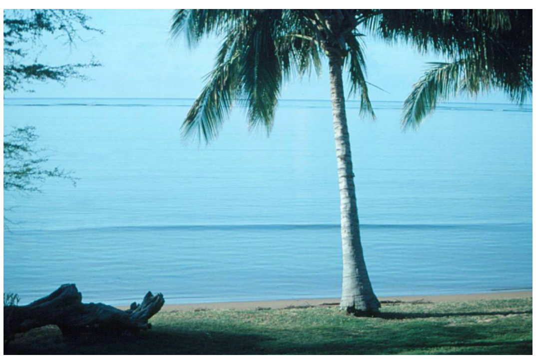
Research Summary

My main research focus is on understanding patterns and coherent structures arising in the natural sciences and engineering. Mathematically, these are described by nonlinear evolution equations, which take the form of partial differential equations (PDEs) or systems of ordinary differential equations (ODEs). Most of the systems I study are nonlinear wave equations, which describe the time evolution of a function representing a wave profile. Coherent structures are spatial patterns which maintain their shape as the system evolves in time. Examples of coherent structures include solitary waves, which are localized disturbances that propagate at a constant velocity; wavetrains, which are periodic disturbances that also propagate at a constant velocity; and breathers, which are disturbances that are localized in space but oscillate in time. Solitary waves have been a topic of interest since the 19th century, when John Scott Russell observed a single, large surface wave on the Edinburgh-Glasgow Union Canal in Scotland; the wave propagated along the canal undisturbed for a few miles before eventually decaying. This phenomenon was explained mathematically by the Korteweg–de Vries (KdV) equation

$$u_t + u_{xxx} + 6uu_x = 0,$$

which has both solitary wave and wavetrain solutions.



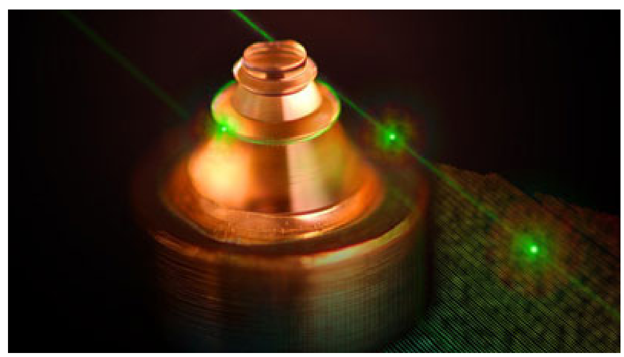


Figure 1: Solitary wave off the coast of Hawaii [1] (left). Optical solitons in a microresonator [7] (right)

Although solitary waves were originally discovered as a water wave phenomenon, they have applications in many fields, including fiber optics, plasma physics, quantum mechanics, molecular biology, and neuroscience (Figure 1). More generally, many nonlinear, dispersive PDEs exhibit solitary wave solutions. My research falls into three broad categories: bifurcations in neural network models, coherent structures in optical lattices, and multi-modal solutions to Hamiltonian systems. In addition, I have done work with undergraduate students in coupled oscillator systems. I will present these in order, starting with my most recent work, and will highlight how I can involve undergraduate students in my research program.

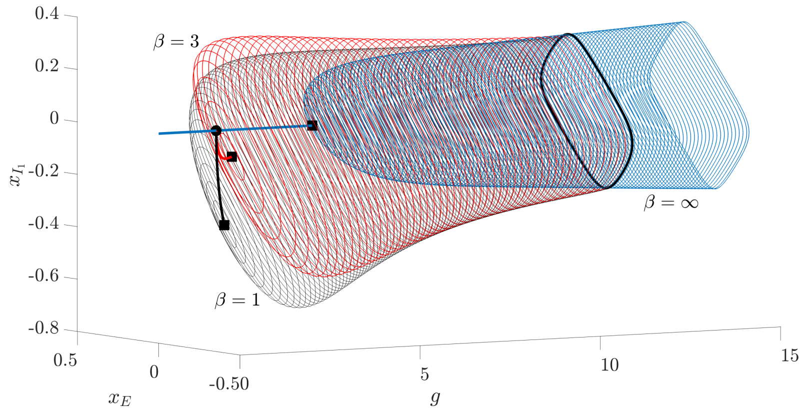


Figure 2: Bifurcation structure of neural network model as the connection strength parameter g is increased, showing fixed points (solid lines) and periodic orbits (rings). Symmetric pitchfork bifurcation indicated with filled circle. Hopf bifurcations indicated with filled squares.

