0^1 \left((1+\sqrt{y})^2 - (1+y)^2 \right) dy$$

Method 3 (Shells)

Note: Cross section is NOT $\perp$ to the axis of rotation.

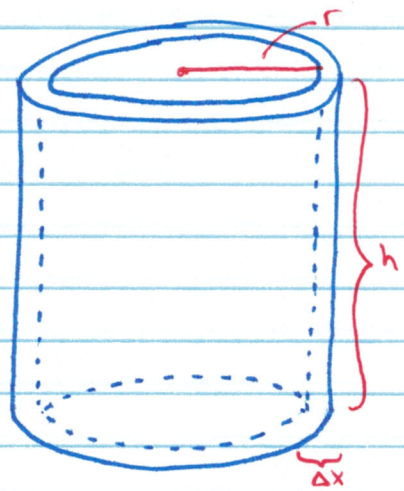


everything w.r.t x

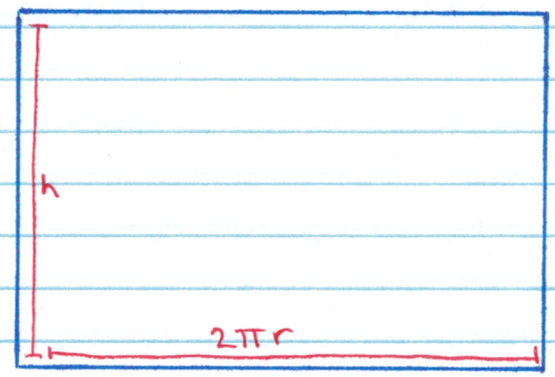
$r = x_{value}$

$h = y_{value}$

$\boxed{\begin{array}{l} r = x \\ h = f(x) \end{array}}$



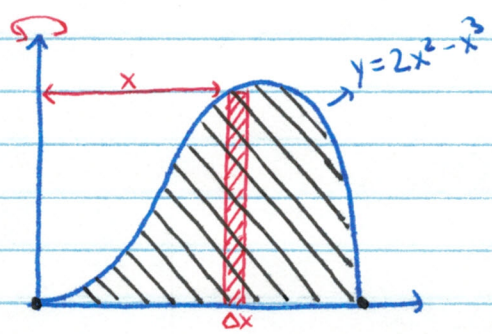
cut $\Rightarrow$



$A(x) = 2\pi r . h$

$$\boxed{V = \int_a^b 2\pi r h . dx}$$

ex: rotate about the y-axis, region bounded by $y = 2x^2 - x^3$ and the x-axis



intercept: $0 = 2x^2 - x^3$

$= x^2(2 - x)$

$x = 0 \mid x = 2$

$r = x_{value} = x$

$h = y_{value} = 2x^2 - x^3$ $\quad y = 2x^2 - x^3$

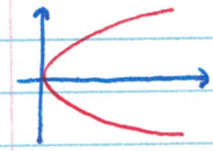
$$\therefore V = \int_0^2 2\pi . (x)(2x^2 - x^3)\, dx$$

ex: ... $x = 1 + (y-2)^2$ and ~~x~~ $x = 2$

around $y = 1$

Note: Values already given as a function of y $\therefore$ probably easy horizontal cut (dy)

$x = y^2 \Rightarrow$ right 1, up 2 $\Rightarrow x = (y-2)^2 + 1$



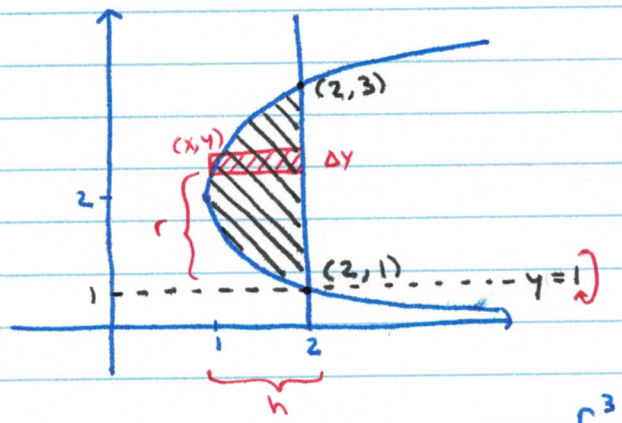
Intercepts: limits of integ.

$2 = (y-2)^2 + 1$

$1 = (y-2)^2$

$y = 2 \pm 1$

$\boxed{\begin{matrix} y = 1 \\ y = 3 \end{matrix}}$



(Shells)

w.r.t y

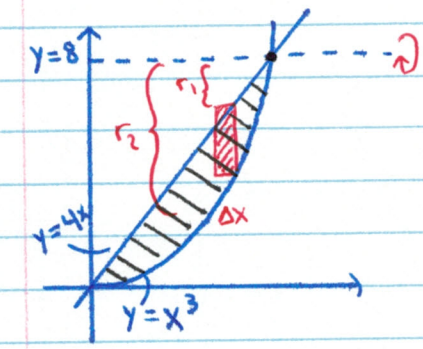
$r = y_{value} - 1$ $\quad$ $r = y - 1$

$h = 2 - x_{value}$ $\quad$ $h = 2 - [(y-2)^2 + 1]$

$h = 1 - (y-2)^2$

$$\therefore V = \int_1^3 2\pi (y-1)(1-(y-2)^2)\, dy$$

ex: ... $y = x^3$, $x \geq 0$, $y = 4x$ around $y = 8$

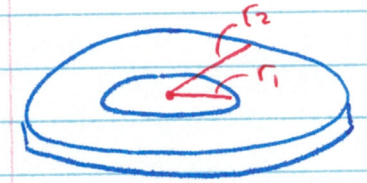


Intercept: $x^3 = 4x$

$\boxed{x = 2} \Rightarrow \boxed{y = 8}$

(washers)

$\boxed{\begin{matrix} r_1 = 8 - 4x \\ r_2 = 8 - x^3 \end{matrix}}$ $\quad$ $r = 8 - y_{value}$



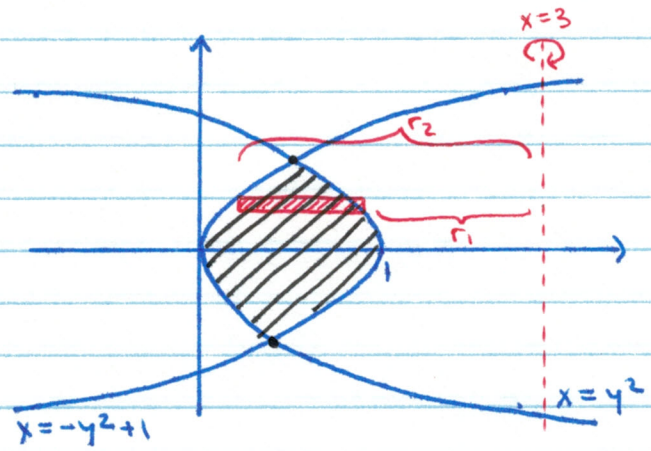
$$V = \pi \int_0^2 \left((8 - x^3)^2 - (8 - 4x)^2 \right) dx$$

ex: ... $x = y^2$, $x = -y^2 + 1$ easy (dy)

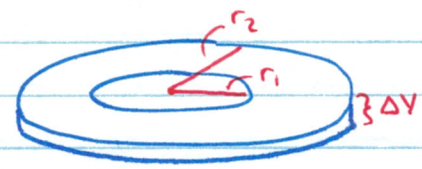
rotated about $x = 3$

$x = -y^2 + 1$

- reflexion
- 1 right



(washer)



Intercept: $y^2 = -y^2 + 1$

$y^2 = 1/2$

$\boxed{y = \pm\sqrt{1/2}}$

$r = 3 - X_{value}$

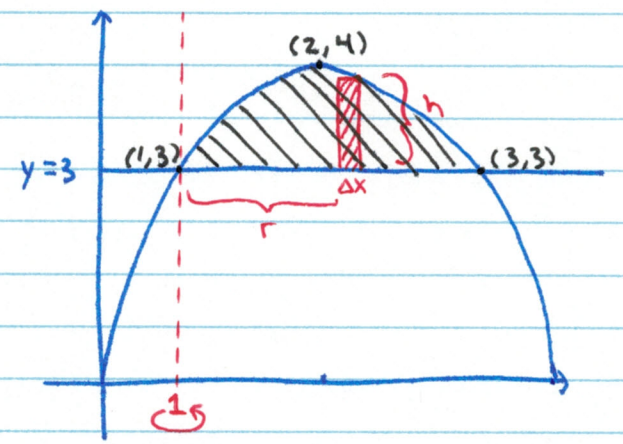
$$\boxed{\begin{aligned} r_1 &= 3 - (-y^2+1) \\ r_2 &= 3 - (y^2) \end{aligned}}$$

$$\therefore \; V = \pi \int_{-\sqrt{1/2}}^{\sqrt{1/2}} \left((3-y^2)^2 - (2+y^2)^2 \right) dy$$

ex: ... $y = -x^2 + 4x$, $y = 3$
rotate about $x = 1$

$$y = -x^2 + 4x$$
$$= -(\underbrace{x^2 - 4x + 4} - 4)$$
$$= -(x-2)^2 + 4 \qquad \therefore V(2,4)$$

Intercept: $3 = -x^2 + 4x$
$x^2 - 4x + 3 = 0$ $\qquad$ $\left.\begin{matrix} S = 4 \\ P = 3 \end{matrix}\right\} 3, 1$



Note: Washers would be too difficult (dy)
→ switch to w.r.t y
→ have 2 fcns ±

Note: Given w.r.t x
∴ use Δx (dx)

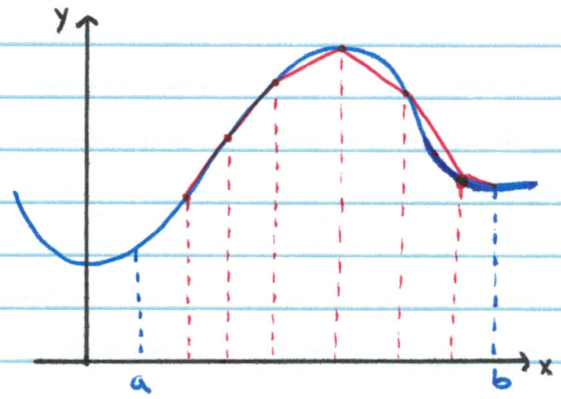
(Shells)

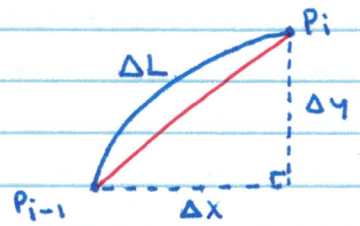
w.r.t x $\begin{cases} r = X_{value} - 1 \\ h = Y_{value} - 3 \end{cases}$

$$\boxed{\begin{matrix} r = x - 1 \\ h = (-x^2 + 4x) - 3 \end{matrix}}$$

$$V = 2\pi \int_1^3 (x-1)(-x^2 + 4x - 3)\, dx$$

Arc Length





Let L be actual distance

$$\Delta L \approx |P_i - P_{i-1}|$$

$$L \approx \sum_{i=1}^{n} |P_i - P_{i-1}|$$

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_i - P_{i-1}|$$

Note: $|P_i - P_{i-1}| = \sqrt{\Delta x^2 + \Delta y^2}$

factor out Δx to resemble Riemann

$$= \sqrt{(\Delta x)^2 \left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right)}$$

$$= \Delta x \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

Note: As $n \to \infty$

$$\frac{\Delta y}{\Delta x} \to \frac{dy}{dx}$$

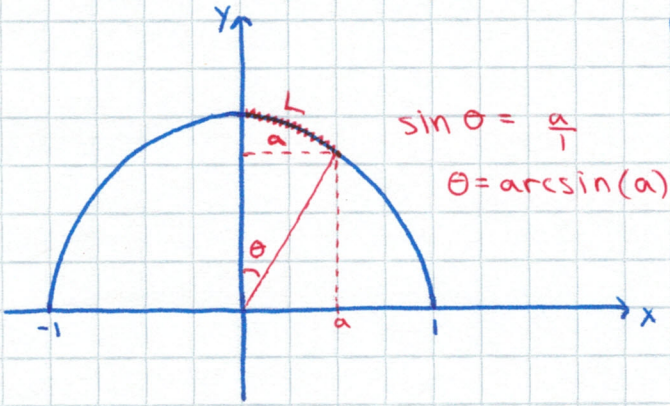
$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ \Delta x$$

$$\therefore L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ \Delta x$$

$$\boxed{L = \int_a^b \sqrt{1 + (f'(x))^2}\ dx}$$

condition:
if $f'(x)$ is continuous on $[a,b]$

ex: Find arclength of $y = \sqrt{1-x^2}$ from $x=0$, $x=a$



$\boxed{r=1}$

$\sin\theta = \frac{a}{1}$

$\theta = \arcsin(a)$

$P = 2\pi r$

$L = \theta \cdot r$ $\quad \boxed{r=1}$

$\boxed{L = \theta} \Rightarrow L = \arcsin(a)$

Use arclength formula: $L = \int_a^b \sqrt{1+(f'(x))^2}\, dx$

$f(x) = \sqrt{1-x^2}$

① $f'(x) = (-\cancel{2}x) \cdot \frac{1}{\cancel{2}\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$

