Research statement

My main research focus is on understanding coherent structures arising in the natural sciences and engineering. Mathematically, these are described by nonlinear evolutions equations which take the form of partial differential equations (PDEs) or systems of ordinary differential equations (ODEs). Most of the systems I study are nonlinear wave equations, which describe the time evolution of a function representing a wave profile. In particular, I study coherent structures in Hamiltonian systems, which are characterized by a conserved quantity, such as energy, that remains constant in time. Coherent structures are spatial patterns which maintain their shape as the system evolves in time. Examples of coherent structures include solitary waves, which are localized disturbances that maintain their shape as they propagate at a constant velocity; wavetrains, which are periodic disturbances that also propagate at a constant velocity; and breathers, which are disturbances which are localized in space but oscillate in time. Solitary waves have been a topic of interest since the 19th century, when John Scott Russell observed a single, large surface wave on the Edinburgh-Glasgow Union Canal in Scotland which propagated along the canal undisturbed for a few miles before eventually decaying. This phenomenon was explained mathematically by the Korteweg–De Vries (KdV) equation

$$u_t + u_{xxx} + 6uu_x = 0,$$

which has both solitary wave and wavetrain solutions.

Figure 1: Solitary wave off the coast of Hawaii [1] (left). Optical solitons in a microresonator [15] (right)

Although solitary waves were originally discovered as a water wave phenomenon, they have applications in many fields such as fiber optics, plasma physics, quantum mechanics, molecular biology, and neuroscience (Figure 1). More generally, many nonlinear, dispersive PDEs exhibit solitary wave solutions. My research falls into two broad categories: multi-pulse solitary waves in Hamiltonian systems, and coherent structures in optical lattices. Beyond those, I have also explored bifurcation structures in a neural network model and systems of coupled oscillators.

Multi-pulse solitary waves in Hamiltonian systems

The prototypical nonlinear wave equation has a primary solitary wave solution, also known as a pulse solution, which looks like a single localized "bump" (Figure 2, left). In many systems, multi-pulse solutions also exist; these are "multi-bump" solitary waves which resembles multiple, well separated copies of the primary pulse solution. (Notably, multi-pulse solutions do not exist to the KdV equation). The entire multi-pulse travels as a unit and maintains its shape unless perturbed. In addition to having applications in nonlinear optics and neuroscience [7], multi-pulses are interesting mathematically. In my research, I explore the existence and stability of these multi-pulse structures. A crucial step in this process is determining the spectrum of the linearization of the underlying system about these multi-pulse solutions. When a multi-pulse is perturbed, interactions between the individual pulses in the structure are revealed, which are a consequence of the inherent nonlinearity of the system. The dynamics of these interactions are determined by the eigenvalues of the linearized system and their corresponding eigenfunctions.
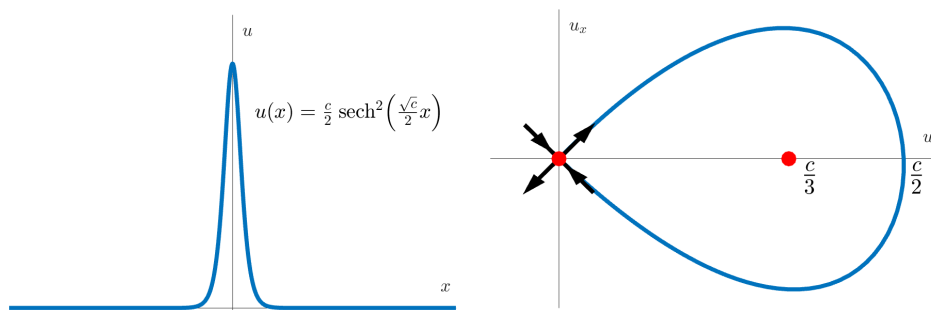

Figure 2: Primary solitary wave solution $u(x)$ to the KdV equation. Left panel is plot of u vs x. Right panel is plot of u_x vs u, showing solitary wave as a homoclinic orbit.

My primary mathematical approach comes from spatial dynamics. From this viewpoint, a solitary wave $u(x)$ is a homoclinic orbit evolving in the spatial variable x. For example, the solitary wave solution $u(x)$ to the KdV equation which moves with constant speed c (Figure 2, left panel) is a solution to the second order ODE $u_{xx} + 3u^2 - cu = 0$. This can be written as the two-dimensional dynamical system

$$\frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ cu - 3u^2 \end{pmatrix}$$

by taking $v = u_x$. From this perspective, the solitary wave solution $(u(x), v(x))$ is a homoclinic orbit connecting the unstable and stable manifolds of the saddle equilibrium at the origin (Figure 2, right panel), which represents the rest state of the system.

