DISCRETE MATHEMATICS 6 (1973) 79–88. © North-Holland Publishing Company

ALGORITHMS FOR CONSTRUCTING GRAPHS AND DIGRAPHS WITH GIVEN VALENCES AND FACTORS*

D.J. KLEITMAN and D.L. WANG

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Mass. 02139, USA

Received 11 August 1972**

Abstract. Given a set of valences $\{v_i\}$ such that $\{v_i\}$ and $\{v_i-k\}$ are both realizable as valences of graphs without loops or multiple edges, an explicit construction method is described for obtaining a graph with valences $\{v_i\}$ having a k-factor. A number of extensions of the result are obtained. Similar results are obtained for directed graphs.

1. Introduction

In this note we obtain a short proof of a result of Kundu [3] (also obtained by Chungphaison in certain cases) that if the sequences $\{v_i\}$ and $\{v_i-k\}$ are both realizable as the valences of the vertices of a graph without loops or multiple edges, then there is a realization of the former which has the one of the latter as a subgraph. This proof gives rise to some simple extensions of the result.

The argument, like that of Kundu, applies when k above is replaced by k_i with k_i either k or $k+1$ for each i; it applies as well if $0 < k_i = k$ or $k+1$ for each i except i_0 with $0 \leq k_{i_0} \leq v_i$. This and other extensions are described in the final section.

It leads as well to a simple algorithm for constructing the graph G described above. An algorithm for the case of directed graphs is also presented in Section 4.

In Section 2 we describe the construction method for realization of a simple sequence, and finally the method as modified to realize $\{v_i\}$ and $\{v_i-k_i\}$ simultaneously. The simple sequence realization method has been known and has appeared in the literature [1, 2].

* Presented in part at the Monterey Conference on Algorithms, January 1972. Supported in part by ONR Contract N00014-67-A-0204-0016.

** Original version received 6 April 1972.

2. The simple sequence realization method

Theorem 2.1. *Suppose the sequence of natural numbers (in non-increasing order) $\{v_j\}$ can be realized as the valences of the vertices (V_j) of graph G. Then they can be realized as the valences of a graph in which vertex V_k is adjacent to the first v_k vertices other than itself.*

Proof. If otherwise, let G be a realization chosen to maximize the number of vertices adjacent to V_k among the first v_k. Let V_m be a vertex not adjacent to V_k in G with $m \leq v_k$ or, if $k \leq v_k$, with $m \leq v_k + 1$, i.e., with V_m among the first v_k vertices. Let V_q be a vertex not among the first v_k that is adjacent to V_k in G. Then $v_m > v_q$ (if $v_m = v_q$, names could be interchanged), and hence V_m is adjacent to some vertex V_p, with $p \neq q$, such that $(V_q\, V_p)$ is not in G. If we alter G by removing edges $(V_m\, V_p)$ and $(V_k\, V_q)$ and replacing them by $(V_m\, V_k)$, $(V_p\, V_q)$, we obtain a graph G' with one more vertex adjacent to V_k among the first v_k, violating the definition of G. This theorem leads to the following algorithm.

Algorithm 2.2. For constructing a graph realizing a set of vertex valences. Choose any vertex V_k. "Lay it off" by connecting it by edges to the first v_k (other) vertices on a list of vertices arranged in non-increasing valence order, removing V_k from the list, and reducing the residual valences of vertices to which it is attached by one. Reorder the vertices if necessary. Repeat with another vertex until there are none.

3. Realizing valences with a factor

Suppose now we are given two sequences $\{v_k\}$ and $\{v_k - p_k\}$, with $p_k = p$ or $p + 1$ for each k, both of which are realizable by graph valences without loops or multiple edges. We define two steps for constructing a realization G_T of $\{v_k\}$ which contains a realization G_S of $\{v_k - p_k\}$ as a subgraph. These are:

Step 1. A "laying off" step, in which a vertex V_k is connected by edges in G_T to the first a_k vertices (excluding itself) with the vertices arranged in non-increasing order of residual a-valence, and to the first b_k vertices (excluding itself) in G_S with the vertices ordered by residual b-valence. Again residual valences are then recomputed for each vertex and they are reordered for the purpose of future "laying off". The difference between this step and the step in Algorithm 2.2 above is that here, if any vertex V_k has $a_k - b_k = 0$, it must be "laid off" before any vertex V_p with $a_p - b_p > 0$, whereas the order of laying off is arbitrary in Algorithm 2.2.