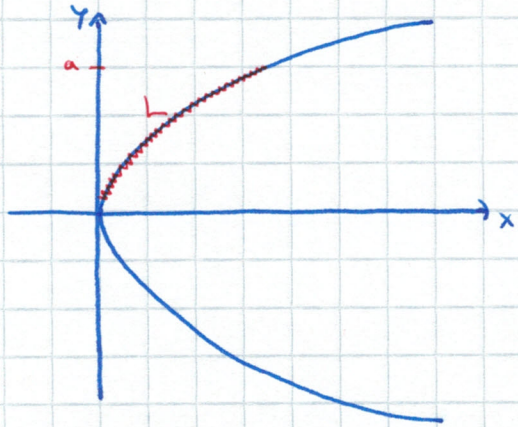
② $1+(f'(x))^2 = 1 + \left(\frac{-x}{\sqrt{1-x^2}}\right)^2 = 1 + \frac{x^2}{1-x^2} = \frac{1-\cancel{x^2}+\cancel{x^2}}{1-x^2}$

③ $L = \int_0^a \sqrt{\frac{1}{1-x^2}}\, dx$

$= \int_0^a \underbrace{\frac{1}{\sqrt{1-x^2}}}_{\text{arcsin}}\, dx$

$= \sin^{-1}(x)\Big|_0^a = \sin^{-1}(a) - \cancel{\sin^{-1}(0)}^{\,0} = \boxed{\sin^{-1}(a)}$

ex: arclength of $x = y^2$ from $y=0$ to $y=a$



$$L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$f(y) = y^2$$

$$f'(y) = 2y$$

$$\therefore L = \int_0^a \sqrt{1 + (2y)^2}\, dy$$

$$L = \int_0^a \sqrt{1 + (2y)^2}\, dy$$

$$= \int_0^{\tan^{-1}(a)} \sqrt{1 + \tan^2\theta} \cdot \frac{\sec^2\theta}{2}\, d\theta$$

$$= \frac{1}{2}\int_0^{\tan^{-1}(a)} \sec^3\theta\, d\theta$$

Trig Sub

You can do this

$$\begin{cases} 2y = (1)\tan\theta \\ 2\,dy = \sec^2\theta\, d\theta \end{cases}$$

$$dy = \frac{\sec^2\theta}{2}\, d\theta$$

if $y = 0$, $\theta = \tan^{-1}(0) = 0$

$y = a$, $\theta = \tan^{-1}(a)$ #

$\theta = \arctan(2y)$

know how to solve (Boomerang)

$$\begin{bmatrix} u = \sec\theta & dv = \sec^2\theta\, d\theta \\ du = \sec\theta\tan\theta\, d\theta & v = \tan\theta \end{bmatrix}$$

$$= \frac{1}{2}\left[\sec\theta\tan\theta \Big|_0^{\tan^{-1}(a)} - \int_0^{\tan^{-1}(a)} \tan^2\theta\sec\theta\, d\theta\right]$$

$$I = \tfrac{1}{2}\left[\sec\theta\tan\theta \Big|_0^{\tan^{-1}(a)} - \left(\underbrace{\int_0^{\tan^{-1}(a)} \sec^3\theta\, d\theta}_{I} - \int_0^{\tan^{-1}(a)} \sec\theta\, d\theta\right)\right]$$

$$I = \tfrac{1}{2}\left(f(x) - I\right)$$

$$3I = f(x)$$

$$\frac{f(x)}{3} = I$$

$$\therefore I = \tfrac{1}{3}\left[\sec\theta\tan\theta \Big|_0^{\tan^{-1}(a)} + \int_0^{\tan^{-1}(a)} \sec\theta\, d\theta\right]$$

ex: arclength of $y = 5x^{3/2}$, $0 \le x \le 5$

$$L = \int_0^5 \sqrt{1 + (f'(x))^2}\, dx$$

$$= \int_0^5 \sqrt{1 + \frac{225}{4}x}\, dx$$

$$= \int_1^{282.25} \frac{4}{225}\sqrt{u}\, du$$

$$= \frac{4}{225}\int_1^{282.25} \sqrt{u}\, du$$

① $f'(x) = 5 \cdot 3/2 \cdot x^{1/2}$

$= \frac{15}{2}\sqrt{x}$

② $1 + (f'(x))^2 = 1 + \left(\frac{15}{2}\sqrt{x}\right)^2$

$= 1 + \frac{225}{4}x$

$u = 1 + \frac{225}{4}x$

$du = dx \cdot \frac{225}{4}$

$\frac{4}{225}du = dx$

---

if $x = 0$, $u = 1$

$x = 5$, $u = 282.25$

ex: $y = \sqrt{x - x^2} + \sin^{-1}(\sqrt{x})$; $0 \le x \le \pi/2$

① $f'(x) = (1-2x) \cdot \frac{1}{2\sqrt{x-x^2}} + \frac{1}{2\sqrt{x}} \cdot \frac{1}{\sqrt{1-(\sqrt{x})^2}}$

$$= \frac{1-2x}{2\sqrt{x-x^2}} + \frac{1}{2\sqrt{x(1-x)}}$$

$$= \frac{1-2x+1}{2\sqrt{x-x^2}} = \frac{2-2x}{2\sqrt{x-x^2}} = \frac{\cancel{2}(1-x)}{\cancel{2}\sqrt{x-x^2}}$$

② $1 + (f'(x))^2 = 1 + \left(\frac{1-x}{\sqrt{x-x^2}}\right)^2$

$$= \frac{x-x^2}{x-x^2} + \frac{(1-x)^2}{x-x^2}$$

$$= \frac{x - \cancel{x^2} + 1 - 2x + \cancel{x^2}}{x-x^2}$$

$$= \frac{(1-x)}{x(1-x)} = 1/x$$

$$\therefore L = \int_0^{\pi/2} \sqrt{1/x}\, dx$$

Improper Integral

→ When an integral does not satisfy the condition of FTC-II :
f must be continuous on $[a,b]$

Type I (Infinite interval)

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^{t} f(x)\,dx$$

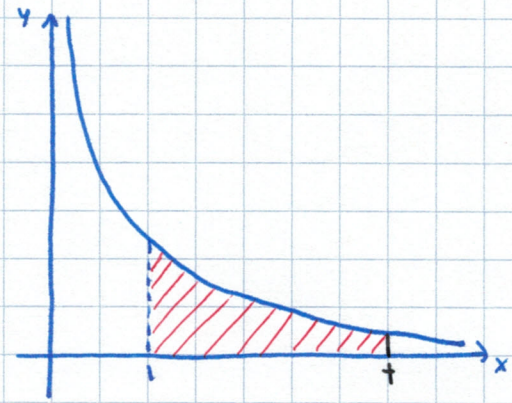
The integral → "converges" if limit exists
→ "diverges" if limit DNE

Note: $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{+\infty} f(x)\,dx$$

$$= \lim_{t\to-\infty} \int_t^a f(x)\,dx + \lim_{s\to\infty} \int_a^s f(x)\,dx$$

* DO NOT : $$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_{-t}^{t} f(x)\,dx$$

ex: consider the area under $y = \frac{1}{x^2}$ above the x-axis from $x = 1 \Rightarrow x = +\infty$



$$\int_1^{\infty} \frac{1}{x^2}\,dx =$$

$$= \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2}\,dx$$

$$= \lim_{t\to\infty} \left(-\frac{1}{x}\right)\Big|_1^t$$

$$= \lim_{t\to\infty} \left(-\cancel{\frac{1}{t}}^{\,0} - -\frac{1}{1}\right) = \boxed{1}$$

ex: $\int_0^{\infty} e^{-kx}\,dx$ $\quad k>0$

$u = -kx$

$du = -k\,dx$

$-\frac{1}{k}\,du = dx$

$x=0,\ u=0$

$x=t,\ u=-kt$

$$= \lim_{t\to\infty} \int_0^{t} e^{-kx}\,dx$$

$$= \lim_{t\to\infty} \int_0^{-kt} -\frac{1}{k} e^{u}\,du$$

$$= \lim_{t\to\infty} -\frac{1}{k}\left(e^{u}\right)\Big|_0^{-kt}$$

$$= \lim_{t\to\infty} -\frac{1}{k}\left(e^{-kt} - e^{0}\right)$$

$$= \lim_{t\to\infty} -\frac{1}{k}\left(e^{-\infty} - 1\right)$$

$$= -\frac{1}{k}\left(0 - 1\right)$$

$$= \boxed{\frac{1}{k}}$$

Type II (discontinuity interval)

(i) If f is discontinuous at $x = b$, then:

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

(ii) If f is discontinuous at $x = a$, then:

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

Consider:

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \left(-\frac{1}{x}\right)\Big|_{-1}^{1} \qquad \text{FTC-II}$$

$$= -\frac{1}{1} - \left(-\frac{1}{-1}\right)$$

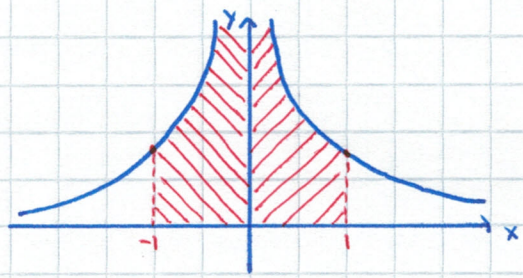
$$= -2 \quad \otimes$$

$\Rightarrow$ FTC-II cannot be applied bc there is a discontinuity

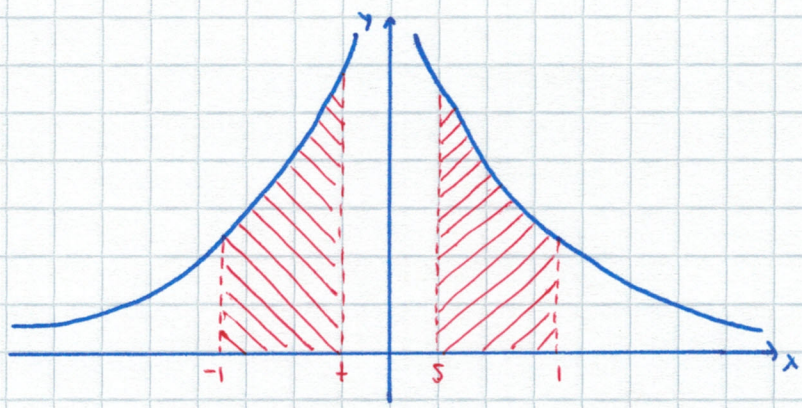
$\frac{1}{x^2}$ Restriction:

$x^2 \neq 0$

$x \neq 0$



* Problem: vertical asymptote at $x=0$
$\rightarrow$ discontinuous



$$\therefore \int_{-1}^{1} \frac{1}{x^2}\,dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{x^2}\,dx + \lim_{s\to 0^+} \int_{s}^{1} \frac{1}{x^2}\,dx$$

$$= \lim_{t\to 0^-} \left(-\frac{1}{x}\right)\Big|_{-1}^{t} + \lim_{s\to 0^+} \left(-\frac{1}{x}\right)\Big|_{s}^{1}$$

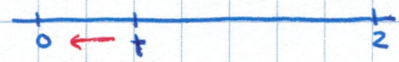
$$= \lim_{t\to 0^-} \left(-\underbrace{\frac{1}{t}}_{-\infty} - -\frac{1}{-1}\right) + \lim_{s\to 0^+} \left(-\frac{1}{1} - -\underbrace{\frac{1}{s}}_{+\infty}\right)$$

$$= -(-\infty) + (+\infty) = \infty \qquad \text{DNE} \therefore \text{divergent}$$

ex: $\displaystyle\int_0^2 x^2 \ln x \, dx$

Restrictions:

$\ln x \Rightarrow x > 0$



$$= \lim_{t \to 0^+} \int_t^2 x^2 \ln x \, dx$$

$$\left[\begin{array}{ll} u = \ln x & dv = x^2\,dx \\ du = \frac{1}{x}\,dx & v = \frac{x^3}{3} \end{array}\right.$$

$$= \lim_{t \to 0^+} \left[\frac{x^3 \ln x}{3}\Bigg|_t^2 - \int_t^2 \frac{x^2}{3}\,dx \right]$$

$$= \lim_{t \to 0^+} \left[\tfrac{1}{3}\left(8 \ln 2 - t^3 \ln(t)\right) - \frac{1}{3}\left(\frac{x^3}{3}\right)\Bigg|_t^2 \right]$$

$$= \lim_{t \to 0^+} \left[\tfrac{1}{3}\left(8 \ln 2 - \underset{0}{t^3}\,\underset{-\infty}{\ln(t)}\right) - \tfrac{1}{9}\left(8 - \underset{0}{t^3}\right) \right]$$

$\circledast$

$$\circledast \lim_{t \to 0^+} t^3 \ln(t) = \lim_{t \to 0^+} \frac{\ln(t)}{t^{-3}} \overset{\frac{\infty}{\infty}}{=} \overset{\text{H}}{=} \lim_{t \to 0^+} \frac{\frac{1}{t}}{-3\,t^{-4}}$$

$$= \lim_{t \to 0^+} \frac{t^3}{-3}$$

$$= \boxed{0}$$

$$= \tfrac{1}{3}\left(8 \ln 2 - 0\right) - \tfrac{1}{9}(8)$$

$\therefore$ convergent.

