[12] von Neumann, J., *Math. Ann.*, **102**, 49 (1929).
[13] Murray, F. J., *An Introduction to Linear Transformations in Hilbert Space* (Princeton University Press, 1941), p. 14.
[14] For a discussion of groups of this type, see, for example Littlewood, D. E., *The Theory of Group Characters* (Oxford University Press, 1950), p. 251.

THE MAXIMUM CONNECTIVITY OF A GRAPH*

By Frank Harary

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN

Communicated by G. E. Uhlenbeck, May 28, 1962

1. *Introduction.*—In the second book on graph theory ever written, Berge[1] lists 14 unsolved problems, one of which is the following: "11. *Quelle est la connexité maximum d'un graphe de n sommets et de m arêtes?* L'intérêt de ce problème est analogue à celui de trouver le diamètre minimum d'un graphe." The purpose of this note is to solve the problem.

In addition to finding the maximum connectivity of any graph with a given number of points and lines, we also obtain the minimum connectivity, the maximum diameter, and the minimum diameter. We refer to Whitney[2] as the original work on the connectivity of a graph.

A *graph* G consists of a finite set V of *points* $v_1, v_2, \ldots, v_p$, together with a prescribed collection X of unordered pairs of distinct points. The elements of X are called *lines*. For brevity, we say that a *p, q graph* is one with p points and q lines. If x is the line containing points u and v, then we write $x = uv$ and say that u and v are *adjacent* and that point u and line x are *incident*. A *subgraph* of G consists of subsets of V and X which form a graph. A *spanning subgraph* of G has the same set V of points.

A *path* joining two points u and v of G is the set of points and lines in a sequence of successively incident points and lines beginning with u and terminating with v, in which all the points are distinct. The *length* of a path is the number of lines in it. A graph is *connected* if there exists a path joining any two points. On *removing a point* v of G we obtain the subgraph $G - v$ which consists of all the points and lines of G except for v and all the lines incident with v. Thus $G - v$ is the maximal subgraph of G not containing v.

A graph G is called *n-connected* if G has at least $n + 1$ points and it is impossible to disconnect G by removing $n - 1$ or fewer points. The *connectivity* of G, denoted $\kappa(G)$, is defined to be n if G is n-connected but not $(n + 1)$-connected. The *complete graph* K_p of p points has every two distinct points adjacent; thus it has $p(p - 1)/2$ lines. Hence, the connectivity of a graph G with p points is $p - 1$ if G is the complete graph and otherwise is the minimum number of points of G whose deletion results in a disconnected graph. We shall show that the maximum connectivity among all p, q graphs is $[2q/p]$ if $q \geq p - 1$ and is 0 otherwise. The minimum connectivity is $q - (p - 1)(p - 2)/2$ or 0, whichever is larger.

The *distance* between two points is the length of a shortest path joining them. The *diameter* of a graph is the maximum distance between any two points. For completeness, if G is a disconnected graph and u and v are points which are not joined by a path, then we say that the distance between u and v is infinite and that the diameter of G is infinity. We will see that the minimum diameter among all p, q graphs is 1, 2, or infinity, depending on the value of q in terms of p, and will find a formula for the maximum diameter occurring among all connected p, q graphs.

The *degree* of a point is the number of lines with which it is incident. A graph is *regular* of degree n if all points have the same degree, n.

2. *The Maximum Connectivity.*—We require some preliminary observations. The *line-connectivity* $\lambda(G)$ is the minimum number of lines whose removal results in a disconnected graph. It is known[2] that for any graph, $\kappa \leq \lambda$.

Let d_i be the degree of the point v_i. It is well known that in any graph $d_1 + d_2 + \ldots + d_p = 2q$. In view of this equation, the average $\bar{d}$ of the degrees of the points of G is given by $\bar{d} = 2q/p$.

LEMMA 1. *If G_1 is a spanning subgraph of G, then $\kappa(G_1) \leq \kappa(G)$.*

LEMMA 2. *The connectivity of a connected graph is at most the minimum of the degrees of its points.*

Proof: If n is the minimum degree and u is a point of degree n, adjacent to points $v_1, \ldots, v_n$, then on removing the points v_i from G, u becomes an isolated point.

THEOREM 1. *Among all p,q graphs, the maximum connectivity is 0 when $q < p - 1$ and is $[2q/p]$ when $q \geq p - 1$.*

Proof: Let G be a p, q graph. When $0 \leq q < p - 1$, G is necessarily disconnected and hence $\kappa(G) = 0$. If $q = p - 1$, then any connected p, q graph is a tree. For any tree with $p > 1$, the connectivity is 1 and $[2q/p] = [2 - 2/p] = 1$, while for the tree K_1 with $p = 1$, we have $q = 0$ and $[2q/p] = 0$, which is the connectivity of K_1 by definition.

In order to show that the maximum connectivity is $[2q/p]$ when $q \geq p$, we will prove the following two statements:

(1) The connectivity of a p, q graph cannot exceed $[2q/p]$.

(2) There exists a p, q graph H whose connectivity is $[2q/p]$.

