A note on a theorem of Erdős & Gallai

Amitabha Tripathi*, Sujith Vijay[1]

Department of Mathematics, Indian Institute of Technology, Hauz Khas, New Delhi 110 016, India

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Abstract

We show that the Erdős-Gallai condition characterizing graphical degree sequences of length p needs to be checked only for as many n as there are distinct terms in the sequence, not all n, 1 6 n 6 p.

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A sequence $d_1;d_2;:::;d_p$ of nonnegative integers is called the *degree sequence* of a graph ^{G} if the vertices of ^{G} can be labeled $v_1;v_2;:::;v_p$ such that degvk = dk for each k, 1 6 k 6 p. A sequence $a_1;a_2;:::;a_p$ of nonnegative integers is called *graphical* if it is the degree sequence of some graph. Any graphical sequence clearly satis:es the two conditions ak 6 p — 1 for each k and XX=i ^{a}k i^s ^{even}. However, these two conditions together do not ensure that a sequence will be graphical. Necessary and sufficient conditions for a sequence of nonnegative integers to be graphical are well known. Two characterizations of graphical sequences are due to Havel, Hakimi and Erdős and Gallai:

Theorem H-H (Havel [3] and Hakimi [2]). *A sequence of nonnegative integers* $a_1;$ $a_{2;:::;}a_p$ *with* a_1 ^ a_2 ^ • • • ^ a_p, a_1 ^ 1 *and* p ^ 3 *is graphical if and only if the sequence* a_2 - \,a^ - \,...,a_{ai+}\ - \,$a_{ai+}2,...,a_p$ *is graphical.*

Theorem EG (Erdős and Gallai [1]). *A sequence of positive integers $a_1, a_2, \ldots, a_p$ with $a_1 \geq a_2 \geq \cdots \geq a_p$ is graphical if and only if $\sum_{k=1}^{p} a_k$ is even and for each integer n, $1 \leqslant n \leqslant p-1$,*

$$\sum_{k=1}^{n} a_k \leqslant n(n-1) + \sum_{k=n+1}^{p} \min(n, a_k).$$

The theorem of Erdős and Gallai (Theorem EG) requires the verification of an inequality for each n, $1 \leqslant n \leqslant p-1$. It is interesting to observe that the inequality need not be checked for $n > s$, where s is the *largest* integer such that $a_s \geq s-1$. For $n > s$, the inequality in Theorem EG reduces to

$$\sum_{k=1}^{n} a_k \leqslant n(n-1) + \sum_{k=n+1}^{p} a_k.$$

Consider the difference between the right- and left-hand sides of the inequality as a function of n. Replacing n by $n+1$, this difference increases by $2(n - a_{n+1}) > 0$ since $n > s$. Thus, assuming the inequality holds for all $n \leqslant s$, it will also hold for all $n \leqslant p-1$. We record this observation as a

Lemma. *Let $\{a_1, a_2, \ldots, a_p\}$ be a sequence of positive integers with $a_1 \geq a_2 \geq \cdots \geq a_p$. Let s be the* largest *integer such that $a_s \geq s-1$. Then the sequence $\{a_1, a_2, \ldots, a_p\}$ is graphical if and only if $\sum_{k=1}^{p} a_k$ is even and for each integer n, $1 \leqslant n \leqslant s$,*

$$\sum_{k=1}^{n} a_k \leqslant n(n-1) + \sum_{k=n+1}^{p} \min(n, a_k).$$

The lemma states that the number of inequalities to check in Theorem EG can be reduced. The purpose of this note is to prove a refined form of Theorem EG mentioned in the introduction. We show that in case of multiple occurrences of numbers in the degree sequence, it suffices to check the inequality in Theorem EG only at the end of each segment of repeated values. Throughout this paper we shall employ the notation $(a)_m$ to denote m occurrences of the integer a. Thus, we may denote a typical degree sequence by

$$s := (a_1)_{m_1}, (a_2)_{m_2}, \ldots, (a_\ell)_{m_\ell}, \tag{1}$$

where $a_1 > a_2 > \cdots > a_\ell$ and each $m_k \geq 1$ with $m_1 + m_2 + \cdots + m_\ell = p$.