Seperable Differential Equations

<u>Def'n</u>: A "differential equation" is an equation involving a fcn and some of its derivatives.

ex': $y' = 2x-1$

$\frac{dy}{dx} = 2x-1$

$\int dy = \int (2x-1)\,dx$

$\boxed{y = x^2 - x + C}$

$P(1,0)$

$0 = 1^2 - 1 + C$

$\boxed{C = 0}$

$\therefore\ y = x^2 - x$

ex: show that the DE: $y = ae^{-x} + be^{-x}$ satisfies

$y'' = -2y' - y$

$y'' + 2y' + y = 0$

$y' = -a.e^{-x} - be^{-x}$

$y'' = ae^{-x} + be^{-x}$

$\therefore\ (ae^{-x} + be^{-x}) + 2(-ae^{-x} - be^{-x}) + (ae^{-x} + be^{-x}) = 0$

Def'n: A "seperable differential equation" is a DE in which:

$\frac{dy}{dx}$ can be factored as

a function of x times a function of y.

$\left(\text{ie. } \frac{dy}{dx} = f(x) \cdot g(y)\right)$

if $\frac{dy}{dx} = (f(x)) \cdot (g(y))$

$$\frac{dy}{g(y)} = f(x)\,dx$$

integrate → $\int \frac{dy}{g(y)} = \int f(x)\,dx$

* The problem of solving DE that satisfy the initial condition:

$$y(\underset{\substack{\uparrow \\ \text{initial time}}}{t_0}) = \underset{\substack{\uparrow \\ \text{initial value}}}{y_0}$$

(ie. the curve $y(t)$ passes through the point $P(t_0, y_0)$)

is called an "initial value problem".

Note:
→ Find C
→ Is fcn + or − ?
(you have a y value so you know if it is + or −)

ex: (a) Solve the DE

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

$$\frac{dy}{dx} = (x^2)\cdot\left(\frac{1}{y^2}\right)$$

$$\int y^2\,dy = \int x^2\,dx$$

$$\frac{y^3}{3} = \frac{x^3}{3} + C$$

This C keeps its value

$$y^3 = x^3 + 3C$$

$$y = \sqrt[3]{x^3 + 3C}$$

(b) Find the sol'n that satisfies the initial condition: $y(0) = 2$ $\therefore$ $P(0,2)$

use: $y^3 = x^3 + 3C$

sub P

$$(2)^3 = (0)^3 + 3C$$

$$\boxed{8 = 3C}$$

$$\boxed{\therefore\ y = \sqrt[3]{x^3 + 8}}$$

ex: Solve DE

$$\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}$$

$$\int (2y + \cos y)\,dy = \int 6x^2\,dx$$

$$y^2 + \sin y = 6\,\frac{x^3}{3} + C$$

$$\boxed{y^2 + \sin y = 2x^3 + C}$$

implicit equation

ex: Solve the DE:

(i) $y' = x^2 y$

$$\frac{dy}{dx} = x^2 y$$

$$\int \frac{1}{y} \cdot dy = \int x^2 \, dx$$

$$\ln|y| = \frac{x^3}{3} + C$$

isolate y

$$e^{(x^3/3 + C)} = |y|$$

$$\boxed{y = \pm e^{(x^3/3 + C)}}$$

(ii) $\frac{dL}{dt} = k L^2 \ln(t)$

$L(1) = -1$

$\therefore P(1, -1)$

$$\int \frac{1}{L^2} \, dL = \int k \ln(t) \, dt$$

$$-L^{-1} = k \cdot (t\ln(t) - t) + C$$

$$\therefore L = -\frac{1}{k(t\ln(t) - t) + C}$$

use $-\frac{1}{L} = k \cdot t(\ln(t) - 1) + C$

sub $P(1, -1)$

$$\frac{-1}{-1} = k \cdot (1)(\cancel{\ln(1)}^{0} - 1) + C$$

$$\cancel{1 = k + C}$$

$$\boxed{1 + k = C} \qquad \therefore L = -\frac{1}{k(t\ln(t) - t) + (1+k)}$$

ex: Solve the DE

(i) $(y^2 + xy^2)\, y' = 1$

$$y^2(1+x)\frac{dy}{dx} = 1$$

$$\int y^2\, dy = \int \frac{1}{1+x}\, dx$$

$$\frac{y^3}{3} = \ln|1+x| + C$$

$$\therefore\ y = \sqrt[3]{3\ln|1+x| + 3C}$$

(ii) $\frac{dp}{dt} = t^2 p - p + t^2 - 1$

$$\frac{dp}{dt} = p(t^2-1) + (t^2-1)$$

$$\frac{dp}{dt} = (t^2-1)(p+1)$$

$$\int \frac{1}{p+1}\, dp = \int (t^2-1)\, dt$$

$$\ln|p+1| = \frac{t^3}{3} - t + C$$

$$|p+1| = e^{(t^3/3 - t + C)}$$

$$\therefore\ p = \pm e^{(t^3/3 - t + C)} - 1$$

(iii) $\frac{dy}{dx} + e^{y+x} = 0$

$$\frac{dy}{dx} = -e^{y+x}$$

$$\frac{dy}{dx} = -e^{y} \cdot e^{x}$$

$$\int e^{-y}\,dy = -\int e^{x}\,dx$$

$$-e^{-y} = -e^{x} + C$$

$$e^{-y} = e^{x} - C$$

$$-y = \ln(e^{x} - C)$$

$$y = -\ln(e^{x} - C)$$

(iv) $\frac{dy}{dx} = \frac{\ln x}{xy}$ $\qquad y(1) = 2 \Rightarrow P(1,2)$

$$\int y\,dy = \int \frac{\ln x}{x}\,dx$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\frac{y^2}{2} = \int u\,du$$

$$\frac{y^2}{2} = \frac{u^2}{2} + C$$

$$y^2 = (\ln x)^2 + 2C$$

$$y = \pm\sqrt{(\ln x)^2 + 2C}$$

C-value

use: $y^2 = (\ln x)^2 + 2C$

$$(2)^2 = (\cancel{\ln(1)}^{\,0})^2 + 2C$$

$$\boxed{4 = 2C}$$

$\therefore\ y = \pm\sqrt{(\ln(x))^2 + 2C}$

pick pos. value bc P(1,2) ∴ "y" positive

$\therefore\ y = \sqrt{(\ln x)^2 + 4}$

ex. Solve the DE:

(i) $\frac{du}{dt} = \frac{2t + \sec^2 t}{2u}$ $\qquad u(0) = -5$

$$\int 2u\,du = \int (2t + \sec^2 t)\,dt$$

$$u^2 = t^2 + \tan(t) + C$$

$$u = \pm\sqrt{t^2 + \tan(t) + C}$$ $\Rightarrow$ pick neg. value bc initial condition

use: $u^2 = t^2 + \tan(t) + C$

$$(-5)^2 = (0)^2 + \tan(0) + C$$

$$25 = 0 + 0 + C$$

$$\boxed{25 = C}$$

$$\therefore\ u = -\sqrt{t^2 + \tan(t) + 25}$$

(ii) $\frac{dy}{dx} = \sqrt{xy}$ $\qquad y(1) = 2$

$$\int y^{-1/2}\,dy = \int x^{1/2}\,dx$$

$$y^{1/2}\cdot\left(\frac{2}{1}\right) = x^{3/2}\cdot\left(\frac{2}{3}\right) + C$$

$$\sqrt{y} = \frac{1}{3}x^{3/2} + \frac{C}{2}$$

$$y = \left(\tfrac{1}{3}x^{3/2} + \frac{C}{2}\right)^2$$

C-value:

use: $\sqrt{y} = \frac{1}{3}x^{3/2} + \frac{C}{2}$

$$\sqrt{2} = \tfrac{1}{3}\cdot(1)^{3/2} + \frac{C}{2}$$

$$\boxed{\sqrt{2} - \tfrac{1}{3} = \frac{C}{2}}$$

$$\therefore\ y = \left(\tfrac{1}{3}x^{3/2} + \sqrt{2} - \tfrac{1}{3}\right)^2$$

ex. Solve the DE:

$$y' \tan x = a + y \qquad y\left(\frac{\pi}{3}\right) = a$$

$$\frac{dy}{dx} \tan x = a + y \qquad x \in (0, \pi/2)$$

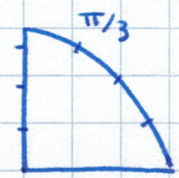
$$\int \frac{1}{a+y}\, dy = \int \frac{1}{\tan x}\, dx$$

$$\ln|a+y| = \int \frac{\cos x}{\sin x}\, dx \qquad u = \sin x, \quad du = \cos x\, dx$$

$$\ln|a+y| = \int \frac{1}{u}\, du$$

$$\ln|a+y| = \ln|\sin x| + C$$

<u>C-value</u> : sub $P\left(\frac{\pi}{3}, a\right)$



$$\ln|a+a| = \ln|\sin \pi/3| + C$$

$$\ln|2a| = \ln|\sqrt{3}/2| + C$$

$$\boxed{\ln|2a| - \ln|\sqrt{3}/2| = C}$$

$$\therefore \ln|a+y| = \ln|\sin x| + C$$

$$|a+y| = e^{\ln|\sin x| + C}$$

$$a + y = \pm e^{\ln|\sin x| + C}$$

$$\therefore y = \pm e^{\ln|\sin x| + \ln|2a| - \ln|\sqrt{3}/2|} - a$$

Sequences and Series

Sequences

$$\{a_n\} = \{a_1, a_2, a_3, a_4, \ldots, a_n\} = \{a_i\}_{i=1}^{n}$$

General form $\Rightarrow$ a_n

ex: Find the general form:

(i) $\overset{n=}{\{}\overset{1}{\tfrac{1}{2}}, \overset{2}{\tfrac{2}{3}}, \overset{3}{\tfrac{3}{4}}, \overset{4}{\tfrac{4}{5}}, \ldots\} \Rightarrow a_n = \frac{n}{n+1}$

(ii) $\{\overset{1}{\tfrac{3}{5}}, -\overset{2}{\tfrac{4}{25}}, \overset{3}{\tfrac{5}{125}}, -\overset{4}{\tfrac{6}{625}}, \ldots\} \Rightarrow a_n = \left(\frac{n+2}{5^n}\right) \cdot \underbrace{(-1)^{n+1}}_{\text{to account for alternating } +/-}$

skip 2 $\therefore$ $2n$

(iii) $\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots\} \Rightarrow a_n = \frac{1}{2n-1}$

(iv) $\{\overset{n=1}{3}, \overset{2}{2}, \overset{3}{\tfrac{4}{3}}, \overset{4}{\tfrac{8}{9}}, \overset{5}{\tfrac{16}{27}}, \ldots\} \Rightarrow a_n = \frac{2^{(n-1)}}{3^{(n-2)}}$

Defn: A sequence $\{a_n\}$ is said to:

