1 Introduction

2 Background

We consider the propagation dynamics in a multi-core fiber consisting of N waveguides arranged in a ring (Figure 1).

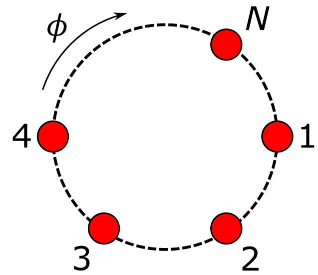

Figure 1: Schematic of N twisted fibers arranged in a ring.

Each fiber is twisted in a uniform fashion along the propagation direction z. The dynamics are given by the coupled system of equations

$$i\partial_z c_n = k\left(e^{-i\phi}c_{n+1} + e^{i\phi}c_{n-1}\right) + i\gamma_n c_n + d|c_n|^2 c_n \tag{1}$$

for $n = 1, \ldots, N$, where $c_0 = c_N$ and $c_{N+1} = c_1$ due to the circular geometry. The quantities $c_n(z)$ are the complex-valued amplitudes of each waveguide, k is the strength of the nearest-neighbor coupling, γ_n is the optical gain or loss at site n, and ϕ is a parameter representing the twist of the fibers. (See [CCSS$^+$, (2.1)] for a description of the parameters in terms of the optical waveguide system). If $\gamma_n = 0$ for all n, i.e. there is no gain or loss at each node, equation (1) becomes

$$i\partial_z c_n = k\left(e^{-i\phi}c_{n+1} + e^{i\phi}c_{n-1}\right) + i\gamma_n c_n + d|c_n|^2 c_n, \tag{2}$$

which is Hamiltonian with energy given by

$$H = \sum_{n=1}^{N} k(c_{n+1}c_n^* e^{-i\phi} + c_n c_{n+1}^* e^{i\phi}) + \frac{d}{2}|c_n|^4. \tag{3}$$

We will consider the Hamiltonian case here, and will comment on the case with loss/gain at the end.

We are interested in standing wave solutions to (2), which are bound states of the form

$$c_n = a_n e^{i(\omega z + \theta_n)}, \tag{4}$$

where $a_n \in \mathbb{R}$, $\theta_n \in (-\pi/2, \pi/2]$, and ω is the frequency of oscillation. (Since a_n can be negative, we can restrict θ_n to that interval). Making this substitution and simplifying, equation (2) becomes

$$k\left(a_{n+1}e^{i((\theta_{n+1}-\theta_n)-\phi)} + a_{n-1}e^{-i((\theta_n-\theta_{n-1})-\phi)}\right) + \omega a_n + d a_n^3 = 0, \tag{5}$$

which can be written as the system of $2n$ equations

$$\begin{aligned} & k\left(a_{n+1}\cos(\theta_{n+1}-\theta_n-\phi)+a_{n-1}\cos(\theta_n-\theta_{n-1}-\phi)\right)+\omega a_n+da_n^3=0 \\ & a_{n+1}\sin(\theta_{n+1}-\theta_n-\phi)-a_{n-1}\sin(\theta_n-\theta_{n-1}-\phi)=0 \end{aligned} \tag{6}$$

by separating real and imaginary parts. We note that the the exponential terms in (5) depend only on the phase differences $\theta_{n+1}-\theta_n$ between adjacent sites. Due to the gauge invariance of (2), if c_n is solution, so is $e^{i\theta}c_n$, thus we may without loss of generality take $\theta_1=0$. If $\phi=0$, i.e. the fibers are not twisted, we can take $\theta_n=0$ for all n, and so (5) reduces to the untwisted case with periodic boundary conditions. Similarly, if we take $\phi=2\pi/N$ and $\theta_n=(n-1)\phi$ for all n, the exponential terms do not contribute, and (5) once again reduces to untwisted case. The interesting case, therefore, occurs when $0<\theta<2\pi/N$.

3 Construction of solutions

In the anti-continuum (AC) limit, which occurs when $k=0$, the sites are decoupled. Each a_n can take on the values $\{0,\pm\sqrt{-\omega/d}\}$, the phases θ_n are arbitrary, and ϕ does not contribute. The amplitudes $\sqrt{-\omega/d}$ are real if d and ω have opposite signs.

