MULTI-BREATHERS IN THE DISCRETE SINE-GORDON EQUATION

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Abstract. We consider the existence and spectral stability of multi-site breathers in the discrete Klein-Gordon equation.

1. Introduction

2. Mathematical background

We will consider the discrete Klein-Gordon equation with on-site nonlinearity $f(u)$

$$\ddot{u}_n = d(\Delta_2 u)_n - f(u_n), \tag{1}$$

where $(\Delta_2 u)_n = u_{n+1} - 2u_n + u_{n-1}$ is the discrete second difference operator, and $f(u) = P'(u)$ for a smooth potential function $P(u)$.

3. Exponential dichotomy

CAPITAL LETTER CONVENTION?

$$A_0 = \begin{pmatrix} \frac{1}{d}\partial_t^2 + \frac{V''(0)}{d} + 2 & -\mathcal{I} \\ \mathcal{I} & 0 \end{pmatrix} \tag{2}$$

Lemma 1. *The eigenvalues and corresponding eigenfunctions of A_0 are given by [INSERT HERE].*

Proof. Consider the eigenvalue problem $A_0 u(t) = \lambda u(t)$ on [SPACE], where $u(t) = (v(t), w(t))^T$. We note that $\lambda = 0$ is not an eigenvalue, since that implies $v = w = 0$. The eigenvalue problem then reduces to the system of equations

$$\left(\frac{1}{d}\partial_t^2 + \frac{V''(0)}{d} + 2 \right) v(t) = \left(\lambda + \frac{1}{\lambda} \right) v(t), \quad w = \frac{1}{\lambda} v(t). \tag{3}$$

Letting $r = \lambda + \frac{1}{\lambda}$ and using the periodic boundary conditions, the set of solutions to (3) is given by

$$v_k(t) = \frac{1}{T} \exp\left(i \frac{2\pi k t}{T} \right), \quad r_k = -\frac{4k^2\pi^2}{dT^2} + \frac{V''(0)}{d} + 2 \qquad k \in \mathbb{Z}, \tag{4}$$

where the functions $v_k(t)$ have been normalized. The eigenvalues of A_0 are then given by $\{\lambda_k, \lambda_k^{-1}\}$, where

$$\lambda_k = \frac{1}{2}\left(r_k + \sqrt{r_k^2 - 4} \right), \tag{5}$$

and the corresponding eigenfunctions are

$$u_k(t) = \begin{pmatrix} v_k(t) \\ \lambda_k^{-1} v_k(t) \end{pmatrix}, \quad u_k^{-1}(t) = \begin{pmatrix} v_k(t) \\ \lambda_k w_k(t) \end{pmatrix} \qquad k \in \mathbb{Z}. \tag{6}$$

□

It follows from Lemma 1 that the spectrum of A_0 is bounded away from the unit circle provided $|r_k| > 2$ for all k. [MAYBE SOMETHING HERE ABOUT HOW THIS IS ALWAYS POSSIBLE FOR SUFFICIENTLY SMALL d]

Lemma 2. *[HYPERBOLICITY ASSUMPTION] The set of eigenfunctions $\{u_k(t), u_k(t) : k \in \mathbb{Z}\}$ of A_0 are a Riesz basis for [HILBERT SPACE], i.e. every function $y(t)$ can be written uniquely as*

$$y(t) = \sum_{k\in\mathbb{Z}} a_k u_k(t) + \sum_{k\in\mathbb{Z}} b_k u_k^{-1}(t), \tag{7}$$

where $a_k, b_k \in \mathbb{C}$ and the sum converges.

Proof. Letting $v_k(t) = \frac{1}{T}\exp\left(i\frac{2\pi kt}{T}\right)$, we note that the set $\{z_k^1(t), z_k^2(t) : k \in \mathbb{Z}\}$, where $z_k^1(t) = (v_k(t), 0)^T$ and $z_k^2(t) = (0, v_k(t))^T$ is an orthonormal basis for [HILBERT SPACE]. Therefore, there exist unique scalars $c_k, d_k \in \mathbb{C}$ such that

$$y(t) = \sum_{k\in\mathbb{Z}} c_k z_k^1(t) + \sum_{k\in\mathbb{Z}} d_k z_k^2(t),$$

and the sum converges. Then (7) follows by taking

$$a_k = \frac{1}{\lambda_k^2 - 1}\left(-c_k + \lambda_k d_k\right), \quad b_k = \frac{1}{\lambda_k^2 - 1}\left(\lambda_k^2 c_k - \lambda_k d_k\right),$$

where λ_k is defined in (5), and $\lambda_k^2 \neq 1$ by [HYPERBOLICITY ASSUMPTION]. □

Theorem 1. *The equation $u(n+1) = A_0 u(n)$ has exponential dichotomies on $\mathbb{Z}^+$ and Z^-.*

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