A brief introduction to the Tutte polynomial

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Abstract

It can be defined for each graph (or matroid) a polynomial invariant called the Tutte polynomial. It is essentially the only invariant which can be defined from a recurrence in terms of the minors of the graph and which behaves multiplicatively for disjoint unions. We will talk about some of its properties and applications, about how it can be used to study knots and spin models, and hopefully we will give a sense of its importance.

1 A taste of motivation and history

How many spanning trees does a graph have? Tutte himself [1] relates that it was motivated by this question that he became acquainted to the theory of polynomial invariants for graphs, where he would be responsible for a huge further development.

In general, many apparently unrelated quantitative questions about graphs will happen to be closely related. In common, we will see that they can all be expressed in terms of a certain two variable polynomial. Furthermore, this polynomial finds applications in unexpected branches of mathematics and physics.

This two variable polynomial, which from now on we refer as the Tutte polynomial, is the canonical representative of a large class of polynomial invariants. It was first considered by Tutte in 1954 [2], although Whitney already studied the coefficients of this polynomial years ago [3].

The first mentions of polynomial invariants for graphs appear to be in Veblen [4] and Birkhoff [5]. During the first half of the 20th century, graph theory was yet strongly motivated by the problem of the four colouring of maps. Quantitative questions about colourings were being considered, therefore it was natural to consider a function which could count **how many proper vertex colourings would a graph have**. Subsequently mentions of this function are found in Whitney [6] and Birkhoff again [7].

On the other hand, and it will be clear later in this text why this is "the other hand", Tutte investigated flows [8] and considered a **function which could count the number of flows in a graph**. Again this function will turn out to be a specialization of the Tutte polynomial. More recently, interconnections between the Tutte polynomial and some spin models have been observed. We will mention some of these facts, as well as the remarkable relation between the Tutte polynomial and the Jones polynomial for knots.

Unfortunately, there will be no room for an introductory approach of every topic treated below. In addition, the sections about knots and spin models have just the purpose of showing that the Tutte polynomial can be applied to these fields, therefore they will definitely look superficial.

If there is further interest on a badly explained subject, we suggest the following list of books:

The last three chapters of *Algebraic Graph Theory* from Godsil & Royle [9] are a great resource about graphs and knots invariants.

Bollobás' *Modern Graph Theory* [10] devotes a chapter to the subject, also exploring spin models and knots.

Aigner's *A Course in Enumeration* [11] writes a great chapter about counting polynomials.

Welsh [12] writes *Complexity: Knots, Colourings and Counting.* Needless to say, it is a book devoted to the subject.

Tutte [13] himself writes a chapter of his textbook on graph theory about the topic.

Finally, Brylawski & Oxley [14] are the authors of a chapter about the Tutte polynomial in *Matroid Applications.* This is certainly the most classical resource on the topic.

2 Introduction

From now on, let $G = (V, E)$ denote a graph where loops and multiple edges are allowed. The concepts of deletion and contraction of an edge will be important. We will denote by $G\backslash e$ the graph where the edge e was simply removed, and this operation will be called *edge deletion*. By G/e we will mean the graph where the edge e was removed, and its incident vertices were identified, and this operation shall be named *edge contraction*.

To the ones familiar with matroid theory, the same notation shall be used to denote the deletion or the contraction of an element in the ground set of a matroid.

2.1 A recursive relation

For the examples below, suppose the edges considered are not loops or bridges.

2.1.1 Spanning trees (or forests)

Given an edge e, there is a bijection of the spanning trees of G/e and the spanning trees of G containing e. Likewise, there is a bijection of the spanning trees of $G\backslash e$ and the spanning trees of G not containing e. Therefore, if $\tau(G)$ stands to count the number of spanning trees of a graph, the following relation holds:

$$\tau(G) = \tau(G\backslash e) + \tau(G/e)$$

2.1.2 Colourings

Suppose the edge e is originally adjacent to u and v. Looking to a proper colouring of $G\backslash e$ with k colours, either the colours of u and v are the same, or they are different. There is a bijection from the former case to the colourings of G/e with k colours, and another bijection from the later case to the colourings of G. If $P_G(k)$ denotes the number of proper k-colourings of G, then:

$$P_G(k) = P_{G\backslash e}(k) - P_{G/e}(k)$$

2.1.3 Flows

Suppose now that G has an arbitrary orientation. Let H be a finite abelian group and let $\overrightarrow{e}$ be a directed edge. Let e_{uv} denote the vertex of G/e which is the identification of the neighbours u and v of e. We consider an H-flow on G/e and we look at the H-function on $G\backslash e$ which attributes the same values on the edges. When e_{uv} is split back to u and v, either both vertices remain with an excess of 0, or one of them keeps an excess which is the inverse of the excess of the other. In the former case, there is a bijection between some H-flows of G/e and the H-flows of $G\backslash e$. In the later case, there is a bijection of the other H-flows of G/e and the H-flows of G, where obviously $\overrightarrow{e}$ will receive the non-zero excess of the vertices u and v.

Let $F_G(H)$ denote the number of H-flows on a graph, hence:

$$F_G(H) = F_{G/e}(H) - F_{G\backslash e}(H)$$

Note that the domain of the function is a set of groups, but we will see that this function depends only on the size of the group.

