Multi-core waveguides composed of N twisted fibers arranged in a ring, propagation dynamics in setting of no gain/loss described by

$$i\partial_z c_n = k\left(e^{-i\phi}c_{n+1} + e^{i\phi}c_{n-1}\right) + d|c_n|^2 c_n, \tag{1}$$

which is [CCSS$^+$, (2.1)]. The degree of twisting is represented by ϕ. As in DNLS, we are interested in standing wave solutions, which are bound states that can be written in the form

$$c_n = a_n e^{i(\omega z + \theta_n)}, \tag{2}$$

where $a_n \in \mathbb{R}$ and $\theta_n \in (-\pi/2, \pi/2]$. (Allowing a_n to be negative lets us restrict θ_n in this way). Making this substitution, equation (1) becomes (after simplification)

$$k\left(a_{n+1}e^{i((\theta_{n+1}-\theta_n)-\phi)} + a_{n-1}e^{-i((\theta_n-\theta_{n-1})-\phi)}\right) + \omega a_n + d a_n^3 = 0 \tag{3}$$

Note that the the exponential terms depend only on the phase differences between adjacent sites. Separating into real and imaginary parts, we have the $2n$ equations

$$\begin{aligned} k\left(a_{n+1}\cos(\theta_{n+1}-\theta_n-\phi) + a_{n-1}\cos(\theta_n-\theta_{n-1}-\phi)\right) + \omega a_n + d a_n^3 &= 0 \\ a_{n+1}\sin(\theta_{n+1}-\theta_n-\phi) + a_{n-1}\sin(\theta_n-\theta_{n-1}-\phi) &= 0 \end{aligned} \tag{4}$$

Due to the gauge invariance of (1), we may without loss of generality take $\theta_1 = 0$. If $\phi = 0$ (no twist), we can take $\theta_n = 0$ for all n, which is the untwisted case with perioidic boundary conditions. Similarly, if we take $\phi = 2\pi/N$ and $\theta_n = (n-1)\phi$ for $n = 1, \dots, N$, the "twist" terms in (3) do not contribute, and the magnitudes a_n are the same as in the untwisted case. The more interesting case is when $0 < \theta < 2\pi/N$.

In the AC limit ($k = 0$), the sites are decoupled: each a_n can take on any of the values $0, \pm\sqrt{\omega}$, the phases θ_n can take on any value, and ϕ does not contribute. To construct solutions, we use AUTO for parameter continuation from the AC limit. For the an initial condition, we take $a = (1, 0, \dots, 0)$ (only one excited site) with $\theta_n = 0$ for all n and $\phi = 0$. We first continue in the coupling parameter k, and then, for fixed k, we continue in the twist parameter ϕ. In doing this, we find that the solutions have the following symmetry:

$$\begin{aligned} a_k &= a_{N-k+2} & k &= 2, \dots, M-1 \\ \theta_k &= -a_{N-k+2} & k &= 2, \dots, M-1 \end{aligned} \tag{5}$$

where $M = (N/2) + 1$ for N even and $M = (N+1)/2$ for N odd. In addition, for N even, $\theta_M = 0$.

For N even, using these symmetries, as well as $\theta_0 = \theta_M = 0$, (4) reduces to the system of equations

$$\begin{aligned}
&2k a_2\cos(\theta_2-\phi) + \omega a_1 + d a_1^3 = 0 \\
&k\left(a_3\cos(\theta_3-\theta_2-\phi) + a_1\cos(\theta_2-\phi)\right) + \omega a_2 + d a_2^3 = 0 \\
&a_3\sin(\theta_3-\theta_2-\phi) + a_1\sin(\theta_2-\phi) = 0 \\
&k\left(a_{n+1}\cos(\theta_{n+1}-\theta_n-\phi) + a_{n-1}\cos(\theta_n-\theta_{n-1}-\phi)\right) + \omega a_n + d a_n^3 = 0 \qquad n = 3, \dots, M-2 \\
&a_{n+1}\sin(\theta_{n+1}-\theta_n-\phi) + a_{n-1}\sin(\theta_n-\theta_{n-1}-\phi) = 0 \qquad n = 3, \dots, M-2 \\
&k\left(a_M\cos(-\theta_{M-1}-\phi) + a_{M-2}\cos(\theta_{M-1}-\theta_{M-2}-\phi)\right) + \omega a_{M-1} + d a_{M-1}^3 = 0 \\
&a_M\sin(-\theta_{M-1}-\phi) + a_{M-2}\sin(\theta_{M-1}-\theta_{M-2}-\phi) = 0 \\
&2k a_{M-1}\cos(\theta_{M-1}+\phi) + \omega a_M + d a_M^3 = 0
\end{aligned} \tag{6}$$