(i) "converge" if $\lim_{n\to\infty} a_n$ exists

(ii) "diverge" if $\lim_{n\to\infty} a_n$ DNE

Note: A sequence has a restricted domain:

$n =$ counted numbers : $\{1, 2, 3, 4, 5, \ldots\}$

ex: (a) $a_n = \ln(1+n^2) - \frac{1}{2}\ln(n^3+4n)$

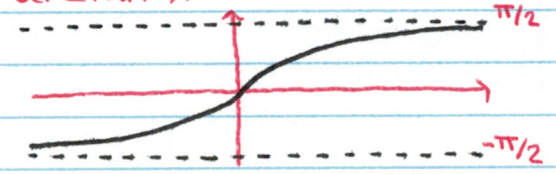
$$\lim_{n\to\infty}\left[\ln(1+\overset{\infty}{n^2}) \overset{-}{-} \tfrac{1}{2}\overset{\infty}{\ln(n^3+4n)}\right]$$

$$= \lim_{n\to\infty} \ln\left(\underbrace{\frac{1+n^2}{\sqrt{n^3+4n}}}_{\frac{\deg(2)}{\deg(3/2)} \Rightarrow +\infty}\right)$$

$$= \ln(+\infty) = +\infty \qquad \therefore \text{ divergent}$$

(b) $a_n = e^{2n/(n+2)} \cdot \tan^{-1}\left(\frac{n}{2}\right)$

Recall: $y = \arctan x$



$$\lim_{n\to\infty} e^{2n/(n+2)\,(*)} \cdot \underbrace{\tan^{-1}\left(\frac{n}{2}\right)}_{\pi/2}$$

$$(*) \lim_{n\to\infty} \frac{2n}{n+2} = \frac{2}{1} \quad \frac{\deg(1)}{\deg(1)}$$

$$= e^2 \cdot \pi/2 \qquad \therefore \text{ convergent}$$

(c) $a_n = \dfrac{(2n-1)!}{(2n+1)!}$

Factorials: $n! = (n)(n-1)(n-2)\ldots(3)(2)(1)$
$= (n)(n-1)\cdot(n-2)!$

$$\lim_{n\to\infty} \frac{\cancel{(2n-1)!}}{(2n+1)\cdot(2n)\cdot\cancel{(2n-1)!}}$$

$$= \lim_{n\to\infty} \frac{1}{2n(2n+1)} = \frac{1}{\infty} = 0 \qquad \therefore \text{ convergent}$$

(d) $a_n = \dfrac{3\sin(2n)}{2+\sqrt{n}}$ (squeeze)

$$-1\left(\frac{3}{2+\sqrt{n}}\right) \leq \frac{3\sin(2n)}{2+\sqrt{n}} \leq 1\left(\frac{3}{2+\sqrt{n}}\right)$$

$$\left.\begin{aligned} &\lim_{n\to\infty} -\frac{3}{2+\sqrt{n}} = -\frac{1}{\infty} = 0 \\ &\lim_{n\to\infty} \frac{3}{2+\sqrt{n}} = \frac{1}{\infty} = 0 \end{aligned}\right\} \therefore \lim_{n\to\infty} \frac{3\sin(2n)}{2+\sqrt{n}} = 0$$

$\therefore$ convergent.

(e) $a_n = \sqrt[n]{2^{1+3n}}$

$$\lim_{n\to\infty} 2^{\left(\frac{1+3n}{n}\right)\circledast}$$

$$= 2^3 = 8$$

$\circledast \lim_{n\to\infty} \dfrac{3n+1}{n} = 3$

$\therefore$ convergent

(f) $a_n = \dfrac{(\ln(n))^2}{n}$

$$\lim_{n\to\infty} \frac{(\ln(n))^2}{n} \frac{\infty}{\infty} \overset{\text{H}}{=} \lim_{n\to\infty} \frac{2.\ln(n).\ 1/n}{1}$$

$$= \lim_{n\to\infty} \frac{2\ln(n)}{n}$$

$$\overset{\text{H}}{=} \lim_{n\to\infty} \frac{2.(1/n)}{1}$$

$$= \frac{2}{\infty} = 0$$

$\therefore$ convergent

For what values of r does $\{r^n\}$ converge

(ie. when does $\lim_{n\to\infty} r^n$ exist)

if...

$\boxed{r>1}$ $\lim_{n\to\infty}(r^n) = +\infty$ DNE $\therefore$ div.

- $\boxed{r=1}$ $\lim_{n\to\infty}(1)^n = 1$ $\therefore$ cgt.
- $\boxed{-1<r<1}$ $\lim_{n\to\infty}(r^n) = 0$ $\therefore$ cgt.

ex: $\lim_{n\to\infty}\left(\frac{1}{4}\right)^n = 0$

$\boxed{r=-1}$ $\lim_{n\to\infty}(-1)^n$ DNE $\therefore$ div.

$\{-1,1,-1,1,-1,1,\ldots\}$

$\boxed{r<-1}$ $\lim_{n\to\infty}(r^n)$ DNE $\therefore$ div.

$\therefore$ $a_n = r^n$ is:

cgt. if $\boxed{-1<r\leq 1}$

and div. elsewhere.

Ex: determine cgt. or div. of the sequences:

(a) $a_n = n^2 e^{-n}$

$$\lim_{n\to\infty} \underset{\infty}{n^2} \cdot \underset{0}{e^{-n}} = \lim_{n\to\infty} \frac{n^2}{e^n} \quad \frac{\infty}{\infty}$$

$$\overset{H}{=} \lim_{n\to\infty} \frac{2n}{e^n}$$

$$\overset{H}{=} \lim_{n\to\infty} \frac{2}{e^n} = \frac{1}{\infty} = 0 \qquad \therefore \text{ cgt}$$

(b) $a_n = 1 + 2\left(\frac{4}{5}\right)^n$

$$\lim_{n\to\infty} 1 + 2\underbrace{\left(\frac{4}{5}\right)^n}_{r<1} = 1 \qquad \therefore \text{ cgt.}$$

$r^n = 0$

(c) $a_n = \dfrac{2\sin(n^2)}{n + \ln(n)}$

$$\lim_{n\to\infty} \frac{2\sin(n^2)}{n+\ln(n)} \qquad \text{(squeeze)}$$

$$-\frac{2}{n+\ln(n)} \leq \frac{2\sin(n^2)}{n+\ln(2)} \leq \frac{2}{n+\ln(n)}$$

$$\left.\begin{aligned} &\lim_{n\to\infty} -\frac{2}{n+\ln(n)} = -\frac{2}{\infty} = 0 \\ &\lim_{n\to\infty} \frac{2}{n+\ln(n)} = \frac{2}{\infty} = 0 \end{aligned}\right\} \therefore \lim_{n\to\infty} \frac{2\sin(n^2)}{n+\ln(n)} = 0$$

$\therefore$ cgt.

(d) $a_n = \left(1 + \frac{2}{n}\right)^n$

$$\lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^n \qquad 1^\infty$$

$$= \lim_{n\to\infty} e^{\ln(1+2/n)^n}$$

$$= \lim_{n\to\infty} e^{n \cdot \ln(1+2/n)} \circledast$$

$$= e^2$$

$\therefore$ cgt.

$$\circledast \lim_{n\to\infty} n \cdot \ln(1+2/n) \qquad \infty \cdot 0$$

$$= \lim_{n\to\infty} \frac{\ln(1+2/n)}{n^{-1}}$$

$$\overset{H}{=} \lim_{n\to\infty} \frac{\cancel{+}2 \cdot \cancel{n^{-2}} \cdot \frac{1}{1+2/n}}{\cancel{+}\cancel{n^{-2}}}$$

$$= \lim_{n\to\infty} \frac{2}{1+\underset{\to 0}{2/n}} = 2$$

Note: Limit definition of e

$$\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a$$

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

Infinite Series

Consider an infinite sum:

$$\boxed{\sum_{n=1}^{\infty} a_n} = \lim_{m\to\infty} \sum_{n=1}^{m} a_n$$

This is called an infinite series

$$\sum_{n=1}^{\infty} a_n \approx \sum_{n=1}^{m} a_n = S_m$$

(partial sum)

increase the accuracy by using larger number of terms (m)

Def'n: The series $\sum_{n=1}^{\infty} a_n$ if $\sum_{n=1}^{m} a_n = S_m$

(i) "converges" if $\lim_{m\to\infty} S_m$ exists

(ii) "diverges" if $\lim_{m\to\infty} S_m$ DNE

Consider:

$$\{a_n\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right\} = \left\{\frac{1}{2^n}\right\}$$

↳ sequence

→ What happens if we add up all values?

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 1$$

↳ series

travel:
- half of length (1)
- half of what is left
- half of what is left

...

As $n \to \infty$: approach 1

But never pass 1 ∴ cgt.

consider: $\sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right) \approx \sum_{n=1}^{m} a_n = S_m$

$m=1$ $\quad S_1 = \frac{1}{2} = \frac{1}{2}$

$m=2$ $\quad S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$m=3$ $\quad S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

$m=4$ $\quad S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$

Pattern $S_m = \frac{2^m - 1}{2^m}$

$$\therefore \boxed{S_m = \frac{2^m - 1}{2^m}}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{m\to\infty} S_m = \lim_{m\to\infty} \frac{2^m - 1}{2^m}$$

$$\lim_{m\to\infty} \frac{2^m - 1}{2^m} \overset{H}{=} \lim_{m\to\infty} \frac{2^m \ln(2)}{2^m \ln(2)} = \boxed{1}$$

Geometric Series

The geometric series (for $a \neq 0$):

$$\boxed{\sum_{n=1}^{\infty} ar^{n-1}} \quad \text{is} \quad \begin{cases} \text{cgt. if} & |r| < 1 \\ \text{div. if} & |r| \geq 1 \end{cases}$$

Also, if cgt. then:

$$\boxed{\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}} = S = \lim_{m \to \infty} S_m$$

Note: convert into this specific form

consider:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots$$

r is common ratio as we go from one term to the next. (ie. divide one number by the previous one)

a is the first term being added

* Change counter to $n=1$

→ if decrease counter increase inside

→ if increase counter decrease inside

Q. For what values of r will geometric series converge?

if: $\boxed{r=1}$ $\sum_{n=1}^{\infty} a(1)^{n-1} = a+a+a+a+\ldots$ div.

$$S_m = m.a$$

$\boxed{r=-1}$ $\sum_{n=1}^{\infty} a(-1)^{n-1} = a-a+a-a+a\ldots$

$$\left.\begin{array}{l} S_1 = a \\ S_2 = 0 \\ S_3 = a \\ S_4 = 0 \end{array}\right\} \lim_{m\to\infty} S_m \text{ DNE}$$ div.

$\boxed{r>1}$
$\boxed{r<-1}$ $\sum_{n=1}^{\infty} ar^{n-1} = a+ar+ar^2+\ldots = \pm\infty$

div.

$\boxed{0<r<1}$ $\sum_{n=1}^{\infty} ar^{n-1}$

$$S_m = a + ar + ar^2 + \ldots + ar^{m-2} + ar^{m-1}$$
$$r.S_m = ar + ar^2 + ar^3 + \ldots + ar^{m-1} + ar^m$$

consider: $S_m - rS_m = a - ar^m$

$$S_m(1-r) = a - ar^m$$

$$\boxed{S_m = \frac{a-ar^m}{1-r}}$$

evaluate: $\lim_{m\to\infty} S_m$

$$= \lim_{m\to\infty} \frac{a(1-r^m)}{1-r}$$

Recall:
$|r|<1$ $\therefore \lim_{m\to\infty} r^m = 0$
$|r|>1$ $\therefore \lim_{m\to\infty} r^m$ DNE

$$\therefore \boxed{\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}}$$ cgt.

ex: convergent or divergent ?

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n}$

put in form $\sum_{n=1}^{\infty} a r^{n-1}$

$= \sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{1}{2^{n-1}}$ $\quad r = 1/2 \Rightarrow |r| < 1 \therefore$ cgt $\quad$ cgt.

$a = 1/2$

$= \frac{a}{1-r} = \frac{1/2}{-1/2+1} = \boxed{1}$

(b) $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$

$= \sum_{n=1}^{\infty} \frac{3}{3^n} \cdot 2^{2n}$

$= \sum_{n=1}^{\infty} 3 \cdot \left(\frac{4}{3}\right)^n$

$= \sum_{n=1}^{\infty} 3 \cdot \frac{4}{3} \cdot \left(\frac{4}{3}\right)^{n-1}$ $\quad |r| = 4/3 > 1 \quad \therefore$ div.