2.1.4 Reliability

Suppose there is a fixed probability p such that each edge of a graph G shall be removed with this probability. Let $R_G(p)$ denote the probability that the number of connected components of G does not increase. It is a measure on how much reliable is a network. As e is not a bridge, its removal does not change the number of connected components. Hence we analyse what happens whether e is removed or not. If it is, which happens with probability p, the chance that the number of connected components of G increases is precisely the chance that the number of connected components of $G\backslash e$ increases. If it is not, G will have the same structure with respect to edge-connectivity of G/e. Therefore:

$$R_G(p) = p \cdot R_{G\backslash e}(p) + (1 - p) \cdot R_{G/e}(p)$$

2.2 Let us generalize

The examples above show that a relation of the form:

$$f_G(x) = a \cdot f_{G\backslash e}(x) + b \cdot f_{G/e}(x)$$

arises commonly when one analyses counting functions on a graph. We also observe that bridges and loops may have a special role, as the arguments above would not work for them. Motivated by these considerations, we define:

Definition 1. For a graph G, we denote by $U_G(x, y, \alpha, \sigma, \tau)$ its *universal polynomial*, which is recursively defined by the following relations:

(i) If there is no edge, $U_{E_n}(x, y, \alpha, \sigma, \tau) = \alpha^n$.

(ii) If e is an edge, $U_G = \begin{cases} x \cdot U_{G\backslash e} & \text{if } e \text{ is a bridge} \\ y \cdot U_{G\backslash e} & \text{if } e \text{ is a loop} \\ \sigma \cdot U_{G\backslash e} + \tau \cdot U_{G/e} & \text{if } e \text{ is neither a bridge nor a loop} \end{cases}$

It is not clear first of all that U is indeed a polynomial and that it is unique. A general fact about graphs and matroids is that properties defined in terms of deletions and contractions usually do not depend on the order. However, before focusing on these matters, we revisit the examples to clarify the definition above:

- Spanning forests: The empty set is a spanning forest for the empty graph on n vertices, therefore $\alpha = 0$. If e is a bridge, it lies on every spanning forest; and every spanning forest of $G\backslash e$ plus e is a spanning forest of G. Hence $x = 1$. Clearly $y = 1$, for loops are never on spanning forests. And by the recursive relation observed, $\sigma = \tau = 1$.
- k-colourings: The empty graph contains precisely k^n possible colourings, hence $\alpha = k$. To write P_G in terms of $P_{G\backslash e}$ if e is a bridge, observe that given the colouring of one of the components incident fo e, the other component can be freely coloured unless the colours incident to e are the same. As the components are completely independent but for e, an argument by symmetry on renaming colours shows that $x = 1 - \frac{1}{k} = \frac{k-1}{k}$. Clearly no proper colouring exists if a graph has a loop, therefore $y = 0$. And the recursive relation found shows that $\sigma = 1$ and $\tau = -1$.
- H-flows: the empty graph has the empty flow, therefore $\alpha = 1$. Note that a graph has no flow if it contains a bridge, thus $x = 0$. If e is a loop, it can receive any non-zero value of H as its flow, hence $y = |H| - 1$. We saw that $\sigma = -1$ and $\tau = 1$. Note in particular that the number of flows is determined by these parameters, and depend only on $|H|$. Therefore we will consider F_G as a polynomial that takes and integer q and says how many H-flows the graph has, where $|H| = q$.
- Reliability: The number of connected components of the empty graph cannot increase, hence $\alpha = 1$. The removal of a bridge implies an increasing on the number of components, hence $x = (1 - p)$. A loop can be removed or not, it doesn't matter, $y = 1$. And we saw $\sigma = p$ and $\tau = (1 - p)$.

2.3 And where is Tutte?

The remarkable fact about this universal polynomial is that it contains too many variables. We will see that a particular specialization of this universal polynomial is sufficient to recover it completely, up to some easily computable factors.

In particular, we introduce a 2-variable polynomial:

Definition 2. For a graph G, we denote by $T_G(x, y)$ its Tutte polynomial, which is recursively defined by he following relations:

(i) If there is no edge, $T_{E_n}(x, y) = 1$.

(ii) If e is an edge, $T_G = \begin{cases} x \cdot T_{G\backslash e} & \text{if } e \text{ is a bridge} \\ y \cdot T_{G\backslash e} & \text{if } e \text{ is a loop} \\ T_{G\backslash e} + T_{G/e} & \text{if } e \text{ is neither a bridge nor a loop} \end{cases}$

In particular, it is the universal polynomial taken for $\alpha = \sigma = \tau = 1$.

Note that the definition above can be extended for matroids at no cost at all, as there is no longer mention to the number of vertices. Note just that it is usual to call the "bridges" of a matroid as *coloops*.

This polynomial by itself is a powerful enumerative tool. But before we move on to see more applications, let us state that it has a closed expression, and consequently, that it is a unique polynomial.

Before, we introduce some notation. Suppose A is a subset of E. We define:

- $\kappa(A)$ is the number of connected components of the subgraph of G which contains only the edges in A, i.e., the number of connected components of $G\backslash(E-A)$. Note that $\kappa(E)$ is the number of connected components of G, and that $\kappa(A) \geq \kappa(E)$ for all $A \subset E$.
- $r(A)$ is the size (number of edges) of a maximal subset of A which contains no cycle. For those familiarized with matroids, this is the rank of A. It is true in general that $r(A) = |V(G)| - \kappa(A)$.
- $n(A) = |A| - r(A)$, known as the nullity of A.

