Scientific Summary

My main research focus is on coherent structures in dynamical systems. Dynamical systems are mathematical models which evolve in time. A prototypical example is a model of a swinging pendulum, which is an ordinary differential equation (ODE) describing the evolution of the angular position and angular velocity of the system. The class of dynamical systems I study is nonlinear wave equations, which are typically expressed mathematically as partial differential equations (PDEs). These equations describe the time evolution of a function representing a wave profile. In particular, I study coherent structures in Hamiltonian systems. Hamiltonian systems are characterized by a conserved quantity, such as energy, that remains constant as the system evolves in time. Coherent structures are spatial patterns which maintain their shape as the system evolves in time. Examples of coherent structures include solitary waves, localized disturbances that maintain their shape as they propagate at a constant velocity, and wavetrains, periodic disturbances that also propagate at a constant velocity. This has been a topic of interest since the 19th century, when John Scott Russell observed a single, large surface wave on the Edinburgh-Glasgow Union Canal in Scotland which propagated along the canal undisturbed for a few miles before eventually decaying. This phenomenon was explained mathematically by the Korteweg–De Vries (KdV) equation

$$u_t + u_{xxx} + 6uu_x = 0,$$

which has both solitary wave and wavetrain solutions.

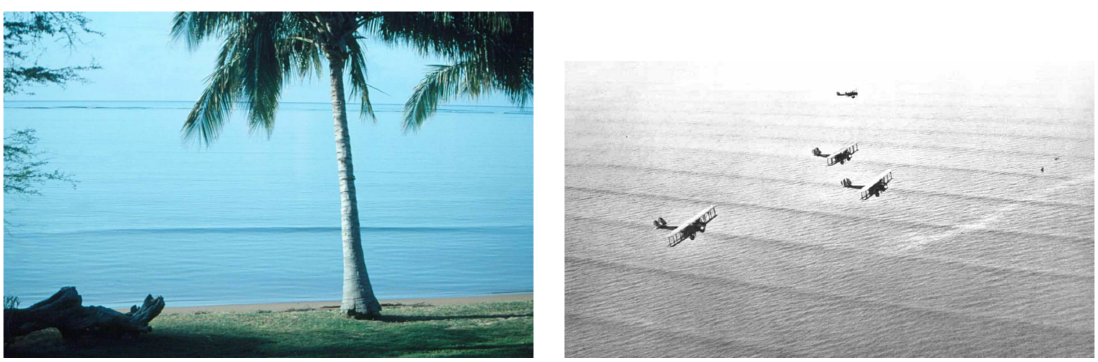

Figure 1: Solitary wave off the coast of Hawaii [1] (left). Periodic wavetrain off the coast of Panama (right, from National Geographic, 1933)

Although solitary waves were originally discovered as a water wave phenomenon, they have applications in many fields such as fiber optics, plasma physics, quantum mechanics, molecular biology, and neuroscience. More generally, many nonlinear, dispersive PDEs exhibit solitary wave solutions.

Multi-pulse solitary waves

The prototypical nonlinear wave equation has a primary solitary wave solution, also known as a pulse solution, which looks like a single localized "bump". In many systems, multi-pulse solutions exist. These are "multi-bump" solitary wave which resembles multiple, well separated copies of the primary pulse solution. (Notably, multi-pulse solutions do not exist to the KdV equation). The entire multi-pulse travels as a unit and maintains its shape unless perturbed. In addition to having applications in nonlinear optics and neuroscience, multi-pulses are interesting mathematically. In my research, I explore the existence and stability of these multi-pulse structures. A crucial step in this process is determining the spectrum of the linearization of the underlying system about these multi-pulse solutions. When a multi-pulse is perturbed, interactions between the individual pulses in the structure are revealed, which are a consequence of the inherent nonlinearity of the system. The dynamics of these interactions are determined by the eigenvalues of the linearization and their corresponding eigenfunctions.

