Free Lattices

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1991 *Mathematics Subject Classification.* Primary 06B25; Secondary 06–04, 68Q25, 06A06, 06B20

This research supported by NSF Grants DMS–9204481 and DMS–9122011.

Contents

Introduction

This monograph covers the fascinating subject of free lattices. The arithmetic of free lattices is interesting and their structure is surprisingly complex. Although free lattices do have several nice properties, some reminiscent of properties of distributive lattices, discovering their nature is not easy. This book presents a thorough development of the structure of free lattices.

A brief history. Although Richard Dedekind found the free *modular* lattice on three generators in 1900, Thoralf Skolem showed the word problem for finitely presented lattices had a uniform solution in 1920, and Garrett Birkhoff showed the free lattice on three generators was infinite, substantial insight into the nature of free lattices really began with the two papers [**132, 133**] of Philip Whitman which appeared in 1941 and 1942. In these he solved the word problem for free lattices and showed that there was a unique shortest term representing each element of the free lattice, known as the canonical form. He also established a strong connection between this syntactic notion and the arithmetic of the lattice. He showed that the canonical form is the best way to represent an element in a very natural lattice theoretical sense. Using this interplay, he was able to prove several strong results about free lattices; for example, the free lattice on a countably infinite generating set is embeddable in the one on a three element generating set.

In the 1950's and 1960's, R. Dean, B. Jónsson, F. Galvin, and J. Kiefer obtained further structural results about free lattices. One of their primary interests was sublattices of free lattices, particularly finite sublattices. As part of his solution to the word problem, Whitman had shown that free lattices satisfy an important property now known as Whitman's condition, which states that the relation $a \wedge b \leq c \vee d$ can hold in a free lattice only in one of the four trivial ways:

$$a \leq c \vee d, \qquad b \leq c \vee d, \qquad a \wedge b \leq c, \qquad \text{or} \qquad a \wedge b \leq d.$$

Jónsson showed that free lattices and all their sublattices satisfy a weak form of distributivity known as semidistributivity. As part of their attack on the problem of characterizing finite sublattices of free lattices, Jónsson and Kiefer showed that if a lattice satisfies Whitman's condition and has a set of five elements, no four of which join above the fifth, then it is infinite. This implies that the breadth of a finite sublattice of a free lattice is at most four and that there is a nontrivial lattice equation true in all finite sublattices of a free lattice.

Chapter I presents these and other classical results after first giving Whitman's solution to the word problem and other basic results. It also shows that free lattices are continuous, not complete, and that free lattices have a unary polynomial which is fixed point free. The first chapter is suitable as an introduction to free lattices in, say, a course on lattice theory or algebra.

In the paper by Jónsson and Kiefer it was conjectured that a finite lattice could be embedded into a free lattice if and only if it satisfies Whitman's condition and is semidistributive. Jónsson worked on this conjecture, producing some unpublished notes that influenced further investigations by others. Ralph McKenzie's important paper [**99**] uses Jónsson's notes in the characterization of finite projective lattices. A great deal of work was put into Jónsson's conjecture, as it was usually termed. J. B. Nation finally confirmed the conjecture in a deep paper [**104**] appearing in 1982. Nation's proof is based on the approach developed in Jónsson's notes. This theorem appears in Chapter V, which also contains a characterization of projective lattices and other results on sublattices of free lattices.

The notion of covering, one element being covered by another, denoted $a \prec b$, naturally leads to several intriguing questions and plays an extremely important role in many of the deeper results on free lattices. Consequently, the theory of the covering relation in free lattices is well developed in this monograph. Whitman was the first to exhibit coverings in free lattices: he showed, for example, that the meet of all but one of the generators covers the meet of all of the generators. R. Dean, in unpublished work, showed that there were elements that covered no other element. When $a \prec b$, we say a is covered by b and a is a lower cover of b.

When $a \prec b$ (in any lattice), there is a unique maximal congruence relation, $\psi(a,b)$, not containing $\langle a,b\rangle$. McKenzie [**99**] showed that the homomorphic image of the free lattice corresponding to this congruence relation is a finite, subdirectly irreducible lattice with the additional property that each element of this lattice has a least and a greatest preimage in the free lattice, i.e., the natural homomorphism is *bounded.* These lattices, which are called splitting lattices, and the corresponding bounded homomorphisms give an effective tool for studying covers in free lattices. For example, with the aid of his techniques, McKenzie was able to show that deciding if $a \prec b$ holds in a free lattice is recursive.

Chapter II gives a detailed coverage of bounded homomorphisms and splitting lattices. Jónsson's unpublished notes studied certain dependency relations in lattices. Although on the face it would not appear so, these relations are closely related to bounded homomorphisms. These and other relations are covered thoroughly in a general setting in this chapter. The chapter includes applications of these ideas and results to several areas of lattice theory and algebra, such as projective lattices, varieties of lattices, and the equational theory of congruence lattices of algebraic systems. The dependency relations of this chapter give an efficient representation of congruences on a finite lattice, facilitating fast algorithms for determining such properties as subdirect irreducibility and simplicity, which are developed in Chapter XI.

In a significant paper [**24**], Alan Day, using his doubling construction, was able to prove that the variety generated by all splitting lattices was in fact all lattices. This implies that free lattices are weakly atomic: every nontrivial interval contains a covering. We present two proofs of this result; the first is in Chapter II. In Chapter IV we prove a somewhat stronger version of Day's Theorem, useful for other results on free lattices.

Several basic questions about covers and about the structure of free lattices remained. Given an element, could one decide if it was covered by any other element? Could one effectively find all such covers? Is there a *coverless* element: one which covers no element and is covered by no element? How long could a chain of covers in a free lattice be? These questions were settled by Freese and Nation in a paper [**62**] which thoroughly developed the theory of covers in free lattices. A computationally efficient and theoretically useful algorithm was given which finds all elements covering a given element. Using this algorithm, it was possible to construct an element with no upper and no lower covers.

In answering some of the deeper questions, two facts played a key role. Of course, if an element a of a free lattice is join irreducible, it can cover at most one element. When this element exists, we denote it a_*. It turns out that a_* is almost always a proper meet and that its canonical expression has a unique meetand, denoted $\kappa(a)$, not above a. This map κ is a bijection between the completely join irreducible elements and the completely meet irreducible elements. While making a table with some examples of elements and their κ's for an early version of [**62**], we observed that if the canonical expression for a was $a_1 \wedge \cdots \wedge a_n$, then the canonical expression for a_* was usually just $\kappa(a) \wedge a_1 \wedge \cdots \wedge a_n$. The first key fact alluded to above was that the canonical expression for a_* is simply the meet of $\kappa(a)$ and those a_i's which are not above $\kappa(a)$. At the time this seemed little more than a cute observation but turns out to play a crucial role in many of the deeper results on free lattices.

Suppose that $a = a_1 \wedge \cdots \wedge a_n$ and suppose that a has both an upper and a lower cover; say $c \prec a \prec b$. Since not every a_i can be above b, by renumbering if necessary, one may assume that $b \nleq a_1$. The second key fact was that, in this situation, a_1 is *totally atomic*, i.e., each element properly above a_1 contains a cover of a_1, and the dual condition holds. Each free lattice has only finitely many totally atomic elements and this severely restricts chains of covers in free lattices. Indeed, except at the very top and bottom of free lattices, chains of covers can have length at most two. With the aid of these tools, it was possible to answer many of the open questions. For example, all lattices isomorphic to finite intervals were found.

The basic theory of covers in free lattices is given in Chapter III. Chapter VI characterizes totally atomic elements and proves several important consequences of this characterization. Chapter VII uses the results of Chapters III and VI to derive some of the deeper results mentioned above on chains of covers, finite intervals, and connected components of the covering relation. A characterization of three element intervals, which is useful in the following chapters, is given.

Perhaps because of the success of [**62**], there was little work on free lattices for a few years except for the characterization of the connected components of the covering relation in [**51**]. But one intriguing question remained. After characterizing all finite intervals of free lattices, Freese and Nation sought to show that every infinite interval in a free lattice contains the free lattice on a countably infinite generating set as a sublattice. By Whitman's work, it would be enough to show that every infinite interval contains a set of three elements which are join and meet irredundant. We felt proving this would be much easier than our work on finite intervals. We were quite surprised when we could not do this and, in fact, were not even able to show the existence of two incomparable elements. That is, we could not rule out the possibility of an infinite interval in a free lattice which is a chain! We were able to derive some unlikely consequences of such an interval, but none produced a contradiction. We showed this problem to Tom Harrison who derived much stronger (and more unlikely) consequences, but still without reaching a contradiction. Harrison showed it to Steve Tschantz, who, after working on it for some time, was able to show that such an interval could not exist, and, in fact, that every infinite interval contains the free lattice on ω generators. Interestingly, most of

Tschantz's proof is devoted to showing that there is no infinite interval which is a chain. A new proof of Tschantz's Theorem is presented in Chapter IX.

Given elements $a < b$ in a free lattice, what are the possible types of maximal chains in the interval sublattice from a to b? In particular, can the ordered set of rationals between 0 and 1 be represented in this way? This question was posed informally to the first author by F. Galvin, J. Mycielski, and W. Taylor. Using Tschantz's Theorem, one can show that every infinite interval of a free lattice contains a coverless element, an element with no upper and no lower cover. In trying to construct a dense chain, we were led to *semisingular* elements. If $a_1 \wedge \cdots \wedge a_n$ is the canonical expression of an element a which is join irreducible and has a lower cover a_*, then a is semisingular if $\kappa(a) \leq a_i$, for some i. As we mentioned earlier, this situation is somewhat exceptional. We conjectured that, with a few trivial exceptions, every semisingular element must be the middle element of a three element interval. The truth of this would allow us to construct dense maximal chains in almost every infinite interval of a free lattice. However, the truth of our conjecture would also prove that no interval of a free lattice could be a chain and thus would substantially shorten the proof of Tschantz's Theorem. This made us doubtful the conjecture could be proved without a great deal of work, but we pressed on anyway and were pleased when we were able to prove it. This result is in Chapter VIII, and the applications to proving Tschantz's Theorem and to constructing maximal chains are in Chapter IX.

In spite of these successes, Tschantz's (and our subsequent) techniques were not strong enough to answer the following question raised in [**131**]: can an interval have an isthmus? Specifically, are there elements $a < c < b$ in a free lattice such that the interval from a to c and the one from c to b are both infinite and every element in the interval from a to b is comparable with c? After a sustained effort we were able to show that such an interval cannot exist. This result and some of its consequences are in Chapter X.

Computing. In their early work on free lattices, Freese and Nation often had to put specific terms into canonical form. Whitman's algorithm for this, while efficient (even polynomial time when implemented correctly), is tedious for humans. Freese used the following alternate procedure: he would stare at the term, looking for any obvious reason it was not in canonical form. If he could not find any such reason, he would take the term to Nation, whose office is one floor higher, and ask him to look at it. Nation used a dual procedure. The inadequacy of this procedure led us to develop computer programs to calculate in free lattices. These programs were later extended into a computer algebra system that deals with all types of lattices. Many of the examples in this monograph were found with the aid of this program. Several of the crucial lemmas were proved after first extensively searching for a counterexample with the program, and of course several conjectured results were disproved with this program.

Chapter XI gives an analysis of many lattice theoretic algorithms. Our original intent was to only analyze those algorithms covered in this book: free lattice algorithms and algorithms for finite lattices which are related to free lattices such as determining if a finite lattice is a splitting lattice. Of course we had to develop the basic algorithms and structures for representing lattices, and as we proceeded, we found there is an interesting body of results spread over both the mathematics and computer science literature. We felt it would be useful to bring this knowledge together, so this chapter covers a much wider area than was originally planned. It covers the basic algorithms for ordered sets such as finding a linear extension. It discusses representations of lattices and proves the result of P. Goralčík, A. Goralčíková, V. Koubek, and V. Rödl [**69**] which shows that one can decide if an ordered set of size n is a lattice in time $O(n^{5/2})$.

The dependence relations of Chapter II give efficient ways of representing and calculating in congruence lattices of finite lattices. For example, it is shown that one can decide if a lattice of size n is simple, or is subdirectly irreducible, or is a splitting lattice, each in time $O(n^2)$. The algorithm of K. Bogart and D. Magagnosc [**13**] to find a chain partition and a maximum sized antichain is presented, as well as an algorithm to find the lattice of all maximum sized antichains of an ordered set.

We give a careful analysis of Whitman's algorithm for testing if $a \leq b$ and for putting terms into canonical form in free lattices, since all other free lattice algorithms use these. The basic algorithms for finitely presented lattices are also covered. Finally some remarks on computer algorithms for automatically drawing lattices are given, including our own new algorithm.

Term rewrite systems were popularized by the work of D. Knuth and P. Bendix in [**91**]. Such systems give rewrite rules of a special form which transform any term into a normal form. In Chapter XII we show that, despite Whitman's very nice algorithm to put terms into canonical form, there is no associative, commutative term rewrite system for lattices. On the other hand we show that several varieties of lattices do have term rewrite systems.

Several of the results of this monograph are new or at least have never been published. The existence of a fixed point free polynomial on the free lattice with three generators and the characterization of fixed point free slim polynomials in Chapter I is new. Many of the results of Chapters II and V are proved in a more general and more natural setting. Certain results in Chapters V, VI and VII are new, such as the example showing that there are infinitely many three element intervals in the free lattice on three generators in Chapter VII and the explicit formula for $\kappa(w)$ of a totally atomic element in Chapter VI. All the results of Chapters VIII and X are new, as are the results on coverless elements and maximal chains in intervals of free lattices in Chapter IX. Several of the theorems and algorithms of Chapter XI are new, for example the $O(n^2 \log_2 n)$ time algorithm to recognize when a partially ordered set is semidistributive or is a splitting lattice, Theorem **??**. The $O(n^2)$ algorithm for testing if a lattice is simple, or if it is subdirectly irreducible, Theorem **??**, is new. The results of Chapter XII are new.

Courses. There are several ways to read this book. As we mentioned before, the first chapter can be used as an introduction to the classical results of free lattices. The first three sections of Chapter I are prerequisites for most of the rest of the book.

Chapter II serves several purposes. It proves various results necessary for Chapters V and XII and some for Chapter III. It also develops several free lattice techniques which are useful in other areas of lattice theory and algebra, particularly varieties. Many free lattice results are proved in the more general setting of semidistributive lattices.

The reader who is more interested in the structure of free lattices, in particular the theory of covers and its consequences, should read Chapter III and Chapters VI through X. Since Chapter II contains a great deal of material, most of which is not needed for Chapter III, we have written Chapter III so that it can be read without first reading Chapter II. Thus some of the results of Chapter II are reproved in the special case of free lattices, where the proof is often easier. On the other hand, the corresponding result from Chapter II is cited for the reader who has read it or who is interested in the more general formulation.

The reader who is interested in sublattices of free lattices and in projective lattices should read Chapters II and V. The reader interested in the computational aspects of lattice theory should read Chapters XI and XII. Most sections of Chapter XI only require a basic knowledge of lattices. For the other sections the reader can refer back to the earlier chapters as needed. Chapter XII requires Chapter I and parts of Chapter II.

The monograph gives several open problems throughout the text. For convenience, they are repeated on page 301.

We would like to thank Bjarni Jónsson, Stan Burris, and the referee for several suggestions. We would also like to thank our editor at the American Mathematical Society, Christine Thivierge, and the AMS technical support group for help with the preparation of the manuscript, which was typeset with $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX.

CHAPTER I

Whitman's Solution to the Word Problem

In this chapter, after first giving a brief introduction to lattice theory including Alan Day's doubling construction, we give the basic properties of free lattices including Whitman's solution to the word problem. Our proof is based on Day's doubling construction. Whitman's solution implies that the elements of a free lattice have a very nice canonical form and we investigate his solution and canonical form carefully. Several very interesting results about free lattice can be derived using this canonical form and the solution to the word problem alone. The existence of the canonical form easily implies that free lattices (and their sublattices) are semidistributive.

We are also able at this point to prove some deeper theorems on free lattices. Some of these are well known such as Whitman's result that the free lattice with 3 generators has a sublattice isomorphic to the free lattice with a countable free generating set. Others are not well known. We prove Whitman's theorem that free lattices are continuous. We prove an unpublished result of the first author that there is a unary polynomial function on the free lattice with three generators without a fixed point. This is relevant to the problem of characterizing order polynomial complete lattices. Using this fixed point free polynomial, it is not hard to prove a little known fact of Whitman that the free lattice with 3 generators contains an ascending chain without a least upper bound. We also prove a result of Jónsson and Kiefer which shows that finite sublattices of free lattices have breadth at most four. This then implies that there is a nontrivial lattice equation which is satisfied by all finite sublattices of free lattices.

1. Basic Concepts from Lattice Theory

We begin this section with a very brief review of the fundamental concepts associated with lattice theory. The reader can consult either Crawley and Dilworth's book [**20**] or McKenzie, McNulty, and Taylor's book [**100**] for more details. Other standard references are Birkhoff [**10**], Davey and Priestley [**22**], and Grätzer [**73**]. For the most part we follow the notation and terminology of [**100**]. We use the notation $A \subseteq B$ for set inclusion, while $A \subset B$ denotes proper set inclusion.

An *order relation* on a set P is a binary relation which satisfies

(1) $x \le x$, for all $x \in P$ (reflexivity)
(2) $x \le y$ and $y \le x$ imply $x = y$, for all $x,\, y \in P$ (anti-symmetry)
(3) $x \le y$ and $y \le z$ imply $x \le z$, for all $x,\, y,\, z \in P$ (transitivity)

An *ordered set* (also known as a *partially ordered set*) is a pair $\mathbf{P} = \langle P, \le \rangle$, where P is a set and $\le$ is an order relation on P. If $\langle P, \le \rangle$ is an ordered set, the relation $\ge$ is defined by $x \ge y$ if and only if $y \le x$. $\langle P, \ge \rangle$ is an ordered set known as the *dual* of $\langle P, \le \rangle$. Each concept and theorem about ordered sets has a dual obtained by reversing the roles of $\le$ and $\ge$. In proofs we often use the phrase 'by duality' to express the symmetry between $\le$ and $\ge$. Of course $x < y$ if and only if $x \le y$ and $x \ne y$, and $>$ is defined dually.

A relation which is reflexive and transitive, but not necessarily anti-symmetric is called a *quasiorder*. If $\le$ is a quasiorder on S, $a \equiv b$ if $a \le b$ and $b \le a$ defines an equivalence relation on S. Then $\le$ induces a natural partial order on $P/{\equiv}$.

A *chain* C in an ordered set $\mathbf{P}$ is a subset of P such that any two elements of C are comparable, i.e., if x and $y \in C$ then either $x \le y$ or $y \le x$. An *antichain* is a subset A of P such that no two elements of A are comparable. The *length* of $\mathbf{P}$ is the supremum of the cardinalities of all chains in $\mathbf{P}$, and the *width* is the supremum of the cardinalities of all antichains in $\mathbf{P}$.

Let S be a subset of P and $a \in P$. We say that a is the *least upper bound* of S if a is an *upper bound* for S, i.e., $s \le a$ for all $s \in S$, and $a \le b$ for any upper bound b of S. If it exists, we denote the least upper bound by $\bigvee S$. The dual concept is called the *greatest lower bound* and is denoted by $\bigwedge S$. If $S = \{a, b\}$ then $\bigvee S$ is denoted by $a \vee b$ and $\bigwedge S$ by $a \wedge b$. The terms *supremum* and *join* are also used for the least upper bound and *infimum* and *meet* are used for the greatest lower bound.

A *lattice* is an algebra $\mathbf{L} = \langle L, \sqcup, \wedge \rangle$, with two binary operations which are both idempotent, commutative, and associative, and satisfy the absorptive laws:

$$x \sqcup (y \wedge x) = x \qquad \text{and} \qquad x \wedge (y \sqcup x) = x$$

for all $x \in L$. If $\mathbf{L} = \langle L, \sqcup, \wedge \rangle$ is a lattice, we can define an order on L by $x \le y$ if and only if $x \wedge y = x$. Under this order, $x \wedge y$ is the greatest lower bound of x and y, and $x \vee y$ is the least upper bound of x and y. Conversely, an ordered set $\langle L, \leq \rangle$ such that each pair of elements of L has both a greatest lower bound and a least upper bound defines a lattice. It is easy to see that the dual of a lattice $\langle L, \vee, \wedge \rangle$ is $\langle L, \wedge, \vee \rangle$.

An element a a lattice $\mathbf{L}$ is *join irreducible* if $a = b \vee c$ implies that either $a = b$ or $a = c$. An element a is *completely join irreducible* if $a = \bigvee S$ implies $a \in S$. An element a is *join prime* if $a \leq b \vee c$ implies that either $a \leq b$ or $a \leq c$; it is *completely join prime* if $a \leq \bigvee S$ implies $a \leq s$ for some $s \in S$. Naturally *meet irreducible*, *completely meet irreducible*, *meet prime*, and *completely meet prime* are defined dually.

If a lattice has a least element, it is denoted by 0 and if it has a greatest element, it is denoted by 1. Note that, technically, 0 is join irreducible but not completely join irreducible. However, we will follow the long-standing convention that in a *finite* lattice, 0 is not regarded as join irreducible. (This is because we treat a finite lattice as a complete lattice.) Dually, in a finite lattice 1 is not considered to be meet irreducible. We let $\mathrm{J}(\mathbf{L})$ denote the join irreducible elements of $\mathbf{L}$ and $\mathrm{M}(\mathbf{L})$ denote the meet irreducible elements of $\mathbf{L}$.

If $a < b$ are elements in a lattice $\mathbf{L}$ and there is no $c \in L$ with $a < c < b$, then we say that *a is covered by b*, and we write $a \prec b$. In this situation we also say that b *covers* a and write $b \succ a$. In addition we say that b is an *upper cover* of a and that a is a *lower cover* of b. We also define a nameless equivalence relation on L by saying c is equivalent to d if there is a finite sequence $c = c_0, c_1, \ldots, c_n = d$ such that c_i either covers or is covered by c_{i+1}. The blocks of the equivalence relation are called the *connected components* of the covering relation of $\mathbf{L}$. The *connected component* of $a \in L$ is the block containing a. If $a \leq b$, then we let b/a denote the interval $\{x : a \leq x \leq b\}$.

An *order ideal* in an ordered set $\mathbf{P}$ is a subset S of P such that whenever $a \leq b$ and $b \in S$ then $a \in S$. An *order filter* is defined dually. A subset I of a lattice $\mathbf{L}$ is called an *ideal* if it is an order ideal and it is closed under finite joins. A *filter* is defined dually. If $S \subseteq L$, then the ideal generated by S, the smallest ideal containing S, consists of all elements $a \in L$ such that $a \leq s_1 \vee \cdots \vee s_k$ for some $s_1, \ldots, s_k \in S$.

Day's doubling construction. A useful construction for free lattice theory is Alan Day's doubling construction. We will use this construction in this chapter to derive one of the basic properties of free lattices, known as Whitman's condition, following Day's approach [**26**]. The doubling construction also plays a crucial role in the proof of Day's important result [**24**] that free lattices are weakly atomic, discussed in Chapter II and Chapter IV.

Let $\mathbf{L}$ be a lattice. A subset C of L is *convex* if whenever a and b are in C and $a \leq c \leq b$, then $c \in C$. Of course an interval of a lattice is a convex set as are lower and upper pseudo-intervals. A subset C of L is a *lower pseudo-interval* if it is a finite union of intervals, all with the same least element. An *upper pseudo-interval* is of course the dual concept.

Let C be a convex subset of a lattice $\mathbf{L}$ and let $L[C]$ be the disjoint union $(L - C) \cup (C \times 2)$. Order $L[C]$ by $x \leq y$ if one of the following holds.

(1) $x, y \in L - C$ and $x \leq y$ holds in $\mathbf{L}$,
(2) $x, y \in C \times 2$ and $x \leq y$ holds in $C \times 2$,
(3) $x \in L - C$, $y = \langle u, i \rangle \in C \times 2$, and $x \leq u$ holds in $\mathbf{L}$, or
(4) $x = \langle v, i \rangle \in C \times 2$, $y \in L - C$, and $v \leq y$ holds in $\mathbf{L}$.

There is a natural map λ from $L[C]$ back onto L given by

$$\lambda(x) = \begin{cases} x & \text{if } x \in L - C \\ v & \text{if } x = \langle v, i \rangle \in C \times 2. \end{cases} \tag{1}$$

The next theorem shows that, under this order, $L[C]$ is a lattice, denoted $\mathbf{L}[C]$.

THEOREM 1.1. *Let C be a convex subset of a lattice $\mathbf{L}$. Then $\mathbf{L}[C]$ is a lattice and $\lambda : \mathbf{L}[C] \to \mathbf{L}$ is a lattice epimorphism.*

Proof: Routine calculations show that $\mathbf{L}[C]$ is a partially ordered set. Let $x_i \in L - C$ for $i = 1, \ldots, n$ and let $\langle u_j, k_j \rangle \in C \times 2$ for $j = 1, \ldots, m$. Let $v = \bigvee x_i \vee \bigvee u_j$ in $\mathbf{L}$ and let $k = \bigvee k_j$ in $\mathbf{2}$. Of course, if $m = 0$, then $k = 0$. Then in $\mathbf{L}[C]$,

$$x_1 \vee \cdots \vee x_n \vee \langle u_1, k_1 \rangle \vee \cdots \vee \langle u_m, k_m \rangle = \begin{cases} v & \text{if } v \in L - C \\ \langle v, k \rangle & \text{if } v \in C \end{cases} \tag{2}$$

To see this let y be the right side of the above equation, i.e., let $y = v$ if $v \in L - C$ and $y = \langle v, k \rangle$ if $v \in C$. It is easy to check that y is an upper bound for each x_i and each $\langle u_j, k_j \rangle$. Let z be another upper bound. First suppose $z = \langle a, r \rangle$ where $a \in C$. Since z is an upper bound, it follows from the definition of the ordering that $a \geq v$ and $r \geq k$ and this implies $z \geq y$. Thus y is the least upper bound in this case. The case when $z \notin C$ is even easier. The formula for meets is of course dual. Thus $\mathbf{L}[C]$ is a lattice and it follows from equation (2) and its dual that λ is a homomorphism which is clearly onto $\mathbf{L}$.

2. Free Lattices

Since the lattice operations are both associative, we define *lattice terms* over a set X, and their associated lengths, in a manner analogous to the way they are defined for rings. Each element of X is a term of length 1. Terms of length 1 are called *variables.* If $t_1, \ldots, t_n$ are terms of lengths $k_1, \ldots, k_n$, then $(t_1 \sqcup \ldots \sqcup t_n)$ and $(t_1 \wedge \cdots \wedge t_n)$ are terms with length $1 + k_1 + \cdots + k_n$. When we write a term we usually omit the outermost parentheses. Notice that if x, y, and $z \in X$ then

$$x \sqcup y \sqcup z \qquad x \sqcup (y \sqcup z) \qquad (x \sqcup y) \sqcup z$$

are all terms (which always represent the same element when interpreted into any lattice) but the length of $x \sqcup y \sqcup z$ is 4, while the other two terms are both of length 5. Thus our length function gives preference to the first expression, i.e., it gives preference to expressions where unnecessary parentheses are removed. Also note that the length of a term (when it is written with the outside parentheses) is the number of variables, counting repetitions, plus the number of pairs of parentheses (i.e., the number of left parentheses). The length of a term is also called its *rank.* It is also useful to have a measure of the depth of a term. The *complexity* of a term t is 0 if $t \in X$, and if $t = t_1 \vee \cdots \vee t_n$ or $t = t_1 \wedge \cdots \wedge t_n$, where $n > 1$, then the complexity of t is one more than the maximum of the complexities of $t_1, \ldots, t_n$. The set of *subterms* of a term t is defined in the usual way: if t is a variable then t is the only subterm of t and, if $t = t_1 \sqcup \ldots \sqcup t_n$ or $t = t_1 \wedge \cdots \wedge t_n$, then the subterms of t consist of t together with the subterms of $t_1, \ldots, t_n$.

By the phrase '$t(x_1, \ldots, x_n)$ is a term' we mean that t is a term and $x_1, \ldots, x_n$ are (pairwise) distinct variables including all variables occurring in t. If $t(x_1, \ldots, x_n)$ is a term and $\mathbf{L}$ is a lattice, then $t^{\mathbf{L}}$ denotes the interpretation of t in $\mathbf{L}$, i.e., the induced n–ary operation on $\mathbf{L}$. If $a_1, \ldots, a_n \in L$, we will usually abbreviate $t^{\mathbf{L}}(a_1, \ldots, a_n)$ by $t(a_1, \ldots, a_n)$. Very often in the study of free lattices, we will be considering a lattice $\mathbf{L}$ with a specific generating set $\{x_1, \ldots, x_n\}$. In this case we will use $t^{\mathbf{L}}$ to denote $t^{\mathbf{L}}(x_1, \ldots, x_n)$.

If $s(x_1, \ldots, x_n)$ and $t(x_1, \ldots, x_n)$ are terms and $\mathbf{L}$ is a lattice in which $s^{\mathbf{L}} = t^{\mathbf{L}}$ as functions, then we say the *equation* $s \approx t$ *holds* in $\mathbf{L}$.

Let $\mathbf{F}$ be a lattice and $X \subseteq F$. We say that $\mathbf{F}$ is *freely generated by* X if X generates $\mathbf{F}$ and every map from X into any lattice $\mathbf{L}$ extends to a lattice homomorphism of $\mathbf{F}$ into $\mathbf{L}$. Since X generates $\mathbf{F}$, such an extension is unique. It follows easily that if $\mathbf{F}_1$ is freely generated by X_1 and $\mathbf{F}_2$ is freely generated by X_2 and $|X_1| = |X_2|$, then $\mathbf{F}_1$ and $\mathbf{F}_2$

are isomorphic. Thus, if X is a set, a lattice freely generated by X is unique up to isomorphism. We will see that such a lattice always exists. It is referred to as the *free lattice over* X and is denoted $\mathbf{FL}(X)$. If n is a cardinal number, $\mathbf{FL}(n)$ denotes a free lattice whose free generating set has size n.

To construct $\mathbf{FL}(X)$, let $\mathrm{T}(X)$ be the set of all terms over X. $\mathrm{T}(X)$ can be viewed as an algebra with two binary operations. Define an equivalence relation $\sim$ on $\mathrm{T}(X)$ by $s \sim t$ if and only if the equation $s \approx t$ holds in all lattices. It is not difficult to verify that $\sim$ restricted to X is the equality relation, that $\sim$ is a congruence relation on $\mathrm{T}(X)$, and that $\mathrm{T}(X)/\sim$ is a lattice freely generated by X, provided we identify each element of $x \in X$ with its $\sim$–class. This is the standard construction of free algebras; see, for example, [**20**] or [**100**].

This construction is much more useful if we have an effective procedure which determines, for arbitrary lattice terms s and t, if $s \sim t$. The problem of finding such a procedure is informally known as the *word problem* for free lattices. In [**132**], Whitman gave an efficient solution to this word problem. Virtually all work on free lattices is based on his solution.

If $w \in \mathbf{FL}(X)$, then w is an equivalence class of terms. Each term of this class is said to *represent* w and is called a *representative* of w. More generally, if $\mathbf{L}$ is a lattice generated by a set X, we say that a term $t \in \mathrm{T}(X)$ represents $a \in L$ if $t^{\mathbf{L}} = a$.

As Jónsson has shown in [**81**], certain basic aspects of Whitman's solution hold in every relatively free lattice and so it is worthwhile developing the theory in this more general context. A *variety* is a class of algebras (such as lattices) closed under the formation of homomorphic images, subalgebras, and direct products. A variety is called nontrivial if it contains an algebra with more than one element. By Birkhoff's Theorem [**9**], varieties are *equational classes*, i.e., they are defined by the equations they satisfy, see [**100**]. If $\mathcal{V}$ is a variety of lattices and X is a set, we denote the free algebra in $\mathcal{V}$ by $\mathbf{F}_{\mathcal{V}}(X)$ and refer to it as the *relatively free lattice* in $\mathcal{V}$ over X. If $\mathcal{L}$ is the variety of all lattices, then, in this notation, $\mathbf{F}_{\mathcal{L}}(X) = \mathbf{FL}(X)$. However, because of tradition, we will use $\mathbf{FL}(X)$ to denote the free lattice. The relatively free lattice $\mathbf{F}_{\mathcal{V}}(X)$ can be constructed in the same way as $\mathbf{FL}(X)$.

Notice that every nontrivial variety of lattices contains the two element lattice, which is denoted by $\mathbf{2}$.

LEMMA 1.2. *Let $\mathcal{V}$ be a nontrivial variety of lattices and let $\mathbf{F}_{\mathcal{V}}(X)$ be the relatively free lattice in $\mathcal{V}$ over X. Then*

(†)

$$\bigwedge S \leq \bigvee T \text{ implies } S \cap T \neq \emptyset \text{ for each pair of finite subsets } S, T \subseteq X.$$

Proof: Suppose that S and T are finite, disjoint subsets of X. As noted above, $\mathbf{2} \in \mathcal{V}$. Let f be the map from X to $\{0, 1\}$ which sends each $x \in S$ to 1 and all other x's to 0. By the defining property of free algebras, f can be extended to a homomorphism from $\mathbf{F}_{\mathcal{V}}(X)$ onto $\mathbf{2}$, which we also denote by f. Then $f(\bigwedge S) = 1 \not\leq 0 = f(\bigvee T)$. This implies that $\bigwedge S \not\leq \bigvee T$, as desired.

The next lemma will be used below to characterize when a subset of a lattice generates a free lattice.

LEMMA 1.3. *Let a be an element of a lattice $\mathbf{L}$ generated by a set X. Suppose that for every finite subset S of X,*

$$(\ddagger) \qquad a \leq \bigvee S \quad \text{implies} \quad a \leq s \quad \text{for some } s \in S.$$

Then (‡) holds for all finite subsets of L.

Proof: Let $\mathcal{K}$ be the collection of all sets U with $X \subseteq U \subseteq L$ such that (‡) holds for every finite subset S of U. By hypothesis, $X \in \mathcal{K}$. Suppose $U \in \mathcal{K}$ and that u and $v \in U$. Then $U \cup \{u \wedge v\} \in \mathcal{K}$. To see this, suppose that $a \leq \bigvee S \vee (u \wedge v)$ for some finite $S \subseteq U$, but $a \not\leq s$ for all $s \in S$. Then, since $a \leq \bigvee S \vee u$, (‡) implies $a \leq u$. Similarly, $a \leq v$ and so $a \leq u \wedge v$. It is trivial that $U \cup \{u \vee v\} \in \mathcal{K}$. Since X generates $\mathbf{L}$, we conclude that $L \in \mathcal{K}$.

LEMMA 1.4. *Let $\mathbf{L}$ be a lattice generated by a set X and let $a \in L$. Then*

(1) *if a is join prime, then $a = \bigwedge S$ for some finite subset $S \subseteq X$,*
(2) *if a is meet prime, then $a = \bigvee S$ for some finite subset $S \subseteq X$.*

If X satisfies condition (†) above, then

(3) *for every finite, nonempty subset $S \subseteq X$, $\bigwedge S$ is join prime and $\bigvee S$ is meet prime,*
(4) *if X is the disjoint union of Y and Z, and F is the filter of $\mathbf{L}$ generated by Y and I is the ideal generated by Z, then L is the disjoint union of F and I.*

Proof: Since $\mathbf{L}$ is generated by X, every element of L can be represented by a term with variables in X. It follows from this and an easy induction on the length of such a term that *if $X = Y \cup Z$, then $L = F \cup I$ where F is the filter generated by Y and I is the ideal generated by Z.*

To prove (1), let F be the filter generated by $Y = \{x \in X : a \leq x\}$ and let I be the ideal generated by $Z = \{x \in X : a \nleq x\}$. Since a is join prime, $a \notin I$, and so by the above observation, $a \in F$. This implies that $a \geq \bigwedge S$, for some finite $S \subseteq Y$. But every element of Y is above a; hence $a = \bigwedge S$, as desired. Of course (2) is proved dually.

Let T be a finite, nonempty subset of X and let $a = \bigwedge T$. It follows from condition (†) that condition (‡) holds for all finite subsets S of X. Hence, by Lemma 1.3, a is join prime.

For (4), we have already observed that $L = F \cup I$. If $F \cap I$ is nonempty, there would be subsets S and T of X with $\bigwedge S \leq \bigvee T$, contrary to our assumption.

Combining Lemmas 1.2 and 1.4, we obtain the following corollary for relatively free lattices.

Corollary 1.5. *Let $\mathcal{V}$ be a nontrivial variety of lattices and let $\mathbf{F}_{\mathcal{V}}(X)$ be the relatively free lattice in $\mathcal{V}$ over X. For each finite, nonempty subset S of X, $\bigwedge S$ is join prime and $\bigvee S$ is meet prime. In particular, every $x \in X$ is both join and meet prime. Moreover, if $x \leq y$ for x and $y \in X$, then $x = y$.*

Unlike the situation for groups and Boolean algebras, the free generating set of a free lattice is uniquely determined, as the next corollary shows.

Corollary 1.6. *If $\mathbf{L}$ is a lattice generated by a set X which satisfies condition (†), then the following hold.*

(1) *If Y generates $\mathbf{L}$ then $X \subseteq Y$.*
(2) *Every automorphism of $\mathbf{L}$ is induced by a permutation of X.*

In particular, these statements hold for the relatively free lattice, $\mathbf{F}_{\mathcal{V}}(X)$, for any nontrivial variety of lattices $\mathcal{V}$. Moreover, the automorphism group of $\mathbf{F}_{\mathcal{V}}(X)$ is isomorphic to the full symmetric group on X.

Proof: By Lemma 1.4(3), each $x \in X$ is both join and meet irreducible. Hence if x is in the sublattice generated by Y, it must be in Y. This proves the first statement. The rest of the theorem follows easily from the first.

The next corollary gives some of the basic coverings in free lattices, including the atoms and coatoms. It follows easily from Lemma 1.4(4).

Corollary 1.7. *Let $\mathbf{L}$ be a lattice generated by a finite set X which satisfies condition (†). If Y is a nonempty, proper subset of X, then we have the following covers in $\mathbf{L}$:*

$$\bigvee Y \prec \bigvee Y \vee \bigwedge (X - Y) \qquad \textit{and} \qquad \bigwedge Y \succ \bigwedge Y \wedge \bigvee (X - Y).$$

Thus these are covers in every nontrivial relatively free lattice $\mathbf{F}_{\mathcal{V}}(X)$*. In particular, the atoms of* $\mathbf{FL}(X)$ *are the elements* $\bigwedge(X - \{x\})$*, for* $x \in X$*. The coatoms are the elements* $\bigvee(X - \{x\})$*, for* $x \in X$*.*

Throughout this book we will use the notation $\underline{x} = \bigwedge(X - \{x\})$ and $\overline{x} = \bigvee(X - \{x\})$ to denote the atoms and coatoms, respectively, of a finitely generated free lattice $\mathbf{FL}(X)$.

Now we turn to Whitman's condition, which is the crux of the solution of the word problem for free lattices.

THEOREM 1.8. *The free lattice* $\mathbf{FL}(X)$ *satisfies the following condition*:

(W) *If* $v = v_1 \wedge \cdots \wedge v_r \leq u_1 \vee \cdots \vee u_s = u$*, then either* $v_i \leq u$ *for some* i*, or* $v \leq u_j$ *for some* j*.*

Proof: Suppose $v = v_1 \wedge \cdots \wedge v_r \leq u_1 \vee \cdots \vee u_s = u$ but that $v_i \nleq u$ and $v \nleq u_j$ for all i and all j. If $v \leq x \leq u$ for some $x \in X$, then since x is meet prime, $v_i \leq x \leq u$ for some i, contrary to our assumption. Let I be the interval u/v and let $\mathbf{FL}(X)[I]$ be the lattice obtained by doubling I. By the above remarks, none of the generators is doubled. This implies that X is a subset of $\mathbf{FL}(X)[I]$ and so the identity map on X extends to a homomorphism $f : \mathbf{FL}(X) \to \mathbf{FL}(X)[I]$. Since $x \notin I$, $\lambda(x) = x$, where λ is the epimorphism defined by (1). Hence $\lambda(f(w)) = w$ for all $w \in \mathbf{FL}(X)$ and this implies $f(w) = w$ if $w \notin I$. Thus it follows from (2) and its dual that

$$\begin{aligned} f(v) = f(v_1) \wedge \cdots \wedge f(v_r) &= v_1 \wedge \cdots \wedge v_r = \langle v, 1 \rangle \\ &\nleq \langle u, 0 \rangle = u_1 \vee \cdots \vee u_s \\ &= f(u_1) \vee \cdots \vee f(u_s) = f(u), \end{aligned}$$

contradicting the fact that $v \leq u$.

The condition (W) is known as *Whitman's condition.* Notice that it does not refer to the generating set and so it is inherited by sublattices. Also note that Day's doubling is a procedure for correcting (W)–failures. As such it has many uses, see [**26**].

COROLLARY 1.9. *Every sublattice of a free lattice satisfies* (W)*. Every element of a lattice which satisfies* (W) *is either join or meet irreducible.*

The next theorem gives a slight variant from [**52**] of Whitman's condition which is more efficient for computation and also useful in theoretical arguments.

THEOREM 1.10. *The free lattice* $\mathbf{FL}(X)$ *satisfies the following condition*:

(W+) *If* $v = v_1 \wedge \cdots \wedge v_r \wedge x_1 \wedge \cdots \wedge x_n \leq u_1 \vee \cdots \vee u_s \vee y_1 \vee \cdots \vee y_m = u$, *where* x_i *and* $y_j \in X$, *then either* $x_i = y_j$ *for some* i *and* j, *or* $v_i \leq u$ *for some* i, *or* $v \leq u_j$ *for some* j.

Proof: Suppose we apply (W) to the inequality $v \leq u$ and obtain $x_i \leq u = u_1 \vee \cdots \vee u_s \vee y_1 \vee \cdots \vee y_m$. Then either $x_i \leq y_j$ for some j, or $x_i \leq u_k$ for some k. The former implies $x_i = y_j$ and the latter implies $v \leq u_k$. Thus (W+) holds in either case. The other possibilities are handled by similar arguments.

Notice that (W+) replaces the test $x_i \leq u$ with the test $x_i \in \{y_1, \ldots, y_m\}$, which is of course much easier. Also notice that in applying (W+) it is permitted that some of the v_i's and u_j's are in X. Thus, for example, if $v = v_1 \wedge \cdots \wedge v_r \wedge x_1 \wedge \cdots \wedge x_n$ and $u = u_1 \vee \cdots \vee u_s$, then $v \leq u$ if and only if $v_i \leq u$ for some i or $v \leq u_j$ for some j.

Theorems 1.5 and 1.8 combine to give a recursive procedure for deciding, for terms s and t, if $s^{\mathbf{FL}(X)} \leq t^{\mathbf{FL}(X)}$ known as Whitman's solution to the word problem.[1] To test if $s \sim t$, the algorithm is used twice to check if both $s^{\mathbf{FL}(X)} \leq t^{\mathbf{FL}(X)}$ and $t^{\mathbf{FL}(X)} \leq s^{\mathbf{FL}(X)}$ hold. In Chapter XI we will give a presentation of this algorithm more suitable for a computer (rather than for a human) and study its time and space complexity.

THEOREM 1.11. *If* $s = s(x_1, \ldots, x_n)$ *and* $t = t(x_1, \ldots, x_n)$ *are terms and* $x_1, \ldots, x_n \in X$, *then the truth of*

$$s^{\mathbf{FL}(X)} \leq t^{\mathbf{FL}(X)} \tag{$*$}$$

can be determined by applying the following rules.

(1) *If* $s = x_i$ *and* $t = x_j$, *then* $(*)$ *holds if and only* $x_i = x_j$.
(2) *If* $s = s_1 \vee \cdots \vee s_k$ *is a formal join then* $(*)$ *holds if and only if* $s_i^{\mathbf{FL}(X)} \leq t^{\mathbf{FL}(X)}$ *holds for all* i.
(3) *If* $t = t_1 \wedge \cdots \wedge t_k$ *is a formal meet then* $(*)$ *holds if and only if* $s^{\mathbf{FL}(X)} \leq t_i^{\mathbf{FL}(X)}$ *holds for all* i.

[1] It is interesting that the word problem for free lattices was first shown to be decidable by Thoralf Skolem in 1920 [**121**], reprinted in [**122**]. In fact, Skolem showed that the universal first order theory of lattices is decidable; in particular, the word problem for finitely presented lattices is (uniformly) solvable. This result of Skolem seems to have gone unnoticed by lattice theorists. It was found by Stan Burris in his studies of the history of logic.

It is also interesting that Skolem's solution to the word problem for finitely presented lattices is polynomial time; see Chapter XI.

(4) *If* $s = x_i$ *and* $t = t_1 \vee \cdots \vee t_k$ *is a formal join, then* $(*)$ *holds if and only if* $x_i \leq t_j^{\mathbf{FL}(X)}$ *for some* j.

(5) *If* $s = s_1 \wedge \cdots \wedge s_k$ *is a formal meet and* $t = x_i$, *then* $(*)$ *holds if and only if* $s_j^{\mathbf{FL}(X)} \leq x_i$ *for some* j.

(6) *If* $s = s_1 \wedge \cdots \wedge s_k$ *is a formal meet and* $t = t_1 \vee \cdots \vee t_m$ *is a formal join, then* $(*)$ *holds if and only if* $s_i^{\mathbf{FL}(X)} \leq t^{\mathbf{FL}(X)}$ *holds for some* i, *or* $s^{\mathbf{FL}(X)} \leq t_j^{\mathbf{FL}(X)}$ *holds for some* j.

Proof: Conditions (1), (4) and (5) hold by Corollary 1.5, while (2) and (3) are trivial. Theorem 1.8 shows that free lattices satisfy (6). It is easy to see that all possibilities are covered by (1)–(6) and that each of these leads to a genuine reduction (except for (1), which gives the answer directly).

We are now in a position to give an easy criterion to determine if a subset of a lattice generates a sublattice isomorphic to a free lattice.

COROLLARY 1.12. *Let* $\mathbf{L}$ *be a lattice which satisfies* (W) *and let* X *generate* $\mathbf{L}$. *Then* $\mathbf{L}$ *is isomorphic to* $\mathbf{FL}(X)$ *if and only if the following condition and its dual hold for all* $x \in X$ *and all finite subsets* $Y \subseteq X$:

$$x \leq \bigvee Y \quad \textit{implies} \quad x \in Y.$$

Proof: If x and $y \in X$ satisfy $x \leq y$ then the condition with $Y = \{y\}$ implies that $x = y$ and, by Lemma 1.3 and its dual, each $x \in X$ is join and meet prime. Of course the identity map on X extends to a homomorphism of $\mathbf{FL}(X)$ onto $\mathbf{L}$. Since X in $\mathbf{L}$ satisfies (1)–(6) of Theorem 1.11, if s and t are terms then $s^{\mathbf{L}} \leq t^{\mathbf{L}}$ if and only if $s^{\mathbf{FL}(X)} \leq t^{\mathbf{FL}(X)}$ and thus this map must be an isomorphism.

COROLLARY 1.13. *A subset* S *of a free lattice* $\mathbf{FL}(X)$ *generates a sublattice isomorphic to a free lattice if and only if for all* $s \in S$ *and all finite subsets* $Y \subseteq S$,

$$s \leq \bigvee Y \quad \textit{implies} \quad s \in Y,$$

and the dual condition holds.

The next corollary shows that relatively free lattices cannot be embedded into free lattices except in the trivial cases. This contrasts greatly with the situation for other varieties. For example, every free distributive lattice can be embedded into a free modular lattice, see [**48**] and [**55**].

COROLLARY 1.14. *If* $\mathcal{V}$ *is a nontrivial variety of lattices, other than the variety of all lattices, and* X *is a set with at least three elements, then* $\mathbf{F}_{\mathcal{V}}(X)$ *is not a sublattice of any free lattice* $\mathbf{FL}(Y)$.

Proof: If X is a subset of $\mathbf{FL}(Y)$ which generates a sublattice isomorphic to $\mathbf{F}_{\mathcal{V}}(X)$, then by Lemmas 1.2 and 1.13, $\mathbf{F}_{\mathcal{V}}(X) \cong \mathbf{FL}(X)$. We will see in Theorem 1.28 below that $\mathbf{FL}(\omega)$ is a sublattice of $\mathbf{FL}(3)$. Thus $\mathbf{FL}(3)$ (and hence $\mathbf{FL}(X)$ if $|X| \geq 3$), generates the variety of all lattices. Of course this means that $\mathbf{F}_{\mathcal{V}}(X) \cong \mathbf{FL}(X)$ is impossible.

3. Canonical Form

The canonical form of an element plays an important role in free lattice theory. In this section we show that each element w of a free lattice has a term of least rank representing it, unique up to commutativity.[2] This term is called the *canonical form* of w. The phrase 'unique up to commutativity' can be made precise by defining *equivalent under commutativity* to be the equivalence relation, $s \equiv t$, given by recursively applying the following rules.

(1) $s, t \in X$ and $s = t$.
(2) $s = s_1 \vee \cdots \vee s_n$ and $t = t_1 \vee \cdots \vee t_n$ and there is a permutation σ of $\{1, \ldots, n\}$ such that $s_i \equiv t_{\sigma(i)}$ for all i.
(3) The dual of (2) holds.

Theorem 1.17 below shows that if two terms both represent the same element of $\mathbf{FL}(X)$ and both have minimal rank among all such representatives, then they are equivalent under commutativity.

We define $s \leq t$ for terms s and t to mean $s^{\mathbf{FL}(X)} \leq t^{\mathbf{FL}(X)}$. Note that this is only a quasiorder.

Later in this section we will derive the semidistributive laws from the existence of the canonical form. We also show that there is a strong connection between the canonical forms and the arithmetic of free lattices. This will allow us to define canonical form in terms of lattice theoretic properties.

The following concept is very important in lattice theory, particularly free lattice theory. Let $\mathbf{L}$ be a lattice and let A and B be finite subsets of L. We say that A *join refines* B and we write $A \ll B$ if for each $a \in A$ there is a $b \in B$ with $a \leq b$. The dual notion is called *meet refinement* and is denoted $A \gg B$. Note, however, that $A \ll B$ does not imply $B \gg A$.

The next lemma lists the basic properties of join refinement, all of which are straightforward to prove.

[2] Usually the canonical form is described as 'unique up to commutativity and associativity.' However, our definition of lattice terms and their ranks, given at the beginning of the previous section, imply that a term of minimal rank can be associated in only one way.

LEMMA 1.15. *The join refinement relation has the following properties.*

(1) *$A \ll B$ implies $\bigvee A \leq \bigvee B$.*
(2) *The relation $\ll$ is a quasiorder on the finite subsets of L.*
(3) *If $A \subseteq B$ then $A \ll B$.*
(4) *If A is an antichain, $A \ll B$, and $B \ll A$, then $A \subseteq B$.*
(5) *If A and B are antichains with $A \ll B$ and $B \ll A$, then $A = B$.*
(6) *If $A \ll B$ and $B \ll A$, then A and B have the same set of maximal elements.*

We use the term 'join refinement' because if $u = \bigvee A = \bigvee B$ and $A \ll B$, then $u = \bigvee A$ is a better join representation of u than $u = \bigvee B$ in that its elements are further down in the lattice. We will see that in free lattices there is a unique best join representation of each element, i.e., a representation that join refines all other join representations.

LEMMA 1.16. *Let $t = t_1 \sqcup \cdots \sqcup t_n$, with $n > 1$, be a term such that*

(1) *each t_i is either in X or formally a meet,*
(2) *if $t_i = \bigwedge t_{ij}$ then $t_{ij} \not\leq t$ for all j.*

If $s = s_1 \sqcup \cdots \sqcup s_m$ and $s \sim t$, then $\{t_1, \dots t_n\} \ll \{s_1, \dots, s_m\}$.

Proof: For each i we have $t_i \leq s_1 \vee \cdots \vee s_m$. Applying (W) if t_i is formally a meet and using join primality if $t_i \in X$, we conclude that either $t_i \leq s_j$ for some j, or $t_i = \bigwedge t_{ij}$ and $t_{ij} \leq s$ for some j. However, since $s \sim t$, the second case would imply $t_{ij} \leq t$, contrary to assumption (2). Hence in all cases there is a j such that $t_i \leq s_j$. Thus $\{t_1, \dots, t_n\} \ll \{s_1, \dots, s_m\}$.