Theorem 1. With the definition above:

$$T_G(x,y) = \sum_{A\subset E} (x-1)^{r(E)-r(A)}(y-1)^{n(A)}$$

This expansion of the Tutte polynomial will be further referred as the rank-nullity expansion.

Even though this theorem seems impressive at a first sight, the proof is simply induction. It is unfortunate that this closed expression is of no help for computing the Tutte polynomial in practice, as the number of terms is an exponential on $|E|$.

The following theorem is known as the *recipe theorem*. It shows how to recover the universal polynomial in terms of the Tutte polynomial.

Theorem 2. Let G be a graph, T_G its Tutte polynomial, U_G its universal polynomial. If $\sigma \neq 0$ and $\tau \neq 0$, then:

$$U_G(x,y,\alpha,\sigma,\tau) = \alpha^{\kappa(E)}\sigma^{n(E)}\tau^{r(E)}T_G\left(\frac{\alpha x}{\tau}, \frac{y}{\sigma}\right)$$

Moreover, if $l(G)$ denotes the number of loops and $b(G)$ the number of bridges, we have:

(A) $U_G(x,y,\alpha,\sigma,0) = \alpha^{|V(G)|}\sigma^{n(E)-l(G)}x^{r(E)}y^{l(G)}$

(B) $U_G(x,y,\alpha,0,\tau) = \alpha^{\kappa(G)+p(G)}\tau^{r(E)-p(G)}x^{p(G)}y^{n(E)}$

(C) $U_G(x,y,\alpha,0,0) = \begin{cases} \alpha^{|V(G)|}x^{p(G)}y^{l(G)} & \text{if } E(G) \text{ has only bridges and loops} \\ 0 & \text{otherwise} \end{cases}$

In particular, all the examples already mentioned can be written in terms of the Tutte polynomial, therefore in terms of its rank-nullity expansion. Observe:

- Spanning forests:

$$\begin{aligned} C(G) = T_G(1,1) &= \sum_{A\subset E}(x-1)^{r(E)-r(A)}(y-1)^{n(A)}\Bigg|_{\substack{x=1\\y=1}} = \\ &= \sum_{\substack{A\subset E\\ r(A)=r(E)\\ r(A)=|A|}} 1 \end{aligned}$$

- k-colourings:

$$\begin{aligned} P_G(k) &= k^{\kappa(E)}(-1)^{r(E)}T_G\left(k\cdot\frac{k-1}{-k}, 0\right) = k^{\kappa(E)}(-1)^{r(E)}T_G\left(1-k,0\right) = \\ &= k^{\kappa(E)}(-1)^{r(E)}\sum_{A\subset E}(-k)^{r(E)-r(A)}(-1)^{n(A)} = \\ &= (-1)^{\kappa(E)}\sum_{A\subset E}k^{\kappa(A)}(-1)^{|A|} \end{aligned}$$

- q-flows:

$$\begin{aligned} F_G(q) &= (-1)^{n(E)}T_G\left(0,1-q\right) = \\ &= (-1)^{n(E)}\sum_{A\subset E}(-1)^{r(E)-r(A)}(-q)^{n(A)} = \\ &= (-1)^{|E|}\sum_{A\subset E}q^{n(A)}(-1)^{|A|} \end{aligned}$$

- Reliability:

$$
\begin{aligned}
R_G(p) &= p^{n(E)}(1-p)^{r(E)}T_G\left(\frac{1-p}{1-p}, \frac{1}{p}\right) = p^{n(E)}(1-p)^{r(E)}T_G\left(1, \frac{1}{p}\right) = \\
&= p^{n(E)}(1-p)^{r(E)} \sum_{A\subset E} (x-1)^{r(E)-r(A)} \left(\frac{1}{p}-1\right)^{n(A)} \Bigg|_{x=1} = \\
&= \sum_{\substack{A\subset E \\ r(A)=r(E)}} (1-p)^{|A|}p^{|E\setminus A|}
\end{aligned}
$$

2.3.1 A concrete example - cycles

Using the recursive definition, we note that given a cycle C_n and an edge e, the deletion of e leaves only bridges, and the contraction reduces to C_{n-1}, with the convention that C_2 is a pair of parallel edges and C_1 a loop. It is not difficult to see now that:

$$T_{C_n}(x,y) = y + x + x^2 + .. + x^{n-1}$$

Specializing to the cases seen above, we note that C_n has n spanning trees. The number of k colourings is $k(-1)^{n-1}\sum_{i=1}^{n-1}(1-k)^i$. Given any orientation and H an abelian group of size q, the number of H-flows is $q-1$. And finally the reliability of a cycle is $(1-p)^n + p(1-p)^{n-1}n$.

2.4 Combinatorial meaning for the coefficients

Let $E = \{e_1, ..., e_m\}$ and consider a total ordering of the edges:

$$e_i \prec e_j \;\Leftrightarrow\; i < j$$

Suppose we are given a spanning forest S. Each edge $e \in S$ defines a cut in the graph, and each edge $f \notin S$ defines a cycle.