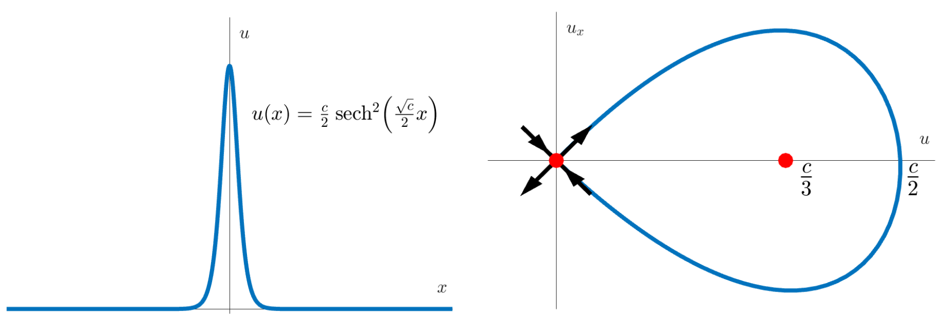

Figure 2: Primary solitary wave solution $u(x)$ to the KdV equation. Left panel is plot of u vs x. Right panel is plot of u_x vs u, showing solitary wave as homoclinic orbit.

My primary mathematical approach comes from spatial dynamics. From this viewpoint, a solitary wave $u(x)$ is a homoclinic orbit evolving in the spatial variable x. For example, the solitary wave for the KdV equation which moves with constant speed c is a solution $u(x)$ to $u_{xx} + 3u^2 - cu = 0$ (Figure 2, left panel). This ODE can be written as the two-dimensional dynamical system

$$\frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ cu - 3u^2 \end{pmatrix}$$

by taking $v = u_x$. From this perspective, the solitary wave is a solution $(u(x), v(x))$ which is a homoclinic orbit connecting the unstable and stable manifolds of the saddle equilibrium at the origin (Figure 2, right panel). This equilibrium point represents the rest state of the system.

Multi-pulses are multi-loop homoclinic orbits. From a theoretical standpoint, multi-pulses

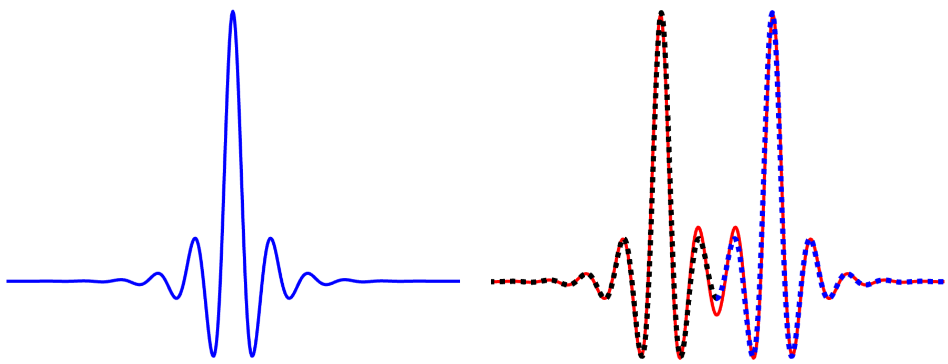

Figure 3: Cartoon illustrating construction of a double pulse solution using Lin's method. Left panel shows primary pulse solution. Right panel shows two copies of the primary pulse solution (black and blue dotted lines) placed end-to-end. Double pulse solution (red solid line) is close to this.

can be constructed using Lin's method [10], a version of the Lyapunov-Schmidt method used to find solutions which remain near a homoclinic orbit. Heuristically, this process involves "gluing together" multiple copies of the primary pulse solution end-to-end Figure 3 using small remainder functions. Lin's method can also be used to construct periodic orbits and multi-loop periodic orbits.