THEOREM 1.17. *For each $w \in \mathbf{FL}(X)$ there is a term of minimal rank representing w, unique up to commutativity. This term is called the canonical form of w.*

Proof: Suppose that s and t are both terms of minimal rank that represent the same element w in $\mathbf{FL}(X)$. If either s or t is in X, then clearly $s = t$.

Suppose that $t = t_1 \sqcup \cdots \sqcup t_n$ and $s = s_1 \sqcup \cdots \sqcup s_m$. If some t_i is formally a join, we could lower the rank of t by removing the parentheses around t_i. Thus each t_i is not formally a join. If there is a t_i such that $t_i = \bigwedge t_{ij}$ and $t_{ij} \leq t$ for some j, then $t_i \leq t_{ij} \leq t$. In this case we could replace t_i with t_{ij} in t, producing a shorter term still representing w, which violates the minimality of the term t. Thus t satisfies the hypotheses of Lemma 1.16, whence $\{t_1, \dots, t_n\} \ll \{s_1, \dots, s_m\}$.

By symmetry, $\{s_1, \dots, s_m\} \ll \{t_1, \dots, t_n\}$. Since both are antichains (by the minimality) they represent the same set of elements of $\mathbf{FL}(X)$. Thus $m = n$ and after renumbering $s_i \sim t_i$. Now by induction s_i and t_i are the same up to commutativity.

If $t = t_1 \sqcup \cdots \sqcup t_n$ and $s = s_1 \wedge \cdots \wedge s_m$, then (W) implies that either $t_i \sim t$ for some i or $s_j \sim s$ for some j, violating the minimality.

The remaining cases can be handled by duality.

Naturally we say a term is in *canonical form* if it is the canonical form of the element it represents. The following theorem gives an effective procedure for transforming a term into canonical form, i.e., if one of the conditions below fails, then t can be transformed into a term of smaller rank representing the same element. So repeated use of these conditions will transform a term into canonical form.

THEOREM 1.18. *A term $t = t_1 \sqcup \cdots \sqcup t_n$, with $n > 1$, is in canonical form if and only if*

(1) *each t_i is either in X or formally a meet,*
(2) *each t_i is in canonical form,*
(3) *$t_i \nleq t_j$ for all $i \neq j$ (the t_i's form an antichain),*
(4) *if $t_i = \bigwedge t_{ij}$ then $t_{ij} \nleq t$ for all j.*

A term $t = t_1 \wedge \cdots \wedge t_n$, with $n > 1$, is in canonical form if and only if the duals of the above conditions hold. A term $x \in X$ is always in canonical form.

Proof: All of these conditions are clearly necessary. For the converse we need to show that if t satisfies (1)–(4) then it has minimal rank among the terms which represent the same element of $\mathbf{FL}(X)$ as t. Suppose that $s = s_1 \sqcup \cdots \sqcup s_m$ is a term of minimal rank representing the same element of $\mathbf{FL}(X)$ as t. Now using (1) and (4) for t, and the arguments of the last theorem for s, Lemma 1.16 yields

$$\{t_1, \dots, t_n\} \ll \{s_1, \dots, s_m\}$$
$$\{s_1, \dots, s_m\} \ll \{t_1, \dots, t_n\}.$$

Since both are antichains, we have that $n = m$ and after renumbering $s_i \sim t_i$, $i = 1, \dots, n$. The proof can now easily be completed with the aid of induction.

Let $w \in \mathbf{FL}(X)$ be join reducible and suppose $t = t_1 \vee \cdots \vee t_n$ (with $n > 1$) is the canonical form of w. Let $w_i = t_i^{\mathbf{FL}(X)}$. Then $\{w_1, \dots, w_n\}$ are called the *canonical joinands* of w. We also say $w = w_1 \vee \cdots \vee w_n$ *canonically* and that $w_1 \vee \cdots \vee w_n$ is the *canonical join representation* (or *canonical join expression*) of w. If w is join irreducible, we define the canonical joinands of w to be the set $\{w\}$. Of course the *canonical* *meet representation* and *canonical meetands* of an element in a free lattice are defined dually. More generally, u is called a *subelement* of w if it is the element of $\mathbf{FL}(X)$ corresponding to some subterm of the canonical representation of w. Although the other terms defined in this paragraph are standard, the term *subelement* is new. When speaking loosely, one could use 'subterm' in place of 'subelement,' but this is not completely correct. Notice that according to this definition (and the definition of subterm), $x \vee y$ is not a subelement of $x \vee y \vee z$.

The next theorem shows a strong connection between the syntactical canonical form and the arithmetic of the free lattice. In some sense, it shows that the canonical join representation of an element of a free lattice is the best way to write it as a join. A join representation $a = a_1 \vee \cdots \vee a_n$ in an arbitrary lattice is said to be a *minimal join representation* if $a = b_1 \vee \cdots \vee b_m$ and $\{b_1, \ldots, b_m\} \ll \{a_1, \ldots, a_n\}$ imply $\{a_1, \ldots, a_n\} \subseteq \{b_1, \ldots, b_m\}$. Equivalently, a join representation $a = a_1 \vee \cdots \vee a_n$ is minimal if it is an antichain and nonrefinable, in the sense that whenever $a = b_1 \vee \cdots \vee b_m$ and $\{b_1, \ldots, b_m\} \ll \{a_1, \ldots, a_n\}$, then $\{a_1, \ldots, a_n\} \ll \{b_1, \ldots, b_m\}$.

THEOREM 1.19. *Let $w = w_1 \vee \cdots \vee w_n$ canonically in $\mathbf{FL}(X)$. If also $w = u_1 \vee \cdots \vee u_m$, then*

$$\{w_1, \ldots, w_n\} \ll \{u_1, \ldots, u_m\}.$$

Thus $w = w_1 \vee \cdots \vee w_n$ is the unique minimal join representation of w.

Proof: Interpreting the terms of Lemma 1.16 in $\mathbf{FL}(X)$ immediately gives this result, since $w_1, \ldots, w_n$ is an antichain.

THEOREM 1.20. *Let $w \in \mathbf{FL}(X)$ and let u be a join irreducible element in $\mathbf{FL}(X)$. Then u is a canonical joinand of w if and only if there is an element a such that $w = u \vee a$ and $w > v \vee a$ for every $v < u$.*

Proof: The previous theorem shows that each canonical joinand w_i satisfies the condition with $a = \bigvee_{j \neq i} w_j$. Conversely, suppose u satisfies the condition with the element a, and let $w = w_1 \vee \cdots \vee w_n$ be the canonical expression for w. Let

$$v = \bigvee_{w_i \leq u} w_i.$$

Since $w = u \vee a$, the previous theorem gives $\{w_1, \ldots, w_n\} \ll \{u, a\}$ and it follows from this that $w = a \vee v$. By hypothesis, this implies $v = u$. Since u is join irreducible, this in turn implies $u = w_i$ for some i, which is the desired conclusion.

A lattice is called *join semidistributive* if it satisfies the following condition.

$$(\mathrm{SD}_\vee) \qquad a \vee b = a \vee c \quad \text{implies} \quad a \vee b = a \vee (b \wedge c).$$

Meet semidistributivity is defined dually and denoted $(\mathrm{SD}_\wedge)$. A lattice is *semidistributive* if it satisfies both of these conditions. The following theorem is due to Jónsson and Kiefer [**82**].

THEOREM 1.21. *Free lattices are semidistributive. Moreover, the condition* $(\mathrm{SD}_\vee)$ *is equivalent to*

$$w = \bigvee_i a_i = \bigvee_j b_j \quad \textit{implies} \quad w = \bigvee_{i,j} (a_i \wedge b_j). \tag{3}$$

In any meet semidistributive lattice, the dual statement holds.

Proof: We will show that (3) holds in every free lattice. It is easy to see that (3), applied to $w = a \vee b = a \vee c$, yields $w = a \vee (b \wedge c)$. Thus we will have shown that free lattices satisfy $(\mathrm{SD}_\vee)$ and, by duality, are semidistributive.

Suppose $w = \bigvee a_i = \bigvee b_j$ in a free lattice, and let $w = w_1 \vee \cdots \vee w_n$ canonically. By Theorem 1.19, for each k there are an i and a j such that $w_k \leq a_i$ and $w_k \leq b_j$. Thus $w_k \leq a_i \wedge b_j$ for this i and j. Therefore $w \leq \bigvee_{i,j}(a_i \wedge b_j)$. The opposite inequality is obvious.

The proof that (3) holds in any join semidistributive lattice is more difficult. To prove it, let $\mathbf{L}$ be an arbitrary lattice satisfying $(\mathrm{SD}_\vee)$ and let $w \in L$ be a given element. We shall prove, by induction on $n+m$, that the following statement, denoted $\mathrm{P}(n, m)$, holds whenever n, and m are positive integers: *for any $v \in L$ and any elements $a_1, \dots, a_n$, $b_1, \dots, b_m$ of L,*

$$w = v \vee \bigvee_{i=1}^{n} a_i = v \vee \bigvee_{j=1}^{m} b_j \quad \textit{implies} \quad w = v \vee \bigvee_{i=1}^{n} \bigvee_{j=1}^{m} (a_i \wedge b_j).$$

Since $\mathrm{P}(1,1)$ is precisely $(\mathrm{SD}_\vee)$, we shall assume that $n + m > 2$. First suppose $n = 1$, and let $b = \bigvee_{j=1}^{m-1} b_j$. Clearly

$$w = (v \vee b) \vee a_1 = (v \vee b) \vee b_m,$$

so that $w = v \vee b \vee (a_1 \wedge b_m)$ by $(\mathrm{SD}_\vee)$. Therefore

$$w = (v \vee (a_1 \wedge b_m)) \vee a_1 = (v \vee (a_1 \wedge b_m)) \vee \bigvee_{j=1}^{m-1} b_j,$$

and it follows by P$(1, m-1)$ that

$$w = v \vee (a_1 \wedge b_m) \vee \bigvee_{j=1}^{m-1} (a_1 \wedge b_j) = v \vee \bigvee_{j=1}^{m} (a_1 \wedge b_j).$$

Now suppose $n > 1$, and let $a = \bigvee_{i=1}^{n-1} a_i$. Clearly

$$w = v \vee a \vee a_n = v \vee a \vee \bigvee_{j=1}^{m} b_j,$$

and it follows by P$(1, m)$ that

$$w = v \vee a \vee \bigvee_{j=1}^{m} (a_n \wedge b_j).$$

Therefore

$$w = (v \vee \bigvee_{j=1}^{m} (a_n \wedge b_j)) \vee \bigvee_{i=1}^{n-1} a_i = (v \vee \bigvee_{j=1}^{m} (a_n \wedge b_j)) \vee \bigvee_{j=1}^{m} b_j,$$

and it follows by P$(n-1, m)$ that

$$w = v \vee \bigvee_{j=1}^{m} (a_n \wedge b_j) \vee \bigvee_{i=1}^{n-1} \bigvee_{j=1}^{m} (a_i \wedge b_j) = v \vee \bigvee_{i=1}^{n} \bigvee_{j=1}^{m} (a_i \wedge b_j).$$

4. Continuity

A subset A of a lattice $\mathbf{L}$ is said to be *up directed* if every finite subset of A has an upper bound in A. It suffices to check this for pairs: A is up directed if for all a, $b \in A$ there exists $c \in A$ with $a \leq c$ and $b \leq c$. Clearly, any join closed set is up directed. A *down directed set* is defined dually.

A lattice is said to be *upper continuous* if whenever A is an up directed set having a least upper bound $u = \bigvee A$, then, for any b,

$$\bigvee_{a \in A} (a \wedge b) = u \wedge b. \tag{4}$$

Lower continuous is defined dually. A lattice is *continuous* if it is both lower and upper continuous. Often it is assumed that continuous lattices are complete, but we do not make that assumption here as free lattices are not complete; see Example 1.24.[3]

[3] In the standard definition of upper and lower continuity for a complete lattice, the set A is assumed to be a chain (rather than a directed set). It is then shown that continuity for chains implies continuity for directed sets. However, this need not be true for lattices which are not complete. Since the stronger condition for directed sets is true in free lattices, and is in fact needed for some arguments, it makes sense

Now let us prove Whitman's result that free lattices are continuous.

THEOREM 1.22. *Free lattices are continuous.*[4]

Proof: We will show that free lattices are lower continuous; the result then follows by duality. Suppose that $u = \bigwedge A$ for a down directed set, and let b be arbitrary. Clearly $u \vee b$ is a lower bound for $\{a \vee b : a \in A\}$. Suppose that it is not the greatest lower bound and let c be an element of minimal rank such that

$$c \not\leq u \vee b, \qquad c \leq a \vee b \quad \text{for all } a \in A. \tag{5}$$

If $c \leq a$ for all $a \in A$, then $c \leq u$ which contradicts (5). Thus $A' = \{a \in A : c \not\leq a\}$ is nonempty, and since A is down directed, so is A' and $\bigwedge A' = u$. Without loss of generality we may assume that $A' = A$, so that $c \not\leq a$ for all $a \in A$. If $c = \bigvee c_j$ is a proper join, then for some j, $c_j \not\leq u \vee b$ and thus c_j violates the minimality of the rank of c. If $c = \bigwedge c_j$ then, for each $a \in A$, we apply (W) to $c = \bigwedge c_j \leq a \vee b$. We obtain a violation of (5) if $c \leq b$, and we have already ruled out the possibility that $c \leq a$. Hence for each a, there is a j such that $c_j \leq a \vee b$. It follows that there exists a j_0 such that $c_{j_0} \leq a \vee b$ for all $a \in A$. For otherwise, for each j we could find a_j with $c_j \not\leq a_j \vee b$. But then, taking a' to be a common lower bound of the a_j's, we would have $c_j \not\leq a' \vee b$ for every j, a contradiction. However, because $c \not\leq u \vee b$ implies $c_{j_0} \not\leq u \vee b$, we now have that (5) holds with c_{j_0} in place of c, contradicting the minimality of the rank of c. Thus c is not a proper meet. Finally, if c is a generator then $c \leq a \vee b$ implies $c \leq a$ or $c \leq b$, both of which cannot occur.

We will give another proof that free lattices are continuous as Corollary 2.27.

5. Fixed Point Free Polynomials and Incompleteness

If $t(x_0, \dots, x_n)$ is a term and $c_1, \dots, c_n$ are elements of a lattice $\mathbf{L}$, the function $f(x) = t^{\mathbf{L}}(x, c_1, \dots, c_n)$ is a (unary) *polynomial* on $\mathbf{L}$. In this section we exhibit a *fixed point free polynomial* on $\mathbf{FL}(3)$, i.e., a polynomial $f(x)$ such that $f(u) = u$ for no $u \in \mathbf{FL}(X)$. Using this

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to take the directed set version as the definition of continuity for lattices which are not necessarily complete.

[4]It is an interesting historical fact that Whitman worried that this theorem might be vacuous, i.e., he did not know if a free lattice could contain an ascending chain with a least upper bound. That such a chain exists was first established by R. A. Dean, who showed, in unpublished work, that $x \wedge (y \vee z)$ has no lower cover and hence is the join of an ascending chain. Whitman did construct an ascending chain without a join; see the next section.

polynomial we give an example, due to Whitman, of an ascending chain in $\mathbf{FL}(3)$ without a least upper bound, showing that free lattices are not complete. Constructing a fixed point free polynomial is not easy on any lattice. An example of a modular lattice with such a polynomial is given in [**50**]. The existence of a lattice with a fixed point free polynomial is important to the study of order polynomial complete lattices.

A lattice $\mathbf{L}$ is called *order polynomial complete* if every order preserving map on $\mathbf{L}$ can be represented as a unary polynomial, see [**119**]. Wille was able to characterize finite order polynomial complete lattices in [**134**], but the problem of characterizing infinite order polynomial complete lattices remains open, and a solution does not appear to be near. It is possible that every such lattice is complete. Anne Davis Morel [**23**] has shown that if $\mathbf{L}$ is not a complete lattice, then it has an order preserving, fixed point free map. (Tarski had proved the converse [**130**].) Thus if it were true that every unary polynomial on a lattice had a fixed point then this would imply that every order polynomial complete lattice is complete. The result of this section defeats this approach. The question if every order polynomial complete lattice is complete remains open.

Another interesting question is *which lattices have the property that every unary polynomial has a fixed point*. This problem is discussed in [**50**], where it is pointed out that every locally complete lattice has this property. A lattice is *locally complete* if every finitely generated sublattice is complete.

Let

$$p(u,x,y,z) = (((((u \wedge y) \vee z) \wedge x) \vee y) \wedge z) \vee x$$

and let $q(u,x,y,z)$ be defined dually. Define unary polynomials

$$\begin{aligned} f(u) = f_1(u) &= p(u,x,y,z) \qquad & f_2(u) &= q(u,y,z,x) \qquad & f_3(u) &= p(u,z,x,y) \\ f_4(u) &= q(u,x,y,z) \qquad & f_5(u) &= p(u,y,z,x) \qquad & f_6(u) &= q(u,z,x,y) \end{aligned}$$

THEOREM 1.23. *f is fixed point free on $\mathbf{FL}(3)$, i.e., $f(w) = w$ holds for no $w \in \mathbf{FL}(3)$.*

Proof: Suppose that s is an element of minimal rank with $f(s) = s$. If $f_i(t) = t$ for some i, then clearly t can be transformed by some permutation of x, y, and z and possibly duality into a fixed point of f. Thus, since such a transformation leaves the rank invariant, the rank of t must be at least that of s.

Let $r = ((((s \wedge y) \vee z) \wedge x) \vee y) \wedge z$, so that $s = f(s) = r \vee x$. Also

$$
\begin{aligned}
(6) \quad f_6(r) &= (((((((((s \wedge y) \vee z) \wedge x) \vee y) \wedge z) \vee x) \wedge y) \vee z) \wedge x) \vee y) \wedge z \\
&= ((((f(s) \wedge y) \vee z) \wedge x) \vee y) \wedge z \\
&= ((((s \wedge y) \vee z) \wedge x) \vee y) \wedge z \\
&= r.
\end{aligned}
$$

We will show that r is a canonical joinand of s and thus the above equation violates the minimality of s.

Since $s \leq 1 = x \vee y \vee z$, $s = f(s) \leq f(1) = ((((y \vee z) \wedge x) \vee y) \wedge z) \vee x$ and thus

$$(7) \qquad s \not\geq y, \qquad s \not\geq z.$$

All the elements in the range of f lie above x, so $x \leq s$. Now $r \not\leq x$, since otherwise $s = r \vee x = x$, but one easily checks that $f(x) \neq x$. Thus r and x are incomparable since the other inequality would imply $z \geq x$. So $s = r \vee x$ is a proper join. Let $s = v_1 \vee \cdots \vee v_n$ be the canonical form of s. By Theorem 1.19, $\{v_1, \ldots, v_n\} \ll \{r, x\}$, i.e., for each i, either $v_i \leq x$ or $v_i \leq r$. Also, since $x \leq s = \bigvee v_i$ and generators are join prime, $x \leq v_i$ for some i. We take $i = 1$. If $v_1 \leq r$ then $x \leq r$, a contradiction. Thus $v_1 \leq x$ and hence $x = v_1$. This in turn implies that $v_i \leq r$ for $i > 1$.

Now apply (W) to

$$[(((s \wedge y) \vee z) \wedge x) \vee y] \wedge z = r \leq s = \bigvee v_i.$$

If $z \leq s$, we contradict (7). If $(((s \wedge y) \vee z) \wedge x) \vee y \leq s$, then $y \leq s$, again violating (7). Hence $r \leq v_i$ for some i. Since $r \not\leq x = v_1$, we must have $i > 1$. But then $v_i \leq r$, and hence $r = v_i$, showing that r is a canonical joinand of s. As pointed out earlier this, together with (6), violates the minimality of s.

Example 1.24. Let f be the unary polynomial on $\mathbf{FL}(3)$ defined above. Then $x \leq f(x)$ and thus

$$x \leq f(x) \leq f^2(x) \leq \cdots$$

is an ascending chain which we denote C. We claim that f does not have a least upper bound. It follows from continuity and an easy inductive argument that *if g is any unary polynomial on a free lattice and $a_0 \leq a_1 \leq a_2 \leq \cdots$ is an ascending chain with a least upper bound $\bigvee a_i$, then $g(\bigvee a_i) = \bigvee g(a_i)$*. Applying this to C we have

$$f(\bigvee f^i(x)) = \bigvee f^{i+1}(x) = \bigvee f^i(x),$$

which implies that f has a fixed point. This contradiction shows that C does not have a least upper bound. We leave it as an exercise for the reader to show that if $\{x, y, z\} \subseteq X$ then C does not have a least upper bound in $\mathbf{FL}(X)$. Hence $\mathbf{FL}(X)$ *is not a complete lattice unless* $|X| \leq 2$.

PROBLEM 1.25. *Which unary polynomials on free lattices are fixed point free? For which unary polynomials f does $\bigvee f^i(a)$ exist for all a?*

We can answer this if f is formed by alternately joining and meeting generators, i.e.,

$$f(u) = \begin{cases} ((u \vee x_1) \wedge x_2) \vee \cdots \wedge x_n & \text{if } n \text{ is even,} \\ ((u \vee x_1) \wedge x_2) \vee \cdots \vee x_n & \text{if } n \text{ is odd} \end{cases} \tag{8}$$

where $x_1, \ldots, x_n$ is a finite sequence of generators. In Example 3.40 the terms corresponding to these polynomials are called *slim*. That example shows that such a term is in canonical form if and only if $x_i \neq x_{i+1}$ for all $i \in \{1, \ldots, n-1\}$ and $(x_i, x_{i+1}) \neq (x_{i+2}, x_{i+3})$ for all $i \in \{1, \ldots, n-3\}$.

THEOREM 1.26. *Let f be a slim polynomial as given in* (8). *Then f has a fixed point if and only if one of the following cases takes place.*

(1) *If $n = 0$, then any element is a fixed point.*
(2) *If $n = 1$, then x_1 is a fixed point.*
(3) *If $n = 2$, then $x_1 \wedge x_2$ is a fixed point.*
(4) *If $n = 3$, then $x_2 \vee x_3$ is a fixed point.*
(5) *If n is even and $x_1 = x_3 = x_5 = \cdots = x_{n-1}$, then $x_1 \wedge x_n$ is a fixed point.*
(6) *If n is even and $x_2 = x_4 = x_6 = \cdots = x_n$, then x_2 is a fixed point.*
(7) *If n is even and $x_1 = x_n$, then $((x_1 \wedge x_2) \vee x_3) \cdots \wedge x_n$ is a fixed point.*
(8) *If n is odd and $x_2 = x_n$, then $((x_2 \vee x_3) \wedge x_4) \cdots \vee x_n$ is a fixed point.*
(9) *If n is odd and $x_2 = \cdots = x_{n-1}$, then $x_2 \vee x_n$ is a fixed point.*
(10) *If n is odd and $x_i \in \{x_1, x_n\}$ for any odd i, then $((x_n \vee x_1) \wedge x_2) \cdots \vee x_n$ is a fixed point.*
(11) *If n is odd and there is an even number j with $x_j, x_{j+1}, x_{j+3}, x_{j+5}, \ldots, x_n \in \{x_1, x_n\}$, then $((x_n \vee x_1) \wedge x_2) \cdots \vee x_n$ is a fixed point.*

The proof will be omitted.

6. Sublattices of Free Lattices

In this section we prove Whitman's theorem that $\mathbf{FL}(\omega)$ is a sublattice of $\mathbf{FL}(3)$, and Jónsson and Kiefer's theorem that there is a nontrivial equation satisfied by all finite sublattices of a free lattice. We begin with the following result of Galvin and Jónsson [**65**], which applies to all varieties of lattices. In fact it even applies to a wider class of algebras which includes Boolean algebras.

Theorem 1.27. *Every relatively free lattice $\mathbf{F}_{\mathcal{V}}(X)$, and hence every sublattice of a relatively free lattice, is a countable union of antichains. Thus relatively free lattices, and sublattices of relatively free lattices, contain no uncountable chains.*

Proof: The result is obvious if X is finite, so assume X is infinite and that X_0 is a countable subset of X. Let $\mathbf{G}$ be the group of automorphisms of $\mathbf{F}_{\mathcal{V}}(X)$ which are induced from the permutations of X fixing all but finitely many $x \in X$. For u and $v \in \mathbf{F}_{\mathcal{V}}(X)$, let $u \sim_{\mathbf{G}} v$ denote the fact that u and v lie in the same orbit, i.e., there is a $\sigma \in G$ such that $\sigma(u) = v$. Notice that $u \sim_{\mathbf{G}} v$ means that u and v can be represented by the same term except that the variables are changed. For every element $u \in \mathbf{F}_{\mathcal{V}}(X)$, there is a v in the sublattice generated by X_0 with $u \sim_{\mathbf{G}} v$. Thus $\mathbf{F}_{\mathcal{V}}(X)$ has only countably many orbits under $\mathbf{G}$. (An orbit is just an equivalence class of $\sim_{\mathbf{G}}$.)

Let $\sigma \in G$ and suppose $u < \sigma(u)$. Then by applying σ repeatedly to this inequality, we obtain

$$u < \sigma(u) < \sigma^2(u) < \cdots . \tag{9}$$

Each element of $\mathbf{G}$ has finite order and thus $\sigma^n(u) = u$ for some positive n. But then (9) implies $u < u$, a contradiction. Thus each orbit is an antichain and there are only countably many orbits, which proves the theorem.

Theorem 1.28. $\mathbf{FL}(3)$ *contains a sublattice isomorphic to* $\mathbf{FL}(\omega)$.

Proof: Let $f = f_1, \dots, f_6$ be the unary polynomials defined before Theorem 1.23. Notice that f_4 is the dual of f. Let $c_n = f^n(x)$ be the n^{th} element of the ascending chain C of Example 1.24, and let $d_n = f_4^n(x)$ be the element dual to c_n. So the d_n's form a descending chain $d_0 = x > d_1 > d_2 > \cdots$. For $n \geq 1$, define

$$w_n = z \vee (c_n \wedge (d_n \vee y)).$$

We will show, *via* Corollary 1.13, that $w_1, w_2, \dots$ generate a free lattice. Our development follows [**20**]. First an easy lemma.

Lemma 1.29. *The following are true for all $n \geq 1$:*

(1) $y \not\leq c_n$,
(2) $y \not\leq w_n$,
(3) $c_n \not\leq z$,
(4) $x \not\leq d_n$,
(5) $x \not\leq w_n$.

Proof: For (1), $c_n = f^n(x)$ and it was shown in the proof of Theorem 1.23 that nothing in the range of f lies above y. For (2), it is easy to see that $y \leq w_n$ implies $y \leq c_n$, which we have already eliminated. Since $x \leq c_n$, (3) holds and (4) follows from $d_n < x$. (5) follows from (W+), (4) and two applications of the fact that x is join prime.

Suppose that $w_m \leq w_n$. Then $c_m \wedge (d_m \vee y) \leq z \vee (c_n \wedge (d_n \vee y)) = w_n$ and we apply (W+). Using the lemma all cases can be eliminated easily except $c_m \wedge (d_m \vee y) \leq c_n \wedge (d_n \vee y)$. If this holds then both of the following hold.

$$c_m \wedge (d_m \vee y) \leq c_n = [((((c_{n-1} \wedge y) \vee z) \wedge x) \vee y) \wedge z] \vee x$$
$$c_m \wedge (d_m \vee y) \leq d_n \vee y$$

Applying (W+) to the first inequality and using Theorem 1.5 and Lemma 1.29 we conclude $c_m \leq c_n$ and hence $m \leq n$. Similarly, the second inequality leads to $d_m \vee y \leq d_n \vee y$.

One easily checks from the definition that $f_4^k(x) \vee y = f_5^k(x \vee y)$. Since f_5 has no fixed points and $x \vee y \geq f_5(x \vee y)$, we see that $f_5^k(x \vee y)$ forms a descending chain, i.e., $f_5^m(x \vee y) \leq f_5^n(x \vee y)$ if and only if $m \geq n$. Since $f_5^m(x \vee y) = d_m \vee y \leq d_n \vee y = f_5^n(x \vee y)$, $m \geq n$. Hence $m = n$ and thus the w_n's form an antichain.

Now suppose that $w_m \leq w_{n_1} \vee \cdots \vee w_{n_k}$ for some distinct, positive $m, n_1, \ldots, n_k$. Then $c_m \wedge (d_m \vee y) \leq w_{n_1} \vee \cdots \vee w_{n_k}$ and we apply (W). By Lemma 1.29, neither x nor y lie below $w_{n_1} \vee \cdots \vee w_{n_k}$, and it follows that the only possibility is $c_m \wedge (d_m \vee y) \leq w_{n_i}$, for some i. But, by joining both sides with z, this gives $w_m \leq w_{n_i}$, a contradiction.

Now suppose $w_{n_1} \wedge \cdots \wedge w_{n_k} \leq w_m$. No $w_n \leq z$ and hence $w_{n_1} \wedge \cdots \wedge w_{n_k} \not\leq z$. This fact, and the incomparability of the w_n's, imply that

$$z \leq w_{n_1} \wedge \cdots \wedge w_{n_k} \leq c_m \wedge (d_m \vee y) \leq d_m \vee y,$$

which cannot occur. Thus, by Theorem 1.13, the sublattice generated by w_n, $n = 1, 2, \ldots$, is isomorphic to $\mathbf{FL}(\omega)$.

Next we present Jónsson and Kiefer's theorem.

THEOREM 1.30. *Let* $\mathbf{L}$ *be a lattice satisfying* (W). *Suppose elements* a_1, a_2, a_3, *and* $v \in L$ *satisfy*

(1) $a_i \not\leq a_j \vee a_k \vee v$ *whenever* $\{i, j, k\} = \{1, 2, 3\}$,

(2) *$v \nleq a_i$ for $i = 1, 2, 3$,*
(3) *v is meet irreducible.*

Then $\mathbf{L}$ *contains a sublattice isomorphic to* $\mathbf{FL}(3)$.

Proof: For $\{i, j, k\} = \{1, 2, 3\}$, let $b_i = a_i \vee [(a_j \vee v) \wedge (a_k \vee v)]$. If $b_i \le b_j \vee b_k$ then $a_i \le b_i \le b_j \vee b_k \le a_j \vee a_k \vee v$, contradicting (1). Thus the b_i's are join irredundant. In particular they are pairwise incomparable. Suppose $b_1 \wedge b_2 \le b_3 = a_3 \vee [(a_1 \vee v) \wedge (a_2 \vee v)]$, and apply (W). Neither b_1 nor b_2 is below b_3 and if $b_1 \wedge b_2 \le a_3$, then $v \le a_3$, contradicting (2). Hence $b_1 \wedge b_2 \le (a_1 \vee v) \wedge (a_2 \vee v) \le a_1 \vee v$, and we apply (W) again. Since $v \le b_1 \wedge b_2$, the inequality $b_1 \wedge b_2 \le a_1$ would imply $v \le a_1$ and so cannot occur. If $b_1 \wedge b_2 \le v$, then $v = b_1 \wedge b_2$ and so would be a proper meet, which contradicts (3). If $b_2 \le a_1 \vee v$, then $a_2 \le a_1 \vee v$, which violates (1). Thus we must have $b_1 \le a_1 \vee v$ which implies

$$(a_2 \vee v) \wedge (a_3 \vee v) \le a_1 \vee v.$$

By (1) neither meetand is contained in $a_1 \vee v$ and by (2) $(a_2 \vee v) \wedge (a_3 \vee v) \nleq a_1$. The last possibility gives $v = (a_2 \vee v) \wedge (a_3 \vee v)$, contradicting (3). Hence $b_1 \wedge b_2 \nleq b_3$, and symmetrically $b_i \wedge b_j \nleq b_k$. Thus, by Corollary 1.13, the sublattice generated by b_1, b_2, and b_3 is isomorphic to $\mathbf{FL}(3)$.

A lattice $\mathbf{L}$ is said to have *breadth at most* n if whenever $a \in L$ and S is a finite subset of L such that $a = \bigvee S$, there is a subset T of S, with $a = \bigvee T$ and $|T| \le n$. The breadth of a lattice is the least n such that it has breadth at most n. The reader can verify that this concept is self dual.

Corollary 1.31. *If* $\mathbf{L}$ *is a finite lattice satisfying* (W)*, then the breadth of* $\mathbf{L}$ *is at most* 4*. The variety generated by finite lattices which satisfy* (W) *is not the variety of all lattices. In particular, finite sublattices of a free lattice have breadth at most four and satisfy a nontrivial lattice equation.*

Proof: Suppose $\mathbf{L}$ is a finite lattice satisfying (W) and $a = a_1 \vee \cdots \vee a_n$ holds for some $a \in L$ and some $n > 4$ and that this join is irredundant. Then, since every element of a lattice which satisfies (W) must be either join or meet irreducible, $v = a_4 \vee \cdots \vee a_n$ is meet irreducible. By Theorem 1.30, $\mathbf{L}$ has a sublattice isomorphic to $\mathbf{FL}(3)$ and hence is infinite, a contradiction. Thus $\mathbf{L}$ has breadth at most 4.

If $\mathcal{B}_4$ is the class of all lattices of breadth at most 4, then, by Jónsson's Theorem [**80**], every subdirectly irreducible lattice in $\boldsymbol{V}(\mathcal{B}_4)$ lies in $\boldsymbol{HSP}_u(\mathcal{B}_4)$. (Here $\boldsymbol{H}$, $\boldsymbol{S}$, $\boldsymbol{P}_u$, and $\boldsymbol{V}$ are the usual closure operations. For example, $\boldsymbol{P}_u$ is the closure under ultraproducts; see [**18**] and [**100**].) But it is easy to see that $\mathcal{B}_4$ is closed under these operators. Thus if $\boldsymbol{V}(\mathcal{B}_4)$ were all lattices, every subdirectly irreducible lattice would have breadth at most 4, which is not the case.

An example of an equation that holds in all finite sublattices of free lattices, but not in every lattice, is 4–distributivity:

$$x \wedge \bigvee_{i=1}^{5} x_i \leq \bigvee_{i=1}^{5}(x \wedge \bigvee_{j \neq i} x_j).$$

It is not hard to see that this equation holds in all breadth 4 lattices, but fails in the lattice of subspaces of a 5–dimensional vector space.

By Corollary 1.7, $\mathbf{FL}(x_1, \ldots, x_n)$ has n atoms: $\underline{x}_i = \bigwedge_{j \neq i} x_j$. It is easy to verify that no $n-1$ of these atoms join above the remaining one. Thus we get the following corollary of Jónsson and Kiefer.

Corollary 1.32. *If $n \geq 5$, the sublattice of* $\mathbf{FL}(n)$ *generated by the atoms is infinite.*

The sublattice generated by the atoms of $\mathbf{FL}(3)$ is the 8 element Boolean algebra. The sublattice generated by the atoms of $\mathbf{FL}(4)$ has 22 elements and is diagrammed in Figure 1.1. To verify this, one needs to label all the elements and show that all the joins and meets are correct. This is left as an exercise for the reader.

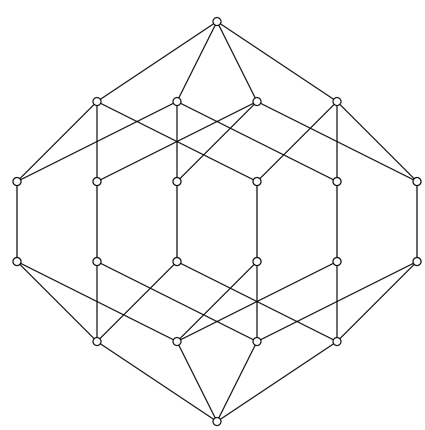


Figure 1.1

CHAPTER II

Bounded Homomorphisms and Related Concepts

If $h : \mathbf{K} \to \mathbf{L}$ is a lattice homomorphism, then for any element $a \in L$ we can consider the sets $h^{-1}(1/a) = \{x \in K : h(x) \geq a\}$ and $h^{-1}(a/0) = \{y \in K : h(y) \leq a\}$. When $h^{-1}(1/a)$ is nonempty, then it is a filter of $\mathbf{K}$, and dually $h^{-1}(a/0)$ is an ideal. If $\mathbf{K}$ is infinite, then $h^{-1}(1/a)$ need not have a least element, nor $h^{-1}(a/0)$ a greatest element. However, investigating when this does occur will lead us to one of the crucial ideas for understanding the structure of free lattices: bounded homomorphisms.

These ideas arose independently in the early 1970's in the work of Ralph McKenzie on nonmodular lattice varieties [**99**] and Bjarni Jónsson on sublattices of free lattices [**83**]. As we shall see, these areas are both closely connected to the structure of free lattices. Alan Day later showed how these ideas could be approached model-theoretically, using his famous doubling construction [**24**]. In this chapter we will present the fundamental ideas concerning upper and lower bounded lattices, and their interconnections with free lattices and varieties of lattices.

Bounded lattices also have ramifications outside of pure lattice theory. Most importantly, they turn up in the theory of congruence varieties, and we will survey some of these applications in the last section of this chapter.

1. Bounded Homomorphisms

We begin with a basic definition. A lattice homomorphism $h : \mathbf{K} \to \mathbf{L}$ is *lower bounded* if for every $a \in L$, $h^{-1}(1/a)$ is either empty or has a least element. The least element of a nonempty $h^{-1}(1/a)$ will be denoted by $\beta_h(a)$, or if the context is clear, simply $\beta(a)$. Thus if h is a lower bounded homomorphism, $\beta_h : \mathbf{L} \to \mathbf{K}$ is a partial mapping whose domain is an ideal of $\mathbf{L}$. Dually, h is an *upper bounded* homomorphism if $h^{-1}(a/0)$ has a greatest element, denoted $\alpha_h(a)$ or just $\alpha(a)$, whenever it is nonempty. For an upper bounded homomorphism, the domain of α_h is clearly a filter of $\mathbf{L}$. A homomorphism which is both upper and lower bounded is called *bounded*.

These definitions simplify considerably when h is an epimorphism. For example, in that case h is lower bounded if and only if each preimage set $h^{-1}(a)$ has a least element.

Likewise, when $\mathbf{L}$ is finite, then $h : \mathbf{K} \to \mathbf{L}$ is lower bounded if and only if $h^{-1}(a)$ has a least element whenever it is nonempty. On the other hand, every homomorphism h from a finite lattice $\mathbf{K}$ is bounded.

Note that β is monotonic and a left adjoint for h, i.e., $a \le h(x)$ iff $\beta(a) \le x$. It then follows from a standard argument that β is a join preserving map on its domain: if $h^{-1}(1/a) \neq \emptyset$ and $h^{-1}(1/b) \neq \emptyset$, then

$$\beta(a \sqcup b) = \beta(a) \sqcup \beta(b).$$

Similarly, α is a right adjoint for h, so that $h(y) \le a$ iff $y \le \alpha(a)$, and for $a, b \in \operatorname{dom} \alpha$,

$$\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b).$$

In particular, if h is an epimorphism, then α and β are respectively meet and join homomorphisms of $\mathbf{L}$ into $\mathbf{K}$.

For future reference, we note that α and β behave correctly with respect to composition.

Theorem 2.1. *Let $f : \mathbf{K} \to \mathbf{L}$ and $g : \mathbf{L} \to \mathbf{M}$ be lattice homomorphisms. If f and g are lower bounded, then $gf : \mathbf{K} \to \mathbf{M}$ is lower bounded, and $\beta_{gf} = \beta_f \beta_g$. Similarly, if f and g are upper bounded, then $\alpha_{gf} = \alpha_f \alpha_g$.*

Proof: For $x \in K$ and $a \in M$, we have

$$a \le gf(x) \quad \text{iff} \quad \beta_g(a) \le f(x) \quad \text{iff} \quad \beta_f \beta_g(a) \le x.$$

The upper bounded case is dual.

We need a way to determine whether a lattice homomorphism $h : \mathbf{K} \to \mathbf{L}$ is upper or lower bounded. The most natural setting for this is when the lattice $\mathbf{K}$ is finitely generated. For the rest of this chapter we will be assuming that $\mathbf{K}$ is generated by a finite set X, postponing the treatment of the infinitely generated case until Chapter V, Section 2 where it arises naturally. For the present, however, there are no special assumptions about $\mathbf{L}$, nor do we assume that h is either upper or lower bounded. We want to analyze the sets $h^{-1}(1/a)$ for $a \in \mathbf{L}$, with the possibility of lower boundedness in mind. The corresponding results for $h^{-1}(a/0)$ are of course obtained by duality. Note that because $\mathbf{K}$ is finitely generated, it has a greatest element $1_{\mathbf{K}} = \bigvee X$, and that $h^{-1}(1/a)$ is nonempty if and only if $a \le h(1_{\mathbf{K}})$. Such elements are naturally the ones in which we are most interested.

We define a pair of closure operators, denoted by the superscripts $^{\wedge}$ and $^{\sqcup}$, on the subsets of an arbitrary lattice $\mathbf{L}$ as follows. For any

$A \subseteq L$, let

$$A^{\wedge} = \{\bigwedge B : B \text{ is a finite subset of } A\}.$$

Let us adopt the convention that if $\mathbf{L}$ has a greatest element, then $\bigwedge \emptyset = 1_{\mathbf{L}}$ and we include it in $A^{\wedge}$, but for lattices without a greatest element $\bigwedge \emptyset$ is undefined. This is certainly arbitrary, but it works reasonably well. The set $A^{\sqcup}$ is defined dually. The lattice properties of these sets will play a more prominent role in Chapter III.

When $\mathbf{K}$ is generated by the finite set X, we can use these operators to write K as a union of a chain of subsets $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n \subseteq \cdots$ defined inductively by

$$H_0 = X^{\wedge},$$
$$H_{k+1} = (H_k)^{\sqcup \wedge}.$$

By induction each $H_n = X^{\wedge(\sqcup \wedge)^n}$ is a finite meet-closed subset of K, and $\bigcup H_n = K$ since X generates $\mathbf{K}$.

We want to analyze $h^{-1}(1/a)$ from the viewpoint of both $\mathbf{K}$ and $\mathbf{L}$. Looking at these sets from inside $\mathbf{K}$, for each $a \in L$ with $a \leq h(1_{\mathbf{K}})$ and $k \in \omega$, define

$$\beta_k(a) = \bigwedge\{w \in H_k : h(w) \geq a\}.$$

(Note that β_h, where h is a homomorphism, is different from β_k, where k is an integer.) For $a \nleq h(1_{\mathbf{K}})$, $\beta_k(a)$ is undefined. We should note that β_k, unlike β, depends on the choice of the generating set X. In situations involving more than one mapping, we will write $\beta_{k,h}(a)$ to indicate the homomorphism. The dual notion is denoted by $\alpha_k(a)$ or $\alpha_{k,h}(a)$. The following result is then an immediate consequence of the definitions.

THEOREM 2.2. *Let $\mathbf{K}$ be finitely generated, and let $h : \mathbf{K} \to \mathbf{L}$ be a lattice homomorphism. For each $a \in L$ with $a \leq h(1_{\mathbf{K}})$, we have*

(1) *$j \leq k$ implies $\beta_j(a) \geq \beta_k(a)$,*
(2) *$\beta_k(a)$ is the least element of $H_k \cap h^{-1}(1/a)$,*
(3) *$h^{-1}(1/a) = \bigcup_{k\in\omega} 1/\beta_k(a)$.*

Viewed from inside $\mathbf{L}$, things take on a different perspective. Let us first introduce some useful terminology. A *join cover* of the element $a \in L$ is any finite subset $S \subseteq L$ such that $a \leq \bigvee S$. A join cover S of a is *nontrivial* if $a \nleq s$ for all $s \in S$. Let $\mathcal{C}(a)$ denote the set of all nontrivial join covers of a in $\mathbf{L}$.

THEOREM 2.3. *Let $\mathbf{K}$ be generated by the finite set X, and let $h : \mathbf{K} \to \mathbf{L}$ be a lattice homomorphism. For each $a \in L$ with $a \leq h(1_{\mathbf{K}})$*

and $k \in \omega$, we have

$$\beta_0(a) = \bigwedge\{x \in X : h(x) \geq a\},$$
$$\beta_{k+1}(a) = \beta_0(a) \wedge \bigwedge{}_{\bigvee S \leq h(1_{\mathbf{K}})}^{S \in \mathcal{C}(a)} \bigvee_{s \in S} \beta_k(s).$$

Proof: Temporarily, for $a \leq h(1_{\mathbf{K}})$ let us define

$$\gamma_0(a) = \bigwedge\{x \in X : h(x) \geq a\},$$
$$\gamma_{k+1}(a) = \gamma_0(a) \wedge \bigwedge{}_{\bigvee S \leq h(1_{\mathbf{K}})}^{S \in \mathcal{C}(a)} \bigvee_{s \in S} \gamma_k(s).$$

Clearly $\gamma_0(a) = \beta_0(a)$, and by induction we have $\gamma_n(a) \in H_n$ for all n. In particular, even though $\mathcal{C}(a)$ may be infinite, there are only finitely many possibilities for $\bigvee_{s \in S} \gamma_k(s)$, so the expression for $\gamma_{k+1}(a)$ makes sense. Also note that, by induction, $\gamma_{k+1}(a) \leq \gamma_k(a)$.

Now we have $\gamma_n(a) \in H_n$ and $h(\gamma_n(a)) \geq a$ for every n. Hence, by Theorem 2.2, $\gamma_n(a) \geq \beta_n(a)$. To prove the reverse inclusion, we will show by induction that if $w \in H_n$ and $h(w) \geq a$, then $w \geq \gamma_n(a)$. Applying this with $w = \beta_n(a)$ then yields $\beta_n(a) \geq \gamma_n(a)$. The statement is trivially true for $n = 0$.

So assume the statement is true for $n \leq k$, and let $w \in H_{k+1}$ with $h(w) \geq a$. Then w can be written in the form

$$w = \bigwedge_i \bigvee_{j < r_i} w_{ij} \wedge \bigwedge_m x_m$$

with each $w_{ij} \in H_k$ and each $x_m \in X$. Clearly, for each m we have $h(x_m) \geq h(w) \geq a$, and thus $x_m \geq \gamma_0(a)$.

For the remaining meetands, each of the form $\bigvee_j w_{ij}$, we distinguish two cases. If $h(w_{ij_0}) \geq a$ for some j_0, then by induction $w_{ij_0} \geq \gamma_k(a)$, and hence

$$\bigvee_j w_{ij} \geq w_{ij_0} \geq \gamma_k(a) \geq \gamma_{k+1}(a).$$

In the other case, $\{h(w_{ij}) : j < r_i\} \in \mathcal{C}(a)$ and $\bigvee_j h(w_{ij}) \leq h(1_{\mathbf{K}})$, and for each j we have again by induction $w_{ij} \geq \gamma_k(h(w_{ij}))$. Thus by the definition of γ_{k+1},

$$\bigvee_j w_{ij} \geq \bigvee_j \gamma_k(h(w_{ij})) \geq \gamma_{k+1}(a).$$

Since each meetand of w is above $\gamma_{k+1}(a)$, we obtain $w \geq \gamma_{k+1}(a)$, as desired.

In the definition of $\mathcal{C}(a)$ we excluded the trivial join covers (those with $a \leq s$ for some $s \in S$) because they would not have really contributed to the expression for $\beta_{k+1}(a)$ in Theorem 2.3. In general this expression still contains some redundant terms which we should try to exclude, and if $\mathbf{L}$ satisfies a weak finiteness condition (given below) we can do so.

Recall from Chapter I that for finite subsets A, $B \subseteq L$ we say that *A join refines B*, written $A \ll B$, if for every $a \in A$ there exists $b \in B$ with $a \leq b$. Theorem 1.19 states that if $w \in \mathbf{FL}(X)$ and $w = \bigvee B$, then the set of canonical joinands of w join refines B. By convention, the abbreviated terminology *A refines B* always refers to join refinement.

Accordingly, we define a *minimal nontrivial join cover* of $a \in L$ to be a nontrivial join cover S with the property that whenever $a \leq \bigvee T$ and $T \ll S$, then $S \subseteq T$. This technical formulation of minimality is useful, and is easily seen to be equivalent to our more intuitive notion of what minimality ought to mean: *a nontrivial join cover S of a is minimal if and only if*

(a) *S is an antichain of join irreducible elements of $\mathbf{L}$, and*
(b) *whenever any element of S is deleted or replaced by a (finite) set of strictly smaller elements, then the resulting set is no longer a join cover of a.*

Let $\mathcal{M}(a)$ denote the set of minimal nontrivial join covers of a in $\mathbf{L}$.

Let us say that $\mathbf{L}$ has the *minimal join cover refinement property* if for each $a \in L$, $\mathcal{M}(a)$ is finite and every nontrivial join cover of a refines to a minimal one. Clearly every finite lattice has the minimal join cover refinement property, but as we will see in the next section, so do free lattices. The property is inherited by projective lattices, and also holds in finitely presented lattices [**53**]. Thus it provides a natural general setting for many of our results.

The following reformulation of Theorem 2.3 is obvious in view of Lemma 1.15(1), but clearly simplifies the calculation of β_k whenever the minimal join cover refinement property holds.

THEOREM 2.4. *Let $\mathbf{K}$ be generated by the finite set X, and let $h : \mathbf{K} \to \mathbf{L}$ be a lattice homomorphism. If $\mathbf{L}$ has the minimal join cover refinement property, then for each $a \in L$ with $a \leq h(1_{\mathbf{K}})$ and $k \in \omega$ we have*

$$\beta_0(a) = \bigwedge\{x \in X : h(x) \geq a\},$$
$$\beta_{k+1}(a) = \beta_0(a) \wedge \bigwedge_{\bigvee S \leq h(1_{\mathbf{K}})}^{S \in \mathcal{M}(a)} \bigvee_{s \in S} \beta_k(s).$$

Now let us look for a condition on the lattice $\mathbf{L}$ which will insure that the homomorphism $h : \mathbf{K} \to \mathbf{L}$ is lower bounded. It is clear from Theorem 2.2 that this will happen if and only if for each $a \leq h(1_{\mathbf{K}})$ there exists an $N \in \omega$ such that $\beta_n(a) = \beta_N(a)$ for all $n \geq N$. In this case, of course, $\beta(a) = \beta_N(a)$ for all $a \in \operatorname{dom}\beta = h(1_{\mathbf{K}})/0$, where N depends upon a.

With this in mind, let $\mathrm{D}_0(\mathbf{L})$ be the set of all join prime elements of $\mathbf{L}$, i.e., the set of all elements which have no nontrivial join cover. Given $\mathrm{D}_k(\mathbf{L})$, define $\mathrm{D}_{k+1}(\mathbf{L})$ to be the set of all $p \in L$ such that every nontrivial join cover of p refines to a join cover contained in $\mathrm{D}_k(\mathbf{L})$, i.e., $p \leq \bigvee S$ nontrivially implies there exists $T \ll S$ with $p \leq \bigvee T$ and $T \subseteq \mathrm{D}_k(\mathbf{L})$. Note that if $\mathbf{L}$ has the minimal join cover refinement property, then $p \in \mathrm{D}_{k+1}(\mathbf{L})$ if and only if every minimal nontrivial join cover of p is contained in $\mathrm{D}_k(\mathbf{L})$.

The definition clearly implies $\mathrm{D}_0(\mathbf{L}) \subseteq \mathrm{D}_1(\mathbf{L}) \subseteq \mathrm{D}_2(\mathbf{L}) \subseteq \cdots$. Let $\mathrm{D}(\mathbf{L}) = \bigcup_{k\in\omega} \mathrm{D}_k(\mathbf{L})$. For $a \in \mathrm{D}(\mathbf{L})$, we define the *D–rank*, $\rho(a)$, to be the least integer N such that $a \in \mathrm{D}_N(\mathbf{L})$; for $a \notin \mathrm{D}(\mathbf{L})$, $\rho(a)$ is undefined. The duals of $\mathrm{D}_k(\mathbf{L})$, $\mathrm{D}(\mathbf{L})$ and $\rho(a)$ are denoted by $\mathrm{D}^d_k(\mathbf{L})$, $\mathrm{D}^d(\mathbf{L})$ and $\rho^d(a)$, respectively.