We say that $e \in S$ is *internally active* if it is the smallest edge on the cut defined by itself. We say that $f \notin S$ is *externally active* if it is the smallest edge on the cycle defined by itself.

The internal activity of S is the number of internally active edges, and the external activity is defined likewise.

Suppose the edges are ordered and let us we write the Tutte polynomial as:

$$T_G(x,y) = \sum_{i,j} t_{ij}x^iy^j$$

The coefficient t_{ij} is exactly the number of spanning forests with internal activity i and external activity j. The proof is again by induction.

The following two consequences are immediate:

- It is not clear at first sight that the coefficients t_{ij} do not depend on the ordering. However, as there is no mention of this ordering in the rank-nullity expansion, this is indeed the case.
- Looking to the rank-nullity expansion $\sum(x-1)^{r(E)-r(A)}(y-1)^{n(A)}$, one cannot guarantee at first sight that the coefficient of x^iy^j is positive. The above relation shows that this is indeed the case.

We mention a relation proved by Brylawski:

Theorem 3. If $h \leq |E|$, then:

$$\sum_{i=0}^{h}\sum_{j=0}^{h-1}(-1)^j\binom{h-i}{j}t_{ij} = 0$$

In particular, if $|E| > 0$, $t_{00} = 1$. And $|E| > 1$ implies $t_{10} = t_{01}$.

2.5 Duality

We mentioned that the definition of the Tutte polynomial could be extended for matroids at no cost. A good point of this extension is that it is easier to work with the concept of duality in matroids rather than in graphs. In particular, every matroid has a dual, what is true in terms of graphs just for planar ones.

It turns out that the Tutte polynomial of the dual can be easily written in terms of the original. More specifically, let M be a matroid and M^* its dual. Therefore:

$$T_M(x, y) = T_{M^*}(y, x)$$

In particular, if G is a planar graph and G^* its dual, we have that:

$$T_G(x, y) = T_{G^*}(y, x)$$

With this expression in view, we can look again to the expressions for the colouring and flow polynomials. Recall that:

$$P_G(k) = k^{\kappa(E)}(-1)^{r(E)}T_G\left(1-k, 0\right) \qquad F_G(q) = (-1)^{n(E)}T_G\left(0, 1-q\right)$$

In particular, if G has just one component, $\kappa(E) = 1$. It is not difficult to prove that $r_G(E) = n_{G^*}(E)$, therefore we have the equality:

$$P_G(x) = F_{G^*}(x)$$

In other words, colourings and flows are dual concepts. More specifically, the polynomial equality related to the famous Tutte's theorem on flow-colouring duality can be obtained just by algebraic manipulations of the Tutte polynomial.

2.6 Convolution formula

There is a famous formula which express the Tutte polynomial in terms of the flows and colouring polynomials of the minors of a graph. We state it:

$$T_G(x, y) = \sum_{A \subset E} T_{G/A}(x, 0)T_{G\setminus(E-A)}(0, y)$$

Unfortunately, there is no room at the present text for further investigations about this and other properties of the Tutte polynomial.

We refer Monaghan & Merino [15] as an excellent updated source for properties, applications and references about the Tutte polynomial.

3 Spin models

In statistical mechanics, there is an interest to study disordered systems. It is desirable to define a probability space, and to do that, one defines a measure which is proportional to the probability and them normalizes it. In many typical examples, it is not trivial to find this normalizing factor. We will see that this problem is related to the problem of computing the Tutte polynomial.

We start this section introducing the notion of a spin model by defining the simplest one.

Roughly speaking, a *spin model* is a network of atoms and bonds in which each atom has a state. Translating to "combinatorish", we are talking about a graph in which each vertex has a state. The *Ising model* is the spin model where there are precisely two possible states, or spins. More precisely, let $G = (V, E)$ be a graph. We say that a function:

$$\omega : V \to \{1, -1\}$$

is a *state of the Ising model*. Usually, we denote $\omega(x) = \omega_x$.

We say that an edge has energy 0 if the states of the vertices incident to it are the same, and 1 otherwise. The *Hamiltonian of a state* (or the total energy of the state) is thus:

$$H(\omega) = \sum_{xy \in E} (1 - \delta(\omega_x, \omega_y))$$

where δ is the Kronecker's Delta, written in this fashion to avoid double subscripts.

We define a measure called the *Boltzmannian measure* by:

$$\mu_G(\omega) = e^{-\beta H(w)}$$

where $\beta = 1/k_B T$, k_B a constant, and T the temperature of the state.

If we denote by Ω the set of all possible states, the probability that the state ω occurs, using this measure, is therefore:

$$\Pr(\omega) = \frac{\mu_G(\omega)}{\sum_{\sigma\in\Omega}\mu_G(\sigma)}$$

Hence it is of great interest to find the value of the sum in the denominator, also known as the *partition function*, and that we denote by $P_G(\beta)$ as the temperature is the only variably quantity.