(c) $\sum_{n=1}^{\infty} 6(0.9)^{n-1}$ $\quad r = 0.9 \Rightarrow |r| < 1 \therefore$ cgt.

$a = 6$

$S = \frac{a}{1-r} = \frac{6}{1-0.9} = 60$

(d) $\sum_{n=3}^{\infty} \frac{10^n}{9^{n-1}}$

change counter

$= \sum_{n=1}^{\infty} \frac{10^{n+2}}{9^{n+1}} = \sum_{n=1}^{\infty} \frac{10^3}{9^2} \left(\frac{10}{9}\right)^{n-1}$ $\quad |r| = \frac{10}{9} > 1 \therefore$ div

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\sqrt{2})^n}$

$$= \sum_{n=1}^{\infty} \frac{1}{\sqrt{2}} \cdot \left(\frac{1}{\sqrt{2}}\right)^{n-1}$$

$|r| = 1/\sqrt{2} < 1 \quad \therefore \quad$ cgt.

$a = 1/\sqrt{2}$

$$S = \frac{a}{1-r} = \frac{1/\sqrt{2}}{1 - 1/\sqrt{2}}$$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{\pi^n}{3^{n+1}}$

$$= \sum_{n=1}^{\infty} \frac{\pi}{3^2} \left(\frac{\pi}{3}\right)^{n-1}$$

$|r| = \pi/3 > 1 \quad \therefore \quad$ div.

ex: express the number $1.53\overline{42}$ as a ratio of 2 integers

$$1.53\overline{42} = 1.53 + 0.0042 + 0.000042 + \ldots$$

$$= \frac{153}{100} + \underbrace{\left(\frac{42}{10^4} + \frac{42}{10^6} + \frac{42}{10^8} + \ldots\right)}$$

$r = \frac{1}{100}$ ← divide any term by previous one

$a = \frac{42}{10^4}$ ← first term

$$S = \frac{a}{1-r} =$$

$$= \frac{42 \cdot 10^{-4}}{1 - 10^{-2}}$$

$$= \frac{7}{1650}$$

$$\therefore 1.53\overline{42} = \frac{153}{100} + \frac{7}{1650}$$

$$= \frac{15189}{9900}$$

Telescoping Series

Form: $\sum_{n=1}^{\infty} (a_n - a_{n+1})$

ex: Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is cgt

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \Rightarrow 1 = A(n+1) + B(n)$$

$\boxed{n=0} \Rightarrow A = 1$

$\boxed{n=-1} \Rightarrow B = -1$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$S_m = \underbrace{\left(1 - \cancel{\tfrac{1}{2}}\right)}_{n=1} + \underbrace{\left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right)}_{n=2} + \underbrace{\left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right)}_{n=3} + \dots + \underbrace{\left(\cancel{\tfrac{1}{m-1}} + \cancel{\tfrac{1}{m}}\right)}_{n=m-1} + \underbrace{\left(\cancel{\tfrac{1}{m}} - \tfrac{1}{m+1}\right)}_{n=m}$$

$$\therefore \boxed{S_m = 1 - \frac{1}{m+1}}$$

$$\therefore \lim_{m\to\infty} S_m = \lim_{m\to\infty} \left(1 - \frac{1}{m+1}\right) = 1$$

$\therefore \sum a_n$ cgt.

$\sum a_n = 1$

ex: $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+4}\right)$

$$S_m = \left(1 - \cancel{\tfrac{1}{5}}\right) + \left(\tfrac{1}{2} - \cancel{\tfrac{1}{6}}\right) + \left(\tfrac{1}{3} - \cancel{\tfrac{1}{7}}\right) + \left(\tfrac{1}{4} - \cancel{\tfrac{1}{8}}\right) + \left(\cancel{\tfrac{1}{5}} - \cancel{\tfrac{1}{9}}\right) +$$
$$\dots + \left(\cancel{\tfrac{1}{m-3}} - \tfrac{1}{m+1}\right) + \left(\cancel{\tfrac{1}{m-2}} - \tfrac{1}{m+2}\right) + \left(\cancel{\tfrac{1}{m-1}} - \tfrac{1}{m+3}\right) + \left(\cancel{\tfrac{1}{m}} - \tfrac{1}{m+4}\right)$$

$$\therefore S_m = 1 + 1/2 + 1/3 + 1/4 - \frac{1}{m+1} - \frac{1}{m+2} - \frac{1}{m+3} - \frac{1}{m+4}$$

ex: $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} \left[\ln(n) - \ln(n+1)\right]$

$$S_m = \underbrace{\left(\ln(1) - \cancel{\ln(2)}\right)}_{n=1} + \underbrace{\left(\cancel{\ln(2)} - \cancel{\ln(3)}\right)}_{n=2} + \dots + \underbrace{\left(\cancel{\ln(m)} - \ln(m+1)\right)}_{n=m}$$

$$S_m = \ln(1) - \ln(m+1)$$

$$\therefore \lim_{m\to\infty} S_m = \lim_{m\to\infty} \left(\cancel{\ln(1)}^{\,0} - \ln(\underbrace{m}_{\infty}+1)\right) = -\infty \qquad \therefore \text{ div.}$$

$\boxed{n^{th} \text{ term test}}$

Theorem: if $\sum_{n=1}^{\infty} a_n$ is cgt. then

$$\lim_{n\to\infty} a_n = 0$$

$\therefore$ Test for divergence

If $\lim_{n\to\infty} a_n \neq 0$ or DNE

then $\sum a_n$ is divergent

Note: the test can only confirm divergence

if $\lim_{n\to\infty} a_n = 0$ $\therefore$ test inconclusive

$\rightarrow$ can be div./cgt.

Note: Rates of growth: $x^x > x! > a^x > x^a$

ex: does $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ converge or diverge

$$\lim_{n\to\infty} \frac{n^2}{5n^2+4} \quad \frac{\deg(2)}{\deg(2)} = \boxed{\frac{1}{5}} \neq 0$$

$\therefore \sum a_n$ cgt. by n^{th}

The Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$

Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ is div

even though $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$

Show that $\lim_{m\to\infty} S_m = +\infty$ DNE

consider only S_{2^m} (subsequence $\Rightarrow$ every term is included in S_m)

$m=1$ $S_2 = 1 + \frac{1}{2}$

$m=2$ $S_4 = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{>1/2} > 1 + 1/2 + 1/2$

$m=3$ $S_8 = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{>1/2} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{>1/2} > 1 + 1/2 + 1/2 + 1/2$

$m=4$ $S_{16} = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{>1/2} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{>1/2}$

$+ \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right)}_{>1/2} > 1 + 1/2 + 1/2 + 1/2 + 1/2$

$\therefore S_{2^m} > 1 + m\left(\frac{1}{2}\right)$ it keeps growing by $\frac{1}{2}$

$\therefore \lim_{m\to\infty} S_{2^m} > \lim_{m\to\infty} 1 + m(1/2) = +\infty$

$\therefore \lim_{m\to\infty} S_{2^m} = +\infty$

$\therefore \lim_{m\to\infty} S_m = +\infty$ $\therefore \sum \frac{1}{n}$ is div.

Integral Test

Suppose $f(x)$ is : ① continuous
② positive
③ decreasing on $[1, \infty)$

Let $a_n = f(n)$, then

$$\boxed{\sum a_n \approx \int_1^\infty f(x)\,dx}$$

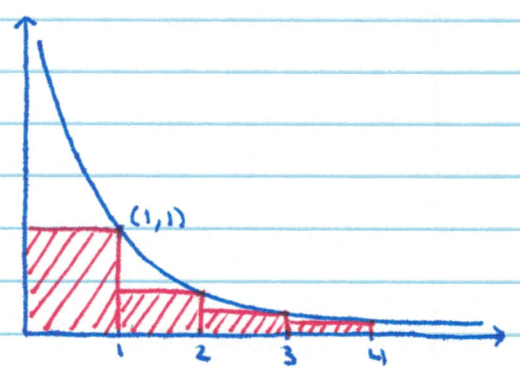
→ behave in similar way

→ converge together.

Consider : $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$

consider : $f(x) = \frac{1}{x^2}$

and $\int_1^\infty \frac{1}{x^2}\,dx = 1$



$$\begin{pmatrix}\text{sum of}\\ \text{rectangles}\end{pmatrix} = \overset{\Delta x \cdot f(x)}{(1)(1)} + (1)(1/4) + (1)(1/9) + (1)(1/16) + \dots$$

$$= 1 + \underbrace{1/4 + 1/9 + 1/16 + \dots}$$

remove bc $\int_1^\infty f(x)\,dx$

$$1/4 + 1/9 + 1/16 + \dots < \int_1^\infty \frac{1}{x^2}\,dx = 1$$

$\therefore\ 1/4 + 1/9 + 1/16 + \dots < 1$

$\therefore\ 1 + 1/4 + 1/9 + 1/16 + \dots < 2$ $\quad\therefore \sum a_n$ cgt.

ex: Test for convergence vs divergence

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$f(x) = \frac{1}{x^2+1}$

① continuous
② pos
③ decreasing $[1,\infty)$

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \left(\tan^{-1}(x)\right)\Big|_1^t$$

$$= \lim_{t\to\infty} \left[\tan^{-1}(t) - \tan^{-1}(1)\right]$$

$$= \pi/2 - \pi/4 = \pi/4$$

$\therefore$ cgt

$\therefore \sum a_n$ cgt.

(b) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

$f(x) = \frac{n^2}{e^{n^3}}$

$\begin{cases} \text{continuous} \\ \text{positive} \\ \text{decreasing} \end{cases}$

$$\int_1^{\infty} \frac{x^2}{e^{x^3}}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{x^2}{e^{x^3}}\,dx$$

$u = x^3$
$du = 3x^2\,dx$
$\frac{1}{3}\,du = x^2\,dx$

$x = 1,\ u = 1$
$x = t,\ u = t^3$

$$= \lim_{t\to\infty} \frac{1}{3}\int_1^{t^3} \frac{1}{e^u}\,du$$

$$= \lim_{t\to\infty} \frac{1}{3}\left(-\frac{1}{e^u}\right)\Big|_1^{t^3}$$

$$= \lim_{t\to\infty} -\frac{1}{3}\left(\cancel{\frac{1}{e^{t^3}}}^{\,0} - \frac{1}{e}\right) = \frac{1}{3e}$$

$\therefore$ cgt

$\therefore \sum a_n$ cgt.

P-Series

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p}} \quad \text{is} \quad \begin{cases} \text{cgt} & \text{if } p>1 \\ \text{div.} & \text{if } p \leq 1 \end{cases}$$

The value of p affects how fast the fcn goes to zero:
- very big $P \Rightarrow$ fast $\Rightarrow$ cgt
- very small $P \Rightarrow$ slow $\Rightarrow$ div.

Q. For what values of P is the series cgt?

$\boxed{P<0}$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^p}$ $\quad n^{th}$ term $\lim_{n\to\infty} \frac{1}{n^p} = +\infty$ $\quad \therefore$ div

$\boxed{P=1}$ $\quad \sum_{n=1}^{\infty} \frac{1}{n}$ $\quad$ harmonic series $\quad \therefore$ div

$\boxed{P>0}$ $\quad f(x) = \frac{1}{x^p}$: continuous, decreasing, positive

$$\int_1^{\infty} x^{-p}\,dx = \lim_{t\to\infty}\left(\frac{x^{-p+1}}{1-p}\right)\Bigg|_1^t$$

$$= \lim_{t\to\infty} \left(t^{1-p}\right)\left(\frac{1}{1-p}\right) - \left(\frac{1}{1-p}\right)$$

(*)

If $0<P<1$ $\quad 1-P>0 \quad \therefore \lim_{t\to\infty} t^{1-p} = +\infty$

(*) DNE $\quad \therefore$ div.

$P>1$ $\quad 1-p<0 \quad \therefore \lim_{t\to\infty} t^{1-p} = 0$

(*) exists $\quad \therefore$ cgt

ex: cgt or div. ?

(a) $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$ $\quad p = \sqrt{2} > 1 \quad \therefore$ cgt.

(b) $\sum_{n=1}^{\infty} \frac{n}{n^{e}}$

$= \sum_{n=1}^{\infty} \frac{1}{n^{e-1}}$ $\quad e-1 > 1 \quad \therefore$ cgt.

(c) $\sum_{n=1}^{\infty} \left(\frac{\sqrt{n} + 4}{n^2} \right)$

$= \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} + \sum_{n=1}^{\infty} \frac{4}{n^2}$

$p = 3/2 \qquad p = 2 \qquad p > 1 \quad \therefore$ cgt

<u>Note</u>: can only split up a series when all series are convergent.