Multi-pulses are multi-loop homoclinic orbits. From a theoretical standpoint, multi-pulses can be constructed using Lin's method [12], a version of the Lyapunov-Schmidt method used to find solutions which remain near a homoclinic orbit. Heuristically, this process involves "gluing together" multiple copies of the primary pulse solution end-to-end using small remainder functions
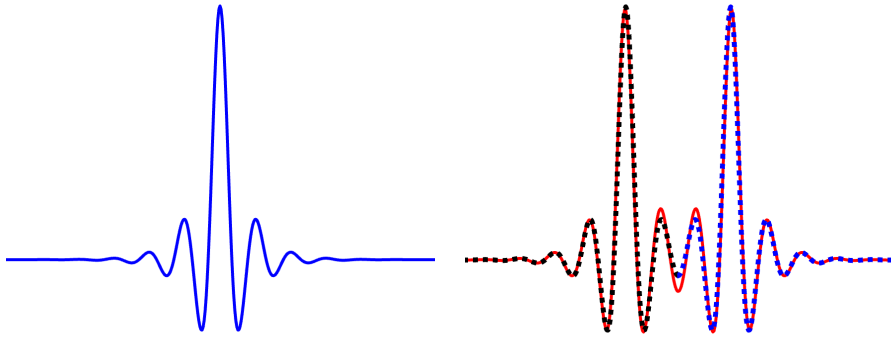
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Figure 3: Cartoon illustrating construction of a double pulse solution using Lin's method. Left panel shows primary pulse solution. Right panel shows two copies of the primary pulse solution (black and blue dotted lines) placed end-to-end. Double pulse solution (red solid line) is close, but not equal, to this.

(Figure 3). Lin's method can also be used to construct periodic orbits and multi-loop periodic orbits. Construction of multi-pulse solutions is constrained by the geometry of the primary pulse and the underlying system. In Figure 3, for example, multi-pulses only exist when the tail oscillations of the individual pulses match up.

To analyze the stability of a mult-pulse solution, the first step is to compute the spectrum of the linearization of the underlying system about the solution. In general, each pulse that is added to a multi-pulse structure is associated with one or more eigenvalues in the spectrum [26], which I call interaction eigenvalues, since they result from nonlinear interactions between the tails of neighboring pulses. The systems I study

Figure 4: Possible interaction eigenvalue patterns in a Hamiltonian system.

are Hamiltonian, which are not covered by the results of [26]. On one hand, the Hamiltonian structure is very helpful, since all eigenvalues must come in quartets of the form $\pm\alpha \pm \beta i$. This means that each additional set of interaction eigenvalues must occur in one of the three patterns in Figure 4. On the other hand, the presence of any eigenvalue with nonzero real part implies that there is an unstable eigenvalue. This means that Hamiltonian systems cannot be dissipative, which makes stability analysis more difficult. My main results relate the spectral pattern of the interaction eigenvalues to the underlying geometry of the multi-pulse. In all cases, the spectral pattern is determined by this geometry.
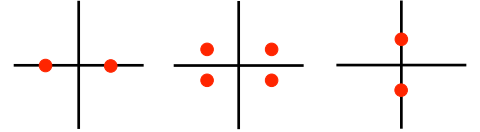
In addition to theoretical work, I make extensive use of numerical analysis, both to generate hypotheses and to verify analytical results. For multi-pulses, I start by constructing the primary solitary wave, which either involves parameter continuation from a known solution using the software package AUTO or an energy minimization method [5]. I then glue together multiple copies of the primary solitary wave and use a root-finding method such as conjugate gradient to construct a multi-pulse. I compute the spectrum using an eigenvalue solver on an appropriate discretization of the linearized system. Four systems I have studied are the discrete nonlinear Schrödinger equation, the Chen-McKenna suspension bridge equation, the fifth-order KdV equation, and a fourth order nonlinear Schrödinger equation.
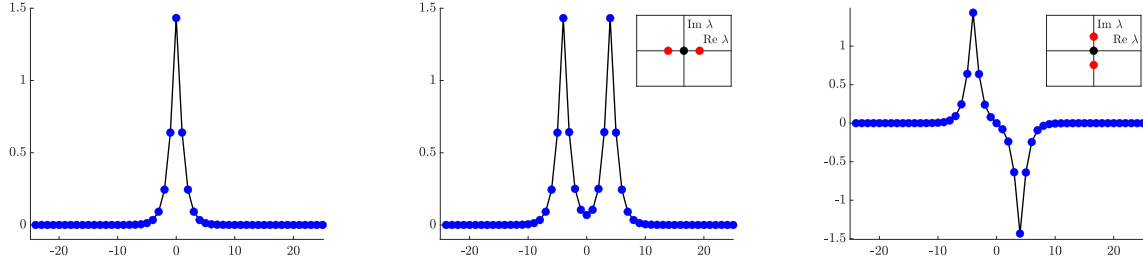
Figure 5: Primary pulse solution for DNLS (left). Out-of-phase (middle) and in-phase (right) double pulse solutions for DNLS. Interaction eigenvalue patterns for double pulses are shown in insets. Black dot is a kernel eigenvalue with algebraic multiplicity 2.