Bifurcations in neural models

My most recent work involves investigating bifurcations in a model of a neural network

$$\dot{\mathbf{x}} = -\mathbf{x} + \frac{1}{\sqrt{N}} H \tanh(g\mathbf{x}),$$

where $\mathbf{x} = (x_1, \ldots, x_N)$ represents the firing rate of each neuron in the network [12, 2]. The individual neurons are connected by a sigmoidal activation function though a connectivity matrix H, which specifies the network topology and neuronal connection weights. Adjusting the parameter g, which represents the global connection strength, leads to a sequence of bifurcations, in which the stability and number of equilibrium points in the system change; periodic solutions may also emerge at these bifurcation points. The specific bifurcation structure depends on symmetries in the matrix H.

For one example, suppose the neurons are grouped into two clusters, one containing excitatory cells and the other containing inhibitory cells. When $g = 0$, the origin is a stable equilibrium point of the system. As g is increased, the origin loses stability in a symmetric pitchfork bifurcation (filled circle in Figure 2). Multiple branches of equilibria emerge from this bifurcation point due to the symmetries of H; the specific configuration of each branch follows from the equivariant bifurcation theorem. As g is further increased, there is a Hopf bifurcation (filled squares in Figure 2) on each branch of equilibria, after which point periodic orbits emerge. At a critical value of g, these limit cycles coalesce (dark band in Figure 2) into a single stable limit cycle. To locate equilibria and bifurcation points, and to compute periodic solutions, I use the technique of numerical parameter continuation. I start with a known solution for a specific value of the parameter g (e.g. $\mathbf{x} = 0$ at $g = 0$), and gradually change g. This can be done by coding an algorithm such as pseudo-arclength continuation in Matlab or Python or by using a specialized software package such as

AUTO [3]. I then use these numerical results as a starting point for theoretical work. For example, the parameter continuation results suggested a perturbation method that I used to prove the Hopf bifurcations in Figure 2 exist, and to determine their location to leading order. Future work with undergraduates includes exploring symmetries and bifurcations in the model resulting from other groupings of neurons, in particular hierarchical clustering models, as well as studying the effects of noise on the model. This research would introduce students to numerical continuation and perturbation methods, which are two essential tools in computational and applied mathematics.

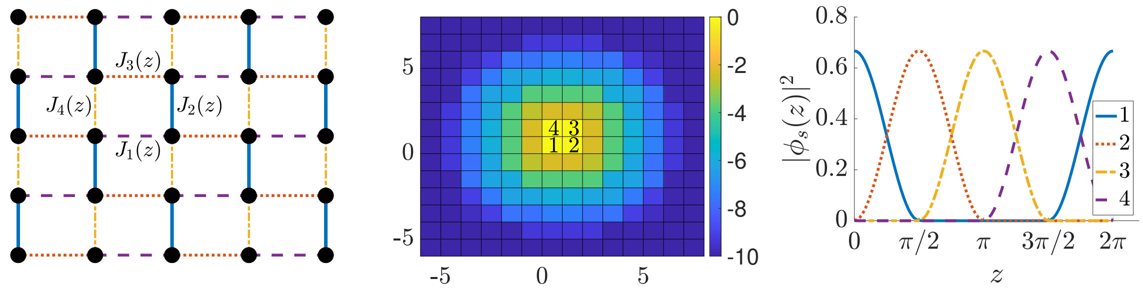


Figure 3: Schematic of square lattice with z-dependent coupling (left). At any value of z, only one of the four coupling functions $J_k(z)$ is nonzero. Log intensity of fundamental breather solution over one period, showing localization to a single unit square in lattice (center). Square intensity of fundamental breather solution on four sites of unit square over one period (right).