We construct solutions to (6) by parameter continuation from the AC limit with no twist using AUTO. As an initial condition, we choose $a=(\sqrt{-\omega/d},0,\ldots,0)$ (a single excited site) and $\theta_n=0$ for all n. We also take $\phi=0$. We first continue in the coupling parameter k, and then, for fixed k, we continue in the twist parameter ϕ. In doing this, we observe that the solutions have the following symmetry:

$$\begin{aligned} a_k &= a_{N-k+2} \qquad && k=2,\ldots,M-1 \\ \theta_k &= -a_{N-k+2} \qquad && k=2,\ldots,M-1 \end{aligned} \tag{7}$$

where $M=(N/2)+1$ for N even and $M=(N+1)/2$ for N odd. See Figure 2 for an illustration of these symmetry relations for $N=6$ and $N=7$.

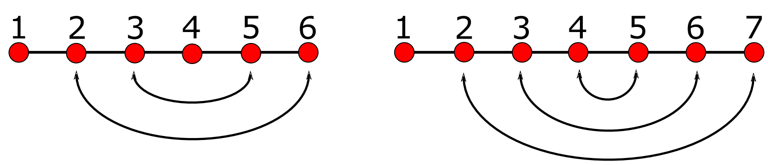

Figure 2: Schematic of symmetry relationship between nodes for $N=6$ and $N=7$. For nodes connected with arrows, the amplitudes a_k are the same and the phases θ_k are opposite.

For N even, node M is the node directly across the ring from node 1, and $\theta_M=0$. For all N, $\theta_1=0$. Figure 3 shows an example of a standing wave solution produced by numerical parameter continuation for $N=6$. The right panel illustrates the symmetry relations (7) among the amplitudes a_k.

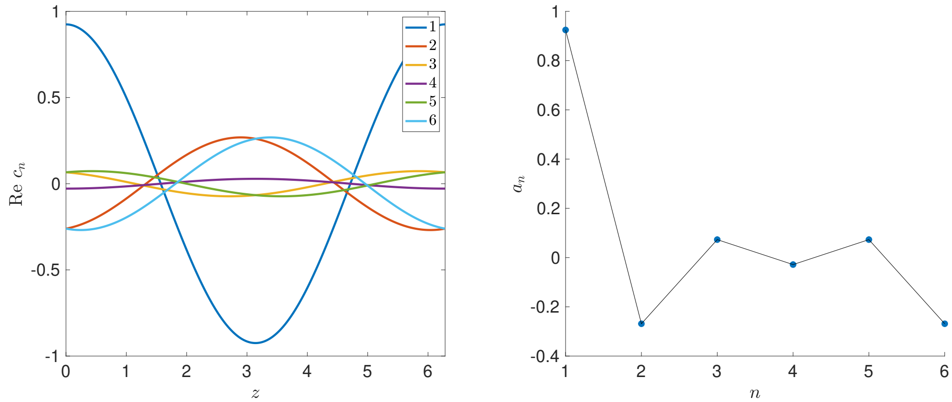

Figure 3: Standing wave solution for $N = 6$, $\omega = 1$, and $\phi = 0.25$. Left is real part of solution c_n versus z for each node over a full period (2π), right is amplitude a_n solution at each node. $k = 0.25$, $d = -1$.

3.1 N even

Numerical parameter continuation for N even, starting from a single excited node at node 1, suggests that when the twist parameter $\phi = \pi/N$, the opposite node is completely dark, i.e. has an amplitude of 0. Using the symmetries (7), as well as $\theta_0 = \theta_M = 0$, the system (6) for N even reduces to the system of equations

$$\begin{aligned}
&2ka_2\cos(\theta_2-\phi)+\omega a_1+da_1^3=0\\
&k(a_3\cos(\theta_3-\theta_2-\phi)+a_1\cos(\theta_2-\phi))+\omega a_2+da_2^3=0\\
&a_3\sin(\theta_3-\theta_2-\phi)-a_1\sin(\theta_2-\phi)=0\\
&k(a_{n+1}\cos(\theta_{n+1}-\theta_n-\phi)+a_{n-1}\cos(\theta_n-\theta_{n-1}-\phi))+\omega a_n+da_n^3=0 \qquad n=3,\ldots,M-2\\
&a_{n+1}\sin(\theta_{n+1}-\theta_n-\phi)-a_{n-1}\sin(\theta_n-\theta_{n-1}-\phi)=0 \qquad n=3,\ldots,M-2\\
&k(a_M\cos(-\theta_{M-1}-\phi)+a_{M-2}\cos(\theta_{M-1}-\theta_{M-2}-\phi))+\omega a_{M-1}+da_{M-1}^3=0\\
&a_M\sin(-\theta_{M-1}-\phi)-a_{M-2}\sin(\theta_{M-1}-\theta_{M-2}-\phi)=0\\
&2ka_{M-1}\cos(\theta_{M-1}+\phi)+\omega a_M+da_M^3=0
\end{aligned} \tag{8}$$