Comparaison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms:

(i) if $\sum b_n$ is cgt. and $a_n \leq b_n$

$\therefore \sum a_n$ is cgt

(ii) if $\sum b_n$ is div. and $a_n \geq b_n$

$\therefore \sum a_n$ is div

Note: to find b_n, use dominant term from numerator and denominator

Limit Comparaison Test

if $\lim_{n \to \infty} \frac{a_n}{b_n} \frac{0}{0} = C$

Note: $C > 0$
C exists

then: $\sum a_n \approx \sum b_n$

→ they behave in the same way
→ converge together

ex:

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

$\sum b_n = \sum \frac{1}{2^n}$

Geometric series: $|r| = 1/2 < 1$

$\therefore$ cgt

$\frac{1}{2^n+1} < \frac{1}{2^n}$

larger denom

$\sum_{n=1}^{\infty} \frac{1}{2^n+1} < \sum_{n=1}^{\infty} \frac{1}{2^n}$

$\therefore \sum \frac{1}{2^n+1}$ also cgt.

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2-1}}$

$b_n = \frac{1}{\sqrt{n^2}} = \frac{1}{n}$

$\sum \frac{1}{n}$ is div (Harmonic)

$\frac{1}{\sqrt{n^2-1}} > \frac{1}{n}$

smaller denom

$\therefore \sum \frac{1}{\sqrt{n^2-1}}$ is also div.

(c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{\underbrace{n^5}_{\text{dominant}}+n^2}}$

$b_n = \frac{1}{\sqrt{n^5}}$

P-series: $P = 5/2 > 1$ $\therefore$ cgt

$\frac{1}{\sqrt{n^5+n^2}} < \frac{1}{\sqrt{n^5}}$

$\therefore \sum \frac{1}{\sqrt{n^5+n^2}}$ is also cgt.

(d) $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

$b_n = \frac{1}{2^n}$

Geometric: $|r| = 1/2 < 1$ $\therefore$ cgt

$\frac{1}{2^n-1} > \frac{1}{2^n}$

smaller denom

$\therefore$ Test inconclusive
(cannot use comparaison test)

ex: Limit comparaison

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ $\qquad b_n = \dfrac{1}{2^n}$

Geometric series: $|r| = 1/2 < 1 \therefore$ cgt

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} \overset{\frac{\infty}{\infty}}{=} \overset{H}{\lim_{n\to\infty}} \frac{2^n \cdot \ln(2)}{2^n \cdot \ln(2)} = \boxed{1}$$

$\therefore \sum a_n \approx \sum b_n$

$\therefore \sum a_n$ is also cgt.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{n^5 + 5}}$ $\qquad b_n = \dfrac{n^2}{\sqrt{n^5}} = \dfrac{1}{n^{1/2}}$

P-series: $p = 1/2 < 1 \therefore$ div.

$$\lim_{n\to\infty} \frac{(2n^2 + 3n)}{\sqrt{n^5 + 5}} \cdot \frac{\sqrt{n^5}}{n^2} \qquad \frac{\deg(5/2)}{\deg(5/2)} = \frac{2}{1} = \boxed{2}$$

$\therefore \sum a_n \approx \sum b_n$

$\therefore \sum a_n$ also div.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(1 + \frac{1}{n})^2}{e^n}$ $\qquad b_n = \dfrac{1}{e^n}$

Geometric: $|r| = 1/e < 1 \quad \therefore$ cgt

$$\lim_{n\to\infty} \frac{(1 + \overset{\to 0}{\frac{1}{n}})^2}{\cancel{e^n}} \cdot \left(\frac{\cancel{e^n}}{1}\right) = \boxed{1}$$

$\therefore \sum a_n \approx \sum b_n$

$\therefore \sum a_n$ also cgt.

(d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n + 4^n}{n^2 + 6^n}\right)$ $\qquad b_n = \dfrac{4^n}{6^n} = \left(\dfrac{4}{6}\right)^n$

Geometric: $|r| = 2/3 < 1 \quad \therefore$ cgt

$$\lim_{n\to\infty} \left(\frac{4^n + n}{6^n + n^2}\right) \cdot \frac{6^n}{4^n} = \lim_{n\to\infty} \frac{\cancel{4^n}(1 + n/4^n)}{\cancel{6^n}(1 + n^2/6^n)} \cdot \frac{\cancel{6^n}}{\cancel{4^n}} = \boxed{1}$$

$\therefore \sum a_n \approx \sum b_n$

$\therefore \sum a_n$ is also cgt.

Ratio Test

comparing one term to the term that comes after

Let $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

(i) if $L < 1$ $\therefore \sum a_n$ cgt.

(ii) if $L > 1$ $\therefore \sum a_n$ div.

(iii) if $L = 1$ $\therefore$ inconclusive

ex: $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

pair off terms

$$\lim_{n\to\infty} \left| \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} \right| = \lim_{n\to\infty} \left| \underbrace{\frac{(n+1)^3}{n^3}}_{\frac{1}{1}} \cdot \frac{\cancel{3^n}}{3^{\cancel{n}+1}} \right|$$

$$= 1 \cdot \frac{1}{3} = \boxed{\frac{1}{3}} \quad \therefore \text{cgt.}$$

ex:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\lim_{n\to\infty} \frac{(n+1)^{(n+1)}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^{(n+\cancel{1})}}{\cancel{(n+1)} \cdot \cancel{n!}} \cdot \frac{\cancel{n!}}{n^n}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \qquad 1^{\infty}$$

$$\lim_{n\to\infty} \left(1+\frac{a}{n}\right)^n = e^a$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \boxed{e}$$

$e > 1 \therefore \sum a_n$ div

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

$$\lim_{n\to\infty} \frac{((n+1)!)^2}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2} = \lim_{n\to\infty} \frac{(2n)!}{(2n+2)!} \cdot \frac{((n+1)\cdot n!)^2}{(n!)^2}$$

$$= \lim_{n\to\infty} \frac{\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}} \cdot \frac{(n+1)^2 \cdot \cancel{(n!)^2}}{\cancel{(n!)^2}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} \qquad \frac{\deg(2)}{\deg(2)}$$

$$= \frac{1}{2 \cdot 2} = \boxed{1/4}$$

$1/4 < 1 \therefore \sum a_n$ cgt.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{100^n}$

$$\lim_{n\to\infty} \frac{(n+1)!}{100^{n+1}} \cdot \frac{\cancel{100^n}}{n!} = \lim_{n\to\infty} \frac{(n+1)\,\cancel{n!}}{\cancel{n!}} \cdot \frac{1}{100} = \lim_{n\to\infty} \frac{n+1}{100} = +\infty$$

$\therefore \sum a_n$ div.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n \cdot n!}{5 \cdot 8 \cdot 11 \cdots (3n+2)}$

$$\lim_{n\to\infty} \frac{2^{n+1} \cdot (n+1)!}{\cancel{5 \cdot 8 \cdot 11} \cdots (3(n+1)+2)} \cdot \frac{\cancel{5 \cdot 8 \cdot 11} \cdots (3n+2)}{\cancel{2^n} \cdot n!}$$

$$= \lim_{n\to\infty} \frac{2 \cdot (n+1) \cdot \cancel{n!}}{\cancel{n!}} \cdot \frac{\cancel{(3n+2)}}{\cancel{(3n+2)} \cdot (3(n+1)+2)}$$

$$= \lim_{n\to\infty} \frac{2 \cdot (n+1)}{3n+5} \quad \frac{\deg(1)}{\deg(1)} = \boxed{\frac{2}{3}} < 1 \quad \therefore \sum a_n \text{ cgt}$$

ex: For which values of $K \in \mathbb{Z}^+$ is the series cgt:

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(Kn)!} \qquad \text{ie. when is } \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

$$\lim_{n\to\infty} \frac{((n+1)!)^2}{(K(n+1))!} \cdot \frac{(Kn)!}{(n!)^2}$$

$$= \lim_{n\to\infty} \frac{((n+1) \cdot n!)^2}{(n!)^2} \cdot \frac{(Kn)!}{(Kn+K)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 \cdot \cancel{(n!)^2}}{\cancel{(n!)^2}} \cdot \frac{\cancel{(Kn)!}}{(Kn+K)(Kn+K-1)(Kn+K-2) \cdots (Kn+1) \cdot \cancel{(Kn)!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(Kn+K)(Kn+K-1) \cdots (Kn+1)} \qquad \frac{\deg(2)}{\deg(K)}$$

if $K < 2 \Rightarrow \infty \therefore$ div

if $K > 2 \Rightarrow 0 \therefore$ cgt

if $K = 2 \Rightarrow \displaystyle\lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \boxed{\frac{1}{4}} < 1$

$\therefore$ cgt when $K \geq 2$

Root Test

Let $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$

(i) if $L < 1$ $\therefore \sum a_n$ cgt.

(ii) if $L > 1$ $\therefore \sum a_n$ div.

(iii) if $L = 1$ $\therefore$ inconclusive

ex: $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \boxed{\frac{2}{3}} < 1 \quad \therefore \sum a_n \text{ is cgt.}$$

ex: $\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^{5n}$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n}{n+1}\right)^{5n}} = \lim_{n\to\infty} \left(\frac{2n}{n+1}\right)^5 = \frac{2^5}{1^5} = \boxed{2^5} > 1 \quad \therefore \text{div.}$$

ex: $\sum_{n=1}^{\infty} \left(1+\frac{3}{n}\right)^{n^2}$

$$\lim_{n\to\infty} \sqrt[n]{\left(1+\frac{3}{n}\right)^{n^2}} = \lim_{n\to\infty} \left(1+\frac{3}{n}\right)^n = \boxed{e^3} > 1 \quad \therefore \text{div.}$$

HARD CASES

(i) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n!}{n^n}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{n!}{n^n}\right)^n} = \lim_{n\to\infty}\frac{n!}{n^n} = \circledast = \lim_{n\to\infty}\frac{\cancel{n}\cdot\overbrace{(n-1)(n-2)(n-3)\dots(3)(2)}^{n-2\text{ terms}}\cancel{(1)}}{\cancel{n}\cdot\underbrace{(n)(n)(n)\dots(n)(n)(n)}_{n-1\text{ terms}}}$$

Statement

$$\boxed{(\overset{<n}{n-1})(\overset{<n}{n-2})(\overset{<n}{n-3})\dots(\overset{<n}{3})(\overset{<n}{2}) < n^{n-2}}$$

$$\therefore \circledast < \lim_{n\to\infty}\left(\frac{n^{n-2}}{n^{n-1}}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$$

$$\therefore \circledast < 0$$

$$\underbrace{0 \le}_{\text{for sure}} \lim_{n\to\infty}\sqrt[n]{\left(\frac{n!}{n^n}\right)^n} < 0$$

$\therefore \displaystyle\lim_{n\to\infty}\sqrt[n]{\left(\frac{n!}{n^n}\right)^n} = 0$ by squeeze theorem

$\therefore 0 < 1 \quad \therefore$ cgt.

(ii) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{6^n}{n!}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{6^n}{n!}\right)^n} = \lim_{n\to\infty}\frac{6^n}{n!} = \circledast \lim_{n\to\infty}\left(\frac{6}{n}\right)\underbrace{\left(\overset{<1}{\frac{6}{n-1}}\right)\left(\overset{<1}{\frac{6}{n-2}}\right)\cdots\left(\overset{<1}{\frac{6}{7}}\right)}_{<1}\underbrace{\left(\frac{6}{6}\right)\left(\frac{6}{5}\right)\left(\frac{6}{4}\right)\left(\frac{6}{3}\right)\left(\frac{6}{2}\right)\left(\frac{6}{1}\right)}_{C}$$

$$\circledast < \lim_{n\to\infty}\frac{6}{n}(1)\cdot C = 0$$

$$\underbrace{0 \le}\ \circledast < 0 \qquad \therefore \lim_{n\to\infty}\frac{6^n}{n!} = 0 \text{ by squeeze}$$

$\therefore \sum a_n$ cgt.

(iii) $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^n}{n^{4n}}$

$$\lim_{n\to\infty}\sqrt[n]{\frac{(n!)^n}{n^{4n}}} = \lim_{n\to\infty}\frac{n!}{n^4} = \lim_{n\to\infty}\frac{n(n-1)(n-2)(n-3)(n-4)!}{n\cdot n\cdot n\cdot n}$$

$$= \lim_{n\to\infty}\left(\overset{=1}{\frac{n}{n}}\right)\left(\overset{=1}{\frac{n-1}{n}}\right)\left(\overset{=1}{\frac{n-2}{n}}\right)\left(\overset{=1}{\frac{n-3}{n}}\right)\cdot(n-4)!$$

$$= +\infty > 1 \qquad \therefore \sum a_n \text{ div}$$

ex: cgt or div. ?

