1 Introduction

2 Background

We consider the propagation dynamics in a multi-core fiber consisting of N waveguides arranged in a ring (Figure 1).

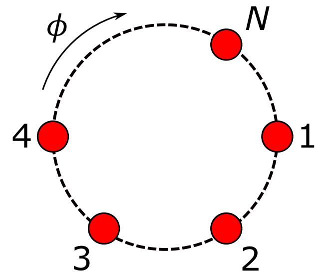

Figure 1: Schematic of N twisted fibers arranged in a ring.

Each fiber is twisted in a uniform fashion along the propagation direction z. The dynamics are given by the coupled system of equations

$$i\partial_z c_n = k\left(e^{-i\phi}c_{n+1} + e^{i\phi}c_{n-1}\right) + i\gamma_n c_n + d|c_n|^2 c_n \tag{1}$$

for $n = 1, \ldots, N$, where $c_0 = c_N$ and $c_{N+1} = c_1$ due to the circular geometry. The quantities $c_n(z)$ are the complex-valued amplitudes of each waveguide, k is the strength of the nearest-neighbor coupling, γ_n is the optical gain or loss at site n, and ϕ is a parameter representing the twist of the fibers. (See [CCSS$^+$, (2.1)] for a description of the parameters in terms of the optical waveguide system). If $\gamma_n = 0$ for all n, i.e. there is no gain or loss at each node, equation (1) becomes

$$i\partial_z c_n = k\left(e^{-i\phi}c_{n+1} + e^{i\phi}c_{n-1}\right) + i\gamma_n c_n + d|c_n|^2 c_n, \tag{2}$$

which is Hamiltonian with energy given by

$$H = \sum_{n=1}^{N} k(c_{n+1}c_n^* e^{-i\phi} + c_n c_{n+1}^* e^{i\phi}) + \frac{d}{2}|c_n|^4. \tag{3}$$

We will consider the Hamiltonian case here, and will comment on the case with loss/gain at the end.

We are interested in standing wave solutions to (2), which are bound states of the form

$$c_n = a_n e^{i(\omega z + \theta_n)}, \tag{4}$$

where $a_n \in \mathbb{R}$, $\theta_n \in (-\pi/2, \pi/2]$, and ω is the frequency of oscillation. (Since a_n can be negative, we can restrict θ_n to that interval). Making this substitution and simplifying, equation (2) becomes

$$k\left(a_{n+1}e^{i((\theta_{n+1}-\theta_n)-\phi)} + a_{n-1}e^{-i((\theta_n-\theta_{n-1})-\phi)}\right) + \omega a_n + d a_n^3 = 0, \tag{5}$$

which can be written as the system of $2n$ equations

$$\begin{aligned} &k\left(a_{n+1}\cos(\theta_{n+1}-\theta_n-\phi)+a_{n-1}\cos(\theta_n-\theta_{n-1}-\phi)\right)+\omega a_n+da_n^3=0 \\ &a_{n+1}\sin(\theta_{n+1}-\theta_n-\phi)-a_{n-1}\sin(\theta_n-\theta_{n-1}-\phi)=0 \end{aligned} \tag{6}$$

by separating real and imaginary parts. We note that the the exponential terms in (5) depend only on the phase differences $\theta_{n+1}-\theta_n$ between adjacent sites. Due to the gauge invariance of (2), if c_n is solution, so is $e^{i\theta}c_n$, thus we may without loss of generality take $\theta_1=0$. If $\phi=0$, i.e. the fibers are not twisted, we can take $\theta_n=0$ for all n, and so (5) reduces to the untwisted case with periodic boundary conditions. Similarly, if we take $\phi=2\pi/N$ and $\theta_n=(n-1)\phi$ for all n, the exponential terms do not contribute, and (5) once again reduces to untwisted case. The interesting case, therefore, occurs when $0<\theta<2\pi/N$.