Proof of (1): We first consider the case that $2q/p$ is an integer r. If G has at least one point v_i such that $d_i > r$, then, since the average of the degrees of all the points of G is exactly r, there must be a point v_j whose degree $d_j < r$. But by Lemma 2, the minimum degree of all the points is an upper bound to the connectivity. Hence for a p, q graph G which is not regular of degree r, the connectivity $\kappa < r$. If, on the other hand, G is a regular graph, then by Lemma 2, $\kappa \leqslant r$.

For a graph G such that $2q/p$ is not an integer, it is also immediate that the minimum degree cannot exceed $[2q/p]$, and the lemma again applies.

Proof of (2): We have already discussed graphs with $p = 1$ or 2. Consider any pair of integers p and q, such that $p \geq 3$ and $p \leq q \leq \binom{p}{2}$. We construct a p, q graph H with connectivity $[2q/p]$. Obviously when the average degree is 2, the graph H with maximum connectivity is a cycle. This is a special case of the following general construction.

Case 1: The average degree $2q/p$ is an integer r: It is convenient to give the construction separately for even and odd values of r. We first consider the even value $r = 2k$. For this case, we construct the following graph, H_{2k}. We begin by drawing a p-gon and labeling its points by the integers $0, 1, 2, \ldots, p - 1$. We then join two points i and j if and only if $|i - j| \equiv m(\text{mod } p)$, where $2 \leq m \leq k$. By construction, H_{2k} is regular of degree $r = 2k$ and has q lines where $q = kp$.

We now construct the regular graph H_{2k+1} for the case that $r = 2k + 1$ is odd. Obviously, any regular graph of odd degree has an even number of points. Hence, we let $p = 2n$. Then H_{2k+1} is constructed as follows. We first draw the graph H_{2k} for which $p = 2n$ by the procedure of the preceding paragraph. We then add to this graph all the n lines which join pairs of points which are diametrically opposite in the $2n$-gon. The resulting graph H_{2k+1} is regular of degree $r = 2k + 1$.

We now prove that $\kappa(H_r) = r$. By Lemma 2, $\kappa(H_r) \leq r$. To show that $\kappa(H_r) \geq r$, we merely have to show that at least r points must be removed to disconnect the graph. In the case of H_{2k}, it is necessary (and sufficient) to remove two separate subsets of k consecutive points each, along the circumference of the polygon. For H_{2k+1}, it is still necessary to remove two such subsets of k points each to break the circumferential connections, but at least one more point must also be removed to break the diameteric connection.

Case 2: $2q/p$ is not an integer: Let $[2q/p] = r$. We first take up the subcase that p is even or r is even and begin by constructing the regular graph H_r with p points and $rp/2$ lines. We obtain a p, q graph H by adjoining any $q - rp/2$ new lines at random. Since H_r is a spanning subgraph of H, $r = \kappa(H_r) \leq \kappa(H)$ by Lemma 1. But by Lemma 2, $\kappa(H) \leq [2q/p] = r$. Hence, $\kappa(H) = r$.

When p is odd and the given value of q results in an odd number $r = [2q/p]$, a modification of this construction must be made. For there is no regular graph of odd degree with an odd number of points. This situation occurs for $p = 5$, $q = 8$, $[2q/p] = 3$, for $p = 7$, $q = 18$, $[2q/p] = 5$, etc. In this subcase, we take the regular graph H_{r-1} (in which $r - 1$ is even) with p points and first adjoin $(p + 1)/2$ new lines as follows. Join all pairs of points i and j of H_{r-1} such that $i - j = (p - 1)/2$. The resulting p, q graph H_r' has $(p - 1)$ points of degree r and one point of degree $(r + 1)$. The argument of Case 1 then applies to show that $\kappa(H_r') \geq r$. Again by Lemma 2, $\kappa(H_r') \leq r$. Hence $\kappa(H_r') = r$, completing the proof of the theorem.

We state without proof the corresponding result for line-connectivity, since its proof is contained in that of Theorem 1.

THEOREM 2. *Among all p, q graphs, the maximum line-connectivity is 0 when $q < p - 1$ and is $[2q/p]$ when $q \geq p - 1$.*

3. *The Minimum Connectivity.*—If $0 \leq q \leq \binom{p-1}{2}$, then there is a disconnected p, q graph. Hence the minimum connectivity over this range of q is zero. If $\binom{p-1}{2} < q \leq \binom{p}{2}$, then any p, q graph of minimum connectivity consists of a complete subgraph K_{p-1} together with exactly one additional point v which is adjacent to any $q - \binom{p-1}{2}$ points of K_{p-1}. Obviously, any such graph has connectivity equal to the degree of point v.

THEOREM 3. *The minimum connectivity among all p, q graphs is 0 or $q - \binom{p-1}{2}$ whichever is larger.*

4. *The Minimum and Maximum Diameter.*—We have already seen that whenever $q < p - 1$, every p, q graph is disconnected. Hence, for any such graph, the diameter is infinite. If $q = \binom{p}{2}$, there is exactly one p, q graph, namely the complete graph K_p, and its diameter is 1.