We are interested in the property $\mathrm{D}(\mathbf{L}) = L$, i.e., when every element is in $\mathrm{D}_k(\mathbf{L})$ for some k. *If A is a finite subset of $\mathrm{D}_k(\mathbf{L})$, then $\bigvee A \in \mathrm{D}_{k+1}(\mathbf{L})$.* Indeed, if $\bigvee A \leq \bigvee S$ for some set S, then, for $a \in A$, S is a join cover of a. Since $a \in \mathrm{D}_k(\mathbf{L})$, if this is a nontrivial join cover, there is a set $U_a \subseteq \mathrm{D}_{k-1}(\mathbf{L})$ with $U_a \ll S$ and $a \leq \bigvee U_a$. For those a's for which S is a trivial join cover, set $U_a = \{a\}$. Let $U = \bigcup U_a$. Then $U \subseteq \mathrm{D}_k(\mathbf{L})$ and is a join cover of $\bigvee A$ and $U \ll S$, which shows that $\bigvee A$ is in $\mathrm{D}_{k+1}(\mathbf{L})$. Hence $\mathrm{D}(\mathbf{L})$ is a join subsemilattice of $\mathbf{L}$. On the other hand, it is easy to see that in a lattice with the minimal join cover refinement property, every element is a join of join irreducibles. Combining these observations, we obtain the following useful equivalence.

LEMMA 2.5. *For a lattice with the minimal join cover refinement property,* $\mathrm{D}(\mathbf{L}) = L$ *if and only if* $\mathrm{J}(\mathbf{L}) \subseteq \mathrm{D}(\mathbf{L})$.

For example, if $\mathbf{K}$ is a finite distributive lattice, then $\mathrm{D}(\mathbf{K}) = \mathrm{D}_1(\mathbf{K}) = K$ because every join irreducible element is join prime. Likewise, it is easy to see that for the pentagon $\mathbf{N}_5$ we have $\mathrm{D}(\mathbf{N}_5) = \mathrm{D}_1(\mathbf{N}_5) = N_5$. On the other hand, for the diamond we have $\mathrm{D}(\mathbf{M}_3) = \{0\} \neq M_3$.

We need to analyze the property $\mathrm{D}(\mathbf{L}) = L$ at length, but first let us show that it does what we need it to do.

THEOREM 2.6. *Let* $\mathbf{K}$ *be a finitely generated lattice and let* $h : \mathbf{K} \to \mathbf{L}$ *be a homomorphism. If* $\mathrm{D}(\mathbf{L}) = L$*, then* h *is lower bounded.*

The proof of this theorem yields a slightly stronger statement which, because we will refer to it a couple of times, we give as a separate lemma.

LEMMA 2.7. *Let* $\mathbf{K}$ *be a finitely generated lattice and let* $h : \mathbf{K} \to \mathbf{L}$ *be a homomorphism. For all* $w \in K$ *and* $a \in \mathrm{D}_n(\mathbf{L})$*,*

$$(*) \qquad h(w) \geq a \text{ if and only if } w \geq \beta_n(a).$$

Thus in the case when $\mathrm{D}(\mathbf{L}) = L$, we obtain that h is lower bounded with $\beta(a) = \beta_{\rho(a)}(a)$ for all $a \in L$.

Proof: [Proof of Lemma 2.7] We of course use Theorem 2.3. In order to prove $(*)$ by induction, for each $n \in \omega$ define

$$W_n = \{w \in K : \text{for all } a \in \mathrm{D}_n(\mathbf{L}),\ h(w) \geq a \text{ iff } w \geq \beta_n(a)\}.$$

We need to show that $W_n = K$ for all n. Clearly the generating set X for $\mathbf{K}$ is contained in W_n, and W_n is closed under meets for each n. So it remains to show that W_n is closed under joins. For $n = 0$ this follows immediately from the definition of $\mathrm{D}_0(\mathbf{L})$ as the set of all join prime elements.

Assume $W_n = K$ for all $n \leq k$, and consider W_{k+1}. Suppose S is a finite subset of W_{k+1}, $a \in \mathrm{D}_{k+1}(\mathbf{L})$ and $h(\bigvee S) \geq a$. If $h(s) \geq a$ for some $s \in S$, then since $s \in W_{k+1}$ we have $\bigvee S \geq s \geq \beta_{k+1}(a)$, as desired. Otherwise, $\bigvee_{s \in S} h(s)$ is a nontrivial join cover of a. As $a \in \mathrm{D}_{k+1}(\mathbf{L})$, there exists $T \ll h(S)$ with $a \leq \bigvee T$ and $T \subseteq \mathrm{D}_k(\mathbf{L})$. Of course T is also a nontrivial join cover of a, so by the definition of $\beta_{k+1}(a)$ we have $\bigvee_{t \in T} \beta_k(t) \geq \beta_{k+1}(a)$. On the other hand, for each $t \in T$ there exists $s \in S$ with $h(s) \geq t$, and hence by induction $s \geq \beta_k(t)$. Thus $\{\beta_k(t) : t \in T\} \ll S$. Putting these together, we obtain

$$\bigvee S \geq \bigvee_{t \in T} \beta_k(t) \geq \beta_{k+1}(a).$$

We conclude that W_{k+1} is also closed under joins, and $W_{k+1} = K$.

Combining Theorem 2.4 and Lemma 2.7, we obtain the following interesting version.

COROLLARY 2.8. *Let* $\mathbf{K}$ *be generated by the finite set* X*, and let* $h : \mathbf{K} \to \mathbf{L}$ *be a lattice homomorphism. If* $\mathbf{L}$ *has the minimal join cover refinement property, then for each* $a \in \mathrm{D}(\mathbf{L}) \cap (h(1_{\mathbf{K}})/0)$ *we have*

$$\beta(a) = \bigwedge \{x \in X : h(x) \geq a\} \wedge \bigwedge_{\substack{S \in \mathcal{M}(a) \\ \bigvee S \leq h(1_{\mathbf{K}})}} \bigvee_{s \in S} \beta(s).$$

Next, let us show that free lattices satisfy $\mathrm{D}(\mathbf{FL}(X)) = \mathrm{FL}(X)$ and the minimal join cover refinement property. In fact, the D–rank $\rho(w)$ of an element $w \in \mathbf{FL}(X)$ is closely connected to the complexity of its canonical form, as defined in Chapter I. We will actually prove a slightly more general theorem after first proving a lemma.

LEMMA 2.9. *Let* $\mathbf{L}$ *be a lattice which is generated by a set* X*. Then* $\mathrm{D}_n(\mathbf{L}) \subseteq X^{\wedge(\ \sqcup\ \wedge)^n}$.

Proof: We will prove this lemma by showing that if $f : \mathbf{FL}(X) \twoheadrightarrow \mathbf{L}$ is an epimorphism, then $\mathrm{D}_n(\mathbf{L}) \subseteq f(H_n)$ where $H_n = X^{\wedge(\ \sqcup\ \wedge)^n}$ in $\mathbf{FL}(X)$. So let f be given and $a \in \mathrm{D}_n(\mathbf{L})$. There is a finite subset $Y \subseteq X$ such that the restricted map $\hat{f} : \mathbf{FL}(Y) \to \mathbf{L}$ has a in its range. Now Lemma 2.7 shows that $\beta_{n,\hat{f}}(a)$, which is an element of H_n, is the least preimage of a with respect to $\hat{f}$. Thus $a = \hat{f}\beta_{n,\hat{f}}(a) \in f(H_n)$, as claimed.

THEOREM 2.10. *If* $\mathbf{F}$ *is a lattice which satisfies Whitman's condition* (W) *and is generated by a set* X *of join prime elements, then*

(1) $\mathrm{D}(\mathbf{F}) = F$, *and*
(2) $\mathbf{F}$ *satisfies the minimal join cover refinement property.*

The join prime elements of $\mathbf{F}$ *are precisely all nonempty meets of generators, and* $\mathrm{D}_n(\mathbf{F}) = X^{\wedge(\ \sqcup\ \wedge)^n}$ *for each* $n > 0$.[1] *If* $a \in \mathrm{D}_k(\mathbf{F})$ *then every minimal, nontrivial join cover of* a *is contained in* $\mathrm{D}_{k-1}(\mathbf{F})$.

Proof: In $\mathbf{F}$, let $K_n = X^{\wedge(\ \sqcup\ \wedge)^n}$. We are claiming that, except for $n = 0$, $\mathrm{D}_n(\mathbf{F}) = K_n$. Nonempty meets of generators are join prime by (W), so $K_0 - \{1_{\mathbf{F}}\} \subseteq \mathrm{D}_0(\mathbf{F})$. In the case when $1_{\mathbf{F}}$ exists and is join irreducible, then it will also be in X and $\mathrm{D}_0(\mathbf{F})$. We have already observed that the join of a finite subset of $\mathrm{D}_k(\mathbf{L})$ is an element of $\mathrm{D}_{k+1}(\mathbf{L})$. Just as easily, in a lattice satisfying (W), each $\mathrm{D}_k(\mathbf{L})$ is closed under meets, so $S \subseteq \mathrm{D}_{k+1}(\mathbf{F})$ implies $\bigwedge S \subseteq \mathrm{D}_{k+1}(\mathbf{F})$. Combining these two statements, we have by induction

$$K_{k+1} = (K_k)^{\ \sqcup\ \wedge} \subseteq (\mathrm{D}_k(\mathbf{F}))^{\ \sqcup\ \wedge} \subseteq \mathrm{D}_{k+1}(\mathbf{F}).$$

Hence, for all n, $K_n \subseteq \mathrm{D}_n(\mathbf{F})$ and so the two sets are equal by Lemma 2.9.

By the first part, any element $a \in F$ belongs to $\mathrm{D}_k(\mathbf{F})$ for some k. By induction on k, we shall construct a finite set, $S(a)$, of join covers of a such that any nontrivial join cover of a refines to one in $S(a)$. This clearly implies that $\mathbf{F}$ has the minimum join cover refinement property.

[1] Actually, $\mathrm{D}_0(\mathbf{F}) = X^{\wedge}$ would hold except that we defined the latter to include 1. Although this definition is not convenient here, it is elsewhere. Note that $1 \in \mathrm{D}_1(\mathbf{FL}(X))$.

If $a \in \mathrm{D}_0(\mathbf{F})$, then a has no nontrivial join cover and we set $S(a) = \emptyset$. Now suppose $a \in \mathrm{D}_k(\mathbf{F})$ with $k > 0$. Since $\mathrm{D}_k(\mathbf{F}) = \mathrm{D}_{k-1}(\mathbf{F})^{\vee\wedge}$, $a = \bigvee V_1 \wedge \cdots \wedge \bigvee V_r$ for a finite collection $V_1, \dots, V_r$ of finite subsets of $\mathrm{D}_{k-1}(\mathbf{F})$. Denote by T the set of the elements belonging to either V_i for some $i = 1, \dots, r$ or to some member of $S(b)$ with $b \in V_i$. Let $S(a)$ be the set of the subsets of T that are nontrivial join covers of a. Clearly, $S(a)$ is finite. Suppose U is a nontrivial join cover of a. By (W), there is an i such that $v \leq \bigvee U$ for all $v \in V_i$. If $v \in V_i$ and U is a nontrivial join cover of v, then, since $v \in \mathrm{D}_{k-1}(\mathbf{F})$ and using induction on k, there is a $U_v \in S(v)$ with $U_v \ll U$. If $v \in V_i$ is such that $v \leq u$ for some $u \in U$, set $U_v = \{v\}$. Let $U_0 = \bigcup_{v \in V_i} U_v$. It is clear that $U_0 \subseteq T$ and $U_0 \ll U$. Since $v \leq \bigvee U_v$ for each $v \in V_i$, U_0 is a join cover of a. Moreover, U_0 is a nontrivial join cover of a, since $U_0 \ll U$. Hence $U_0 \in S(a)$. By induction, $T \subseteq \mathrm{D}_{k-1}(\mathbf{F})$, and therefore $V \subseteq \mathrm{D}_{k-1}(\mathbf{F})$ for all $V \in S(a)$.

Corollary 2.11. *Every free lattice* $\mathbf{FL}(X)$ *satisfies the minimal join cover refinement property and* $\mathrm{D}(\mathbf{FL}(X)) = \mathrm{FL}(X)$. *In fact, we have* $\mathrm{D}_0(\mathbf{FL}(X)) = X^{\wedge} - \{1_{\mathbf{FL}(X)}\}$, *and* $\mathrm{D}_n(\mathbf{FL}(X)) = X^{\wedge(\,\sqcup\,\wedge)^n}$ *for each* $n > 0$. *If* $a \in \mathrm{D}_k(\mathbf{FL}(X))$ *then every minimal, nontrivial join cover of* a *is contained in* $\mathrm{D}_{k-1}(\mathbf{FL}(X))$.

Corollary 2.12. *For* $w \in \mathbf{FL}(X)$, $\rho(w)$ *and* $\rho^d(w)$ *differ by at most* 1.

This brings us to the first fundamental result on lower bounded lattices. The theorem was proved for finite lattices by Ralph McKenzie in his work on splitting varieties and splitting lattices [**99**]. Similar ideas were developed simultaneously by Bjarni Jónsson in connection with finite projective lattices, which were partly included in [**99**] and appeared more fully in [**83**]. The extension to the finitely generated case was done by Alan Kostinsky [**92**].

Theorem 2.13. *For a finitely generated lattice* $\mathbf{L}$, *the following are equivalent.*

(1) *There exists a finite set* X *and a lower bounded epimorphism* $f : \mathbf{FL}(X) \twoheadrightarrow \mathbf{L}$.
(2) *For every finitely generated lattice* $\mathbf{K}$, *every homomorphism* $h : \mathbf{K} \to \mathbf{L}$ *is lower bounded.*
(3) $\mathrm{D}(\mathbf{L}) = L$.

Proof: The implication (3) implies (2) is exactly Theorem 2.6, and (2) implies (1) trivially, so it remains to show (1) implies (3).

Let X be finite, and let $f : \mathbf{FL}(X) \twoheadrightarrow \mathbf{L}$ be a lower bounded epimorphism. We want to show that for all $a \in L$ and $n \in \omega$, $\beta(a) \in \mathrm{D}_n(\mathbf{FL}(X))$ implies $a \in \mathrm{D}_n(\mathbf{L})$. In view of Corollary 2.11, this will prove that $\mathrm{D}(\mathbf{L}) = L$.

First assume $\beta(a) \in \mathrm{D}_0(\mathbf{FL}(X))$ and $a \leq \bigvee S$ in $\mathbf{L}$. Since β preserves joins, $\beta(a) \leq \bigvee \beta(S)$, and hence $\beta(a) \leq \beta(s)$ for some $s \in S$. As f is an epimorphism, this implies $a = f(\beta(a)) \leq f(\beta(s)) = s$. Thus a is join prime, i.e., $a \in \mathrm{D}_0(\mathbf{L})$.

Now assume the statement holds for $n \leq k$, and let $\beta(a) \in \mathrm{D}_{k+1}(\mathbf{FL}(X))$ and $a \leq \bigvee S$ nontrivially. Then $\beta(a) \leq \bigvee \beta(S)$; arguing as above, we see that this is also a nontrivial join cover. By Corollary 2.11, there exists a minimal join cover T of $\beta(a)$ with $T \ll \beta(S)$. Since $\beta(a) \in \mathrm{D}_{k+1}(\mathbf{FL}(X))$, we have $T \subseteq \mathrm{D}_k(\mathbf{FL}(X))$. Now $\beta(a) \leq \bigvee T$ implies $\beta(a) = \beta f \beta(a) \leq \bigvee \beta f(T)$. But $\beta f(T) \ll T$, and T was a minimal join cover, so $T = \beta f(T)$. Thus for each $t \in T$, $\beta f(t) \in \mathrm{D}_k(\mathbf{FL}(X))$, and by induction $f(t) \in \mathrm{D}_k(\mathbf{L})$. On the other hand, $T \ll \beta(S)$ implies $f(T) \ll f\beta(S) = S$. Therefore $a \in \mathrm{D}_{k+1}(\mathbf{L})$, as desired.

We call a finitely generated lattice having the three equivalent properties of Theorem 2.13 a *lower bounded lattice*. For lattices which may not be finitely generated, but for which there exists a lower bounded epimorphism $h : \mathbf{FL}(X) \twoheadrightarrow \mathbf{L}$ for some X, we use the fuller terminology *lower bounded homomorphic image of a free lattice*; these are characterized in Theorem 5.33. The dual of lower bounded is again called *upper bounded*, and a lattice which is both upper and lower bounded is called *bounded*.[2] Clearly any result about lower bounded lattices has, in addition to the dual statement for upper bounded lattices, a combined form for bounded lattices. Most often we will give only one form of a given theorem.

Corollary 2.14. *Every finitely generated sublattice of a lower bounded lattice is itself lower bounded.*

Proof: Let $\mathbf{L}$ be lower bounded, and let $\mathbf{S}$ be a finitely generated sublattice of $\mathbf{L}$. There is of course a homomorphism h from a finitely generated free lattice into $\mathbf{L}$ whose image is $\mathbf{S}$. By Theorem 2.13(2), h is lower bounded, which makes $\mathbf{S}$ a lower bounded lattice by Theorem 2.13(1).

This Corollary is so important that we allow ourselves the luxury of a second proof, which adds a little information about the D–rank of the elements involved.

[2] The term *lower bounded lattice* has also been used to refer to any lattice with a least element. We will always use *lower bounded* in the sense defined here, and similarly for *upper bounded* and *bounded*.

Corollary 2.15. *Let* $\varepsilon : \mathbf{S} \to \mathbf{L}$ *be an embedding of a finitely generated lattice into a lower bounded lattice. If* $u \leq \varepsilon(1_{\mathbf{S}})$ *and* $u \in \mathrm{D}_k(\mathbf{L})$*, then* $\beta_\varepsilon(u) \in \mathrm{D}_k(\mathbf{S})$*. Consequently,* $\mathrm{D}(\mathbf{S}) = S$ *and* $\mathbf{S}$ *is lower bounded.*

Proof: Observe that for $u \leq \varepsilon(1_{\mathbf{S}})$ we have $u \leq \varepsilon\beta(u)$, by the definition of β. Because ε is an embedding, we also have $s = \beta\varepsilon(s)$ for $s \in S$. Thus for $s \in S$ and $u \in L$, $u \leq \varepsilon(s)$ if and only if $\beta(u) \leq s$. Applying the claim of the Corollary with $u = \varepsilon(s)$ will show that $\varepsilon(s) \in \mathrm{D}_k(\mathbf{L})$ implies $s \in \mathrm{D}_k(\mathbf{S})$, and hence that the D–ranks satisfy $\rho_{\mathbf{S}}(s) \leq \rho_{\mathbf{L}}(\varepsilon(s))$.

The proof of the implication is a straightforward induction. If $u \in \mathrm{D}_0(\mathbf{L})$ and $\beta(u) \leq \bigvee A$ in $\mathbf{S}$, then $u \leq \varepsilon\beta(u) \leq \bigvee \varepsilon(A)$. Thus $u \leq \varepsilon(a_0)$ for some $a_0 \in A$, whence $\beta(u) \leq \beta\varepsilon(a_0) = a_0$. Hence $\beta(u) \in \mathrm{D}_0(\mathbf{S})$.

Now let $u \in \mathrm{D}_k(\mathbf{L})$ with $k > 0$, and assume that $\beta(u) \leq \bigvee A$ nontrivially in $\mathbf{S}$. Again we obtain $u \leq \bigvee \varepsilon(A)$ in $\mathbf{L}$. Then there exists $B \subseteq \mathrm{D}_{k-1}(\mathbf{L})$ such that $B \ll \varepsilon(A)$ and $u \leq \bigvee B$. By induction $\beta(B) \subseteq \mathrm{D}_{k-1}(\mathbf{S})$, while $\beta(B) \ll \beta\varepsilon(A) = A$ and $\beta(u) \leq \bigvee \beta(B)$. Thus $\beta(u) \in \mathrm{D}_k(\mathbf{S})$.

Corollary 2.16. *Every finitely generated sublattice of a free lattice is both lower and upper bounded.*

Using the fact that $\mathrm{D}(\mathbf{L})$ is closed under joins, we can see that the class of lower bounded lattices is closed under finite direct products. It is evident from the definition that if $\mathbf{K}$ is a lower bounded lattice and $h : \mathbf{K} \twoheadrightarrow \mathbf{L}$ is a lower bounded epimorphism, then $\mathbf{L}$ is also a lower bounded lattice. Since any epimorphism between finite lattices is both upper and lower bounded, we obtain the following.

Corollary 2.17. *The class of finite lower bounded lattices is closed under* $\boldsymbol{H}$*,* $\boldsymbol{S}$ *and* $\boldsymbol{P}_f$*, and hence is a pseudovariety.*

This means that there is an ideal $\mathcal{I}$ in the lattice $\mathbf{\Lambda}$ of lattice varieties such that a finite lattice $\mathbf{L}$ is lower bounded if and only if $\mathbf{L} \in \mathcal{V}$ for some $\mathcal{V} \in \mathcal{I}$. This ideal is described explicitly in [**108**], and the result is generalized in [**30**]. It is interesting to note that the class of finite lattices satisfying ($\mathrm{SD}_\vee$) (or ($\mathrm{SD}_\wedge$), or both) is also a pseudovariety. (The argument in the proof of Theorem 2.20 below shows that such a class is closed under $\boldsymbol{H}$.) A sequence of weaker and weaker equations is given in Jónsson and Rival [**84**] with the property that a finite lattice is join semidistributive if and only if it satisfies one of them. Being a pseudovariety has a nice consequence.

Corollary 2.18. *If* $\mathbf{L}$ *is a finite lattice, then there exists a least congruence* $\lambda \in \mathbf{Con}\,\mathbf{L}$ *such that* $\mathbf{L}/\lambda$ *is lower bounded.*

Finally, for future reference, we note one further consequence of Theorem 2.13.

COROLLARY 2.19. *Every lower bounded lattice has the minimal join cover refinement property.*

Proof: Let $\mathbf{L}$ be a lower bounded lattice. By definition, $\mathbf{L}$ is generated by a finite set X. By Theorem 2.13, $\mathrm{D}(\mathbf{L}) = L$. Thus every $a \in L$ is in $\mathrm{D}_k(\mathbf{L})$ for some k, and so every nontrivial join cover of a refines to one in $\mathrm{D}_{k-1}(\mathbf{L})$, which is finite by Lemma 2.9. This clearly implies that $\mathbf{L}$ has the minimal join cover refinement property.

The lattice of Figure 5.2 on page 134, which is not finitely generated, satisfies $\mathrm{D}(\mathbf{L}) = L$ but not the minimal join cover refinement property. In Theorem 5.33 we will prove that lower bounded homomorphic images of a free lattice satisfy the minimal join cover refinement property.

Throughout this chapter and the next we will develop many examples of upper and lower bounded lattices. In particular, in the third section of this chapter, and again in Chapter III (from a different viewpoint), we will see how to construct many examples of finite lower bounded lattices, which need not be upper bounded, e.g., every lattice except the first ($\mathbf{2} \times \mathbf{2}$) in Figure 2.2 on page 58. It is always good to keep in mind two extreme cases: every finite distributive lattice is bounded, and of course every finitely generated free lattice is bounded.

It is important to know that lower bounded lattices inherit ($\mathrm{SD}_{\sqcup}$) from free lattices.

THEOREM 2.20. *Every lower bounded homomorphic image of a free lattice is join semidistributive.*

Proof: Let f be a lower bounded epimorphism of $\mathbf{FL}(n)$ onto a lattice $\mathbf{L}$. Then $u = a \sqcup b = a \sqcup c$ in $\mathbf{L}$ implies

$$\beta(u) = \beta(a) \sqcup \beta(b) = \beta(a) \sqcup \beta(c) = \beta(a) \sqcup (\beta(b) \wedge \beta(c))$$

since $\mathbf{FL}(n)$ is join semidistributive and β preserves joins. Thus

$$u = f\beta(u) = f(\beta(a) \sqcup (\beta(b) \wedge \beta(c))) = a \sqcup (b \wedge c).$$

Next we embark upon the proof of a generalization of Theorem 2.20. In fact, some of the results below are more general than we will need, but they are nice. Perhaps more importantly, they afford us a natural opportunity to introduce some ideas and connections which will play a significant role later.

We want to extend the notion of a canonical join representation to lattices satisfying ($\mathrm{SD}_\vee$) and the minimal join cover refinement property. The critical structural property of the canonical join representation in free lattices is given by Theorem 1.19.

Recall, as in Chapter I, that in any lattice $\mathbf{L}$ we say that the finite set $C \subseteq L$ is a *minimal join representation* of the element a if

(1) $a = \bigvee C$ and
(2) if $a = \bigvee B$ and $B \ll C$, then $C \subseteq B$.

The second condition implies that $a = \bigvee C$ irredundantly. Note that in a lattice with the minimal join cover refinement property, every element has at least one, and at most finitely many, minimal join representations. Of course, join irreducible elements have only the trivial representation.

Similarly, we say that an element $a \in L$ has a *canonical join representation* if there exists a finite set $C \subseteq L$ such that

(1) $a = \bigvee C$ irredundantly, and
(2) if $a = \bigvee B$ then $C \ll B$.

Thus a canonical join representation is a unique, irredundant, minimum (in the sense of refinement) expression of an element as a join of join irreducible elements. The elements of a canonical join representation are called the *canonical joinands* of a.

Of course, a *canonical meet representation* is defined dually, and its elements are called *canonical meetands*.

THEOREM 2.21. *If* $\mathbf{L}$ *is a lattice satisfying* $\mathrm{D}(\mathbf{L}) = L$*, then every element of* L *has a canonical join representation.*

Proof: Let P denote the set of all sequences $\mathbf{s}$ of nonnegative integers which are eventually 0, i.e., for each $\mathbf{s} \in P$ there exists N such that $s_n = 0$ for all $n \geq N$. We order P by putting $\mathbf{s} < \mathbf{t}$ if there exists an index m such that $s_m < t_m$ and $s_n = t_n$ for all $n > m$. It is not hard to see that $\mathbf{P} = \langle P, \leq \rangle$ is a totally ordered set satisfying the descending chain condition.

Now fix an element a in a lattice $\mathbf{L}$ with $\mathrm{D}(\mathbf{L}) = L$. With each finite set $B \subseteq L$ we can associate a sequence $\mathbf{b} = \sigma(B) \in P$ defined by $b_k = |\{b \in B : \rho(b) = k\}|$. Amongst all the finite subsets of L which join to a, pick one C such that $\sigma(C)$ is minimal in $\mathbf{P}$. We claim that $a = \bigvee C$ canonically. Clearly (1) holds.

Suppose $a = \bigvee B$ and C does not refine B, i.e., there exists $c \in C$ such that $c \nleq b$ for all $b \in B$, and hence $c \leq \bigvee B$ nontrivially. We can certainly choose c such that $\rho(c)$ is maximal for elements with this property. Let $\rho(c) = m$, and note that $m > 0$ since c is not join prime.

Then there exists $E \subseteq \mathrm{D}_{m-1}(\mathbf{L})$ such that $E \ll B$ and $c \leq \bigvee E$. Now let $C' = C - \{c\} \cup E$. Because $c \leq \bigvee E \leq \bigvee B = a$, we see that $\bigvee C' = a$. But if $\sigma(c) = \mathbf{c}$ and $\sigma(C') = \mathbf{c}'$, then $c'_m = c_m - 1$ and $c'_n = c_n$ for $n > m$, whence $\mathbf{c}' < \mathbf{c}$ in $\mathbf{P}$. This contradicts the choice of C. Thus (2) holds, and C is a canonical join representation of a.

Recall from Theorem 1.21 that $(\mathrm{SD}_\vee)$ is equivalent to the apparently stronger implication

$$w = \bigvee_i a_i = \bigvee_j b_j \quad \text{implies} \quad w = \bigvee_{i,j} (a_i \wedge b_j).$$

The proof of the part of that theorem which says that free lattices satisfy this join semidistributive law clearly uses only the properties we have used to define a canonical join representation. Thus we have the following results.

Lemma 2.22. *If every element of a lattice* $\mathbf{L}$ *has a canonical join representation, then* $\mathbf{L}$ *is join semidistributive.*

Corollary 2.23. *Every lattice* $\mathbf{L}$ *satisfying* $\mathrm{D}(\mathbf{L}) = L$ *is join semidistributive.*

Now despite the fact that every lattice satisfying $\mathrm{D}(\mathbf{L}) = L$ has a canonical join representation, $\mathrm{D}(\mathbf{L}) = L$ does not imply the minimal join cover refinement property; a counterexample is given in Figure 5.2 on page 134. Nonetheless, for lattices which do satisfy the minimal join cover refinement property, we have the following important characterization of join semidistributivity.

Theorem 2.24. *Let* $\mathbf{L}$ *be a lattice satisfying the minimal join cover refinement property. Then* $\mathbf{L}$ *satisfies* $(\mathrm{SD}_\vee)$ *if and only if every element of* L *has a canonical join representation.*

Proof: One direction is of course given by Lemma 2.22. Conversely, assume $\mathbf{L}$ has the minimal join cover refinement property. Then every element of $\mathbf{L}$ has at least one minimal join representation, and every representation $a = \bigvee A$ refines to a minimal one. Suppose an element $a \in L$ has two distinct minimal, irredundant join representations, say $a = \bigvee B$ and $a = \bigvee C$. Then, by the minimality and non-refinement of each, $a > \bigvee\{b \wedge c : b \in B,\ c \in C\}$. This is a violation of the strong form of the join semidistributive law given above.

2. Continuity

In this section we show that the minimal join cover refinement property implies lower continuity. This has as a corollary that several types of lattices are continuous.

THEOREM 2.25. *Every lattice with the minimal join cover refinement property is lower continuous.*

Proof: Assume that $\mathbf{L}$ has the minimal join cover refinement property, and let A be a down directed set in $\mathbf{L}$ with $u = \bigwedge A$, and let $b \in L$. Suppose $c \in L$ and $c \leq a \vee b$ for all $a \in A$. Since the set of minimal join covers $\mathcal{M}(c)$ is finite, there exists $E \in \mathcal{M}(c)$ such that $E \ll \{a, b\}$ for all $a \in A$. (Otherwise, for each $F \in \mathcal{M}(c)$ we could find a_F such that F does not refine $\{a_F, b\}$. Taking $a' \in A$ to be a common lower bound of the a_F's, $\{a', b\}$ would be refined by no member of $\mathcal{M}(c)$, a contradiction.) If $e \in E$ and $e \nleq b$, then $e \leq a$ for every $a \in A$, and hence $e \leq u$. Thus $E \ll \{u, b\}$ and $c \leq \bigvee E \leq u \vee b$ and hence $b \vee \bigwedge A = \bigwedge_{a \in A} b \vee a$, as desired.

COROLLARY 2.26. *Every lower bounded lattice is lower continuous.*

COROLLARY 2.27. *All of the following types of lattices are* (*both upper and lower*) *continuous*:

(1) *free lattices* $\mathbf{FL}(X)$,
(2) *projective lattices,*
(3) *lattices* $\mathbf{FL}(\mathbf{P})$ *freely generated by an ordered set,*
(4) *finitely presented lattices.*

The proofs that projective lattices and $\mathbf{FL}(\mathbf{P})$ have the minimal join cover refinement property are in Chapter V. For finitely presented lattices, see Freese [**53**] and [**54**].

3. Doubling and Congruences on a Finite Lattice

Alan Day's doubling construction was devised as a means of correcting (W)–failures in lattices, and hence provided a simple proof that (W) holds in free lattices (in fact, projective lattices; see Chapter V). Alan always claimed there was more to the construction than that, but for a while no one else was convinced. Then, in [**24**] and [**25**], he found the connection he was looking for: finite bounded lattices are precisely the lattices which can be obtained from finite distributive lattices by doubling a sequence of intervals.

In this section we will prove this nice result in its 'lower bounded' form. Along the way we shall establish some interesting facts about congruences on finite lattices which will be used in Chapter XI to give efficient computer representation and fast algorithms for congruences. The results of this section will give us a means of constructing lots of examples of lower bounded lattices. We need to recall from Chapter I that a *lower pseudo-interval* in a lattice is a union of intervals with a common least element, $C = \bigcup_{i \in I} a_i/b$. *Upper pseudo-intervals* are defined dually. Of course, either type of pseudo-interval is a convex set, and hence the doubling construction may be applied to it to form the lattice $\mathbf{L}[C]$.

LEMMA 2.28. *If* $\mathbf{L}$ *is a finite lower bounded lattice and* C *is a lower pseudo-interval in* $\mathbf{L}$*, then* $\mathbf{L}[C]$ *is also lower bounded.*

Proof: By Lemma 2.5, it suffices to show that $\mathrm{J}(\mathbf{L}[C]) \subseteq \mathrm{D}(\mathbf{L}[C])$, given that $\mathrm{J}(\mathbf{L}) \subseteq \mathrm{D}(\mathbf{L})$. It is easy to see that there are three types of join irreducibles in $\mathbf{L}[C]$:

$$\mathrm{J}(\mathbf{L}[C]) = J_{-1} \cup J_0 \cup J_1$$

with

$$\begin{aligned} J_{-1} &= \mathrm{J}(\mathbf{L}) - C, \\ J_0 &= \{\langle p, 0\rangle : p \in \mathrm{J}(\mathbf{L}) \cap C\}, \\ J_1 &= \{\langle b, 1\rangle\}, \end{aligned}$$

where b is the unique minimum element of C.

Now the canonical map $\lambda : \mathbf{L}[C] \twoheadrightarrow \mathbf{L}$ is bounded (as is every homomorphism between finite lattices), and for any $p \in \mathrm{J}(\mathbf{L})$, $\beta_\lambda(p) \in J_{-1} \cup J_0$. Using this, it is not hard to show that if $p \in J_{-1} \cup J_0$ and $p \le \bigvee B$, then there exists $A \subseteq J_{-1} \cup J_0$ with $A \ll B$ and $p \le \bigvee A$; in fact, $A = \{\beta_\lambda(b) : b \in B\}$ works. This allows us to prove by induction simultaneously that

(1) if $p \in J_{-1} \cap \mathrm{D}_j(\mathbf{L})$, then $p \in \mathrm{D}_j(\mathbf{L}[C])$, and
(2) if $\langle p, 0\rangle \in J_0$ and $p \in \mathrm{D}_j(\mathbf{L})$, then $\langle p, 0\rangle \in \mathrm{D}_j(\mathbf{L}[C])$.

Since $\mathbf{L}$ is finite, for some $n \in \omega$ we have $\mathrm{J}(\mathbf{L}) \subseteq \mathrm{D}_n(\mathbf{L})$, and thus $J_{-1} \cup J_0 \subseteq \mathrm{D}_n(\mathbf{L}[C])$.

But there is only one more element in $\mathrm{J}(\mathbf{L}[C])$, *viz.* $\langle b, 1\rangle$, and it is evident that any minimal nontrivial join cover of $\langle b, 1\rangle$ must be contained in $J_{-1} \cup J_0$. Thus $\langle b, 1\rangle \in \mathrm{D}_{n+1}(\mathbf{L}[C])$, and $\mathbf{L}[C]$ is lower bounded.

The case when $\mathbf{L}$ is both upper and lower bounded, and C is an interval, is particularly important.

COROLLARY 2.29. *If* $\mathbf{L}$ *is a finite bounded lattice and* C *is an interval in* $\mathbf{L}$*, then* $\mathbf{L}[C]$ *is also bounded.*

We would like to show, conversely, that every finite lower bounded lattice can be obtained by a sequence of doublings, starting from a one-element lattice. This requires a little analysis of how congruences work on finite lattices. Rather than take the shortest direct path to the converse, we will travel the scenic route, and thereby gain a substantial amount of useful new information about finite lattices in general, and finite lower bounded lattices in particular.

If $\mathbf{L}$ is a finite lattice and u is a join irreducible element of $\mathbf{L}$, then u has a unique lower cover, which we will denote by u_*. Dually, if p is meet irreducible in $\mathbf{L}$, then the unique upper cover of p is denoted by p^*. Pairs of the form $\langle u, u_* \rangle$ with $u \in \mathrm{J}(\mathbf{L})$ play a special role in the theory of congruences. (We could equally well use the pairs $\langle p, p^* \rangle$ with $p \in \mathrm{M}(\mathbf{L})$.) This is reflected in a basic theorem due to R. P. Dilworth [**40**].

Theorem 2.30. *Let* $\mathbf{L}$ *be a finite lattice. A congruence* θ *is join irreducible in* $\mathbf{Con\, L}$ *if and only if* $\theta = \mathrm{Cg}\,(u, u_*)$ *for some* $u \in \mathrm{J}(\mathbf{L})$. *A congruence* ψ *is meet irreducible if and only if, for some* $u \in \mathrm{J}(\mathbf{L})$, ψ *is the unique largest congruence separating* u *and* u_*.

The correspondence between join irreducible congruences and join irreducible elements of $\mathbf{L}$ is in general not one-to-one, e.g., as in $\mathbf{M}_3$. Without reconstructing the proof of Theorem 2.30, it is perhaps worthwhile to recall the most elementary part of it, which is what we need. Clearly any congruence relation on a finite lattice is determined by the covering pairs which it collapses. Assuming $a \succ b$ and $a\,\theta\,b$ with $\theta \in \mathrm{Con}\,\mathbf{L}$, choose u minimal in $\mathbf{L}$ such that $u \leq a$ and $u \nleq b$. Then $u \in \mathrm{J}(\mathbf{L})$, and u/u_* is projective with a/b, so in particular $u\,\theta\,u_*$. Thus every congruence relation on a finite lattice is determined by the set of pairs $\langle u, u_* \rangle$ with $u \in \mathrm{J}(\mathbf{L})$ which it collapses.

To analyze this connection further, we introduce a *join dependency relation* D on the set of elements of a (possibly infinite) lattice $\mathbf{L}$, defined by

$$\begin{aligned} a\ D\ b \quad & \text{if } a \neq b,\ b \text{ is join irreducible, and there is} \\ & \text{a } p \in L \text{ with } a \leq b \sqcup p \text{ and } a \nleq c \vee p \text{ for } c < b. \end{aligned}$$

In this case we will say that $a\ D\ b$ *via* p, or a depends on b *via* p. If b has a lower cover b_*, for example if $\mathbf{L}$ is finite, the last part of the definition can be replaced with '$a \nleq b_* \vee p$.' The *meet dependency relation* D^d is defined dually.[3]

The join dependency relation is easily understood in terms of minimal nontrivial join covers.

[3]The original definition of $a\ D\ b$ in Day [**25**] included the additional assumption that a is join irreducible. Indeed, when one is working with congruences, it is the restriction of D to $\mathrm{J}(\mathbf{L})$ that is important. But for many of the results in Chapters III and XII, we will find the present definition more useful.

LEMMA 2.31. *Let* $\mathbf{L}$ *be a lattice with the minimal join cover refinement property. For an element* a *of* $\mathbf{L}$ *we have* $a\ D\ b$ *if and only if* b *belongs to a minimal nontrivial join cover of* a.

Proof: If b is in a minimal nontrivial join cover U of a, we can take $p = \bigvee(U - \{b\})$ and check that $a\ D\ b$. On the other hand, if $a\ D\ b$ *via* p, the join cover $\{b, p\}$ of a can be refined to a minimal nontrivial join cover U, and it is easy to see that b belongs to the set U, as follows. Since $U \ll \{b, p\}$, we have $\bigvee U \leq c \vee p$ where $c = \bigvee\{u \in U : u \leq b\}$. As $a \leq \bigvee U$, the definition of D implies $c = b$. Because b is join irreducible and U is finite, that yields $b \in U$, as desired.

Before proceeding further, we pause for an elementary, but important, lemma.

LEMMA 2.32. *Let* $\mathbf{L}$ *be a finite lattice and fix* $\theta \in \mathbf{Con\,L}$. *Let*

$$R_\theta = \{a \in \mathrm{J}(\mathbf{L}) : a\,\theta\,a_*\} = \{a \in \mathrm{J}(\mathbf{L}) : \mathrm{Cg}\,(a, a_*) \leq \theta\},$$
$$S_\theta = \mathrm{J}(\mathbf{L}) - R_\theta \ = \ \{a \in \mathrm{J}(\mathbf{L}) : \langle a, a_* \rangle \notin \theta\}.$$

Then for all u, $v \in L$, $u\,\theta\,v$ *if and only if* $u/0 \cap S_\theta = v/0 \cap S_\theta$.

Proof: First assume $u\,\theta\,v$, and let $c \in u/0 \cap S_\theta$. Then $c = c \wedge u\ \theta\ c \wedge v$. Since $\langle c, c_* \rangle \notin \theta$, this implies $c \leq v$, whence $c \in v/0 \cap S_\theta$. Thus $u/0 \cap S_\theta \subseteq v/0 \cap S_\theta$, and by symmetry they are equal.

Conversely, suppose $\langle u, v \rangle \notin \theta$. Then one of the pairs $\langle u, u \wedge v \rangle$ or $\langle v, u \wedge v \rangle$ is not related by θ, say the former. As L is finite, we can find s, $t \in L$ with $u \geq s \succ t \geq u \wedge v$ and $\langle s, t \rangle \notin \theta$. Let c be minimal in L such that $c \leq s$ and $c \not\leq t$. As before, we find that $c \in \mathrm{J}(\mathbf{L})$ and that c/c_* is projective to s/t, whence $\langle c, c_* \rangle \notin \theta$. Also, $c \leq s \leq u$, while $c \not\leq v$ because $c \not\leq t$. Hence $c \in (u/0 \cap S_\theta) - (v/0 \cap S_\theta)$, and so they are not equal.

Next, we need a result originally found in Jónsson and Nation [**83**]. (A refined version will appear later in Theorem 3.15.)

LEMMA 2.33. *Let* $\mathbf{L}$ *be a finite lattice, and let* T *be a set of join irreducible elements of* $\mathbf{L}$. *The following are equivalent.*

(1) *For some* $\theta \in \mathrm{Con}\,\mathbf{L}$, $T = R_\theta = \{c \in \mathrm{J}(\mathbf{L}) : c\,\theta\,c_*\}$.
(2) *If* a *is join irreducible,* $a\ D\ b$ *and* $b \in T$, *then* $a \in T$.

Proof: Fix $\theta \in \mathrm{Con}\,\mathbf{L}$, and suppose $T = R_\theta$. Suppose a, $b \in \mathrm{J}(\mathbf{L})$ with $b \in T$ and $a\ D\ b$ *via* p, i.e., $a \leq b\ \sqcup\ p$ but $a \not\leq b_*\ \sqcup\ p$. Then we claim $\langle a, a_* \rangle \in \theta$, because

$$a = a \wedge (b\ \sqcup\ p)\ \theta\ a \wedge (b_*\ \sqcup\ p) \leq a_*.$$

Hence $a \in T$, and condition (2) holds.

Conversely, let $T \subseteq \mathrm{J}(\mathbf{L})$ satisfy (2). We want to construct a congruence relation θ on $\mathbf{L}$ such that, for $c \in \mathrm{J}(\mathbf{L})$, $c\,\theta\,c_*$ holds if and only if $c \in T$. Lemma 2.32 tells us exactly how to do this.

Accordingly, let $S = \mathrm{J}(\mathbf{L}) - T$, and note that S has the property that $d \in S$ whenever $c\,D\,d$ and $c \in S$. Define

$$u\,\theta\,v \quad \text{if} \quad u/0 \cap S = v/0 \cap S.$$

It is immediate that for $c \in \mathrm{J}(\mathbf{L})$ we have $c\,\theta\,c_*$ if and only if $c \notin S$, i.e., $c \in T$. So if we can show that θ is a congruence relation, then we will have condition (1), the desired conclusion. It should be clear that θ is an equivalence relation which preserves meets: $u\,\theta\,v$ implies $u \wedge z\,\theta\,v \wedge z$. It remains to show that θ respects joins.

Assume $u\,\theta\,v$, and let $z \in L$. We want to show $(u \sqcup z)/0 \cap S \subseteq (v \sqcup z)/0 \cap S$, so let $c \in S$ and $c \leq u \sqcup z$. If $c \leq u$ or $c \leq z$, then (using $u\,\theta\,v$ in the former case) we get $c \leq v \sqcup z$, which is what we want. So suppose $c \nleq u$ and $c \nleq z$. Then there exists a minimal nontrivial join cover Q of c with $Q \ll \{u, z\}$. If $q \in Q$ and $q \leq z$, then of course $q \leq v \sqcup z$. Otherwise $q \leq u$, and since $c \in S$ and $c\,D\,q$ (by Lemma 2.31), we have $q \in S$. Thus $q \in u/0 \cap S = v/0 \cap S$, so $q \leq v \leq v \sqcup z$. It follows that $p \leq \bigvee Q \leq v \sqcup z$. This shows $(u \sqcup z)/0 \cap S \subseteq (v \sqcup z)/0 \cap S$; by symmetry, they are equal. Hence $u \sqcup z\,\theta\,v \sqcup z$.

COROLLARY 2.34. *Let* $\mathbf{L}$ *be a finite lattice, and let* U *be a set of join irreducible elements of* $\mathbf{L}$. *Then* $U = S_\theta$ *for some* $\theta \in \operatorname{Con}\mathbf{L}$ *if and only if* $b \in U$ *whenever* $a\,D\,b$ *and* $a \in U$.

Lemma 2.33 gives us a nice construction for the congruence lattice of a finite lattice. Let $\trianglelefteq$ denote the reflexive and transitive closure of D restricted to $\mathrm{J}(\mathbf{L})$. Then $\trianglelefteq$ is a quasiorder (reflexive and transitive), and so it induces an equivalence relation $\equiv$ on $\mathrm{J}(\mathbf{L})$, modulo which $\trianglelefteq$ is a partial order, *viz.*, $a \equiv b$ if and only if $a \trianglelefteq b$ and $b \trianglelefteq a$. If we let $\mathbf{Q_L}$ denote the ordered set $\langle \mathrm{J}(\mathbf{L})/\equiv, \trianglelefteq \rangle$, then the lemma translates as follows.

THEOREM 2.35. *If* $\mathbf{L}$ *is a finite lattice, then* $\mathbf{Q_L} \cong \mathrm{J}(\mathbf{Con\ L})$ *and thus* $\mathbf{Con\ L}$ *is isomorphic to the lattice of order ideals of* $\mathbf{Q_L}$.

In Chapter XI we will give an efficient procedure to find $\mathbf{Q_L}$ and thus $\mathbf{Con\ L}$. This procedure is used to give fast tests for subdirect irreducibility and other properties.

Now every congruence $\theta \in \operatorname{Con}\mathbf{L}$ is the join of all join irreducible congruences $\operatorname{Cg}(b, b_*)$ with $b\,\theta\,b_*$. We would like to know, for an arbitrary pair of join irreducible elements a, $b \in \mathrm{J}(\mathbf{L})$, when $\operatorname{Cg}(a, a_*) \leq \operatorname{Cg}(b, b_*)$ holds. The next lemma answers this.

Lemma 2.36. *Let* $\mathbf{L}$ *be a finite lattice and let* a, $b \in \mathrm{J}(\mathbf{L})$. *Then we have* $\mathrm{Cg}\,(a, a_*) \leq \mathrm{Cg}\,(b, b_*)$ *if and only if* $a \trianglelefteq b$, *i.e., either* $a = b$ *or there exist* $c_1, \ldots, c_{k-1} \in \mathrm{J}(\mathbf{L})$ *such that*

$$(\dagger) \qquad a \; D \; c_1 \; D \ldots D \; c_{k-1} \; D \; b.$$

Proof: The set $R_{\mathrm{Cg}\,(b,b_*)} = \{a \in \mathrm{J}(\mathbf{L}) : \langle a, a_* \rangle \in \mathrm{Cg}\,(b, b_*)\}$ is the smallest subset T of $\mathrm{J}(\mathbf{L})$ such that $b \in T$ and condition (2) of Lemma 2.33 holds. The condition (†) clearly describes exactly those elements.

For example, we obtain a nice characterization of finite subdirectly irreducible lattices, which will be used in Chapter III.

Corollary 2.37. *A finite lattice* $\mathbf{L}$ *is subdirectly irreducible if and only if there exists* $a \in \mathrm{J}(\mathbf{L})$ *such that, for all* $b \in \mathrm{J}(\mathbf{L})$, *there exist* $c_1, \ldots, c_{k-1}$ *such that* (†) *holds.*

Now let us again restrict our attention to lower bounded lattices. In the first place, we can now formulate a very useful version of the condition $\mathrm{D}(\mathbf{L}) = L$.

By a *D–sequence* in a lattice $\mathbf{L}$, we mean a sequence a_i $(i < n)$ of join irreducible elements of $\mathbf{L}$ such that $a_i \; D \; a_{i+1}$ for all i, where $2 \leq n \leq \omega$. A D–sequence is *infinite* if $n = \omega$. The elements of a D–sequence are not required to be distinct (although $a_i \neq a_{i+1}$ by the definition of D), so we define a *D–cycle* to mean a finite sequence $a_0, \ldots, a_{n-1}$ $(n \geq 2)$ of join irreducible elements of $\mathbf{L}$ such that $a_i \; D \; a_{i+1}$ for all $i < n$, where the subscripts are computed modulo n. A D–cycle can obviously be regarded as a special type of infinite D–sequence (even though the indexing is different). The notions of D^d-sequence and D^d–cycle are of course defined analogously.

Theorem 2.38. *A finitely generated lattice* $\mathbf{L}$ *is lower bounded if and only if it contains no infinite D–sequence and satisfies the minimal join cover refinement property.*

Proof: First assume that $\mathbf{L}$ is lower bounded so $\mathrm{D}(\mathbf{L}) = L$ by Theorem 2.13. By Lemma 2.19, $\mathbf{L}$ has the minimal join cover refinement property. If a, $b \in \mathrm{J}(\mathbf{L})$ and $a \; D \; b$, then b is in a minimal nontrivial join cover of a, so $a \in \mathrm{D}_n(\mathbf{L})$ implies $b \in \mathrm{D}_{n-1}(\mathbf{L})$. Hence a D–sequence starting with a can contain at most $\rho(a) + 1$ elements, where $\rho(a)$ denotes the D–rank of a.

Conversely, assume that $\mathbf{L}$ satisfies the minimal join cover refinement property, but is not lower bounded so $\mathrm{D}(\mathbf{L}) \neq L$. By Lemma 2.5, this implies $\mathrm{J}(\mathbf{L}) \nsubseteq \mathrm{D}(\mathbf{L})$, so we can choose an element $b_0 \in \mathrm{J}(\mathbf{L}) - \mathrm{D}(\mathbf{L})$. Now for any element $b \in \mathrm{J}(\mathbf{L})$, the set $\mathcal{M}(b)$ is finite. So given $b_i \in \mathrm{J}(\mathbf{L}) - \mathrm{D}(\mathbf{L})$, some minimal nontrivial join cover of b_i contains an element $b_{i+1} \in \mathrm{J}(\mathbf{L}) - \mathrm{D}(\mathbf{L})$, and for this pair we have $b_i\ D\ b_{i+1}$. In this way we can construct an infinite D–sequence b_i $(i \in \omega)$.

Of course, we are particularly interested in the finite case.

COROLLARY 2.39. *A finite lattice* $\mathbf{L}$ *is lower bounded if and only if it contains no* D*–cycle.*

This corollary makes it easy to show that certain lattices are not lower bounded. For example, if we let a_i $(i < n)$ denote the atoms of the 2-dimensional modular lattice $\mathbf{M}_n$, then for $n \geq 3$ we have the D–cycle

$$a_0\ D\ a_1\ D \ldots D\ a_{n-1}\ D\ a_0\ ,$$

so $\mathbf{M}_n$ is not lower bounded. Of course, $\mathbf{M}_n$ does not satisfy (SD$_\sqcup$) for $n \geq 3$, so this could be derived from Theorem 2.20.

The lattice of convex subsets of a four element chain (Figure 2.1) provides an example of a lattice which satisfies (SD$_\sqcup$) but is not lower bounded. Lattices which satisfy both semidistributive laws, but are not lower bounded, are harder to come by. In Chapter V we will see how to construct examples with those properties, as in Figure 5.5 on page 146.

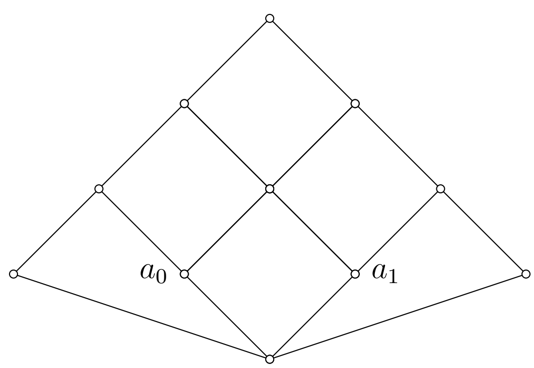


FIGURE 2.1

Combining Lemma 2.36 and Corollary 2.39, we obtain a nice property of lower bounded lattices.