If $a_M = 0$, then it follows from (8) that $a_n = 0$ for all n unless

$$\begin{aligned}
&\cos(\theta_{M-1}+\phi)=0\\
&\sin(\theta_n-\theta_{n-1}-\phi)=0 \quad n=3,\ldots,M-1\\
&\sin(\theta_2-\phi)=0
\end{aligned} \tag{9}$$

One solution to this is

$$\begin{aligned}
&\theta_{M-1}+\phi=\pi/2\\
&\theta_n-\theta_{n-1}-\phi=0 \quad n=3,\ldots,M-1\\
&\theta_2-\phi=0
\end{aligned} \tag{10}$$

from which it follows that we can have a single dark node when $\phi = \pi/N$, which agrees with the numerical results. For this case, $a_M = 0$, and (8) reduces to the simpler system of equations

$$\begin{aligned} &2ka_2 + \omega a_1 + da_1^3 = 0 \\ &k\left(a_{n+1} + a_{n-1}\right) + \omega a_n + da_n^3 = 0 \quad n = 2, \dots, M-2 \\ &ka_{M-2} + \omega a_{M-1} + da_{M-1}^3 = 0 \end{aligned} \tag{11}$$

For the full solution, $a_M = 0$, $a_{M+k} = a_{M-k}$ for $k = 1, \dots, M-2$, $\theta_0 = 0$, $\theta_M = 0$, $\theta_n = (n-1)\phi$ for $n = 2, \dots, M-1$, and $\theta_{M+k} = -\theta_{M-k}$ for $k = 1, \dots, M-2$.

This system of equations is of the form $F(a,k) = 0$, where $a = (a_1, \dots, a_{M-1})$. $F(\tilde{a}, 0) = 0$, where $\tilde{a} = (\sqrt{-\omega/d}, 0, \dots, 0)$. Since $D_F(\tilde{a}, 0) = \text{diag}(-2\omega, \omega, \dots, \omega)$, which is invertible for $\omega \neq 0$, the system (11) has a solution for sufficiently small k by the implicit function theorem. Figure 4 shows this solution for $N = 6$. This observation of a dark node for $N = 6$ when $\phi = \pi/6$ agrees with what was shown in [CCSS+].

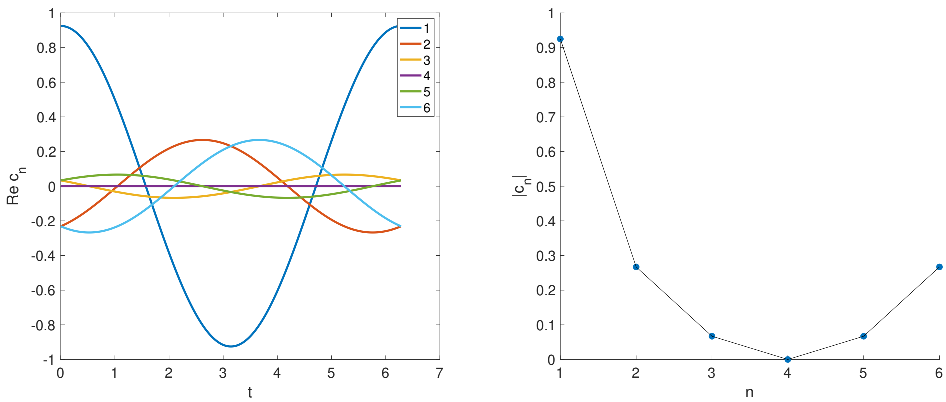

Figure 4: Standing wave solution for $N = 6$ and $\phi = \pi/6$. Left is real part of solution for each node, right is absolute value of solution at each node (this is constant in t). Node 1 has maximum amplitude, and node 4 is a dark node. $\omega = 1$, $k = 0.25$, $d = -1$.

