Multi-core waveguides composed of N twisted fibers arranged in a ring, propagation dynamics in setting of no gain/loss described by

$$i\partial_z c_n = k\left(e^{-i\phi}c_{n+1} + e^{i\phi}c_{n-1}\right) + d|c_n|^2 c_n, \tag{1}$$

which is [CCSS$^+$, (2.1)]. The degree of twisting is represented by ϕ. $d < 0$ is the defocusing case, and $d > 0$ is the focusing case. We consider $d < 0$ here. As in DNLS, we are interested in standing wave solutions, which are bound states that can be written in the form

$$c_n = a_n e^{i(\omega z + \theta_n)}, \tag{2}$$

where $a_n \in \mathbb{R}$ and $\theta_n \in (-\pi/2, \pi/2]$. (Allowing a_n to be negative lets us restrict θ_n in this way). Making this substitution, equation (1) becomes (after simplification)

$$k\left(a_{n+1}e^{i((\theta_{n+1}-\theta_n)-\phi)} + a_{n-1}e^{-i((\theta_n-\theta_{n-1})-\phi)}\right) + \omega a_n + d a_n^3 = 0 \tag{3}$$

Note that the the exponential terms depend only on the phase differences between adjacent sites. Separating into real and imaginary parts, we have the $2n$ equations

$$\begin{aligned} k\left(a_{n+1}\cos(\theta_{n+1}-\theta_n-\phi) + a_{n-1}\cos(\theta_n-\theta_{n-1}-\phi)\right) + \omega a_n + d a_n^3 &= 0 \\ a_{n+1}\sin(\theta_{n+1}-\theta_n-\phi) - a_{n-1}\sin(\theta_n-\theta_{n-1}-\phi) &= 0 \end{aligned} \tag{4}$$

Due to the gauge invariance of (1), we may without loss of generality take $\theta_1 = 0$. If $\phi = 0$ (no twist), we can take $\theta_n = 0$ for all n, which is the untwisted case with perioidic boundary conditions. Similarly, if we take $\phi = 2\pi/N$ and $\theta_n = (n-1)\phi$ for $n = 1, \dots, N$, the "twist" terms in (3) do not contribute, and the magnitudes a_n are the same as in the untwisted case. The more interesting case is when $0 < \theta < 2\pi/N$.

In the AC limit ($k = 0$), the sites are decoupled: each a_n can take on any of the values $0, \pm\sqrt{-\omega/d}$ (which is real since we are taking $d < 0$), the phases θ_n can take on any value, and ϕ does not contribute. To construct solutions, we use AUTO for parameter continuation from the AC limit. For the an initial condition, we take $a = (1, 0, \dots, 0)$ (only one excited site) with $\theta_n = 0$ for all n and $\phi = 0$. We first continue in the coupling parameter k, and then, for fixed k, we continue in the twist parameter ϕ. In doing this, we find that the solutions have the following symmetry:

$$\begin{aligned} a_k &= a_{N-k+2} & k &= 2, \dots, M-1 \\ \theta_k &= -a_{N-k+2} & k &= 2, \dots, M-1 \end{aligned} \tag{5}$$

where $M = (N/2) + 1$ for N even and $M = (N+1)/2$ for N odd. In addition, for N even, $\theta_M = 0$.

For N even, using these symmetries, as well as $\theta_0 = \theta_M = 0$, the system (4) reduces to the system of equations