Construction of multi-pulse solutions is constrained by the geometry of the underlying system. In Figure 3, for example, multi-pulses only exist when the tail oscillations of the individual pulses match up. Once a multi-pulse has been constructed, the nex

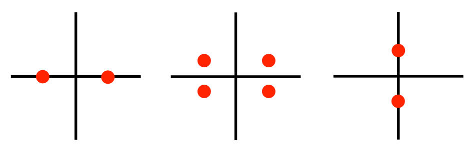

Figure 4: Possible interaction eigenvalue patterns.

The first step in this process is to analyze the spectrum of the linearization of the underlying system about a multi-pulse solution. In general, each pulse that is added to a multi-pulse structure is associated with one or more eigenvalues in the spectrum [22]. I refer to these as interaction eigenvalues, since they result from nonlinear interactions between the tails of neighboring pulses. The systems I study are Hamiltonian, which are not covered by the results of [22]. On one hand, the Hamiltonian structure is very helpful, since all eigenvalues must come in quartets of the form $\pm\alpha \pm \beta i$. This means that each additional set of interaction eigenvalues must occur in one of the three patterns in Figure 4. On the other hand, the presence of any eigenvalue with nonzero real part implies that there is an unstable eigenvalue. This means that Hamiltonian systems cannot be dissipative, which makes stability analysis more difficult. My main results relate the spectral pattern of the interaction eigenvalues to the underlying geometry of the multi-pulse. In all cases, the spectral pattern is determined by the geometry of the underlying multi-pulse solution.

In addition to theoretical work, I also use numerical analysis, both to generate hypotheses and to verify analytical results. For multi-pulses, I start by constructing the primary solitary wave, which either involves parameter continuation from a known solution using the software package AUTO or an energy minimization method such as the string method [4] or the mountain pass method [5]. I then glue together multiple copies of the primary solitary wave and use a Newton solver such as Matlab's `fsolve` function or a conjugate gradient method to construct a multi-pulse.

Discrete nonlinear Schrödinger equation

The discrete nonlinear Schrödinger equation (DNLS)

$$i\partial_t u_n + d(u_{n+1} - 2u_n + u_{n-1}) + |u_n|^2 u_n \qquad\qquad n \in \mathbb{Z}$$

is the discrete analogue to the nonlinear Schrödinger equation (NLS) on the integer lattice. The parameter d quantifies the degree of coupling between adjacent lattice sites. In addition to being a fundamental model of a nonlinear dynamical system on a lattice, DNLS has applications to nonlinear optics and condensed matter physics [8]. For all values of d, DNLS has a stable, primary pulse solution (Figure 5, left panel). Provided that the individual peaks are separated by a sufficiently large number of lattice points, we prove using Lin's method that multi-pulse solutions exist as long as a geometric constraint is satisfied: neighboring peaks must either be in-phase or out-of-phase [16]. Furthermore, we prove this geometry determines the stability of multi-pulses. For double pulses, the in-phase double pulse is unstable, since there is eigenvalue with positive real part, and the out-of-phase double pulse is neutrally stable, since the entire spectrum lies on the imaginary axis (see Figure 5). For general multi-pulses, the entire structure is unstable if any pair of neighboring peaks are in-phase. The only neutrally stable multi-pulses are those where every pair of neighboring peaks is out-of-phase. The proof of this result adapts the method of [22] to the Hamiltonian case, and uses Lin's method to construct the eigenfunctions corresponding to the interaction eigenvalues. This reduces the spectral problem for an n-pulse to finding the eigenvalues of an $n \times n$ matrix.

Chen-McKenna suspension bridge equation

The Chen-McKenna suspension bridge equation

$$\partial_t^2 u + \partial_x^4 u + e^{u-1} - 1 = 0$$

is a smooth approximation to a model for waves propagating on an infinitely long suspended beam, and is motivated by observations of traveling waves on suspension bridges [13, 5].