LEMMA 2.40. *Let* $\mathbf{L}$ *be a finite lattice. Then* $\mathbf{L}$ *is lower bounded if and only if the correspondence between join irreducible elements of* $\mathbf{L}$ *and join irreducible congruences on* $\mathbf{L}$ *is one-to-one, i.e.,* $a \neq b$ *implies* $\mathrm{Cg}\,(a, a_*) \neq \mathrm{Cg}\,(b, b_*)$. *Thus* $\mathbf{L}$ *is lower bounded if and only if* $|\,\mathrm{J}(\mathbf{L})| = |\mathbf{Q_L}|$, *which holds if and only if* $|\,\mathrm{J}(\mathbf{L})| = |\,\mathrm{J}(\mathbf{Con\ L})|$. *It is* *upper bounded if and only if* $|\,\mathrm{M}(\mathbf{L})| = |\mathbf{Q_L}|$ *and this holds if and only if* $|\,\mathrm{M}(\mathbf{L})| = |\,\mathrm{J}(\mathbf{Con\ L})|$.

Proof: By Theorem 2.35 $|\mathbf{Q_L}| = |\,\mathrm{J}(\mathbf{Con\ L})|$ and, for a pair of join irreducible elements in any finite lattice, $\mathrm{Cg}\,(a, a_*) = \mathrm{Cg}\,(b, b_*)$ if and only if a and b are on a common D–cycle. The theorem follows from this and duality.

Actually, we have already proved more. Define the *depth* of an element x in a finite ordered set $\mathbf{Q}$ to be the length of a maximal chain in the filter generated by x. For example, maximal elements have depth 0. The proof of Theorem 2.38, when combined with Theorem 2.35, gives the following result of Day [**25**] about the structure of $\mathbf{Con\ L}$ when L is lower bounded.

THEOREM 2.41. *Let* $\mathbf{L}$ *be a finite lower bounded lattice. Then* $\mathbf{Q_L} \cong \mathrm{J}(\mathbf{Con\ L})$, *and for* $a \in \mathrm{J}(\mathbf{L})$, *the depth of* $\mathrm{Cg}\,(a, a_*)$ *in* $\mathrm{J}(\mathbf{Con\ L})$ *is equal to the* D*–rank* $\rho(a)$. *In particular, the length of* $\mathbf{Q_L}$ *is equal to the maximum of the* D*–ranks* $\rho(a)$ *for* $a \in \mathrm{J}(\mathbf{L})$.

Now we are ready to prove the crucial lemma to provide the converse to Lemma 2.28.

LEMMA 2.42. *Let* $\mathbf{L}$ *be a finite lower bounded lattice, and let* θ *be an atom in* $\mathrm{Con}\,\mathbf{L}$. *Then* $\mathbf{L}/\theta$ *is lower bounded, and there is a lower pseudo-interval* C *in* $\mathbf{L}/\theta$ *such that* $\mathbf{L} \cong \mathbf{L}/\theta[C]$.

Proof: Since finite lower bounded lattices form a pseudovariety, $\mathbf{L}/\theta$ is lower bounded. By the argument just after Theorem 2.30, there is an $a \in \mathrm{J}(\mathbf{L})$ such that $\theta = \mathrm{Cg}\,(a, a_*)$. Suppose there is an element $c \in \mathrm{J}(\mathbf{L})$ with $c\ D\ a$. As we observed in the proof of Lemma 2.33, this implies $\mathrm{Cg}\,(c, c_*) \le \mathrm{Cg}\,(a, a_*)$. Since $\mathrm{Cg}\,(a, a_*)$ is an atom, it follows that $\mathrm{Cg}\,(c, c_*) = \mathrm{Cg}\,(a, a_*)$, whence $c = a$ by Lemma 2.40. Thus $c\ D\ a$ holds for no $c \in \mathrm{J}(\mathbf{L})$. Hence by Lemma 2.36, $R_\theta = \{c \in \mathrm{J}(\mathbf{L}) : c\,\theta\,c_*\} = \{a\}$, and so $S_\theta = \mathrm{J}(\mathbf{L}) - \{a\}$. Applying Lemma 2.32, we conclude that

$$u\,\theta\,v \quad \text{iff} \quad u/0 \cap (\mathrm{J}(\mathbf{L}) - \{a\}) = v/0 \cap (\mathrm{J}(\mathbf{L}) - \{a\})\ .$$

Thus, if u and v are distinct and θ–related, then they both contain the same set of join irreducible elements except one of them, say u, contains a and the other does not. It follows that $a_* \le u \prec v = u \sqcup a$. Hence each θ–class contains only one or two elements. It follows that $a_* \le u < a \sqcup u$ implies $a \sqcup u \succ u$, because congruence classes are convex and $a_* \le u$ implies $u = a_* \sqcup u\ \theta\ a \sqcup u$.

Since $\mathbf{L}$ is finite, the natural map $\mathbf{L} \to \mathbf{L}/\theta$ is bounded, and so $\mathbf{L}/\theta$ is meet isomorphic with the meet subsemilattice $\mathbf{S} = \alpha(\mathbf{L})$ consisting of the largest member of each θ–class. In view of the preceding remarks, we can describe $\mathbf{S}$ rather easily: its universe is

$$S = \{x \in L : a_* \nleq x \text{ or } a \leq x\},$$

with the order (and meet operation) inherited from $\mathbf{L}$. Of course, $\mathbf{S}$ is a lattice, and we can identify the set, C, of those members of S which come from two-element θ–classes, i.e., the upper elements of each two element class:

$$C = \{x \in L : \text{there exists } y \in L \text{ with } a_* \leq y < x = y \sqcup a\} .$$

The element y in the description of C is uniquely determined by x, since $x \; \theta \; y$, so we can denote it by $y = \zeta(x)$. We want to show that C is a lower pseudo-interval in $\mathbf{S}$, and that $\mathbf{L} \cong \mathbf{S}[C]$.

Clearly a is the least element of C. The claim is that if $a \leq t \leq x \in C$, then $t \in C$. Certainly $a_* \leq t \wedge \zeta(x)$ and $t = t \wedge x \; \theta \; t \wedge \zeta(x)$. Hence $t \succ t \wedge \zeta(x)$, and because $a \nleq \zeta(x)$ we have $t = a \sqcup (t \wedge \zeta(x))$. Therefore $t \in C$ with $\zeta(t) = t \wedge \zeta(x)$. Thus C is convex with a unique minimal element, which makes it a lower pseudo-interval.

Now we define our mapping $h : \mathbf{L} \to \mathbf{S}[C]$ as follows.

(1) If $x \notin C \cup \zeta(C)$, then $h(x) = x$.
(2) If $x \in C$, then $h(x) = \langle x, 1\rangle$ and $h(\zeta(x)) = \langle x, 0\rangle$.

It is not hard to see that h is well-defined on L, one-to-one and onto. But we must check that the order on $\mathbf{L}$ corresponds to the order on $\mathbf{S}[C]$ given by the doubling construction, i.e., $x \leq y$ if and only if $h(x) \leq h(y)$. For the forward direction, we note that $x \leq \zeta(y)$ never holds for x, $y \in C$, because $a \leq x$ and $a \nleq \zeta(y)$. The cases which do occur are easy, because if $x \in C$ and $y \in S$, then $\zeta(x) \leq y$ implies $x = \zeta(x) \sqcup a \leq y$, and similarly $\zeta(x) \leq \zeta(y)$ implies $x \leq y$. For the reverse implication, recall that it was shown above that for t and $x \in C$ with $t \leq x$ we have $\zeta(t) = t \wedge \zeta(x)$. Hence $\langle t, 0\rangle \leq \langle x, 0\rangle$ in $\mathbf{S}[C]$ implies $\zeta(t) \leq \zeta(x)$, and the remaining cases are trivial. We conclude that h is a lattice isomorphism, as desired.

Combining Lemmas 2.28 and 2.42, we obtain Day's characterization of lower bounded lattices in terms of the doubling construction.

Theorem 2.43. *Finite lower bounded lattices are the smallest nonempty class of lattices closed under the doubling of lower pseudo-intervals. In other words, a finite lattice is lower bounded if and only if it can be obtained from a one-element lattice by a sequence of doubling constructions using lower pseudo-intervals.*

Proof: This is a straightforward induction on either the cardinality or length of **Con L**, using Lemma 2.42 for the induction step.

Naturally, the dual version of Theorem 2.43 characterizes upper bounded lattices as the smallest nonempty class closed under the doubling of upper pseudo-intervals. Even more important for our immediate purposes is the self-dual version of this theorem, which is obtained in the same way by combining Lemmas 2.28 and 2.42 and their duals.

Corollary 2.44. *Finite bounded lattices are the smallest nonempty class of lattices closed under the doubling of intervals. Thus a finite lattice is bounded if and only if it can be obtained from a one-element lattice by a sequence of doubling constructions using intervals.*

It is a little harder to describe the smallest class closed under doubling arbitrary convex sets, but this has been done by Winfried Geyer [**67**] (see also Day [**27**]). This class is referred to as *congruence normal* lattices. Congruence normal lattices are interesting, but play no role in our study of free lattices.

Producing examples is just a matter of doubling a few lower pseudo-intervals in succession. There is no need to start with a one-element lattice, because we know that every finite distributive lattice is lower bounded. (Distributive lattices can be obtained from a one-element lattice by doubling only filters.) Figure 2.2 shows one such sequence, and Figure 2.3 shows another. The doubled lower pseudo-intervals are indicated by the solid points. In the second sequence we have doubled only intervals, so all the lattices in Figure 2.3 are in fact bounded.

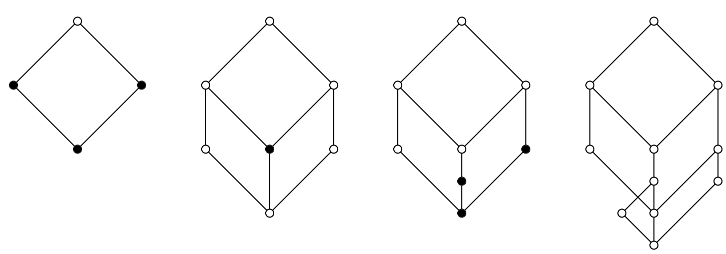


Figure 2.2

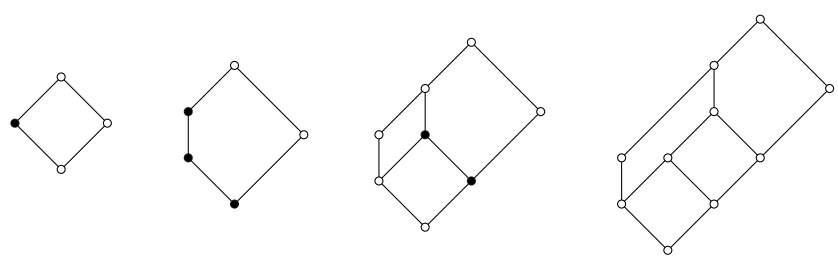


Figure 2.3

4. A Refinement of the D Relation

In practice, it sometimes turns out to be useful to consider relations which may be properly contained in the D relation. In particular, this will be the case for both free lattices and finite semidistributive lattices. In order to include both of these applications, throughout this section, $\mathbf{L}$ will be a lattice with the minimal join cover refinement property. (Recall that every finite lattice has this property.)

For $a \in L$, $\mathcal{M}(a)$ denotes the set of minimal nontrivial join covers of a. Let $\mathcal{M}^*(a)$ denote the set of those $U \in \mathcal{M}(a)$ such that $\bigvee U$ is minimal, i.e., there does not exist $V \in \mathcal{M}(a)$ with $\bigvee V < \bigvee U$. Thus the members of $\mathcal{M}^*(a)$ are minimal in two senses: refinement and their join is minimal. Of course, it follows that if $U \in \mathcal{M}^*(a)$, then $\bigvee V \not< \bigvee U$ for any nontrivial join cover V of a.

As before, for p, $q \in L$ we have $p \mathrel{D} q$ if $q \in Q$ for some $Q \in \mathcal{M}(p)$. Define $p \mathrel{E} q$ if $q \in Q$ for some $Q \in \mathcal{M}^*(p)$. Clearly $E \subseteq D$. Our goal is to clarify the roles of D, E, and the relation $C = A \cup B$ (defined in Section 5) which is special to semidistributive lattices.

To begin with, it is *not* true that every join cover in $\mathcal{M}(a)$ refines to one in $\mathcal{M}^*(a)$. Nor can $\mathcal{M}(a)$ be replaced by $\mathcal{M}^*(a)$ in the formula for calculating $\beta_{k+1}(a)$ from Theorem 2.4,

$$\beta_{k+1}(a) = \beta_0(a) \wedge \bigwedge_{\bigvee S \leq h(1_{\mathbf{K}})}^{S \in \mathcal{M}(a)} \bigvee_{s \in S} \beta_k(s).$$

Both these statements can be seen in Figure 2.4. There we have $\mathcal{M}^*(a) = \{\{b, d\}\}$, which does not refine $\{b, c\} \in \mathcal{M}(a)$. Hence $a \mathrel{D} c$ but not $a \mathrel{E} c$. Also, if we use $\mathcal{M}^*(a)$ in place of $\mathcal{M}(a)$ in the above formula, the value $\beta_1(a)$ is not correct. However, the formula using $\mathcal{M}^*(a)$ does calculate the correct value of $\beta(a)$, as is shown by Theorem 2.47 below.

For a lattice with the minimal join cover refinement property, there are two ways of looking at $\mathrm{D}_k(\mathbf{L})$. For $p \in L$, we have

(1) $p \in \mathrm{D}_k(\mathbf{L})$ iff $Q \in \mathcal{M}(p)$ implies $Q \subseteq \mathrm{D}_{k-1}(\mathbf{L})$, or equivalently,
(2) $p \in \mathrm{D}_k(\mathbf{L})$ iff $p \mathrel{D} q$ implies $q \in \mathrm{D}_{k-1}(\mathbf{L})$.

We define $\rho_D(p)$ to be the least nonnegative integer k such that $p \mathrel{D} q$ implies $\rho_D(q) < k$ if such a k exists; otherwise (i.e., if $p \notin \mathrm{D}(\mathbf{L})$) $\rho_D(p)$ is undefined. Note that this agrees with our previous definition of $\rho(p)$ as the least integer n such that $p \in \mathrm{D}_n(\mathbf{L})$. The degenerate case of these definitions makes $\mathrm{D}_0(\mathbf{L})$ the set of join prime elements, and gives these elements rank 0.

Correspondingly, we define $\mathrm{E}_0(\mathbf{L})$ to be the set of join prime elements, and $p \in \mathrm{E}_k(\mathbf{L})$ iff $p \mathrel{E} q$ implies $q \in \mathrm{E}_{k-1}(\mathbf{L})$. Likewise,

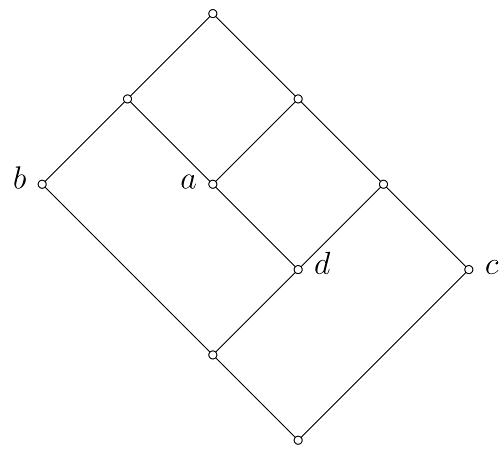


FIGURE 2.4

$\rho_E(p) = 0$ if p is join prime, and $\rho_E(p)$ is the least integer such that $p \mathrel{E} q$ implies $\rho_E(q) < k$ if such exists, otherwise $\rho_E(p)$ is undefined. As before, this makes $\rho_E(p)$ the least integer n such that $p \in \mathrm{E}_n(\mathbf{L})$.

We want to show that $\mathrm{E}_k(\mathbf{L}) = \mathrm{D}_k(\mathbf{L})$ for all k, and that $\rho_E = \rho_D$.

LEMMA 2.45. *Let* $\mathbf{L}$ *be a lattice with the minimal join cover refinement property. If* $p \in L - \mathrm{D}_k(\mathbf{L})$, *then there exists* $q \in L - \mathrm{D}_{k-1}(\mathbf{L})$ *with* $p \mathrel{E} q$.

Proof: In the statement of the theorem, take $\mathrm{D}_{-1}(\mathbf{L}) = \emptyset$. The case $k = 0$ is easy, using of course the minimal join cover refinement property. So let $k > 0$. Since $p \notin \mathrm{D}_k(\mathbf{L})$, there exists $Q \in \mathcal{M}(p)$ with $Q \nsubseteq \mathrm{D}_{k-1}(\mathbf{L})$. If $Q \in \mathcal{M}^*(p)$ we are done. Otherwise, there exists $R \in \mathcal{M}^*(p)$ with $\bigvee R < \bigvee Q$, and we may assume $R \subseteq \mathrm{D}_{k-1}(\mathbf{L})$. For each $r \in R$, either $r \leq q$ for some $q \in Q$, in which case we can take $S_r = \{q\}$, or there exists $S_r \in \mathcal{M}(r)$ with $S_r \ll Q$. In the latter case, of course $S_r \subseteq D_{k-2}(\mathbf{L})$. Now $\bigcup_{r \in R} S_r \ll Q$ and $p \leq \bigvee R \leq \bigvee(\bigcup_{r \in R} S_r)$. Hence by the minimality of Q we have $Q \subseteq \bigcup_{r \in R} S_r \subseteq \mathrm{D}_{k-1}(\mathbf{L})$, a contradiction.

THEOREM 2.46. *Let* $\mathbf{L}$ *be a lattice with the minimal join cover refinement property. Then* $\mathrm{E}_k(\mathbf{L}) = \mathrm{D}_k(\mathbf{L})$ *for all* k*, and hence* $\rho_E = \rho_D$.

Proof: For $k = 0$ this is easy, so let $k > 0$ and assume $\mathrm{E}_{k-1}(\mathbf{L}) = \mathrm{D}_{k-1}(\mathbf{L})$. Because $E \subseteq D$ we have $\mathrm{D}_k(\mathbf{L}) \subseteq \mathrm{E}_k(\mathbf{L})$, while the lemma shows that $L - \mathrm{D}_k(\mathbf{L}) \subseteq L - \mathrm{E}_k(\mathbf{L})$, which is the reverse inclusion.

In view of this result, we can drop the subscripts and return to the original notation $\rho(a)$. Similarly, we will only refer to $\mathrm{D}_k(\mathbf{L})$.

The next theorem shows that it *is* sufficient to use $\mathcal{M}^*(a)$ in calculating β for a lower bounded lattice, even though the intermediate steps of the limit table may not be correct, as illustrated above.

THEOREM 2.47. *Let* $\mathbf{K}$ *be generated by the finite set* X*, and let* $h : \mathbf{K} \to \mathbf{L}$ *be a lattice homomorphism, where* $\mathbf{L}$ *has the minimal join cover refinement property. For* $a \le h(1_{\mathbf{K}})$ *define*

$$\gamma_0(a) = \bigwedge\{x \in X : h(x) \ge a\}$$
$$\gamma_{k+1}(a) = \gamma_0(a) \wedge \bigwedge_{\bigvee S \le h(1_{\mathbf{K}})}^{S \in \mathcal{M}^*(a)} \bigvee_{s \in S} \gamma_k(s).$$

If $a \in \mathrm{D}_k(\mathbf{L})$*, then* $\gamma_k(a) = \beta(a)$.

Proof: For each $a \in \mathrm{D}(\mathbf{L}) \cap (h(1_{\mathbf{K}})/0)$, Corollary 2.8 and the fact that β preserves joins give

$$\beta(a) = \bigwedge\{x \in X : h(x) \ge a\} \wedge \bigwedge_{\bigvee S \le h(1_{\mathbf{K}})}^{S \in \mathcal{M}^*(a)} \bigvee_{s \in S} \beta(s).$$

Using that, a straightforward induction shows that $\gamma_k(a) = \beta(a)$ for $a \in \mathrm{D}(\mathbf{L}) \cap (h(1_{\mathbf{K}})/0)$.

Now with Corollary 2.34 in mind, and looking ahead to Theorem 3.15, we define a subset $A \subseteq L$ to be D*–closed* if $p \mathrel{D} q$ and $p \in A$ implies $q \in A$. Similarly, define $A \subseteq L$ to be E–closed if $p \mathrel{E} q$ and $p \in A$ implies $q \in A$.

The significance of D–closed sets is contained in the following easily proved observation, which is really the critical property of D–closed sets (see Section 2 of Chapter III).

LEMMA 2.48. *Let* $\mathbf{L}$ *be a lattice with the minimal join cover refinement property. A subset* A *of* L *is* D*–closed if and only if for each* $a \in A$*, every join cover of* a *can be refined to a join cover contained in* A.

The next lemma shows that D–closed sets have another nice closure property.

LEMMA 2.49. *Let A be a D–closed set, and let $b = a_1 \vee \cdots \vee a_n$ with $a_i \in A$ for each i. If $b = \bigvee U$ is a minimal join representation, then $U \subseteq A$.*

Proof: Fix $u \in U$. As u is join irreducible, there exists an i such that $a_i \nleq c \vee \bigvee(U - \{u\})$ whenever $c < u$. Hence $a_i \; D \; u$, which implies $u \in A$.

Now let $\mathbf{L}$ be a lattice with the minimal join cover refinement property, and let $p \in L$. Let $\mathrm{J}(p)$ denote the smallest D–closed set containing every minimal join representation of p (including $\{p\}$ if it is join irreducible). Note that $\mathrm{J}(p) \subseteq \mathrm{J}(\mathbf{L})$, and that a set A is D–closed if and only if $p \in A$ implies $\mathrm{J}(p) \subseteq A$. Also, by the preceding lemma, if a D–closed set contains one minimal join representation of p then it contains them all.

For lattices with the minimal join cover refinement property and $\mathrm{D}(\mathbf{L}) = L$, we can describe $\mathrm{J}(p)$ inductively as follows.

$$\mathrm{J}(p) = \begin{cases} \{p\} & \text{if } p \text{ is join prime,} \\ \{p\} \cup \bigcup_{p \, D \, q} \mathrm{J}(q) & \text{if } p \text{ is join irreducible but not join prime,} \\ \bigcup_i \mathrm{J}(p_i) & \text{if } p = \bigvee p_i \text{ canonically.} \end{cases}$$

Using Theorem 2.21 and the fact that $p \; D \; q$ implies $\rho(q) < \rho(p)$, it is easy to check that this description makes sense and is correct.

Analogously, let $\mathrm{J}^*(p)$ be the smallest E–closed set containing every minimal join representation of p. For lattices with the minimal join cover refinement property and $\mathrm{D}(\mathbf{L}) = L$, in view of Theorem 2.46, we again have an inductive description of $\mathrm{J}^*(p)$, obtained by replacing D with E and J with J^* in the formula above, or, to write it in exactly the form in which it will be used,

$$\mathrm{J}^*(p) = \begin{cases} \{p\} & \text{if } p \text{ is join prime,} \\ \{p\} \cup \bigcup_{Q \in \mathcal{M}^*(p)} \bigcup_{q \in Q} \mathrm{J}^*(q) & \text{if } p \text{ is join irreducible but not join prime,} \\ \bigcup_i \mathrm{J}^*(p_i) & \text{if } p = \bigvee p_i \text{ canonically.} \end{cases}$$

We want to show that these notions are equivalent for lower bounded lattices.

THEOREM 2.50. *If a lattice $\mathbf{L}$ has the minimal join cover refinement property and $\mathrm{D}(\mathbf{L}) = L$, then every join cover of p in $\mathbf{L}$ refines to one contained in $\mathrm{J}^*(p)$.*

Proof: We induct on the D–rank of p. If $\rho(p) = 0$, all join covers are trivial and there is nothing to prove.

Let $\rho(p) = k > 0$, and suppose $p \leq \bigvee Q$ nontrivially. If $Q \in \mathcal{M}^*(p)$ we are done. Otherwise, there exists $R \in \mathcal{M}^*(p)$ with $\bigvee R < \bigvee Q$. Of course $R \subseteq \mathrm{D}_{k-1}(\mathbf{L})$. For each $r \in R$, either $r \leq q$ for some $q \in Q$, in which case we can take $S_r = \{q\}$, or else by induction there exists $S_r \subseteq \mathrm{J}^*(r)$ with $S_r \ll Q$ and $r \leq \bigvee S_r$. In either case, $S_r \subseteq \mathrm{J}^*(p)$. Now $\bigcup_{r \in R} S_r \ll Q$ and $p \leq \bigvee R \leq \bigvee(\bigcup_{r \in R} S_r)$, as desired.

THEOREM 2.51. *If a lattice* $\mathbf{L}$ *has the minimal join cover refinement property and* $\mathrm{D}(\mathbf{L}) = L$*, then the following conditions are equivalent for a subset* A *of* L*.*

(1) *A is D–closed.*
(2) *If $p \in A$, then $\mathrm{J}(p) \subseteq A$.*
(3) *A is E–closed.*
(4) *If $p \in A$, then $\mathrm{J}^*(p) \subseteq A$.*

In particular, for $p \in L$ we have $\mathrm{J}(p) = \mathrm{J}^*(p)$.

Proof: In any lattice with the minimal join cover refinement property, (1) and (2) are equivalent, (3) and (4) are equivalent, and (1) implies (3). The last theorem says that (4) implies (1) when $\mathrm{D}(\mathbf{L}) = L$ also holds.

In an arbitrary finite lattice, an E–closed set need not be D–closed; an example is easily found in the finite modular lattice $\mathbf{M}_{3,3}$. It would be interesting to know whether this is true in finite semidistributive lattices.

For an element w in a free lattice, it is easy to determine $\mathrm{J}(w)$ from the canonical form of w. This formula is given on page 95; in fact, it gives $\mathrm{J}^*(w)$, but $\mathrm{J}(w) = \mathrm{J}^*(w)$ in a free lattice by the preceding theorem. Even though finitely presented lattices need not satisfy $\mathrm{D}(\mathbf{L}) = L$, Freese has extended the notion of canonical form to these in such a way that if $\mathbf{L}$ is finitely presented and $w \in L$, then $\mathrm{J}(w)$ can be found from the canonical form of w. This is sketched briefly in Chapter XI, Section 9, and given in detail in [**53**]. Thus, in practice, to construct D–closed sets in free or finitely presented lattices, we use the condition '$w \in A$ implies $\mathrm{J}(w) \subseteq A$'. For this reason, it has become customary to refer to these sets as J–*closed* rather than D–closed. With the proof of Theorem 2.51 under our belts, we shall adopt this change in terminology.

The next pair of results show that it suffices to use E to determine lower boundedness. This will prove to be particularly useful when we investigate finite sublattices of free lattices.

LEMMA 2.52. *Let* $\mathbf{L}$ *be a lattice with the minimal join cover refinement property. If $p \in L - \mathrm{D}(\mathbf{L})$, then there exists $r \in L - \mathrm{D}(\mathbf{L})$ such that $p \mathrel{E} r$.*

Proof: Let $Q \in \mathcal{M}(p)$ with $Q \nsubseteq \mathrm{D}(\mathbf{L})$. Again, if $Q \in \mathcal{M}^*(p)$ we are done, so assume $R \in \mathcal{M}^*(p)$ with $\bigvee R < \bigvee Q$. If $R \subseteq \mathrm{D}(\mathbf{L})$, then the argument of Theorem 2.50 provides a refinement of Q contained in $\mathrm{D}(\mathbf{L})$, a contradiction. Hence we have $p\ E\ r$ for some $r \in R - \mathrm{D}(\mathbf{L})$.

E–sequences and E–cycles are defined analogously to D–sequences and D–cycles as sequences of join irreducible elements with $a_i\ E\ a_{i+1}$. Lemma 2.52 allows us to extend Corollary 2.39 as follows.

Corollary 2.53. *The following three conditions are equivalent for a finite lattice* $\mathbf{L}$:

(1) $\mathbf{L}$ *is lower bounded*;
(2) $\mathbf{L}$ *contains no D–cycle*;
(3) $\mathbf{L}$ *contains no E–cycle.*

Again, there is the corresponding version for infinite lattices with infinite E–sequences, extending Theorem 2.38.

5. Semidistributive Lattices

In this section we will concentrate on finite semidistributive lattices. For these lattices we will define a relation C on $\mathrm{J}(\mathbf{L})$ with $E|_{\mathrm{J}(\mathbf{L})} \subseteq C \subseteq D|_{\mathrm{J}(\mathbf{L})}$ which is particularly well behaved. Many of the same results apply more generally to completely join irreducible elements in semidistributive lattices with the minimal join cover refinement property and its dual. Of course, this includes free lattices, and in Chapter III we will put these results to good use. With that in mind, we will develop the results in the more general context whenever they will be used in that form later.

For this purpose, we need to note that a join irreducible element a in an arbitrary lattice $\mathbf{L}$ is completely join irreducible if and only if it has a unique lower cover, which we will again denote by a_*. In other words, $\bigvee\{x \in L : x < a\}$ always exists and is either a or a_*. Dually, the unique upper cover of a completely meet irreducible element b is denoted by b^*.

Now let $\mathbf{L}$ be an arbitrary lattice, and let a be a completely join irreducible element of $\mathbf{L}$. Let

$$K(a) = \{u \in L : u \geq a_* \text{ but } u \ngeq a\}.$$

Now $K(a)$ may or may not have maximal elements; of course, if $\mathbf{L}$ is finite then there is at least one maximal element, which is in general not unique. If there is a unique largest element in $K(a)$, i.e., if $\bigvee K(a)$ exists and is itself in $K(a)$, then we will denote that element by $\kappa_{\mathbf{L}}(a)$. Clearly $a \vee \kappa_{\mathbf{L}}(a)$ is the unique upper cover of $\kappa_{\mathbf{L}}(a)$, whence $\kappa_{\mathbf{L}}(a)$ is completely meet irreducible. Thus $\kappa_{\mathbf{L}}$ is a partial map from the completely join irreducible elements of $\mathbf{L}$ to the completely meet irreducible elements. The relationship between a and $\kappa_{\mathbf{L}}(a)$ is illustrated in Figure 2.5.

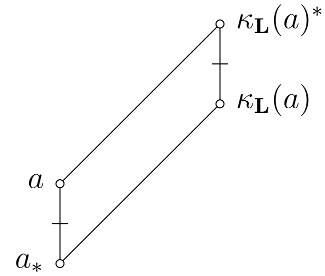


FIGURE 2.5

Dually, on the completely meet irreducible elements of $\mathbf{L}$ we have $\kappa_{\mathbf{L}}^d$ partially defined.

Recall from Theorem 2.24 that in a lattice which satisfies $(\mathrm{SD}_\vee)$ and the minimal join cover refinement property, every element has a canonical join representation. Our next result applies the dual of that theorem.

THEOREM 2.54. *Let* $\mathbf{L}$ *be a lattice satisfying* $(\mathrm{SD}_\wedge)$ *and the dual of the minimal join cover refinement property. Then* $\kappa_{\mathbf{L}}(a)$ *exists for every completely join irreducible element of* L*. Indeed,* $\kappa_{\mathbf{L}}(a)$ *is the unique canonical meetand of* a_* *not above* a*.*

Proof: Let a be completely join irreducible in $\mathbf{L}$. There must be a canonical meetand b of a_* such that $b \not\geq a$. (If a_* is meet irreducible, take $b = a_*$.) Of course, $a \wedge b = a^*$ as $a \succ a_*$. Thus the canonical meet representation C of a_* satisfies $C \gg \{a, b\}$. This means that b is the only canonical meetand not above a. If u is any element such that $u \geq a_*$ but $u \not\geq a$, then $a \wedge u = a_*$ and so, again by the refinement property of canonical representations, $b \geq u$. Hence $b = \kappa_{\mathbf{L}}(a)$ by the definition.

Corollary 2.55. *If* $\mathbf{L}$ *is semidistributive and satisfies the minimal join cover refinement property and its dual, then* $\kappa_{\mathbf{L}}$ *and* $\kappa^d_{\mathbf{L}}$ *are bijections.*

In fact, for a completely join irreducible, $\kappa_{\mathbf{L}}(a)$ is the unique completely meet irreducible element b such that $a \wedge b = a_*$ and $a \vee b = b^*$.

For finite lattices, there is a converse to Theorem 2.54.

Theorem 2.56. *A finite lattice* $\mathbf{L}$ *satisfies* $(\mathrm{SD}_\wedge)$ *if and only if* $\kappa_{\mathbf{L}}(a)$ *exists for every* $a \in \mathrm{J}(\mathbf{L})$.

Proof: One direction is given by the theorem above. Conversely, suppose that $\mathbf{L}$ fails meet semidistributivity with $d = a \wedge b = a \wedge c < a \wedge (b \vee c)$. Let e be an element of $\mathbf{L}$ which is minimal with respect to $e \leq a \wedge (b \vee c)$ but $e \nleq d$. Clearly, e is join irreducible. By the minimal property of e we have $e_* \leq d$. Using $e \leq a$, we calculate $e \wedge b = e \wedge c = e_*$ whereas $e \wedge (b \vee c) = e$, which means that $\kappa_{\mathbf{L}}(e)$ does not exist.

The following simple criterion is extremely useful.

Lemma 2.57. *Let* a *be a completely join irreducible element such that* $\kappa_{\mathbf{L}}(a)$ *exists. For any element* $b \in L$, $b \leq \kappa_{\mathbf{L}}(a)$ *if and only if* $a_* \vee b \ngeq a$.

Now again let $\mathbf{L}$ be a semidistributive lattice satisfying the minimal join cover refinement property and its dual. Define binary relations A, B, and C on the set of completely join irreducible elements of $\mathbf{L}$ by

$$\begin{aligned} &a\ A\ b \quad \text{if } b < a \text{ and } a \leq b \vee \kappa_{\mathbf{L}}(b), \\ &a\ B\ b \quad \text{if } a \neq b,\ a \leq a_* \vee b \text{ and } a \nleq a_* \vee b_*, \\ &a\ C\ b \quad \text{if either } a\ A\ b \text{ or } a\ B\ b. \end{aligned}$$

For comparison, when b is completely join irreducible the relation D can be written as

$$a\ D\ b \quad \text{if } a \neq b \text{ and there is a } p \in L \text{ with } a \leq b \vee p \text{ and } a \nleq b_* \vee p.$$

Dually, we have the relations A^d, B^d, and C^d defined on the completely meet irreducible elements of $\mathbf{L}$. These relations were introduced by Bjarni Jónsson in [**83**].

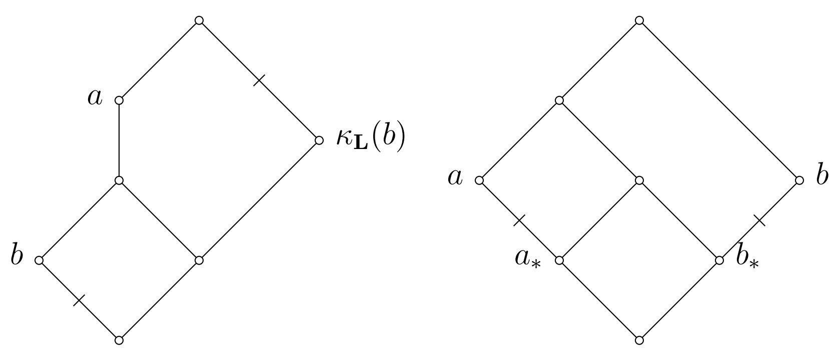
If $a\ A\ b$, then $b < a < b \vee \kappa_{\mathbf{L}}(b) = \kappa_{\mathbf{L}}(b)^*$ since a is join irreducible. Thus the elements a, b and $\kappa_{\mathbf{L}}(b)$ generate either the seven element sublattice pictured in Figure 2.6(1), or the pentagon obtained from that figure by collapsing b_* to $a \wedge \kappa_{\mathbf{L}}(b)$.

Note that by Lemma 2.57 we have $a\ B\ b$ if and only if $a \neq b$, $b_* \leq \kappa_{\mathbf{L}}(a)$ and $b \nleq \kappa_{\mathbf{L}}(a)$. Moreover, if $a\ B\ b$ then a and b are incomparable.

LEMMA 2.58. *Let a and b be completely join irreducible elements in a semidistributive lattice* $\mathbf{L}$ *satisfying the minimal join cover refinement property and its dual. If $a\ B\ b$, then $a \leq \kappa_{\mathbf{L}}(b)$.*

Proof: Suppose $a \not\leq \kappa_{\mathbf{L}}(b)$, so that $b \leq b_* \vee a$. Since $a \leq a_* \vee b$ also holds, we have $a \vee b = a \vee b_* = a_* \vee b$. Applying the strong form of $(\mathrm{SD}_\vee)$ (Theorem 1.21), we get $a \vee b = a_* \vee b_* \vee (a \wedge b) = a_* \vee b_*$, which contradicts the definition of $a\ B\ b$.

It follows that if $a\ B\ b$, then the elements a, b, a_*, and b_* generate the eight element sublattice pictured in Figure 2.6(2), or the pentagon obtained when $b_* = b_* \wedge a$.



(1) $a\ A\ b$ (2) $a\ B\ b$

FIGURE 2.6

To see that the results of the previous section apply to the relation $C = A \cup B$, we need to know the following.

THEOREM 2.59. *Let* $\mathbf{L}$ *be a semidistributive lattice which satisfies the minimal join cover refinement property and its dual. Let a be completely join irreducible in* $\mathbf{L}$.

(1) *If $a\ D\ b$, then b is completely join irreducible.*
(2) *If $a\ E\ b$, then $a\ C\ b$.*
(3) *If $a\ C\ b$, then $a\ D\ b$.*

Thus in a finite semidistributive lattice $E|_{\mathrm{J}(\mathbf{L})} \subseteq C \subseteq D|_{\mathrm{J}(\mathbf{L})}$*, and both these containments can be proper.*

Proof: Let $\mathbf{L}$ and a be as given in the statement of the theorem.

To prove (1), assume $a\ D\ b$, which means that $b \in Q$ for some $Q \in \mathcal{M}(a)$. Hence b is join irreducible. If b is not completely join irreducible, then $b = \bigvee R$ with $r < b$ for all $r \in R$. We may assume that R is an up directed set (by taking all joins of finite subsets of R). For each $r \in R$ we have $\bigvee(Q - \{b\}) \vee r \ngeq a$, and hence

$$a \wedge \left[\bigvee(Q - \{b\}) \vee r\right] \leq a_*.$$

On the other hand,

$$a \wedge \left[\bigvee(Q - \{b\}) \vee \bigvee R\right] = a \wedge \bigvee Q = a.$$

This violates the continuity of $\mathbf{L}$ (Theorem 2.25). Hence b is completely join irreducible.

Part (2) is a consequence of the next result, which plays such an important role in the structure of free lattices; see Theorem 3.4.

THEOREM 2.60. *Assume* $\mathbf{L}$ *is semidistributive and has the minimal join cover refinement property and its dual. If* a *is completely join irreducible in* $\mathbf{L}$ *and* $Q \in \mathcal{M}^*(a)$*, then* $a\ C\ q$ *for all* $q \in Q$*. In fact, there is a unique* $q_1 \in Q$ *with* $a\ B\ q_1$*, and* $a\ A\ q$ *for* $q \in Q - \{q_1\}$*.*

Proof: By $(\mathrm{SD}_\wedge)$ there exists $q_1 \in Q$ such that $a \leq a_* \vee q_1$, since we cannot have $a \wedge (a_* \vee q) = a_*$ for all $q \in Q$. Since $Q \in \mathcal{M}^*(a)$, we have $a_* \vee q_1 = \bigvee Q$, whence by $(\mathrm{SD}_\vee)$ we have $Q \ll \{q_1, a_*\}$, and thus $q < a$ for all $q \in Q - \{q_1\}$. We claim that $a\ B\ q_1$ and $a\ A\ q$ for $q \leq a_*$.

Clearly $a \neq q_1$, $a \leq a_* \vee q_1$ and $a \nleq a_* \vee r$ whenever $r < q_1$. Since q_1 is completely join irreducible by (1), this is equivalent to $a\ B\ q_1$.

Now let $q \in Q$ with $q < a$. Note that $\kappa_{\mathbf{L}}(q)$ exists since q is completely join irreducible by (1). It is easy to see that $q' \leq \kappa_{\mathbf{L}}(q)$ for all $q' \in Q - \{q\}$, for otherwise Theorem 2.57 would imply $q \leq q_* \vee q'$ for some q', whence $\bigvee Q = q_* \vee \bigvee(Q - \{q\})$, contrary to canonical form. So $q < a \leq \bigvee Q \leq q \vee \kappa_{\mathbf{L}}(q)$, whence $a\ A\ q$.

Part (3) is easy. If $a\ A\ b$, take $p = \kappa_{\mathbf{L}}(b)$. If $a\ B\ b$, take $p = a_*$. With these choices of p it is easy to check that $a\ D\ b$ holds in both cases.

To see that $E|_{\mathrm{J}(\mathbf{L})}$ can be properly contained in C, consider the lattice in Figure 2.7. It is semidistributive and has $a\ B\ b$, but $\mathcal{M}^*(a) = \{\{c, d\}\}$ so $a\ E\ b$ fails.

In the lattice of Figure 2.4 on page 60 we have $a\ D\ c$ and not $a\ C\ c$, so the containment $C \subseteq D|_{\mathrm{J}(\mathbf{L})}$ can be proper.

This completes the proof of Theorem 2.59.

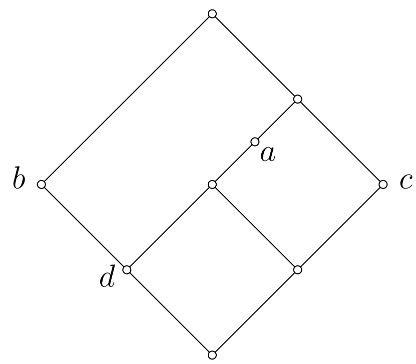


FIGURE 2.7

Temporarily, we restrict our attention to finite semidistributive lattices.[4] Combining Lemma 2.52 with Theorem 2.59(2), we obtain a lemma of Jónsson from [**83**].

LEMMA 2.61. *Let* $\mathbf{L}$ *be a finite semidistributive lattice. For any join irreducible* $a \in L - \mathrm{D}(\mathbf{L})$ *there exists a join irreducible* $b \in L - \mathrm{D}(\mathbf{L})$ *such that* $a \; C \; b$.

We naturally define C*–sequences* and C*–cycles* as sequences of completely join irreducible elements with $a_i \; C \; a_{i+1}$, with the subscripts taken modulo n for cycles. The previous lemma gives the next result which forms the basis for our proof of the characterization of finite sublattices of free lattices in Chapter V.

LEMMA 2.62. *A finite semidistributive lattice is lower bounded if and only if it contains no* C*–cycle.*

[4]Because the relation C (unlike D and E) applies to completely join irreducible elements only, we cannot use it to test an infinite lattice for boundedness. The problem is that while Lemma 2.5 says that if $\mathrm{D}(\mathbf{L}) \neq L$, then $\mathrm{J}(\mathbf{L}) - \mathrm{D}(\mathbf{L})$ is nonempty, it does not guarantee that there is a completely join irreducible element not in $\mathrm{D}(\mathbf{L})$. On the other hand, the C relation does give a lot of information about the structure surrounding completely join irreducible elements in an infinite lattice satisfying $\mathrm{D}(\mathbf{L}) = L = \mathrm{D}^d(\mathbf{L})$, which includes free lattices.

However, the preceding lemma can still be considerably improved. Recall from Theorem 2.20 that every lower bounded lattice is join semidistributive. On the other hand, a finite and lower bounded lattice is not necessarily meet semidistributive. Our next goal is to prove Alan Day's theorem that a finite, lower bounded lattice is bounded if and only if it is meet semidistributive. This result was proved in [**25**], [**29**], [**83**] and [**103**]. We will give two proofs, the first following that in [**103**], and the second closer to the original in [**25**].

We begin with a nice duality result due to Nation [**103**], which is also useful in the infinite case.

LEMMA 2.63. *Let* $\mathbf{L}$ *be a semidistributive lattice which satisfies the minimal join cover refinement property and its dual. Let* a *and* b *be completely join irreducible elements in* $\mathbf{L}$*. Then*

(1) $a\ A\ b$ *if and only if* $\kappa_{\mathbf{L}}(a)\ B^d\ \kappa_{\mathbf{L}}(b)$,
(2) $a\ B\ b$ *if and only if* $\kappa_{\mathbf{L}}(a)\ A^d\ \kappa_{\mathbf{L}}(b)$.

Proof: By definition, $a\ A\ b$ if and only if $b < a \leq \kappa_{\mathbf{L}}(b)^*$. As we have already observed, the definition of $a\ B\ b$ is equivalent to $a \neq b$, $b_* \leq \kappa_{\mathbf{L}}(a)$ and $b \nleq \kappa_{\mathbf{L}}(a)$. Since $\kappa_{\mathbf{L}}$ is a bijection, $a \neq b$ is equivalent to $\kappa_{\mathbf{L}}(a) \neq \kappa_{\mathbf{L}}(b)$. Also, for any element w with $b_* \leq w$ we have $w \leq \kappa_{\mathbf{L}}(b)$ if and only if $b \nleq w$. Thus the conditions for $a\ B\ b$ can be rewritten as $b_* \leq \kappa_{\mathbf{L}}(a) < \kappa_{\mathbf{L}}(b)$.

Dually, using $\kappa^d_{\mathbf{L}}(\kappa_{\mathbf{L}}(b)) = b$, we have $\kappa_{\mathbf{L}}(a)\ A^d\ \kappa_{\mathbf{L}}(b)$ if and only if $\kappa_{\mathbf{L}}(b) > \kappa_{\mathbf{L}}(a) \geq b_*$, and similarly $\kappa_{\mathbf{L}}(a)\ B^d\ \kappa_{\mathbf{L}}(b)$ if and only if $\kappa_{\mathbf{L}}(b)^* \geq a > b$. The equivalences are now obvious.

This brings us to Day's theorem.

THEOREM 2.64. *A finite, lower bounded lattice is bounded if and only if it is meet semidistributive.*

Proof: Recall from Theorem 2.20 that a lower bounded lattice satisfies ($\mathrm{SD}_\vee$). Hence a bounded lattice satisfies both semidistributive laws, and in particular is meet semidistributive.

Suppose that there is a finite, lower bounded and semidistributive lattice $\mathbf{L}$ which is not upper bounded. By the dual of Lemma 2.62, there is a C^d–cycle in $\mathbf{L}$. Now we can apply the dual of Lemma 2.63 to see that there is also a C–cycle in $\mathbf{L}$. Applying Lemma 2.62 once more we obtain that $\mathbf{L}$ is not lower bounded, a contradiction.

COROLLARY 2.65. *For a finite semidistributive lattice* $\mathbf{L}$*, the following conditions are equivalent.*

(1) $\mathbf{L}$ *is bounded.*
(2) $\mathbf{L}$ *is lower bounded.*

(3) **L** *contains no C–cycle.*

As another application of Lemma 2.63, we have the following useful fact.

THEOREM 2.66. *Assume* **L** *is semidistributive and has the minimal join cover refinement property and its dual. If a is completely join irreducible in* **L**, *then $\rho(a) = \rho^d(\kappa_{\mathbf{L}}(a))$.*

Proof: By Theorems 2.46 and 2.59, $\rho(a)$ is defined if and only if a lies on no C–cycle or infinite C–sequence, and in that case it is equal to the length of the longest sequence

$$a \; C \; b_1 \; C \; \dots \; C \; b_k.$$

By Lemma 2.63, the exact corresponding situation holds dually for $\kappa_{\mathbf{L}}(a)$, which makes $\rho^d(\kappa_{\mathbf{L}}(a)) = \rho(a)$.

The hard direction of Day's theorem also follows from the next characterization of finite bounded lattices, along with the fact that for a finite semidistributive lattice $|\,\mathrm{J}(\mathbf{L})| = |\,\mathrm{M}(\mathbf{L})|$, with $\kappa_{\mathbf{L}}$ providing the bijection.

THEOREM 2.67. *A finite lattice* **L** *is bounded if and only if* $|\,\mathrm{J}(\mathbf{L})| = |\,\mathrm{M}(\mathbf{L})| = |\,\mathrm{J}(\mathbf{Con\ L})|$.

Proof: By Lemma 2.40, a finite lattice is lower bounded if and only if $|\,\mathrm{J}(\mathbf{L})| = |\,\mathrm{J}(\mathbf{Con\ L})|$. Dually, it is upper bounded if and only if $|\,\mathrm{M}(\mathbf{L})| = |\,\mathrm{J}(\mathbf{Con\ L})|$.

We close this section with some useful observations about join prime elements in semidistributive lattices. These results have applications both to finite lattices and to intervals in finitely generated free lattices.

THEOREM 2.68. *Let* **L** *satisfy* $(\mathrm{SD}_\wedge)$ *and the dual of the minimal join cover refinement property, and let p be completely join irreducible in* **L** *(so $\kappa_{\mathbf{L}}(p)$ exists). Then p is join prime if and only if $L = 1/p \,\dot\cup\, \kappa_{\mathbf{L}}(p)/0$. Consequently, if p is join prime, then it is completely join prime and $\kappa_{\mathbf{L}}(p)$ is completely meet prime.*

Proof: If a completely join irreducible element p is join prime and $u \ngeq p$, then $p_* \vee u \ngeq p$ and hence $u \leq \kappa_{\mathbf{L}}(p)$ by Lemma 2.57. Conversely, if $u \ngeq p$ implies $u \leq \kappa_{\mathbf{L}}(p)$, then clearly p is join prime.

THEOREM 2.69. *If a lattice* **L** *satisfies* $(\mathrm{SD}_\wedge)$ *and a is an atom of* **L**, *then a is join prime.*

Proof: For suppose $a \succ 0$, $a \nleq b$ and $a \nleq c$. Then $a \wedge b = a \wedge c = 0$, whence $a \wedge (b \sqcup c) = 0$ by $(\mathrm{SD}_\wedge)$. Thus $a \nleq b \sqcup c$, as desired.

Now we consider the dual situation.

COROLLARY 2.70. *Assume* $\mathbf{L}$ *satisfies* $(\mathrm{SD}_\vee)$ *and the minimal join cover refinement property. If* b *is a coatom of* $\mathbf{L}$*, then* $\kappa^d_{\mathbf{L}}(b)$ *is completely join prime and a canonical joinand of* $1_{\mathbf{L}}$.

Proof: By the dual of Theorem 2.69, the coatoms of $\mathbf{L}$ are meet prime. By by the dual of Theorem 2.68, each coatom b is then completely meet prime and each $\kappa^d_{\mathbf{L}}(b)$ is completely join prime. Since $b \vee \kappa^d_{\mathbf{L}}(b) = 1_{\mathbf{L}}$ and $\kappa^d_{\mathbf{L}}(b)_* \leq b$, these elements are easily seen to be the canonical joinands of $1_{\mathbf{L}}$.

THEOREM 2.71. *Assume* $\mathbf{L}$ *satisfies* $(\mathrm{SD}_\vee)$*, the minimal join cover refinement property and its dual. If* $\mathbf{L}$ *has a largest element, then the canonical joinands of* $1_{\mathbf{L}}$ *are join prime. If a canonical joinand* a *of* $1_{\mathbf{L}}$ *is completely join irreducible, then* $\kappa_{\mathbf{L}}(a)$ *is a coatom.*

Proof: The hypotheses insure that $1_{\mathbf{L}}$ has a canonical join representation $1_{\mathbf{L}} = \bigvee A$. Let $a \in A$, and suppose that $a \leq \bigvee B$ for some finite subset B of L. Then clearly $1_{\mathbf{L}} = \bigvee(A - \{a\}) \vee \bigvee B$. But this implies $A \ll (A - \{a\}) \cup B$, and hence $a \leq b$ for some $b \in B$. Thus each $a \in A$ is join prime.

Now further assume that a is completely join irreducible. Then $\kappa_{\mathbf{L}}(a)$ exists. To see this, note that for each maximal meet representation $a_* = \bigwedge E$ there exists $e \in E$ with $e \ngeq a$. There are only finitely many such maximal meet representations, so amongst all these elements we can pick one e_0 which is maximal in $\mathbf{L}$. We claim that $e_0 = \kappa_{\mathbf{L}}(a)$. For suppose there exists an element v with $v \geq a_*$, $v \ngeq a$ and $v \nleq e_0$. Then $v \vee e_0 \ngeq a$ since the latter is join prime, whence $a \wedge (v \vee e_0) = a_*$. But there is a maximal meet representation $a_* = \bigwedge F$ with $F \gg \{a, v \vee e_0\}$, which contradicts the choice of e_0.