Step 2. A "switching" step, which is to be used after each laying off step that attempts to connect V_k by an edge in G_S which is not in G_T.

After such connections are eliminated by switching steps, each vertex laid off will be connected to vertices in G_S only if they are so connected in G_T. If the layoff vertex V_k had $a_k - b_k = 0$, the values of $a_p - b_p$ for remaining vertices will be unchanged after the layoff and switching. Otherwise, they will be decreased by one, by the procedure, for any vertex which is connected to V_k in G_T and not in G_S. It follows that if we always lay off vertices in Step 1 which have $a_k - b_k = 0$ first, $a_p - b_p$ can never become negative for residual vertices.

Let, at the beginning, the vertices be arranged in non-increasing order of a-valence and b-valence simultaneously. A vertex V_k, being laid off in G_T and G_S, can then get connected to a vertex V_q in G_S and not in G_T when $a_k - b_k \geq 0$ only if non-increasing order of the remaining vertices becomes altered in G_S or in G_T so that V_q appears closer to the beginning for G_S than it does for G_T. Since $a_k - b_k \geq 0$ when V_k is laid off, for each V_q so connected we can find a V_m such that V_k lays off against V_m in G_T and not in G_S.

If $a_q - a_m = 0$ when V_k is laid off, then the order of V_q and V_m in G_T is immaterial – so we can reverse it by inserting $(V_q V_k)$ rather than $(V_m V_k)$ in G_T. Similarly, if $b_q - b_m = 0$, we can replace $(V_q V_k)$ in G_S by $(V_m V_k)$ since the ordering of V_q and V_m, when laying out V_k in G_S, was immaterial.

We need therefore only consider cases in which $a_q - a_m < 0$ and $b_q - b_m > 0$, so that $a_q - b_q \leq (a_m - b_m) - 2$. Since at the beginning we had $(a_q - b_q) \geq (a_m - b_m) - 1$, we must have laid off at least one vertex (V_s below) which diminished $(a_q - b_q)$ without changing $(a_m - b_m)$.

There remain two significant subcases; if V_s is adjacent to V_m in both G_T and G_S as constructed so far, we have the following edges present.

$(V_m V_s)$ in G_T and G_S,

$(V_q V_s)$ in G_T only,

$(V_q V_k)$ in G_S only,
$(V_m V_k)$ in G_T only,
which we switch by replacing $(V_m V_s)$ and $(V_q V_k)$ in G_S by $(V_m V_k)$ and $(V_q V_s)$. This switch preserves all valences achieved so far.

If V_s is adjacent to V_m neither in G_T nor G_S, the following edges are present:
$(V_q V_s)$ in G_T only,
$(V_q V_k)$ in G_S only,
$(V_m V_k)$ in G_T only,
and we replace the two edges in G_T by $(V_q V_k)$ and $(V_m V_s)$. Again all valences are preserved.

By successive application of laying off, and switching when necessary, one can construct the entire graph.

The algorithm for same can be summarized as follows.

Algorithm 3.1. If there are no two residual vertices, stop. If there are, then:

Step 1. Compute the residual valences a_k and b_k of vertex V_k for each k, in G_S and G_T. Proceed to Step 2.

Step 2. If there are residual vertices with $a_k - b_k = 0$, choose one otherwise choose any residual vertex V_k. Remove V_k from the list of residual vertices. Go to Step 3.

Step 3. Connect V_k (tentatively) to the a_k vertices other than V_k having largest a value in G_T and to the b_k vertices having largest b value in G_S. Proceed to Step 4.

Step 4. If $G_S \subset G_T$ so far, go to Step 1. If an edge $(V_k V_q)$ is in G_S and not in G_T, find an edge $(V_k V_m)$ in G_T and not in G_S. If $a_q - a_m = 0$, remove $(V_k V_m)$ from G_T and insert $(V_k V_q)$. If $b_q - b_m = 0$, remove $(V_k V_q)$ from G_S and insert $(V_k V_m)$. If $a_q - a_m < 0$ and $b_q - b_m > 0$, find a vertex V_s such that $(V_s V_q)$ is currently assigned to $G_T - G_S$, and $(V_s V_m)$ is currently in neither or both G_S and G_T. Go to Step 5.