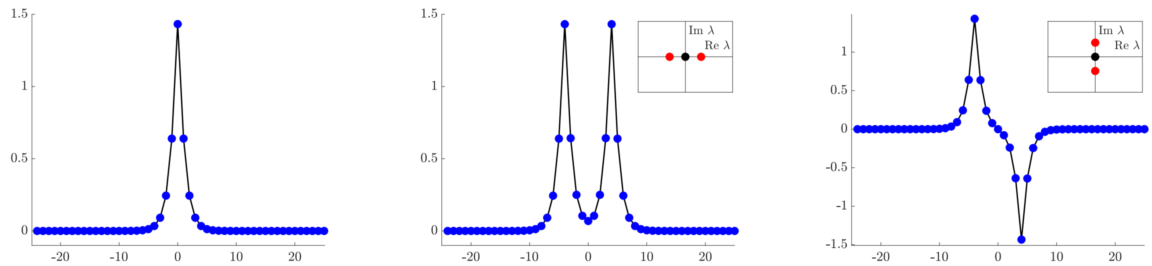

Figure 5: Primary pulse solution for DNLS (left). Out-of-phase (middle) and in-phase (right) double pulse solutions for DNLS. Interaction eigenvalue patterns for double pulses are shown in insets. Black dot is a kernel eigenvalue with algebraic multiplicity 2.

For wave speeds c between 0 and $\sqrt{2}$, a primary solitary wave solution exists [24]. This solution has exponentially decaying, oscillatory tails (Figure 6, top left). Provided that the individual peaks are sufficiently well separated, multi-pulse solutions again exist as long as a geometric constraint is satisfied: the tail oscillations of neighboring peaks must be "overlap", i.e. must be in-phase (see the cartoon illustration in Figure 3). This constraint is a consequence of the specific alignment of the unstable and stable manifolds which is necessary for multi-pulses to occur. As a result, the distance between consecutive peaks is, to leading order, an integer multiple of a phase parameter. This is illustrated in the top right panel of Figure 6, which plots the first four double pulse solutions on the same graph. As with DNLS, the stability of multi-pulses depends on their geometry. Double pulses, for example, alternate between unstable (bottom left of Figure 6, corresponding to even integers) and neutrally stable (bottom right of Figure 6, corresponding to odd integers). I proved these spectral results using an extension of the Krein matrix [6], a tool which projects the infinite-dimensional spectral problem onto a finite-dimensional space.

Fourth order nonlinear Schrödinger equation

The fourth-order nonlinear Schrödinger equation (NLS4)

$$iu_t + \frac{\beta_4}{24} u_{xxxx} - \frac{\beta_2}{2} u_{xx} + \gamma |u|^2 u = 0$$

is a variant of the nonlinear Schrödinger equation which was introduced to account for the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity [7, 23]. There is particular interest in the case where $\beta_2 = 0$, in which case the system exhibits pure quartic dispersion. The corresponding solitary wave, known as pure quartic solitons, have been created experimentally. In [14], we prove that while multi-pulse solitary wave solutions exist, they unstable to the presence of at least one eigenvalue with positive real part.