Moreover, for each $b \in A - \{a\}$ we have $b \leq \kappa_{\mathbf{L}}(a)$, because $a \leq a_* \vee b$ would imply $a_* \vee \bigvee(A - \{a\}) = 1_{\mathbf{L}}$, contrary to canonical form (Theorem 2.24). Thus $\kappa_{\mathbf{L}}(a)^* = a \vee \kappa_{\mathbf{L}}(a) \geq \bigvee A = 1_{\mathbf{L}}$, which makes $\kappa_{\mathbf{L}}(a)$ a coatom.

In fact, the conclusion of Theorem 2.71 makes sense and is true if we only assume that $\mathbf{L}$ is finitely generated and join semidistributive. As we will not need this stronger form, we leave its proof to the reader.

6. Splitting Lattices

In this section we develop the connections between bounded lattices, covering pairs in free lattices, and lattice varieties. The ideas and results in this section all come from Ralph McKenzie's important paper *Equational bases and non-modular lattice varieties* [**99**]. It is these connections which motivated McKenzie to introduce the notions of bounded homomorphism and bounded lattice in this paper.

Recall that if $\mathbf{L}$ is a subdirectly irreducible lattice and μ its monolith (minimal nonzero congruence), then a *critical pair* for $\mathbf{L}$ is any pair of distinct elements a, b with $a\ \mu\ b$. For any critical pair, we have of course $\mu = \mathrm{Cg}\,(a,b)$. For a critical pair with $a > b$, we call a/b a *critical quotient*. If in addition $a \succ b$, then a/b is a *prime critical quotient*. These turn out to be of particular importance. Note that every finite subdirectly irreducible lattice has a prime critical quotient.

Let s and t be elements of $\mathbf{FL}(X)$. We say that a lattice $\mathbf{L}$ *satisfies the inclusion* $s \le t$ if $f(s) \le f(t)$ for every homomorphism $f : \mathbf{FL}(X) \to \mathbf{L}$. A class $\mathcal{K}$ satisfies $s \le t$ if every lattice $\mathbf{L} \in \mathcal{K}$ does. It is easy to see that a variety $\mathcal{V}$ satisfies $s \le t$ if and only if $h(s) \le h(t)$ for the natural homomorphism $h : \mathbf{FL}(X) \twoheadrightarrow \mathbf{F}_{\mathcal{V}}(X)$ which is the identity on the generators. Clearly, the inclusion $s \le t$ can be rephrased in terms of the identities $s \approx s \wedge t$ or $t \approx s \sqcup t$ whenever it is convenient to do so.

Theorem 2.72. *Let $\mathbf{L}$ be a finitely generated subdirectly irreducible lattice, and let $f : \mathbf{FL}(X) \twoheadrightarrow \mathbf{L}$ be an epimorphism, with X finite. Assume there exists a pair of elements u, $v \in L$ such that $u/u \wedge v$ is a prime critical quotient. Form the sequences $\beta_{n,f}(u)$ and $\alpha_{n,f}(v)$. Then for any lattice variety $\mathcal{V}$, $\mathbf{L} \notin \mathcal{V}$ if and only if, for some n, $\mathcal{V}$ satisfies the inclusion $\beta_{n,f}(u) \le \alpha_{n,f}(v)$.*

The inclusions $\beta_{n,f}(u) \le \alpha_{n,f}(v)$ $(n \in \omega)$ are called a set of *conjugate equations* for $\mathbf{L}$. (Of course, a particular set of conjugate equations for $\mathbf{L}$ depends on the choice of X and f.)

Proof: The lattice $\mathbf{L}$ does not satisfy any of the inclusions $\beta_n(u) \le \alpha_n(v)$, since for all n we have $u \le f(\beta_n(u))$ and $f(\alpha_n(v)) \le v$, while $u \not\le v$. So if $\mathcal{V}$ satisfies one of these inclusions, then $\mathbf{L} \notin \mathcal{V}$.

For the converse, let $\mathcal{V}$ be any variety with $\mathbf{L} \notin \mathcal{V}$. Let $w = u \wedge v$, and recall our assumption that u/w is a prime critical quotient in $\mathbf{L}$. Since $w \le v$ in $\mathbf{L}$, we have $\alpha_n(w) \le \alpha_n(v)$ in $\mathbf{FL}(X)$ for all n. Thus it will suffice to show that $\mathcal{V}$ satisfies $\beta_n(u) \le \alpha_n(w)$ for all n sufficiently large, for by the preceding inequality this implies that $\mathcal{V}$ satisfies $\beta_n(u) \le \alpha_n(v)$ for the same values of n.

We have two homomorphisms, and consequently two congruences on $\mathbf{FL}(X)$, to consider. The first is the given homomorphism $f : \mathbf{FL}(X) \twoheadrightarrow \mathbf{L}$; let us denote $\ker f = \phi$. ($\ker f$ denotes the kernel of f, i.e., the congruence associated with f.) The other is the natural homomorphism $h : \mathbf{FL}(X) \twoheadrightarrow \mathbf{F}_{\mathcal{V}}(X)$ which is the identity on the generators; let $\eta = \ker h$. We know that $\eta \nleq \phi$ since $\mathbf{L} \notin \mathcal{V}$. Therefore $\mathbf{FL}(X)/\phi \sqcup \eta$ is isomorphic to a proper homomorphic image of $\mathbf{L}$. Thus for any pair of elements $r \in f^{-1}(u)$ and $s \in f^{-1}(w)$, we have $r \; \phi \sqcup \eta \; s$. Pick such a pair $r_0, s_0 \in \mathbf{FL}(X)$ with $f(r_0) = u$, $f(s_0) = w$ and $r_0 > s_0$. Using Dilworth's standard description of lattice congruences ([**40**], see e.g. [**20**]), there exist elements $t_1, \dots, t_{k-1}$ such that $r_0 = t_0 \geq t_1 \geq \cdots \geq t_k = s_0$ and

$$t_0 \; \phi \; t_1 \; \eta \; t_2 \; \dots \; \phi \; t_{k-1} \; \eta \; t_k \, .$$

Since $u \succ w$, there exists an i for which $f(t_0) = \cdots = f(t_i) = u$ and $f(t_{i+1}) = \cdots = f(t_k) = w$; of course, $t_i \; \eta \; t_{i+1}$ must hold (rather than $t_i \; \phi \; t_{i+1}$) because $f(t_i) \neq f(t_{i+1})$. Let $r_1 = t_i$ and $s_1 = t_{i+1}$. Thus we have found elements r_1, $s_1 \in \mathbf{FL}(X)$ such that $r_1 > s_1$, $f(r_1) = u$, $f(s_1) = w$ and $r_1 \; \eta \; s_1$.

Because $f(r_1) = u$, we have $r_1 \geq \beta_p(u)$ for some p, and hence also $r_1 \geq \beta_m(u)$ for all $m \geq p$. Similarly, $f(s_1) = w$ implies that there exists q such that $s_1 \leq \alpha_m(w)$ for all $m \geq q$. For n the larger of p and q, we then have

$$\beta_n(u) \leq r_1 \; \eta \; s_1 \leq \alpha_n(w) \, .$$

Hence $h(\beta_n(u)) \leq h(r_1) = h(s_1) \leq h(\alpha_n(w))$, so that $\mathcal{V}$ satisfies the inclusion $\beta_n(u) \leq \alpha_n(w)$. As noted above, this implies that $\mathcal{V}$ satisfies $\beta_m(u) \leq \alpha_m(v)$ for all $m \geq n$.

There are two essential tools for the study of finitely generated lattice varieties. The first is, of course, Jónsson's Lemma [**80**], and the second is Theorem 2.72. Let $\mathbf{\Lambda}$ denote the lattice of all lattice varieties. Jónsson's Lemma says, among other things, that $\mathbf{\Lambda}$ is distributive, and that the variety generated by a finite subdirectly irreducible lattice is join irreducible in $\mathbf{\Lambda}$. Now in any distributive lattice $\mathbf{D}$, for any nonzero join irreducible element $p \in \mathrm{J}(\mathbf{D})$, the set

$$\mathcal{J}_p = \{x \in D : x \ngeq p\}$$

is an ideal. Thus $\mathbf{D}$ splits into the disjoint union of the principal filter $1/p$ and the ideal $\mathcal{J}_p$. Applying this to $\mathbf{\Lambda}$ and the variety generated by a finite subdirectly irreducible lattice, we get the ideal

$$\mathcal{J}_{\boldsymbol{V}(\mathbf{L})} = \{\mathcal{X} \in \mathbf{\Lambda} : \mathbf{L} \notin \mathcal{X}\}$$

in $\mathbf{\Lambda}$. Theorem 2.72 gives us an equational description of $\mathcal{J}_{\boldsymbol{V}(\mathbf{L})}$.

More particularly, let $\mathbf{L}$, f, u and v be as in the setup of the theorem, and let ε_n denote the inclusion $\beta_n(u) \leq \alpha_n(v)$. These inclusions can be rewritten as lattice equations, so the class of all lattices satisfying ε_n is a lattice variety $\mathcal{V}_n$, with of course $\mathbf{L} \notin \mathcal{V}_n$. Because $\beta_{n+1}(u) \leq \beta_n(u)$ and $\alpha_{n+1}(v) \geq \alpha_n(v)$, the inclusion ε_{n+1} is weaker than its predecessor

ε_n, and hence $\mathcal{V}_{n+1} \geq \mathcal{V}_n$. Thus $\{\mathcal{V}_n : n \in \omega\}$ is a cofinal ascending chain in $\mathcal{J}_{\mathbf{V(L)}}$.

Some concrete examples will demonstrate these principles clearly.

EXAMPLE 2.73. Let a, b and c denote the atoms of $\mathbf{M}_3$. Then we can take $u = a$ and $v = b$ in our algorithm. The most natural homomorphism to use is $f : \mathbf{FL}(3) \to \mathbf{M}_3$ with $f(x) = a$, $f(y) = b$ and $f(z) = c$. In this case, we have the recursive formulas

$$\begin{aligned} \beta_0(a) &= x & \beta_{n+1}(a) &= x \wedge (\beta_n(b) \sqcup \beta_n(c)) \\ \beta_0(b) &= y & \beta_{n+1}(b) &= y \wedge (\beta_n(a) \sqcup \beta_n(c)) \\ \beta_0(c) &= z & \beta_{n+1}(c) &= z \wedge (\beta_n(a) \sqcup \beta_n(b)) \end{aligned}$$

and

$$\begin{aligned} \alpha_0(a) &= x & \alpha_{n+1}(a) &= x \sqcup (\alpha_n(b) \wedge \alpha_n(c)) \\ \alpha_0(b) &= y & \alpha_{n+1}(b) &= y \sqcup (\alpha_n(a) \wedge \alpha_n(c)) \\ \alpha_0(c) &= z & \alpha_{n+1}(c) &= z \sqcup (\alpha_n(a) \wedge \alpha_n(b)). \end{aligned}$$

Thus a set of conjugate equations for $\mathbf{M}_3$ is given by

$$(\varepsilon_n)\ \ \beta_n(a) = a \wedge (\beta_{n-1}(b) \sqcup \beta_{n-1}(c)) \leq b \sqcup (\alpha_{n-1}(a) \wedge \alpha_{n-1}(c)) = \alpha_n(b)$$

which of course have increasing complexity as n increases, and one can show that the corresponding varieties $\mathcal{V}_n$ form a properly ascending chain in $\mathbf{\Lambda}$.

As you can imagine, the conjugate equations can get more complicated for larger lattices, but otherwise this example is reasonably generic. Next let us look at an example with a lower bounded lattice.

EXAMPLE 2.74. Now take $\mathbf{L}_1$ to be our familiar friend in Figure 2.8, as labeled. Recall that this lattice is lower bounded, but not upper bounded. The natural homomorphism to use is $g : \mathbf{FL}(3) \twoheadrightarrow \mathbf{L}_1$ with $g(x) = a$, $g(y) = b$ and $g(z) = c$. We can take $u = a$ and say $v = b$.

Since $\mathrm{J}(\mathbf{L}_1) = \{a, b, c\}$ with b, $c \in \mathrm{D}_0(\mathbf{L}_1)$ and $a \in \mathrm{D}_1(\mathbf{L}_1)$, we calculate easily that

$$\begin{aligned} \beta(b) &= \beta_0(b) = y \\ \beta(c) &= \beta_0(c) = z \\ \beta(a) &= \beta_1(a) = x \wedge (y \sqcup z) \end{aligned}$$

and the left hand sides of the conjugate equations stabilize quickly at $\beta(a) = x \wedge (y \sqcup z)$. (The complexity of calculating α and β is discussed in Chapter XI.)

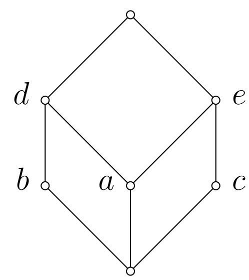


FIGURE 2.8

To calculate the upper maps α_n, we note that $\mathrm{M}(\mathbf{L}_1) = \{b, c, d, e\}$ with d, $e \in \mathrm{D}_0^d(\mathbf{L}_1)$ but b, $c \notin \mathrm{D}^d(\mathbf{L}_1)$. Hence

$$\alpha(d) = \alpha_0(d) = x \sqcup y$$
$$\alpha(e) = \alpha_0(e) = x \sqcup z$$

while

$$\alpha_0(b) = y \qquad \alpha_{n+1}(b) = y \sqcup (\alpha(d) \wedge \alpha_n(c)) = y \sqcup ((x \sqcup y) \wedge \alpha_n(c))$$
$$\alpha_0(c) = z \qquad \alpha_{n+1}(c) = z \sqcup (\alpha(e) \wedge \alpha_n(b)) = z \sqcup ((x \sqcup z) \wedge \alpha_n(b)).$$

Thus a set of conjugate equations for $\mathbf{L}_1$ is given by expanding

$$(\varepsilon_n) \qquad x \wedge (y \sqcup z) \leq \alpha_n(b),$$

which again corresponds to an infinite ascending chain in $\mathbf{\Lambda}$.

EXAMPLE 2.75. The preceding example makes it easy to predict what will happen when we have a bounded lattice like $\mathbf{N}_5$. Let $\{a, b, c, 0, 1\}$ with $a > b$ be the universe of $\mathbf{N}_5$, and let $h : \mathbf{FL}(3) \twoheadrightarrow \mathbf{N}_5$ be the homomorphism with $h(x) = a$, $h(y) = b$ and $h(z) = c$. Choose $u = a$ and $v = b$. Then

$$\beta(a) = \beta_1(a) = x \wedge ((x \wedge y) \sqcup z)$$

and

$$\alpha(b) = \alpha_1(b) = y \sqcup ((x \sqcup y) \wedge z).$$

The conjugate equations for $\mathbf{N}_5$ then reduce to a single inclusion

$$x \wedge ((x \wedge y) \sqcup z) \leq y \sqcup ((x \sqcup y) \wedge z)$$

which is, of course, equivalent to the modular law. Correspondingly, the variety of all modular lattices is the unique largest lattice variety not containing $\mathbf{N}_5$.

The situation with $\mathbf{N}_5$ and the modular law is just the best known instance of the natural corollary to Theorem 2.72.

COROLLARY 2.76. *Let* $\mathbf{L}$ *be a finite, bounded, subdirectly irreducible lattice, and let* $f : \mathbf{FL}(X) \twoheadrightarrow \mathbf{L}$ *be an epimorphism, with* X *finite. Choose a pair of elements* $u, v \in L$ *such that* $u/u \wedge v$ *is a prime critical quotient. Then there is a unique lattice variety* $\mathcal{C}_{\mathbf{L}}$ *which is maximal with respect to not containing* $\mathbf{L}$*, determined by the inclusion* $\beta_f(u) \leq \alpha_f(v)$.

Following McKenzie, we call a finite, bounded, subdirectly irreducible lattice a *splitting lattice*. The corresponding inclusion $\beta_f(u) \leq \alpha_f(v)$ is called a *conjugate equation* for $\mathbf{L}$, and $\mathcal{C}_{\mathbf{L}}$ is called the *conjugate variety* for $\mathbf{L}$. Since the inclusion depends on the choice of X, f, u and v, it is not unique, but the conjugate variety $\mathcal{C}_{\mathbf{L}}$ is unique and is determined by each conjugate equation for a splitting lattice $\mathbf{L}$.

In any complete lattice $\mathbf{L}$, an ordered pair of elements $\langle u, v\rangle$ is called a *splitting pair* if L is the disjoint union of the filter $1/u$ and the ideal $v/0$. Clearly, this means that $u \not\leq v$ and every element of L is either above u or below v. In any splitting pair $\langle u, v\rangle$, the element u is completely join prime and v is completely meet prime. Conversely, given any completely join prime element $u \in L$, the element $v = \bigvee\{a \in L : a \not\geq u\}$ is completely meet prime, and $\langle u, v\rangle$ is a splitting pair for $\mathbf{L}$. Dually, any completely meet prime element determines a splitting pair. In terms of our earlier notation, $v = \kappa_{\mathbf{L}}(u)$ and $u = \kappa^d_{\mathbf{L}}(v)$.[5]

In particular, in the lattice $\mathbf{\Lambda}$ of all lattice varieties, the ordered pair of varieties $\langle \mathcal{U}, \mathcal{V}\rangle$ is a splitting pair if Λ is the disjoint union of the filter $1/\mathcal{U}$ and the ideal $\mathcal{V}/0$. Corollary 2.76 says that if $\mathbf{L}$ is a finite, bounded, subdirectly irreducible lattice, then $\langle \boldsymbol{V}(\mathbf{L}), \mathcal{C}_{\mathbf{L}}\rangle$ is a splitting pair in $\mathbf{\Lambda}$. We would like to show that there are no others, thereby justifying the terminology *splitting lattice*.

A lattice inclusion $u \leq v$ is *nontrivial* if it fails in some lattice, or equivalently, if $u \not\leq v$ in $\mathbf{FL}(X)$.

[5]However, the situation is not entirely self-dual. The lattice Λ is distributive and dually algebraic. These conditions insure that any completely meet irreducible element is completely meet prime, but a completely join irreducible element need not be completely join prime.

LEMMA 2.77. *Every nontrivial lattice inclusion ε fails in some finite lower bounded lattice.*[6]

Proof: The proof uses a construction found in Dean [**32**] and McKinsey [**101**]. Let X be finite and $u \not\leq v$ in $\mathbf{FL}(X)$. Then we can find an integer $n \geq 0$ such that both u and v are in the subset $S_n = X^{(\wedge\ \sqcup\ )^{n+1}} = X^{\wedge(\ \sqcup\ \wedge)^n\ \sqcup}$ of $\mathbf{FL}(X)$. Now S_n is a finite join subsemilattice of $\mathbf{FL}(X)$ containing $0 = \bigwedge X$, and hence it forms a lattice $\mathbf{S}_n$ in its own right. The join operation on $\mathbf{S}_n$ is inherited from $\mathbf{FL}(X)$, while the meet is given by

$$a \wedge_{\mathbf{S}_n} b = \bigvee \{s \in S_n : s \leq a \wedge_{\mathbf{FL}(X)} b\},$$

whence $a \wedge_{\mathbf{S}_n} b \leq a \wedge_{\mathbf{FL}(X)} b$.

Consider the homomorphism $h : \mathbf{FL}(X) \to \mathbf{S}_n$ with $h(x) = x$ for all $x \in X$. An easy induction shows that $h(w) \leq w$ for all $w \in \mathbf{FL}(X)$, while $h(t) = t$ if and only if $t \in S_n$. Thus, for $w \in \mathbf{FL}(X)$ and $t \in S_n$, $h(w) \geq h(t)$ implies $w \geq h(w) \geq h(t) = t$. Hence h is a lower bounded homomorphism with β_h the identity on S_n, making $\mathbf{S}_n$ a lower bounded lattice.[7]

Clearly the inclusion $u \leq v$ fails in $\mathbf{S}_n$ *via* the homomorphism h.

COROLLARY 2.78. *The variety of all lattices is generated by its finite members.*

For comparison, the variety of all modular lattices is not generated by its finite members, or even by finite dimensional modular lattices [**48**]. Thus there is a lattice inclusion which is satisfied by all finite dimensional modular lattices, but not all modular lattices.

We can now easily characterize splitting pairs.

THEOREM 2.79. *If $\langle \mathcal{U}, \mathcal{V} \rangle$ is a splitting pair in $\mathbf{\Lambda}$, then $\mathcal{U} = \boldsymbol{V}(\mathbf{L})$ and $\mathcal{V} = \mathcal{C}_{\mathbf{L}}$ for some splitting lattice $\mathbf{L}$.*

Proof: Let $\langle \mathcal{U}, \mathcal{V} \rangle$ be a splitting pair. Then $\mathcal{V}$ is not the variety of all lattices, so we can find a nontrivial lattice inclusion ε which holds in $\mathcal{V}$. By Lemma 2.77, ε fails in some finite lower bounded lattice $\mathbf{K}_0$, and consequently $\mathcal{U} \subseteq \boldsymbol{V}(\mathbf{K}_0)$. By Jónsson's Lemma, this means that $\mathcal{U} = \boldsymbol{V}(\mathbf{L})$ for some finite lattice $\mathbf{L}$. Since $\mathcal{U}$ is join irreducible in $\mathbf{\Lambda}$, $\mathbf{L}$ can be taken to be subdirectly irreducible. Because $\mathbf{L}$ is a finite lattice in $\boldsymbol{V}(\mathbf{K}_0)$, $\mathbf{L}$ is lower bounded by Corollary 2.17.

[6] Later in this chapter, we will prove Day's theorem that every nontrivial lattice inclusion fails in some finite bounded lattice.

[7] In fact, $\mathbf{S}_n$ is a relatively free lattice; see [**108**].

However, ε also fails in a finite upper bounded lattice $\mathbf{K}_1$, whence $\mathcal{U} \subseteq \boldsymbol{V}(\mathbf{K}_1)$, and $\mathbf{L}$ is also upper bounded. Thus $\mathcal{U} = \boldsymbol{V}(\mathbf{L})$ where $\mathbf{L}$ is a finite bounded subdirectly irreducible lattice. Clearly then $\mathcal{V}$ must be the corresponding conjugate variety $\mathcal{C}_{\mathbf{L}}$.

We now make the connection between the preceding results and the structure of free lattices. For any lattice inclusion ε, let $\mathrm{V}(\varepsilon)$ denote the variety of all lattices which satisfy ε.

THEOREM 2.80. *A variety $\mathcal{V}$ is completely meet prime in $\mathbf{\Lambda}$ if and only if there exist a finite set X and elements s, $t \in \mathbf{FL}(X)$ such that $s \succ s \wedge t$ in $\mathbf{FL}(X)$ and $\mathcal{V} = \mathrm{V}(s \leq t)$.*

Proof: By Theorem 2.79, if $\mathcal{V}$ is completely meet prime in $\mathbf{\Lambda}$, then $\mathcal{V} = \mathcal{C}_{\mathbf{L}}$ for some splitting lattice $\mathbf{L}$. Thus $\mathcal{V} = \mathrm{V}(\varepsilon)$ where ε is the conjugate equation $\beta_f(u) \leq \alpha_f(v)$, with X, f, u and v chosen as in Corollary 2.76. In particular, f is an epimorphism and $u \succ u \wedge v$, and we claim that this implies $\beta_f(u) \succ \beta_f(u) \wedge \alpha_f(v)$. Suppose on the contrary that there exists $r \in \mathbf{FL}(X)$ with $\beta_f(u) > r > \beta_f(u) \wedge \alpha_f(v)$. Note that

$$u = f(\beta_f(u)) \geq f(r) \geq f(\beta_f(u) \wedge \alpha_f(v)) = u \wedge v.$$

Since $\beta_f(u)$ is the least preimage of u, $f(r) < u$, and similarly $r \nleq \alpha_f(v)$ implies $f(r) \nleq v$. Thus $u > f(r) > u \wedge v$, a contradiction.

Now let X be finite and $s \succ s \wedge t$ in $\mathbf{FL}(X)$. Let $t' = s \wedge t$, so that $s \succ t'$, and of course the inclusion $s \leq t'$ is equivalent to $s \leq t$. We want to show that $\mathrm{V}(s \leq t')$ is completely meet prime in $\mathbf{\Lambda}$. Let Y be a countable set with $X \subseteq Y$. Then $\mathbf{\Lambda}$ is dually isomorphic to the lattice of fully invariant congruences on $\mathbf{FL}(Y)$, so it suffices to show that the fully invariant congruence Θ generated by s and t' is completely join prime there. Of course, we no longer have $s \succ t'$ in $\mathbf{FL}(Y)$.

Let Φ_i $(i \in I)$ be a collection of fully invariant congruences on $\mathbf{FL}(Y)$ with $\Theta \leq \bigvee \Phi_i$, i.e., $\langle s, t' \rangle \in \bigvee \Phi_i$. Then there exist $r_1, \ldots, r_k \in \mathbf{FL}(Y)$ and $i_0, \ldots, i_k \in I$ such that

$$s = r_0 \geq r_1 \geq \cdots \geq r_k \geq r_{k+1} = t'$$

and $r_j \; \Phi_{i_j} \; r_{j+1}$ for $0 \leq j \leq k$. Let h be any endomorphism of $\mathbf{FL}(Y)$ such that $h(x) = x$ for $x \in X$ and $h(y) \in \mathbf{FL}(X)$ for $y \in Y - X$. Then

$$s = h(s) = h(r_0) \geq h(r_1) \geq \cdots \geq h(r_k) \geq h(r_{k+1}) = h(t') = t'$$

and $h(r_j) \; \Phi_{i_j} \; h(r_{j+1})$ for $0 \leq j \leq k$. However, all of this is taking place in $\mathbf{FL}(X)$, where $s \succ t'$, so there must be an m such that $s = h(r_m)$ and $t' = h(r_{m+1})$. Hence $\langle s, t' \rangle \in \Phi_{i_m}$ and $\Theta \leq \Phi_{i_m}$, as desired.

Theorem 2.79 connects splitting pairs in $\mathbf{\Lambda}$ with splitting lattices, while Theorem 2.80 gives the link between splitting pairs and covers in free lattices. This implicitly establishes a connection between covers in free lattices and finite bounded subdirectly irreducible lattices, which will be developed more fully in Chapter III.

Corollary 2.81. *If $p \succ p \wedge q$ in $\mathbf{FL}(X)$, then there exists a splitting lattice for which $p \leq q$ is a conjugate equation.*

So far, we have talked about covers in free lattices $\mathbf{FL}(X)$, but we don't know very much about them. If X is infinite, it is easy to see that there is none, so we can restrict our attention to finitely generated free lattices. Recall that a lattice $\mathbf{L}$ is *weakly atomic* if, for every pair $s > t$ in L, there exist p, $q \in L$ with $s \geq p \succ q \geq t$. An important question raised by McKenzie: *Is every finitely generated free lattice $\mathbf{FL}(X)$ weakly atomic?* The following result relates this question to our previous theorems.

Lemma 2.82. *The following are equivalent.*

1. *For every finite set X, $\mathbf{FL}(X)$ is weakly atomic.*
2. *Every nontrivial lattice inclusion fails in a finite bounded lattice.*
3. *Every nontrivial lattice inclusion fails in a splitting lattice.*
4. *Splitting lattices generate the variety of all lattices.*

In the next section we will present Alan Day's beautiful proof that the statements in Lemma 2.82 are indeed true, and in Chapter IV we will prove a strong version of Day's theorem.

Proof: [Proof of Lemma 2.82] The equivalence of (2), (3) and (4) is basic universal algebra, so we will show that (1) is equivalent to (2).

Let X be finite, and assume that $\mathbf{FL}(X)$ is weakly atomic. Given $s \nleq t$ in $\mathbf{FL}(X)$, with $t' = s \wedge t$ we can find p, q such that $s \geq p \succ q \geq t'$. By Corollary 2.81, there is a splitting lattice $\mathbf{L}$ for which $p \leq q$ is a conjugate equation. In particular, $p \leq q$ fails in $\mathbf{L}$, i.e., there exists a homomorphism $h : \mathbf{FL}(X) \to \mathbf{L}$ with $h(p) \nleq h(q)$. Clearly then $h(s) \nleq h(t')$ and $h(s) \nleq h(t)$, so $s \leq t$ also fails in $\mathbf{L}$.

Conversely, let $s > t$ in $\mathbf{FL}(X)$ with X finite, and assume that $s \leq t$ fails in some finite bounded lattice. This means that there exist a finite bounded lattice $\mathbf{L}$ and a homomorphism $h : \mathbf{FL}(X) \to \mathbf{L}$ with $h(s) > h(t)$. Since $\mathbf{L}$ is bounded, h is a bounded homomorphism, and we can assume that it is onto by Corollary 2.14. Since $\mathbf{L}$ is finite, we can find u and v in L such that $h(s) \geq u \succ v \geq h(t)$. Note that $h(s) \geq u$ implies $s \geq \beta_h(u)$, and $h(t) \leq v$ implies $t \leq \alpha_h(v)$. We claim that $s \geq p \succ p \wedge q \geq t$ holds with $p = \beta_h(u) \sqcup t$ and $q = \alpha_h(v)$. By the preceding observation, $s \geq p \geq p \wedge q \geq t$. If $p > r > p \wedge q$ in $\mathbf{FL}(X)$, then

$$h(p) = h(\beta_h(u) \sqcup t) = u \geq h(r) \geq h(p \wedge q) = u \wedge h(\alpha_h(v)) = v.$$

But, since $r \geq t$, $r \ngeq \beta_h(u)$ implies $u > h(r)$, while $r \nleq q = \alpha_h(v)$ implies $h(r) > v$. Together, these contradict the choice of u covering v.

In closing this section, we remark that McKenzie's original paper [**99**] was about nonmodular lattice varieties, and we have omitted many nice applications of the results of this section along those lines. Besides [**99**], the appropriate references include [**84**], [**94**], [**106**], [**107**], [**118**] and [**137**].

7. Day's Theorem: Free Lattices are Weakly Atomic

In this section we will present Alan Day's beautiful construction of the finitely generated free lattice $\mathbf{FL}(X)$ from the corresponding free distributive lattice $\mathbf{F}_{\mathcal{D}}(X)$ using the doubling construction. Day later formulated a more elegant version of the construction [**26**], and Freese and Nation proved a slightly stronger result in [**62**], which will be the subject of Chapter IV, but here we will follow the inherently simple original version of [**24**].

The idea motivating the argument is found in the characterization of free lattices given by Corollary 1.12: *Let* $\mathbf{L}$ *be a lattice which satisfies* (W) *and let* X *generate* $\mathbf{L}$. *Then* $\mathbf{L}$ *is isomorphic to* $\mathbf{FL}(X)$ *if and only if the following condition and its dual hold for all* $x \in X$ *and all finite subsets* $Y \subseteq X$*:*

$$x \leq \bigvee Y \quad \text{implies} \quad x \in Y.$$

In an arbitrary lattice $\mathbf{L}$, call a set $X \subseteq L$ *join irredundant* if for all $x \in X$ and all finite subsets $Y \subseteq X$, $x \leq \bigvee Y$ implies $x \in Y$. *Meet irredundant* is defined dually. The following lemma collects some simple but useful facts about these concepts.

Lemma 2.83. *Let* $\mathbf{L}$ *be a lattice and* $X \subseteq L$.

(1) X *is join irredundant if and only if it is an antichain and each* $x \in X$ *is join prime in the sublattice* $\mathrm{Sg}(X)$ *generated by* X.
(2) *If* $a \wedge b \leq c \sqcup d$ *is a* (W)*–failure in* $\mathbf{L}$*, then the interval* $c \sqcup d/a \wedge b$ *contains no join prime or meet prime element.*
(3) *Assume* X *is join irredundant and* $h : \mathbf{K} \to \mathbf{L}$ *is a homomorphism. If* A *is a system of distinct preimages of* X *in* K*, then* A *is join irredundant.*

Proof: Assume that X is join irredundant. Then $x \nleq \bigvee Y$ for all finite subsets $Y \subseteq X - \{x\}$; in particular, X is an antichain. Let $x \in X$ and put

$$G = \{w \in L : x \leq w \text{ or } w \leq \bigvee Y \text{ for some } Y \subseteq X - \{x\}\}.$$

Then $X \subseteq G$, and G is closed under meets and joins, so $G = \mathrm{Sg}(X)$. Now if $w_1, \ldots, w_m \in \mathrm{Sg}(X)$ and $x \nleq w_i$ for all i, then there exist $Y_i \subseteq X - \{x\}$ with $w_i \leq \bigvee Y_i$. But then, if $Y = \cup Y_i$, $\bigvee w_i \leq \bigvee Y$, so $x \nleq \bigvee w_i$.

This proves half of (1); the converse is trivial. Parts (2) and (3) are completely straightforward.

Fix a finite set X; to avoid the trivial cases, we assume $|X| \geq 3$. Our objective is to construct $\mathbf{FL}(X)$ from a sequence of finite bounded lattices. By Lemma 1.2, the generators in any relatively free lattice are join and meet irredundant. Indeed, if Y, $Z \subseteq X$ with $Y \cap Z = \emptyset$, then $\bigwedge Y \leq \bigvee Z$ fails in a two-element lattice, and hence in $\mathbf{F}_{\mathcal{V}}(X)$ for any nontrivial lattice variety $\mathcal{V}$. In particular, this condition holds in $\mathbf{F}_{\mathcal{D}}(X)$. Also note that $\mathbf{F}_{\mathcal{D}}(X)$ is a finite bounded lattice, so it will make a good starting place for our construction.

On the other hand, $\mathbf{F}_{\mathcal{D}}(X)$ does not satisfy (W) as the lattice of Figure 2.9 shows. Recall that in the proof of Theorem 1.8, the doubling construction was used to show that free lattices satisfy (W). Now we want to use the doubling process to eliminate the (W)–failures in $\mathbf{F}_{\mathcal{D}}(X)$. However, fixing the old (W)–failures can create new ones; in fact, that is what we expect to happen, since $\mathbf{F}_{\mathcal{D}}(X)$ is finite and $\mathbf{FL}(X)$ is infinite. Thus our construction of $\mathbf{FL}(X)$ from $\mathbf{F}_{\mathcal{D}}(X)$ will have to involve some sort of limit. The details go as follows.

Let $X = X^0 = \{x_1^0, \ldots, x_n^0\}$. We construct a sequence of lattices $\mathbf{L}_i$ $(i \in \omega)$ with the following properties.

(1) $\mathbf{L}_0 = \mathbf{F}_{\mathcal{D}}(X)$.
(2) Each $\mathbf{L}_i$ is a finite bounded lattice.
(3) Each $\mathbf{L}_i$ is generated by a join and meet irredundant set $X^i = \{x_1^i, \ldots, x_n^i\}$.
(4) For each $i > 0$, the natural map $h_i : X^i \to X^{i-1}$ with $h_i(x_j^i) = x_j^{i-1}$ extends to a homomorphism $h_i : \mathbf{L}_i \to \mathbf{L}_{i-1}$.
(5) If $i > 0$ and the elements a, b, c, d generate a (W)–failure in $\mathbf{L}_{i-1}$, i.e., $a \wedge b \leq c \sqcup d$ but $a, b \nleq c \sqcup d$ and $a \wedge b \nleq c, d$, then for any $p \in h_i^{-1}(a)$, $q \in h_i^{-1}(b)$, $r \in h_i^{-1}(c)$, and $s \in h_i^{-1}(d)$, we have $p \wedge q \nleq r \sqcup s$.

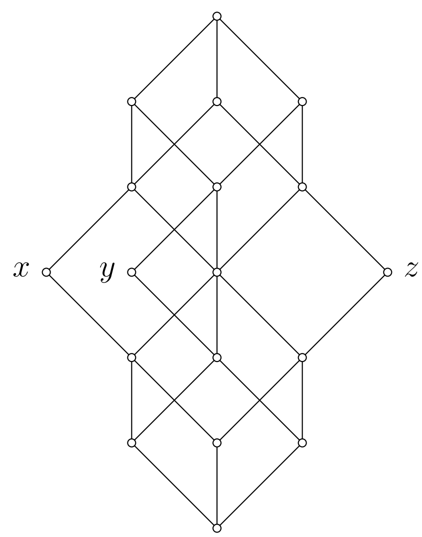


FIGURE 2.9. $\mathbf{F}_{\mathcal{D}}(3)$

The last condition is interpreted as saying that the (W)–failure $a \wedge b \leq c \sqcup d$ in $\mathbf{L}_{i-1}$ is 'fixed' in $\mathbf{L}_i$.[8]

Assume that $i > 0$ and we are given $\mathbf{L}_{i-1}$. Let $\langle a_1, b_1, c_1, d_1 \rangle, \ldots, \langle a_k, b_k, c_k, d_k \rangle$ be a listing of all the (W)–failures in $\mathbf{L}_{i-1}$. For each j with $1 \leq j \leq k$, let I_j be the interval $c_j \sqcup d_j / a_j \wedge b_j$ in $\mathbf{L}_{i-1}$, and let $\mathbf{L}_{ij}$ denote the lattice obtained by doubling I_j, $\mathbf{L}_{ij} = \mathbf{L}_{i-1}[I_j]$. By Corollary 2.29, each $\mathbf{L}_{ij}$ is a (finite) bounded lattice.

Form the direct product $\mathbf{P}_i = \prod_{j=1}^{k} \mathbf{L}_{ij}$. There are canonical homomorphisms $\lambda_{ij} : \mathbf{L}_{ij} \to \mathbf{L}_{i-1}$ collapsing the doubling of I_j. Now by part (2) of Lemma 2.83, no x_m^{i-1} is in any (W)–failure interval I_j. Thus each x_m^{i-1} has a unique preimage under each λ_{ij}, *viz.*, itself (under the notation of the doubling construction). Therefore we can define x_m^i to be the element of $\mathbf{P}_i$ such that $(x_m^i)_j = x_m^{i-1}$ for $1 \leq j \leq k$, and $\mathbf{L}_i = \mathrm{Sg}^{\mathbf{P}_i}(x_1^i, \ldots, x_n^i)$.

[8]Alternatively, we could correct more general (W)–failures of the form $\bigwedge a_j \leq \bigvee c_k$ at each step.

Since $\mathbf{L}_i \in \boldsymbol{SP}_f(\mathbf{L}_{i1}, \ldots, \mathbf{L}_{ik})$, $\mathbf{L}_i$ is again a bounded lattice by Corollary 2.17. Also, $\{x_1^i, \ldots, x_n^i\}$ is join and meet irredundant because it is so in each component by Lemma 2.83(3). This leaves properties (4) and (5) to be checked for $\mathbf{L}_i$.

Let

$$\mathbf{S} = \{u \in P_i : \forall j, l \quad \lambda_{ij}(u_j) = \lambda_{il}(u_l)\}.$$

It is not hard to see that $\mathbf{S}$ is a sublattice of $\mathbf{P}_i$, and that the map $h : \mathbf{S} \to \mathbf{L}_{i-1}$ by $h(u) = \lambda_{ij}(u_j)$ is a well-defined lattice homomorphism. Moreover, $X^i \subseteq S$ because for all j, l and m we have $\lambda_{ij}((x_m^i)_j) = x_m^{i-1} = \lambda_{il}((x_m^i)_l)$. Hence $\mathbf{L}_i$ is a sublattice of $\mathbf{S}$, and $h_i = h|_{L_i} : \mathbf{L}_i \to \mathbf{L}_{i-1}$ is a homomorphism with $h_i(x_m^i) = x_m^{i-1}$, as desired for (4).

Any (W)–failure in $\mathbf{L}_{i-1}$ is of the form $a_j \wedge b_j \leq c_j \sqcup d_j$ for some j. Let $p \in h_i^{-1}(a_j)$, $q \in h_i^{-1}(b_j)$, $r \in h_i^{-1}(c_j)$, and $s \in h_i^{-1}(d_j)$. As we saw in the preceding paragraph, $h_i(p) = a_j = \lambda_{ij}(p_j)$, and $a_j \notin I_j$ implies $p_j = a_j$. Similarly, $q_j = b_j$, $r_j = c_j$ and $s_j = d_j$. Hence in $\mathbf{L}_{ij}$,

$$(p \wedge q)_j = (a_j \wedge b_j, 1) \not\leq (c_j \sqcup d_j, 0) = (r \sqcup s)_j.$$

Therefore $p \wedge q \not\leq r \sqcup s$ in $\mathbf{L}_i$, yielding (5).

Thus we can construct $\mathbf{L}_i$ $(i \in \omega)$ satisfying (1)–(5). Next, we form the inverse limit $\mathbf{K}$ of the lattices $\mathbf{L}_i$ using the homomorphisms h_i, i.e., $\mathbf{K}$ is the set of all elements $v \in \prod \mathbf{L}_i$ such that $h_{i+1}(v_{i+1}) = v_i$ for all $i \in \omega$. In particular, K contains all the elements $\hat{x}_m$ $(1 \leq m \leq n)$ with $(\hat{x}_m)_i = x_m^i$ for all i. Let $\mathbf{F}$ be the sublattice generated by $\hat{X} = \{\hat{x}_1, \ldots, \hat{x}_n\}$ in $\mathbf{K}$.

We claim that $\hat{X}$ generates $\mathbf{F}$ freely by virtue of Corollary 1.12. Since $\mathbf{L}_0$ is a homomorphic image of $\mathbf{K}$, the set $\hat{X}$ is join and meet irredundant by Lemma 2.83(3). On the other hand, the inverse limit $\mathbf{K}$ satisfies (W), so *a fortiori* $\mathbf{F}$ does. For suppose $a \wedge b \leq c \sqcup d$ were a (W)–failure in $\mathbf{K}$. Then $a_i \wedge b_i \leq c_i \sqcup d_i$ for all $i \geq 0$, but for some sufficiently large j we would have $a_j, b_j \not\leq c_j \sqcup d_j$ and $a_j \wedge b_j \not\leq c_j, d_j$. But then $a_j \wedge b_j \leq c_j \sqcup d_j$ would be a (W)–failure in $\mathbf{L}_j$, so that by property (5) we would have $a_{j+1} \wedge b_{j+1} \not\leq c_{j+1} \sqcup d_{j+1}$, a contradiction. Hence $\mathbf{F}$ satisfies (W), and we conclude that $\mathbf{F} \cong \mathbf{FL}(X)$.

Since Day's construction gives $\mathbf{FL}(X)$ as a sublattice of a direct product of finite bounded lattices, we actually get a slightly better result than promised in Section 3.

THEOREM 2.84. *Every free lattice* $\mathbf{FL}(X)$ *is in the quasivariety generated by all finite bounded lattices. Thus every universal Horn sentence which holds in all finite bounded lattices, also holds in free lattices. In particular, every nontrivial lattice equation fails in a splitting lattice.*

COROLLARY 2.85. *Every finitely generated free lattice is weakly atomic.*

It is not hard to strengthen Day's theorem to show that *every finitely generated projective lattice is weakly atomic.* This is done in Theorem 5.18. The analogue of Day's Theorem does not hold in every lattice variety. By Freese [**48**], the variety of modular lattices is not generated by its finite dimensional members. Thus, either free modular lattices fail to be weakly atomic, or there is an infinite dimensional splitting modular lattice.

8. Applications to Congruence Varieties

Certainly upper and lower boundedness look like pure lattice theoretic concepts, far removed from any applications to general algebra. Amazingly, this is not the case. In this section we want to briefly survey this connection, omitting most of the proofs.

Let us start by looking at the congruence lattice of a finite semilattice. These provide the simplest example of the general result we want to describe. Different notations are used for semilattices, so let us agree to write them multiplicatively, denoting the operation by juxtaposition, and order the elements by

$$x \leq y \quad \text{iff} \quad xy = x.$$

Thus xy is the greatest lower bound (meet) of x and y, and if $\mathbf{S}$ is finite then it has a least element, which we will denote by 0. There is also a naturally induced partial join operation $+$ on a finite meet semilattice, which gives the least upper bound of any pair of elements which have a common upper bound:

$$x + y = \prod \{z \in S : z \geq x \text{ and } z \geq y\}$$

whenever x and y have a common upper bound; otherwise, $x + y$ is undefined. We say that the sum is *proper* if x and y are incomparable.

So let $\mathbf{S}$ be a finite semilattice. The following observations are elementary.

(1) Each element $a \in S$ determines a congruence ψ_a with $\langle x, y \rangle \in \psi_a$ if and only if either both x and y are above a, or both are not above a.

(2) For any congruence $\phi \in \mathbf{Con\ S}$, each ϕ-class has a unique minimal element. Let B_ϕ denote the collection of least elements of congruence classes of ϕ. Then in $\mathbf{Con\ S}$,

$$\phi = \bigwedge_{a \in B_\phi} \psi_a.$$

Thus the congruences ψ_a $(a \neq 0)$ are all coatoms of **Con S**, and they are precisely the meet irreducible congruences. It turns out that the right question to ask at this point is: *How does the dual dependency relation* D^d *work on* $\mathrm{M}(\mathbf{Con\ S})$*?*

LEMMA 2.86. *Let* **S** *be a finite semilattice, and let* a*,* $b \in S - \{0\}$*. The relation* $\psi_a\ \mathrm{D}^d\ \psi_b$ *holds in* **Con S** *if and only if there exists* $c \in S$ *such that* $a = b + c$ *properly. In particular,* $\psi_a\ \mathrm{D}^d\ \psi_b$ *implies* $a > b$*.*

Proof: Consider a meet of meet irreducible congruences in **Con S**, say $\theta = \bigwedge_{b \in B} \psi_b$ where $B \subseteq S - \{0\}$. We have $(x, y) \in \theta$ if and only if $\{b \in B : b \leq x\} = \{b \in B : b \leq y\}$, i.e., $B \cap x/0 = B \cap y/0$. Hence a nonzero element a is the least element of a/θ, or equivalently $\psi_a \geq \theta$, if and only if $a = \sum A$ for some $A \subseteq B$. Since each ψ_b is a coatom, $\psi_a \ngeq \psi_b$ whenever $a \neq b$. We conclude that $\psi_a \geq \bigwedge_{b \in B} \psi_b$ represents a minimal nontrivial meet cover precisely when $a \notin B$, $a = \sum B$ and $a \neq \sum C$ for every proper subset $C \subset B$.

The statement of the lemma now follows, using the dual of Lemma 2.31.

As a consequence of the last claim of Lemma 2.86, the congruence lattice of a finite semilattice cannot contain a D^d–cycle. In view of Corollary 2.39, this proves a nice theorem due to K. V. Adaricheva [**1**].

THEOREM 2.87. *If* **S** *is a finite semilattice, then* **Con S** *is an upper bounded lattice.*

This strengthens, in the finite case, a much earlier result of D. Papert [**110**], that *congruence lattices of semilattices satisfy* $(\mathrm{SD}_\wedge)$. Freese and Nation also discovered Theorem 2.87 independently, and found a generalization, which was improved by Keith Kearnes to the following result [**59**].

THEOREM 2.88. *If* $\mathcal{V}$ *is any variety of algebras such that the congruence lattices of algebras in* $\mathcal{V}$ *satisfy* $(\mathrm{SD}_\wedge)$*, then the congruence lattices of finite algebras in* $\mathcal{V}$ *are upper bounded.*

The proof makes heavy use of tame congruence theory (see [**74**]), and we will not attempt to reproduce it here. The technical statement of Kearnes' theorem is somewhat stronger, and with a little more tame congruence theory has the following consequence for congruence join semidistributive varieties.

THEOREM 2.89. *If* $\mathcal{W}$ *is any variety of algebras such that the congruence lattices of algebras in* $\mathcal{W}$ *satisfy* $(\mathrm{SD}_\sqcup)$*, then the congruence lattices of finite algebras in* $\mathcal{W}$ *are both upper and lower bounded.*

On the other hand, we also know that congruence lattices of semilattices do not satisfy any nontrivial lattice identity [**60**]. The variety

$\mathcal{P}$ described by Polin in [**112**] does satisfy nontrivial congruence identities. The lattice $\mathbf{Con\,F}_{\mathcal{P}}(n)$ is a splitting lattice [**28**], and the splitting equations play an important role in Day and Freese's characterization of identities that imply congruence modularity.

Lower bounded lattices also arise naturally in algebra, as in the following result of K. V. Adaricheva, Wieslaw Dziobiak and Viktor Gorbunov [**2**].

THEOREM 2.90. *If $\mathcal{K}$ is a locally finite quasivariety of algebras and the lattice of subquasivarieties of $\mathcal{K}$ is finite, then the lattice of subquasivarieties of $\mathcal{K}$ is a lower bounded lattice.*

Some deeper results along these lines are contained in Freese, Kearnes and Nation [**59**] and Gorbunov [**71**]. But it is time for us to return to free lattices.

CHAPTER III

Covers in Free Lattices

In this chapter we study covers in a finitely generated free lattice $\mathbf{FL}(X)$, with the aim of giving an algorithm which finds all the lower covers of a given element. Of course, upper covers can be treated dually. In Section 1 we collect the material that can be derived by elementary methods and reduce the problem of finding the lower covers of an element to the case when the element is join irreducible. In Sections 2 and 3 we build the basic theory and in Section 4 we use this theory to formulate and prove a fundamental result characterizing when a join irreducible element w has a lower cover in $\mathbf{FL}(X)$. For each join irreducible element w, we construct a finite lattice $\mathbf{L}^{\vee}(w)$. The element w has a lower cover if and only if this lattice is semidistributive. This shows that the question of deciding if w has a lower cover is recursive. In Section 5 we then proceed to obtain a syntactic version of the algorithm, extracting the crucial information from $\mathbf{L}^{\vee}(w)$ without requiring that the lattice be constructed. This syntactic algorithm is more suitable for computer language implementations, and the techniques employed make it also possible to obtain more detailed information on the canonical representation of a completely join irreducible element and its lower cover. We will show in Chapter XI that these algorithms are polynomial time.

Most of the results in this chapter are from Freese and Nation's paper *Covers in free lattices* [**62**].

Throughout this chapter let X be a finite set with at least three elements.

1. Elementary Theorems on Covers in $\mathbf{FL}(X)$

Recall that a join irreducible element w of $\mathbf{FL}(X)$ is completely join irreducible if and only if it has a lower cover; this lower cover is then unique and is denoted by w_*. Dually, the upper cover of a completely meet irreducible element w is denoted by w^*.

Of course w_* must have a canonical meetand v which is not above w. Then $w_* = w \wedge v$ and so by canonical form, Theorem 1.19, all other canonical meetands of w_* are above w. This shows that there is a unique canonical meetand of w_* not above w. We denote this meetand by $\kappa(w)$. An argument like the one just given shows that each element above w_* is either above w or below $\kappa(w)$. We record this fact in the next theorem. This can also be derived from Theorem 2.54, which shows that in any semidistributive lattice $\mathbf{L}$ with the minimal join cover refinement property and its dual (free lattices have these properties by Theorem 1.21 and Corollary 2.11), every completely join irreducible element a has such a $\kappa_{\mathbf{L}}(a)$.

THEOREM 3.1. *Let w be a completely join irreducible element of $\mathbf{FL}(X)$. Then there is a unique canonical meetand $\kappa(w)$ of w_* which is not above w. Every element of $\mathbf{FL}(X)$ which is above w_* is either above w or below $\kappa(w)$. Dually, if v is completely meet irreducible in $\mathbf{FL}(X)$, then $\kappa^d(v)$ is the unique canonical joinand of v^* which is not below v and every element below v^* is either below v or above $\kappa^d(v)$.*

If w is not completely join irreducible, then $\kappa(w)$ is undefined. It is possible that $w_* = \kappa(w)$, but this occurs only near the top and bottom of free lattices, as we will see in Chapter VIII.

The next two results record some easy consequences of the previous theorem.

COROLLARY 3.2. *Let w be completely join irreducible in $\mathbf{FL}(X)$. If $u \geq w_*$, then either $u \geq w$ or $u \leq \kappa(w)$. Thus the interval $1/w_*$ is the disjoint union of $1/w$ and $\kappa(w)/w_*$. Moreover, for any element u, we have $u \leq \kappa(w)$ if and only if $w_* \vee u \ngeq w$.*

THEOREM 3.3. *If w is a completely join irreducible element of $\mathbf{FL}(X)$, then $\kappa(w)$ is completely meet irreducible and*

$$w_* = w \wedge \kappa(w), \qquad \kappa(w)^* = w \vee \kappa(w), \qquad \kappa^d(\kappa(w)) = w.$$

Thus the mapping κ is a bijection from the set of completely join irreducible elements onto the set of completely meet irreducible elements of $\mathbf{FL}(X)$, and κ^d is its inverse.