Step 5. If $(V_s V_m)$ is in both G_S and G_T, replace edges $(V_m V_s)$ and $(V_q V_k)$ in G_S by $(V_m V_k)$ and $(V_q V_s)$. If $(V_m V_s)$ is in neither, replace edges $(V_q V_s)$ and $(V_m V_k)$ in G_T by $(V_q V_k)$ and $(V_m V_s)$. Go to Step 4.

This procedure constructs G_T and G_S simultaneously, so long as $\{v_k\}$, $\{v_k - p_k\}$ and $\{p_k\}$ were all realizable as valence sequences and $p_k = p$ or $p + 1$. It extends however to many other cases. Thus if, for some particular m, $v_m \geq p_m \geq 0$ and we omit the restriction $p_m = p$

or $p + 1$, we can lay off V_m first, and the procedure will work as above so long as there are no other vertices having $p_k = 0$. Thus if $p \geq 1$, one p_m may be chosen arbitrarily. Another extension is that if $b_m = 0$, then whenever the switching procedure is needed at V_m, we can only encounter the situation that $a_m - a_q > 0$, $b_q - b_m > 0$ (since $b_m = 0$, $b_m - b_q > 0$ is impossible for any q). So all we need is $p_m \leq p + 1$ at V_m to make the switching step possible. We state all these in the following theorem.

Theorem 3.2. *Suppose that the sequences $\{a_i\}$ and $\{b_i\}$, $i = 1, ..., n$, are all graphical (i.e., realizable as the valences of a graph without loops or multiple edges), and $a_i \geq b_i$. Assume that there exists a non-negative integer p such that*

(i) *if $b_i \neq 0$, then $a_i - b_i = p$ or $p + 1$;*

(ii) *if $b_i = 0$, then $a_i - b_i = a_i \leq p + 1$;*

(iii) *if $p > 0$, then conditions* (i) *and* (ii) *are not required at vertex* 1, *i.e. $a_1 - b_1$ may be arbitrary.*

Then there exists a graph with valences $\{a_i\}$ which has a subgraph with valences $\{b_i\}$.

4. Directed graphs

We shall allow only digraphs having no multiple arcs and no loops and all digraphs are to be drawn on vertices $V_1, V_2, ..., V_n$. However, a pair of arcs $\overrightarrow{V_iV_j}$, $\overrightarrow{V_jV_i}$ is allowed. (An arc from V_i to V_j is written as $\overrightarrow{V_iV_j}$.) Given a sequence of ordered pairs of non-negative integers $\langle\langle(a_i^+, a_i^-)\rangle\rangle$, we say that it is graphical if there exists a digraph G with the outdegree and indegree of vertex V_i being equal respectively to a_i^+ and a_i^-. We say that G has degree sequence $\langle\langle(a_i^+, a_i^-)\rangle\rangle$. We shall identify G with the set of arcs in G.

Kundu proved the following theorem.

Theorem 4.1. *Assume that the degree sequences $\langle\langle(a_i^+, a_i^-)\rangle\rangle$, $\langle\langle(b_i^+, b_i^-)\rangle\rangle$ are all graphical. Also, assume that $a_i^+ - b_i^+ = c$, c a non-negative constant, and that $a_i^- \geq b_i^-$. Then there exists a digraph G_T with degree sequence $\langle\langle(a_i^+, a_i^-)\rangle\rangle$ containing a subgraph G_S with degree sequence $\langle\langle(b_i^+, b_i^-)\rangle\rangle$.*

We first give a constructive proof of the same result by using a similar laying-off and switching procedure to that used above in the case of graphs.

First, we need two algorithms for constructing a digraph with a given graphical sequence.

We define $(x, y) \geq_\ell (x', y')$ if $x > x'$, or $x = x'$ and $y \geq y'$ (i.e., the ordinary lexicographic order), and we also define $(x, y) \geq_r (x', y')$ if $y > y'$, or $y = y'$ and $x \geq x'$ (i.e., lexicographic order from right to left).

Algorithm 4.2. Let the sequence $\langle(a_i^+, a_i^-)\rangle$ be graphical. Then we can construct a digraph G with the given degree sequence $\langle(a_i^+, a_i^-)\rangle$ as follows:

Step 1. Set $L = \emptyset$, $G = \emptyset$.