We note that we can start with more excited nodes at the AC limit, but these do not appear to be stable.

3.2 N odd

For N odd, using the symmetries above, the system (6) reduces to the system of equations

$$
\begin{aligned}
&2ka_2\cos(\theta_2-\phi)+\omega a_1+da_1^3=0 \\
&k\left(a_3\cos(\theta_3-\theta_2-\phi)+a_1\cos(\theta_2-\phi)\right)+\omega a_2+da_2^3=0 \\
&a_3\sin(\theta_3-\theta_2-\phi)-a_1\sin(\theta_2-\phi)=0 \\
&k\left(a_{n+1}\cos(\theta_{n+1}-\theta_n-\phi)+a_{n-1}\cos(\theta_n-\theta_{n-1}-\phi)\right)+\omega a_n+da_n^3=0 \quad n=3,\dots,M-1 \\
&a_{n+1}\sin(\theta_{n+1}-\theta_n-\phi)-a_{n-1}\sin(\theta_n-\theta_{n-1}-\phi)=0 \quad n=3,\dots,M-1 \\
&k(a_M\cos(-2\theta_M-\phi)+a_{M-1}\cos(\theta_M-\theta_{M-1}-\phi))+\omega a_M+da_M^3=0 \\
&a_M\sin(-2\theta_M-\phi)-a_{M-1}\sin(\theta_M-\theta_{M-1}-\phi)=0
\end{aligned}
\tag{12}
$$

For this symmetry, we can have a solution with a dark node at a_1. In this case, it will be opposite a pair of bright nodes at a_M and a_{M+1} with the same amplitude. We observe this as well using numerical parameter continuation. If $a_1=0$, then it follows from (12) that $a_n=0$ for all n unless

$$
\begin{aligned}
&\cos(\theta_2-\phi)=0 \\
&\sin(\theta_n-\theta_{n-1}-\phi)=0 \quad n=3,\dots,M-1 \\
&\sin(2\theta_M+\phi)=0
\end{aligned}
\tag{13}
$$

One solution to this is

$$
\begin{aligned}
&\theta_2-\phi=-\pi/2 \\
&\theta_n-\theta_{n-1}-\phi=0 \quad n=3,\dots,M-1 \\
&2\theta_M+\phi=0
\end{aligned}
\tag{14}
$$

from which it follows that we can have a single dark node when $\phi=\pi/N$. This is the same condition as for N even case, and it agrees with the numerical results. For this case, $a_1=0$, and (12) reduces to the simpler system of equations

$$
\begin{aligned}
&ka_3+\omega a_2+da_2^3=0 \\
&k(a_{n+1}+a_{n-1})+\omega a_n+da_n^3=0 \quad n=3,\dots,M-1 \\
&k(a_M+a_{M-1})+\omega a_M+da_M^3=0
\end{aligned}
\tag{15}
$$

Figure 5 shows this solution for $N=7$.

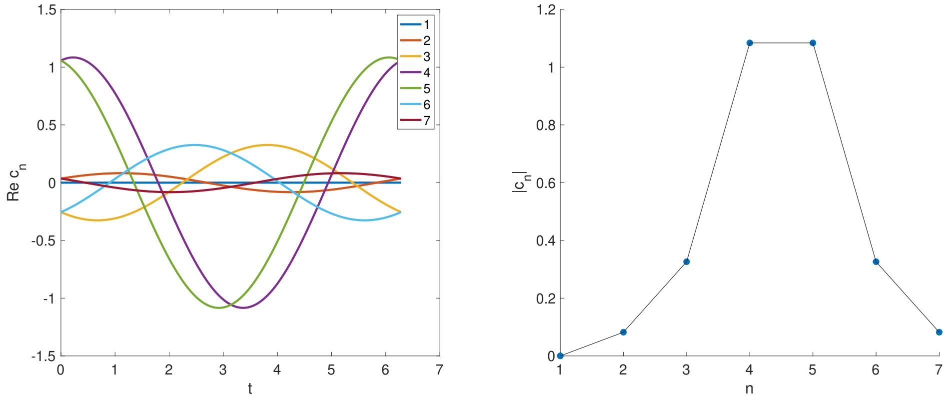

Figure 5: Standing wave solution for $N = 7$ and $\phi = \pi/7$. Left is real part of solution for each node, right is absolute value of solution at each node (this is constant in t). Nodes 4 and 5 have equal and maximum amplitude, and node 1 is a dark node. $\omega = 1$, $k = 0.25$, $d = -1$.

