

2 Discrete Random Variables

2.1 Random Variables

A *random variable* is a real-valued function on a sample space¹. The random variable generally is a quantity we wish to measure; the output of the random variable depends on which element of the sample space is chosen. Random variables are designated by uppercase letters, e.g. X and Y .

In this section, we will look at discrete random variables.

A *discrete* random variable Y is a random variable which can only take on a finite or countable set of distinct values.

Here are some examples of discrete random variables:

1. The number of voters in Rhode Island who prefer Hillary Clinton.
2. The number of defective lightbulbs out of a shipment of 1000 lightbulbs.
3. The number of times I play a slot machine in Las Vegas until I win.

We will use the following simple example to illustrate features of discrete random variables.

Example. Let \mathcal{S} be the sample space representing the flip of two fair coins. Let Y be the number of heads flipped. Then Y is a discrete random variable, since it can only have the values 0, 1, or 2. We can illustrate it graphically below.

	H	T
H	2	1
T	1	0

Uppercase letters, such as Y , are used to designate random variables. We use lowercase letters, such as y , to designate a value that a random variable can take. The expression $(Y = y)$ is shorthand for the set of all points in our sample space \mathcal{S} for which the random variable Y outputs the value y . Since $(Y = y)$ is a subset of \mathcal{S} , it is an event in our sample space. In the two-coin-toss problem, for example, the possible values of Y are 0, 1, and 2, so we have:

- $(Y = 0) = \{(T, T)\}$
- $(Y = 1) = \{(H, T), (T, H)\}$
- $(Y = 2) = \{(H, H)\}$

¹It's not really a variable at all, but we are stuck with the terminology.—

Since $(Y = y)$ is an event in our sample space, we can talk about its probability, i.e. $\mathbb{P}(Y = y)$. In fact, the point of random variables is to do just this!

Probability of a discrete random variable

The probability that a discrete random variable Y takes the value y , denoted $\mathbb{P}(Y = y)$, or $p(y)$ for short, is the probability of the event $(Y = y)$, the set of all points in the sample space \mathcal{S} which output the value y .

$\mathbb{P}(Y = y)$ is the sum of the probabilities of all the simple events in \mathcal{S} which are assigned the value y by the random variable Y .

Back to our two-coin-toss problem, let's look at the probabilities of the random variable Y . For convenience, here is a picture of the sample space probabilities next to a graphical representation of Y . Since we are using the discrete uniform distribution for coin tosses, each simple event in our sample space has probability $1/2$.

Probabilities of simple events in sample space \mathcal{S}		Output of random variable Y	
	H	T	
H	$1/4$	$1/4$	H
T	$1/4$	$1/4$	T

Using the rule above, we can compute the following probabilities for Y by adding up the probabilities of the underlying simple events. In this case, since we are using the discrete uniform distribution, we could also just count the number of simple events which lead to each output of Y and divide by 4, the size of the sample space.

- $\mathbb{P}(Y = 0) = 1/4$
- $\mathbb{P}(Y = 1) = 1/4 + 1/4 = 1/2$
- $\mathbb{P}(Y = 2) = 1/4$

Probability mass function

The *probability mass function (pmf)* is a function which gives the probability that a discrete random variable Y is exactly equal to a value y . The pmf can be represented as a function, table, or graph which gives the values $p(y) = \mathbb{P}(Y = y)$ for all possible values y which Y can take.

In the two-coin-flip example, we can represent the pmf of Y in a table:

y	$p(y)$
0	1/4
1	1/2
2	1/4

A discrete random variable induces a probability distribution on the sample space of all possible values the random variable can take. This is a different sample space from the original sample space. Back to our two-coin-flip example, the random variable Y induces a probability distribution on a new sample space $\mathcal{T} = \{0, 1, 2\}$. The probabilities of the sample points in \mathcal{T} are the probabilities $p(y)$ for $y = 0, 1, 2$. We can illustrate this new sample space in a picture.

Probabilities of points in sample space \mathcal{T}
induced by random variable Y

1/4	1/2	1/4
0	1	2

Often (as we shall see), we care much more about the sample space induced by a random variable than the underlying sample space. Since a discrete random variable induces a probability distribution, the following must be true.

For any discrete random variable Y :

$$0 \leq p(y) \leq 1 \text{ for all } y$$

$$\sum_{\text{all } y} p(y) = 1$$

where $p(y) = \mathbb{P}(Y = y)$. Since we have a discrete sample space, the sum is finite or countable.

Let's look at two more examples, this time involving the rolls of two six-sided dice.

Example. Let \mathcal{S} be the sample space representing the rolls of two six-sided dice. Consider the following two random variables:

1. X = the sum of the two dice
2. Y = the larger of the two die rolls (if they are the same, then it's just equal to both die rolls)

Let's look at these random variables graphically.

Two random variables on the sample space of two die rolls

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

X

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

Y