

1 Introduction

2 Background

We consider the propagation dynamics in a multi-core fiber consisting of N waveguides arranged in a ring (Figure 1).

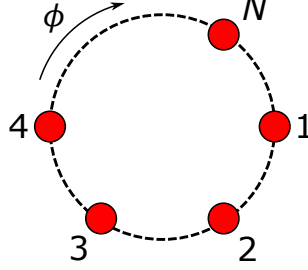


Figure 1: Schematic of N twisted fibers arranged in a ring.

Each fiber is twisted in a uniform fashion along the propagation direction z . The dynamics are given by the coupled system of equations

$$i\partial_z c_n = k (e^{-i\phi} c_{n+1} + e^{i\phi} c_{n-1}) + i\gamma_n c_n + d|c_n|^2 c_n \quad (1)$$

for $n = 1, \dots, N$, where $c_0 = c_N$ and $c_{N+1} = c_1$ due to the circular geometry. The quantities $c_n(z)$ are the complex-valued amplitudes of each waveguide, k is the strength of the nearest-neighbor coupling, γ_n is the optical gain or loss at site n , and ϕ is a parameter representing the twist of the fibers. (See [CCSS⁺, (2.1)] for a description of the parameters in terms of the optical waveguide system). If $\gamma_n = 0$ for all n , i.e. there is no gain or loss at each node, equation (1) becomes

$$i\partial_z c_n = k (e^{-i\phi} c_{n+1} + e^{i\phi} c_{n-1}) + d|c_n|^2 c_n, \quad (2)$$

which is Hamiltonian with energy given by

$$H = \sum_{n=1}^N k(c_{n+1}c_n^* e^{-i\phi} + c_n c_{n+1}^* e^{i\phi}) + \frac{d}{2}|c_n|^4. \quad (3)$$

We will consider the Hamiltonian case here, and will comment on the case with loss/gain at the end.

We are interested in standing wave solutions to (2), which are bound states of the form

$$c_n = a_n e^{i(\omega z + \theta_n)}, \quad (4)$$

where $a_n \in \mathbb{R}$, $\theta_n \in (-\pi/2, \pi/2]$, and ω is the frequency of oscillation. (Since a_n can be negative, we can restrict θ_n to that interval). Making this substitution and simplifying, equation (2) becomes

$$k (a_{n+1} e^{i((\theta_{n+1} - \theta_n) - \phi)} + a_{n-1} e^{-i((\theta_n - \theta_{n-1}) - \phi)}) + \omega a_n + d a_n^3 = 0, \quad (5)$$

which can be written as the system of $2n$ equations

$$\begin{aligned} k(a_{n+1} \cos(\theta_{n+1} - \theta_n - \phi) + a_{n-1} \cos(\theta_n - \theta_{n-1} - \phi)) + \omega a_n + d a_n^3 &= 0 \\ a_{n+1} \sin(\theta_{n+1} - \theta_n - \phi) - a_{n-1} \sin(\theta_n - \theta_{n-1} - \phi) &= 0 \end{aligned} \quad (6)$$

by separating real and imaginary parts. We note that the exponential terms in (5) depend only on the phase differences $\theta_{n+1} - \theta_n$ between adjacent sites. Due to the gauge invariance of (2), if c_n is solution, so is $e^{i\theta} c_n$, thus we may without loss of generality take $\theta_1 = 0$. If $\phi = 0$, i.e. the fibers are not twisted, we can take $\theta_n = 0$ for all n , and so (5) reduces to the untwisted case with periodic boundary conditions. Similarly, if we take $\phi = 2\pi/N$ and $\theta_n = (n-1)\phi$ for all n , the exponential terms do not contribute, and (5) once again reduces to untwisted case. The interesting case, therefore, occurs when $0 < \theta < 2\pi/N$.

3 Construction of solutions

In the anti-continuum (AC) limit, which occurs when $k = 0$, the sites are decoupled. Each a_n can take on the values $\{0, \pm\sqrt{-\omega/d}\}$, the phases θ_n are arbitrary, and ϕ does not contribute. The amplitudes $\sqrt{-\omega/d}$ are real if d and ω have opposite signs.

