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A SIMPLE PROOF OF THE ERDOS-GALLAI THEOREM ON GRAPH SEQUENCES

S.A. CHOUDUM

A central theorem in the theory of graphic sequences is due to P. Erdos and T. Gallai. Here, we give a simple proof of this theorem by induction on the sum of the sequence.

THEOREM (Erdos and Gallai [2]):

A sequence $\pi : d_1 \geq d_2 \geq \ldots \geq d_p$ *of non-negative integers, whose sum (say* s*) is even is graphic if and only if*

$$\text{(EG)}: \sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{p} \min(d_i, k) \text{ , for every } k \text{ , } 1 \leq k \leq p \text{ .}$$

The known direct proofs are lengthy (see Harary [3]) while short proofs use the theory of flows in networks (see Berge [1]). Here, we give a simple direct proof. Since the necessary part is easy (see Harary [3]) we prove only sufficiency.

Proof. By induction on s . The theorem holds when $s = 0$ or 2 . Suppose that the theorem is true for sequences whose sum is $s - 2$ and let $\pi : d_1 \geq d_2 \geq \ldots \geq d_p$ be a sequence whose sum s is even and which

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satisfies (EG). There is no loss of generality in assuming $d_p \geq 1$. Let $t(\geq 1)$ be the smallest integer such that $d_t > d_{t+1}$; if π is regular then define t to be $p-1$. Consider the sequence $\pi^*: d_1 \geq \ldots \geq d_{t-1} > d_t-1 \geq d_{t+1} \geq \ldots \geq d_{p-1} > d_p-1$. We verify that π^* satisfies (EG). So, let k be an integer such that $1 \leq k \leq p$. We split the proof into five cases and prove in each case that π^* satisfies (EG); we use repeatedly the inequality: $\min(a,b) - 1 \leq \min(a-1,b)$.

(1) $k \geq t$.

$$\sum_{i=1}^{k} d_i - 1 \leq k(k-1) + \sum_{j=k+1}^{p} \min(d_j,k) - 1 \quad \text{[by (EG)]}$$

$$\leq k(k-1) + \sum_{j=k+1}^{p-1} \min(d_j,k) + \min(d_p-1,k) .$$

(2) $1 \leq k \leq t-1$ and $d_k \leq k-1$.

Clearly, $\sum_{i=1}^{k} d_i = k\, d_k \leq k(k-1) + \sum_{j=k+1}^{p} \min(d_j,k)$.

(3) $1 \leq k \leq t-1$ and $d_k = k$.

We first observe that $d_{k+2} + \ldots + d_p \geq 2$. This is obvious if $k+2 \leq p-1$. If $k+2 \geq p$, then $t = p-1$ and so π is $(p-2)^{p-1}, d_p$. But then, $s = (p-2)(p-1) + d_p$ is even, and hence $d_p \geq 2$. So,

$$\sum_{i=1}^{k} d_i = k^2 - k + d_{k+1} \leq k^2 - k + d_{k+1} + d_{k+2} + \ldots + d_p - 2$$

$$\leq k(k-1) + \sum_{j=k+1, \neq t}^{p-1} \min(d_j,k) + \min(d_t-1,k) + \min(d_p-1,k) .$$

(4) $1 \leq k \leq t-1$, $d_k \geq k+1$, and $d_p \geq k+1$.

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{p} \min(d_j,k) \quad \text{[by (EG)]}$$

$$= k(k-1) + \sum_{j=k+1,\neq t}^{p-1} \min(d_j,k) + \min(d_t-1,k) + \min(d_p-1,k) \; .$$

$$(\text{since, } \min(d_j,k) = \min(d_j-1,k) = k) \; .$$

(5) $1 \le k \le t-1$, $d_k \ge k+1$ and $d_p < k+1$.

Let r be the smallest integer such that $d_{t+r+1} \le k$. If $\sum_{i=1}^{k} d_i = k(k-1) + \sum_{i=k+1}^{p} \min(d_j,k)$, then we arrive at a contradiction to (EG) as follows.

We first have,

$$k\, d_k = \sum_{i=1}^{k} d_i = k(k-1) + (t+r-k)k + \sum_{j=t+r+1}^{p} d_j = k(t+r-1) + \sum_{j=t+r+1}^{p} d_j \; .$$

So,

$$\sum_{i=1}^{k+1} d_i = (k+1)d_k = (k+1)(t+r-1) + \frac{k+1}{k} \sum_{j=t+r+1}^{p} d_i .$$

$$> (k+1)k + (t+r-k-1)(k+1) + \sum_{j=t+r+1}^{p} d_j \; , \; (\text{since } \frac{1}{k} \sum_{j=t+r+1}^{p} d_j > 0)$$

$$= (k+1)k + \sum_{j=k+2}^{p} \min(d_j,k+1) \; .$$

Hence,

$$\sum_{i=1}^{k} d_i \le k(k-1) + \sum_{j=k+1}^{p} \min(d_j,k) - 1 \quad [\text{by (EG)}]$$

$$\le k(k-1) + \sum_{j=k+1,\neq t}^{p-1} \min(d_j,k) + \min(d_t-1,k) + \min(d_p-1,k) \; .$$

Thus in each case π^* satisfies (EG) and hence by the induction hypothesis it is graphic. Let G be a realization of π^* on the vertices $v_1,v_2,\ldots,v_p$. If $(v_t,v_p) \notin E(G)$, then $G + (v_t,v_p)$ is a realization of π . So, let $(v_t,v_p) \in E(G)$. Since

$\deg_G(v_t) = d_t - 1 \le p - 2$, there is a v_m such that $(v_m, v_t) \notin E(G)$. Since $\deg_G(v_m) \ge \deg_G(v_p)$, there is a v_n such that $(v_m, v_n) \in E(G)$ and $(v_n, v_p) \notin E(G)$. Deleting the edges (v_t, v_p) , (v_m, v_n) and adding the edges (v_t, v_m) , (v_n, v_p) we get a new realization G^* of π^* in which v_t and v_p are non-adjacent. Then $G^* + (v_t, v_p)$ is a realization of π . □

References

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School of Mathematical Sciences,
Madurai Kamaraj University,
Madurai - 625 021,
India.