Discrete nonlinear Schrödinger equation

The discrete nonlinear Schrödinger equation (DNLS)

$$i\frac{d}{dt}u_n + d(u_{n+1} - 2u_n + u_{n-1}) + |u_n|^2 u_n \qquad n \in \mathbb{Z}$$

is the discrete analogue to the nonlinear Schrödinger equation (NLS) on the integer lattice. The parameter d quantifies the degree of coupling between adjacent lattice sites. In addition to being a fundamental model of a nonlinear dynamical system on a lattice, DNLS has applications to nonlinear optics and condensed matter physics [10]. For all values of d, DNLS has a stable, primary pulse solution (Figure 5, left panel). Provided that the individual peaks are separated by a sufficiently large number of lattice points, I use Lin's method to prove that multi-pulse solutions exist as long as the following geometric constraint is satisfied: neighboring peaks must either be out-of-phase or in-phase [20]. Furthermore, I prove this geometry determines the stability of multi-pulses. For double pulses, the in-phase double pulse is unstable, since there is eigenvalue with positive real part, and the out-of-phase double pulse is neutrally stable, since the entire spectrum lies on the imaginary axis (see Figure 5). For general multi-pulses, the entire structure is unstable if any pair of neighboring peaks are in-phase. The only neutrally stable multi-pulses are those in which every pair of neighboring peaks is out-of-phase. The proof of this result [20, Theorem 2] adapts the method of [26] to the Hamiltonian case, and uses Lin's method to construct the eigenfunctions corresponding to the interaction eigenvalues. This reduces the spectral problem for an n-pulse to finding the eigenvalues of an $n \times n$ matrix.

Chen-McKenna suspension bridge equation

The Chen-McKenna suspension bridge equation

$$u_{tt} + u_{xxxx} + e^{u-1} - 1 = 0$$

is a smooth approximation to a model for waves propagating on an infinitely long suspended beam, and is motivated by observations of traveling waves on suspension bridges [16, 6]. For wave
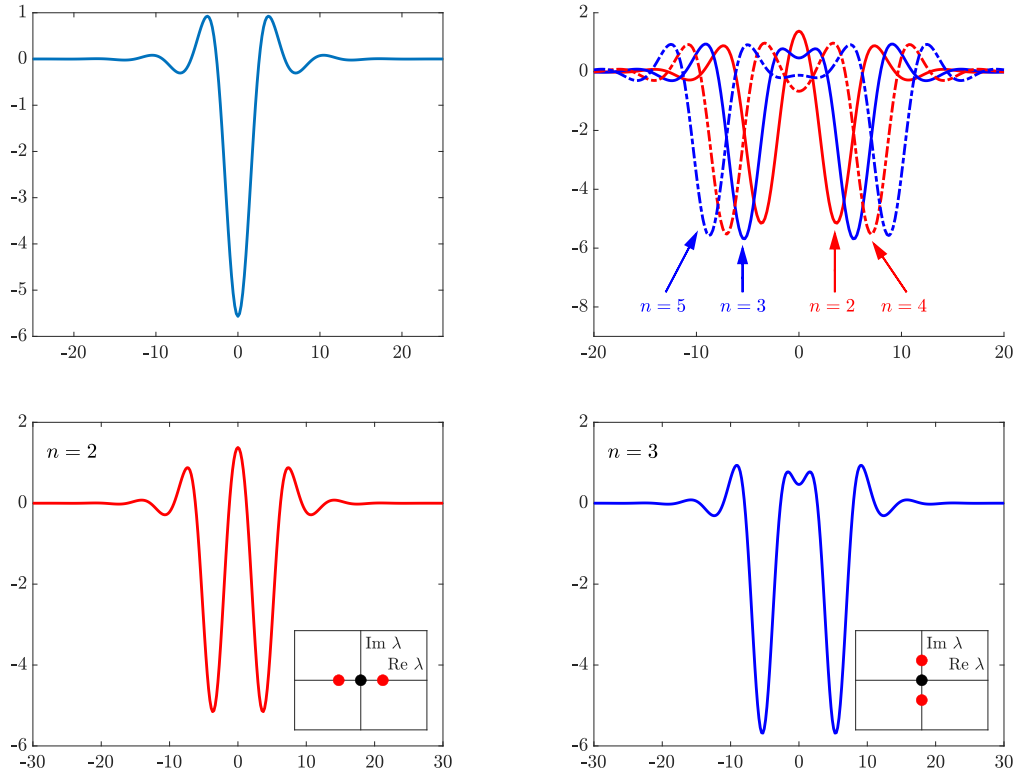
Figure 6: Primary pulse solution for Chen-Mckenna (top left). First four double pulse solutions (top right). Unstable double pulse for $n = 2$ (bottom left) and neutrally stable double pulse for $n = 3$ (bottom right). Interaction eigenvalue patterns for double pulses are shown in insets. Black dot is a kernel eigenvalue with algebraic multiplicity 2.