It is interesting to note that, by Theorem 2.66, w and $\kappa(w)$ have the same complexity, i.e., $\rho(w) = \rho^d(\kappa(w))$. Thus if $w \in X^{\wedge(\vee\wedge)^n}$, then $\kappa(w) \in X^{\vee(\wedge\vee)^n}$. On the other hand, Chapter XI gives examples which show that the lengths of these two elements can differ greatly. Table **??** on page **??** gives several examples of completely join irreducible elements and their κ's.

The following nice result will prove quite useful later. Theorem 2.60 gives a strong generalization of this result; it shows that a stronger conclusion holds in a much more general setting.

Theorem 3.4. *Let w be a completely join irreducible element in $\mathbf{FL}(X)$. If*

$$w = \bigwedge_i \bigvee_j w_{ij} \wedge \bigwedge_k x_k$$

canonically, then for each i there is exactly one j with $w_{ij} \nleq w$.

Proof: Fix i. Since $w \leq w_i$ and $w \nleq \kappa(w)$, there is a j such that $w_{ij} \nleq \kappa(w)$. By renumbering we may assume $j = 1$. Then by Corollary 3.2, $w \leq w_* \vee w_{i1}$. Applying Whitman's condition (W) to this inequality and taking into account the canonical representation of w, we obtain, by Theorem 1.19 on canonical forms, that $w_{i'} \leq w_* \vee w_{i1}$ for some i'. But then $w_{i'} \leq w_* \vee w_{i1} \leq w_i$ and hence $i = i'$, since distinct canonical meetands cannot be comparable. This shows $w_* \vee w_{i1} = w_i$, which implies, by Theorem 1.19, that each w_{ij}, $j \geq 2$, is below w_*.

In our next result we consider the possible lower covers of a join reducible element. This result can be easily derived from Corollary 2.70 and Theorem 2.71, applied to the interval $w/0$. However, the direct proof is also quite simple, so we will give it. The situation where w_1 is completely join irreducible is illustrated in Figure 3.1.

Theorem 3.5. *Let w be a join reducible element of $\mathbf{FL}(X)$. Then there is a bijection γ of the set of lower covers of w onto the set of the canonical joinands of w that are completely join irreducible, defined as follows.*

(1) *If v is a lower cover of w, then $\gamma(v)$ is the unique canonical joinand w_i of w such that $w_i \nleq v$.*
(2) *If w_i is a completely join irreducible canonical joinand of w, then $\gamma^{-1}(w_i) = \kappa(w_i) \wedge w$, and this is the only lower cover of w not above w_i.*

Moreover, each canonical joinand w_i is join prime in $w/0$, and if w_i is completely join irreducible then it is completely join prime in $w/0$. Consequently, if w_i is completely join irreducible, then $w/0$ is the disjoint union of the intervals w/w_i and $w \wedge \kappa(w_i)/0$.

Proof: Let $v \prec w$. Then at least one canonical joinand w_i of w is not below v. On the other hand, if w_i and w_j were distinct and both not below v, then we would have $w = v \vee w_i = v \vee w_j$, whence by $(\mathrm{SD}_\vee)$ $w = v \vee (w_i \wedge w_j)$. Since the canonical joinands of w do not refine the latter expression, this is impossible. Hence there is a unique canonical joinand w_i not below v. We claim that $w_i \wedge v \prec w_i$. For if $w_i \wedge v < u \leq w_i$ for an element u, then since $u \nleq v$, we have $u \vee v = w$. Consequently every canonical joinand of w must be either below u or below v, which implies $u = w_i$. Thus w_i has a lower cover, which makes it completely join irreducible.

Conversely, let w_i be a canonical joinand of w with a lower cover w_{i*}. Put $v = \kappa(w_i) \wedge w$. Denote by t the join of all the canonical joinands of w other than w_i. We have $w_{i*} \vee t < w$, so that $w_{i*} \vee t \not\geq w_i$ and consequently $t \leq \kappa(w_i)$. This shows that every canonical joinand of w other than w_i is below v. If $v < u \leq w$, then $u \geq w_{i*}$ but $u \not\leq \kappa(w_i)$, so that $u \geq w_i$ and hence $u \geq w_i \vee t = w$. Thus $v \prec w$. Also, if v' is any other lower cover of w not above w_i, then by the proof of (1), $w_{i*} = w_i \wedge v = w_i \wedge v'$. By semidistributivity, $w_{i*} = w_i \wedge (v \vee v') = w_i \wedge w = w_i$, a contradiction.

Finally, note that for any canonical joinand w_i of w, if $w_i \leq \bigvee U \leq w$, then $\bigvee U \vee \bigvee_{j \neq i} w_j = w$, whence $w_i \leq u$ for some $u \in U$. Thus w_i is join prime in $w/0$. If w_i is completely join irreducible, we have further that for all $u \leq w$, $u \not\geq w_i$ implies $w_* \vee u \not\geq w_i$ and hence $u \leq w \wedge \kappa(w_i)$. This makes w_i completely join prime in $w/0$.

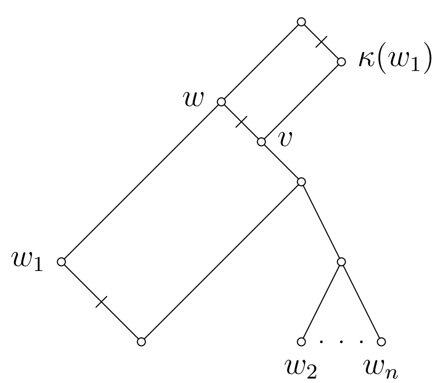


FIGURE 3.1

An element w is called *lower atomic* if for every element u such that $u < w$ there exists an element v with $u \leq v \prec w$. An *upper atomic* element is defined dually.

Corollary 3.6. *Let $w = w_1 \vee \cdots \vee w_n$ be the canonical form of a join reducible element w. Then the number of the lower covers of w coincides with the number of those canonical joinands of w that are completely join irreducible. Consequently, w has at most n lower covers.*

Corollary 3.7. *A join reducible element has a lower cover in $\mathbf{FL}(X)$ if and only if at least one of its canonical joinands is completely join irreducible.*

Corollary 3.8. *Let $w = w_1 \vee \cdots \vee w_n$ be the canonical form of a join reducible element w. The following are equivalent.*

(1) *w is lower atomic.*
(2) *w has precisely n lower covers.*
(3) *Each w_i is completely join irreducible.*

Proof: By Theorem 3.5, (2) and (3) are equivalent and (1) implies (3). Suppose (3) holds and let $u < w$. Then $u \ngeq w_i$ for some i and hence $u \leq \kappa(w_i) \wedge w \prec w$.

Corollary 3.9. *Let w be a join reducible element of $\mathbf{FL}(X)$. Denote by v the join of all completely join irreducible canonical joinands of w and by u the join of all the canonical joinands that are not completely join irreducible. An interval w/t is coatomic if and only if $t \geq u$. Also, an interval w/t contains no coatom if and only if $t \geq v$.*

Proof: Let $t \leq w$. If the interval w/t is coatomic, then $t \vee v = w$, since there is no lower cover of w above all the completely join irreducible canonical joinands of w. But $t \vee v = w$ means that $t \geq u$. Conversely, if t lies above all the canonical joinands that are not completely join irreducible, and if $t \leq q < w$, then there is a completely join irreducible canonical joinand w_i not below q, and we have $q \leq w \wedge \kappa(w_i) \prec w$.

The interval w/t contains no coatom if and only if t is not below any lower cover $w \wedge \kappa(w_i)$ of w, where w_i are all the completely join irreducible canonical joinands of w. But $t \nleq w \wedge \kappa(w_i)$ if and only if $t \nleq \kappa(w_i)$, which holds if and only if $t \geq w_i$.

We see now that the question, which elements of $\mathbf{FL}(X)$ have a lower cover, is decidable if only the same question for join irreducible elements of $\mathbf{FL}(X)$ is decidable. Also, in order to find all the lower covers of a join reducible element, by Theorem 3.5 it is sufficient to know how to decide which join irreducible elements have a lower cover, and how to construct $\kappa(w)$ for the completely join irreducible elements w. This will be accomplished later in this chapter.

EXAMPLE 3.10. For a generator $x \in X$ we have $\kappa(x) = \overline{x} = \bigvee(X - \{x\})$ and $x_* = x \wedge \overline{x}$. Dually, $\kappa^d(x) = \underline{x} = \bigwedge(X - \{x\})$ and $x^* = x \vee \underline{x}$. If w is a meet of generators, $w = \bigwedge Y$ for $\emptyset \subset Y \subset X$, then $\kappa(w) = \bigvee(X - Y)$ and $w_* = w \wedge \kappa(w)$. The element w is also upper atomic and its upper covers are the elements $w \vee \underline{y}$ for each $y \in Y$; see Corollary 1.7.

2. J–closed Sets and the Standard Epimorphism

Let p be an element of a lattice $\mathbf{L}$. Recall from Chapter II that a finite subset $S \subseteq L$ is a *join cover* of p if $p \leq \bigvee S$, and it is nontrivial if $p \leq s$ for no $s \in S$. A join cover S is a *minimal join cover* if $p \leq \bigvee T$ and $T \ll S$ imply $S \subseteq T$. $\mathcal{M}(p)$ is the set of all minimal nontrivial join covers of p. A lattice $\mathbf{L}$ has the *minimal join cover refinement property* if, for each $p \in L$, $\mathcal{M}(p)$ is finite and every nontrivial join cover of p refines to one in $\mathcal{M}(p)$.

$\mathcal{M}^*(p)$ is the set of those $U \in \mathcal{M}(p)$ such that $\bigvee U$ is minimal. $\mathrm{J}(p)$ is the smallest subset of L such that

(1) if $p = \bigvee U$ is a minimal join representation, then $U \subseteq \mathrm{J}(p)$, and
(2) if $q \in \mathrm{J}(p)$ and $V \in \mathcal{M}(q)$, then $V \subseteq \mathrm{J}(p)$.

The set $\mathrm{J}^*(p)$ is defined similarly with $\mathcal{M}^*(q)$ in place of $\mathcal{M}(q)$ in (2). It is not hard to see that, in a lattice with the minimal join cover refinement property, $\mathrm{J}(p)$ is the smallest set such that every join cover of p refines to one in $\mathrm{J}(p)$; see Lemma 2.48.

Recall that a subset A of L is J*–closed* if $p \in A$ implies $\mathrm{J}(p) \subseteq A$, or equivalently, if $p \in A$ implies each $V \in \mathcal{M}(p)$ is a subset of A. (These sets were also called D–closed in Chapter II.) Much of this section will be devoted to exploring the intimate connection between finite J–closed subsets of a lattice $\mathbf{L}$ with the minimal join cover refinement property, and lower bounded epimorphisms from $\mathbf{L}$ onto a finite lattice.

For an element w in a free lattice, the next lemma shows that it is very easy to find $\mathcal{M}^*(w)$. If w is join reducible, then $\mathcal{M}^*(w)$ just consists of the set of canonical joinands of w. The critical case is when w is join irreducible.

LEMMA 3.11. *Let w be a join irreducible element in* $\mathbf{FL}(X)$*. If*

$$w = \bigwedge_i \bigvee_{1 \leq j \leq r_i} w_{ij} \wedge \bigwedge_k x_k$$

canonically with $x_i \in X$, then $U \in \mathcal{M}^*(w)$ if and only if $U = \{w_{i1}, \dots, w_{ir_i}\}$ for some i.

Proof: First suppose that V is a nontrivial join cover of w. Apply Whitman's condition (W) to the inclusion $w = \bigwedge_i w_i \wedge \bigwedge_k x_k \leq \bigvee V$ (where of course $w_i = \bigvee_j w_{ij}$). If $x_k \leq \bigvee V$ then, since generators are join prime, $w \leq x_k \leq v$ for some $v \in V$, contradicting the nontriviality of the join cover V. Again by the nontriviality of V, we conclude that $w_i = \bigvee_j w_{ij} \leq \bigvee V$ for some i. Thus every nontrivial join cover of w joins above some w_i.

Next we show that each $W_i = \{w_{i1}, \ldots, w_{ir_i}\}$ is in $\mathcal{M}(w)$. By the characterization of canonical form in Theorem 1.18, each W_i is a nontrivial join cover. Fixing i, suppose that $V \ll W_i$ and $w \leq \bigvee V$. Then $\bigvee V \leq \bigvee W_i = w_i$, while by the preceding argument $w_{i'} \leq \bigvee V$ for some i'. Since the canonical meetands of w form an antichain by Theorem 1.18, we must have $i' = i$ and $\bigvee V = w_i$. However, $w_i = \bigvee W_i$ canonically, and therefore $W_i \ll V$ by Theorem 1.19. Since W_i is an antichain, this implies $W_i \subseteq V$. Hence the canonical join expression of w_i is a minimal nontrivial join cover of w. By the first argument, its join is minimal, and hence $W_i \in \mathcal{M}^*(w)$.

If $V \in \mathcal{M}^*(w)$ then the first part of the argument shows that $\bigvee V = w_i$, for some i, and it follows that $V = W_i$ by canonical form, Theorem 1.19.

In free lattices, as in finite lattices, $\mathcal{M}^*(w)$ may be properly contained in $\mathcal{M}(w)$. For example, if $w = x \wedge (y \vee (x \wedge (z \vee (x \wedge y))))$, then $\mathcal{M}^*(w) = \{\{y, x \wedge (z \vee (x \wedge y))\}\}$ by the preceding lemma, but it is easy to check that $\{y, z\}$ is also in $\mathcal{M}(w)$.

The next theorem shows that $\mathrm{J}^*(w) = \mathrm{J}(w)$ and gives a simple formula for it. The fact that $\mathrm{J}^*(w) = \mathrm{J}(w)$ also follows from Theorem 2.51, which proves this under much weaker hypotheses. Moreover, the next theorem shows that free lattices have the minimal join cover refinement property and, in fact, gives a strengthening of Corollary 2.11.

THEOREM 3.12. *If w is an element of a free lattice, then $\mathrm{J}^*(w) = \mathrm{J}(w)$ and*

$$\mathrm{J}(w) = \begin{cases} \{w\} & \text{if } w \text{ is a meet of generators,} \\ \{w\} \cup \bigcup_{i,j} \mathrm{J}(w_{ij}) & \text{if } w = \bigwedge_i \bigvee_j w_{ij} \wedge \bigwedge_k x_k \text{ canonically,} \\ \bigcup_i \mathrm{J}(w_i) & \text{if } w = \bigvee_i w_i \text{ canonically.} \end{cases}$$

Proof: We induct on the length of w. If $w = \bigvee w_i$ canonically is join reducible, then it is easy to see that $\mathrm{J}(w) = \bigcup \mathrm{J}(w_i)$ and a similar formula for J^* holds and the theorem follows in this case. Hence we may assume that w is join irreducible. The result also holds if w is a meet of generators, so we may assume that $w = \bigwedge_i \bigvee_j w_{ij} \wedge \bigwedge_k x_k$

canonically. Let S denote the right hand side of the displayed formula, i.e., $S = \{w\} \cup \bigcup_{i,j} \mathrm{J}(w_{ij})$.

It follows from induction and Lemma 3.11 that $S = \mathrm{J}^*(w)$. We will show that every join cover of w refines to one lying in S. Since $\mathrm{J}^*(w) \subseteq \mathrm{J}(w)$ always holds, this will prove the theorem. So let V be a join cover of w. If $w \leq v$ for some $v \in V$ then $\{w\} \ll V$; hence we may assume that V is a nontrivial join cover. This implies $w_i \leq \bigvee V$, for some i, as the simple (W) argument of Lemma 3.11 shows. Thus V is a join cover of w_{ij}, for each j. By induction there is a join cover $V_j \subseteq \mathrm{J}(w_{ij}) = \mathrm{J}^*(w_{ij})$ of w_{ij} refining V. Let $U = \bigcup_j V_j$. Then $U \ll V$ and $\bigvee U = \bigvee_j \bigvee V_j \geq \bigvee_j w_{ij} = w_i \geq w$, proving the claim and thus the theorem.

The set $\mathrm{M}(w)$ is defined dually to $\mathrm{J}(w)$; for practical purposes, we can use the dual of the above recursive description. By an M–*closed set* we mean a subset A of $\mathbf{FL}(X)$ such that $w \in A$ implies $\mathrm{M}(w) \subseteq A$.

Recall that a homomorphism $h : \mathbf{F} \to \mathbf{L}$ is *lower bounded* if $h^{-1}(1/a)$ is either empty or has a least element, which is denoted $\beta_h(a)$ (or simply $\beta(a)$) when it exists, for each $a \in L$. In what follows the set $\{\beta(a) : a \in L\}$ denotes the set of $\beta(a)$'s which exist. We now establish the connection between J–closed sets and lower bounded homomorphisms. This can be done in some generality, reserving the applications to free lattices and finite lower bounded lattices for the next section.

Recall that, for any subset A of a lattice $\mathbf{F}$ with zero 0, we denote by $A^\vee$ the subset of $\mathbf{F}$ consisting of all joins of finite subsets of A, including $\bigvee \emptyset = 0$. If A is finite, then $A^\vee$ is a finite lattice, with the join operation coinciding with that of the lattice $\mathbf{F}$ and the meet operation $\wedge'$ defined by

$$a \wedge' b = \bigvee \{c \in A : c \leq a \wedge b \text{ in } \mathbf{F}\}.$$

If A is a finite set of join irreducible elements of $\mathbf{F}$, then A is the set of join irreducible elements of the lattice $A^\vee$.

Theorem 3.13. *Let h be a lower bounded homomorphism of a lattice $\mathbf{F}$ with the minimal join cover refinement property into a finite lattice $\mathbf{L}$. Let*

$$A = \{\beta(a) : a \in L\} \cap \mathrm{J}(\mathbf{F}).$$

Then A is J*–closed and the image $h(\mathbf{F})$ is isomorphic to $A^\vee$.*

Proof: Let $\mathbf{S}$ denote the sublattice $h(\mathbf{F})$ of $\mathbf{L}$. Since $h\beta(s) = s$ for all $s \in S$, the restriction of β to S is a join embedding of $\mathbf{S}$ into $\mathbf{F}$. Let $B = \{\beta(s) : s \in S\}$. Then B is a finite join subsemilattice of $\mathbf{F}$ including 0. As such it is a lattice in its own right, though with a different meet operation from that in $\mathbf{F}$, and $B \cong \mathbf{S}$.

If $a \in L$, then $\beta(a)$ is defined if and only if $a \leq h(w)$ for some $w \in F$. In that event, $h(w) \in S$, and $\beta(a) = \beta(b)$ where $b = \bigwedge\{s \in S : s \geq a\}$. Thus we can also write $B = \{\beta(a) : a \in L\}$.

We claim that B is J–closed. To see this, let $b \in B$ and $U \in \mathcal{M}(b)$. Since β preserves joins, $b = \beta h(b) \leq \bigvee_{u \in U} \beta h(u)$. But $\beta h(u) \leq u$ for every $u \in U$, whence the minimality of U implies that $\beta h(u) = u$ for all $u \in U$. Thus $U \subseteq B$.

It follows easily that $A = B \cap \mathrm{J}(\mathbf{F})$ is also J–closed, and that $B = A^{\vee}$.

Some comments are in order. The reason for isolating the set denoted by A in Theorem 3.13 is that it corresponds to $\mathrm{J}(h(\mathbf{F}))$, and hence is the smallest set with $A^{\vee} = \{\beta(a) : a \in L\}$. Also, some of the results which follow are most naturally stated for J–closed sets of join irreducible elements, so we need to know that there is no loss of generality in making this reduction. On the other hand, given a lower bounded homomorphism $h : \mathbf{F} \to \mathbf{L}$, the natural thing to calculate is $C = \{\beta(c) : c \in \mathrm{J}(\mathbf{L})\}$. The corollary below says that if h is onto, then $A = C$. In Chapter XII, however, we must also deal with the case when h is not an epimorphism. In that situation, A may be properly contained in C, but it is still true that $A \subseteq C \subseteq A^{\vee}$. This is enough to insure that C is J–closed and $C^{\vee} \cong h(\mathbf{F})$, which are the properties that we require.

By far the most important case, and indeed the only case we will use in this chapter, is when h is onto. It is worth stating this case separately.

Corollary 3.14. *Let h be a lower bounded epimorphism of a lattice $\mathbf{F}$ onto a finite lattice $\mathbf{L}$, and let $C = \{\beta(c) : c \in \mathrm{J}(\mathbf{L})\}$. Then $C = \{\beta(a) : a \in L\} \cap \mathrm{J}(\mathbf{F})$. Hence C is J–closed and $\mathbf{L} \cong C^{\vee}$.*

Proof: As before, let $A = \{\beta(a) : a \in L\} \cap \mathrm{J}(\mathbf{F})$. The containment $A \subseteq C$ holds regardless of whether h is onto: if $\beta(a) \in \mathrm{J}(\mathbf{F})$, then, because β preserves joins, there exists $c \in \mathrm{J}(\mathbf{L})$ with $\beta(c) = \beta(a)$. Now assuming that h is onto, let $u = \beta(c)$ where c is join irreducible in $\mathbf{L}$. If $u = p \vee q$ in $\mathbf{F}$, then $c = h(p) \vee h(q)$ in $\mathbf{L}$, so that either $c = h(p)$ or $c = h(q)$. But then either $u = \beta(c) = \beta h(p) \leq p \leq u$, whence $u = p$, or similarly $u = q$. Thus $C \subseteq A$, and so $C = A$, when h is an epimorphism. Now apply Theorem 3.13 for the remaining conclusions.

Now let us prove the converse of Theorem 3.13.

Theorem 3.15. *Let $\mathbf{F}$ be a lattice with zero and the minimal join cover refinement property, and let A be a finite J–closed subset of $\mathbf{F}$.*

Then the mapping $f : \mathbf{F} \to A^\vee$ *defined by*

$$f(u) = \bigvee \{a \in A : a \leq u\}$$

is a lower bounded epimorphism.

Whenever $\mathbf{F}$ has the minimal join cover refinement property and A is a finite J–closed subset of F we call the mapping f defined in the statement of the theorem the *standard epimorphism* of $\mathbf{F}$ onto $A^\vee$. It will play a crucial role in our study of covers in free lattices.

Proof: Clearly, f is an order preserving mapping. Moreover, the restriction of f to $A^\vee$ is the identity, and $f(u) \leq u$ for all $u \in F$. This means that each element of $A^\vee$ is the least preimage of itself under f. Thus, once we have shown that f is a homomorphism, it will follow that it is a lower bounded epimorphism. Also note that if $a \in A$, then $a \leq f(u)$ if and only if $a \leq u$.

To see that f preserves meets, we calculate

$$\begin{aligned} f(u) \wedge' f(v) &= \bigvee \{a \in A : a \leq f(u) \wedge f(v)\} \\ &= \bigvee \{a \in A : a \leq u \wedge v\} = f(u \wedge v). \end{aligned}$$

It remains to show that f is a join homomorphism. First note that if $a \in A$ and $a \leq u \vee v$ for u and $v \in F$, then $\{u, v\}$ is a join cover of a. Since A is J–closed, there is a $T \subseteq A$ with $T \ll \{u, v\}$ and $a \leq \bigvee T$. Because $f(t) = t$ for all $t \in T$, and every $t \in T$ is either below u or v, we have

$$a \leq \bigvee T = \bigvee_{t \in T} f(t) \leq f(u) \vee f(v).$$

Hence,

$$f(u \vee v) = \bigvee \{a \in A : a \leq u \vee v\} \leq f(u) \vee f(v).$$

Since f is order preserving, $f(u \vee v) = f(u) \vee f(v)$.

Theorems 3.13 and 3.15 combine to give the following interesting result, which is useful in finite lattices and finitely presented lattices, as well as free lattices.

Corollary 3.16. *Let* $\mathbf{F}$ *be a lattice with zero and the minimal join cover refinement property. Then there is a natural one-to-one correspondence between finite* J*–closed subsets of* $\mathrm{J}(\mathbf{F})$ *and congruences* θ *on* $\mathbf{F}$ *such that* $\mathbf{F}/\theta$ *is finite and each* θ*-class has a least element.*

Given a finite J–closed subset of $\mathrm{J}(\mathbf{F})$, the corresponding congruence is, of course, $\theta = \ker f$ where f is the standard epimorphism of $\mathbf{F}$ onto $A^\vee$. Conversely, for a congruence θ with the stated properties, the corresponding J–closed set consists of those elements of $\mathbf{F}$ which are the least element of a θ-class and are join irreducible. When $\mathbf{F}$ is a finite lattice, this corollary is just a restatement of Corollary 2.34.

3. Finite Lower Bounded Lattices

Corollary 3.16, applied to free lattices, yields the following nice characterization of finite, lower bounded lattices. (Recall that a lattice is *lower bounded* if it is finitely generated and a lower bounded homomorphic image of a free lattice.)

THEOREM 3.17. *A finite lattice* $\mathbf{L}$ *is lower bounded if and only if* $\mathbf{L} \cong A^\vee$ *for a finite* J*–closed subset* A *of a finitely generated free lattice.*

An important aspect of this representation is that it gives us another way to construct finite, lower bounded lattices. In particular, recall that for any element w in a finitely generated free lattice, $\mathrm{J}(w)$ is a finite J–closed set. We define $\mathbf{L}^\vee(w)$ to be the lattice $\mathrm{J}(w)^\vee$.

COROLLARY 3.18. *For any element* $w \in \mathbf{FL}(X)$*, the mapping* f *defined by*

$$f(u) = \bigvee \{v \in \mathrm{J}(w) : v \leq u\}$$

is a lower bounded homomorphism (*the standard epimorphism*) *of* $\mathbf{FL}(X)$ *onto the lattice* $\mathbf{L}^\vee(w)$.

These lattices of the form $\mathbf{L}^\vee(w)$ will be the topic of Section 4 of this chapter.

We want to consider the homomorphic images of a finite lower bounded lattice $\mathbf{L}$ in terms of the preceding representation. Observe that, for a J–closed subset A of $\mathbf{FL}(X)$, the minimal nontrivial join covers of an element of A are the same in $A^\vee$ as in $\mathbf{FL}(X)$. Therefore a subset $B \subseteq A$ is J–closed in $A^\vee$ if and only if it is J–closed in $\mathbf{FL}(X)$.

This brings us to the fundamental theorem on homomorphic images of $A^\vee$.

THEOREM 3.19. *Let* A *be a finite,* J*–closed subset of join irreducible elements of* $\mathbf{FL}(X)$*. If a subset* B *of* A *is also* J*–closed, then the restriction to* $A^\vee$ *of the standard epimorphism of* $\mathbf{FL}(X)$ *onto* $B^\vee$ *is a homomorphism of* $A^\vee$ *onto* $B^\vee$*. Moreover, every homomorphic image of* $A^\vee$ *is isomorphic to* $B^\vee$ *for a* J*–closed subset* B *of* A*.*

Proof: As A and B are J–closed in $\mathbf{FL}(X)$, and B is also J–closed in $A^\vee$, Theorem 3.15 gives us three standard epimorphisms

$$
\begin{aligned}
f &: \mathbf{FL}(X) \twoheadrightarrow A^\vee \\
g &: \mathbf{FL}(X) \twoheadrightarrow B^\vee \\
h &: A^\vee \twoheadrightarrow B^\vee
\end{aligned}
$$

each defined by the appropriate version of the formula in that theorem. For $u \in \mathbf{FL}(X)$, we have $hf(u) = \bigvee\{b \in B : b \leq f(u)\}$. However, because $B \subseteq A$, $b \leq f(u)$ if and only if $b \leq u$. Thus $hf = g$. Moreover, for $a \in A^\vee$ we have $f(a) = a$, and hence $h(a) = g(a)$. Thus h is just the restriction of g to $A^\vee$.

The last claim is a straightforward application of Theorem 3.13 with $\mathbf{F} = A^\vee$ (since every homomorphism between finite lattices is lower bounded).

The next result extends Corollary 2.18 to free lattices.

THEOREM 3.20. *The set of the congruences ϕ of $\mathbf{FL}(X)$ for which $\mathbf{FL}(X)/\phi$ is a finite, lower bounded lattice is a filter in the congruence lattice of $\mathbf{FL}(X)$. This filter is dually isomorphic to the distributive lattice of finite* J*–closed subsets of* $\mathrm{J}(\mathbf{FL}(X))$ (*which is a lattice with respect to the operations of union and intersection*).

Proof: The first assertion follows from the facts that a homomorphic image of a finite, lower bounded lattice is again finite and lower bounded, and that a subdirect product of finitely many lower bounded lattices is lower bounded. On the other hand, the set of finite J–closed subsets of $\mathrm{J}(\mathbf{FL}(X))$ is clearly closed under finite unions and intersections, so that it is a distributive lattice. The dual isomorphism between the two lattices is established by Corollary 3.16.

It also follows that the subdirect decompositions of a given finite, lower bounded lattice $A^\vee$ correspond to the decompositions of the set A into unions of J–closed subsets in the following sense.

COROLLARY 3.21. *If A_i $(1 \leq i \leq n)$ are finite* J*–closed subsets of $\mathbf{FL}(X)$, then the lattice $B^\vee$, where $B = \bigcup_{i=1}^n A_i$, is a subdirect product of the lattices $A_i^\vee$. Conversely, if A is a* J*–closed subset of $\mathbf{FL}(X)$ and the lattice $A^\vee$ is isomorphic to a subdirect product of lattices $\mathbf{L}_i$ $(1 \leq i \leq n)$, then there exist finite* J*–closed subsets A_i such that $A = \bigcup_{i=1}^n A_i$ and $\mathbf{L}_i \cong A_i^\vee$ for $1 \leq i \leq n$.*

COROLLARY 3.22. *Let A be a finite* J*–closed subset of $\mathbf{FL}(X)$. If $A^\vee$ is subdirectly irreducible, then $A \cap \mathrm{J}(\mathbf{FL}(X)) = \mathrm{J}(w)$ for some join irreducible element $w \in A$.*

Now in general a lattice of the form $A^\vee$ for a J–closed subset of $\mathbf{FL}(X)$ is lower bounded, but it need not be upper bounded. However, the next result, which is important both for the theory of covers and for the theory of bounded lattices, gives a very nice characterization of when $A^\vee$ is bounded. It is based on Alan Day's result, Theorem 2.64, that if a finite semidistributive lattice is lower bounded, it is also upper bounded. Chapter II gives two proofs of this fact. One is given in Section 5. The other is obtained by reading from Theorem 2.30 through Lemma 2.40. This shows that a finite lattice is lower bounded if and only if $|\,\mathrm{J}(\mathbf{L})| = |\,\mathrm{J}(\mathbf{Con\ L})|$ and it is upper bounded if and only if $|\,\mathrm{M}(\mathbf{L})| = |\,\mathrm{J}(\mathbf{Con\ L})|$. Day's result follows since in a semidistributive lattice $\kappa_{\mathbf{L}}$ gives a bijection between $\mathrm{J}(\mathbf{L})$ and $\mathrm{M}(\mathbf{L})$; see Theorem 2.67.

Theorem 3.23. *Let A be a finite* J*–closed subset of* $\mathbf{FL}(X)$*. Then the following are equivalent*:

1. *the lattice $A^\vee$ is bounded*;
2. *the lattice $A^\vee$ is meet semidistributive*;
3. *every nonzero element of $A^\vee$ is lower atomic*;
4. *every join irreducible element of A is completely join irreducible.*

Proof: By Theorem 3.17, the lattice $A^\vee$ is lower bounded; thus (1) is equivalent to (2) by Theorem 2.64. Conditions (3) and (4) are equivalent by Theorem 3.5.

Next, let us show that (1) implies (4). Suppose that $A^\vee$ is a bounded lattice. If $w \in A \cap \mathrm{J}(\mathbf{FL}(X))$, then w is also join irreducible in the finite lattice $A^\vee$, so w has a unique lower cover u in $A^\vee$. The standard epimorphism $f : \mathbf{FL}(X) \to A^\vee$ is bounded, and we have the associated mappings $\alpha_f, \beta_f : A^\vee \to \mathbf{FL}(X)$. Since $w = \beta_f f(w)$, if $v < w$ in $\mathbf{FL}(X)$ then $f(v) \leq u$, whence $v \leq \alpha_f(u)$. Thus $w \succ w \wedge \alpha_f(u)$ in $\mathbf{FL}(X)$, which makes w completely join irreducible.

It remains to show that (4) implies (2). Assume that every join irreducible element in A is completely join irreducible. Let $w \in A \cap \mathrm{J}(\mathbf{FL}(X))$, denote again by u the unique lower cover of w in $A^\vee$, and let $K = \{s \in A^\vee : s \geq u \text{ and } s \ngeq w\}$. Let $f : \mathbf{FL}(X) \to A^\vee$ be the standard epimorphism, and note that $f(w_*) = u$. Thus, for every $s \in K$ we have $f(w_* \vee s) = u \vee s = s \ngeq w$, whence $w_* \vee s \ngeq w$ and so $s \leq \kappa(w)$. Therefore $\bigvee K \leq \kappa(w)$, and in particular $\bigvee K \ngeq w$. This means that, for every join irreducible element w of the lattice $A^\vee$, the element $\kappa_{A^\vee}(w)$ exists. The lattice is then meet semidistributive by Theorem 2.56.

4. The Lattice $\mathbf{L}^{\sqcup}(w)$

Recall that, for an element w of $\mathbf{FL}(X)$, we denote by $\mathbf{L}^{\vee}(w)$ the lattice $\mathrm{J}(w)^{\vee}$. When w is join irreducible, the lower cover of w in this lattice will be denoted by $w_{\dagger}$. Thus

$$w_{\dagger} = \bigvee \{u \in \mathrm{J}(w) : u < w\}. \tag{1}$$

The lattice $\mathbf{L}^{\wedge}(w) = \mathrm{M}(w)^{\wedge}$ and, for w meet irreducible, the element $w^{\dagger}$ are defined dually.

Theorem 3.24. *Let w be a join irreducible element of $\mathbf{FL}(X)$. Then $\mathbf{L}^{\vee}(w)$ is a finite, lower bounded and subdirectly irreducible lattice with $w/w_{\dagger}$ as a critical prime quotient. The kernel of the standard epimorphism $f : \mathbf{FL}(X) \to \mathbf{L}^{\vee}(w)$ is the unique largest congruence ϕ of $\mathbf{FL}(X)$ with the property that $\langle u, w\rangle \notin \phi$ whenever $u < w$.*

Conversely, if $\mathbf{L}$ is a finite, lower bounded, subdirectly irreducible lattice, then $\mathbf{L} \cong \mathbf{L}^{\vee}(w)$ for some $w \in \mathbf{FL}(X)$.

Proof: The lattice $\mathbf{L}^{\vee}(w)$ is lower bounded by Theorem 3.17. Let ϕ be a nontrivial congruence of $\mathbf{L}^{\vee}(w)$. Then, by Theorem 3.19, ϕ is the kernel of a homomorphism g of $\mathbf{L}^{\vee}(w)$ onto $B^{\vee}$ for a J–closed set B properly contained in $\mathrm{J}(w)$, and $g(u) = \bigvee\{v \in B : v \leq u\}$ for all u. Since B is a J–closed proper subset of $\mathrm{J}(w)$, $w \notin B$ and so

$$g(w) = \bigvee\{v \in B : v \leq w\} = \bigvee\{v \in B : v < w\} = g(w_{\dagger}).$$

Thus $\mathbf{L}^{\vee}(w)$ is subdirectly irreducible with $w/w_{\dagger}$ as a critical prime quotient.

Since w is the least preimage of itself under f, we have $\langle u, w\rangle \notin \ker f$ whenever $u < w$. Now, $\ker f$ is a maximal congruence with this property, since $\mathbf{FL}(X)/\ker f$ is subdirectly irreducible with the critical prime quotient $w/w_{\dagger}$. To see that $\ker f$ is the unique largest such congruence, it remains to show that the join of an arbitrary family of congruences ϕ with $\langle u, w\rangle \notin \phi$ for all $u < w$ has again this property. However, this is an easy consequence of the well known fact that if $u < w$ and $\langle u, w\rangle \in \bigvee \phi_i$, then there is a finite chain $u = u_0 < \cdots < u_n = w$ such that each pair $\langle u_{k-1}, u_k\rangle$ belongs to some ϕ_i.

For the converse, assume $\mathbf{L}$ is a finite, lower bounded, subdirectly irreducible lattice. Then there is a lower bounded epimorphism $h : \mathbf{FL}(X) \twoheadrightarrow \mathbf{L}$ for some finite set X. By Corollary 3.14 we have $\mathbf{L} \cong C^{\vee}$, where $C = \{\beta_h(p) : p \in \mathrm{J}(\mathbf{L})\}$. We can write C as a union of J–closed subsets of $\mathbf{FL}(X)$, $C = \bigcup_{c \in C} \mathrm{J}(c)$. Then by Corollary 3.22, as $\mathbf{L}$ is subdirectly irreducible, $C = \mathrm{J}(w)$ for some $w \in C$. Thus $\mathbf{L} \cong \mathrm{J}(w)^{\vee} = \mathbf{L}^{\vee}(w)$.

We should remark that if w is join reducible in $\mathbf{FL}(X)$, then $\mathbf{L}^{\vee}(w)$ is still a finite, lower bounded lattice. If $w = \bigvee w_i$ canonically, then $\mathrm{J}(w) = \bigcup \mathrm{J}(w_i)$, and hence $\mathbf{L}^{\vee}(w)$ is a subdirect product of the $\mathbf{L}^{\vee}(w_i)$'s by Corollary 3.21. However, this representation need not be proper, as happens when $w = x \vee (y \wedge (x \vee z))$ and $\mathbf{L}^{\vee}(w) \cong \mathbf{L}^{\vee}(y \wedge (x \vee z))$. Nor can every finite, lower bounded lattice be represented as $\mathbf{L}^{\vee}(w)$: the three element chain provides an easy counterexample. These facts tend to limit the usefulness of these lattices for w join reducible, but in later chapters we will have some occasion to use them.

Let us recall that, by Dilworth characterization of lattice congruences [**40**], for any prime quotient u/v of $\mathbf{FL}(X)$ (or of any lattice) there exists a largest congruence separating the elements u and v; this congruence is denoted by $\psi(u, v)$.

The following result was proved in McKenzie [**99**]. Compare it with Theorems 2.79, 2.80 and Corollary 2.81.

THEOREM 3.25. *Let w be a completely join irreducible element of $\mathbf{FL}(X)$. Let f be the standard epimorphism of $\mathbf{FL}(X)$ onto $\mathbf{L}^{\vee}(w)$ and let g be the dual standard epimorphism of $\mathbf{FL}(X)$ onto $\mathbf{L}^{\wedge}(\kappa(w))$. Then*

$$\ker f = \psi(w, w_*) = \psi(\kappa(w)^*, \kappa(w)) = \ker g. \tag{2}$$

For $u \in \mathbf{FL}(X)$, the congruence class of u for this congruence is the interval $\alpha_g g(u)/\beta_f f(u)$. The lattice $\mathbf{FL}(X)/\psi(w, w_*)$ is a splitting lattice and it is isomorphic to $\mathbf{L}^{\vee}(w)$.

Proof: Since $w \wedge \kappa(w) = w_*$ and $w \vee \kappa(w) = \kappa(w)^*$, the quotients w/w_* and $\kappa(w)^*/\kappa(w)$ are projective and a congruence separates w from w_* if and only if it separates $\kappa(w)^*$ from $\kappa(w)$. Hence $\psi(w, w_*) = \psi(\kappa(w)^*, \kappa(w))$. It follows from Theorem 3.24 and its dual that this congruence is the kernel of f and at the same time the kernel of g, and that the factor is isomorphic to both $\mathbf{L}^{\vee}(w)$ and $\mathbf{L}^{\wedge}(\kappa(w))$, which lattice is then both lower bounded and upper bounded and, of course, subdirectly irreducible and finite.

By definition, $\beta_f f(u)$ is the least element of the $\ker f$–congruence class of u and $\alpha_g g(u)$ is the greatest element of the $\ker g$–congruence class of u. Thus the statement about the congruence classes follows easily from (2).

THEOREM 3.26. *The following are equivalent for a join irreducible element $w \in \mathbf{FL}(X)$:*

(1) *w is completely join irreducible in $\mathbf{FL}(X)$;*
(2) *every element of $\mathrm{J}(w)$ is completely join irreducible;*

(3) *every subelement of* w *is lower atomic*;
(4) $\mathbf{L}^{\vee}(w)$ *is meet semidistributive*;
(5) $\mathbf{L}^{\vee}(w)$ *is a splitting lattice.*

Proof: Clearly, (3) implies (2) and (2) implies (1). By Theorem 3.25, (1) implies (5). Also, (5) implies (4) because bounded lattices are semidistributive. By Theorem 3.23, (2) is equivalent to (4). Finally, as every subelement of w not belonging to $\mathrm{J}(w)$ is either a generator or a join reducible element whose all canonical joinands belong to $\mathrm{J}(w)$, (2) implies (3) by Corollary 3.8.

Combining this result with Corollary 3.8 one easily obtains the following characterization of lower atomic elements.

Corollary 3.27. *The following are equivalent for an element* $w \in \mathbf{FL}(X)$:
(1) w *is lower atomic in* $\mathbf{FL}(X)$;
(2) *every element of* $\mathrm{J}(w)$ *is completely join irreducible*;
(3) *every subelement of* w *is lower atomic*;
(4) *the number of lower covers of* w *equals the number of canonical joinands of* w.

We close this section with a theorem which will be useful in later chapters.

Theorem 3.28. *Suppose that* w *is a proper join in* $\mathbf{FL}(X)$, *and that* w_1 *is a canonical joinand of* w. *For any element* u *such that* $w_1 \leq u \leq w$, *we have*
(1) $\mathrm{J}(w_1) \subseteq \mathrm{J}(u)$, *and*
(2) *if* w_1 *is not a generator,* $\mathrm{M}(w_1) \subseteq \mathrm{M}(u)$.
Consequently, if there exists a completely join irreducible element u *with* $w_1 \leq u \leq w$, *then* w_1 *is also completely join irreducible. Similarly, if there exists a completely meet irreducible element in* w/w_1, *then* w_1 *is upper atomic.*

Proof: We will prove (1) by induction on the complexity of w. Since $\mathrm{J}(w_1) \subseteq \mathrm{J}(w)$, we may assume $w_1 < u < w$. If u were a generator, we could replace w_1 by u in the canonical expression of w, obtaining a shorter expression. If u is a proper meet, we can apply (W) to $u = \bigwedge u_i \leq \bigvee w_k = w$. Since $w_1 < u$ we have $u \nleq w_k$ for all k, and hence there is an i such that $u_i \leq w$. Then $w_1 \leq u_i \leq w$, whence by induction $\mathrm{J}(w_1) \subseteq \mathrm{J}(u_i)$. As before, u_i is not a generator, so it is a proper join and $\mathrm{J}(u_i) = \bigcup_j \mathrm{J}(u_{ij}) \subseteq \mathrm{J}(u)$. Thus $\mathrm{J}(w_1) \subseteq \mathrm{J}(u)$, as desired, in this case. On the other hand, suppose u is a proper join. If w_1 is a generator, then $w_1 \leq u = \bigvee u_i$ implies $w_1 \leq u_i$ for some i. If w_1

is not a generator, then we can apply (W) to $w_1 = \bigwedge w_{1j} \leq \bigvee u_i = u$. As $w_{1j} \nleq w$ for all j by canonical form (Theorem 1.18), this also yields $w_1 \leq u_i \, (\leq w)$ for some i. Again by induction $\mathrm{J}(w_1) \subseteq \mathrm{J}(u_i) \subseteq \mathrm{J}(u)$, so we obtain the desired conclusion in this case also.

The proof of (2) is similar, as follows. Looking carefully at the formula for $\mathrm{M}(w)$, one sees that $\mathrm{M}(w_i) \subseteq \mathrm{M}(w)$ for a canonical joinand w_i of w except possibly when w_i is a generator. Thus in our case, $\mathrm{M}(w_1) \subseteq \mathrm{M}(w)$. Hence if $u = w$ then $\mathrm{M}(w_1) \subseteq \mathrm{M}(w) = \mathrm{M}(u)$. Clearly the result also holds if $w_1 = u$. Thus we may assume $w_1 < u < w$. If u were a generator, we could replace w_1 by u in the canonical expression of w, obtaining a shorter expression. If $u = \bigvee u_i$ is a join, then applying (W) to $w_1 = \bigwedge w_{1j} \leq u = \bigvee u_i$ gives an i such that $w_1 \leq u_i$, since $w_{1j} \leq u \leq w$ would violate the canonical form for w, see Theorem 1.18. If $u = \bigwedge u_i$, applying (W) to $u = \bigwedge u_i \leq w = \bigvee w_k$ and using $u \neq w_1$ yields that $w_1 \leq u_i \leq w$ for some i. Now (2) follows easily by induction on the rank of u.

The consequences are obtained by applying Theorem 3.26 to (1) and (2).

5. Syntactic Algorithms

The results of the last section give a conceptually (and visually) easy way to determine if a join irreducible element w of a free lattice is completely join irreducible: find $\mathbf{L}^{\vee}(w) = \mathrm{J}(w)^{\vee}$ and see if it is semidistributive. Several examples are given in the next section. However, this technique is not computationally efficient since $\mathrm{J}(w)^{\vee}$ may be exponentially large compared to $\mathrm{J}(w)$. In this section we will give a polynomial time algorithm which determines if w is completely join irreducible directly from $\mathrm{J}(w)$. We also give an algorithm to find $\kappa(w)$ when w is completely join irreducible. The efficiency of these algorithms is analyzed in Chapter XI.

Let w be a join irreducible element of $\mathbf{FL}(X)$ and let

$$\mathrm{K}(w) = \{v \in \mathrm{J}(w) : w_\dagger \vee v \ngeq w\} \tag{3}$$

where $w_\dagger$ is defined by equation (1) on page 102. The join $\bigvee \mathrm{K}(w)$ is an element of $\mathbf{L}^{\vee}(w)$ with the property that for every $v \in \mathbf{L}^{\vee}(w)$ with $v \geq w_\dagger$, either $v \geq w$ or $v \leq \bigvee \mathrm{K}(w)$. This means that if $\bigvee \mathrm{K}(w) \ngeq w$, then $\bigvee \mathrm{K}(w) = \kappa_{\mathbf{L}^{\vee}(w)}(w)$.

Lemma 3.29. *If w is completely join irreducible in $\mathbf{FL}(X)$, then the following are equivalent for any $v \in \mathbf{FL}(X)$:*

(1) $w \nleq w_\dagger \vee v$,
(2) $w \nleq w_* \vee v$,

(3) $v \leq \kappa(w)$.

Consequently, $w \nleq \bigvee \mathrm{K}(w)$ *whenever* w *is completely join irreducible.*

Proof: Let $f : \mathbf{FL}(X) \to \mathbf{L}^\vee(w)$ be the standard epimorphism. Then $f(w) = w$ and $f(w_*) = w_\dagger$, so $w_* \vee v \geq w$ implies $w_\dagger \vee v \geq w_\dagger \vee f(v) = f(w_*) \vee f(v) = f(w_* \vee v) \geq f(w) = w$. This shows (1) implies (2) and the reverse implication is obvious. (2) and (3) are easily seen to be equivalent; see Corollary 3.2. Since (1) and (3) are equivalent, each element of $\mathrm{K}(w)$ is below $\kappa(w)$ when w is completely join irreducible. The last statement follows since otherwise $w \leq \bigvee \mathrm{K}(w) \leq \kappa(w)$, a contradiction.

The next theorem gives a simple, efficient procedure to decide if a join irreducible element of a free lattice is completely join irreducible.

Theorem 3.30. *Let* w *be a join irreducible element of* $\mathbf{FL}(X)$. *Then* w *is completely join irreducible in* $\mathbf{FL}(X)$ *if and only if the following two conditions are satisfied*:

(1) *every* $u \in \mathrm{J}(w) - \{w\}$ *is completely join irreducible,*
(2) $w \nleq \bigvee \mathrm{K}(w)$.

Proof: The direct implication is a consequence of Theorem 3.26 and the preceding lemma. Conversely, assume that the two conditions are satisfied. By Theorems 3.26 and 2.56 (the proof of the latter is easy), we need only show that $\kappa_{\mathbf{L}^\vee(w)}(u)$ exists for each $u \in \mathrm{J}(w)$. For $u = w$ it follows from (2) that $\bigvee \mathrm{K}(w) = \kappa_{\mathbf{L}^\vee(w)}(w)$. Let $u \in \mathrm{J}(w) - \{w\}$. Since $\mathbf{L}^\vee(u)$ is meet semidistributive, $\kappa_{\mathbf{L}^\vee(u)}(u)$ exists; denote this element by q. Let h be the standard homomorphism of $\mathbf{L}^\vee(w)$ onto $\mathbf{L}^\vee(u)$. We shall show that $\alpha_h(q)$, the largest preimage of q under h, is $\kappa_{\mathbf{L}^\vee(w)}(u)$. If $v \in \mathbf{L}^\vee(w)$ is above the unique lower cover of u in $\mathbf{L}^\vee(w)$, then $h(v)$ is above the unique lower cover of u in $\mathbf{L}^\vee(u)$ and so either $h(v) \geq u$ or $h(v) \leq q$. If $h(v) \geq u$, then $v \geq u$, since u is the least preimage of itself under h. If $h(v) \leq q$, then clearly $v \leq \alpha_h(q)$.

Theorem 3.31. *Let* w *be a completely join irreducible element of* $\mathbf{FL}(X)$. *Then* $\mathrm{M}(\kappa(w)) = \{\kappa(u) : u \in \mathrm{J}(w)\}$.

Proof: Let $f : \mathbf{FL}(X) \to \mathbf{L}^\vee(w)$ be the standard epimorphism and let $g : \mathbf{FL}(X) \to \mathbf{L}^\wedge(\kappa(w))$ be the dual standard epimorphism. By Theorem 3.25, $\ker f = \psi(w, w_*) = \psi(\kappa(w)^*, \kappa(w)) = \ker g$. Let ψ denote this congruence.

By the definition of the standard homomorphism, $u \in \mathrm{J}(w)$ implies $f(u) = u$ and $\beta_f(u) = u$. Recall $\mathbf{L}^\vee(w) = \mathrm{J}(w)^\vee$, from which it easily follows that $\mathrm{J}(w)$ is the set of join irreducible elements of $\mathbf{L}^\vee(w)$. Thus $\mathrm{J}(w)$ may be described as the least elements of the those congruence classes of $\psi = \ker f$ which are join irreducible in the quotient $\mathbf{FL}(X)/\psi$. But if $u \in \mathbf{FL}(X)$ is the least element of its ψ–class then this class is join irreducible in $\mathbf{FL}(X)/\psi$ if and only if u is join irreducible in $\mathbf{FL}(X)$. For if u is a proper join then $f(u)$ will be also since u is the least element of its class, and, if u is join irreducible and $f(u) = a \vee b$, then, since β preserves joins, $u = \beta f(u) = \beta(a) \vee \beta(b)$, which implies $u \leq a$ or $u \leq b$, so that either $f(u) = a$ or $f(u) = b$.

Now let $u \in \mathrm{J}(w)$. Since w is completely join irreducible, $\kappa(u)$ exists. If $\kappa(u)$ is not the greatest element of its ψ–class then $\langle \kappa(u), \kappa(u)^* \rangle \in \psi$, which implies $\langle u, u_* \rangle \in \psi$, contradicting the fact that u is the least element of its class. By the dual of the remarks in the previous paragraph and the fact that $\psi(w, w_*) = \psi(\kappa(w)^*, \kappa(w))$, this implies $\kappa(u) \in \mathrm{M}(\kappa(w))$. Thus $\{\kappa(u) : u \in \mathrm{J}(w)\} \subseteq \mathrm{M}(\kappa(w))$, and dually $\{\kappa^d(v) : v \in \mathrm{M}(\kappa(w))\} \subseteq \mathrm{J}(w)$. Applying κ to this last inclusion and using the first gives

$$\mathrm{M}(\kappa(w)) = \{v : v \in \mathrm{M}(\kappa(w))\} \subseteq \{\kappa(u) : u \in \mathrm{J}(w)\} \subseteq \mathrm{M}(\kappa(w)),$$

proving the result.