(a) $\sum_{n=1}^{\infty} \frac{1}{n+3^n}$

$b_n = \frac{1}{3^n}$

Geometric: $|r| = 1/3 < 1$ $\therefore$ cgt.

$\frac{1}{n+3^n} < \frac{1}{3^n}$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n+3^n}$ is cgt (comparaison)

(b) $\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$

$\lim_{n\to\infty} \sqrt[n]{\frac{(2n+1)^n}{n^{2n}}} = \lim_{n\to\infty} \frac{2n+1}{n^2}$ $\frac{\deg(1)}{\deg(2)} = \boxed{0} < 1$

$\therefore \sum a_n$ cgt (root test)

(c) $\sum_{n=1}^{\infty} \frac{n}{n+2}$

$\lim_{n\to\infty} \frac{n}{n+2}$ $\frac{\deg(1)}{\deg(1)} = \boxed{1} \neq 0$ $\therefore \sum a_n$ div (n^{th} term)

(d) $\sum_{n=1}^{\infty} n^2 e^{-n}$

$\lim_{n\to\infty} \left| \frac{(n+1)^2}{e^{\cancel{n}+1}} \cdot \frac{\cancel{e^n}}{n^2} \right| = \lim_{n\to\infty} \underbrace{\frac{(n+1)^2}{n^2}}_{1} \cdot \frac{1}{e} = \boxed{\frac{1}{e}} < 1$

$\therefore \sum a_n$ cgt (ratio)

(e) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$

$b_n = \frac{1}{n.\sqrt{n^2}} = \frac{1}{n^2}$

P-series: $p = 2 > 1$ $\therefore$ cgt

$\frac{1}{n\sqrt{n^2+1}} < \frac{1}{n^2}$

larger denom

$\therefore \sum a_n$ cgt (comparaison)

(f) $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n^2}\right)$

$$\lim_{n\to\infty} \cos\left(\frac{1}{n^2}\right) = \cos(0) = \boxed{1} \neq 0 \quad \therefore \sum a_n \text{ div } (n^{th} \text{ term})$$

(g) $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

$$\lim_{n\to\infty} \frac{(n+1)!}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n!} = \lim_{n\to\infty} \frac{(n+1).\cancel{n!}}{\cancel{n!}} \cdot \frac{\cancel{e^{n^2}}}{e^{\cancel{n^2}+2n+1}}$$

$$= \lim_{n\to\infty} \frac{(n+1)}{e^{2n+1}} \quad \frac{\infty}{\infty}$$

$$\overset{H}{=} \lim_{n\to\infty} \frac{1}{2.e^{2n+1}} = \boxed{0} < 1$$

$\therefore \sum a_n$ cgt (ratio)

(h) $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+2n^2)^n}$

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^{2n}}{(1+2n^2)^n}} = \lim_{n\to\infty} \frac{n^2}{1+2n^2} \quad \frac{\deg(2)}{\deg(2)} = \boxed{\frac{1}{2}} < 1$$

$\therefore \sum a_n$ cgt (root)

(i) $\sum_{n=1}^{\infty} \frac{n.\ln(n)}{(n+1)^3}$

Power Series

Def'n: A series of form

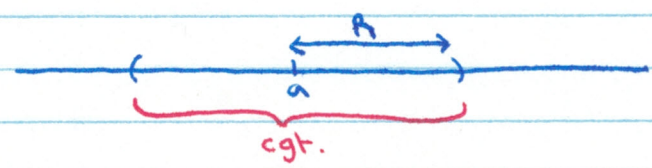
$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

is called a "<u>power series</u>" that is centered at $x=a$.

Note: For some values of x $\sum_{n=1}^{\infty} c_n (x-a)^n$ is cgt. and for others it is div.

3 possibilities:

1. The series converges for all $x \in \mathbb{R}$
2. There is some positive value R s.t the series converges for $|x-a| < R$, and diverges for $|x-a| > R$.
3. The series converges only for $x=a$ $R=0$ (no radius, only one value)



For the values of x where $\sum_{n=1}^{\infty} c_n x^n$ is cgt. it is equal to $f(x)$.

ie. $f(x) = \sum_{n=1}^{\infty} c_n (x-a)^n$ only within "<u>Radius of Convergence</u>"

ex: What is the radius of convergence for the given series?

→ Always use Ratio Test

→ Treat x like constant until getting rid of $\lim_{n\to\infty}$

Ratio Test : $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ ⇒ for convergence

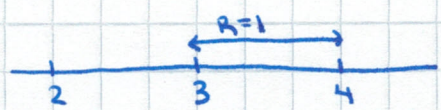
(a) $\sum_{n=1}^{\infty}\frac{(x-3)^n}{n+1}$

$$\lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{(n+2)}\cdot\frac{(n+1)}{(x-3)^n}\right| < 1$$

$$= \lim_{n\to\infty}\left|\underbrace{\frac{\cancel{(n+1)}}{\cancel{(n+2)}}}_{1}\cdot\frac{(x-3)^{\cancel{n}+1}}{\cancel{(x-3)^n}}\right|$$

$$= |x-3|$$

⇒ Need $|x-3| < 1$ ∴ $R=1$ centered at $a=3$



(b) $\sum_{n=1}^{\infty}(-1)^n\cdot\frac{x^{2n}}{(2n)!}$

$$\lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+1}}\cdot x^{2(n+1)}}{(2(n+1))!}\cdot\frac{(2n)!}{\cancel{(-1)^n}\cdot x^{2n}}\right|$$

Cancel out (-1) bc absolute value.

$$= \lim_{n\to\infty}\left|\frac{x^{\cancel{2n}+2}}{\cancel{x^{2n}}}\cdot\frac{\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(2n+1)}\right| = \left|\frac{x^2}{\infty}\right| = 0 \quad \forall x\in\mathbb{R}$$

∴ No matter what value of x, it always equals 0

∴ Need $0 < 1$ ⇒ always true

∴ Radius of convergence is $R=\infty$

(ie. $\sum a_n$ cgt. $\forall x\in\mathbb{R}$)

(c) $\sum_{n=1}^{\infty} n!\, x^n$

$$\lim_{n\to\infty} \left| \frac{(n+1)! \cdot x^{\cancel{n}+1}}{n! \cdot \cancel{x^n}} \right|$$

$$\lim_{n\to\infty} \left| (n+1) \cdot x \right|$$ Need <1 for cgt.

<u>Note:</u> if $\boxed{x=0}$ $\therefore \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 < 1$ (cgt)

for all other values, $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = +\infty$ (div)

$\therefore$ Radius of convergence $R=0$,
cgt. only at $x=0$

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{4^n} (x+3)^n$

$$\lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}} \cdot (n+1) \cdot (x+3)^{\cancel{n}+1}}{4^{\cancel{n}+1}} \cdot \frac{\cancel{4^n}}{\cancel{(-1)^n} \cdot n \cdot \cancel{(x+3)^n}} \right| < 1$$

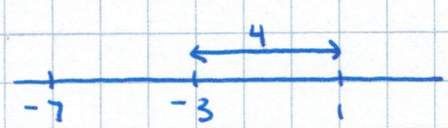
$$= \lim_{n\to\infty} \left| \underbrace{\frac{(n+1)}{n}}_{1} \cdot \frac{(x+3)}{4} \right|$$

$$= \left| \frac{x+3}{4} \right|$$

$\Rightarrow$ Need $\left| \frac{1}{4}(x+3) \right| < 1$

$$\boxed{|x+3| < 4}$$

$\therefore$ Radius of cgt $R=4$,
centered at $a=-3$



Taylor Series / Machaurin Series

$f(x) \approx T_n(x)$

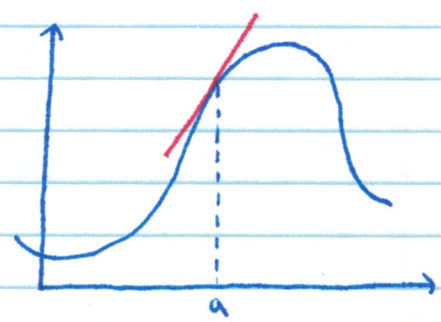
$f(x) = \lim_{n \to \infty} T_n(x)$

<u>Recall</u>: The Nth degree Taylor Polynomial is:

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!} \cdot (x-a) + \frac{f''(a)}{2!} \cdot (x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \ldots$$

consider:



$T_0(x) = f(a)$

$T_1(x) = f(a) + f'(a)(x-a)$

Require: ①. $T_1(a) = f(a)$

② $T_1'(a) = f'(a)$

$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2} \cdot (x-a)^2$

Now require: ① $T_2(a) = f(a)$ → share point

② $T_2'(a) = f'(a)$ → share slope of tan.

③ $T_2''(a) = f''(a)$ → share concavity

consider:

$$T_3(x) = \overset{\text{constant}}{f(a)} + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$T'_3(x) = 0 + f'(a) + \frac{f''(a)}{\cancel{2}} \cdot \cancel{2}(x-a) + \frac{f'''(a)}{\cancel{3!}\,2} \cdot \cancel{3}(x-a)^2$$

$$\therefore \boxed{T_3'(a) = f'(a)}$$

$$T''_3(x) = 0 + 0 + f''(a) + \frac{f'''(a)}{\cancel{2}} \cdot \cancel{2}(x-a)$$

$$\therefore \boxed{T''(a) = f''(a)}$$

$$T'''_3(x) = 0 + 0 + 0 + f'''(a)$$

$$\therefore \boxed{T'''(a) = f'''(a)}$$

Def'n The "<u>Taylor Series</u>" of $y = f(x)$ at $x = a$ is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

<u>Note</u>: if $a=0$, then we call it the "<u>MacLaurin Series</u>"

$$f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

ex: Find the <u>MacLaurin</u> Series of $f(x) = e^x$

$a = 0$

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$$

$$\left.\begin{array}{l|l} f(x) = e^x & f(0) = e^0 = 1 \\ f'(x) = e^x & f'(0) = e^0 = 1 \\ f''(x) = e^x & f''(0) = e^0 = 1 \\ f'''(x) = e^x & f'''(0) = e^0 = 1 \end{array}\right\} f^n(0) = 1$$

$$\therefore e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

<u>Note</u>: only equal to eachother within radius of convergence (where cgt.)

ex: Find the Taylor Series of $f(x) = e^x$ centered at $a = 3$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!} (x-3)^n$$

$$\left.\begin{array}{l} f(3) = e^3 \\ f'(3) = e^3 \\ f''(3) = e^3 \\ f'''(3) = e^3 \end{array}\right\} f^n(3) = e^3$$

$$\therefore e^x = \sum_{n=0}^{\infty} \frac{e^3}{n!} (x-3)^n$$

<u>Note</u>: Series alternating +/-

$(-1)^n \Rightarrow$ n "even" (+)
n "odd" (−)

$(-1)^{n+1} \Rightarrow$ n "even" (−)
n "odd" (+)

ex: Find the Taylor series for $\ln(x)$ centered at $a=2$

$$\sum_{n=0}^{\infty} \frac{f^{n}(2)}{n!}(x-2)^n$$

$f(x) = \ln(x)$	$f(2) = \ln(2)$	$n=0$
$f'(x) = 1/x$	$f'(2) = +(2)^{-1}$	$n=1$
$f''(x) = -x^{-2}$	$f''(2) = -(2)^{-2}$	$n=2$
$f'''(x) = 2x^{-3}$	$f'''(2) = 2(2)^{-3}$	$n=3$
$f''''(x) = -3.2\,x^{-4}$	$f^{4}(2) = -3.2(2)^{-4}$	$n=4$
$f^{5}(x) = 4.3.2\,x^{-5}$	$f^{5}(2) = 4.3.2(2)^{-5}$	$n=5$

Pattern does not apply to $n=0$

$\therefore$

Start pattern at $n=1$

$$f^{(n)} = (-1)^{n+1}(n-1)!\,.\,(2)^{-n}$$

$$\therefore \ln(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \underbrace{\ln(2)}_{n=0} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}.(n-1)!}{(2)^n} \cdot \frac{(x-2)^n}{n!}$$

$$\frac{(n-1)!}{n.(n-1)!} = \frac{1}{n}$$

$$\therefore \boxed{\ln(x) = \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\,(x-2)^n}{(2)^n.\,n}}$$