speeds c between 0 and $\sqrt{2}$, a primary solitary wave solution exists [30], which has exponentially decaying, oscillatory tails (Figure 6, top left). Provided that the individual peaks are sufficiently well separated, multi-pulse solutions again exist as long as the following geometric constraint is satisfied: the tail oscillations of neighboring peaks must overlap in-phase (see the cartoon illustration in Figure 3). This constraint is a consequence of the specific alignment of the unstable and stable manifolds which is necessary for a multi-loop homoclinic orbit to be possible. As a result, the distance between consecutive peaks is, to leading order, an integer multiple of a phase parameter. This is illustrated in the top right panel of Figure 6, which plots the first four double pulse solutions on the same graph. As with DNLS, the stability of multi-pulses depends on their geometry. Double pulses, for example, alternate between unstable (bottom left of Figure 6, corresponding to even integers) and neutrally stable (bottom right of Figure 6, corresponding to odd integers). I prove these spectral results using an extension of the Krein matrix [8], a tool which projects the infinite-dimensional spectral problem onto a finite-dimensional space.
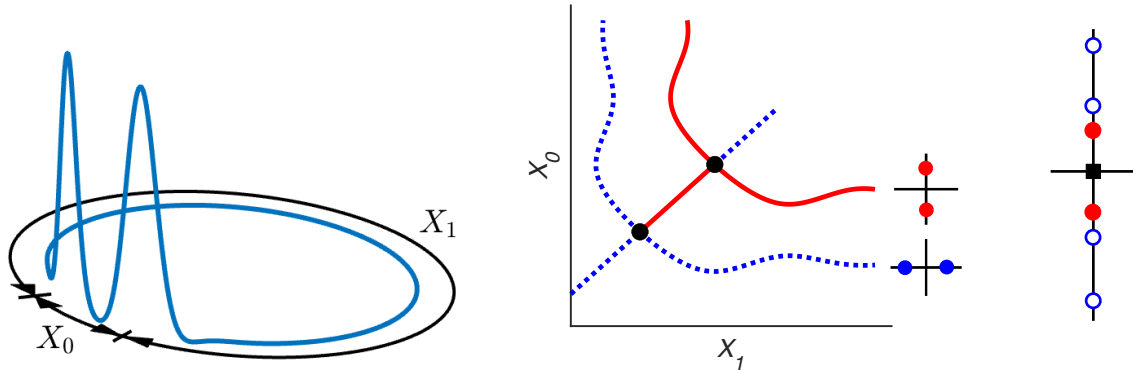
Figure 7: Construction (left) and bifurcation diagram (center) of periodic double pulse solutions to KdV5. Spectrum of neutrally stable periodic double pulses (right) comprising interaction eigenvalues (red filled circles), essential spectrum eigenvalues (blue open circles), and kernel eigenvalue with algebraic multiplicity 2 (black square).