We construct solutions to (6) by parameter continuation from the AC limit with no twist using AUTO. As an initial condition, we choose a single excited site at node 1, i.e. $a_1 = \sqrt{-\omega/d}$ and $a_n = 0$ for all other n ; for the phases, $\theta_n = 0$ for all n . We also take $\phi = 0$. We first continue in the coupling parameter k , and then, for fixed k , we continue in the twist parameter ϕ . In doing this, we observe that the solutions have the following symmetry:

$$\begin{aligned} a_k &= a_{N-k+2} & k &= 2, \dots, M-1 \\ \theta_k &= -\theta_{N-k+2} & k &= 2, \dots, M-1, \end{aligned} \quad (7)$$

where $M = (N/2) + 1$ for N even and $M = (N+1)/2$ for N odd. See Figure 2 for an illustration of these symmetry relations for $N = 6$ and $N = 7$.

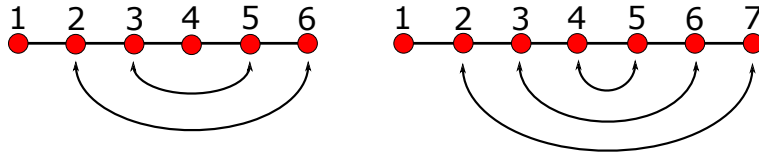


Figure 2: Schematic of symmetry relationship between nodes for $N = 6$ and $N = 7$. For nodes connected with arrows, the amplitudes a_k are the same and the phases θ_k are opposite.

For N even, node M is the node directly across the ring from node 1, and $\theta_M = 0$. For all N , $\theta_1 = 0$. Figure 3 shows an example of a standing wave solution produced by numerical parameter continuation for $N = 6$. The right panel illustrates the symmetry relations (7) among the amplitudes a_k .

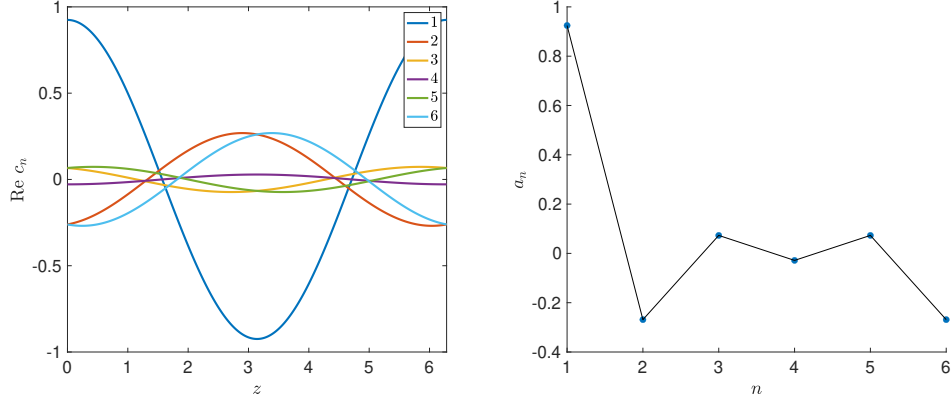


Figure 3: Standing wave solution for $N = 6$, $\omega = 1$, and $\phi = 0.25$. Left is real part of solution c_n versus z for each node over a full period (2π), right is amplitude a_n solution at each node. $k = 0.25$, $d = -1$.

3.1 N even

Numerical parameter continuation for N even, starting from a single excited node at node 1, suggests that when the twist parameter $\phi = \pi/N$, the opposite node is completely dark, i.e. has an amplitude of 0. Using the symmetries (7), when $a_M = 0$, the system (6) reduces to

$$\begin{aligned}
2ka_2 \cos(\theta_2 - \phi) + \omega a_1 + da_1^3 &= 0 \\
k(a_{n+1} \cos(\theta_{n+1} - \theta_n - \phi) + a_{n-1} \cos(\theta_n - \theta_{n-1} - \phi)) + \omega a_n + da_n^3 &= 0 \quad n = 2, \dots, M-1 \\
a_{n+1} \sin(\theta_{n+1} - \theta_n - \phi) - a_{n-1} \sin(\theta_n - \theta_{n-1} - \phi) &= 0 \quad n = 2, \dots, M-1 \\
2ka_{M-1} \cos(\theta_{M-1} + \phi) &= 0 \\
\theta_1 = \theta_M &= 0.
\end{aligned} \tag{8}$$