Step 2. If all the residual indegrees of L^c are 0, stop. G is then the required digraph. Otherwise, choose a vertex V_i from $L^c = \{V_1, ..., V_n\} - L$ such that $a_i^- > 0$. Choose a_i^- vertices $V_{l_1}, ..., V_{l_{a_i^-}}$ other than V_i with the biggest degree with respect to the lexicographic order $\geq_\ell$. I.e., for any vertex V_q other than V_i, $V_{l_1}, ..., V_{l_{a_i^-}}$, we always have $(a_{l_j}^+, a_{l_j}^-) \geq_\ell (a_q^+, a_q^-)$ for all $j = 1, ..., a_i^-$.

Step 3. Replace G by $G \cup \{\overrightarrow{V_{l_j}V_i}: j = 1, ..., a_i^-\}$. Replace L by $L \cup \{V_i\}$.

Step 4. If $L^c = \emptyset$, stop. Now G is the required digraph. If $L^c \neq \emptyset$, replace (a_i^+, a_i^-) by $(a_i^+, 0)$ and replace $(a_{l_j}^+, a_{l_j}^-)$ by $(a_{l_j}^+ - 1, a_{l_j}^-)$, $j = 1, ..., a_i^-$. Other degrees remain unchanged. Go to Step 2.

That Algorithm 4.2 works follows from the following theorem.

Theorem 4.3. *Let the sequence* $\langle(a_i^+, a_i^-)\rangle$ *be graphical, and let* V_k *be a fixed vertex. Let* $S(V_k) = \{V_{l_1}, ..., V_{l_{a_k^-}}\}$ *be a fixed set of vertices other than* V_k *such that the degree of each vertex in* $S(V_k)$ *is bigger* ($\geq_\ell$ *in the lexicographic order) than the degree of each vertex in* $\{V_1, ..., V_n\} - \{V_k, V_{l_1}, ..., V_{l_{a_k^-}}\}$. *Then there exists a digraph G with the given* $\langle(a_i^+, a_i^-)\rangle$ *and with the property that* $\overrightarrow{V_{l_i}V_k} \in G$ *for all* $i = 1, ..., a_k^-$.

Proof. If otherwise, let G be chosen to maximize the number of vertices V in $S(V_k)$ such that $\overrightarrow{VV_k}$ is in G. Then there exists a vertex $V_m \in S(V_k)$ and a vertex $V_q \notin S(V_k)$ such that $\overrightarrow{V_mV_k} \notin G$ and $\overrightarrow{V_qV_k} \in G$. By assumption, we have $(a_m^+, a_m^-) \geq_\ell (a_q^+, a_q^-)$.

Case 1. There exists a vertex V_p, $p \neq k, q, m$, such that $\overrightarrow{V_mV_p} \in G$,

$\overrightarrow{V_q V_p} \notin G$. By removing $\overrightarrow{V_m V_p}$, $\overrightarrow{V_q V_k}$ and replacing them by $\overrightarrow{V_m V_k}$, $\overrightarrow{V_q V_p}$, we get a new digraph G' with the given graphical sequence $\langle\langle(a_i^+, a_i^-)\rangle\rangle$.

Case 2. If Case 1 does not hold, then we must have $a_m^+ = a_q^+$, $a_m^- \geq a_q^-$ and $\overrightarrow{V_m V_q} \in G$, $\overrightarrow{V_q V_m} \notin G$. Since $a_m^- \geq a_q^-$, there exists a vertex V_r, $r \neq m, q$, such that $\overrightarrow{V_r V_m} \in G$, $\overrightarrow{V_r V_q} \notin G$. By removing $\overrightarrow{V_r V_m}$, $\overrightarrow{V_m V_q}$, $\overrightarrow{V_q V_k}$ and replacing them by $\overrightarrow{V_r V_q}$, $\overrightarrow{V_q V_m}$, $\overrightarrow{V_m V_k}$, we get a new digraph G'' with the given $\langle\langle(a_i^+, a_i^-)\rangle\rangle$.

The new digraphs G', G'' all have one more vertex in $S(V_k)$, namely V_m, such that $\overrightarrow{V_m V_k}$ is the new digraph, a contradiction to the assumption on G.

The other algorithm for constructing a digraph with a given graphical sequence is:

Algorithm 4.4. All steps except Step 2 are the same as those in Algorithm 4.2, so we need state Step 2 only.