So far, there is absolutely no indicative that this partition function may be related to the Tutte polynomial. However, it turns out that it satisfies a recursive relation like the ones presented before:

- If there is no edge, the energy of every state will be zero, thus its measure will be 1, and the sum will count just this 1 for each state. Therefore:

$$P_G(\beta) = 2^n$$

- We now expand the expression of P_G. Observe:

$$\begin{aligned} P_G(\beta) &= \sum_{\omega\in\Omega}\mu_G(\omega) = \sum_{\omega\in\Omega} e^{-\beta H(w)} = \sum_{\omega\in\Omega} e^{-\beta\sum_{xy\in E}(1-\delta(\omega_x,\omega_y))} = \\ &= \sum_{\omega\in\Omega} e^{-\beta+\beta\sum_{xy\in E}\delta(\omega_x,\omega_y)} = \sum_{\omega\in\Omega} e^{-\beta}\prod_{xy\in E} e^{\beta\delta(\omega_x,\omega_y)} = \\ &= \sum_{\omega\in\Omega}\prod_{xy\in E}(e^{-\beta} + (1-e^{-\beta})\delta(\omega_x,\omega_y)) \end{aligned}$$

  Now for any edge ab, either $\omega(a) = \omega(b)$ or $\omega(a) \neq \omega(b)$. We split the sum into two, taking care of the factor appearing:

$$\begin{aligned} P_G(\beta) &= \sum_{\omega\in\Omega}\prod_{xy\in E}(e^{-\beta} + (1-e^{-\beta})\delta(\omega_x,\omega_y)) = \\ &= \sum_{\substack{\omega\in\Omega\\ \omega_a=\omega_b}}\prod_{xy\in E\setminus ab}(e^{-\beta} + (1-e^{-\beta})\delta(\omega_x,\omega_y)) + \\ &\qquad + e^{-\beta}\sum_{\substack{\omega\in\Omega\\ \omega_a\neq\omega_b}}\prod_{xy\in E\setminus ab}(e^{-\beta} + (1-e^{-\beta})\delta(\omega_x,\omega_y)) \end{aligned}$$

  Now for $G\backslash e$, they can be equal of different, but certainly for G/e they must be equal. Therefore we have found the following relation:

$$P_G(\beta) = e^{-\beta}P_{G\backslash e}(\beta) + (1-e^{-\beta})P_{G/e}(\beta)$$

  which, in particular, is true for any edge, including loops and bridges.

Theorem 4. The partition function of the Ising model is the universal polynomial with parameters $\alpha = 2$, $x = y = 1$, $\sigma = e^{-\beta}$ and $\tau = (1 - e^{-\beta})$.

In particular, it can be expressed in terms of the Tutte polynomial as:

$$P_G(\beta) = 2^{\kappa(G)}e^{-\beta|E|}(e^{\beta}-1)^{r(E)}T_G\left(\frac{2}{(1-e^{-\beta})}, \frac{1}{e^{-\beta}}\right)$$

The natural generalization of the Ising model is the Potts model, where the vertices are no longer required to assume states in a 2-set, but in a q-set. Replacing 2 by q gives no extra difficulty, and the resulting polynomial, now in 2-variables, is strongly related with a polynomial called dichromatic polynomial. This dichromatic polynomial is in much similar to the Tutte polynomial, and only deserves a name for it was introduced by Tutte himself in an earlier date. In particular, if $v = e^{\beta} - 1$, the partition function of the q-states Potts model is:

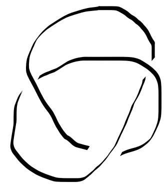
$$P_G(q,v) = e^{-\beta|E|}\sum_{A\subset E} q^{\kappa(A)}v^{|A|}$$

Further generalization to these models was proposed by Fortuin and Kasteleyn leading to the study of a multivariate Tutte polynomial which is essentially the algebraic codification of a matroid. We make no more comments about these topics, reinforcing that much better introductions to the relation of spin models and the Tutte polynomial can be found on the referred bibliography.

4 Knots

Now we introduce a mathematical structure called a *knot*. A knot is a closed curve (and why not smooth?) without self-intersections in $\mathbb{R}^3$. The relevant property of knots is whether or not a given knot can be deformed into some other by an isotopy, and in that case they are called equivalent. Thus we will actually consider two knots to be the same if they are equivalent and, with no surprise, different otherwise.

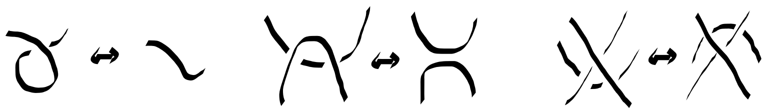
Unfortunately, we cannot draw things in 3 dimensions, or perhaps we could in a computer, but certainly not in a paper. There is absolutely no problem in representing a knot in the plan, except by the fact that we must take care to inform which part of the curve goes above the other. Observe the following example:



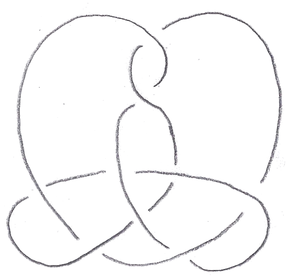
We call such a drawing a *diagram*. The theorem which makes the study of knots possible for a combinatorialist is the following:

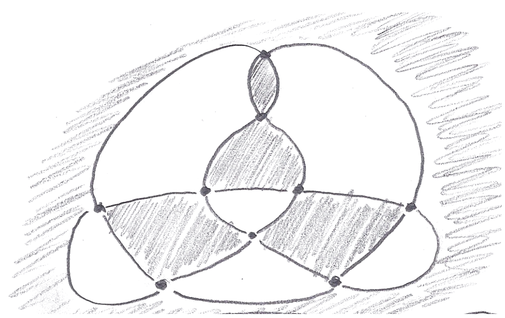
Theorem 5. Two knots are equivalent if and only if there is an isotopy of the plan and a sequence of Reidemeister moves that transform the diagram of one into the diagram of the other.