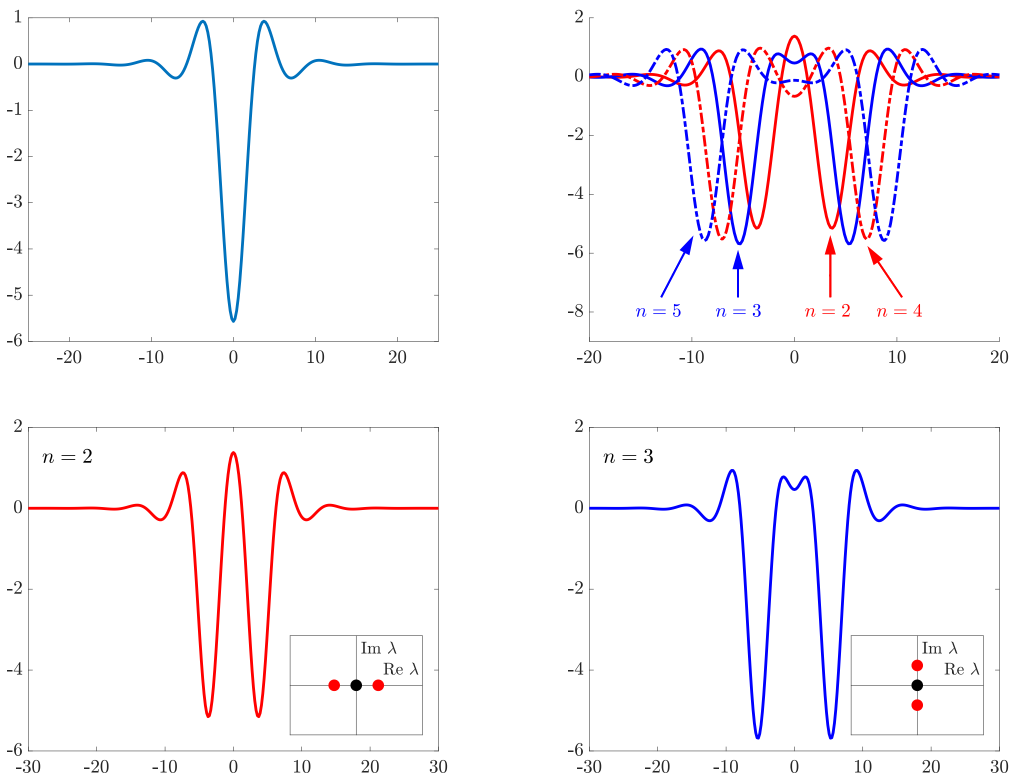

Figure 6: Primary pulse solution for Chen-Mckenna (top left). First four double pulse solutions (top right). Unstable double pulse for $n = 2$ (bottom left) and neutrally stable double pulse for $n = 3$ (bottom right). Interaction eigenvalue patterns for double pulses are shown in insets. Black dot is a kernel eigenvalue with algebraic multiplicity 2.

Fifth order KdV equation

The fifth-order Korteweg de-Vries equation (KdV5)

$$\partial_t u - \partial_x^5 u + \partial_x^3 u + 2u\partial_x u = 0$$

is a weakly nonlinear long wave approximation to capillary-gravity wave problem which also has applications to plasma physics and laser optics [20]. Multi-pulse solutions to KdV5 exist [21], but their stability analysis is complicated due to the fact that the essential spectrum for all localized solutions comprises the entire imaginary axis. In particular, this means that any purely imaginary interaction eigenvalues would be embedded in the essential spectrum, which makes the analysis difficult.

To simplify the situation I impose periodic boundary conditions on the problem and look at periodic multi-pulses, which are periodic orbits containing multiple peaks. From a spatial dynamics perspective, a periodic multi-pulse is a multi-loop periodic orbit which is close to the primary homoclinic orbit. A periodic double pulse, for example, is constructed by gluing two single pulses together at both ends. Since this construction involves two length parameters X_0 and X_1 (see Figure 7, left), there is an extra degree of freedom when compared to double pulses on the real line. As a consequence, I prove that periodic double pulses exist in a continuous family in which asymmetric periodic double pulses (those with $X_0 \neq X_1$) bifurcate from symmetric periodic 2-pulses (those with $X_0 = X_1$) in a series of pitchfork bifurcations (Figure 7, center) [18]. The advantage of looking at periodic solutions is that the essential spectrum becomes a discrete set of eigenvalues on the imaginary axis (blue open circles in Figure 7, right). Purely imaginary interaction eigenvalues can then lie in gaps between essential spectrum eigenvalues, which avoids the problem of embedded eigenvalues.