Step 2. If all the residual indegrees of L^c are 0, stop. G is then the required digraph. Otherwise, choose a vertex V_i from L^c such that $a_i^- > 0$ and such that $(a_i^+, a_i^-) \geq_r (a_j^+, a_j^-)$ for all $j = 1, \ldots, n$. (Here we use the lexicographic order from right to left).

Choose a_i^- vertices $V_{l_1}, \ldots, V_{l_{a_i^-}}$ other than V_i with the biggest outward degree. (I.e., for any $V_j \in \{V_1, \ldots, V_n\} - \{V_i, V_{l_1}, \ldots, V_{l_{a_i^-}}\}$, we have $a_m^+ \geq a_j^+$ for all $m = l_1, \ldots, l_{a_i^-}$.)

Algorithm 4.4 is justified by the following theorem.

Theorem 4.5. *Let the sequence $\langle\langle(a_i^+, a_i^-)\rangle\rangle$ be graphical, and let V_k be a vertex such that $(a_k^+, a_k^-) \geq_r (a_j^+, a_j^-)$ for all $j = 1, \ldots, n$. Let $S(V_k) = \{V_{l_1}, \ldots, V_{l_{a_k^-}}\}$ be a fixed set of a_k^- vertices other than V_k such that the outward degree of each vertex in $S(V_k)$ is bigger than the outward degree of each vertex in $\{V_1, \ldots, V_n\} - \{V_k, V_{l_1}, \ldots, V_{l_{a_k^-}}\}$. Then there exists a graph G with the given $\langle\langle(a_i^+, a_i^-)\rangle\rangle$ and with the property that $\overrightarrow{V_{l_i} V_k} \in G$ for all $i = 1, \ldots, a_k^-$.*

Proof. If otherwise, let G be chosen to maximize the number of vertices V in $S(V_k)$ such that $\overrightarrow{V V_k}$ is in G. Then there exists a vertex $V_m \in S(V_k)$ and a vertex $V_q \notin S(V_k)$ such that $\overrightarrow{V_m V_k} \notin G$ and $\overrightarrow{V_q V_k} \in G$. Also, by assumption $a_m^+ \geq a_q^+$.

Case 1. There exists a vertex V_p, $p \neq m, q, k$, such that $\overrightarrow{V_m V_p} \in G$ and $\overrightarrow{V_q V_p} \notin G$. By removing $\overrightarrow{V_m V_p}$, $\overrightarrow{V_q V_k}$ and replacing them by $\overrightarrow{V_m V_k}$, $\overrightarrow{V_q V_p}$, we get a new digraph G' with $\langle\langle(a_i^+, a_i^-)\rangle\rangle$.

Case 2. If Case 1 does not hold, we must have $a_m^+ = a_q^+$, $\overrightarrow{V_m V_q} \in G$ and $\overrightarrow{V_q V_m} \notin G$. Since $(a_k^+, a_k^-) \geq_{\mathrm{r}} (a_q^+, a_q^-)$, we still have two subcases to consider.

Case 2.1. There exists a vertex V_r such that $\overrightarrow{V_r V_k} \in G$ and $\overrightarrow{V_r V_q} \notin G$. If we replace $\overrightarrow{V_m V_q}$, $\overrightarrow{V_r V_k}$ by $\overrightarrow{V_m V_k}$, $\overrightarrow{V_r V_q}$, we get a new digraph G'' having degree sequence $\langle\langle(a_i^+, a_i^-)\rangle\rangle$.

Case 2.2. If no such V_r exists as in Case 2.1, we must have $a_k^- = a_q^-$ and $a_k^+ \geq a_q^+$, and also $\overrightarrow{V_k V_q} \notin G$. Since $a_k^+ \geq a_q^+$, there must exist a vertex V_s such that $s \neq q, k$ and $\overrightarrow{V_k V_s} \in G$, $\overrightarrow{V_q V_s} \notin G$. By removing $\overrightarrow{V_m V_q}$, $\overrightarrow{V_q V_k}$, $\overrightarrow{V_k V_s}$ and replacing them by $\overrightarrow{V_m V_k}$, $\overrightarrow{V_k V_q}$, $\overrightarrow{V_q V_s}$, we get a new digraph G''' with degree sequence $\langle\langle(a_i^+, a_i^-)\rangle\rangle$.

The new digraphs G', G'', G''' all have one more vertex in $S(V_k)$, namely V_m, such that $\overrightarrow{V_n V_k}$ is in the new graph. This is a contradiction.