Reidemeister moves:

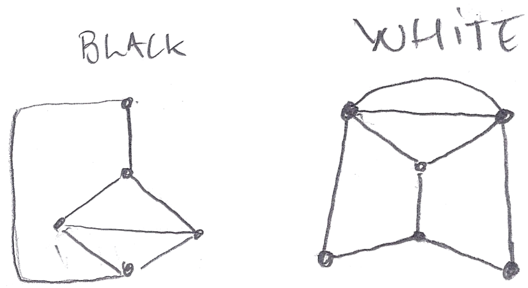


But before talking more about Reidemeister moves, let us see how we can interpret knots as graphs:

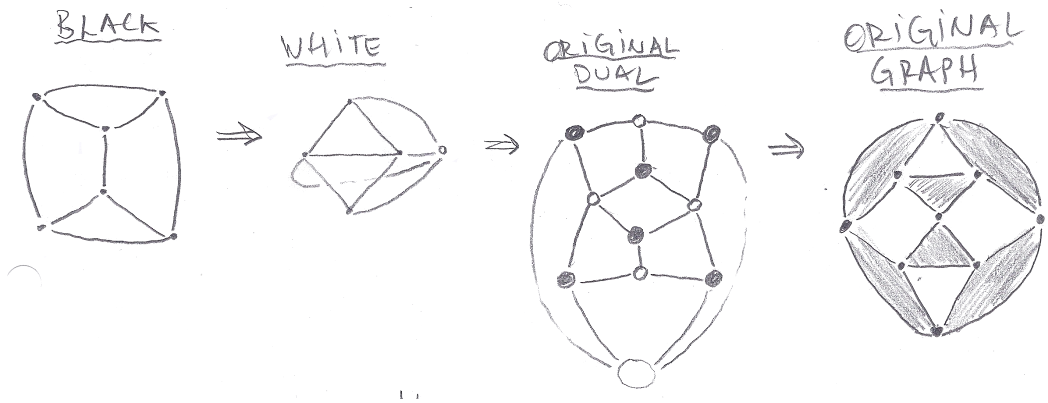




Note that this is a 4-regular graph. If we look to its dual, it is easy to see that if all faces are even, then so are all the cycles. But the faces of the dual correspond to 4-valent vertices, hence they are indeed even. So the dual of this graph is bipartite, as the two colouring above shows. We define two new graphs for each component of the dual, by making two vertices adjacent if their correspondent faces on the original graph share a vertex. Observe:

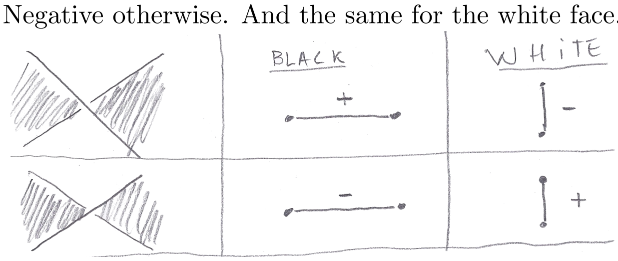


It is not difficult to see that the process can be reversed by any of the black or white graphs. In particular, observe that they are a pair of dual graphs. Observe this other example:

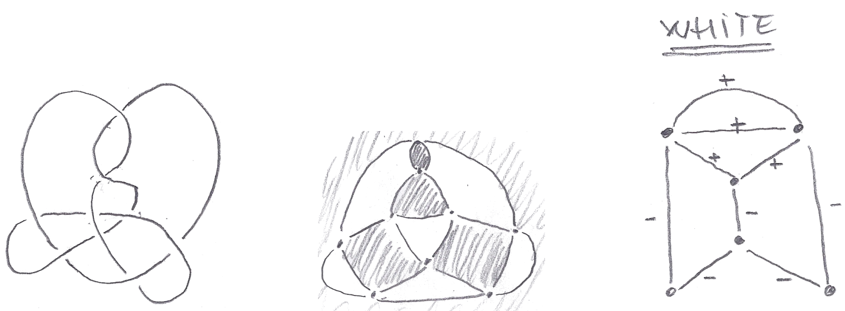


But how could we have recovered the knot if the information on each crossing was lost when they became vertices?! We do this by attributing signs to the edges of the black (or white) graphs following the rule:

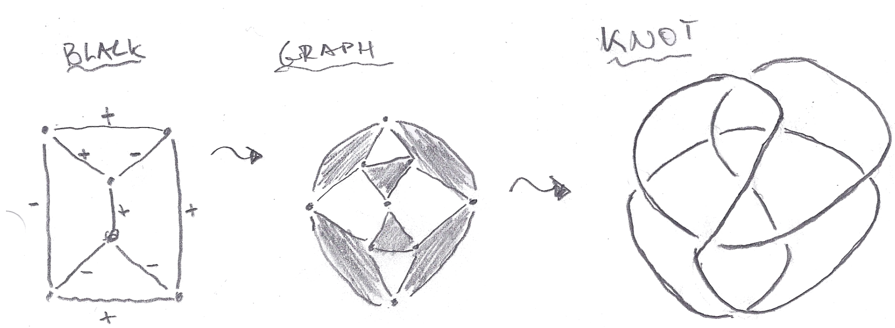
If a person over the black face looking to the crossing sees the top cord on his left, the sign shall be positive. Negative otherwise. And the same for the white face.