We now restrict ourselves to connected p, q graphs for which $p - 1 \leq q < \binom{p}{2}$ and show that the minimum diameter among all such graphs is 2 by exhibiting one with diameter 2. Consider points $u, v_1, v_2, \ldots, v_{p-1}$ and form all $p - 1$ lines of the form uv_i. The remaining $q - p + 1$ lines are drawn joining any points v_i and v_j arbitrarily. The resulting graph cannot have diameter 1 since there are too few lines. Hence, its diameter is 2 by construction, for there exist points v_i and v_j which are not adjacent but are at distance 2 from each other by means of the path v_iuv_j. Combining these observations, we obtain the following statement.

THEOREM 4. *The minimum diameter of a p, q graph is 1, 2, or* ∞ *when* $q = \binom{p}{2}$, $p - 1 \leq q < \binom{p}{2}$, *or* $q < p - 1$ *respectively.*

The maximum diameter of a p, q graph with $0 \leq q \leq \binom{p-1}{2}$ is infinite, since there are disconnected graphs over this range of q. The more interesting question is to determine the maximum diameter among all connected p, q graphs with $p - 1 \leq q \leq \binom{p}{2}$. It is easy to see that when $q = p - 1 + \binom{n}{2}$ for some n, the largest diameter is realized by a graph in which as many lines as possible are used to form a complete subgraph K_n with the remaining points and lines forming a single path.

When the number q of lines is not of the form $q = p - 1 + \binom{n}{2}$ for any n, then the subgraph containing as many lines as possible is not complete, and the remaining points and lines still form a single path terminating at one of the points of the subgraph. We require the following notation to express the general formula for the maximum diameter among all connected p, q graphs.

Let $q = p - 1 + m$ where $m \geq 0$. The number m is called the *index* of G and is the number of independent cycles. We denote by $[x]^+$ the smallest integer such that $0 \leq [x]^+ - x < 1$; so that $[x]^+ = -[-x]$. Let $y(m) = 1$ if $m = \binom{n}{2}$ for some n, and $y(m) = 2$ otherwise. Let $x(m) = [1 + (1 + 8m)/2]^+$

THEOREM 5. *The maximum diometer among all connected p,q graphs with index* $m = q - p + 1$ *is* $(p - 1) - x(m) + y(m)$.

Proof: The term $(p - 1)$ is the length of a path with p points. The quantity $x(m)$ is subtracted from this in order to account for the number of lines of this path which now lie in the subgraph described above. Then $y(m) = 1$ or 2 is added to the resulting number since it is the diameter of this subgraph: 1 if the subgraph is complete and 2 otherwise.

* The first draft of this article was written at the Los Alamos Scientific Laboratory in August, 1959, the final version at the RAND Corporation in February 1962.

[1] Berge, C., Théorie des graphes et ses applications (Paris: Dunod, 1958).

[2] Whitney, H., "Congruent graphs and the connectivity of graphs," *Amer. J. Math.*, **54**, 150–168 (1932).

*LOCALLY PERIPHERALLY EUCLIDEAN SPACES ARE LOCALLY EUCLIDEAN, II**

BY O. G. HARROLD, JR.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE

Communicated by N. E. Steenrod, May 14, 1962

1. *Introduction.*—A set-theoretic problem that has been of interest for some time is that of characterizing the locally Euclidean spaces among a more general class of topological spaces. In the one- and two-dimensional cases, a possible candidate for the more general class of topological spaces has been the Peano spaces. An expository paper by Van Kampen in the first volume of the *Duke Mathematical Journal* summarizes some well-known results in this situation. In the higher dimensional cases, the results attained so far have had to avoid the difficulty that the Schoenflies theorem of plane topology does not extend in the most general form. One method in the case $n = 3$ has been to assume the existence of a family of distinguished 2-spheres satisfying a system of five postulates and then prove that a sequence of partitionings exist that permit a homeomorphism between a 3-sphere and the topological space to be established.[2, 4]

It turns out that the axiom system in reference 4 may be carried over more or less directly to the n-dimensional case. However, the proof that a compact metric continuum satisfying our axiom system is topologically the n-sphere rests on the observation that the combinatorial steps in references 5 and 4 give one a scheme for approximating in a bi-uniform way certain subdivisions of the boundary of an $n + 1$ simplex and the appropriately chosen subdivisions (or partitions) of the compact metric continuum X.

There has been a revival of interest in problems relating to Cartesian products. Much of this may be traced to examples by Bing.[3] The present paper has some connections with the existence of Cartesian products but from a different angle than that in the examples referred to. The examples (together with their refinements) show that if a Cartesian product of two spaces and one factor are each locally Euclidean, the second factor may fail to be locally Euclidean at any point. In fact, the second factor may fail to be even a homotopy manifold. The result in this form has been announced by Kwun.[6] In this paper, one may think of our problem in the following light: Suppose a locally Euclidean space A is embedded in a larger space B. Will there be conditions that will guarantee that A has a neighborhood in B that is locally Euclidean?

In the study of the embedding of a k-manifold in a $(k + p)$-manifold, the local nature of the embedding depends strongly on the concepts of local unknottedness and local peripheral unknottedness.[5] The Axiom of Deformation below seems to provide a link between the problem mentioned in this paragraph and that of the