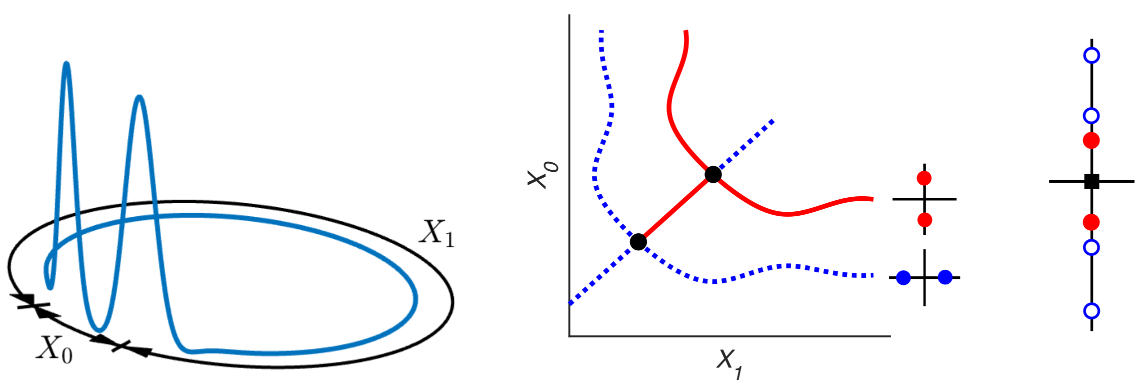

Figure 7: Construction, bifurcation, and spectrum of periodic 2-pulses for KdV5.

The eigenvalues associated with a periodic multi-pulse can be found by solving a block matrix equation [18, Theorem 5.3] for the eigenvalues λ. To leading order, this is given by

$$\det \begin{pmatrix} K(\lambda) - \frac{1}{2}\lambda \tilde{M} K^+(\lambda) & \lambda^2 M_c I \\ -\frac{1}{2}\lambda M_c K^+(\lambda) & A - \lambda^2 M I \end{pmatrix} = 0. \tag{1}$$

The essential spectrum eigenvalues are encoded by the matrix $K(\lambda)$, which only depends on the background state and size of the periodic domain; in particular, it is independent of the periodic multi-pulse solution being studied. The interaction eigenvalues are encoded by the matrix A, which depends on the geometry of the periodic multi-pulse. As long as the periodic domain size is not too large, the interaction eigenvalues and essential spectrum eigenvalues do not interfere with each other. For a periodic double pulse, there is a pair of interaction eigenvalues which is either real or purely imaginary depending on the geometry of the periodic 2-pulse (Figure 7, center); the eigenvalue pattern switches between real and imaginary at the pitchfork bifurcation points.

There is, however, an additional complication in the periodic case. As the periodic domain size X is increased, the essential spectrum eigenvalues move towards the origin. At a critical value of X, there will be a collision between one of the essential spectrum eigenvalues and a purely imaginary interaction eigenvalue. As X is further increased, I prove that a brief instability bubble is formed, wherein the two eigenvalues collide, move off the imaginary axis, trace an approximate circle in the complex plane, and recombine on the imaginary axis in a "reverse" collision (see left panel of Figure 8 for a cartoon) [18]. This instability bubble, which we call a Krein bubble, is a direct consequence of the block matrix reduction (1). A numerical simulation of the Krein bubble, performed using parameter continuation with AUTO, is shown in Figure 8 (right). The location and size of the Krein bubble in the simulation agree with that predicted by the theory [18].