3 Construction of solutions

In the anti-continuum (AC) limit, which occurs when $k=0$, the sites are decoupled. Each a_n can take on the values $\{0,\pm\sqrt{-\omega/d}\}$, the phases θ_n are arbitrary, and ϕ does not contribute. The amplitudes $\sqrt{-\omega/d}$ are real if d and ω have opposite signs.

We construct solutions to (6) by parameter continuation from the AC limit with no twist using AUTO. As an initial condition, we choose a single excited site at node 1, i.e. $a_1=\sqrt{-\omega/d}$ and $a_n=0$ for all other n; for the phases, $\theta_n=0$ for all n, and we also take $\phi=0$. (We can start with more than once excited state, but, in general, these solutions will not be stable.) We first continue in the coupling parameter k, and then, for fixed k, we continue in the twist parameter ϕ. In doing this, we observe that the solutions have the following symmetry:

$$\begin{aligned} a_k &= a_{N-k+2} \qquad && k=2,\dots,M-1 \\ \theta_k &= -a_{N-k+2} \qquad && k=2,\dots,M-1, \end{aligned} \tag{7}$$

where $M=(N/2)+1$ for N even and $M=(N+1)/2$ for N odd. See Figure 2 for an illustration of these symmetry relations for $N=6$ and $N=7$.

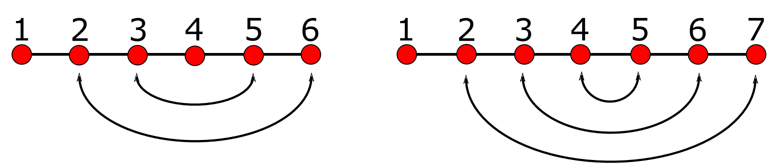

Figure 2: Schematic of symmetry relationship between nodes for $N=6$ and $N=7$. For nodes connected with arrows, the amplitudes a_k are the same and the phases θ_k are opposite.

For N even, node M is the node directly across the ring from node 1, and $\theta_M=0$. For all N, $\theta_1=0$. Figure 3 shows an example of a standing wave solution produced by numerical parameter continuation for $N=6$. The right panel illustrates the symmetry relations (7) among the amplitudes a_k.

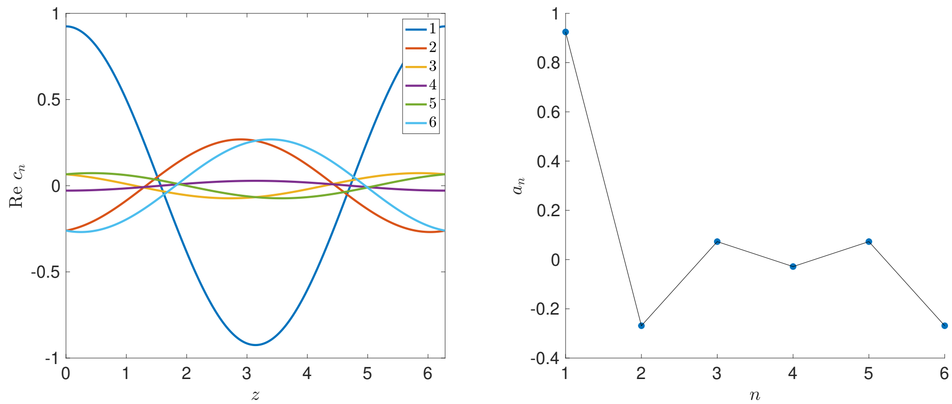

Figure 3: Standing wave solution for $N = 6$, $\omega = 1$, and $\phi = 0.25$. Left is real part of solution c_n versus z for each node over a full period (2π), right is amplitude a_n solution at each node. $k = 0.25$, $d = -1$.