For the full solution, $a_1 = 0$, $a_{M+k} = a_{M-k+}$ for $k = 1, \dots, M-1$, $\theta_0 = 0$, $\theta_n = (n-1)\phi - \pi/2$ for $n = 2, \dots, M$, and $\theta_{M+k} = -\theta_{M-k+1}$ for $k = 1, \dots, M-1$.

This system of equations is again of the form $F(a, k) = 0$, where $a = (a_2, \dots, a_M)$. $F(\tilde{a}, 0) = 0$, where $\tilde{a} = (0, \dots, 0, \sqrt{-\omega/d}, 0)$. Since $D_F(\tilde{a}, 0) = \text{diag}(\omega, \dots, \omega, -2\omega)$, which is invertible for $\omega \neq 0$, the system (15) has a solution for sufficiently small k by the implicit function theorem.

4 Stability

The linearization about a standing wave solution $c_n = a_n e^{i(\omega z + \theta_n)} = (v_n + iw_n)e^{i\omega z}$ is the $2N \times 2N$ block matrix

$$A(\phi) = k \begin{pmatrix} S & C \\ -C & S \end{pmatrix} + \omega \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} - \begin{pmatrix} \text{diag}(2v_n w_n) & \text{diag}(v_n^2 + 3w_n^2) \\ -\text{diag}(3v_n^2 + w_n^2) & -\text{diag}(2v_n w_n) \end{pmatrix}$$

where each block is $N \times N$, C is the periodic banded matrix with $\cos\phi$ on the first upper and lower diagonals, and S is the periodic banded matrix with $\sin\phi$ on the first lower diagonal and $-\sin\phi$ on the first upper diagonal.

For spectral stability, all solutions generated this way (from AC limit with single excited node or the solutions with a single dark node) are spectrally neutrally stable, for both N even and N odd. In other words, the spectrum is purely imaginary. In particular, this is true for the two case described above with a single dark node. This is verified by timestepping for a perturbations of the standing wave solutions. Results of this for $N = 6$ is shown in Figure 6. Similar results are obtained for $N = 7$

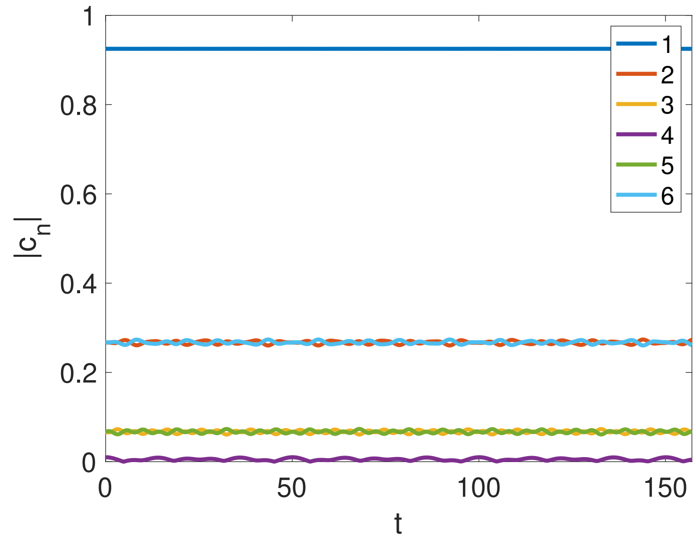

Figure 6: $|c_n|$ versus t. Solution with $N = 6$ with dark node, perturbed by adding 0.01 to initial condition at dark node. RK4 for timestepping, $k = 0.25$, $d = -1$.

References

[CCSS+] Claudia Castro-Castro, Yannan Shen, Gowri Srinivasan, Alejandro B Aceves, and Panayotis G Kevrekidis, *Light dynamics in nonlinear trimers and twisted multicore fibers* (2016), 11 (en).