Now, we can prove Theorem 4.1 by the following algorithm.

Algorithm 4.6. With the assumptions of Theorem 4.1, we construct G_T, G_S via the following steps:

Step 1. Set $G_T = \emptyset$, $G_S = \emptyset$, $L = \emptyset$.

Step 2. If for all vertices V_k in L^c, $(a_k^+, a_k^-) = (0, 0)$, stop. Then G_T, G_S are the required digraphs. Otherwise, choose a vertex V_k in L^c such that $(b_k^+, b_k^-) \geq_{\mathrm{r}} (b_j^+, b_j^-)$ for all $j = 1, ..., n$.

Choose b_k^- vertices $V_{n_1}, ..., V_{n_{b_k^-}} = S(V_k)$ such that $V_k \notin S(V_k)$ and such that for all $V_i \notin S(V_k) \cup \{V_k\}$, we have $a_j^+ \geq a_i^+$ for all $j = n_1, ..., n_{b_k^-}$.

Choose a_k^- vertices $\{V_{l_1}, ..., V_{l_{a_k^-}}\} = T(V_k)$ such that $V_k \notin T(V_k)$ and such that for all $V_i \notin T(V_k) \cup \{V_k\}$, we have $(a_j^+, a_j^-) \geq_{\mathrm{Q}} (a_i^+, a_i^-)$ for all $j = l_1, ..., l_{a_k^-}$.

Replace G_S by $G_S \cup \{\overrightarrow{V_{n_i} V_k}: i = 1, .., b_k^-\}$.

Replace G_T by $G_T \cup \{\overrightarrow{V_{l_i} V_k}: i = 1, ..., a_k^-\}$.

Step 3. (i) If $G_S \subset G_T$, remove flag on next line. Go to Step 6.

(ii) If $G_S \not\subset G_T$, go to Step 4 if no flag here; to Step 5 if flag.

Step 4. Choose vertex $V_m \in T(V_k)$, $V_q \in S(V_k)$ such that $\overrightarrow{V_m V_k} \in G_T - G_S$, $\overrightarrow{V_q V_k} \in G_S - G_T$.

(i) If $(a_m^+, a_m^-) = (a_q^+, a_q^-)$, go to Step 3 with $T(V_k)$ replaced by $T(V_k) \cup \{V_q\} - \{V_m\}$.

(ii) If $b_q^+ = b_m^+$, go to Step 3 with $S(V_k)$ replaced by $S(V_k) \cup \{V_m\} - \{V_q\}$.

(iii) If for all choices of $V_m \in T(V_k)$, $V_q \in S(V_k)$ such that $\overrightarrow{V_m V_k} \in G_T - G_S$, $\overrightarrow{V_q V_k} \in G_S - G_T$, we have neither $(a_m^+, a_m^-) = (a_q^+, a_q^-)$ nor $b_q^+ = b_m^+$, insert flag on second line of Step 3, go to Step 5.

Step 5. Choose a vertex V_p, $p \neq m, k, q$, such that $\overrightarrow{V_q V_p} \in G_T$ and such that $\overrightarrow{V_m V_p}$ lies either in both G_T and G_S or in neither of them.

(i) If $\overrightarrow{V_m V_p}$ is in both G_T and G_S, replace G_S by $G_S \cup \{\overrightarrow{V_m V_k}, \overrightarrow{V_q V_p}\} - \{\overrightarrow{V_m V_p}, \overrightarrow{V_q V_k}\}$.

(ii) If $\overrightarrow{V_m V_p}$ is neither of G_T and G_S, replace G_T by $G_T \cup \{\overrightarrow{V_q V_k}, \overrightarrow{V_m V_p}\} - \{\overrightarrow{V_m V_k}, \overrightarrow{V_q V_p}\}$.

Go to Step 3.

Step 6. Replace L by $L \cup \{V_k\}$.

(i) If $L^c = \emptyset$, stop. Then G_T, G_S are the required digraphs.

(ii) If $L^c \neq \emptyset$, replace (a_k^+, a_k^-) by $(a_k^+, 0)$ and (b_k^+, b_k^-) by $(b_k^+, 0)$, and replace $(a_{l_i}^+, a_{l_i}^-)$ by $(a_{l_i}^+ - 1, a_{l_i}^-)$, $i = 1, \ldots, a_k^-$ and $(b_{n_i}^+, b_{n_i}^-)$ by $(b_{n_i}^+ - 1, b_{n_i}^-)$, $i = 1, \ldots, b_k^-$.