3.1 N even

Numerical parameter continuation for N even, starting from a single excited node at node 1, suggests that when the twist parameter $\phi = \pi/N$, the opposite node is completely dark, i.e. has an amplitude of 0. Using the symmetries (7), when $a_M = 0$, the system (6) reduces to

$$\begin{aligned}
&2ka_2\cos(\theta_2 - \phi) + \omega a_1 + da_1^3 = 0 & \\
&k(a_{n+1}\cos(\theta_{n+1} - \theta_n - \phi) + a_{n-1}\cos(\theta_n - \theta_{n-1} - \phi)) + \omega a_n + da_n^3 = 0 & n = 2, \ldots, M-1 \\
&a_{n+1}\sin(\theta_{n+1} - \theta_n - \phi) - a_{n-1}\sin(\theta_n - \theta_{n-1} - \phi) = 0 & n = 2, \ldots, M-1 \\
&2ka_{M-1}\cos(\theta_{M-1} + \phi) = 0 & \\
&\theta_1 = \theta_M = 0. &
\end{aligned} \tag{8}$$

It follows that $a_n = 0$ for all n unless

$$\begin{aligned}
&\cos(\theta_{M-1} + \phi) = 0 \\
&\sin(\theta_n - \theta_{n-1} - \phi) = 0 \qquad n = 3, \ldots, M-1 \\
&\sin(\theta_2 - \phi) = 0.
\end{aligned} \tag{9}$$

One solution to this is

$$\begin{aligned}
&\theta_{M-1} + \phi = \pi/2 \\
&\theta_n - \theta_{n-1} - \phi = 0 \quad n = 3, \ldots, M-1 \\
&\theta_2 - \phi = 0,
\end{aligned} \tag{10}$$

from which it follows that we can have a single dark node at site M when $\phi = \pi/N$. If this is the case, the system of equations (8) reduces to the simpler system

$$\begin{aligned}
&2ka_2 + \omega a_1 + d a_1^3 = 0 \\
&k\left(a_{n+1} + a_{n-1}\right) + \omega a_n + d a_n^3 = 0 \qquad\qquad n = 2, \dots, M-2 \\
&k a_{M-2} + \omega a_{M-1} + d a_{M-1}^3 = 0.
\end{aligned} \tag{11}$$

This system is of the form $F(a,k) = 0$, where $a = (a_1, \dots, a_{M-1})$. $F(\tilde{a}, 0) = 0$, where $\tilde{a} = (\sqrt{-\omega/d}, 0, \dots, 0)$. Since $D_F(\tilde{a}, 0) = \text{diag}(-2\omega, \omega, \dots, \omega)$, which is invertible for $\omega \neq 0$, the system (11) has a solution for sufficiently small k by the implicit function theorem. Once (11) has been solved, we obtain the full solution to (6) using

$$\begin{aligned}
&a_M = 0 \\
&a_{M+k} = a_{M-k} && k = 1, \dots, M-2 \\
&\theta_0 = 0 \\
&\theta_k = (k-1)\phi && k = 2, \dots, M-1 \\
&\theta_M = 0 \\
&\theta_{M+k} = -\theta_{M-k} && k = 1, \dots, M-2.
\end{aligned}$$

Figure 4 shows this solution for $N = 6$. This observation of a dark node for $N = 6$ when $\phi = \pi/6$ agrees with what was shown in [CCSS$^+$]. Numerical parameter continuation suggests that for $N = 6$, $\omega = 1$, and $d = -1$, these standing wave solutions exist for approximately $0 \leq k \leq 0.57735$.