We shall write

$$\sigma_k := \sum_{i=1}^{k} m_i, \quad \text{with } \sigma_0 := 0, \quad \text{and } S_{r,t} := \sum_{i=r}^{t} a_i m_i.$$

Our main result is the

Theorem. *A sequence* (1) *is graphical if and only if $S_{1,\ell}$ is even and the inequality in Theorem EG holds for $n = \sigma_k$, $1 \leqslant k \leqslant \ell$.*

Proof. By Theorem EG, we only need to prove that checking the inequality at each a_k implies the inequality holds at each n. Suppose the inequality holds at each a_k, but is not valid for some $n=N$. Let $N0$ be the *least* such N, and write

$$N0 = (Jk + n', \text{ where } 1 \; 6 \; n' < m_{k+}1 \text{ and } 0 \; 6 \; k < `_{:}$$

Thus,

$$S\backslash,k + a_{k+}\backslash n' > (a_k + n')(a_k + n' - 1) + \{m_k + \backslash -n')mm\{a_k + \backslash,Gk + «')$$
$$+ m_{k+2} \min(a_{i+2}, a_k + n') + \text{———} m_{\cdot} \min(a/, a_k + n') \quad (2)$$

and

$$S1_{;k} + a_{k+}i(n' - 1)$$
$$\wedge(<j_k + n' - l)(a_k + n'-2) + (m_{k+}1 - n' + 1)mm(a_{k+u}a_k + n' - 1)$$
$$+ m_{k+2} \min(a_{i+2}, a_k + n' - 1) -\backslash\text{———}Ym_{(} \min(a/, a_k + n' - 1). \quad (3)$$

Suppose now that $a_k+1 < <r_k + n'$. Then subtracting (3) from (2) gives the inequality

$$a_{k+}1 > 2(a_k + ri - 1) - a_{k+}1_{;}$$

which contradicts our assumption. Thus,

$$ak+1 >< ?k + n' \quad (4)$$

and (2) reduces to

$$S1_{;}k + a_{k+}\backslash ri > (a_k + ri)((j_{k+}\backslash - 1) + m_{k+2} \min(a_{k+2}, a_k + n')$$
$$+ \cdots + m_\ell \min(a_\ell, a_k + n'). \quad (5)$$

Let r be such that $a_r < <r_k + n' \; 6 \; a_r-\backslash$. Such an r exists because $a_k + n' \; 6 \; a`$ together with (5) would imply

$$ai(a_k + n') \wedge Si_{>k} + a_{k+}m' > (a_k + n')(a«? - 1)_{;}$$

which is impossible since $a1 \; 6 \; p — 1 = <J\{ — 1$. From (4), $r \wedge k + 2$, and (5) further reduces to

$$S1_{;k} + a_{k+x}n' > (a_k + n')(\wedge\text{-}i - 1) + S_{r;}` \quad (6)$$

and (3) similarly to

$$S1_{;k} + a_{k+1}(n' - 1) \; 6 \; (a_k + n' - l)(a_r\text{-}i - 1) + S_{r;}` \quad (7)$$

Now subtracting (7) from (6) yields

$$a_{k+}1 \wedge oy\_i. \quad (8)$$

Let n'' be the *largest* integer $6 \; m_{k+}1$ for which the inequality in Theorem EG is not valid for $o_k + n''$; from the de:nition, $n' \; 6 \; n'' < m_{k+}\backslash$. Furthermore, analogous to (2) and (3), we have

$$\begin{aligned}\sigma_{1,k} &+ a_{k+1}n'' \\ &\geq (a_k + n'')(a_k + n'' - 1) + (m_{k+1} - n'')\min(a_{k+1}, a_k + n'') \\ &\quad + m_{k+2}\min(a_{k+2}, a_k + n'') + \cdots + m_t \min(a_t, a_k + n'')\end{aligned} \tag{9}$$

and

$$\begin{aligned}\sigma_{1,k} &+ a_{k+1}(n'' + 1) \\ &< (a_k + n'')(a_k + n'' + 1) + (m_{k+1} - n'' - 1)\min(a_{k+1}, a_k + n'' + 1) \\ &\quad + m_{k+2}\min(a_{k+2}, a_k + n'' + 1) + \cdots + m_t \min(a_t, a_k + n'' + 1).\end{aligned} \tag{10}$$

By (8), $a_k + n'' < a_{k+1} \leq a_{r-1} \leq a_{k+1}$. Define s such that $a_s \leq a_k + n'' < a_{s-1}$, and note that $k + 1 < s \leq r$. Now, the difference between the right-hand sides of (9) and (10) equals

$$2(a_k + n'') + (m_{k+1} - a_k - 2n'' - 1) + m_{k+2} + \cdots + m_{s-1} = \sigma_{s-1} - 1.$$

Since a_{k+1} is the difference between the left-hand sides of (9) and (10), the inequalities

$$\sigma_{s-1} - 1 < \sigma_{s-1} \leq \sigma_{r-1} \leq a_{k+1}$$

lead to a contradiction. This completes the proof of our result. •

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