Optical lattices

There has been much recent interest in the field of topological photonics, as experimental physicists and engineers have developed new techniques for controlling light propagation in photonic crystals and optical fibers. One specific application is light transmission through multi-core optical fibers. In particular, optical transmission properties can be tuned by introducing a twist to the fiber bundle. The propagation of light through a twisted optical fiber comprising N waveguides arranged in a circle is described by the coupled mode equations

$$i\frac{d}{dz}c_n = k\left(e^{-i\phi}c_{n+1} + e^{i\phi}c_{n-1}\right) + |c_n|^2 c_n \qquad\qquad n = 1, \dots, N,$$

where z is the axis of propagation, k is the strength of the nearest-neighbor coupling, and ϕ is a nondimensional parameter representing the twist of the fiber. When the twist parameter ϕ and the number of fibers N are related by $\phi = \pi/N$ (for N even), I prove that there is a stable standing wave solution of the form $c_n(z) = a_n e^{i\omega z}$ which has a "dark node" with no optical activity opposite a "bright node" of maximum intensity [10]. I also use an asymptotic approach to show that the same phenomenon occurs if the model incorporates a second-order temporal dispersion term [13]. More recent work involves a model of a waveguide consisting of a square lattice of fibers, in which there are periodic variations along the waveguide axis causing the nearest-neighbor coupling to vary periodically in z [8]. In particular, for any z, a waveguide is coupled to only one of its four neighbors (Figure 3, left). This configuration gives rise to localized periodic breather solutions, in which the bulk of the optical intensity is confined to a single square in the lattice but "jumps" around that square counterclockwise (Figure 3, center and right) [11].

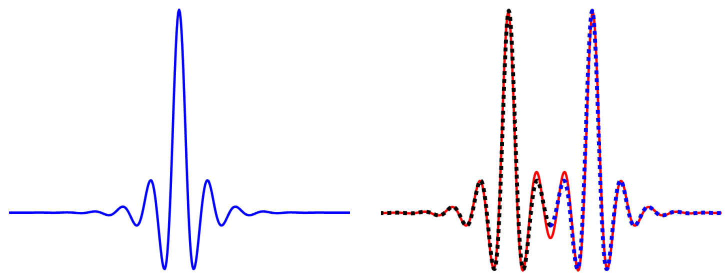


Figure 4: Schematic illustrating construction of a double pulse solution using Lin's method. Left panel shows primary pulse solution. Right panel shows two copies of the primary pulse solution (black and blue dotted lines) placed end-to-end. Double pulse solution (red solid line) is close, but not equal, to this.

Future research with undergraduates involves studying coherent structures in more complicated arrays of optical fibers and correlating the results from the mathematical models with experimental data. In two dimensions, many lattice geometries are possible, including square, triangular, and honeycomb, and these different geometries may exhibit qualitatively different behavior. This research would introduce students to many different computational techniques, including numerical parameter continuation, shooting methods, and energy minimization methods to generate solutions; eigenvalue solvers to compute the spectrum of the linearization of the system about that solution; and numerical ODE and PDE solvers to study how perturbations of the solution evolve in time.

Coherent structures in Hamiltonian systems

The bulk of my published research concerns coherent structures in Hamiltonian systems. At a high level, I start with a simple coherent structure, such as a solitary wave, and use it as a building block to construct more elaborate structures. I then study the stability of these more complicated structures in terms of their underlying geometry, together with properties of the simpler structure.

The prototypical nonlinear wave equation has a primary solitary wave solution, also known as a primary pulse solution, which looks like a single localized "bump" (Figure 4, left). In many systems, including those with applications to nonlinear optics and neuroscience, multi-pulse solutions also exist; these are "multi-bump" solitary waves, which resemble a sequence of multiple, well-separated copies of the primary pulse (Figure 4, right). The entire multi-pulse structure travels as a unit, and it maintains its shape unless perturbed. In my research, I explore the existence and stability of these multi-pulse structures. A crucial step in this process is determining the spectrum of the linearization of the underlying system about a multi-pulse. When a multi-pulse is perturbed, interactions between the individual pulses in the structure are revealed, which are a consequence of the inherent nonlinearity of the system. The dynamics of these interactions are determined by the eigenvalues of the linearized system and their corresponding eigenfunctions.

My primary mathematical approach comes from spatial dynamics. From this viewpoint, a solitary wave is a homoclinic orbit evolving in the spatial variable x. For example, the solitary wave solution $u(x)$ to the KdV equation which travels to the right with constant speed c (Figure 5, left)

