IS THE NULL-GRAPH A POINTLESS CONCEPT?

Frank Harary
University of Michigan
and Oxford University

Ronald C. Read
University of Waterloo

ABSTRACT

The graph with no points and no lines is discussed critically. Arguments for and against its official admittance as a graph are presented. This is accompanied by an extensive survey of the literature. Paradoxical properties of the null-graph are noted. No conclusion is reached.

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1. INTRODUCTION

It is well-known in set theory that the introduction of the seemingly paradoxical null-set, having no elements, results in a considerable simplification of the statement of theorems, relieving an author of having to make special provision for the possibility of there being no objects possessing the property which defines the elements of a certain set. Some writers in graph theory, presumably by imitation, have introduced the concept of a (or the) "null-graph" - having no points and no lines; see Figure 1. The advantages of such a concept are less obvious than those of the null-set, and we can reasonably question whether such advantages as it may have are not outweighed by extra complications that it introduces. We discuss here the *pros* and *cons* of the use of the null-graph. Note that it is not a question of whether the null-graph "really exists"; it is simply a question of whether there is any point in it.

Figure 1. The Null Graph

2. CURRENT USAGE

Let us start with a brief survey of how some well-known writers on graph theory have handled the null-graph, noting in particular that if a writer does not explicitly state that the set of points of a graph is non-empty, he has, by default, admitted the null-graph, even though he may subsequently pay no attention to it.

Ore [7] makes no mention of the null-graph in the sense that we are using the term, but instead uses the adjective "null" to describe a graph consisting only of isolated points. One presumes, therefore, that since he regards as trivial a graph whose set of lines is empty, he would, *a fortiori*, discount the graph whose set of points is empty, even though he does not actually say so.

Berge [1] and Wagner [13] do not explicitly state that the set of points is non-empty, and hence, by default, admit the null-graph. However, the null-graph is never mentioned as such, and hence one may assume that they do not intend it to be regarded as a graph.

König [5] distinguishes between the null-graph and "proper" graphs (eigentlichen Graphen). He adds that the adjective "proper" will be omitted when the context makes it clear what is meant. He thus recognizes the problem, but surmounts it by what are perhaps somewhat unsatisfactory methods. Sedlaček [11], Zykov [14], and Busacker and Saaty [2] all define the set of vertices of a graph to be non-empty, thus explicitly excluding the null-graph.

In complete contrast, Tutte [12] specifically admits the null-graph and refers to it as a graph. Many of his theorems apply to the null-graph as well as other graphs, but many do not. In consequence, the statements of a large number of theorems in his book contain the proviso that the graphs to which they refer must not be the null-graph. Sachs [10] also explicitly defines the null-graph.

Perhaps the most enthusiastic proponents of the null-graph are Maxwell and Reed [6], who in their own words, give to the null-graph "full-fledged membership in the 'society' of graphs." They realize, however, that "the acceptance of the null-graph has the aspect of both simplifying and complicating the development of the theory of graphs." In Section 2.9 of their book, they present an amusing but somewhat confused discussion of "this paradoxical beast - the null-graph." In fact, they ask, "Does the null-graph possess <u>all</u> properties, or in fact, does it have <u>no</u> properties?", and infer that such questions "lead to a situation of absolute and utter chaos." Despite their apparent affection for it, they conclude by saying, "except on specific occasions, ... the null-graph is ignored."

As for the authors of this paper, one of us [4] is decisively opposed to the null-graph; the other is willing to afford it grudging recognition in certain contexts, but nevertheless, when called upon to give a definition of a graph [9] has required the point-set to be non-empty.

3. PRO

We present three arguments in favor of the null-graph involving set theoretic considerations, the chromatic polynomial, and graphical enumeration.

The most convincing argument in favor of admitting the null-graph is that this assures that the intersection of any two subgraphs of a given graph is always a subgraph. In particular, the collection of all subgraphs of G then forms a boolean algebra, in which the null-graph is the zero element.

It is convenient to introduce a notation for the null-graph. By definition, a graph is <u>complete</u> if every pair of points are adjacent. Thus the null-graph is obviously, though vacuously, complete. It is becoming standard notation to write K_p for the complete graph with p points; hence the null-graph is K_0. A good example of this set theoretical use of K_0 occurs in the theory of chromatic polynomials. In the expository paper [8], the following results are given:

(1) If a graph G has two components A and B, then

$$P_G(\lambda) = P_A(\lambda) \cdot P_B(\lambda)$$

where $P_G(\lambda)$ denotes the chromatic polynomial of G.