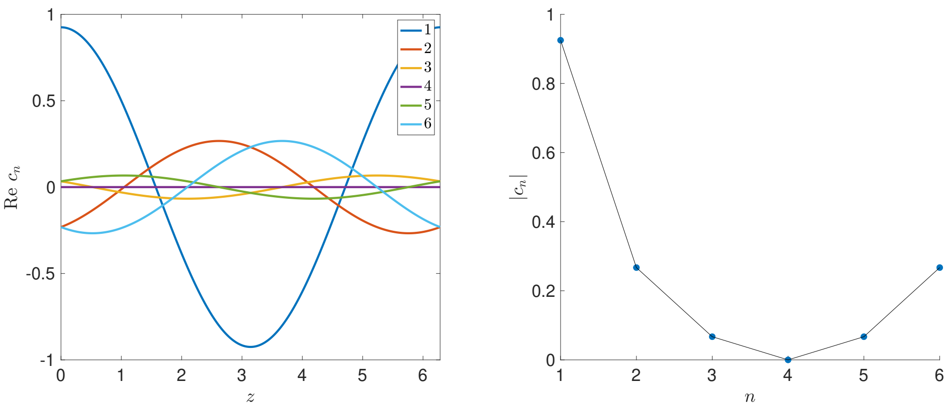

Figure 4: Standing wave solution for $N = 6$ and $\phi = \pi/6$. Left is real part of solution for each node, right is absolute value of solution at each node. Node 1 has maximum amplitude, and node 4 is a dark node. $\omega = 1$, $k = 0.25$, $d = -1$.

3.2 N odd

We can also obtain a dark node when N is odd by taking node 1 to be the dark node; in this case, the dark node will be opposite a pair of bright nodes at a_M and a_{M+1} with the same amplitude. Using the symmetries (7), when $a_1 = 0$, the system (6) reduces to

$$
\begin{aligned}
&2ka_2\cos(\theta_2-\phi)=0 \\
&ka_3\cos(\theta_3-\theta_2-\phi)+\omega a_2+da_2^3=0 \\
&a_3\sin(\theta_3-\theta_2-\phi)=0 \\
&k\left(a_{n+1}\cos(\theta_{n+1}-\theta_n-\phi)+a_{n-1}\cos(\theta_n-\theta_{n-1}-\phi)\right)+\omega a_n+da_n^3=0 \quad && n=3,\dots,M-1 \\
&a_{n+1}\sin(\theta_{n+1}-\theta_n-\phi)-a_{n-1}\sin(\theta_n-\theta_{n-1}-\phi)=0 && n=3,\dots,M-1 \\
&k(a_M\cos(-2\theta_M-\phi)+a_{M-1}\cos(\theta_M-\theta_{M-1}-\phi))+\omega a_M+da_M^3=0 \\
&a_M\sin(-2\theta_M-\phi)-a_{M-1}\sin(\theta_M-\theta_{M-1}-\phi)=0.
\end{aligned}
\tag{12}
$$

It follows that $a_n = 0$ for all n unless

$$
\begin{aligned}
&\cos(\theta_2-\phi)=0 \\
&\sin(\theta_n-\theta_{n-1}-\phi)=0 \qquad n=3,\dots,M-1 \\
&\sin(2\theta_M+\phi)=0.
\end{aligned}
\tag{13}
$$

One solution to this is

$$
\begin{aligned}
&\theta_2-\phi=-\pi/2 \\
&\theta_n-\theta_{n-1}-\phi=0 \qquad n=3,\dots,M-1 \\
&2\theta_M+\phi=0,
\end{aligned}
\tag{14}
$$

from which it follows that we can have a single dark node at a_1 when $\phi = \pi/N$. This condition for a dark node is the same as for the N even case. For this case, (12) reduces to the simpler system of equations

$$
\begin{aligned}
&ka_3+\omega a_2+da_2^3=0 \\
&k(a_{n+1}+a_{n-1})+\omega a_n+da_n^3=0 \qquad n=3,\dots,M-1 \\
&k(a_M+a_{M-1})+\omega a_M+da_M^3=0.
\end{aligned}
\tag{15}
$$

This system of equations is again of the form $F(a,k)=0$, where $a=(a_2,\dots,a_M)$. $F(\tilde{a},0)=0$, where $\tilde{a}=(0,\dots,0,\sqrt{-\omega/d},0)$. Since $D_F(\tilde{a},0)=\mathrm{diag}(\omega,\dots,\omega,-2\omega)$, which is invertible for $\omega \neq 0$, the system (15) has a solution for sufficiently small k by the implicit function theorem. Once (15) has been solved, we obtain the full solution to (6) using

$$
\begin{aligned}
&a_1=0 \\
&a_{M+k}=a_{M-k+1} \qquad && k=1,\dots,M-1 \\
&\theta_0=0 \\
&\theta_k=(k-1)\phi-\pi/2 && k=2,\dots,M \\
&\theta_{M+k}=-\theta_{M-k+1} && k=1,\dots,M-1
\end{aligned}
$$

Figure 5 shows this solution for $N = 7$.