Now we come to one of the most important theorems of this chapter. It gives a formula for $\kappa(w)$ for a completely join irreducible element w which is both theoretically and computationally useful.

Theorem 3.32. *Let w be a completely join irreducible element of $\mathbf{FL}(X)$. Then*

$$\begin{aligned}\kappa(w) = \bigvee &\{x \in X : w_\dagger \vee x \ngeq w\} \\ &\vee \bigvee \{k^\dagger \wedge \kappa(v) : v \in \mathrm{J}(w) - \{w\},\ w \nleq \kappa(v)\}\end{aligned}$$

where

$$k^\dagger = \bigwedge \left\{\kappa(v) : v \in \mathrm{J}(w) - \{w\},\ \kappa(v) \geq \bigvee \mathrm{K}(w)\right\}$$

and

$$\mathrm{K}(w) = \{v \in \mathrm{J}(w) : w_\dagger \vee v \ngeq w\}.$$

Proof: Put $A = \{\kappa(v) : v \in \mathrm{J}(w) - \{w\}\}$, so that by Theorem 3.31 the set A is M–closed and $A = \mathrm{M}(\kappa(w)) - \{\kappa(w)\}$. We have the standard epimorphism $f : \mathbf{FL}(X) \to \mathbf{L}^\vee(w)$, the dual standard epimorphism $g : \mathbf{FL}(X) \to \mathbf{L}^\wedge(\kappa(w))$ and the dual standard epimorphism $h : \mathbf{FL}(X) \to A^\wedge$. Since A is a proper subset of $\mathrm{M}(\kappa(w))$, $\ker h$ properly contains the congruence $\ker g = \ker f$. For this reason, if $\kappa(v) \in A$ is an element such that $\kappa(v) \geq \bigvee \mathrm{K}(w) = \kappa_{\mathbf{L}^\vee(w)}(w)$, then

$$\kappa(v) = h(\kappa(v)) \geq h(\kappa_{\mathbf{L}^\vee(w)}(w)) = h(\kappa(w)) = h(\kappa(w)^*) \geq \kappa(w)^*.$$

So, for an element $\kappa(v) \in A$ we have $\kappa(v) \geq \bigvee \mathrm{K}(w)$ if and only if $\kappa(v) \geq \kappa(w)^*$ (the converse implication is obvious). This shows that $k^\dagger \geq \kappa(w)^* \geq \kappa(w)$ and, in fact, $k^\dagger = \alpha_g g(\kappa(w)^*) = \alpha_f f(\kappa(w)^*)$.

To prove the formula for $\kappa(w)$, observe first that all the joinands of the right hand side are below $\kappa(w)$. Indeed, if $x \in X$ and $w_\dagger \vee x \not\geq w$, then $w_* \vee x \not\geq w$ and hence $x \leq \kappa(w)$. Suppose $w \not\leq \kappa(v)$. Note that $k^\dagger$ is the unique upper cover of $\kappa(w)$ in $\mathbf{L}^\wedge(\kappa(w))$ and that $g(w) = \kappa^d_{\mathbf{L}^\wedge(\kappa(w))}(\kappa(w))$. Thus, since $k^\dagger \wedge \kappa(v) \leq k^\dagger$, it is either above $g(w)$ or below $\kappa(w)$. But the former implies $w \leq g(w) \leq \kappa(v)$, a contradiction.

To prove the opposite inequality, we need only show that the right hand side is above any canonical joinand of $\kappa(w)$. Let u be a canonical joinand of $\kappa(w)$. If $u \in X$, then $u \in \{x \in X : w_\dagger \vee x \not\geq w\}$. So, let $u \notin X$. By the dual of Theorem 3.4 there is a unique canonical meetand of u not above $\kappa(w)$. Of course, this canonical meetand belongs to $\mathrm{M}(\kappa(w)) - \{\kappa(w)\}$ and thus equals $\kappa(v)$ for some $v \in \mathrm{J}(w) - \{w\}$. Since $\kappa(v) \not\geq \kappa(w)^*$, we have $\kappa(v) \not\geq w$ (otherwise all canonical meetands of u, and hence u itself, would be above w). So, $u \leq k^\dagger \wedge \kappa(v)$, which is one of the joinands of the right hand side of the formula for $\kappa(w)$.

Now we consider the canonical form of w_*, with the *caveat* that $w_* = \kappa(w)$ can occur, though only finitely often in each free lattice, as we shall see in Section 2 of Chapter VIII. First we require a lemma.

LEMMA 3.33. *If w is completely join irreducible, then w_* is not a canonical joinand of any canonical meetand of w.*

Proof: Suppose the lemma fails and let w be a counterexample of minimal rank. Then w is a completely join irreducible element and w_* is a canonical joinand of some canonical meetand w_1 of w. It follows from Theorem 3.4, that $w_1 = w_{11} \vee w_*$ canonically for some w_{11}. This implies that w_* belongs to $\mathrm{J}(w)$ and so is completely join irreducible by Theorem 3.26, whence w_{**} and $\kappa(w_*)$ exist. Moreover, since w_* is a subelement of w, it has lower rank.

First suppose that $\kappa(w_*) \leq \kappa(w)$. Now, by Corollary 3.2 applied to w_*, either $w_{**} \vee w_{11} \leq \kappa(w_*)$ or $w_{**} \vee w_{11} \geq w_*$. In the former case, $w_{11} \leq \kappa(w_*)$ and so

$$w \leq w_1 = w_{11} \vee w_* \leq \kappa(w_*) \vee w_* \leq \kappa(w),$$

a contradiction. In the latter case,

$$w_{11} \vee w_{**} = w_{11} \vee w_* \vee w_{**} = w_1.$$

Since $w_1 = w_{11} \vee w_*$ canonically, this contradicts the fact that canonical expression cannot be refined, see Theorem 1.19.

So we may assume that $\kappa(w_*) \nleq \kappa(w)$. Now $w \succ w_* \succ w_{**}$ and since w is join irreducible, the interval w/w_{**} contains only these three elements. Thus $w \wedge \kappa(w_*) = w_{**}$. Let $\kappa(w) = v_1 \vee \cdots \vee v_m$ canonically. By Theorem 3.4, we may assume that $v_i \leq w_{**}$, for $i \geq 2$. Applying (W) to

$$w_{**} = w \wedge \kappa(w_*) \leq \kappa(w) = \bigvee v_i$$

yields that $w_{**} \leq v_i$ for some i. If $i = 1$, then v_1 is above all other v_i's. This implies that $\kappa(w) = v_1$. Of course v_1 is join irreducible and $\kappa(w)$ is meet irreducible, and so $\kappa(w)$ must be a generator x. By Corollary 1.7, $w = \kappa^d(\kappa(w)) = \kappa^d(x) = \bigwedge X - \{x\} \succ 0$, and so w_{**} cannot exist.

Thus we must have $w_{**} \leq v_i$ for some $i \geq 2$. But then $v_i = w_{**}$, i.e., w_{**} is a canonical joinand of $\kappa(w)$, which is a canonical meetand of w_*. Thus w_* is also a counterexample to the lemma. Since w_* has lower rank than w, this is a contradiction.

The following theorem will prove to be very important in our study of chains of covers and finite intervals in free lattices.

Theorem 3.34. *Let w be a completely join irreducible element of $\mathbf{FL}(X)$ with $w = w_1 \wedge \cdots \wedge w_m$ canonically. Then*

$$\{\kappa(w)\} \cup \{w_i : w_i \ngeq \kappa(w)\}$$

is the set of canonical meetands of w_*.

Proof: By Theorem 3.1, $\kappa(w)$ is the unique canonical meetand of w_* not above w. Denote by $u_1, \ldots, u_k$ all the remaining canonical meetands of w_* and (after renumbering) let $w_1, \ldots, w_n$ be all the canonical meetands of w not above $\kappa(w)$. It follows easily from Theorem 1.18 that the element $w_1 \wedge \cdots \wedge w_n$ is in canonical form. Since $w_* = w \wedge \kappa(w) = w_1 \wedge \cdots \wedge w_n \wedge \kappa(w)$, we have $\{u_1, \ldots, u_k, \kappa(w)\} \gg \{w_1, \ldots, w_n, \kappa(w)\}$ and so, by the dual of Theorem 1.19, $\{u_1, \ldots, u_k\} \gg \{w_1, \ldots, w_n\}$. In order to show that the two sets are equal, it remains to prove $u_1 \wedge \cdots \wedge u_k = w_1 \wedge \cdots \wedge w_n$, as this will then imply $\{w_1, \ldots, w_n\} \gg \{u_1, \ldots, u_k\}$. Suppose, on the contrary, that $u_1 \wedge \cdots \wedge u_k \nleq w_i$ for some $i \leq n$. Then the condition (W) applied to the inequality $u_1 \wedge \cdots \wedge u_k \wedge \kappa(w) = w_* \leq w_i$, where w_i is to be expressed as the join of its canonical joinands, gives us that there exists a canonical joinand of w_i above w_*. However, by Theorem 3.4 all but one canonical joinands are below w_*; consequently, w_* is a canonical joinand of w_i. But this contradicts Lemma 3.33.

Theorem 3.35. *Let Y be a subset of X, and let $w \neq \bigwedge Y$ be a join irreducible element of the lattice $\mathbf{FL}(Y)$ (which we can consider to be the the sublattice of $\mathbf{FL}(X)$ generated by Y). Then w is completely* *join irreducible in* $\mathbf{FL}(Y)$ *if and only if it is completely join irreducible in* $\mathbf{FL}(X)$*.*

Proof: This is a consequence of Theorem 3.26, as the lattice $\mathbf{L}^{\vee}(w)$ depends only on the elements in the sublattice of $\mathbf{FL}(X)$ generated by the elements of X that occur in the canonical expression of w.

Let us remark that while the existence of a lower cover of w in $\mathbf{FL}(X)$ depends only on the set $\mathbf{var}\,(w)$ of generators occurring in w, the element w_* actually covered by w does depend on the set X. Using Theorem 3.32, it is not hard to see that if $\kappa(w) = p(x_1, \ldots, x_n)$ in $\mathbf{FL}(\,\mathbf{var}\,(w))$, then in $\mathbf{FL}(X)$ the new $\kappa(w)$ is given (not necessarily in canonical form) by $p(x_1 \vee s, \ldots, x_n \vee s)$ where $s = \bigvee(X - \mathbf{var}\,(w))$.

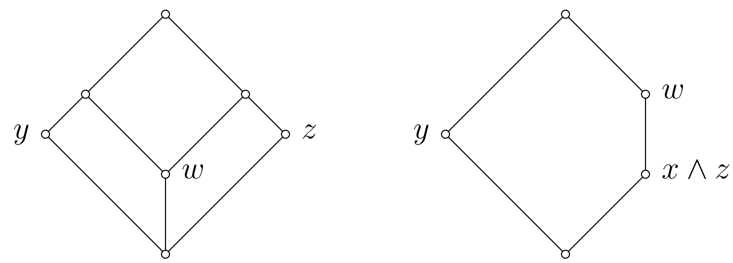
6. Examples

Example 3.36. In $\mathbf{FL}(X)$ with $X = \{x, y, z\}$ take $w = x \wedge (y \vee z)$. Then $\mathrm{J}(w) = \{x \wedge (y \vee z), y, z\}$ and $\mathbf{L}^{\vee}(w)$ is the lattice drawn in Figure 3.2(1). This lattice fails meet semidistributivity, so by Theorem 3.26 we conclude that w does not have a lower cover. (This was the first example of an element in $\mathbf{FL}(X)$ with no lower cover, found by R. A. Dean around 1960 using *ad hoc* methods.) By the dual of Theorem 3.5, the upper covers of w are of the form $w \vee \kappa^d(w_i)$ for each w_i in the canonical meet representation of w which has an upper cover. In this case we have $w_1 = x$ and $\kappa^d(x) = y \wedge z$, yielding the upper cover $(x \wedge (y \vee z)) \vee (y \wedge z)$, and $w_2 = y \vee z$ with $\kappa^d(y \vee z) = x$, yielding the upper cover $(x \wedge (y \vee z)) \vee x = x$.

Example 3.37. Again in $\mathbf{FL}(X)$ with $X = \{x, y, z\}$ take $w = x \wedge (y \vee (x \wedge z))$. Then $\mathrm{J}(w) = \{w, y, x \wedge z\}$. The lattice $\mathbf{L}^{\vee}(w)$, which is drawn in Figure 3.2(2), is a five element nonmodular lattice. This lattice is meet semidistributive, so we conclude that w has a lower cover. To find $\kappa(w)$, apply Theorem 3.32: $\kappa(w) = z \vee ((x \vee z) \wedge y)$. Then $w_* = w \wedge \kappa(w) = x \wedge (y \vee (x \wedge z)) \wedge (z \vee ((x \vee z) \wedge y))$.

As in the preceding example, we obtain an upper cover of w in $w \vee \kappa^d(x) = (x \wedge (y \vee (x \wedge z))) \vee (y \wedge z)$. However, this is the only upper cover, since the second canonical meetand of w, the element $y \vee (x \wedge z)$, has no upper cover by the argument dual to that given in Example 3.36. This means that the element w has no upper cover below x.

Example 3.38. In Table **??** we give $\kappa(w)$ and the lower and upper covers for six selected join irreducible elements in $\mathbf{FL}(X)$ with $X = \{x, y, z\}$. The corresponding lattices $\mathbf{L}^{\vee}(w)$ are shown in Figure 3.3.



(1) $\mathbf{L}^{\vee}(x \wedge (y \vee z))$ (2) $\mathbf{L}^{\vee}(x \wedge (y \vee (x \wedge z)))$

FIGURE 3.2

EXAMPLE 3.39. An element $w \in \mathbf{FL}(X)$ is called *coverless* if it has no lower and no upper covers. Here is an example of a coverless element in $\mathbf{FL}(X)$ with $X = \{x, y, z\}$:

$$w = ((x \wedge (y \vee z)) \vee (y \wedge z)) \wedge ((y \wedge (x \vee z)) \vee (x \wedge z)).$$

To see that w has no lower cover, take the subelement $x \wedge (y \vee z)$, which is without lower cover by Example 3.36, and apply Theorem 3.26. On the other hand, by the dual of (3) in Example 3.38, $(x \wedge (y \vee z)) \vee (y \wedge z)$ has no upper cover and symmetrically the same is true for $(y \wedge (x \vee z)) \vee (x \wedge z)$. Since these are the elements in the canonical meet representation of w, we can conclude that w has no upper cover.

We will show in Chapter IX that every infinite interval in a free lattice contains a coverless element.

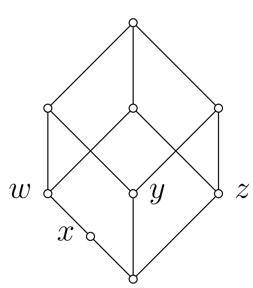
EXAMPLE 3.40. Recall from Chapter I that an element w of $\mathbf{FL}(X)$ is said to be *slim* if, in the case that it is join irreducible, its canonical form is either

$$w = x_n \wedge (x_{n-1} \vee (x_{n-2} \wedge (\cdots \vee (x_2 \wedge x_1)))) \quad (n \geq 2,\ n \text{ even})$$

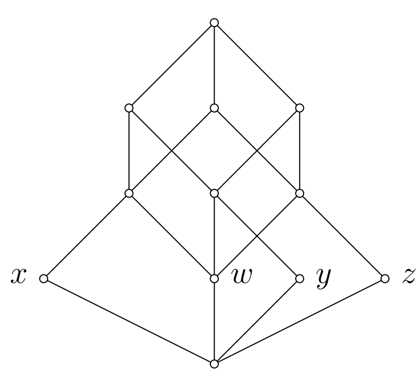
or

$$w = x_n \wedge (x_{n-1} \vee (x_{n-2} \wedge (\cdots \wedge (x_2 \vee x_1)))) \quad (n \geq 1,\ n \text{ odd})$$

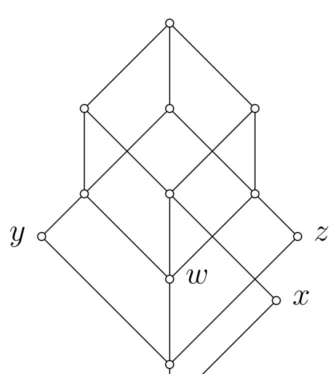
where the x_i's are generators, not necessarily distinct. For w meet irreducible, take the dual expressions. In both cases, the condition on



(1) $\mathbf{L}^{\vee}((x \vee y) \wedge (x \vee z))$



(2) $\mathbf{L}^{\vee}((x \vee y) \wedge (x \vee z) \wedge (y \vee z))$



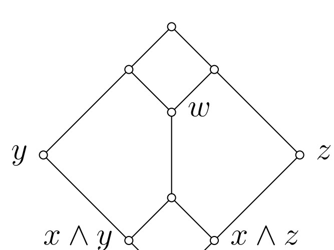


TABLE 3.1

(1)	w	$(x \ \sqcup \ y) \wedge (x \vee z)$
	$\kappa(w)$	$x \vee ((x \vee y) \wedge (x \vee z) \wedge (y \vee z))$
	w_*	$x \vee ((x \vee y) \wedge (x \vee z) \wedge (y \vee z))$
	upper covers	$x \vee y$ & $x \vee z$
(2)	w	$(x \vee y) \wedge (x \vee z) \wedge (y \vee z)$
	$\kappa(w)$	not defined
	w_*	not defined
	upper covers	$x \vee w$ & $y \vee w$ & $z \vee w$
(3)	w	$(x \vee (y \wedge z)) \wedge (y \vee z)$
	$\kappa(w)$	not defined
	w_*	not defined
	upper covers	$x \vee (y \wedge z)$
(4)	w	$x \wedge (y \vee (x \wedge z)) \wedge (z \vee (x \wedge y))$
	$\kappa(w)$	$(y \wedge (x \vee z)) \vee (z \wedge (x \vee y))$
	w_*	$w \wedge \kappa(w)$
	upper covers	$w \vee (y \wedge z)$
(5)	w	$(x \vee (y \wedge z)) \wedge (y \vee (x \wedge z))$
	$\kappa(w)$	$z \vee (x \wedge (y \vee z)) \vee (y \wedge (x \vee z))$
	w_*	$w \wedge \kappa(w)$
	upper covers	none
(6)	w	$x \wedge ((x \wedge y) \vee (x \wedge z) \vee (y \wedge z))$
	$\kappa(w)$	$(x \wedge y) \vee (x \wedge z)$
	w_*	$(x \wedge y) \vee (x \wedge z)$
	upper covers	$(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$

w to be in canonical form is that $x_i \neq x_{i+1}$ and $(x_i, x_{i+1}) \neq (x_{i+2}, x_{i+3})$ for all i, $x_1 \neq x_3$ and $(x_1, x_3) \neq (x_4, x_5)$.

Here we are going to decide which join irreducible slim elements have a lower cover in $\mathbf{FL}(X)$. If $n \leq 2$, then clearly w does have a lower cover. We have found in Example 3.36 that this is not the case when $n = 3$. From this it follows by Theorem 3.26 that if $n \geq 3$ is odd, then w does not have a lower cover.

In the following let $n \geq 4$ and n be an even number. For $i = 2, 4, \dots, n$ put $v_i = x_i \wedge (x_{i-1} \vee (x_{i-2} \wedge (\cdots \vee (x_2 \wedge x_1))))$, so that $w = v_n$ and

$$\mathrm{J}(w) = \{v_2, v_4, \dots, v_n, x_3, x_5, \dots, x_{n-1}\}.$$

We can assume that if $x_4 \in \{x_1, x_2\}$, then $x_4 = x_2$.

Suppose that $x_k \notin \{x_1, x_2\}$ for some even number $k \geq 4$ and take the least such number k. For any even number $i \leq k-2$ we have $v_2 \leq v_i$ and thus $v_i \not\leq v_k$, since $v_2 \not\leq v_k$. Also, no generator can be below v_k. We see that in $\mathbf{L}^\vee(v_k)$, $v_k \wedge' x_{k-1} = v_k \wedge' v_{k-2} = 0$. On the other hand, $v_k \wedge' (x_{k-1} \vee v_{k-2}) = v_k$ and consequently the lattice $\mathbf{L}^\vee(v_k)$ is not meet semidistributive, which implies that the element v_k and then also the element w do not have a lower cover.

Next suppose that $x_k \in \{x_1, x_2\}$ for all even numbers k and that there is an even $k \geq 4$ with $x_k \neq x_{k-2}$. Taking again the least such number k, we have $k \geq 6$, $x_k = x_1$ and $x_{k-2} = x_{k-4} = \cdots = x_2$. Then $v_2 < v_4 < \cdots < v_{k-2}$, $v_2 < v_k$ but (as it is easy to see) $v_4 \not\leq v_k$. Also, no generator can be below v_k. We see that in $\mathbf{L}^\vee(v_k)$, $v_k \wedge' (x_{k-1} \vee v_2) = v_k \wedge' v_{k-2} = v_2$. On the other hand, $v_k \wedge' (x_{k-1} \vee v_2 \vee v_{k-2}) = v_k \wedge' (x_{k-1} \vee v_{k-2}) = v_k \neq v_2$. As before, w does not have a lower cover.

In the case when $x_2 = x_4 = \cdots = x_n$ the element w does have a lower cover. In order to prove it by induction on n, according to Theorem 3.30 it is sufficient to verify that $w \not\leq \bigvee \mathrm{K}(w)$. We have $v_2 < v_4 < \cdots < v_n$, $w_\dagger = v_{n-2}$ and $\mathrm{K}(w)$ is the union of $\{v_2, v_4, \dots, v_{n-2}\}$ with the set $Y = \{x_3, x_5, \dots, x_{n-1}\} - \{x_{n-1}\}$. Then $\bigvee \mathrm{K}(w) = v_{n-2} \vee \bigvee Y$ and now it is easy to check that $w \not\leq \bigvee \mathrm{K}(w)$.

Thus we have found all completely join irreducible slim elements, which can be summarized as follows.

A join irreducible slim element has a lower cover in $\mathbf{FL}(X)$ *if and only if it is either a generator or an element of the canonical form*

$$w = x_n \wedge (x_{n-1} \vee (x_{n-2} \wedge (\cdots \vee (x_2 \wedge x_1))))$$

where $n \geq 2$ *is even and* $x_2 = x_4 = \cdots = x_n$.

For any positive integer n let $f(n)$ stand for the number of join irreducible slim elements having a lower cover whose canonical form has length n. It follows that if $X = \{x, y, z\}$, then f is upper bounded by a linear function, while $f(n)$ grows exponentially if X contains more than three elements.

Example 3.41. In $\mathbf{FL}(X)$ with $X = \{x, y, z, t\}$, the element

$$w = (x \vee y) \wedge (x \vee ((y \vee z) \wedge (z \vee t)))$$

does not have a lower cover. On the other hand, every canonical meetand of w has exactly one canonical joinand not below w. This example shows that the converse of Theorem 3.4 is false. However, in the following example we show that for elements of very low complexity the converse of Theorem 3.4 does hold.

EXAMPLE 3.42. Let $w \in X^{\wedge\vee\wedge}$ be a join irreducible element of $\mathbf{FL}(X)$. Then w has a lower cover in $\mathbf{FL}(X)$ if and only if every canonical meetand of w which is not a generator has exactly one canonical joinand not below w.

To prove this assertion, let $w = \bigwedge_i \bigvee_j w_{ij} \wedge \bigwedge_k x_k$ canonically. Then $\mathrm{J}(w)$ consists of w and of the elements w_{ij}, each of which is a meet of generators. If for any i there is exactly one j with $w_{ij} \nleq w$, then clearly $w_{ij} \in \mathrm{K}(w)$ if and only if $w_{ij} < w$ and we can apply Theorem 3.30.

7. Connected Components and the Bottom of $\mathbf{FL}(X)$

Let us recall from Chapter I that the connected component of an element $a \in \mathbf{FL}(X)$ is the set of the elements b for which there exists a finite sequence $a = c_0, c_1, \ldots, c_k = b$ such that each c_i either covers or is covered by c_{i+1}. In this section we are going to describe the connected components of a generator, of the zero element and of a few elements contained in the sublattice generated by the atoms. (Of course, the connected component of 1 is described by duality.) Verification of the facts is a straightforward application of the algorithm developed in Section 5, so we only formulate the results.

EXAMPLE 3.43. The connected component of a generator x of $\mathbf{FL}(X)$ is a five element set, which as a lattice is isomorphic to the five element nonmodular lattice $\mathbf{N}_5$. Its elements are

$$x, \qquad x_* = x \wedge \overline{x}, \qquad x^* = x \vee \underline{x}, \qquad x_* \vee \underline{x}, \qquad x^* \wedge \overline{x},$$

where, as usual, $\overline{x} = \bigvee(X - \{x\})$ and $\underline{x} = \bigwedge(X - \{x\})$. These are ordered as follows:

$$x_* \prec x \prec x^*,$$
$$x_* \prec x_* \vee \underline{x} < x^* \wedge \overline{x} \prec x^*.$$

This is diagrammed in Figure 7.5 on page 212.

EXAMPLE 3.44. The connected component of 0 in $\mathbf{FL}(x, y, z)$ is the eleven element sublattice pictured in Figure 3.4. The three atoms are denoted by $\underline{x} = y \wedge z$, $\underline{y} = x \wedge z$, $\underline{z} = x \wedge y$ and their join by m.

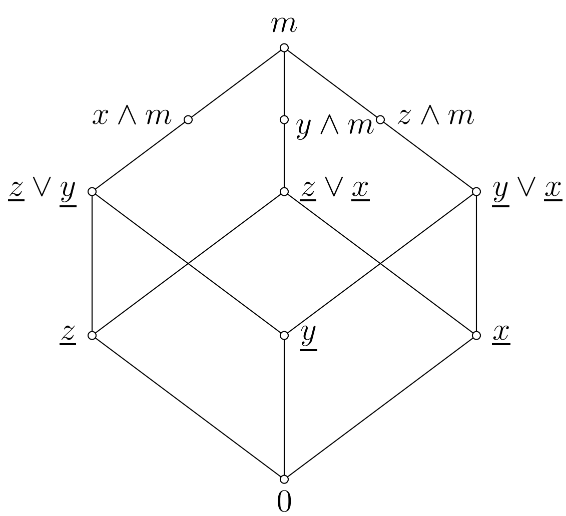


Figure 3.4

Example 3.45. The sublattice $\mathbf{L}_n$ generated by the atoms of $\mathbf{FL}(x_1, \ldots, x_n)$ is finite for $n \leq 4$ and infinite (as we have seen in Theorem 1.32) for $n \geq 5$. For $n = 3$ the sublattice contains 8 elements, and for $n = 4$ it contains 22 elements; see Figure 1.1 on page 33. For $n \geq 4$, the connected component of 0 consists of 0, the n atoms of $\mathbf{FL}(n)$, the $\binom{n}{2}$ pairwise joins of atoms, and, for each pair of atoms x_i and x_j, the element $\bigwedge_{k \neq i,j} (\underline{x}_i \vee \underline{x}_j \vee \underline{x}_k)$. Note this is a subset of $\mathbf{L}_n$ and is the union of (not pairwise disjoint) chains

$$0 \prec \underline{x}_i \prec \underline{x}_i \vee \underline{x}_j \prec \bigwedge_{k \neq i,j} (\underline{x}_i \vee \underline{x}_j \vee \underline{x}_k)$$

for i and $j \in \{1, \ldots, n\}$, $i \neq j$, see Figure 3.5. It contains $1 + n^2$ elements.

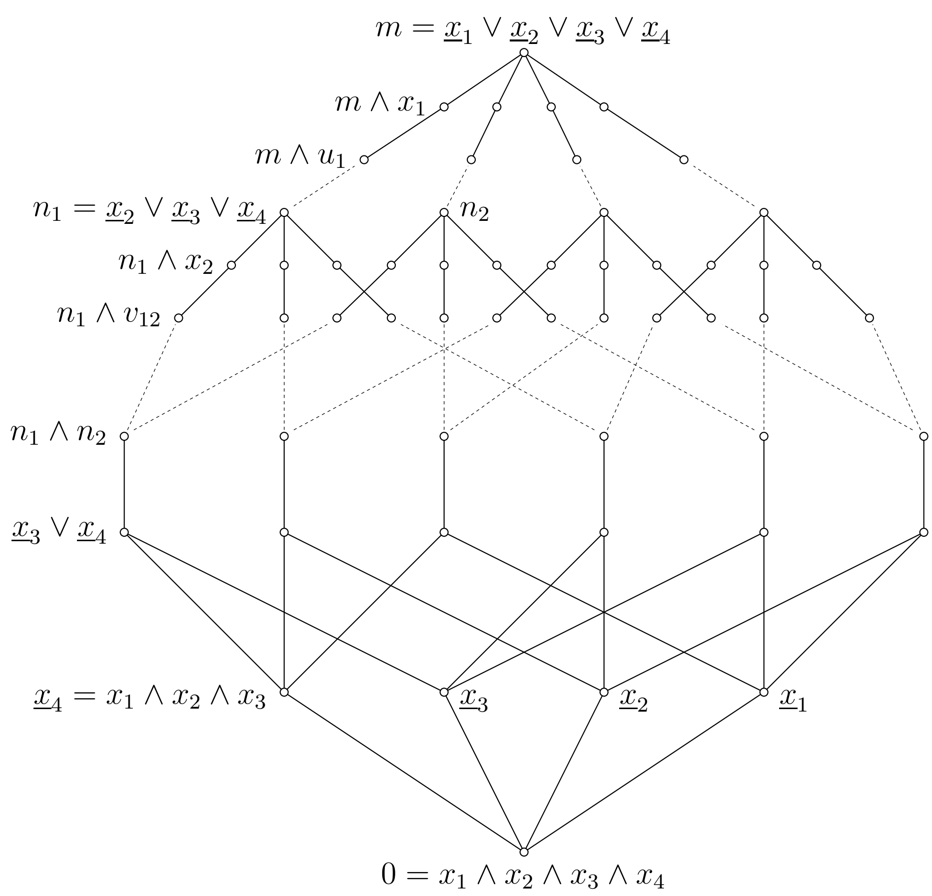

$m = \underline{x}_1 \vee \underline{x}_2 \vee \underline{x}_3 \vee \underline{x}_4$
$m \wedge x_1$
$m \wedge u_1$
$n_1 = \underline{x}_2 \vee \underline{x}_3 \vee \underline{x}_4$
n_2
$n_1 \wedge x_2$
$n_1 \wedge v_{12}$
$n_1 \wedge n_2$
$\underline{x}_3 \vee \underline{x}_4$
$\underline{x}_4 = x_1 \wedge x_2 \wedge x_3$
$\underline{x}_3$
$\underline{x}_2$
$\underline{x}_1$
$0 = x_1 \wedge x_2 \wedge x_3 \wedge x_4$

Figure 3.5

Let $n \geq 4$. For any subset J of $\{1, \ldots, n\}$ with $|J| \geq 3$, the connected component of the element

$$a_J = \bigvee_{j \in J} \underline{x}_j$$

is the union of the chains

$$a_J \wedge \bigvee_{j \in J - \{i\}} (x_i \wedge x_j) \prec a_J \wedge x_i \prec a_J$$

for $i \in J$; it contains $1 + 2|J|$ elements.

The bottom of $\mathbf{FL}(4)$ is pictured in Figure 3.5, where solid lines indicate coverings, dotted lines indicate noncoverings, $u_1 = (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (x_1 \wedge x_4)$ and $v_{12} = (x_2 \wedge x_3) \vee (x_2 \wedge x_4)$. None of the elements in Figure 3.5 covers, or is covered by, any element not in the picture.

CHAPTER IV

Day's Theorem Revisited

Alan Day's theorem, that finitely generated free lattices are weakly atomic, is a fundamental result. The methods developed in Chapter III provide us with an alternate proof of it which, while perhaps not as elegant as the original version presented in Section 5 of Chapter II, is constructive and carries more information. This proof first appeared in Freese and Nation [**62**].

The new and improved version of Day's theorem gives estimates on the complexities of the elements involved in terms of the D–rank ρ and the dual D–rank ρ^d. Recall that, by Theorem 2.11, for an element $u \in \mathbf{FL}(X)$ with $u \neq 1_{\mathbf{FL}(X)}$, $\rho(u)$ is the least nonnegative integer n such that $u \in X^{\wedge(\vee\wedge)^n}$. The dual D–rank $\rho^d(u)$ can be described dually.

THEOREM 4.1. *If $u \nleq v$ in $\mathbf{FL}(X)$ with X finite, then there exists a completely join irreducible element $q \in \mathbf{FL}(X)$ such that $q \leq u$ and $v \leq \kappa(q)$. Moreover, if $\rho^d(u) = m$ and $\rho(v) = n$ with $m + n \geq 1$, then q may be chosen so that $\rho(q) \leq m + n - 1$.*

It is easy to see that the element q given by the theorem satisfies

$$v \leq \kappa(q) \wedge (q \sqcup v) \prec q \sqcup v \leq u \sqcup v.$$

If $u > v$, this produces a cover in the interval u/v. Conversely, for any covering pair with $u \geq a \succ b \geq v$ in $\mathbf{FL}(X)$, by Theorem 3.5 there is a canonical joinand q of a with $q \nleq b$, and $q \succ q \wedge b$. Hence q is completely join irreducible, so $\kappa(q)$ exists, $q \leq a \leq u$ and $v \leq b \leq \kappa(q)$. Thus every cover in u/v arises in this way.

Again, the result has an equivalent formulation in terms of lattice inclusions satisfied by finite lower bounded lattices.

COROLLARY 4.2. *If $u \nleq v$ in $\mathbf{FL}(X)$ with X finite, then there exists a finite bounded lattice $\mathbf{L}$ and an epimorphism $f : \mathbf{FL}(X) \twoheadrightarrow \mathbf{L}$ such that $f(u) \nleq f(v)$. Moreover, if $\rho^d(u) = m$ and $\rho(v) = n$ with $m+n \geq 1$, then $\mathbf{L}$ may be chosen so that $\mathrm{J}(\mathbf{L}) \subseteq \mathrm{D}_{m+n-1}(\mathbf{L})$.*

Of course, the lattice $\mathbf{L}$ of the corollary is just $\mathbf{L}^{\sqcup}(q)$ for the element q given by Theorem 4.1, which is in fact a splitting lattice.

Proof: [Proof of Theorem 4.1] For each pair (u, v) of elements of $\mathbf{FL}(X)$ with $u \not\leq v$, we will define an element $q(u, v) \in \mathrm{J}(\mathbf{FL}(X))$ with the following properties:

(1) $q(u, v)$ is completely join irreducible in $\mathbf{FL}(X)$,
(2) $q(u, v) \leq u$ and $v \leq \kappa(q(u, v))$,
(3) if $\rho^d(u) = m$ and $\rho(v) = n$ with $m + n \geq 1$, then $\rho(q(u, v)) \leq m + n - 1$.

Let $\mathbf{K}(u, v)$ be the lattice $\mathbf{L}^{\sqcup}(q(u, v))$. Condition (1) makes $\mathbf{K}(u, v)$ a bounded lattice by Theorem 3.26, while (2), which implies $q \not\leq v$, insures that $f(u) \not\leq f(v)$ for the standard epimorphism (see Theorem 3.15). The third property takes care of the rank condition. Our proof will use induction on the complexity of the canonical form of u and v.

Case 1. If $u \in X$, then $u \not\leq v$ implies $v \leq \bigvee(X - \{u\}) = \kappa(u)$. In this case, let $q(u, v) = u$. On the other hand, if $v \in X$, then $u \not\leq v$ implies $u \geq \bigwedge(X - \{v\}) = \kappa^d(v)$. In this case, take $q(u, v) = \bigwedge(X - \{v\})$. In either case, it is easy to check that properties (1)–(3) hold.

Case 2. If $u = \bigvee u_i$, then $u \not\leq v$ implies $u_{i_0} \not\leq v$ for some i_0. Let $q(u, v) = q(u_{i_0}, v)$, and note that conditions (1) and (2) still hold since $u_i \leq u$. By the dual of Theorem 2.11, for any $w \neq 0_{\mathbf{FL}(X)}$, $\rho^d(w)$ is the least k such that $w \in X^{(\sqcup \wedge)^k \sqcup}$. Thus $\rho^d(\bigvee u_i) = \max_i \rho^d(u_i) \geq \rho^d(u_{i_0})$, whence (3) also holds.

If $v = \bigwedge v_i$, we can proceed dually.

Case 3. Let $u \not\leq v$ with $u = \bigwedge u_i$ and $v = \bigvee u_j$. Then for all i we have $u_i \not\leq v$, and likewise $u \not\leq v_j$ for all j. Thus by induction we have completely join irreducible elements $q(u_i, v)$ for each i, and $q(u, v_j)$ for each j, satisfying (1)–(3) for the appropriate elements. Let $J_0 = \bigcup \mathrm{J}(q(u_i, v)) \cup \bigcup \mathrm{J}(q(u, v_j))$, and note that J_0 is a J-closed set of completely join irreducible elements. Let $\mathbf{K}$ be the lattice $J_0^{\sqcup}$, and let $f : \mathbf{FL}(X) \twoheadrightarrow \mathbf{K}$ be the standard epimorphism, so that

$$f(w) = \bigvee \{p \in J_0 : p \leq w\}$$

for all $w \in \mathbf{FL}(X)$.

It could happen that for some $p \in J_0$ we have $p \leq u$ and $p \not\leq v$. If this occurs, choose q minimal in J_0 with respect to these properties. We claim that $v \leq \kappa(q)$. Now $\mathbf{K}$ is a bounded lattice by Theorem 3.23, so $\kappa_{\mathbf{K}}(q)$ exists. Since $q \not\leq f(v)$ but $q_\dagger \leq f(v)$, we have $f(v) \leq \kappa_{\mathbf{K}}(q)$. Therefore $v \leq \alpha_f(\kappa_{\mathbf{K}}(q)) = \kappa(q)$, as claimed. In this case, let $q(u, v) = q$. The above argument shows that (2) holds. Condition (1) is satisfied because every element in J_0 is completely join irreducible, and (3) is not hard once we observe that $\rho^d(u) = \max \rho^d(u_i) + 1$ and $\rho(v) = \max \rho(v_j) + 1$.

Thus we may assume that for all $p \in J_0$, $p \leq u$ implies $p \leq v$. In $\mathbf{K}$, let $\overline{u} = f(u)$ and $\overline{v} = f(v)$. Note that $\overline{u} \leq u$ and $\overline{v} \leq v$, and by our assumption $\overline{u} \leq \overline{v}$. In principle, we obtain $q(u, v)$ by doubling the interval $\overline{v}/\overline{u}$ in $\mathbf{K}$, and setting $q = \beta_h((\overline{u}, 1))$ where $h : \mathbf{FL}(X) \twoheadrightarrow \mathbf{K}[\overline{v}/\overline{u}]$ is the standard epimorphism.[1] In fact, it is easy to write down the result of this calculation, using the definitions of $\mathbf{K}[\overline{v}/\overline{u}]$, h and Corollary 2.8:

$$q = \bigwedge\{x \in X : x \geq \overline{u},\ x \nleq \overline{v}\} \wedge \bigwedge\{\bigvee A : A \subseteq J_0,\ \bigvee A \geq \overline{u},\ \bigvee A \nleq \overline{v}\}.$$

Note $\overline{u} \leq q$. Only minimal nontrivial join covers A of $\overline{u}$ will actually contribute to the second term, but it does not hurt to include them all. In order to verify that this really works, we need to check that q satisfies (1)–(3).

First, let us show that $q \leq u$. Now $u = \bigwedge u_i$, and for each i the element $q_i = q(u_i, v) \in J_0$ is such that $q_i \leq u_i$ and $v \leq \kappa(q_i)$. Also, $\overline{u}$ is a join of elements in J_0. Thus $q_i \sqcup \overline{u}$ is an element of the form $\bigvee A_i$ with $A_i \subseteq J_0$, which satisfies $\overline{u} \leq q_i \sqcup \overline{u} \leq u_i \sqcup u = u_i$, while $q_i \sqcup \overline{u} \nleq \overline{v}$ because $q_i \nleq v$ and $\overline{v} \leq v$. Therefore $q \leq \bigwedge(q_i \sqcup \overline{u}) \leq \bigwedge u_i = u$.

Next, we show that $q \nleq v$. For otherwise, we could apply (W) to the inclusion

$$\bigwedge\{x \in X : x \geq \overline{u},\ x \nleq \overline{v}\} \wedge \bigwedge\{\bigvee A : A \subseteq J_0,\ \bigvee A \geq \overline{u},\ \bigvee A \nleq \overline{v}\} \leq \bigvee v_j.$$

If $q \leq v_{j_0}$ for some j_0, then $\overline{u} \leq q \leq v_{j_0}$. However, $q(u, v_{j_0}) \leq \overline{u}$ and $q(u, v_{j_0}) \nleq v_{j_0}$, so this is impossible. Likewise, if $x \leq \bigvee v_j$ for some $x \geq \overline{u}$, then $\overline{u} \leq x \leq v_{j_0}$ for some j_0, which gives the same contradiction. But if $\bigvee A \leq v$ for some $A \subseteq J_0$, then $\bigvee A \leq f(v) = \overline{v}$ by the definition of f; hence no term of the second type is below v. Therefore $q \nleq v$.

Our main work comes in showing that q is completely join irreducible. At least it is easy to see that q is join irreducible in $\mathbf{FL}(X)$. Suppose $q = r \sqcup s$ with $r, s < q$. Then, by applying Whitman's condition (W) to

$$\bigwedge\{x \in X : x \geq \overline{u},\ x \nleq \overline{v}\} \wedge \bigwedge\{\bigvee A : A \subseteq J_0,\ \bigvee A \geq \overline{u},\ \bigvee A \nleq \overline{v}\} \leq r \sqcup s$$

we easily obtain that $q = \bigvee A_0$ for some $A_0 \subseteq J_0$ with $\bigvee A_0 \geq \overline{u}$ and $\bigvee A_0 \nleq \overline{v}$. However, since $\bigvee A_0 = q \leq u$, our assumption in this

[1] Alternately, this whole proof could be recast in terms of the doubling construction, yielding something intermediate between this syntactic version and the one in Section 4 of Chapter II.

case would imply $\bigvee A_0 \leq v$, whence $\bigvee A_0 \leq \overline{v}$, a contradiction. Thus $q \in \mathrm{J}(\mathbf{FL}(X))$.

We need to verify that $\mathrm{J}(q) \subseteq \{q\} \cup J_0$. It suffices to show that if $q = \bigwedge x_k \wedge \bigwedge(\bigvee q_{mn})$ canonically, then each $q_{mn} \in J_0$. This is equivalent to showing that $f(q_{mn}) = q_{mn}$, where we are still using the standard epimorphism $f : \mathbf{FL}(X) \twoheadrightarrow \mathbf{K}$. Now we know $f(q_{mn}) \leq q_{mn}$, and each term $\bigvee_n q_{mn}$ is a minimal nontrivial join cover of q (Lemma 3.11), so if we can show that $q \leq \bigvee_n f(q_{mn})$, then it will follow that $f(q_{mn}) = q_{mn}$ for every m and n. To do this, note that we have $\bigvee_n f(q_{mn}) \geq f(q) = \overline{u}$. On the other hand, applying (W) to

$$\bigwedge\{x \in X : x \geq \overline{u},\, x \not\leq \overline{v}\} \wedge \bigwedge\{\bigvee A : A \subseteq J_0,\, \bigvee A \geq \overline{u},\, \bigvee A \not\leq \overline{v}\} \leq \bigvee_n q_{mn}$$

yields $\bigvee A \leq \bigvee_n q_{mn}$ for some $A \subseteq J_0$ with $\bigvee A \geq \overline{u}$ and $\bigvee A \not\leq \overline{v}$. Then $\bigvee A \leq f(\bigvee_n q_{mn}) = \bigvee_n f(q_{mn})$, whence $\bigvee_n f(q_{mn}) \not\leq \overline{v}$. This makes $\bigvee_n f(q_{mn})$ one of the meetands in the definition of q, and we conclude that $\bigvee_n f(q_{mn}) \geq q$, as required.

Now let $\mathbf{L} = \mathbf{L}^{\sqcup}(q)$, and let $g : \mathbf{FL}(X) \twoheadrightarrow \mathbf{L}$ be the standard epimorphism, i.e.,

$$g(w) = \bigvee\{p \in \mathrm{J}(q) : p \leq w\}.$$

Since $\mathrm{J}(q) \subseteq \{q\} \cup J_0$ and $\overline{u} \leq q \leq u$, we have $g(u) = q$, while $g(v) \leq f(v) = \overline{v}$ because $q \not\leq v$. In order to prove that q is completely join irreducible in $\mathbf{FL}(X)$, it suffices, by Theorem 3.30, to show that $\kappa_{\mathbf{L}}(q)$ exists, and that $g(v) \leq \kappa_{\mathbf{L}}(q)$.

In fact, $g(v) = \kappa_{\mathbf{L}}(q)$. Note that the lower cover of q in $\mathbf{L}$ is given by $q_\dagger = g(\overline{u})$. Together with $q \not\leq v$, this implies that $q_\dagger \leq g(\overline{v}) = g(v)$ and $q \not\leq g(v)$. Let w be any element in $\mathbf{L}$ with $w \geq q_\dagger$ and $w \not\geq q$. Then $w = \bigvee B$ for some set $B \subseteq \mathrm{J}(q) - \{q\} \subseteq J_0$. Hence also $\overline{u} \sqcup w \not\geq q$, as otherwise we would have

$$w = q_\dagger \sqcup w = g(\overline{u} \sqcup w) \geq g(q) = q,$$

a contradiction. By the definition of q, this means that $\overline{u} \sqcup w \leq \overline{v}$, which in turn implies that $w = g(w) \leq g(\overline{v}) = g(v)$. Thus $g(v)$ is the largest element in $\mathbf{L}$ which is above $q_\dagger$ and not above q, i.e., $g(v) = \kappa_{\mathbf{L}}(q)$.

Finally, for the rank condition, assume $\rho^d(u) = m$ and $\rho(v) = n$ with $m + n \geq 1$. Note that in this case $\rho^d(u) = 1 + \max \rho^d(u_i)$ and $\rho(v) = 1 + \max \rho(v_j)$. Hence, by induction, $\rho(p) \leq m + n - 2$ for all

$p \in J_0$. Thus, by the definition of q,

$$\begin{aligned}\rho(q) &\leq \max\{\rho(p) : p \in J_0\} + 1 \\ &\leq m + n - 1\end{aligned}$$

which is the desired estimate.

Therefore conditions (1)–(3) hold for $q(u, v) = q$.

We can also show that the bound given for the rank of q in Theorem 4.1, *viz.*, $\rho(q) \leq \rho^d(u) + \rho(v) - 1$, is sometimes the best possible.

Define two sequences of elements in $\mathbf{FL}(X)$ with $X = \{x, y, z, \dots\}$ by

$$\begin{array}{rclrcl} s_0 &=& z & t_0 &=& x \\ s_1 &=& x \wedge (y \sqcup z) & t_1 &=& y \sqcup (z \wedge x) \\ &\vdots& & &\vdots& \\ s_{2k} &=& z \wedge (y \sqcup s_{2k-1}) & t_{2k} &=& x \sqcup (z \wedge t_{2k-1}) \\ s_{2k+1} &=& x \wedge (y \sqcup s_{2k}) & t_{2k+1} &=& y \sqcup (z \wedge t_{2k}). \end{array}$$

Note that $\rho^d(s_m) = m$ and $\rho(t_n) = n$. It is not hard to show, using induction and Whitman's condition (W), that $s_m \not\leq t_n$ if m is even or n is odd (and not otherwise).

THEOREM 4.3. *Let s_m and t_n be members of the above sequences with $m > 1$, $n > 0$, and either m even or n odd. If q is a completely join irreducible element of $\mathbf{FL}(X)$ such that $q \leq s_m$ and $t_n \leq \kappa(q)$, then $\rho(q) \geq m + n - 1$.*

Proof: We will use induction on the sum $m + n$. The induction is begun with the observation that there is no $p \in \mathrm{D}_0(\mathbf{FL}(X)) = X^\wedge$ such that $p \leq s_1$ but $p \not\leq t_1$. Hence $q \leq s_1$ and $t_1 \leq \kappa(q)$ imply $\rho(q) \geq 1$. (However, the conclusion of the theorem is false for $m = 1$ and $n > 1$, so we must use some care in our induction.)

Now assume $m > 1$, $n > 0$, and either m even or n odd, so that $s_m \not\leq t_n$. Let q be a completely join irreducible element in $\mathbf{FL}(X)$ with $q \leq s_m$ and $t_n \leq \kappa(q)$. Then the lattice $\mathbf{L}^{\sqcup}(q)$ is bounded, and $f(s_m) \not\leq f(t_n)$ for the standard epimorphism. We wish to show that $\rho(q) \geq m + n - 1$. There are three cases to consider.

Case 1: m **even,** n **odd.** In this case $s_m = z \wedge (y \sqcup s_{m-1})$ and $t_n = y \sqcup (z \wedge t_{n-1})$. Since $f(s_m) \not\leq f(t_n)$, we have $f(s_{m-1}) \not\leq f(t_n)$ and $f(s_m) \not\leq f(t_{n-1})$. Thus there exist $p_1, p_2 \in \mathrm{J}(q)$ such that $p_1 \leq s_{m-1}$, $p_1 \not\leq t_n$ and $p_2 \leq s_m$, $p_2 \not\leq t_{n-1}$. Taking p_1 and p_2 minimal in $\mathrm{J}(q)$ with these properties, we obtain $t_n \leq \kappa(p_1)$ and $t_{n-1} \leq \kappa(p_2)$.

Now $p_1 \neq q$, because $p_1 \leq s_m$ would imply $p_1 \leq s_{m-1} \wedge s_m \leq x \wedge z \leq t_n$, a contradiction. Likewise, $p_2 \neq q$, or else we would have $t_n \leq \kappa(p_2)$, whence $p_2 \le s_m \le x \sqcup y = t_{n-1} \sqcup t_n \le \kappa(p_2)$, a contradiction. Thus $\rho(q) \ge \max(\rho(p_1), \rho(p_2)) + 1$.

If $m > 2$, the inductive hypothesis implies $\rho(p_1) \ge m + n - 2$. Similarly, if $n > 1$ we have $\rho(p_2) \ge m+n-2$. This leaves the possibility $m = 2$ and $n = 1$, and our observation in the first paragraph of the proof implies that for this case $\rho(p_1) \ge 1$. Thus we conclude that $\rho(q) \ge m + n - 1$, as desired.

Case 2: m **odd,** n **odd.** In this case $s_m = x \wedge (y \sqcup s_{m-1})$ and $t_n = y \sqcup (z \wedge t_{n-1})$. Since $f(s_m) \nleq f(t_n)$, we have $f(s_{m-1}) \nleq f(t_n)$, which as above gives us an element $p \in \mathrm{J}(q)$ such that $p \le s_{m-1}$ and $t_n \le \kappa(p)$. Now $p \ne q$, because $p \le s_m$ would imply $p \le s_{m-1} \wedge s_m \le x \wedge z \le t_n$, a contradiction. By induction we have $\rho(p) \ge m + n - 2$, and therefore $\rho(q) \ge m + n - 1$.

Case 3: m **even,** n **even.** In this case $s_m = z \wedge (y \sqcup s_{m-1})$ and $t_n = x \sqcup (z \wedge t_{n-1})$. Thus $f(s_m) \nleq f(t_n)$ implies $f(s_m) \nleq f(t_{n-1})$, so there is an element $p \in \mathrm{J}(q)$ with $p \le s_m$ and $t_{n-1} \le \kappa(p)$. Again $p \ne q$, for otherwise we would have $p \le s_m \le x \sqcup y = t_{n-1} \sqcup t_n \le \kappa(p)$, a contradiction. Since $\rho(p) \ge m + n - 2$ by induction, we again conclude that $\rho(q) \ge m + n - 1$.

CHAPTER V

Sublattices of Free Lattices and Projective Lattices

The class of lattices that are embeddable into a free lattice is a naturally interesting class of lattices. Two obvious necessary conditions for a lattice to be a sublattice of a free lattice are semidistributivity and the condition (W). Except for finite lattices, reasonably applicable necessary and sufficient conditions for embeddability into a free lattice are not yet known.