Fifth order KdV equation

The fifth-order Korteweg de-Vries equation (KdV5)

$$u_t - u_{xxxxx} + u_{xxx} + 2uu_x = 0$$

is a weakly nonlinear long wave approximation to capillary-gravity wave problem which also has applications to plasma physics and laser optics [24]. Multi-pulse solutions to KdV5 exist [25], but their stability analysis is complicated due to the fact that the essential spectrum for all localized solutions comprises the entire imaginary axis. In particular, this means that any purely imaginary interaction eigenvalues would be embedded in the essential spectrum, which makes them difficult to locate.

To simplify the situation, I impose periodic boundary conditions on the problem and look instead at periodic multi-pulses, which are periodic orbits containing multiple peaks. From a spatial dynamics perspective, a periodic multi-pulse is a multi-loop periodic orbit which is close to the primary homoclinic orbit. A periodic double pulse, for example, is constructed by gluing two single pulses together at both ends. Since this construction involves two length parameters X_0 and X_1 (Figure 7, left), there is an extra degree of freedom when compared to double pulses on the real line. As a consequence, I prove that periodic double pulses exist in a continuous family, in which asymmetric periodic double pulses (those with $X_0 \neq X_1$) bifurcate from symmetric periodic double pulses (those with $X_0 = X_1$) in a series of pitchfork bifurcations (Figure 7, center) [22]. The advantage of looking at periodic solutions is that the essential spectrum becomes a discrete set of eigenvalues on the imaginary axis (blue open circles in Figure 7, right). Purely imaginary interaction eigenvalues can then lie between essential spectrum eigenvalues, which avoids the problem of embedded eigenvalues.

Using Lin's method, I prove that the eigenvalues associated with a periodic multi-pulse can be
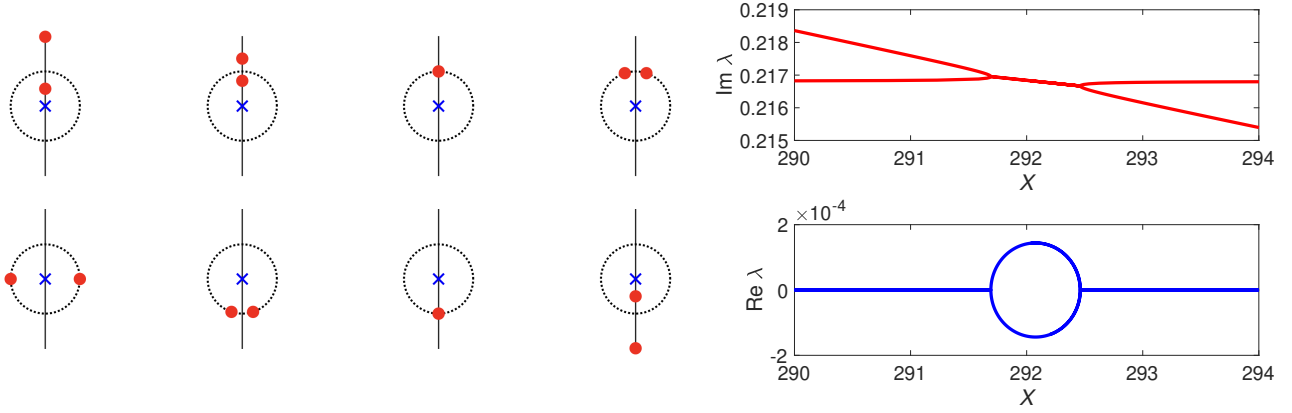
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Figure 8: Krein instability bubble cartoon (left) and numerical simulation for KdV5 (right).

found by solving a block matrix equation [22, Theorem 5.3] for the eigenvalues λ. To leading order, this is given by

$$\det \begin{pmatrix} K(\lambda) - \frac{1}{2}\lambda\tilde{M}K^+(\lambda) & \lambda^2 M_c I \\ -\frac{1}{2}\lambda M_c K^+(\lambda) & A - \lambda^2 M I \end{pmatrix} = 0. \tag{1}$$

The essential spectrum eigenvalues are encoded by the matrix $K(\lambda)$, which, to leading order, only depends on the background state and size of the periodic domain; in particular, it is independent of the periodic multi-pulse solution. The interaction eigenvalues are encoded by the matrix A, which depends on the geometry of the periodic multi-pulse. M, M_c, and $\tilde{M}$ are fundamental constants associated with the primary pulse solution. As long as the periodic domain size is not too large, the interaction eigenvalues and essential spectrum eigenvalues do not interfere with each other. As a consequence, for a periodic double pulse, there is a pair of interaction eigenvalues which is either real or purely imaginary depending on the geometry of the periodic 2-pulse (Figure 7, center). The eigenvalue pattern switches between real and imaginary at the pitchfork bifurcation points.

