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Graph-polynomials

W.T. Tutte

Faculty of Mathematics, University of Waterloo, Waterloo, ON, Canada N2L 3G1

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Abstract

This paper describes how I became acquainted with the Tutte polynomial, and how I was led to the theorems about its represention as a sum over spanning trees and about its invariance under the flipping of a rotor of order less than 6.

Once upon a time there were four undergraduates of Trinity College Cambridge, and they took as their hobby the study of Perfect Rectangles. By a perfect rectangle they meant a rectangle that is dissected into unequal squares. A rectangle that is dissected into squares, not necessarily all different, was just a "squared rectangle." They hoped to find a Perfect Square, a perfect rectangle that was itself a square. A conjecture was going around that such a figure was impossible. See [1].

In connection with the linear equations occurring in their research those undergraduates were led to study the spanning trees of a general graph G. For those equations were simply Kirchhoff's equations for a network of unit resistances. Kirchhoff had solved these equations in terms of the spanning trees and spanning double trees of the network concerned. In their writings the undergraduates denoted the number of spanning trees of G by $C(G)$ and called it the "complexity" of G. They soon noticed a recursion formula relating the complexity of a graph G having a link A to the complexities of two related graphs G'_A and G''_A. The first of these is derived from G by deleting the link A, and the second by contracting A, with its two ends, into a single new vertex. Their formula was

$$C(G) = C\left(G'_A\right) + C\left(G''_A\right). \tag{1}$$

E-mail address: kung@unt.edu.

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To prove it they noted that the spanning trees of the deletion-graph are those of G that do not contain A, while the spanning trees of the contraction-graph are in one-to-one correspondence with those of G that do contain A.

Now I was one of the four, and I often applied formula (1) in the course of our research. I wondered if there were any other interesting functions of graphs, invariant under isomorphism, with similar recursion formulae. I happened to come across one such in a footnote to one of Hassler Whitney's papers [8]. R.M. Foster, I gathered, had observed in effect that the chromatic polynomial $P(G,x)$ of a graph G satisfied a rule like (1) but with the plus sign replaced by minus. We can prove this by pointing out that, when x is a positive integer, the x-colourings of G'_A in which the two ends of A have different colours are the x-colourings of G, while the others are in one-to-one correspondence with the x-colourings of G''_A.

It is true that the equation for chromatic polynomials has a minus sign where Eq. (1) has a plus. But this difference can be eliminated by replacing $P(G,x)$ by the trivially different $e(G)P(G,x)$, where $e(G)$ is $+1$ or -1 according as the number of vertices of G is even or odd. Here was an exciting connection between the theory of squared rectangles and that of coloured graphs!

Then there was the dual of the chromatic polynomial, the flow-polynomial $F(G,y)$. When y is a positive integer it gives the number of nowhere-zero y-flows. It too is found to satisfy a recursion like (1), but now the first term on the right has a minus sign and the second a plus. We can reduce this equation to the exact form of (1) by replacing the function $F(G,y)$ by $i(G)F(G,y)$, where $i(G)$ is $+1$ or -1 according as the number of vertices of G plus the number of edges is even or odd.

Some graph-functions are multiplicative over components, that is, the value for a given graph G is the product of the values for the components of G. The chromatic polynomial and the flow-polynomial are multiplicative over components but the complexity or tree-number is not. However if we count, not spanning trees but subgraphs intersecting each component in a spanning tree of that component, we get a multiplicative function, and this function is equal to the complexity for all connected graphs.

In [3] a graph-function satisfying an equation of type (1) is called a W-function. If multiplicative over components it is called a V-function.

It seems convenient to define here the operation of suppression of a divalent vertex v. Suppose first that v is incident with two distinct edges A and B, whose other ends are a and b, respectively. Then the operation deletes v and replaces A and B by a single new edge E joining a and b. It may happen that $a=b$. In that case E is a loop on a. It is sometimes convenient to allow the operation even when A and B coincide as a loop on v. Then v is deleted and A is retained as a "loose" edge, an edge with no incident vertex. A V-function is said to be "topologically invariant" if its value does not alter when one or more divalent vertices are suppressed. The flow-polynomial is an example.

For a topologically invariant V-function $f(G)$ it is possible to find a recursion formula that involves only cubic graphs (if a "cubic" graph is allowed to have loose edges). Suppose a cubic graph G has an edge X with distinct ends U and v. Let u be incident with the half-edges A and B, and v with the half-edges C and D. Such a half-edge may be the hither portion of an edge that goes to a third vertex. But any two of the half-edges could be the two halves of a single edge joining u and v, or two parts of a loop on u or v.

Let G' be the graph derived from G by deleting X and then suppressing its two ends, which have become divalent. Evidently G' is cubic. The operation may introduce a loose edge. This happens when A and B constitute a loop on u, or C and D a loop on v. Let Z be the graph obtained from G by contracting X, with its two ends, into a single new vertex w. This graph is not cubic, the new vertex being incident with all four of the given half-edges. By the recursion formula for a V-function we have

$$f(G) = f(G') + f(Z).$$

"Twisting" is an operation that rearranges the incidences of the four half-edges with u and v. It makes A and C incident with u, and B and D with v. The edge X still joins u and v. Call the resulting graph H. Let H' be derived from it by deleting X and then suppressing its two ends, and Y by contracting X with its two ends into a single new vertex. Evidently H and H' are cubic and Y can be identified with Z. We have

$$f(H) = f(H') + f(Z).$$

Eliminating $f(Z)$ from these two equations we find that

$$f(G) - f(G') = f(H) - f(H').$$

This is the required recursion formula involving cubic graphs only.