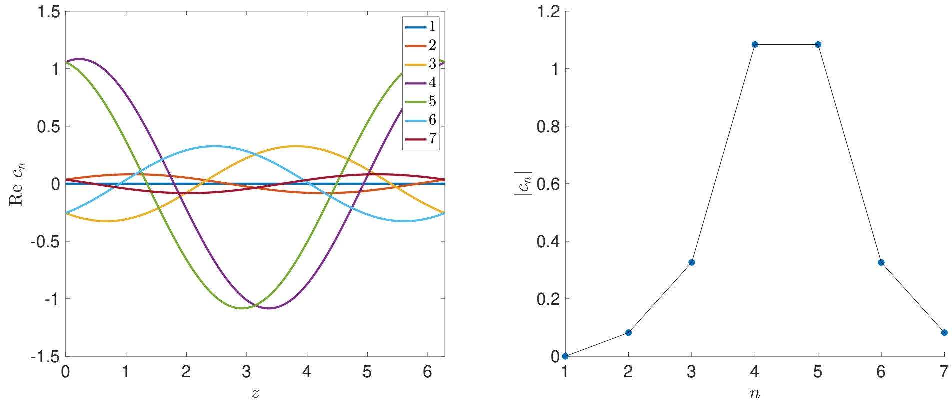

Figure 5: Standing wave solution for $N = 7$ and $\phi = \pi/7$. Left is real part of solution for each node, right is absolute value of solution at each node. Nodes 4 and 5 have equal and maximum amplitude, and node 1 is a dark node. $\omega = 1$, $k = 0.25$, $d = -1$.

4 Stability

We now look at the stability of the standing wave solutions we constructed in the previous section. The linearization of (2) about a standing wave solution $c_n = a_n e^{i(\omega z + \theta_n)} = (v_n + iw_n)e^{i\omega z}$ is the $2N \times 2N$ block matrix

$$A(c_n) = k \begin{pmatrix} S & C \\ -C & S \end{pmatrix} + \omega \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} - \begin{pmatrix} \operatorname{diag}(2v_n w_n) & \operatorname{diag}(v_n^2 + 3w_n^2) \\ -\operatorname{diag}(3v_n^2 + w_n^2) & -\operatorname{diag}(2v_n w_n) \end{pmatrix}$$

where each block is $N \times N$, C is the periodic banded matrix with $\cos\phi$ on the first upper and lower diagonals, and S is the periodic banded matrix with $\sin\phi$ on the first lower diagonal and $-\sin\phi$ on the first upper diagonal. The spectrum of all solutions generated from the AC limit starting with a single excited node is purely imaginary, thus we expect these solutions to be neutrally stable. In particular, this is the case for solutions with even N and a single dark node opposite the bright node. The left panel of Figure 6 shows this spectrum for the $N = 6$. There is a double eigenvalue at 0 from the gauge invariance of (2), and the remainder of the spectrum is purely imaginary. The right panel shows the results of timestepping for a small perturbation of the standing wave solution when $N = 6$. The solutions show small oscillations but no growth, suggesting neutral stability. Similar results are obtained for other values of N for N even, as well as N odd with a dark node opposite a pair of bright nodes. In addition, if we start with a neutrally stable standing wave solution and perturb the system by a small change in k or ϕ, the time evolution resembles that in Figure 6.