(2) If two graphs A and B intersect in a complete graph K_p, then the chromatic polynomial of their union is

$$P_A(\lambda)\ P_B(\lambda)\ /\ P_{K_p}(\lambda)\ .$$

If K_0 is admitted as a graph, these two results can be combined, (1) being the special case of (2) for $p = 0$. All that is required is that we define $P_{K_0}(\lambda)$ to be 1. This is perfectly natural; no matter what the number λ of colors, there is only one way to color the points of K_0, namely, do nothing! The chromatic polynomial of K_p is

$$\lambda^{(p)} = \lambda(\lambda - 1)\ \ldots\ (\lambda - p + 1)$$

and we are therefore led to write $\lambda^{(0)} = 1$. This agrees with standard practice in combinatorial analysis.

Perhaps the most striking way in which the null-graph arises "naturally" in graph-theoretical research occurs in the theory of graphical enumeration. Consider, for example, the exponential generating function $T(x)$ for labeled trees, defined by

$$(3) \qquad T(x) = \sum_{r=1}^{\infty} T_n x^n / n!$$

where T_n is the number of trees on n labeled points. It is well-known that $T_n = n^{n-2}$. Another result, not quite so well-known, is that the generating function $F(x)$ for labeled forests is obtained by taking the formal exponential of $T(x)$:

$$(4) \qquad F(x) = \exp\ T(x)$$

Now the left-hand side of (4) contains the term "1", and if this refers to anything at all, it can only refer to the null-graph.

A relation similar to (4) holds in more general applications between the numbers of labeled graphs and of connected labeled graphs of a certain kind. Let $G(x)$ be the exponential generating function for such graphs and $C(x)$ the corresponding function for the connected graphs. Then these two functions are related by an equation which is often written as

$$(5) \qquad 1 + G(x) = \exp C(x),$$

or equivalently as

$$(6) \qquad C(x) = \log(1 + G(x)).$$

Here the terms in both G and C correspond to non-null graphs only.

In most enumeration problems, it is the counting result for all graphs of a specified type (not necessarily connected) which is obtained first, the result for connected graphs being derived from it by use of (6). The decision to take powers from 1 upwards is then a natural one; it avoids introducing the null-graph, but at the expense of having to insert a "1" into equation (5) in order to "make the answer come out right." On the other hand, when it is the connected result which is obtained first (as in the labeled tree example above), the null-graph can be ignored only by hustling it out of sight after it has already made its appearance.

A similar phenomenon occurs with enumeration problems [3] for unlabeled graphs; for example, with the relation between the numbers g_{pq} for graphs that are not necessarily connected. This relation is

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} g_{pq} x^p y^q = \prod_{\substack{m=1 \\ n=0}}^{\infty} (1 - x^m y^n)^{-g_{mn}}$$

Here again the null-graph $(p = q = 0)$ occurs naturally on the left-hand side, but not on the right-hand side (we cannot have $m = n = 0$, or the whole right-hand side vanishes!)

These examples show that the null-graph can arise naturally, without being specifically introduced. They also indicate that the null-graph must be regarded as not being connected! We shall consider below the paradox which this statement seems to imply.

4. CONTRA

The main objection to admitting the null-graph is that it has flagrantly contradictory properties. By one definition, a graph G is <u>connected</u> if every pair of points are joined by a path. As K_0 has no points, this definition is vacuously satisfied. Thus K_0 is connected.

Clearly K_0 is also acyclic since it contains no cycles; hence K_0 is a tree. This opinion is shared by Tutte [12] who asserts in his Theorem 3.31 that "All null-graphs, vertex-graphs, and link-graphs are trees," by which he means K_0 , K_1 , and K_2 are trees.

However, by another definition, a graph is a tree if it is connected, and $p = q + 1$. For K_0 , $p = q = 0$; hence K_0 is <u>not</u> a tree. But the graph K_0 is acyclic and so by definition is a forest, but not a tree. One naturally concludes that K_0 is in fact <u>disconnected</u>! This has already been suggested by the enumeration argument in the preceding section. Confirmation of this conclusion is obtained if we take the definition of connected to mean that a graph has exactly one component; but K_0 has no components at all.

Perhaps surprisingly there is also an enumerational argument against the recognition of a null-graph. This is entailed in equation (6) above, where fortunately, the explicit presence of the term "1" enables one to expand the right-hand member and thus express $C(x)$ in powers of $G(x)$.

5. CONCLUDING REMARKS

It can be seen from these examples that the main reason for the paradoxical nature of the null-graph is as follows. Many concepts in graph theory can be defined in several ways. When there are several possible definitions for a concept, it is necessary that they be consistent, and for non-null graphs this is so. But since some authors admit the null-graph while others do not, it often happens that different definitions of a concept disagree when applied to the null-graph. It would presumably be possible to frame a set of definitions that would be consistent even when applied to the null-graph, and would give to the null-graph the properties that it requires for those applications when it occurs naturally. To achieve this would be of limited usefulness, and the present authors have no intention of attempting to do so.

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