In my search for graph-functions with simple recursion formulae I came upon the number $p(G)$ of 1-factors of a cubic graph G. I found that it satisfied an equation very like the preceding one. In fact

$$p(G) + p(G') = p(H) + p(H').$$

To make this equation valid in all cases it is necessary to ascribe two 1-factors to the "cubic" graph U made up of one loose edge and no vertices. A cubic graph G has an even number $2n(G)$ of vertices. It is easily verified that the function $P(G)$ of cubic graphs, obtained from $p(G)$ by multiplying by $+1$ or -1 according as $n(G)$ is even or odd, satisfies the same recursion as any specialization to cubic graphs of a topologically invariant V-function. This implies, as is shown in [3], that $P(G)$ is such a specialization.

I was pleased to have found another V-function that seemed important in graph theory. However for non-cubic graphs it was not in general the number of 1-factors. An interpretation of it is given in [6].

Playing with my W-functions I obtained a two-variable polynomial from which either the chromatic polynomial or the flow-polynomial could be obtained by setting one of the variables equal to zero, and adjusting signs. With minor simplifications this became a function $T(G; x, y)$, multiplicative over blocks. For the vertex-graph, link-graph and loop-graph it is 1, x, and y respectively, where x and y are the variables. Moreover it satisfies a recursion formula. Let A be any link, not an isthmus, of a connected graph G. Then the polynomial for G is the sum of the polynomials for G'_A and G''_A. I was able to show that this definition is consistent. In my papers I called this function the dichromate, but it is now generally known as the Tutte polynomial. This may be unfair to Hassler Whitney who knew and used analogous coefficients without bothering to affix them to two variables [8].

It follows inductively from the above definition that $T(G;1,1)$ is the complexity, the number of spanning trees of the graph G. I wondered if the general function $T(G;x,y)$ might be a sum of something simple in x and y over all the spanning trees of G. This simple hypothesis seemed to come to grief on the triangle or 3-circuit, for which $T(G;x,y)$ is the sum of y, x, and x^2. There are indeed just three spanning trees to associate with the three terms of the polynomial. But those three trees are equivalent under the symmetry of the triangle, so how can the three terms of the polynomial be different?

I tried to evade this refutation by enumerating the three edges of the triangle as $A(1)$, $A(2)$, and $A(3)$. Then I was able to find a rule by which one could derive each term of the polynomial from a corresponding spanning tree in its relation to the enumeration. Surprisingly, the rule worked for other graphs than the triangle. It is stated and proved for all finite graphs, in [4]. Non-negative integers, called internal and external activities with respect to the arbitrary enumeration are defined for each spanning tree. They serve as the indices of x and y in the product that is the corresponding term of the polynomial.

I marvelled that all the different possible enumerations should give rise to the same polynomial $T(G;x,y)$, even though different enumerations usually gave different internal and external activities for a given spanning tree. But I recalled that Hassler Whitney, giving the chromatic polynomial in terms of broken circuits, had encountered a similar phenomenon [7].

With this formula I began to see $T(G;x,y)$ as a polynomial of special interest. I had already noticed one of its remarkable properties in connection with pairs G, G^* of dual planar graphs. These satisfy

$$T(G^*;x,y) = T(G;y,x). \tag{2}$$

In search of another theorem I went back to the paper of 1940 on dissections of rectangles into squares, wherein there is a discussion of subgraphs called rotors. Consider a graph G which is the union of two subgraphs R and S having just n vertices, and no edges, in common. Call these n vertices the vertices of attachment of R. Suppose further that R has a rotational symmetry with respect to which the vertices of attachment form a complete set of equivalent vertices. Then we call R a rotor of G of order n, and S is the corresponding stator. To "flip" the rotor is to detach it and then replace it going the other way round. Let this operation change G into a graph H. In the cases of interest G and H are not isomorphic.

In [1], G is regarded as an electrical network of unit resistances. Current enters G at one vertex and leaves at another, both vertices being in the stator. It is shown that flipping the rotor does not alter the complexity: $C(G) = C(H)$. Moreover it does not alter the currents in S though it may alter some or all of those in R.

Since $C(G)$ is a special case of $T(G;x,y)$ one naturally asks if the general polynomial is invariant under rotor-flipping? In [5] this question is answered in the affirmative for rotors of orders 3, 4, and 5, but the proof given does not extend to rotors of higher order. S. Foldes found a non-planar counter-example for order 6 [2]. The result for orders less than 6 gave a new way to construct different graphs with the same Tutte polynomial. With F. Bernhart I once used it to answer a query of Ruth Bari. Yes, there are two different planar maps with equal chromatic polynomials and with dual maps having equal chromatic polynomials too!

That completes my account of my work on the polynomial. Later I was astonished to hear that it had found applications in other branches of mathematics, even in knot theory. I learned something of these applications at the workshop in Barcelona.

References

[1] R.L. Brooks, C.A.B. Smith, A.H. Stone, W.T. Tutte, The dissection of rectangles into squares, Duke Math. J. 7 (1940) 312–340.
[2] S. Foldes, The rotor effect can alter the chromatic polynomial, J. Combin. Theory Ser. B 25 (1978) 237–239.
[3] W.T. Tutte, A ring in graph theory, Proc. Cambridge Philos. Soc. 43 (1947) 26–40.
[4] W.T. Tutte, A contribution to the theory of chromatic polynomials, Canad. J. Math. 6 (1954) 80–91.
[5] W.T. Tutte, Codichromatic graphs, J. Combin. Theory Ser. B 16 (1974) 168–174.
[6] W.T. Tutte, 1-factors and polynomials, European J. Combin. 1 (1980) 77–87.
[7] H. Whitney, A logical expansion in mathematics, Bull. Amer. Math. Soc. 38 (1932) 572–579.
[8] H. Whitney, The coloring of graphs, Ann. of Math. (2) 33 (1932) 688–718.