$$
\begin{aligned}
&2ka_2\cos(\theta_2-\phi)+\omega a_1+da_1^3=0 \\
&k\left(a_3\cos(\theta_3-\theta_2-\phi)+a_1\cos(\theta_2-\phi)\right)+\omega a_2+da_2^3=0 \\
&a_3\sin(\theta_3-\theta_2-\phi)-a_1\sin(\theta_2-\phi)=0 \\
&k\left(a_{n+1}\cos(\theta_{n+1}-\theta_n-\phi)+a_{n-1}\cos(\theta_n-\theta_{n-1}-\phi)\right)+\omega a_n+da_n^3=0 \qquad n=3,\dots,M-2 \\
&a_{n+1}\sin(\theta_{n+1}-\theta_n-\phi)-a_{n-1}\sin(\theta_n-\theta_{n-1}-\phi)=0 \qquad n=3,\dots,M-2 \\
&k\left(a_M\cos(-\theta_{M-1}-\phi)+a_{M-2}\cos(\theta_{M-1}-\theta_{M-2}-\phi)\right)+\omega a_{M-1}+da_{M-1}^3=0 \\
&a_M\sin(-\theta_{M-1}-\phi)-a_{M-2}\sin(\theta_{M-1}-\theta_{M-2}-\phi)=0 \\
&2ka_{M-1}\cos(\theta_{M-1}+\phi)+\omega a_M+da_M^3=0
\end{aligned}
\tag{6}
$$

From the numerical parameter continuation, we see that when $\phi=\pi/N$, node M (the node opposite the node with maximum excitation) has an amplitude of 0, i.e. is a dark node. If $a_M=0$, then it follows from (6) that $a_n=0$ for all n unless

$$
\begin{aligned}
&\cos(\theta_{M-1}+\phi)=0 \\
&\sin(\theta_n-\theta_{n-1}-\phi)=0 \quad n=3,\dots,M-1 \\
&\sin(\theta_2-\phi)=0
\end{aligned}
\tag{7}
$$

One solution to this is

$$
\begin{aligned}
&\theta_{M-1}+\phi=\pi/2 \\
&\theta_n-\theta_{n-1}-\phi=0 \quad n=3,\dots,M-1 \\
&\theta_2-\phi=0
\end{aligned}
\tag{8}
$$

from which it follows that we can have a single dark node when $\phi=\pi/N$, which agrees with the numerical results. For this case, $a_M=0$, and (6) reduces to the simpler system of equations

$$
\begin{aligned}
&2ka_2+\omega a_1+da_1^3=0 \\
&k\left(a_{n+1}+a_{n-1}\right)+\omega a_n+da_n^3=0 \quad n=2,\dots,M-2 \\
&ka_{M-2}+\omega a_{M-1}+da_{M-1}^3=0
\end{aligned}
\tag{9}
$$

This system of equations is of the form $F(a,k)=0$, where $a=(a_1,\dots,a_{M-1})$. $F(\tilde{a},0)=0$, where $\tilde{a}=(\sqrt{-\omega/d},0,\dots,0)$. Since $D_F(\tilde{a},0)=\operatorname{diag}(-2\omega,\omega,\dots,\omega)$, which is invertible for $\omega\neq 0$, the system (9) has a solution for sufficiently small k by the implicit function theorem. Figure 1 shows this solution for $N=6$. This observation of a dark node for $N=6$ when $\phi=\pi/6$ agrees with what was shown in [CCSS$^+$].

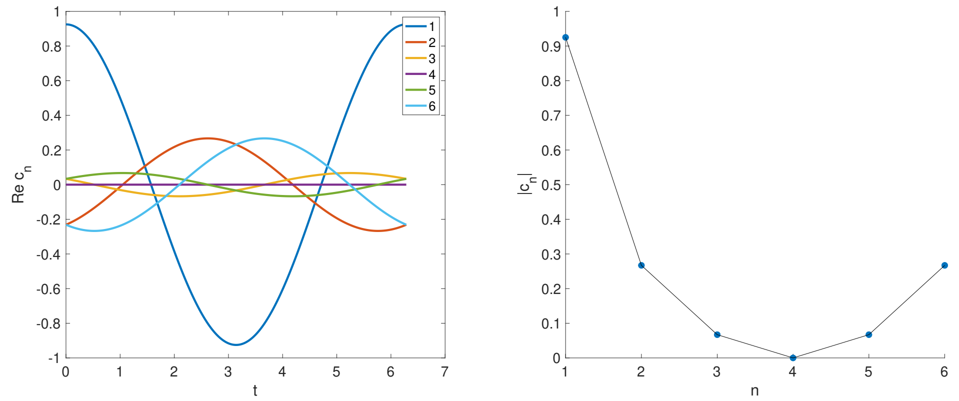

Figure 1: Standing wave solution for $N = 6$ and $\phi = \pi/6$. Left is real part of solution for each node, right is absolute value of solution at each node (this is constant in t). Node 1 has maximum amplitude, and node 4 is a dark node. $\omega = 1$, $k = 0.25$, $d = -1$.