There is, however, an additional complication in the periodic case. As the periodic domain size X is increased, the essential spectrum eigenvalues move towards the origin. At a critical value of X, there will be a collision between one of the essential spectrum eigenvalues and a purely imaginary interaction eigenvalue. As X is further increased, I prove that a brief instability bubble is formed, wherein the two eigenvalues collide, move off the imaginary axis, trace an approximate circle in the complex plane, and recombine on the imaginary axis in a "reverse" collision (see left panel of Figure 8 for a cartoon) [22]. This instability bubble, which we call a Krein bubble since the eigenvalues involved in the collision have opposite Krein signature, is a direct consequence of the block matrix reduction (1). A numerical simulation of the Krein bubble, computed using parameter continuation with AUTO, is shown in the right panel of Figure 8. The location and size of the Krein bubble in the simulation agree with that predicted by the theory [22].

Fourth order nonlinear Schrödinger equation

The following fourth-order nonlinear Schrödinger equation (NLS4)

$$iu_t + \frac{\beta_4}{24}u_{xxxx} - \frac{\beta_2}{2}u_{xx} + \gamma|u|^2 u = 0$$

is a variant of the nonlinear Schrödinger equation which was introduced to account for the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity [9, 29]. Motivated by recent experiments [28], there is particular interest in the case where $\beta_2 = 0$, in which case the system exhibits pure quartic dispersion. I prove that while multi-pulse solitary wave solutions exist, they are all unstable due to the presence of at least one eigenvalue with positive real part [18, Theorems 1 and 2].

Other systems and future directions

Other systems of interest include the discrete sine-Gordon equation, nonlocal lattice equations, and equations on higher dimensional lattices. The discrete sine-Gordon equation

$$\frac{d^2}{dt^2}u_n = d(\Delta_2 u)_n - \sin(u_n) \qquad\qquad n \in \mathbb{Z}$$

was introduced to describe the dynamics of crystal lattices, and has since been used in numerous applications, including a mechanical model for a chain of pendulums coupled with elastic springs, arrays of Josephson junctions, and DNA dynamics [3]. A particular class of coherent structures of interest in this system are kinks, which are exponentially localized stationary solutions connect adjacent minima of the potential $V(u) = \cos u$. In [21], I prove the existence of stationary multi-kink solutions as well as analyze their spectral stability. Another important class of coherent structures in this system are breathers, which are localized, oscillatory patterns. Future work involves using Lin's method to construct multi-site breathers by splicing together multiple, sequential copies of the primary breather and then analyzing their spectral stability.

One nonlocal version of DNLS [11] is

$$i\frac{d}{dt}u_n = \frac{1}{h^{2s}}\sum_{m\neq n}\frac{u_m - u_n}{|m-n|^{1+2s}} + |u_n|^2 u_n \qquad\qquad n \in \mathbb{Z},$$