$\rightarrow$ Find radius of convergence

$$\lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+2}}.\,(x-2)^{\cancel{n+1}}}{(2)^{\cancel{n+1}}.\,(n+1)} \cdot \frac{\cancel{(2)^n}.\,n}{\cancel{(-1)^{n+1}}.\,\cancel{(x-2)^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-2)}{2} \cdot \underbrace{\frac{n}{(n+1)}}_{1} \right|$$

$$= \left| \frac{1}{2}(x-2) \right|$$

$\Rightarrow$ Need $\left| \frac{1}{2}(x-2) \right| < 1$

$$|x-2| < 2$$

$\therefore$ $R = 2$, centered at $a=2$

ex: Find the MacLaurin Series for $f(x) = \sin(x)$

$$\sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!} x^{n}$$

$n=0$	$f(x) = \sin(x)$	$f(0) = 0$
$n=1$	$f'(x) = +\cos(x)$	$f'(0) = 1$
$n=2$	$f''(x) = -\sin(x)$	$f''(0) = 0$
$n=3$	$f'''(x) = -\cos(x)$	$f'''(0) = -1$
$n=4$	$f^{4}(x) = +\sin(x)$	$f^{4}(0) = 0$

$$\therefore \sin(x) = \overset{n=0}{0} + \overset{n=1}{1.(x)^{1}} + \overset{n=2}{0} + \overset{n=3}{\frac{(-1)}{3!}(x)^{3}} + \overset{n=4}{0} + \overset{n=5}{\frac{(1)}{5!}(x)^{5}} + \overset{n=6}{0} + \overset{n=7}{\frac{(-1)}{7!}(x)^{7}} \ldots$$

Drop all zeros, change counter

$$= \overset{n=0}{(x)^{1}} + \overset{n=1}{\frac{(-1)}{3!}(x)^{3}} + \overset{n=2}{\frac{(1)}{5!}(x)^{5}} + \overset{n=3}{\frac{(-1)}{7!}(x)^{7}} + \overset{n=4}{\frac{(1)}{9!}(x)^{9}} \ldots$$

$$= \cancel{(-1)^{n}(2n+1)} \quad \boxed{\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)!}(x)^{2n+1}}$$

ex: Find the MacLaurin Series for $\frac{1}{(x-2)^2}$

$f(x) = (x-2)^{-2}$	$f(0) = \frac{1}{(-2)^2}$ $n=0$
$f'(x) = -2(x-2)^{-3}$	$f'(0) = -\frac{2}{(-2)^3}$ $n=1$
$f''(x) = +3.2(x-2)^{-4}$	$f''(0) = \frac{3.2}{(-2)^4}$ $n=2$
$f'''(x) = -4.3.2(x-2)^{-5}$	$f'''(0) = -\frac{4.3.2}{(-2)^5}$ $n=3$

$$f^n(0) = \frac{(n+1)!}{2^{n+2}}$$

<u>or</u>

$$f^n(0) = \frac{(-1)^n \cdot (n+1)!}{(-2)^{n+2}} = \frac{\cancel{(-1)^n} \cdot (n+1)!}{\cancel{(-1)^{n+2}} \cdot (2)^{n+2}}$$

$$\therefore \frac{1}{(x-2)^2} = \sum_{n=0}^{\infty} \frac{(n+1)!}{(2)^{n+2}} \cdot \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{(n+1)}{2^{n+2}} \cdot x^n$$

→ Find R

$$\lim_{n\to\infty} \left| \frac{(n+2) \cdot x^{\cancel{n}+1}}{2^{\cancel{n}+3}} \cdot \frac{2^{\cancel{n}+2}}{(n+1) \cdot \cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} \left| \underbrace{\frac{(n+2)}{(n+1)}}_{1} \cdot \frac{1}{2} \cdot x \right|$$

$$= \left| \tfrac{1}{2} x \right|$$

$\Rightarrow$ Need $\left| \tfrac{1}{2} x \right| < 1$

$|x| < 2$ $\therefore R = 2$, centered at $a = 0$

ex: Find the MacLaurin Series for $\sin(\pi x)$

$f(x) = \sin(\pi x)$	$f(0) = 0$	$n=0$
$f'(x) = \pi \cdot \cos(\pi x)$	$f'(0) = \pi$	$n=1$
$f''(x) = -\pi^2 \sin(\pi x)$	$f''(0) = 0$	$n=2$
$f'''(x) = -\pi^3 \cos(\pi x)$	$f'''(0) = -\pi^3$	$n=3$
$f^4(x) = \pi^4 \sin(\pi x)$	$f^4(0) = 0$	$n=4$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (x)^n$$

$$\therefore \sin(\pi x) = \overset{n=0}{0} + \overset{1}{\pi x} + \overset{2}{0} + \overset{3}{\frac{(-\pi^3)}{3!} x^3} + \overset{4}{0} + \overset{5}{\frac{(\pi^5)}{5!} x^5} + \cdots$$

change counter

$$= \overset{n=0}{\pi x} + \overset{n=1}{\frac{(-\pi^3)}{3!} x^3} + \overset{n=2}{\frac{\pi^5}{5!} x^5} + \overset{n=3}{\frac{(-\pi^7)}{7!} x^7} + \cdots$$

$$\therefore \boxed{\sin(\pi x) = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{(2n+1)}}{(2n+1)!} \cdot x^{(2n+1)}}$$

Integrative Activity - Rectilinear Motion

Def'n: Motion along a straight line is classified as "Rectilinear Motion"

CAL 2 CAL 1

Position fcn: $s(t) = \int v(t)\,dt$ $\qquad v(t) = s'(t) = \frac{ds}{dt}$

Velocity fcn: $v(t) = \int a(t)\,dt$

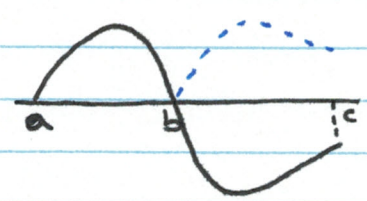
Acceleration fcn: $a(t) = \frac{dv}{dt}$

Speed: $|v(t)|$

FTC-II

Displacement: from t_1 to t_2 $\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$
$(s(t_2) - s(t_1))$

Total distance travelled: From t_1 to t_2: $\int_{t_1}^{t_2} |v(t)|\,dt$



$$|v(t)| = \begin{cases} v(t) & \text{if } v(t) \geq 0 \Rightarrow [a, b] \\ -v(t) & \text{if } v(t) < 0 \Rightarrow (b, c] \end{cases}$$

$$\therefore \int_a^c |v(t)|\,dt = \int_a^b v(t)\,dt - \int_b^c v(t)\,dt$$

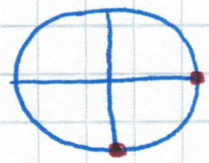
substract when neg.

ex: Let $v(t) = \cos\left(\frac{\pi}{2}t\right)$ [cm/s]

(a) Find displacement over interval $[0,3]$

$$\text{displacement} = \int_0^3 \cos\left(\frac{\pi}{2}t\right)dt = \frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right)\Big|_0^3$$

$$= \frac{2}{\pi}\Big(\underbrace{\sin\left(\frac{3\pi}{2}\right)}_{-1} - \underbrace{\sin(0)}_{0}\Big)$$



$$= \boxed{-\frac{2}{\pi}}$$

(b) Find total distance travelled on $[0,3]$

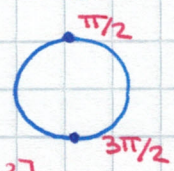
$$\int_0^3 \left|\cos\left(\frac{\pi}{2}t\right)\right| dt$$

When is $\cos\left(\frac{\pi}{2}t\right)$ pos or neg? value must be between $[0,3]$

$$\cos\left(\frac{\pi}{2}t\right) = 0$$

$$t = 1, 3, 5, \ldots$$

not between $[0,3]$



	0		1		3
$\cos\left(\frac{\pi}{2}t\right)$		+		−	

$$\therefore \left|\cos\left(\frac{\pi}{2}t\right)\right| = \begin{cases} \cos\left(\frac{\pi}{2}t\right) & \text{on } [0,1] \\ \cos\left(\frac{\pi}{2}t\right) & \text{on } (1,3] \end{cases}$$

$$\therefore \int_0^3 \left|\cos\left(\frac{\pi}{2}t\right)\right| dt = \int_0^1 \cos\left(\frac{\pi}{2}t\right)dt - \int_1^3 \cos\left(\frac{\pi}{2}t\right)dt$$

$$= \frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right)\Big|_0^1 - \frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right)\Big|_1^3$$

$$= \frac{2}{\pi}\Big(\underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} - \underbrace{\sin(0)}_{0}\Big) - \frac{2}{\pi}\Big(\underbrace{\sin\left(\frac{3\pi}{2}\right)}_{-1} - \underbrace{\sin\left(\frac{\pi}{2}\right)}_{1}\Big)$$

$$= \frac{2}{\pi}(1) - \frac{2}{\pi}(-2)$$

$$= \boxed{6/\pi}$$

Integrative Activity - Exponential Growth/Decay

<u>Theorem</u>: If $y(t)$ is a quantity at time t and the rate of change of y is proportional to $y(t)$, then:

$$\boxed{\frac{dy}{dt} = K \cdot y(t)}$$

- $\boxed{\frac{dy}{dt}}$ is always changing
  → it's the rate of change at instant t
- $\boxed{K}$ is a constant
  → growth factor
  → percentage of population (if pop. increase ∴ X% of bigger number)

consider solving this DE:

$$\int \frac{1}{y}\, dy = K \int dt$$

$$\ln|y| = K.(t) + C$$

$$|y| = e^{Kt + C}$$

$$y = \pm e^{Kt + C}$$

$$y = \pm e^{C} . e^{Kt}$$

Let $e^{C} = A$

$$y = \pm A . e^{Kt}$$

↳ for this application, only consider positive

$$\therefore \boxed{y = Ae^{Kt}}$$

Consider $y(0) = A . e^{K(0)} = A$

$\therefore A = y_0$ ⇒ initial quantity.

$$\therefore \boxed{y = y_0 . e^{Kt}}$$

ex: Suppose the initial population of 10000 bacteria grows exponentially at a rate of 1% per hour and that $y(t) = y$ is the number of bacteria present after t hours.

K (growth factor)

$K = 0.01$
$y_0 = 10\,000$

(a) What is the differential equation that models population growth of bacteria.

$$\frac{dy}{dt} = K \cdot y(t)$$

$$\frac{dy}{dt} = (0.01) \cdot y(t)$$

(b) Find the formula for $y(t)$

$$y = y_0 \cdot e^{kt}$$

$$y(t) = 10\,000 \cdot e^{(0.01)t}$$

(c) How long does it take for population to double?

$$y(t) = 2y_0$$

$$\therefore \; 2\cancel{y_0} = \cancel{y_0}\, e^{(0.01)t}$$

$$2 = e^{(0.01)t}$$

$$\ln(2) = (0.01)t$$

$$\boxed{t = \frac{\ln(2)}{0.01}}$$

[hours]

(d) How long does it take for population to reach 45000 bacteria?

$$45000 = 10000 \cdot e^{(0.01) \cdot t}$$

$$4.5 = e^{(0.01)t}$$

$$\ln(4.5) = (0.01)t$$

$$\boxed{t = \frac{\ln(4.5)}{0.01}}$$

ex: An E-coli cell divides into 2 cells every 20 min
The initial population is 60 cells

(a) Find the expression for the number of cells after t hours:

$K \Rightarrow$ Solve for

$$y(t) = y_0 e^{Kt}$$

$$y(1/3) = 2 \cdot 60 = 120 \qquad [1/3 \text{ hours} = 20 \text{ min}]$$

$$120 = 60 \cdot e^{K \cdot 1/3}$$
$$\ln(2) = K \cdot 1/3$$
$$K = 3 \cdot \ln(2)$$

$$\therefore \quad y(t) = 60 \cdot e^{3 \cdot \ln(2) t}$$

(b) Find the number of cells after 8 hours

$$y(8) = 60 \cdot e^{3\ln(2) \cdot (8)}$$
$$y(8) = 1.01 \times 10^9$$

(c) When will the population be 20 000 cells

$$20000 = 60 \cdot e^{3\ln(2)t}$$

$$\frac{1000}{3} = e^{3\ln(2) \cdot t}$$

$$\ln\left(\frac{1000}{3}\right) = 3\ln(2) \cdot t$$

$$\frac{\ln\left(\frac{1000}{3}\right)}{3\ln(2)} = t$$

$$t = 2.79 \text{ hours}$$