For N odd, using the symmetries above, the system (4) reduces to the system of equations

$$\begin{aligned}
&2ka_2\cos(\theta_2-\phi)+\omega a_1+da_1^3=0\\
&k\left(a_3\cos(\theta_3-\theta_2-\phi)+a_1\cos(\theta_2-\phi)\right)+\omega a_2+da_2^3=0\\
&a_3\sin(\theta_3-\theta_2-\phi)-a_1\sin(\theta_2-\phi)=0\\
&k\left(a_{n+1}\cos(\theta_{n+1}-\theta_n-\phi)+a_{n-1}\cos(\theta_n-\theta_{n-1}-\phi)\right)+\omega a_n+da_n^3=0 \quad && n=3,\dots,M-1\\
&a_{n+1}\sin(\theta_{n+1}-\theta_n-\phi)-a_{n-1}\sin(\theta_n-\theta_{n-1}-\phi)=0 && n=3,\dots,M-1\\
&k(a_M\cos(-2\theta_M-\phi)+a_{M-1}\cos(\theta_M-\theta_{M-1}-\phi))+\omega a_M+da_M^3=0\\
&a_M\sin(-2\theta_M-\phi)-a_{M-1}\sin(\theta_M-\theta_{M-1}-\phi)=0
\end{aligned} \tag{10}$$

For this symmetry, we can have a solution with a dark node at a_1. In this case, it will be opposite a pair of bright nodes at a_M and a_{M+1} with the same amplitude. We see this occur in the numerical parameter continuation. If $a_1 = 0$, then it follows from (10) that $a_n = 0$ for all n unless

$$\begin{aligned}
&\cos(\theta_2-\phi)=0\\
&\sin(\theta_n-\theta_{n-1}-\phi)=0 \quad n=3,\dots,M-1\\
&\sin(2\theta_M+\phi)=0
\end{aligned} \tag{11}$$

One solution to this is

$$\begin{aligned}
&\theta_2-\phi=-\pi/2\\
&\theta_n-\theta_{n-1}-\phi=0 \quad n=3,\dots,M-1\\
&2\theta_M+\phi=0
\end{aligned} \tag{12}$$

from which it follows that we can have a single dark node when $\phi = \pi/N$. This is the same as for the case of N even, and it agrees with the numerical results. For this case, $a_1 = 0$, and (10) reduces to the simpler system of equations

$$
\begin{aligned}
&ka_3 + \omega a_2 + da_2^3 = 0 \\
&k(a_{n+1} + a_{n-1}) + \omega a_n + da_n^3 = 0 \qquad n = 3, \dots, M-1 \\
&k(a_M + a_{M-1}) + \omega a_M + da_M^3 = 0
\end{aligned} \tag{13}
$$

Figure 2 shows this solution for $N = 7$.