Go to Step 2.

Justification of Algorithm 4.6. We need only justify Step 5. Suppose that there exist $V_m \in T(V_k)$, $V_q \in S(V_k)$ such that $\overrightarrow{V_m V_k} \in G_T - G_S$, $\overrightarrow{V_q V_k} \in G_S - G_T$, and $(a_m^+, a_m^-) \neq (a_q^+, a_q^-)$, $b_q^+ \neq b_m^+$. We must then have $(a_m^+, a_m^-) \geq_{\ell} (a_q^+, a_q^-)$ and $b_q^+ > b_m^+$.

(i) $a_m^+ > a_q^+$, $b_q^+ > b_m^+$. Then $a_m^+ - b_m^+ \geq a_q^+ - b_q^+ + 2$. Since $a_m^+ - b_m^+ = a_q^+ - b_q^+ =$ constant c in the beginning, there must exist a vertex $V_p \in L$, $p \neq k, m, q$, such that $\overrightarrow{V_q V_p} \in G_T$ and $\overrightarrow{V_m V_p}$ is in both G_T and G_S or in neither of them.

(ii) $a_m^+ = a_q^+$, $a_m^- > a_q^-$, $b_q^+ > b_m^+$. In this case, we have $a_m^+ - b_m^+ \geq a_q^+ - b_q^+ + 1$ and $a_m^- > 0$. The latter implies that $V_m \notin U$, so that $\overrightarrow{V_q V_m} \notin G_T$. Thus $a_m^+ - b_m^+ \geq a_q^+ - b_q^- + 1$ implies the existence of V_p as in (i). The existence of such a V_p justifies Step 5.

From the justifications of Algorithm 4.6, we see that Kundu's result for digraph can be extended to the following situation. If for some i, $b_i^+ = 0$, then V_i can only take the role of V_m and never of V_q in (i) and (ii). The argument thus remains intact if we relax the condition $a_i^+ - b_i^+ = c$ to $a_i^+ - b_i^+ \leq c$ at this vertex. We therefore have the following theorem.

Theorem 4.7. *Assume that the degree sequences $\langle\langle a_i^+, a_i^-\rangle\rangle$, $\langle\langle b_i^+, b_i^-\rangle\rangle$ are all graphical; $a_i^+ \geq b_i^+$, $a_i^- \geq b_i^-$. Also, assume that there is a non-negative integer c such that $a_i^+ - b_i^+ = c$ if $b_i^+ \neq 0$, $a_i^+ - b_i^+ \leq c$ if $b_i^+ = 0$. Then there exists a graph G_T with degree sequence $\langle\langle a_i^+, a_i^-\rangle\rangle$ and containing a subgraph G_S with degree sequence $\langle\langle b_i^+, b_i^-\rangle\rangle$.*

Added in proof. Kundu has raised the question: "What similar results hold for more general k_i, in particular if all k_i but two are equal to k or $k + 1$, and two, k_1 and k_2, are different?" The method of section 3 above can easily be shown to apply for $k \geqslant 2$, whenever it is possible to lay off the two odd vertices first so that s is a subgraph of T on arcs containing them and so that the remaining residual degree sequences are both realizable. This will be so if and only if either of

$$(v_i - \delta_{i1} - \delta_{i2})$$

or

$$(v_i - k_i + \delta_{i1} + \delta_{i2})$$

(Kronecker δ's) are realizable given that $\{v_i\}$ and $\{v_i - k_i\}$ are realizable. In other words, the sequence $\{v_i\}$ will be realizable by a graph possessing a subgraph with degrees $\{v_i - k_i\}$ when both these sequences are realizable and there is a realization of the former containing an arc joining the first two vertices, or one of the latter not containing that arc.

References

[1] S.L. Hakimi, On realizability of a set of integers as degrees of the vertices of a linear graph I, II, J. Soc. Indust. Appl. Math. 10 (1962) 496–506, 11 (1963) 135–147.

[2] D.J. Kleitman, Minimal number of multiple edges in realization of an incidence sequence without loops, SIAM J. Appl. Math. 18 (1970) 25–28.

[3] S. Kundu, The K factor conjecture is true, to be published.