From the numerical parameter continuation, we notice that when $\phi = \pi/N$, node M (the nod) opposite the node with maximum excitation) has an amplitude of 0, i.e. is a dark node. If $a_M = 0$, then it follow from (6) that $a_n = 0$ for all n unless

$$\begin{aligned} &\theta_{M-1} + \phi = \pi/2 \\ &\theta_n - \theta_{n-1} - \phi = 0 \quad n = 3, \dots, M-1 \\ &\theta_2 - \phi = 0 \end{aligned} \tag{7}$$

from which it follows that $\phi = \pi/N$, which agrees with the numerical results. For this case, $a_M = 0$, $\theta_n = (n-1)\phi$, and (6) reduces to the simpler system of equations

$$\begin{aligned} &2ka_2 + \omega a_1 + da_1^3 = 0 \\ &k\left(a_{n+1} + a_{n-1}\right) + \omega a_n + da_n^3 = 0 \quad n = 2, \dots, M-2 \\ &ka_{M-2} + \omega a_{M-1} + da_{M-1}^3 = 0 \end{aligned} \tag{8}$$

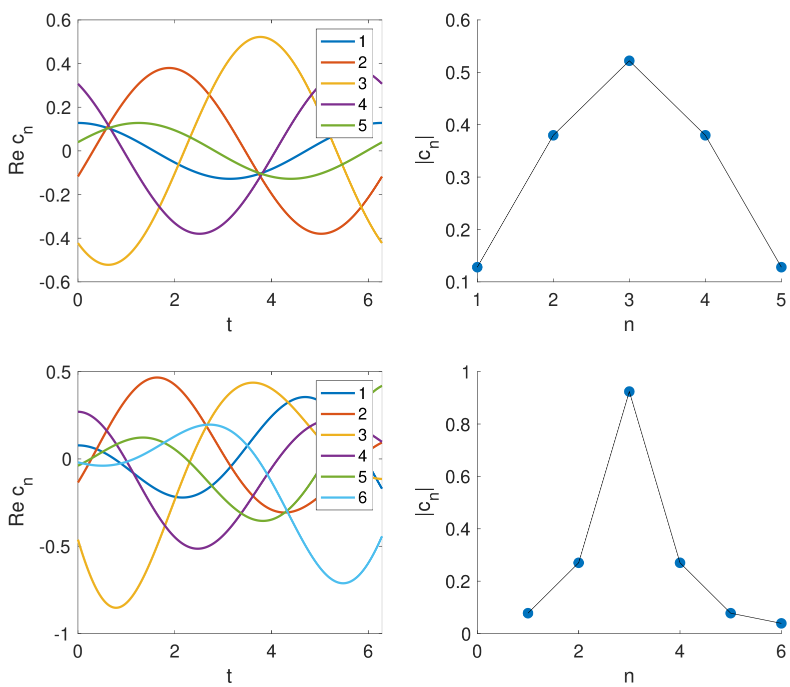

Figure 1: Examples of standing wave solutions for $N = 5$ (top) and $N = 6$ (bottom) obtained by parameter continuation with AUTO from AC limit. Left is real part of solution at each node, right is absolute value of solution at each node (this is constant in t). $\omega = 1$, $k = 0.25$.

Figure 1 shows examples of these standing wave solutions starting at the AC limit with only one node excited to $\sqrt{\omega}$.

We can also obtain solutions which are always 0 at one node. For N odd, let $M = (N+1)/2$ be the label of the "center" node. (Since we are on a ring, the nodes are not distinguished, so this is purely for notation convenience.) We will take $a_M = 0$. Then, by symmetry, we have $a_n = -a_{N-n+1}$ for $n = 1, \dots, M-1$. This reduces (3) to the system of M equations (with the last one trivial).

$$\begin{aligned} &k\left(-a_1 + a_2\right) + \omega a_1 - a_1^3 = 0 \\ &k\left(a_{n+1} + a_{n-1}\right) + \omega a_n - a_n^3 = 0 \quad n = 2, \dots, M-1 \\ &a_M = 0 \end{aligned} \tag{9}$$

This can be solved numerically (Figure 2).