It follows that $a_n = 0$ for all n unless

$$\begin{aligned}
\cos(\theta_{M-1} + \phi) &= 0 \\
\sin(\theta_n - \theta_{n-1} - \phi) &= 0 \quad n = 3, \dots, M-1 \\
\sin(\theta_2 - \phi) &= 0.
\end{aligned} \tag{9}$$

One solution to this is

$$\begin{aligned}
\theta_{M-1} + \phi &= \pi/2 \\
\theta_n - \theta_{n-1} - \phi &= 0 \quad n = 3, \dots, M-1 \\
\theta_2 - \phi &= 0,
\end{aligned} \tag{10}$$

from which it follows that we can have a single dark node at site M when $\phi = \pi/N$. If this is the case, the system of equations (8) reduces to the simpler system

$$\begin{aligned} 2ka_2 + \omega a_1 + da_1^3 &= 0 \\ k(a_{n+1} + a_{n-1}) + \omega a_n + da_n^3 &= 0 \quad n = 2, \dots, M-2 \\ ka_{M-2} + \omega a_{M-1} + da_{M-1}^3 &= 0. \end{aligned} \quad (11)$$

This system is of the form $F(a, k) = 0$, where $a = (a_1, \dots, a_{M-1})$. $F(\tilde{a}, 0) = 0$, where $\tilde{a} = (\sqrt{-\omega/d}, 0, \dots, 0)$. Since $D_F(\tilde{a}, 0) = \text{diag}(-2\omega, \omega, \dots, \omega)$, which is invertible for $\omega \neq 0$, the system (11) has a solution for sufficiently small k by the implicit function theorem. Once (11) has been solved, we obtain the full solution to (6) using

$$\begin{aligned} a_M &= 0 \\ a_{M+k} &= a_{M-k} & k &= 1, \dots, M-2 \\ \theta_0 &= 0 \\ \theta_k &= (k-1)\phi & k &= 2, \dots, M-1 \\ \theta_M &= 0 \\ \theta_{M+k} &= -\theta_{M-k} & k &= 1, \dots, M-2. \end{aligned}$$

Figure 4 shows this solution for $N = 6$. This observation of a dark node for $N = 6$ when $\phi = \pi/6$ agrees with what was shown in [CCSS⁺].

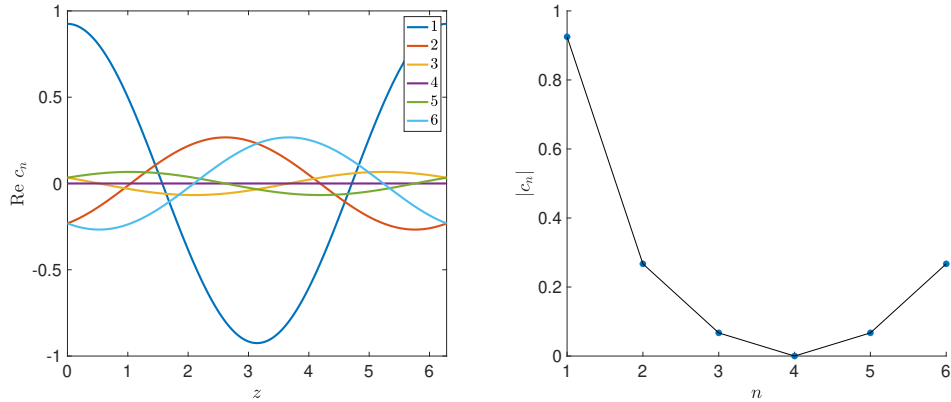


Figure 4: Standing wave solution for $N = 6$ and $\phi = \pi/6$. Left is real part of solution for each node, right is absolute value of solution at each node. Node 1 has maximum amplitude, and node 4 is a dark node. $\omega = 1$, $k = 0.25$, $d = -1$.