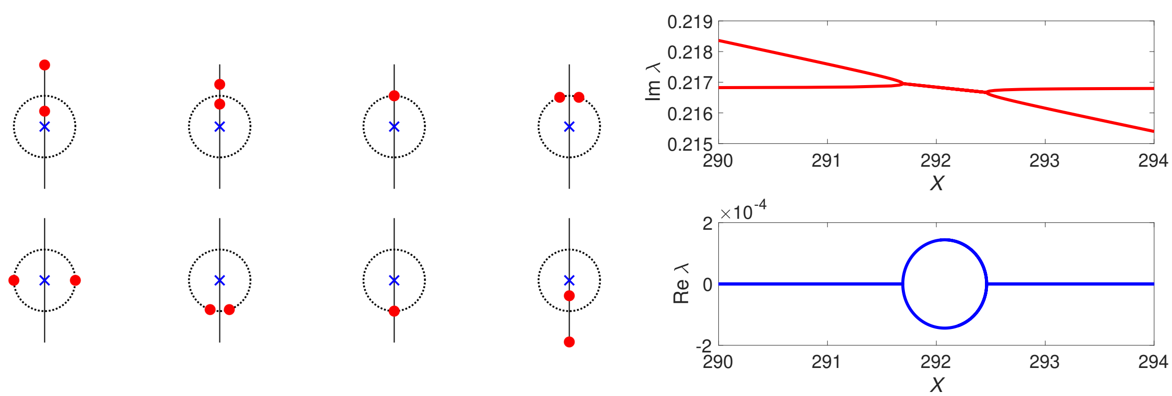

Figure 8: Krein instability bubble cartoon (left) and numerical simulation for KdV5 (right).

Other systems and future directions

Other systems of interest include the discrete sine-Gordon equation, nonlocal lattice equations, and equations in higher dimensions. The discrete sine-Gordon equation

$$\ddot{u}_n = d(\Delta_2 u)_n \pm \sin(u_n) \qquad\qquad n \in \mathbb{Z}$$

was introduced to describe the dynamics of crystal lattices, and has since been used in numerous applications, including a mechanical model for a chain of pendula coupled with elastic springs, arrays of Josephson junctions, and DNA dynamics [2]. A particular class of coherent structures of interest in this system are kinks, which are exponentially localized stationary solutions connect adjacent minima of the potential $V(u) = \cos u$. In [17], we prove the existence of stationary multi-kink solutions as well as analyze their spectral stability. Another class of coherent structures for this system are breathers, which are localized, oscillatory patterns. Future work involves using Lin's method to construct multi-site breathers by splicing together multiple, sequential copies of the primary breather and to then analyze their stability.

The following version of DNLS with nonlocal interactions

$$i\partial_t u_n = \frac{1}{h^{2s}} \sum_{m \neq n} \frac{u_m - u_n}{|m-n|^{1+2s}} + |u_n|^2 u_n \qquad\qquad n \in \mathbb{Z},$$

where $s > 0$ is a fixed parameter specifying the decay of the nonlocal interactions, is considered in [9]. They prove that in the continuum limit $h \to 0$, solutions converge to solutions of an NLS equation with fractional Laplacian. Applications include a model for charge transport in DNA polymers. Future directions include studying whether this system, like the ordinary DNLS equation, possesses solitary wave and multi-pulse solutions, as well as whether the nonlocal interaction term permits other solutions which are not found in DNLS. A final, related area of interest is coherent structures in DNLS on the square integer lattice $\mathbb{Z}^2$. This could include nonlocal interactions in one or both directions.

Optical lattices

There has been much recent interest, both by experimental physicists and applied mathematicians, in the propagation dynamics of light pulses through arrays of optical fibers. One particular application is light transmission through multi-core optical fibers. In particular, optical transmission properties can be tuned by introducing a twist to the entire fiber bundle [11, 3, 19] (Figure 9).

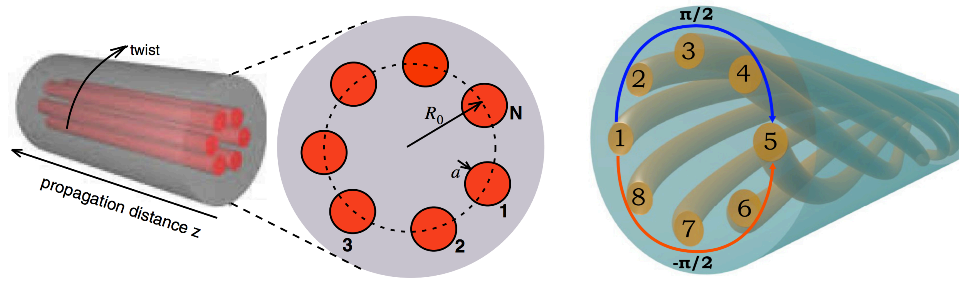

Figure 9: Schematic of circular multi-core fiber [11] (left) and twisted multi-core fiber with eight waveguides [19] (right).