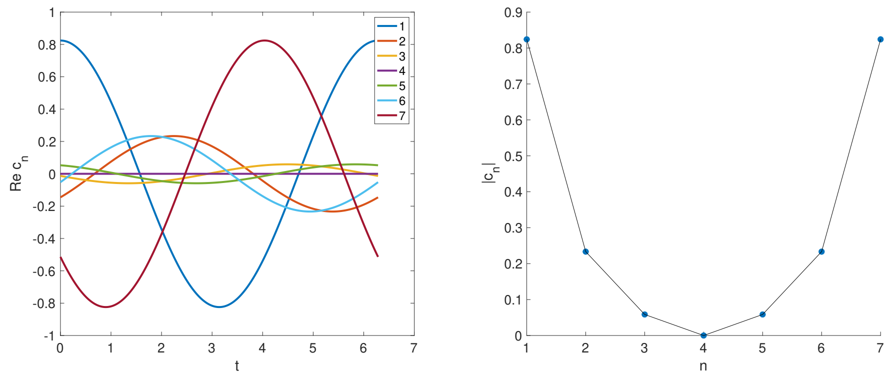

Figure 2: Standing wave solution for $N = 7$ with a hole at $n = 4$. $\omega = 1$, $k = 0.25$.

For N even, let $M = N/2$. If we take $a_1 = 0$, then, by symmetry, the opposite node in the ring must also be 0, i.e. $a_{M+1} = 0$. We also have the symmetry relations $a_n = -a_{N-n+2}$ for $n = 2, \dots, M$. This reduces to the following system of $M+1$ equations (the first and last one are trivial).

$$\begin{aligned} &a_1 = 0 \\ &k\left(a_{n+1} + a_{n-1}\right) + \omega a_n - a_n^3 = 0 \quad n = 2, \dots, M \\ &a_{M+1} = 0 \end{aligned} \tag{10}$$

This can also be solved numerically (Figure 3).

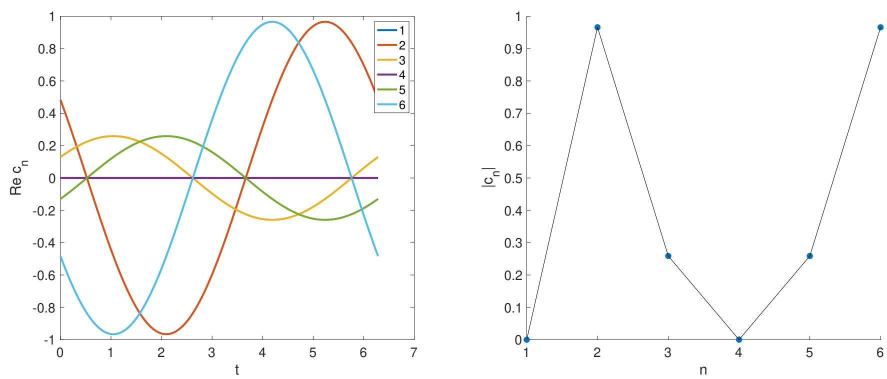

Figure 3: Standing wave solution for $N = 6$ with holes at $n = 1$ and $n = 4$. $\omega = 1$, $k = 0.25$.

The linearization about a standing wave solution $c_n = a_n e^{i(\omega z + \theta_n)} = (v_n + i w_n) e^{i\omega z}$ is the $2N \times 2N$ block matrix

$$A(\phi) = k \begin{pmatrix} S & C \\ -C & S \end{pmatrix} + \omega \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} - \begin{pmatrix} \operatorname{diag}(2v_n w_n) & \operatorname{diag}(v_n^2 + 3w_n^2) \\ -\operatorname{diag}(3v_n^2 + w_n^2) & -\operatorname{diag}(2v_n w_n) \end{pmatrix}$$

where each block is $N \times N$, C is the periodic banded matrix with $\cos\phi$ on the first upper and lower diagonals, and S is the periodic banded matrix with $\sin\phi$ on the first lower diagonal and $-\sin\phi$ on the first upper diagonal. For the special case where $\phi = 0$ (in which case the standing wave solution becomes $c_n = v_n e^{i\omega z}$,

$$A(0) = k \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix} + \omega \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} - \begin{pmatrix} 0 & \operatorname{diag}(v_n^2) \\ -\operatorname{diag}(3v_n^2) & 0 \end{pmatrix}$$

where B is the periodic banded matrix with 1 on the first upper and lower diagonals. For $\phi = 2\pi/N$ (which is the case we are considering), the eigenvalues of $A(\phi)$ and $A(0)$ are the same (when computed numerically), which suggests that the two matrices are similar. This can probably be proved, as it is just finite dimensional linear algebra, but it might be annoying to do. There is always an eigenvalue at 0 from the gauge invariance (same as DNLS).