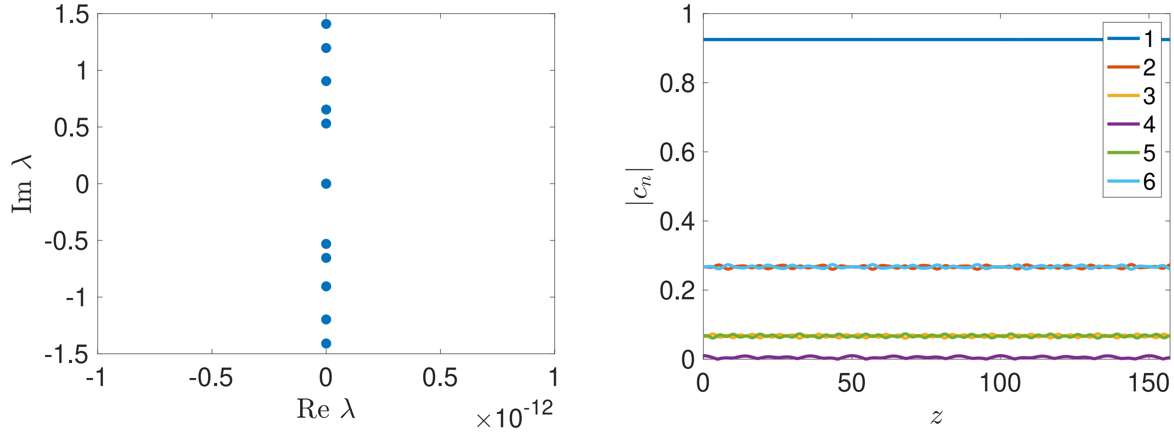

Figure 6: Left panel is spectrum of linearization of (2) about solution with $N = 6$ and a single dark node opposite a single bright node. Right panel shows $|c_n|$ versus z, where initial condition is perturbed by adding 0.01 to dark node. Timestepping using a fourth order Runge-Kutta scheme. $k = 0.25$, $d = -1$, $\phi = \pi/6$.

If the strength of the nearest-neighbor coupling is allowed to differ between pairs of nodes, equation (2) becomes

$$i\partial_z c_n = k_{n+1}e^{-i\phi}c_{n+1} + k_{n-1}e^{i\phi}c_{n-1} + i\gamma_n c_n + d|c_n|^2 c_n. \tag{16}$$

This allows for asymmetric solutions, as shown in Figure 7. (Contrast to the symmetric solutions for uniform k in Figure 3). These asymmetric solutions are also neutrally stable.

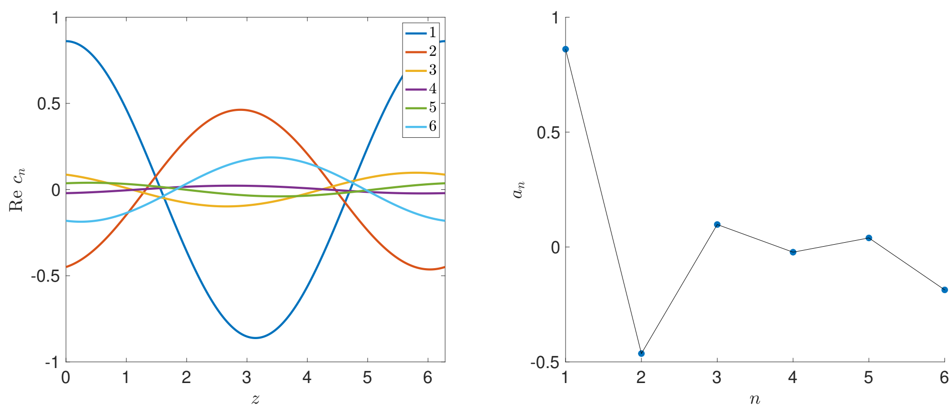

Figure 7: Standing wave solution to (16) for $N = 6$. $\omega = 1$, $k_1 = 0.4$, and $k_n = 0.2$ for all other n. Left is real part of solution c_n versus z for each node over a full period (2π), right is amplitude a_n solution at each node. $\phi = 0.25$, $d = -1$.

5 Conclusions

References

[CCSS+] Claudia Castro-Castro, Yannan Shen, Gowri Srinivasan, Alejandro B Aceves, and Panayotis G Kevrekidis, *Light dynamics in nonlinear trimers and twisted multicore fibers* (2016), 11 (en).