3.2 N odd

We can also obtain a dark node when N is odd by taking node 1 to be the dark node; in this case, the dark node will be opposite a pair of bright nodes at a_M and a_{M+1} with the same

amplitude. Using the symmetries (7), when $a_1 = 0$, the system (6) reduces to

$$\begin{aligned}
2ka_2 \cos(\theta_2 - \phi) &= 0 \\
ka_3 \cos(\theta_3 - \theta_2 - \phi) + \omega a_2 + da_2^3 &= 0 \\
a_3 \sin(\theta_3 - \theta_2 - \phi) &= 0 \\
k(a_{n+1} \cos(\theta_{n+1} - \theta_n - \phi) + a_{n-1} \cos(\theta_n - \theta_{n-1} - \phi)) + \omega a_n + da_n^3 &= 0 \quad n = 3, \dots, M-1 \\
a_{n+1} \sin(\theta_{n+1} - \theta_n - \phi) - a_{n-1} \sin(\theta_n - \theta_{n-1} - \phi) &= 0 \quad n = 3, \dots, M-1 \\
k(a_M \cos(-2\theta_M - \phi) + a_{M-1} \cos(\theta_M - \theta_{M-1} - \phi)) + \omega a_M + da_M^3 &= 0 \\
a_M \sin(-2\theta_M - \phi) - a_{M-1} \sin(\theta_M - \theta_{M-1} - \phi) &= 0.
\end{aligned} \tag{12}$$

It follows that $a_n = 0$ for all n unless

$$\begin{aligned}
\cos(\theta_2 - \phi) &= 0 \\
\sin(\theta_n - \theta_{n-1} - \phi) &= 0 \quad n = 3, \dots, M-1 \\
\sin(2\theta_M + \phi) &= 0.
\end{aligned} \tag{13}$$

One solution to this is

$$\begin{aligned}
\theta_2 - \phi &= -\pi/2 \\
\theta_n - \theta_{n-1} - \phi &= 0 \quad n = 3, \dots, M-1 \\
2\theta_M + \phi &= 0,
\end{aligned} \tag{14}$$

from which it follows that we can have a single dark node at a_1 when $\phi = \pi/N$. This condition for a dark node is the same as for N even case. For this case, (12) reduces to the simpler system of equations

$$\begin{aligned}
ka_3 + \omega a_2 + da_2^3 &= 0 \\
k(a_{n+1} + a_{n-1}) + \omega a_n + da_n^3 &= 0 \quad n = 3, \dots, M-1 \\
k(a_M + a_{M-1}) + \omega a_M + da_M^3 &= 0.
\end{aligned} \tag{15}$$

This system of equations is again of the form $F(a, k) = 0$, where $a = (a_2, \dots, a_M)$. $F(\tilde{a}, 0) = 0$, where $\tilde{a} = (0, \dots, 0, \sqrt{-\omega/d}, 0)$. Since $D_F(\tilde{a}, 0) = \text{diag}(\omega, \dots, \omega, -2\omega)$, which is invertible for $\omega \neq 0$, the system (15) has a solution for sufficiently small k by the implicit function theorem. Once (15) has been solved, we obtain the full solution to (6) using

$$\begin{aligned}
a_1 &= 0 \\
a_{M+k} &= a_{M-k+1} \quad k = 1, \dots, M-1 \\
\theta_0 &= 0 \\
\theta_k &= (k-1)\phi - \pi/2 \quad k = 2, \dots, M \\
\theta_{M+k} &= -\theta_{M-k+1} \quad k = 1, \dots, M-1
\end{aligned}$$

Figure 5 shows this solution for $N = 7$.