Here are some spectral results (computed numerically) as well as results for timestepping of perturbations of the solutions we discussed above.

1. Single peak (continued from AC limit). Spectrum is imaginary (both N even and N odd).

2. N odd, hole in center. Continued from AC limit with single peak at $n = 1$. Pair of real eigenvalues, unstable. Timestepping confirms this.

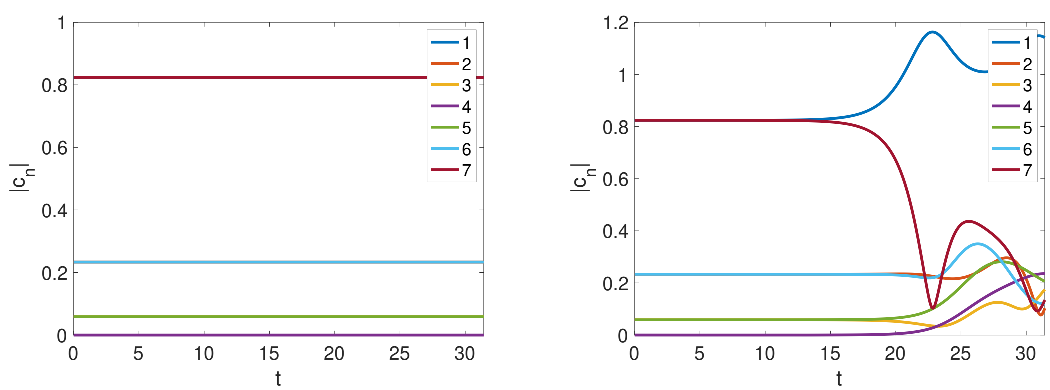

Figure 4: $|c_n|$ versus t. Standing wave solution for $N = 7$ with holes at $n = 4$. Left is unperturbed. On right we perturb by adding $1e-4$ to the hole.

3. N even, two holes in opposite postions in ring. Continued from AC limit with single peak between two holes. Spectrum is imaginary. Timestepping confirms a neutral

stability (best we can do for Hamiltonian systems). Perturbed solution remains close to unperturbed solution.

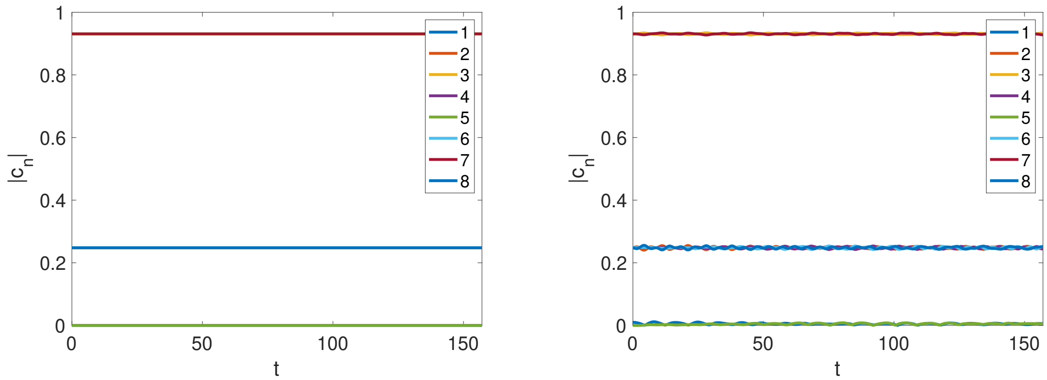

Figure 5: $|c_n|$ versus t. Standing wave solution for $N = 8$ with holes at $n = 1$ and $n = 5$. Left is unperturbed. On right we perturb by adding $1e - 2$ to the hole.

References

[CCSS+] Claudia Castro-Castro, Yannan Shen, Gowri Srinivasan, Alejandro B Aceves, and Panayotis G Kevrekidis, *Light dynamics in nonlinear trimers and twisted multicore fibers* (2016), 11 (en).