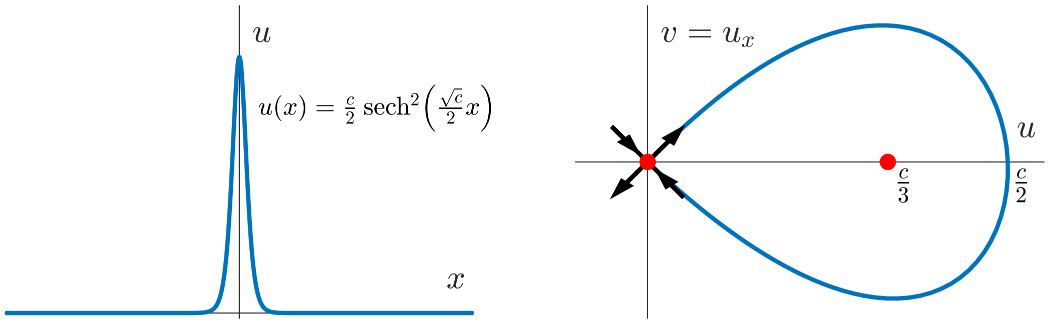


Figure 5: Primary solitary wave solution $u(x)$ to the KdV equation. Left panel is plot of u vs x. Right panel is plot of u_x vs u, showing solitary wave as a homoclinic orbit.

is a solution to the second order ODE

$$u_{xx} + 3u^2 - cu = 0.$$

This can be written as the two-dimensional dynamical system

$$\frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ cu - 3u^2 \end{pmatrix}$$

by taking $v = u_x$. From this perspective, the solitary wave solution is a homoclinic orbit $(u(x), v(x))$ connecting the unstable and stable manifolds of the saddle equilibrium point at the origin (Figure 5, right), which represents the rest state of the system.

From a spatial dynamics perspective, multi-pulses are multi-loop homoclinic orbits. (Notably, multi-pulses do not exist in the KdV equation, since the phase space for the primary homoclinic orbit is two-dimensional). Multi-pulses can be constructed using Lin's method [6], a version of the Lyapunov-Schmidt reduction, which can be used to find solutions that are close to a homoclinic orbit. Heuristically, this process involves "gluing together" multiple copies of the primary pulse solution end-to-end using small remainder functions (Figure 4, right). The existence of multi-pulse solutions is constrained by the geometry of the primary pulse and the underlying system. In Figure 4, for example, multi-pulses can only be constructed when the tail oscillations of the individual pulses overlap in-phase. In general, each pulse that is added to a multi-pulse structure is associated with one or more eigenvalues in the spectrum of the linearized system. I call these interaction eigenvalues, since they result from nonlinear interactions between the tails of neighboring pulses. Since the systems I study are Hamiltonian, all eigenvalues must come in quartets of the form $\pm\alpha \pm \beta i$. Although this additional structure is helpful, it also means that the presence of any eigenvalue with nonzero real part implies that there is an unstable eigenvalue with positive real part. As a consequence, Hamiltonian systems cannot be dissipative, which makes stability analysis more difficult. My main results relate the spectral pattern of the interaction eigenvalues to the underlying geometry of the multi-pulse. In all cases, the spectral pattern is determined by this geometry.