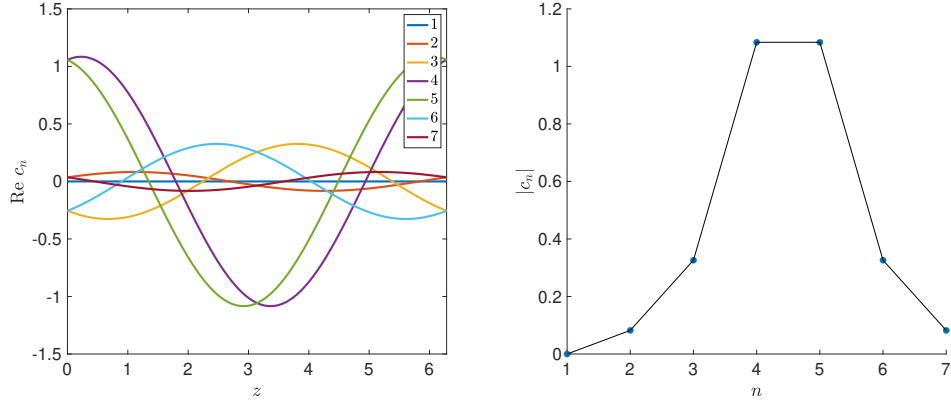


Figure 5: Standing wave solution for $N = 7$ and $\phi = \pi/7$. Left is real part of solution for each node, right is absolute value of solution at each node. Nodes 4 and 5 have equal and maximum amplitude, and node 1 is a dark node. $\omega = 1$, $k = 0.25$, $d = -1$.

4 Stability

The linearization about a standing wave solution $c_n = a_n e^{i(\omega z + \theta_n)} = (v_n + iw_n) e^{i\omega z}$ is the $2N \times 2N$ block matrix

$$A(\phi) = k \begin{pmatrix} S & C \\ -C & S \end{pmatrix} + \omega \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} - \begin{pmatrix} \text{diag}(2v_n w_n) & \text{diag}(v_n^2 + 3w_n^2) \\ -\text{diag}(3v_n^2 + w_n^2) & -\text{diag}(2v_n w_n) \end{pmatrix}$$

where each block is $N \times N$, C is the periodic banded matrix with $\cos \phi$ on the first upper and lower diagonals, and S is the periodic banded matrix with $\sin \phi$ on the first lower diagonal and $-\sin \phi$ on the first upper diagonal.

For spectral stability, all solutions generated this way (from AC limit with single excited node or the solutions with a single dark node) are spectrally neutrally stable, for both N even and N odd. In other words, the spectrum is purely imaginary. In particular, this is true for the two case described above with a single dark node. This is verified by timestepping for a perturbations of the standing wave solutions. Results of this for $N = 6$ is shown in Figure 6. Similar results are obtained for $N = 7$

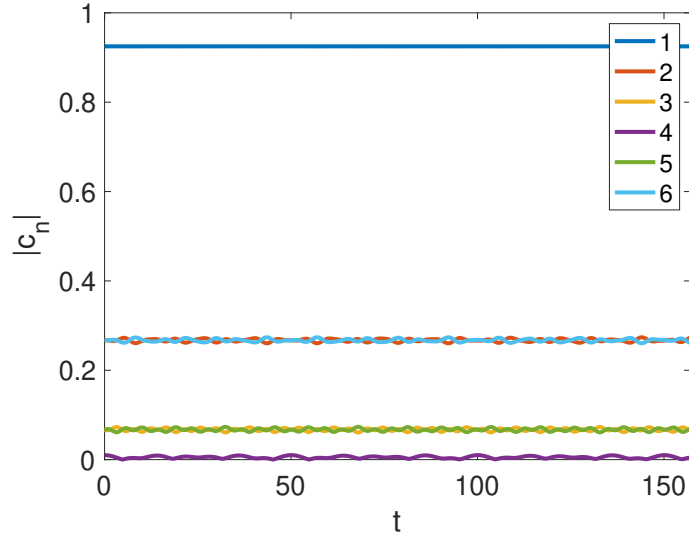


Figure 6: $|c_n|$ versus t . Solution with $N = 6$ with dark node, perturbed by adding 0.01 to initial condition at dark node. RK4 for timestepping, $k = 0.25$, $d = -1$.

References

- [CCSS⁺] Claudia Castro-Castro, Yannan Shen, Gowri Srinivasan, Alejandro B Aceves, and Panayotis G Kevrekidis, *Light dynamics in nonlinear trimers and twisted multicore fibers* (2016), 11 (en).