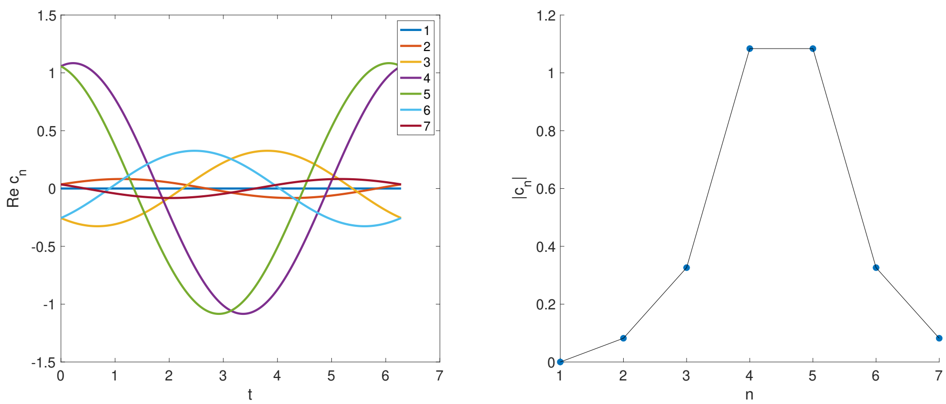

Figure 2: Standing wave solution for $N = 7$ and $\phi = \pi/7$. Left is real part of solution for each node, right is absolute value of solution at each node (this is constant in t). Nodes for and 4 have equal and maximum amplitude, and node 1 is a dark node. $\omega = 1$, $k = 0.25$, $d = -1$.

This system of equations is of the form $F(a, k) = 0$, where $a = (a_2, \dots, a_M)$. $F(\tilde{a}, 0) = 0$, where $\tilde{a} = (0, \dots, 0, \sqrt{-\omega/d}, 0)$. Since $D_F(\tilde{a}, 0) = \operatorname{diag}(\omega, \dots, \omega, -2\omega)$, which is invertible for $\omega \neq 0$, the system (13) has a solution for sufficiently small k by the implicit function theorem.

The linearization about a standing wave solution $c_n = a_n e^{i(\omega z + \theta_n)} = (v_n + i w_n) e^{i\omega z}$ is the $2N \times 2N$ block matrix

$$
A(\phi) = k \begin{pmatrix} S & C \\ -C & S \end{pmatrix} + \omega \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} - \begin{pmatrix} \operatorname{diag}(2v_n w_n) & \operatorname{diag}(v_n^2 + 3w_n^2) \\ -\operatorname{diag}(3v_n^2 + w_n^2) & -\operatorname{diag}(2v_n w_n) \end{pmatrix}
$$

where each block is $N \times N$, C is the periodic banded matrix with $\cos\phi$ on the first upper and lower diagonals, and S is the periodic banded matrix with $\sin\phi$ on the first lower diagonal and $-\sin\phi$ on the first upper diagonal.

For spectral stability, all solutions generated this way (from AC limit with single excited node) are spectrally neutrally stable, for both N even and N odd. In other words, the spectrum is purely imaginary. In particular, this is true for the two case described above with a single dark node. This is verified by timestepping for a perturbations of the standing wave solutions. Results of this for $N = 6$ is shown in Figure 3. Similar results are obtained for $N = 7$

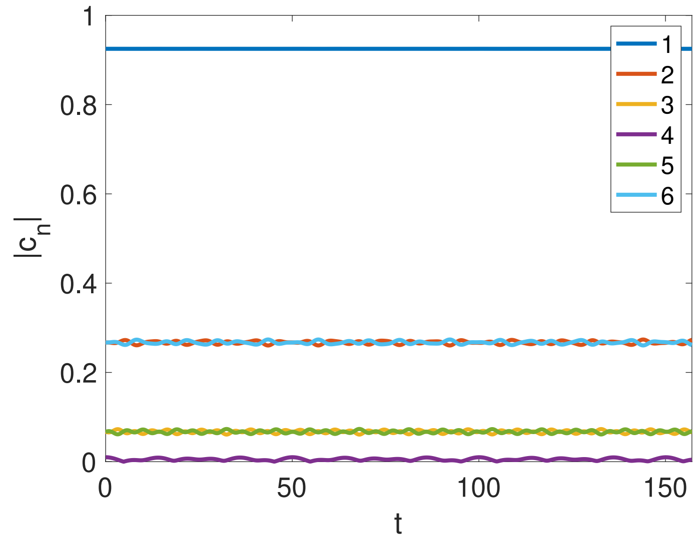

Figure 3: $|c_n|$ versus t. Solution with $N = 6$ with dark node, perturbed by adding 0.01 to initial condition at dark node. RK4 for timestepping, $k = 0.25$, $d = -1$.

References

[CCSS$^+$] Claudia Castro-Castro, Yannan Shen, Gowri Srinivasan, Alejandro B Aceves, and Panayotis G Kevrekidis, *Light dynamics in nonlinear trimers and twisted multicore fibers* (2016), 11 (en).