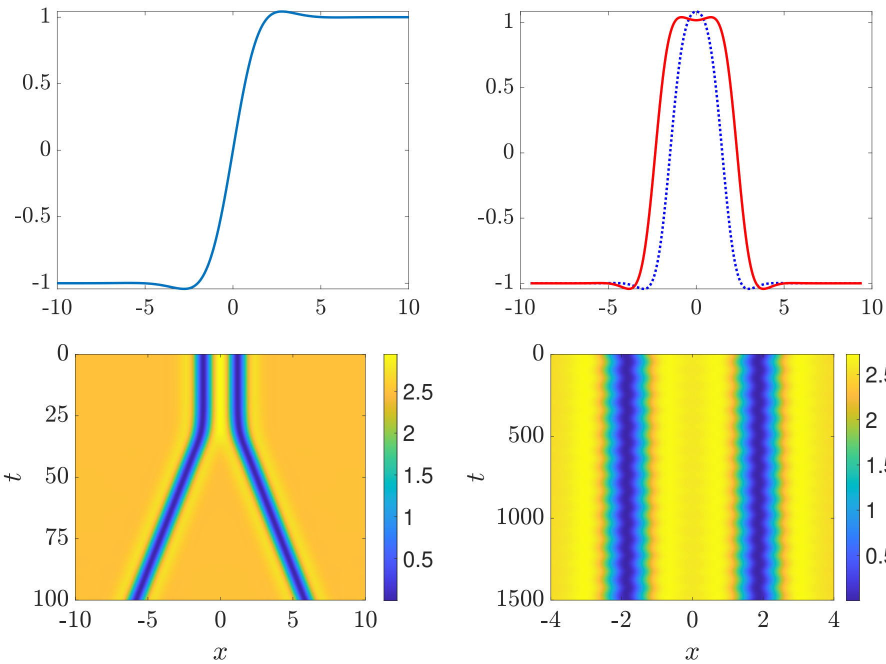


Figure 6: Top left: kink solution to NLS4 for $\beta_4 > 0$. Top right: two kink-antikink solutions; blue dotted line is unstable, red solid line is spectrally neutrally stable. Bottom: time evolution of power of unstable (left) and neutrally stable (right) kink-antikink.

A specific equation I have studied recently is the fourth-order generalization of the nonlinear Schrödinger equation (NLS4)

$$iu_t + \frac{\beta_4}{24}u_{xxxx} - \frac{\beta_2}{2}u_{xx} + \gamma|u|^2 u = 0,$$

which was introduced to account for the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity [5, 16]. Motivated by recent experiments [15], there is particular interest in the case where $\beta_2 = 0$, in which case the system exhibits pure quartic dispersion; the resulting bright solitary wave solutions are known as pure quartic solitons (PQS), which exist when $\beta_4 < 0$. I prove that while multi-pulse solitary wave solutions exist, they are all unstable due to the presence of at least one interaction eigenvalue with positive real part [9, Theorems 1 and 2].

Some recent work explores the $\beta_4 > 0$ parameter regime, in which the primary coherent structure is a “kink” (heteroclinic orbit) connecting two nonzero background states (Figure 6, top left). Since the kink solution does not decay to 0, it is natural to pose the problem on a finite domain with Neumann boundary conditions. Kink-antikink solutions can then be constructed, in which a kink solution $u(x)$ and an anti-kink solution $-u(x)$ are spliced together (Figure 6, top right). In contrast to the bright solitary wave solutions, numerical computations suggest that the family of kink-antikinks alternates between unstable solutions (Figure 6, bottom left), in which the kink centers repel each other, and neutrally stable solutions (Figure 6, bottom right), in which the kink centers exhibit oscillatory behavior. Both behaviors can be explained by the spectrum of the corresponding solutions.

Another avenue of future research involves extending the model to incorporate terms corresponding to gain and loss of energy in the laser cavity, as in done in the Lugiato-Lefever equation, as well as Raman scattering, which leads to symmetry distortion and reduced pulse amplitude. I am currently mentoring a graduate student who is exploring this problem. We have found a family of stable multi-pulse solutions, and it would interesting to see if similar solutions could be created experimentally. Another direction to explore would be to extend the model to incorporate nonlocal effects using a convolution term.

Coupled oscillators

Coupled oscillators are an active area of research, and projects on coupled oscillators are well suited for undergraduates, as they employ both analytical and computational techniques. In the summer of 2021, I supervised an REU in which undergraduate students learned about different coupled oscillator models and then designed their own research projects to explore these models computationally. One example is the Kuramoto model

$$\frac{d\theta_j}{dt} = \omega_j + \frac{K}{N}\sum_{k=1}^{N} \sin(\theta_k - \theta_j) \qquad\qquad j = 1, \ldots N,$$

which was developed to study synchronization in systems of chemical and biological oscillators; it has numerous applications, including circadian rhythms, chirping crickets, and flashing fireflies. The model describes a set of N oscillators with phases θ_j and natural frequencies ω_j. Every oscillator is connected to every other oscillator using a nonlinear sinusoidal function, and the parameter K controls the strength of this coupling. For a critical value of K, which depends on the initial distribution of natural frequencies ω_j, the oscillators will start to synchronize, despite their different natural frequencies [14]. As the model evolves in time, the degree of synchrony can be measured by computing the complex order parameter

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{i\theta_j},$$

which characterizes the "collective rhythm" of the oscillators [14]. The modulus r quantifies the phase coherence, with $r = 0$ representing no synchrony, and $r = 1$ representing complete synchrony. My students found that as the coupling parameter K is rapidly varied, the system gains synchrony faster when K is increased than it loses synchrony when K is decreased. They presented this result on a poster at the regional SIAM TX-LA conference. Future work with undergraduates includes examining models in which the connections between oscillators are specified by an adjacency matrix, and exploring a second-order variant of the Kuramoto model [4] which has applications to electrical power grids.

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