where $s > 0$ is a fixed parameter specifying the decay of the nonlocal interactions. In the continuum limit $h \to 0$, this equation converges to an NLS equation with fractional Laplacian [11]. Applications include a model for charge transport in DNA polymers. Future directions include studying solitary wave and multi-pulse solutions in this system, as well as whether the nonlocal interaction term permits other solutions which are not found in DNLS. A final, related area of interest is coherent structures in DNLS on the square integer lattice $\mathbb{Z}^2$, which could include nonlocal interactions in one or both directions.
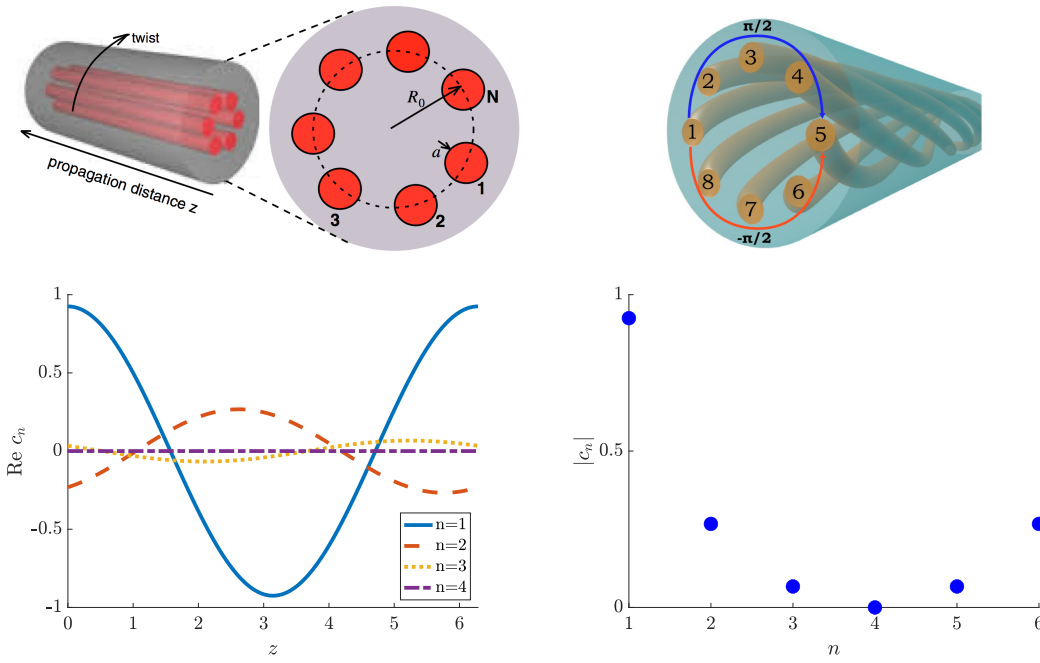
Figure 9: Circular multi-core fiber [13] (top left) and twisted multi-core fiber with eight waveguides [23] (top right). Bottom panel shows standing wave solution for twisted multi-core fiber with $N = 6$ and $\phi = \pi/6$, with evolution of real part of solution for first four sites on bottom left and magnitude of solution at the sites on bottom right. Site 1 has maximum intensity, and opposite site 4 has zero intensity.

Coherent structures in optical lattices

There has been much recent interest in the field of topological photonics, as experimental physicists and engineers have developed new techniques for controlling light propagation. One particular application is light transmission through multi-core optical fibers. In particular, optical transmission properties can be tuned by introducing a twist to the fiber bundle [13, 4, 23] (Figure 9, top). The propagation of light through a system of N waveguides arranged in a circle can be described by the coupled mode equations

$$i\frac{d}{dz}c_n = k\left(e^{-i\phi}c_{n+1} + e^{i\phi}c_{n-1}\right) + |c_n|^2 c_n \qquad\qquad n = 1, \ldots, N,$$

where z is the axis of propagation, $c_0 = c_N$ and $c_{N+1} = c_1$ due to the circular geometry, k is the strength of the nearest-neighbor coupling, and ϕ is a parameter representing the twist of the fiber. When the twist parameter ϕ and the number of fibers in the bundle N are related by $\phi = \pi/N$, I prove that there is a stable standing wave solution of the form $c_n(z) = a_n e^{i\omega z}$ which has a "dark node" with no optical activity opposite a "bright node" of maximum intensity [19] (Figure 9, bottom).

Future research involves studying coherent structures in two-dimensional arrays of optical fibers and correlating the results from the mathematical models with experimental data. One example
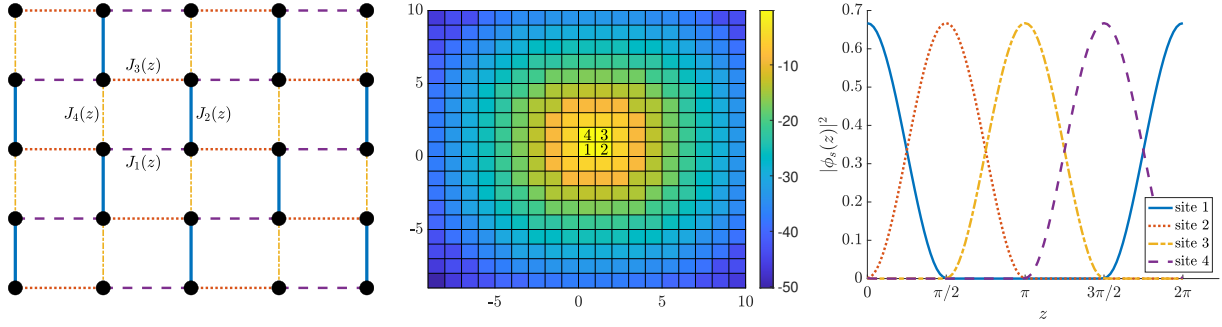
Figure 10: Cartoon of square lattice with z-dependent coupling (left). Log intensity of fundamental breather solution over one period, showing localization to a single unit square in lattice. Square intensity of fundamental breather solution on the unit square over one period.