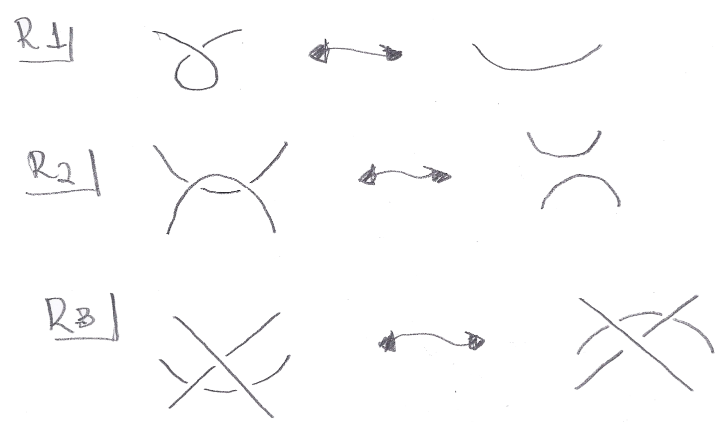
We revisit our examples...



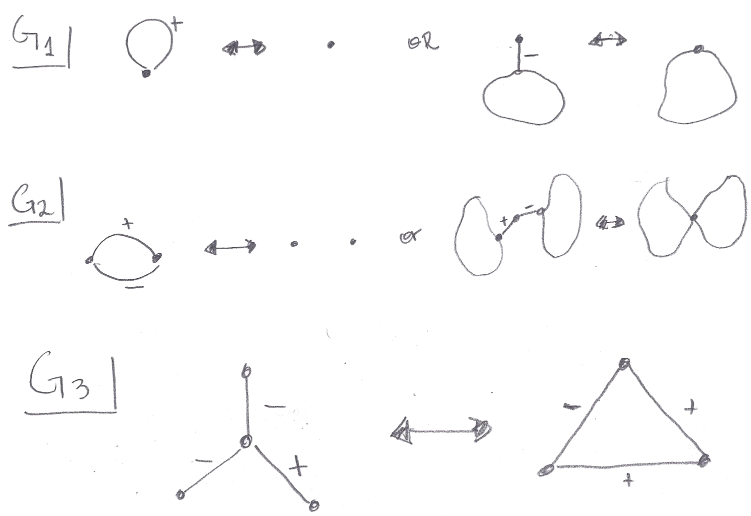
...to see that the reverse process is now well defined:



There are graphic movements associated with the Reidemeister moves:



And they are:



where the "or" is related to the fact that one of them occurs on the white and the other on the black graph.

4.1 Polynomial for knots

The big problem in knot theory is to decide whether or not two knots are the same. The ideal solution would be some invariant which completely determines the knot up to isotopies. No such invariant is known. However, we will define a polynomial which is invariant under Reidemeister moves. This implies that it is useful to decide that two knots are not the same, for that it is enough that they don't have the same polynomial invariant. Sadly, it can be the case that two non-equivalent knots have the same polynomial.

We will define a polynomial for signed graphs which is invariant under the moves above at certain values:

Definition 3. Let G be a graph. We define a polynomial $R_G(\alpha, \beta, x, y)$ by:

$$R_G = \begin{cases} (\alpha + \beta y) \cdot R_{G\setminus e} & \text{if } e \text{ is a positive loop} \\ (\alpha y + \beta) \cdot R_{G\setminus e} & \text{if } e \text{ is a negative loop} \\ \alpha \cdot R_{G\setminus e} + \beta \cdot R_{G/e} & \text{if } e \text{ is positive and not a loop} \\ \beta \cdot R_{G\setminus e} + \alpha \cdot R_{G/e} & \text{if } e \text{ is negative and not a loop} \end{cases}$$

and $R_{E_n} = x^{n-1}$.

A rank nullity expression for this polynomial is:

$$R_G(\alpha,\beta,x,y) = x^{\kappa(E)-1}\sum_{A\subset E}\alpha^{|\overline{A}(+)|+|A(-)|}\beta^{|A(+)|+|\overline{A}(-)|}x^{r(E)-r(A)}y^{|A|-r(A)}$$

where $A(+)$ represents the positive edges of A, and analogously for $A(-), \overline{A}(+)$ and $\overline{A}(-)$.

It satisfies the duality relation if the graphs are connected:

$$R_G(\alpha,\beta,x,y) = R_{G^*}(\alpha,\beta,y,x)$$

We note that this polynomial is neither in the form of the universal polynomial nor can be written directly in terms of the Tutte polynomial. Nonetheless, the resemblance is evident and we shall make a further comment about the topic.

For now, we state the following theorem:

Theorem 6. Let G be a signed graph. If $\alpha\beta = 1$ and $y = -(\alpha^2+\alpha^{-2})$, then the polynomial $R_G(\alpha,\beta,x,y)$ is invariant under the Reidemeister moves G2 and G3.