The propagation of light through a system of N waveguides arranged in a circle can be described by the coupled mode equations

$$i\partial_z c_n = k\left(e^{-i\phi}c_{n+1} + e^{i\phi}c_{n-1}\right) + |c_n|^2 c_n \qquad\qquad n = 1, \ldots, N,$$

where $c_0 = c_N$ and $c_{N+1} = c_1$ due to the circular geometry, k is the strength of the nearest-neighbor coupling, and ϕ is a parameter representing the twist of the fiber. When the twist parameter ϕ and the number of fibers in the bundle N are related by $\phi = \pi/N$, we prove that there is a stable standing wave solution of the form $c_n(z) = a_n e^{i\omega z}$ which as a "dark node" with no optical activity opposite a "bright node" of maximum intensity [15] (Figure 10).

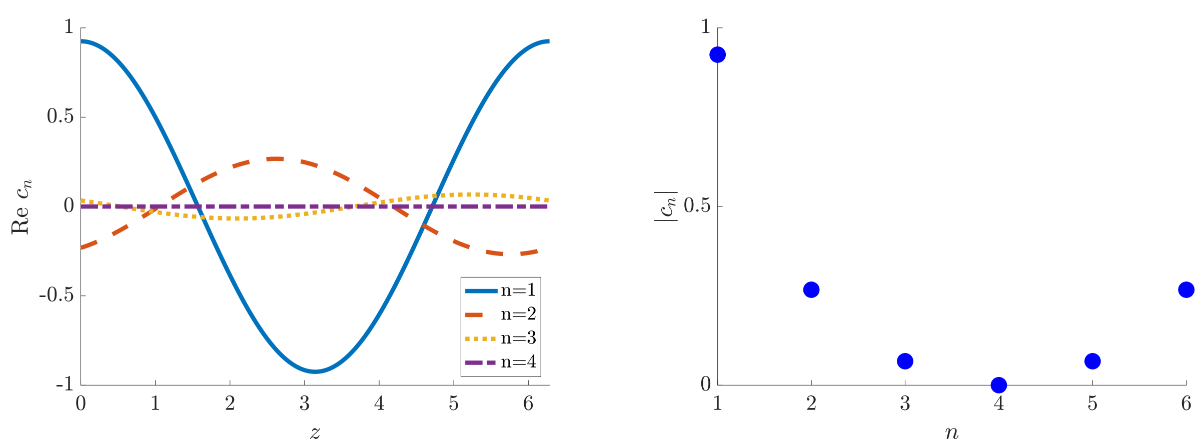

Figure 10: Standing wave solution $c_n(z)$ for twisted multi-core fiber with $N = 6$ and $\phi = \pi/6$. Evolution of real part of solution of first four sites (left) and magnitude of solution at the six lattice points (right). Node 1 has maximum intensity, and opposite node 4 has zero intensity.

Future research involves studying other configurations of optical fibers and which have been employed by experimentalists. One example is a waveguide consisting of a square lattice of fibers in which there is periodic variations along the waveguide axis. These cause the nearest-neighbor coupling in the lattice to vary periodically, and give rise to periodic breather solutions in which bulk of the optical intensity is confined to a single square in the lattice and "jumps" around that square. No systematic mathematical study of the existence and stability of these solutions has been done to date. Other, more complicated arrangements of fibers which are of interest to experimentalists include concentric rings and honeycomb lattices [12].

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