is a waveguide consisting of a square lattice of fibers in which there are periodic variations along the waveguide axis causing the nearest-neighbor coupling to vary periodically in z [17]. In particular, for any z, a waveguide is only coupled to one of its four neighbors (Figure 10, top). This configuration gives rise to periodic breather solutions in which the optical intensity is confined to a single square in the lattice but "jumps" around that square. No systematic mathematical study of the existence and stability of these solutions has been done to date. Other, more complicated arrangements of fibers which are of interest to experimentalists include concentric rings and honeycomb lattices [14].

Beyond optics

I have recently been studying a neural network model comprising N neurons.

$$\dot{\mathbf{x}} = -\mathbf{x} + \frac{1}{\sqrt{N}} H \tanh(g\mathbf{x}),$$

where $\mathbf{x} = (x_1, \ldots x_N)$ represents the firing rate of each neuron [2]. The network topology and neuronal connection weights are specified by the connectivity matrix H, and the individual neurons are connected via the tanh sigmoidal activation function. Adjusting the parameter g, which represents the global connection strength, leads to bifurcations in the system. The specific bifurcation structure depends on symmetries in the matrix H. For one example, suppose the neurons are grouped into two clusters, one containing excitatory cells and the other containing inhibitory cells. When $g = 0$, the origin is a stable equilibrium point of the system. As g is increased, the origin loses stability in a symmetric pitchfork bifurcation (filled circles in Figure 11). Multiple branches of equilibria emerge from this bifurcation point due to the symmetries of H. As g is further increased, there is a Hopf bifurcation on each branch of equilibria (filled squares in Figure 11), after which point there are limit cycle solutions. At a critical value of g, these limit cycle solutions coalesce (dark band in Figure 11), after which point the system only has a single stable limit cycle.
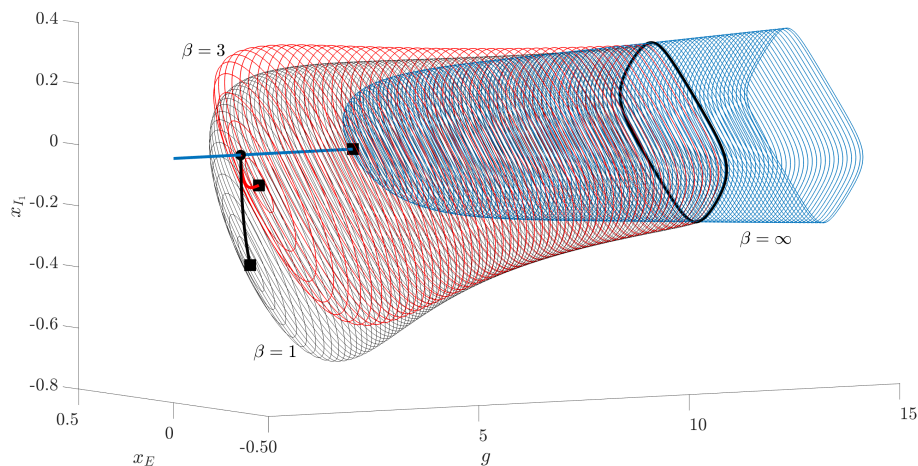
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Figure 11: Bifurcation structure of neural network model as parameter g is increased. Symmetric pitchfork bifurcation indicated with filled circle. Hopf bifurcations indicated with filled squares.

Finally, I have done research on coupled oscillators in collaboration with undergraduate students in a summer REU. A specific example is the Kuromoto model [27], which was developed to study synchronization and has numerous physical and biological applications, including circadian rhythms and flashing fireflies. The model is

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) \qquad\qquad i = 1, \ldots N,$$

which describes a set of N oscillators with phases θ_i and natural frequencies ω_i. In the original model, every oscillator is connected to every other oscillator, and the parameter K controls the strength of this coupling. Notably, for a critical value of K, which depends on the initial distribution of natural frequencies ω_i, the oscillators will start to synchronize [27]. My students examined the case where not all the oscillators are connected; in particular, the connections between oscillators are specified by an adjacency matrix. They explored the degree of synchrony in the network as the number of connections is varied, ans in particular how this depends on the coupling parameter K. They presented a poster of their results at the regional SIAM conference.

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