Remember that the black graph and white graph of a knot are a pair of duals. By imposing $x = y$ to guarantee that taking either the black graph or the white graph of a knot yields the same polynomial, we define the *Kauffman bracket* of a knot:

$$[L] = R_G(\alpha,\alpha^{-1},-(\alpha^2+\alpha^{-2}),-(\alpha^2+\alpha^{-2}))$$

where G is either the signed black graph or the signed white graph.

We compute the Kauffman bracket of the first example of knot given at the beginning of this section, the left handed trifoil. Its graph is the triangle with double edges. Color the outer face white. Therefore its black graph will be an all negative triangle. Using the rank nullity expression, one gets:

$$[L] = \alpha^{-7} - \alpha^{-3} - \alpha^{5}$$

4.2 Further comments

To overcome the fact this polynomial is not an invariant under G1, we orient the knot in any direction, and we define the writhe of a knot L to be:

$$\texttt{wr}(L) = p(L) - n(L)$$

where $p(L)$ is the number of crossings where the top edge comes from the left and leaves to the right, and $n(L)$ is the number of crossings where the top edge comes from the right and leaves to the left. The writhe does not depend on the chosen orientation.

Theorem 7. The polynomial:

$$(-\alpha)^{3\texttt{wr}(L)}[L]$$

is invariant under all Reidemeister moves.

By this reason, with a slight adjustment, it receives a name:

Definition 4. The Jones polynomial of a link L is:

$$V_L(t) = (-\alpha)^{3\texttt{wr}(L)}[L]$$

by making $\alpha = t^{1/4}$.

The left handed trifoil mentioned above has Jones polynomial:

$$-t^{-4}+t^{-3}+t^{-1}$$

as its writhe is -3.

Unfortunately this polynomial is not a complete invariant, in the sense that two non-equivalent knots may have the same Jones polynomial. There are other ways to present this theory, in particular, it is possible to define all these recursive relations just based on knot moves.

We drive our last comments to another direction. We observed that the left handed trifoil had a signed graph all negative. When such phenomenon happens, the knot is called *alternating*. It is not a joke: the signs are the same, but if you decide to walk over the knot, you will be crossing over and under, over and under, that is why the name alternating.

Now let G be the positive signed graph of an alternating knot. Let $a(L)$ be the number of vertices in G, and $b(L)$ be the number of vertices in the dual of G. If you prefer, if G is the white positively signed graph, let $a(L)$ be the number of white faces and $b(L)$ the number of black faces. Therefore the Jones polynomial of the knot satisfies:

$$V_L(t) = (-1)^{\mathtt{wr}(L)} t^{(b(L)-a(L)+3\mathtt{wr}(L))/4} T_G(-t, -1/t)$$

where T is the Tutte polynomial! We revisit the left handed trifoil:

$\mathtt{wr}(L) = -3$

$b(L) = 3$

$a(L) = 2$

G is three parallel edges. $T_G(x, y) = x + y + y^2$

Therefore:

$$\begin{aligned} V_L(t) &= (-1)^{\mathtt{wr}(L)} t^{(b(L)-a(L)+3\mathtt{wr}(L))/4} T_G(-t, -1/t) = (-1)t^{-2}\left(-t - \frac{1}{t} + \frac{1}{t^2}\right) = \\ &= -t^{-4} + t^{-3} + t^{-1} \end{aligned}$$

as wished.

References

[1] William Tutte. Graph-polynomial. *Advances in Applied Mathematics*, 32:5–9, 2004.

[2] William Tutte. A contribution to the theory of chromatic polynomials. *Canadian Journal of Mathematics*, 6:80–91, 1954.

[3] H. Whitney. The coloring of graphs. *Ann. Math. (2)*, 33:688–718, 1933.

[4] O. Veblen. An applications of modular equations in analysis situs. *Ann. Math. (2)*, 2:86–64, 1912.

[5] G. D. Birkhoff. A determinant formula for the number of ways of coloring a map. *Ann. Math*, 13:42–46, 1912-1913.

[6] H. Whitney. A set of topological invariants for graphs. *Amer. J. Math.*, 55:231–5, 1933.

[7] G. D. Birkhoff and D. C. Lewis. Chromatic polynomials. *Trans. Amer. Math. Soc.*, 60:355–451, 1946.

[8] William Tutte. A ring in graph theory. *Proc. Camb. Phil. So.*, 43:26–40, 1947.

[9] C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer Verlag, New York, 2004.

[10] Bla Bollobs. *Modern Graph Theory*. Springer Verlag, New York, 1998.

[11] Martin Aigner. *A Course in Enumeration*. Springer Verlag, New York, 2007.

[12] D. J. A. Welsh. *Complexity: Knots, Colourings and Couting*. Cambridge University Press, New York, 1993.

[13] William Tutte. *Graph Theory*. Cambridge University Press, New York, 1984.

[14] T. Brylawski and J. Oxley. The Tutte polynomial and its applications. In N. White, editor, *Matroid Applications*, volume 40, chapter Matroid Applications. Cambridge University Press, Cambridge, 1992.

[15] J. A. Ellis-Monaghan and C. Merino. Graph Polynomials and Their Applications i: The Tutte Polynomial. *arXiv:math*, 0803.3079v2, 2008.