Projective lattices form an important subclass of this class. They are discussed in Section 1, where the most prominent result is a characterization theorem for projective lattices due to Freese and Nation [**61**]. This extends earlier results due to McKenzie [**99**] and Kostinsky [**92**].

In the second section we discuss the lattice $\mathbf{FL}(\mathbf{P})$ freely generated by an ordered set, and the related question of which ordered sets can be embedded into free lattices.

Section 3 contains a proof that for finite lattices, semidistributivity and the condition (W) together are sufficient to characterize sublattices of free lattices. This surprising result is due to Nation [**103**]. It was conjectured by Jónsson, who suggested a line of approach to the solution in unpublished notes written in the 1960's. The approach was later described in [**83**] and taken by Nation in his final attack. Here we follow Nation's original proof, with some minor modifications.

Some more results on finite sublattices of a free lattice and related topics, including a characterization in terms of forbidden lattices and an extension to lattices without infinite chains, are proved in Section 4.

Section 5 brings a collection of examples of finite subdirectly irreducible sublattices of a free lattice. The collection is exhaustive, as proved in Ježek and Slavík [**76**].

We conclude with a summary of the lattice properties of free lattices, sublattices of free lattices, projective lattices and finitely presented lattices.

1. Projective Lattices

A lattice $\mathbf{L}$ is said to be *projective* if for any epimorphism $f : \mathbf{M} \twoheadrightarrow \mathbf{N}$ and any homomorphism $h : \mathbf{L} \to \mathbf{N}$ there exists a homomorphism

$g : \mathbf{L} \to \mathbf{M}$ such that $h = fg$. (In the category of lattices, epimorphisms are just onto homomorphisms.)

Let f be a mapping of a set K onto a set L. Then a mapping g of L into K is called a *transversal* of f if fg is the identity on L. In other words, a transversal picks for each element $x \in L$ a preimage $g(x) \in f^{-1}(x)$. A homomorphism f of a lattice $\mathbf{K}$ onto a lattice $\mathbf{L}$ is said to be a *retraction* if it has a transversal which is a homomorphism; the transversal is then an embedding of $\mathbf{L}$ into $\mathbf{K}$. A lattice $\mathbf{L}$ is called a *retract* of a lattice $\mathbf{K}$ if there exists a retraction of $\mathbf{K}$ onto $\mathbf{L}$, i.e., if there exist homomorphisms $f : \mathbf{K} \to \mathbf{L}$ and $g : \mathbf{L} \to \mathbf{K}$ with $fg = i_{\mathbf{L}}$.

The following standard result explains the connection between these ideas.

Theorem 5.1. *The following are equivalent for a lattice* $\mathbf{L}$.

(1) $\mathbf{L}$ *is projective.*
(2) $\mathbf{L}$ *is a retract of a free lattice.*
(3) *For any lattice* $\mathbf{K}$, *any epimorphism* $f : \mathbf{K} \twoheadrightarrow \mathbf{L}$ *is a retraction.*

Proof: (1) implies (3) by the definition of a projective lattice: let $\mathbf{N} = \mathbf{L}$ and take h to be the identity on $\mathbf{L}$. As any lattice is a homomorphic image of a free lattice, (3) implies (2). In order to prove that (2) implies (1), let $f : \mathbf{M} \twoheadrightarrow \mathbf{N}$ and $h : \mathbf{L} \to \mathbf{N}$ be given. By assumption, there exists a retraction f_0 of a free lattice $\mathbf{F}$ onto $\mathbf{L}$, with a transversal $g_0 : \mathbf{L} \to \mathbf{F}$ so that $f_0 g_0 = i_{\mathbf{L}}$. Since f maps M onto N, for every free generator x of $\mathbf{F}$ we can find an element $h_0(x) \in M$ such that $f(h_0(x)) = h(f_0(x))$. The mapping h_0 can be extended to a homomorphism of $\mathbf{F}$ into $\mathbf{M}$. We have $hf_0 = fh_0$, as these two homomorphisms coincide on the set of free generators. The mapping $g = h_0 g_0$ is a homomorphism of $\mathbf{L}$ into $\mathbf{M}$ and $fg = fh_0 g_0 = hf_0 g_0 = h$, as desired.

Corollary 5.2. *Every projective lattice is a sublattice of a free lattice.*

Recall from Corollary 2.11 that every free lattice has the minimal join cover refinement property. The next lemma shows that this property is inherited by projective lattices.

Lemma 5.3. *Let f be a homomorphism of a lattice* $\mathbf{K}$ *onto a lattice* $\mathbf{L}$ *such that there exists a join preserving transversal g of f. If* $\mathbf{K}$ *has the minimal join cover refinment property, then so does* $\mathbf{L}$.

Proof: For each $a \in \mathbf{L}$, let $\mathcal{S}(a)$ be the set of the nontrivial join covers of a that are of the form $f(V)$ for some minimal join cover $V \in \mathcal{M}(g(a))$. Suppose U is an arbitrary nontrivial join cover of a in

L. Then $g(U)$ is a nontrivial join cover of $g(a)$ in **K**, since g is join preserving. Consequently, there exists a $V \in \mathcal{M}(g(a))$ with $V \ll g(U)$. The set $f(V)$ is a join cover of a in **L** and $f(V) \ll fg(U) = U$. Since U is a nontrivial join cover of a, $f(V)$ is also nontrivial and so $f(V) \in \mathcal{S}(a)$. Thus every nontrivial join cover of a refines to one contained in $\mathcal{S}(a)$, so **L** has the minimal join cover refinement property.

Lemma 5.4. *If* **L** *is a projective lattice, then* $\mathrm{D}(\mathbf{L}) = L$.

Note that if **L** is finitely generated, then this is an immediate consequence of Corollary 2.16.

Proof: Let $f : \mathbf{FL}(X) \twoheadrightarrow \mathbf{L}$ be a retraction and $g : \mathbf{L} \to \mathbf{FL}(X)$ be a transversal of f. It is sufficient to prove by induction on k that if $b \in \mathrm{D}_k(\mathbf{FL}(X))$ and $b \le gf(b)$, then $f(b) \in \mathrm{D}_k(\mathbf{L})$. The lemma will follow from this, since if $a \in L$, then by Corollary 2.11 the element $b = g(a)$ belongs to some $\mathrm{D}_k(\mathbf{FL}(X))$, so that $b \le g(a) = gf(b)$ will imply $a = f(b) \in \mathrm{D}_k(\mathbf{L})$.

Let b be an element of $\mathrm{D}_k(\mathbf{FL}(X))$ such that $b \le gf(b)$, and let U be a nontrivial join cover of $f(b)$ in **L**. Then $g(U)$ is a join cover of $gf(b)$ in $\mathbf{FL}(X)$. Since $b \le gf(b)$, $g(U)$ is a join cover of b. Moreover, $g(U)$ is a nontrivial join cover of b, since if $b \le g(u)$ for some $u \in U$, then $f(b) \le fg(u) = u$. In particular, if $b \in \mathrm{D}_0(\mathbf{FL}(X))$ then $f(b) \in \mathrm{D}_0(\mathbf{L})$. Thus we may suppose $k > 0$. It follows from Corollary 2.11 and Lemma 5.3 that there is a minimal nontrivial join cover U' of $f(b)$ with $U' \ll U$. As before, $g(U')$ is a nontrivial join cover of b, and by Corollary 2.11 there is a $V \in \mathcal{M}(b)$ with $V \ll g(U')$ and $V \subseteq \mathrm{D}_{k-1}(\mathbf{FL}(X))$. Now $f(V)$ is a join cover of $f(b)$ and $f(V) \ll fg(U') = U'$. At this point, if we had $v \le gf(v)$ for each $v \in V$, then we could apply the inductive hypothesis and be done. Instead, we have the weaker statement $V \ll gf(V)$. This holds because the minimality of U' implies $U' \subseteq f(V)$, and thus $V \ll g(U') \subseteq gf(V)$. Denote by $v_1, \dots, v_r$ the elements of V. There exists a mapping σ of $\{1, \dots, r\}$ into itself such that $v_i \le gf(v_{\sigma(i)})$ for $i = 1, \dots, r$. Applying f, we obtain $f(v_i) \le fgf(v_{\sigma(i)}) = f(v_{\sigma(i)})$. Suppose say $1 \ne \sigma(1)$. As the range of the sequence $1 = \sigma^0(1), \sigma^1(1), \sigma^2(1), \dots$ is finite, there are two nonnegative integers s, t such that $s < t$, $\sigma^s(1) = \sigma^t(1)$ and $1, \sigma^1(1), \dots, \sigma^s(1)$ are distinct. Then

$$f(v_{\sigma^s(1)}) \le f(v_{\sigma^{s+1}(1)}) \le \cdots \le f(v_{\sigma^t(1)}) = f(v_{\sigma^s(1)}).$$

Hence all of these inequalities must be equalities, and we have

$$v_{\sigma^s(1)} \le gf(v_{\sigma^{s+1}(1)}) = gf(v_{\sigma^s(1)}).$$

Similarly

$$v_{\sigma^{s+1}(1)} \le gf(v_{\sigma^{s+1}(1)}), \quad \dots, \quad v_{\sigma^{t-1}(1)} \le gf(v_{\sigma^{t-1}(1)}).$$

Moreover, we have

$$f(v_1) \leq f(v_{\sigma(1)}) \leq f(v_{\sigma^2(1)}) \leq \cdots \leq f(v_{\sigma^s(1)}).$$

Hence the set $f(V - \{v_1, v_{\sigma(1)}, \ldots, v_{\sigma^{s-1}(1)}\})$ still is a join cover of $f(b)$. These arguments show that if we set $V' = \{v \in V : v \leq gf(v)\}$, then $f(V')$ is a join cover of $f(b)$. Since $V \subseteq \mathrm{D}_{k-1}(\mathbf{FL}(X))$, by induction we have $f(v) \in \mathrm{D}_{k-1}(\mathbf{L})$ for each $v \in V'$. Since $f(V) \ll U$, we have $f(V') \ll U$. Hence $f(b) \in \mathrm{D}_k(\mathbf{L})$ by definition.

A lattice (or more generally, an ordered set) $\mathbf{L}$ is said to be *finitely separable* if there exist two mappings A and B with domain L such that for any $a, b \in L$, $A(a)$ is a finite subset of $\{c \in L : c \geq a\}$, $B(b)$ is a finite subset of $\{c \in L : c \leq b\}$, and if $a \leq b$, then the intersection $A(a) \cap B(b)$ is nonempty.

LEMMA 5.5. *Every free lattice is finitely separable.*

Proof: Let $\mathbf{F} = \mathbf{FL}(X)$ be a free lattice. Recall that by Corollary 2.11, $\mathrm{D}_0(\mathbf{F}) = X^\wedge - \{1_\mathbf{F}\}$ and $\mathrm{D}_k(\mathbf{F}) = \mathrm{D}_{k-1}(\mathbf{F})^{\vee\wedge}$ for $k > 0$; the sets $\mathrm{D}_l^d(\mathbf{F})$ can be characterized dually. For $a \in \mathbf{F}$ denote by **var** (a) the set of the generators that are subelements of a, so that a belongs to the finitely generated free lattice $\mathbf{FL}(\textbf{ var } (a))$. When k is the least number such that $a \in \mathrm{D}_k(\mathbf{F})$, let $A(a) = \mathrm{D}_k(\mathbf{FL}(\textbf{ var } (a))) \cap \{c \in F : c \geq a\}$. Similarly, with l being the least number such that $b \in \mathrm{D}_l^d(\mathbf{F})$, let $B(b) = \mathrm{D}_l^d(\mathbf{FL}(\textbf{ var } (b))) \cap \{c \in F : c \leq b\}$. Clearly, both $A(a)$ and $B(b)$ are finite subsets. We shall show by induction on $k + l$ that if $a \leq b$, then $A(a) \cap B(b)$ is nonempty.

If $k = l = 0$, then $a = x_1 \wedge \cdots \wedge x_n$ and $b = y_1 \vee \cdots \vee y_m$ for some generators x_i, y_j and $a \leq b$ implies $x_i = y_j$ for some i, j. Clearly, $x_i \in A(a) \cap B(b)$.

Let $k = 0$ and $l > 0$. Then a is a meet of generators and $b = \bigwedge U_1 \vee \cdots \vee \bigwedge U_m$ for some subsets $U_1, \ldots, U_m$ of $\mathrm{D}_{l-1}^d(\mathbf{F})$; we can assume that this is a canonical representation of b. As a is join prime, $a \leq b$ implies $a \leq \bigwedge U_j$ for some j. For any $u \in U_j$ we have $a \leq u$; since $u \in \mathrm{D}_{l-1}^d(\mathbf{F})$, by induction there is an element $c_u \in A(a) \cap B(u)$. Put $c = \bigwedge\{c_u : u \in U_j\}$, so that $a \leq c \leq b$. Since $A(a)$ is closed under meets, $c \in A(a)$. For any $u \in U_j$ we have **var** $(u) \subseteq$ **var** (b), so that $\mathrm{D}_{l-1}^d(\mathbf{FL}(\textbf{ var } (u))) \subseteq \mathrm{D}_{l-1}^d(\mathbf{FL}(\textbf{ var } (b)))$ and thus $c_u \in B(u) \subseteq \mathrm{D}_{l-1}^d(\mathbf{FL}(\textbf{ var } (u))) \subseteq \mathrm{D}_{l-1}^d(\mathbf{FL}(\textbf{ var } (b)))$. But then $c \in \mathrm{D}_l^d(\mathbf{FL}(\textbf{ var } (b)))$ and thus $c \in B(b)$.

If $k > 0$ and $l = 0$, the proof of $A(a) \cap B(b) \neq \emptyset$ is similar.

Finally, let $k > 0$ and $l > 0$. Then $a = \bigvee V_1 \wedge \cdots \wedge \bigvee V_n$ for some subsets $V_1, \ldots, V_n$ of $\mathrm{D}_{k-1}(\mathbf{F})$ and $b = \bigwedge U_1 \vee \cdots \vee \bigwedge U_m$ for $U_1, \ldots, U_m \subseteq \mathrm{D}_{l-1}^d(\mathbf{F})$. By (W) either there exists an i such that $v \leq b$ for all $v \in V_i$, or there exists a j such that $a \leq u$ for all $u \in U_j$. In the former case induction yields that for every $v \in V_i$ there is an element $c_v \in A(v) \cap B(b)$. Setting $c = \bigvee\{c_v : v \in V_i\}$, we obtain $c \in A(a) \cap B(b)$. The other case is handled dually.

LEMMA 5.6. *A lattice* $\mathbf{L}$ *is finitely separable if and only if there exists a homomorphism of a free lattice onto* $\mathbf{L}$ *with an order preserving transversal.*

Proof: Suppose first that there exists a homomorphism f of a free lattice $\mathbf{FL}(X)$ with an order preserving transversal g. By Lemma 5.5, $\mathbf{FL}(X)$ is finitely separable with respect to a pair of mappings A, B. For $a \in L$ set $A'(a) = f(A(g(a)))$ and $B'(a) = f(B(g(a)))$. If $a \leq b$ in $\mathbf{L}$, then $g(a) \leq g(b)$ in $\mathbf{FL}(X)$ and hence there exists an element $c \in A(g(a)) \cap B(g(b))$; the element $f(c)$ belongs to $A'(a) \cap B'(b)$.

Now let $\mathbf{L}$ be finitely separable with respect to a pair of mappings A, B. We can define a transfinite sequence $(M_\gamma : \gamma < \delta)$ of pairwise disjoint, nonempty and at most countable subsets of L in the following way. Suppose that the sets M_γ are already defined for all γ less than an ordinal number γ_0 and put $M = \bigcup_{\gamma<\gamma_0} M_\gamma$. If $M = L$, put $\delta = \gamma_0$ and stop. If $M \subset L$, choose an element $a \in L - M$ arbitrarily and define M_{γ_0} to be the least subset of L containing a, disjoint from M, and such that $A(b) \cup B(b) \subseteq M \cup M_{\gamma_0}$ for any $b \in M_{\gamma_0}$.

The sets M_γ define a partition of L. We shall write $a \sim b$ if $\{a, b\} \subseteq M_\gamma$ for some γ. There exists a well ordering $\sqsubseteq$ of L such that $a \in M_{\gamma_1}$, $b \in M_{\gamma_2}$ and $\gamma_1 < \gamma_2$ imply $a \sqsubseteq b$ and each M_γ is a chain of type $\leq \omega$ with respect to a restriction of $\sqsubseteq$. This well ordering has the special property that if $a, b \in L$ and $b \in A(a) \cup B(a)$, then either $b \sqsubseteq a$ or else $a \sim b$ and b comes only finitely many places after a.

Let $X = \{x_a : a \in L\}$ be a set of elements in one-to-one correspondence with the elements of L and take $f : \mathbf{FL}(X) \to \mathbf{L}$ to be the extension of $f(x_a) = a$. For $a \in L$ define $g(a) \in \mathbf{FL}(X)$ by transfinite induction with respect to $\sqsubseteq$ as follows:

$$\begin{aligned} g(a) =& (x_a \wedge \bigwedge\{g(b) : b \sim a,\ b \sqsubseteq a,\ a < b\} \wedge \bigwedge\{g(b) : b \sqsubset a,\ b \in A(a)\}) \\ &\vee \bigvee\{g(b) : b \sim a,\ b \sqsubseteq a,\ b < a\} \vee \bigvee\{g(b) : b \sqsubset a,\ b \in B(a)\}. \end{aligned}$$

(Empty joins or meets are simply omitted. By construction, there are only finitely many elements with $b \sim a$ and $b \sqsubseteq a$; hence all the meets and joins are finite.) A straightforward induction shows that $fg(a) = a$ for any a.

Let us prove that $a \leq b$ implies $g(a) \leq g(b)$. We shall proceed by transfinite induction on the larger of the elements a, b with respect to $\sqsubseteq$. There is an element $c \in A(a) \cap B(b)$.

First consider the case when $a \sqsubseteq b$. Since $g(c)$ is one of the joinands in the definition of $g(b)$, we have $g(c) \leq g(b)$. If $c \sqsubseteq b$, then induction yields $g(a) \leq g(c)$, and hence $g(a) \leq g(b)$. If $b \sqsubseteq c$, then $a \sqsubseteq c$, and the special property of $\sqsubseteq$ implies $a \sim c$. This makes $g(a)$ one of the joinands of $g(c)$, and again we get $g(a) \leq g(c)$.

Next consider the case when $b \sqsubseteq a$ and either $a \sim b$ or $b \in A(a)$. Then $g(b)$ is a meetand of the first joinand of $g(a)$, so that the first joinand is $\leq g(b)$. Any other joinand of $g(a)$ is of the form $g(u)$ and one can easily see, using induction, that $g(u) \leq g(b)$. Consequently, $g(a) \leq g(b)$.

Finally, let $b \sqsubseteq a$ and neither $a \sim b$ nor $b \in A(a)$. The special property of $\sqsubseteq$ then yields $c \sqsubseteq a$. We have $g(a) \leq g(c)$ by the previous case and $g(c) \leq g(b)$ by induction.

Now we are in a position to state the main result of this section, the characterization of projective lattices from Freese and Nation [**61**].

THEOREM 5.7. *A lattice* $\mathbf{L}$ *is projective if and only if it satisfies the following four conditions*:

(1) *Whitman's condition* (W),
(2) $L = \mathrm{D}(\mathbf{L}) = \mathrm{D}^d(\mathbf{L})$,
(3) $\mathbf{L}$ *has the minimal join cover refinement property and its dual,*
(4) $\mathbf{L}$ *is finitely separable.*

Proof: The necessity of these conditions is established by the preceding lemmas. Conversely, suppose that the four conditions are satisfied. Since $\mathbf{L}$ is finitely separable, by Lemma 5.6 there exist a free lattice $\mathbf{FL}(X)$ and a homomorphism $f : \mathbf{FL}(X) \twoheadrightarrow \mathbf{L}$ with an order preserving transversal g. For each $k \geq 0$ we define a mapping g_k of L into $\mathbf{FL}(X)$ as follows:

$$g_0(a) = g(a),$$
$$g_{k+1}(a) = g(a) \wedge \bigwedge\{\bigvee g_k(U) : U \in \mathcal{M}(a)\}.$$

Because $\mathbf{L}$ has the minimal join cover refinement property, this definition makes sense. Moreover, if $a \in \mathrm{D}_k(\mathbf{L})$, then any minimal nontrivial join cover $U \in \mathcal{M}(a)$ is contained in $\mathrm{D}_{k-1}(\mathbf{L})$. Using this, an easy induction shows that if $a \in \mathrm{D}_k(\mathbf{L})$, then $g_k(a) = g_n(a)$ for all $n \geq k$. Hence for any $a \in L$, the sequence $g_0(a), g_1(a), \ldots$ is eventually constant. Let $g_-(a)$ denote the final value of that sequence. Since g is order preserving, so are each g_k and g_-. Furthermore, we claim that g_- preserves joins. To see this, let a and b be two incomparable elements of L and choose k large enough so that $g_k(a) = g_-(a)$, $g_k(b) = g_-(b)$ and $g_k(a \vee b) = g_-(a \vee b)$. Since $\{a, b\}$ is a nontrivial join cover of $a \vee b$, there is a $U_0 \in \mathcal{M}(a \vee b)$ with $U_0 \ll \{a, b\}$. Because g_k is monotonic, this implies $g_k(U_0) \ll \{g_k(a), g_k(b)\}$ and hence $\bigvee g_k(U_0) \ll g_k(a) \vee g_k(b)$. Examining the definition of $g_{k+1}(a \vee b)$ we see that

$$g_-(a \vee b) = g_{k+1}(a \vee b) \leq \bigvee g_k(U_0) \leq g_k(a) \vee g_k(b) = g_-(a) \vee g_-(b).$$

The reverse inequality follows from the monotonicity of g_-, and we get $g_-(a \vee b) = g_-(a) \vee g_-(b)$.

Since $\mathbf{L}$ satisfies (W), every nontrivial join cover of $a \wedge b$ is a nontrivial join cover of a or of b. Hence for each $U \in \mathcal{M}(a \wedge b)$ there is a $U' \in \mathcal{M}(a) \cup \mathcal{M}(b)$ with $U' \ll U$. Using this observation one can easily prove that if g preserves meets, then so does each g_k, and hence g_- does also.

Let g^- be the mapping defined dually to g_-. Then let $h = (g^-)_-$. By the dual of the argument above, g^- preserves meets. The preceding arguments, with g replaced by g^-, then show that h preserves joins and inherits the meet-preserving property of g^-. Therefore h is a homomorphism of $\mathbf{L}$ into $\mathbf{FL}(X)$. It is easy to check that fh is the identity on L, so that f is a retraction. By Theorem 5.1, $\mathbf{L}$ is projective.

Let us examine some of the consequences of this characterization.

Corollary 5.8. *A countable lattice is projective if and only if it satisfies the first three conditions of* Theorem *5.7.*

Proof: Every countable lattice $\mathbf{L}$ is finitely separable. For if $L = \{a_0, a_1, \dots\}$, then we can set $A(a_n) = \{a_i : i \leq n, a_i \geq a_n\}$ and $B(a_n) = \{a_i : i \leq n, a_i \leq a_n\}$. Then $a \leq b$ implies that the element with the smaller index is in $A(a) \cap B(b)$.

An important special case, which predates Theorem 5.7, is due to Alan Kostinsky [**92**]

Corollary 5.9. *A finitely generated lattice* $\mathbf{L}$ *is projective if and only if it is a bounded lattice satisfying* (W)*.*

Proof: By Theorem 2.13 and its dual, a finitely generated lattice $\mathbf{L}$ is bounded if and only if condition (2) holds. By Corollary 2.19, every bounded lattice satisfies (3); alternately, one can use Lemma 5.3 and its dual. As finitely generated lattices are countable, Corollary 5.8 now applies.

Corollary 5.10. *A finitely generated lattice is projective if and only if it is a sublattice of a free lattice.*

Proof: By Theorem 2.14, a finitely generated sublattice of a free lattice is bounded, and of course it inherits (W).

The original result along these lines goes back to Ralph McKenzie [**99**] and, independently, Bjarni Jónsson (see [**99**] and [**83**]).

COROLLARY 5.11. *A finite lattice is projective if and only if it is a sublattice of a free lattice.*

A rather different type of example comes from Freese and Nation [**62**].

THEOREM 5.12. *If $v \leq u$ in $\mathbf{FL}(X)$, then u/v is a projective lattice.*

Proof: It is immediate that conditions (1) and (4) of Theorem 5.7 are inherited. Let $f : \mathbf{FL}(X) \twoheadrightarrow 1/v$ by $f(w) = w \vee v$. Then f is a join homomorphism, and the identity map serves as a join preserving transversal. Straightforward adaptations of the proofs of Theorem 2.13 and Lemma 5.3 show that $\mathrm{D}(u/v) = u/v$ and u/v has the minimal join cover refinement property, respectively. We obtain the dual properties using $g(w) = w \wedge u$, whence (2) and (3) also hold.

Now we want to show that the four conditions of Theorem 5.7 are independent.

EXAMPLE 5.13. To show that conditions (2), (3) and (4) do not suffice, we can use any finite bounded lattice not satisfying (W), e.g., the lattice $\mathbf{L}^{\vee}((x \vee (y \wedge z)) \wedge (y \vee z))$ from Figure 3.3(3) on page 112.

EXAMPLE 5.14. The lattice in Figure 5.1 is finitely generated (by the two doubly irreducible elements at the extreme sides of the diagram, the join of the two atoms, and the meet of the two coatoms) and satisfies the conditions (1), (3) and (4) of Theorem 5.7, yet it is not projective because $\mathrm{D}(\mathbf{L}) \neq L$. This lattice was constructed from the one in Figure 5.5 on page 146 by repeatedly using the doubling construction to correct (W)–failures. The C–cycle in the original lattice is inherited by this one (indicated by solid points).

EXAMPLE 5.15. For any chain $\mathbf{C}$ with a least element z, denote by $\mathbf{C}^*$ the lattice obtained by adding a new element p to the direct product $\mathbf{C} \times \mathbf{2}$ in such a way that $\langle z, 0\rangle < p < \langle z, 1\rangle$. Now let $\mathbf{D}$ be a chain isomorphic to $(\omega + 1)^d$, say

$$z < \cdots < c_2 < c_1 < c_0$$

so that $\mathbf{D}^*$ is the lattice pictured in Figure 5.2. Then $\mathbf{D}^*$ satisfies the conditions (1), (2) and (4) of Theorem 5.7, but it fails the minimal join cover refinement property because $\langle z, 1\rangle < p \vee \langle c_k, 0\rangle = \langle c_k, 1\rangle$ for every $k < \omega$. Hence $\mathbf{D}^*$ is not projective. It is, however, a sublattice of a free lattice, since it is isomorphic to a sublattice of $\mathbf{E}^*$, where

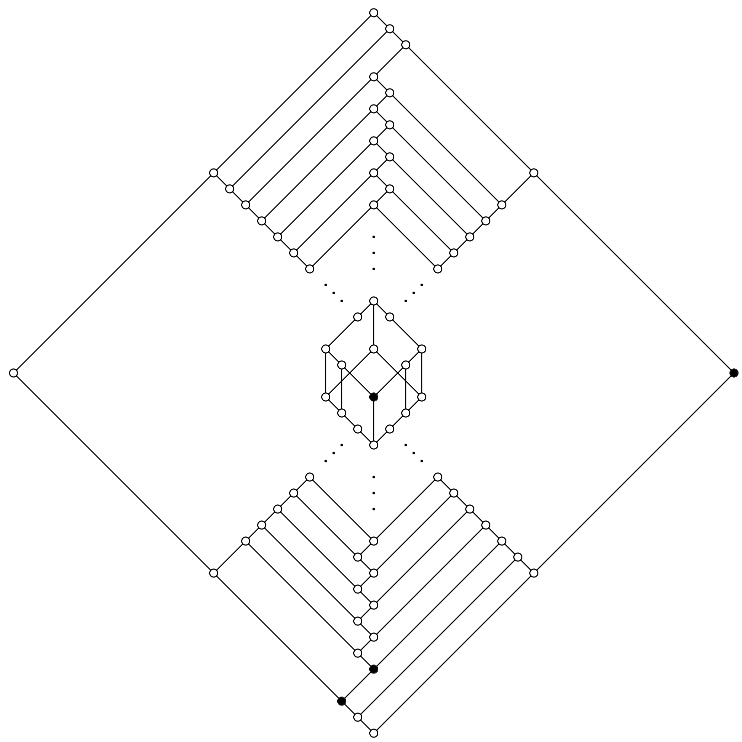


FIGURE 5.1

$\mathbf{E}$ is a chain isomorphic to $(\omega + 2)^d$ and $\mathbf{E}^*$ is projective by Theorem 5.8. This example shows that conditions (1) and (2) alone are not sufficient to characterize countable projective lattices, even though, by Theorem 5.9, they characterize finitely generated projective lattices.

In order to see that (1), (2) and (3) do not suffice, we will use the following theorem, which is of interest in itself.

THEOREM 5.16. *Let $\mathbf{L} = \mathbf{L}_0 + \mathbf{L}_1$ be the ordinal sum of two lattices $\mathbf{L}_0$, $\mathbf{L}_1$. Then $\mathbf{L}$ is projective if and only if both $\mathbf{L}_0$ and $\mathbf{L}_1$ are and one of the following holds*:

(1) *$\mathbf{L}_0$ has a greatest element.*
(2) *$\mathbf{L}_1$ has a least element.*
(3) *$\mathbf{L}_0$ has a countable cofinal chain and $\mathbf{L}_1$ has a countable coinitial chain.*

Proof: Let $\mathbf{L}$ be projective. A homomorphism $f : \mathbf{K} \twoheadrightarrow \mathbf{L}_0$ can be extended to a homomorphism $f' : \mathbf{K} + \mathbf{L}_1 \twoheadrightarrow \mathbf{L}$ by $f'(a) = a$ for all $a \in L_1$, and since $\mathbf{L}$ is projective, there is a homomorphism $g : \mathbf{L} \to \mathbf{K} + \mathbf{L}_1$ which is a transversal of f'; the restriction of g to

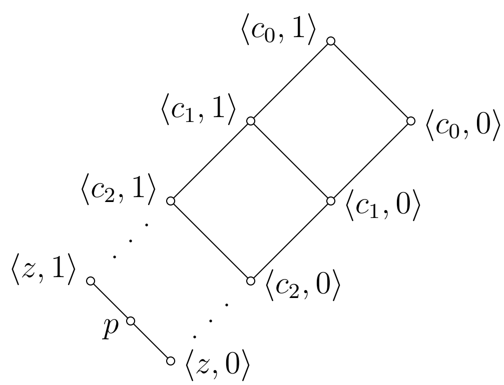


FIGURE 5.2

$\mathbf{L}_0$ is a homomorphism and a transversal of f. This shows that $\mathbf{L}_0$ is projective. Similarly, $\mathbf{L}_1$ is projective.

There are two mappings A, B with respect to which $\mathbf{L}$ is finitely separable. Suppose that $\mathbf{L}$ satisfies neither (1) nor (2) nor (3). Without loss of generality we shall assume that $\mathbf{L}_0$ has no countable cofinal chain.

It is possible to construct inductively an infinite descending sequence $d_0 > d_1 > d_2 > \dots$ of elements of L_1 such that if $i > j$, then d_i does not belong to the filter generated by $B(d_j) \cap L_1$. This follows from the fact that since $\mathbf{L}_1$ has no least element, any finitely generated filter in $\mathbf{L}_1$ is a proper subset of $\mathbf{L}_1$. Clearly, $B(d_i) \cap B(d_j) \cap L_1 = \emptyset$ whenever $i \neq j$.

The ideal generated by $\bigcup_{i=0}^{\infty}(B(d_i) \cap L_0)$ is a proper subset of L_0, for otherwise $\mathbf{L}_0$ would have a countable cofinal chain. Take an element c not belonging to this ideal.

For any i we have $c < d_i$, as $c \in L_0$ and $d_i \in L_1$. Consequently, there exists an element $e_i \in A(c) \cap B(d_i)$. By our choice of c, $e_i \notin L_0$ and thus $e_i \in L_1$. As the sets $B(d_i) \cap L_1$ are pairwise disjoint, it follows that the elements e_i are pairwise distinct. But this forces $A(c)$ to be infinite. The contradiction finishes the proof of the direct implication of the theorem.

Now suppose that $\mathbf{L}_0$ and $\mathbf{L}_1$ are both projective and either (1) or (2) or (3) takes place. We shall, of course, apply Theorem 5.7 to show that $\mathbf{L}$ is projective. It is not difficult to verify that the first three conditions of Theorem 5.7 are satisfied in $\mathbf{L}$, when they are satisfied in both $\mathbf{L}_0$ and $\mathbf{L}_1$. So all we need to do is to prove that $\mathbf{L}$ is finitely separable. For $i = 0, 1$ let A_i, B_i be a pair of mappings making $\mathbf{L}_i$ finitely separable.

If $\mathbf{L}_0$ has a greatest element u, then for $a \in L_0$ and $b \in L_1$ we can define

$$\begin{aligned} A(a) &= A_0(a) \cup \{u\}, \\ B(a) &= B_0(a), \\ A(b) &= A_1(b), \\ B(b) &= B_1(b) \cup \{u\} \end{aligned}$$

to see that $\mathbf{L}$ is finitely separable with respect to A, B.

If $\mathbf{L}_1$ has a least element, then the dual construction applies.

If (3) holds, let $c_0 < c_1 < c_2 \dots$ be a cofinal chain in $\mathbf{L}_0$ and $d_0 > d_1 > d_2 > \dots$ be a coinitial chain in $\mathbf{L}_1$. For any $a \in L_0$ and $b \in L_1$ we can find nonnegative integers i, j such that $a \leq c_i$ and $b \geq d_j$. Put

$$\begin{aligned} A(a) &= A_0(a) \cup \{c_i, d_0, d_1, \dots, d_i\}, \\ B(a) &= B_0(a), \\ A(b) &= A_1(a), \\ B(b) &= B_1(b) \cup \{d_j, c_0, c_1, \dots, c_j\}. \end{aligned}$$

Again, it is easy to see that $\mathbf{L}$ is finitely separable with respect to A, B.

COROLLARY 5.17. *The ordinal sum of two free lattices is a projective lattice if and only if either one of the two free lattices is finitely generated or both are countable.*

Interestingly, Theorem 5.16 and its Corollary 5.17 are true in any lattice variety; see Freese [**56**].

As the ordinal sum of any two free lattices is easily seen to satisfy conditions (1), (2) and (3) of Theorem 5.7, it is a consequence of Corollary 5.17 that the three conditions alone are not sufficient to characterize arbitrary projective lattices.

In Theorem 1.27 it was shown that any free lattice, and thus any sublattice of a free lattice, is a countable union of antichains. In particular, a free lattice has no uncountable chains. The ordinal sum of two free lattices has then the same property. This shows that it is not sufficient to replace the condition (4) in Theorem 5.7 by the requirement that $\mathbf{L}$ is a countable union of antichains.

We conclude this section with a theorem of Alan Day which first appeared in [**62**].

Theorem 5.18. *Every finitely generated projective lattice is weakly atomic.*

Proof: Let $\mathbf{P}$ be a finitely generated projective lattice. By Corollary 5.2, we can regard $\mathbf{P}$ as a sublattice of $\mathbf{FL}(X)$ for some finite set X. Assume $v < u$ in $\mathbf{P}$. By Day's theorem (Theorem 2.84 or Corollary 4.2), there are a finite bounded lattice $\mathbf{L}$ and a homomorphism $g : \mathbf{FL}(X) \to \mathbf{L}$ with $g(v) < g(u)$.

Now $\mathbf{L}' = g(\mathbf{P})$ is a sublattice of $\mathbf{L}$, and hence also a bounded lattice. Therefore $h = g|_P$ is a bounded homomorphism of $\mathbf{P}$ onto $\mathbf{L}'$ with $h(v) < h(u)$. Choose $w \in P$ such that $v < w \leq u$ and $h(v) \prec h(w) \leq h(u)$. Then

$$u \geq w \geq v \sqcup \beta_h h(w) \succ \alpha_h h(v) \wedge (v \sqcup \beta_h h(w)) \geq v$$

by the same argument as that in the proof of Lemma 2.82, which gives the desired covering in $\mathbf{P}$.

2. The Free Lattice Generated by an Ordered Set

Let $\mathbf{P}$ be an ordered set. A lattice $\mathbf{F}$ is *freely generated by* $\mathbf{P}$ if

(1) F has a subset P such that P with the induced order is isomorphic to $\mathbf{P}$,
(2) P generates $\mathbf{F}$,
(3) every order preserving map $h_0 : P \to \mathbf{L}$, where $\mathbf{L}$ is a lattice, can be extended to a lattice homomorphism $h : \mathbf{F} \to \mathbf{L}$.

It is not hard to see that these conditions determine $\mathbf{F}$ up to isomorphism, and hence we denote this lattice by $\mathbf{FL}(\mathbf{P})$. If $\mathbf{P}$ is an antichain, then $\mathbf{FL}(\mathbf{P})$ is of course just a free lattice in the ordinary sense.

Lattices freely generated by ordered sets were studied by R. A. Dean in [**34**], who showed that these lattices closely mimic the theory of free lattices developed by Whitman, except for the order on the generators. The results of this section will justify and clarify this assertion.

Implicit in the definition of $\mathbf{FL}(\mathbf{P})$ is the fact that any ordered set can be embedded in a lattice. Indeed, we can use the lattice of order ideals, or dually order filters, of $\mathbf{P}$, which are distributive lattices. Thus it makes sense to talk about the free distributive lattice generated by $\mathbf{P}$, denoted $\mathbf{FD}(\mathbf{P})$. This has the advantage that it is easily constructed, and is finite when $\mathbf{P}$ is. To get $\mathbf{FD}(\mathbf{P})$, we first form the free meet semilattice generated by $\mathbf{P}$, which is a straightforward elementary construction, and then take the free join semilattice over that. One then checks that this is a lattice, in which the meets are given by distributing.[1] It is not hard to see that the generators (elements of P) are both meet and join prime; more generally, meets of generators are join prime and joins of generators are meet prime.

Now we follow Day's construction of the free lattice $\mathbf{FL}(X)$ from $\mathbf{FD}(X)$ given in Section 7 of Chapter II. Let $\mathbf{F}_0 = \mathbf{FD}(\mathbf{P})$, and use the doubling construction to form a new lattice $\mathbf{F}_1$ in which the (W)–failures in $\mathbf{F}_0$ are corrected. Again, because the generators are both join and meet prime, they will not be in any doubled interval. Iterating this process gives us a sequence of lattices $\mathbf{F}_n$, which either terminates (e.g., when $\mathbf{FL}(\mathbf{P})$ is finite) or else has $\mathbf{FL}(\mathbf{P})$ naturally embedded in its limit. If $\mathbf{P}$ is finite, then each $\mathbf{F}_n$ is a finite, bounded lattice, but the construction works regardless.

As a consequence of this construction, the solution of the word problem for $\mathbf{FL}(\mathbf{P})$ is analogous to that for $\mathbf{FL}(X)$, as shown by Dean [**31**].

THEOREM 5.19. *Let $\mathbf{P}$ be an ordered set, and let $s = s(x_1, \dots, x_n)$ and $t = t(x_1, \dots, x_n)$ be lattice terms with $x_1, \dots, x_n \in P$. Then the truth of*

$$(*) \qquad s^{\mathbf{FL}(\mathbf{P})} \le t^{\mathbf{FL}(\mathbf{P})}$$

can be determined by applying the rule

(1′) *If $s = x_i$ and $t = x_j$, then $(*)$ holds if and only $x_i \le x_j$ in $\mathbf{P}$,*

and the direct analogues of rules (2)–(6) *of* Theorem *1.11, obtained by replacing $\mathbf{FL}(X)$ by $\mathbf{FL}(\mathbf{P})$ in the statements. In particular, $\mathbf{FL}(\mathbf{P})$ satisfies* (W).

[1] The idea is most clearly seen in the finite case, where one takes the lattice of order filters of $\mathbf{P}$, and then the lattice of order ideals of that. The 0 and 1 of this lattice, corresponding to the empty ideal and filter, respectively, are not in the sublattice generated by P, and hence must be removed. Clearly we get a distributive lattice this way. It *is* isomorphic to $\mathbf{FD}(\mathbf{P})$ (see Chapter XII, Section 4), but the relative freeness is not essential to the subsequent construction of $\mathbf{FL}(\mathbf{P})$.

For a concrete example, let $\mathbf{n}_1 + \cdots + \mathbf{n}_k$ denote the ordered set which is a disjoint union of k chains $\mathbf{n}_i$ with n_i elements each. Figure 5.3 shows the diagrams of $\mathbf{FD}(\mathbf{2}+\mathbf{2})$ and $\mathbf{FL}(\mathbf{2}+\mathbf{2})$. J. I. Sorkin originally proved that $\mathbf{FL}(\mathbf{2}+\mathbf{2})$ and $\mathbf{FL}(\mathbf{1}+\mathbf{4})$ are infinite [**124**], and H. L. Rolf constructed the diagrams for these lattices [**117**]. Rolf also proved that $\mathbf{FL}(\mathbf{n}_1+\mathbf{n}_2)$ contains a sublattice isomorphic to $\mathbf{FL}(3) = \mathbf{FL}(\mathbf{1}+\mathbf{1}+\mathbf{1})$ exactly when $n_1 n_2 \geq 5$. For the best discussion of the structure of $\mathbf{FL}(\mathbf{P})$ when $\mathbf{P}$ is finite, see Rival and Wille [**116**].

Returning to the general theory, note that by Theorem 5.19, $\mathbf{FL}(\mathbf{P})$ satisfies (W) and has generators which are both meet and join prime. Thus Theorem 2.10 applies.

THEOREM 5.20. *For any ordered set* $\mathbf{P}$*, the lattice* $\mathbf{FL}(\mathbf{P})$ *satisfies*

$$\mathrm{D}(\mathbf{FL}(\mathbf{P})) = \mathrm{FL}(\mathbf{P}) = \mathrm{D}^d(\mathbf{FL}(\mathbf{P})).$$

Moreover, $\mathbf{FL}(\mathbf{P})$ *has the minimal join cover refinement property and its dual.*

Therefore $\mathbf{FL}(\mathbf{P})$ satisfies the first three conditions for projectivity.

COROLLARY 5.21. *If* $\mathbf{P}$ *is a countable ordered set, then* $\mathbf{FL}(\mathbf{P})$ *is a projective lattice.*

We will be interested in knowing which ordered sets can be embedded in a free lattice. That $\mathbf{FL}(\mathbf{P})$ can be embedded in $\mathbf{FL}(X)$ whenever $|P| \leq \aleph_0$ and $|X| \geq 3$ was first shown by Crawley and Dean [**19**]; an easy argument using weaving and the fact that $\mathbf{FL}(\omega)$ can be embedded in $\mathbf{FL}(3)$ (Theorem 1.28) is given in Dean [**34**]. This is of course weaker than Corollary 5.21.

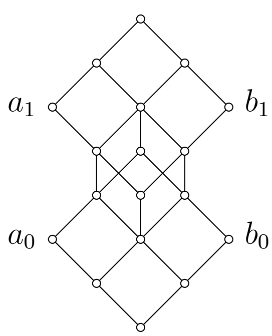
It remains to consider when $\mathbf{FL}(\mathbf{P})$ is finitely separable. This cannot always be the case; for example, it must fail when $\mathbf{P}$ contains an uncountable chain. Freese and Nation showed that $\mathbf{FL}(\mathbf{P})$ is finitely separable exactly when $\mathbf{P}$ is [**61**].

THEOREM 5.22. $\mathbf{FL}(\mathbf{P})$ *is projective if and only if* $\mathbf{P}$ *is finitely separable.*

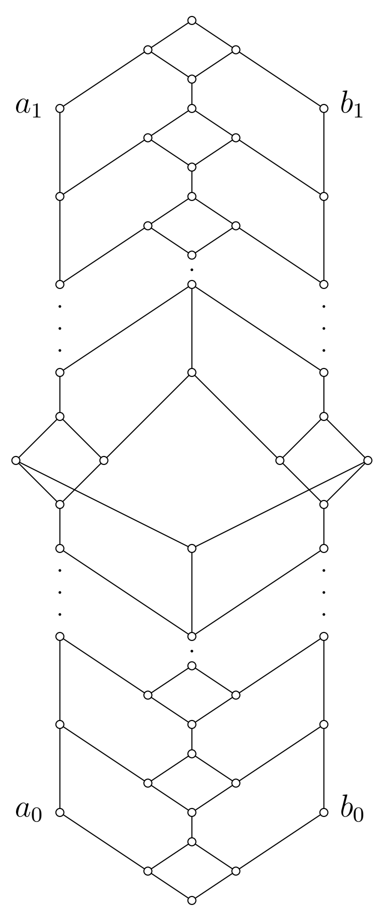
Proof: Suppose $\mathbf{FL}(\mathbf{P})$ is projective. Then $\mathbf{FL}(\mathbf{P})$ is finitely separable *via* operators A and B, say. For $p \in P$, let

$$A'(p) = \{r \in P : r \geq p \text{ and } r \in \mathrm{var}(a) \text{ for some } a \in A(p)\}\,.$$

The set $B'(p)$ is defined dually. If $p \leq q$ in $\mathbf{P}$, then there exists an $a \in A(p) \cap B(q)$. Using the fact that p is join prime and q is meet prime, a routine induction on the complexity of a shows that there is an $r \in \mathrm{var}(a)$ with $p \leq r \leq q$. Hence $A'(p) \cap B'(q)$ is nonempty.



$\mathbf{FD}(\mathbf{2}+\mathbf{2})$



$\mathbf{FL}(\mathbf{2}+\mathbf{2})$

Conversely, assume **P** is finitely separable. Let $X = \{x_p : p \in P\}$ and let $f : \mathbf{FL}(X) \to \mathbf{FL}(\mathbf{P})$ be the homomorphism extending the map $f(x_p) = p$. Exactly as in the proof of Lemma 5.6, we can arrange P into a doubly indexed sequence and, with the aid of this sequence, we can find an order preserving map $g : P \to \mathbf{FL}(X)$ such that $fg(p) = p$ for all $p \in P$. By the universal property of $\mathbf{FL}(\mathbf{P})$, the map g can be extended to a homomorphism from $\mathbf{FL}(\mathbf{P})$ into $\mathbf{FL}(X)$. The extended map satisfies $fg = i_{\mathbf{FL}(\mathbf{P})}$, so it is a transversal of f, and this makes $\mathbf{FL}(\mathbf{P})$ projective by Theorem 5.1(2).

Corollary 5.23. *The following are equivalent for an ordered set* **P**.

(1) **P** *is embeddable in a free lattice.*
(2) $\mathbf{FL}(\mathbf{P})$ *is embeddable in a free lattice.*
(3) **P** *is embeddable in an ordered set* **Q** *which is finitely separable.*

Proof: Clearly, (2) implies (1) implies (3). If (3) holds, then $\mathbf{FL}(\mathbf{P})$ is embeddable in $\mathbf{FL}(\mathbf{Q})$, which is in turn projective, so (2) holds.

At this point, some examples are in order.

Example 5.24. Let **P** be a bipartite ordered set consisting of two antichains C and D, with $c \leq d$ for all $c \in C$ and $d \in D$. Then $\mathbf{FL}(\mathbf{P})$ is the ordinal sum of two free lattices. Hence by Corollary 5.17, $\mathbf{FL}(\mathbf{P})$ is projective if and only if either C or D is finite, or both are countable.

Example 5.25. Now let C and D be antichains as in the previous example, and let $Q = C \cup D \cup \{r\}$ be ordered by $c \leq r \leq d$ for all $c \in C$ and $d \in D$. It is a trivial exercise to show that **Q** is finitely separable, and hence $\mathbf{FL}(\mathbf{Q})$ is projective. Since every **P** in Example 5.24 is contained in the corresponding Q, they can all be embedded in free lattices (regardless of $|C|$ and $|D|$).

Example 5.26. Let X be an uncountable set, and let **R** be the bipartite ordered set isomorphic to all atoms and coatoms of $\mathbf{2}^X$. It was shown in [**61**] that **R** cannot be embedded into a finitely separable ordered set, and hence not into a free lattice. This argument was generalized by the following result of Nation and J. Schmerl [**109**], whose proof we omit. Recall that the *order dimension* of an ordered set **P** is the least cardinal κ such that the order of **P** is the intersection of κ total orders.

Theorem 5.27. *If* $\mathcal{V}$ *is a nontrivial lattice variety and* X *is an infinite set, then the order dimension of the relatively free lattice* $\mathbf{F}_{\mathcal{V}}(X)$ *is the smallest cardinal* λ *such that* $|X| \leq 2^{\lambda}$.

PROBLEM 5.28. *Characterize those ordered sets which can be embedded into a free lattice.*

Corollary 5.23, while a significant reduction of the problem, hardly counts as a characterization. For reference, we only know two general necessary conditions: if $\mathbf{P}$ can be embedded in a free lattice, then

(1) $\mathbf{P}$ is a countable union of antichains,
(2) if $\mathbf{P}$ is infinite, then $\dim\ \mathbf{P} \leq \lambda$ where λ is the smallest cardinal with $|P| \leq 2^\lambda$.

Some further restrictions necessary for a bipartite ordered set to be embeddable in a free lattice can be found in Nation [**105**]. In particular, the following example appears there.

EXAMPLE 5.29. Let U be an uncountable collection of subsets of ω. Let $\mathbf{S}$ be the bipartite ordered set consisting of two antichains $C = \{c_i : i \in \omega\}$ and $D = \{d_J : J \in U\}$, with $c_i \leq d_J$ iff $i \in J$. Then $\mathbf{S}$ is the union of two antichains, and has countable order dimension, but cannot be embedded in a free lattice. Thus the two conditions above are not sufficient.

Let us close this section with a few results which shed a somewhat different light on the results of Section 1. In Chapter II, the definition of a lower bounded lattice required that $\mathbf{L}$ be finitely generated. Let us now drop that restriction, and consider lattices which are lower bounded homomorphic images of $\mathbf{FL}(\mathbf{P})$ for some ordered set $\mathbf{P}$.

Suppose $h : \mathbf{K} \to \mathbf{L}$ is a homomorphism. In Chapter II, the assumption that $\mathbf{K}$ is generated by a finite set X was used to make each $H_k = X^{\wedge(\vee\wedge)^k}$ finite, and thus assure that $\beta_k(a) = \bigwedge\{w \in H_k : h(w) \geq a\}$ is defined for $a \leq h(1_\mathbf{K})$. However, there is another way to accomplish this. Again let $\mathbf{K}$ be generated by X, which now may be infinite. Assume

(1) if $a \in L$ and $a \leq h(x)$ for some $x \in X$, then $\bigwedge\{x \in X : h(x) \geq a\}$ exists and is in $X^\wedge$ (i.e., is the meet of a finite subset), and
(2) $\mathbf{L}$ has the minimal join cover refinement property.

The first condition means that β_0 is defined, except that if $a \leq h(w)$ for some $w \in K$ but $a \nleq h(x)$ for all $x \in X$, the value $\beta_0(a) = 1$ may be artificial (if $\mathbf{K}$ has no largest element). The second condition insures that the formulas for $\beta_k(a)$ in Theorems 2.3 and 2.4 make sense and agree. The argument of Theorem 2.3 then shows that these formulas do represent the least element of $\{w \in H_k : h(w) \geq a\}$, as desired.

Likewise, the analogues of Theorems 2.6–2.8 with these hypotheses are valid, with the same proofs.[2]

Assuming these results, let us proceed.

LEMMA 5.30. *Let* $\mathbf{F}$ *be a lattice with the minimal join cover refinement property and* $\mathrm{D}(\mathbf{F}) = F$. *If there exists an epimorphism* $h : \mathbf{F} \twoheadrightarrow \mathbf{L}$ *with a join-preserving transversal* g, *then* $\mathbf{L}$ *has the minimal join cover refinement property and* $\mathrm{D}(\mathbf{L}) = \mathbf{L}$.

This is just a combination of Lemma 5.3 and the proof of Lemma 5.4 (or for lower bounded homomorphisms, the simpler argument of Theorem 2.13).

THEOREM 5.31. *A lattice* $\mathbf{L}$ *is a lower bounded homomorphic image of some* $\mathbf{FL}(\mathbf{P})$ *if and only if* $\mathbf{L}$ *has the minimal join cover refinement property and* $\mathrm{D}(\mathbf{L}) = \mathbf{L}$.

Proof: The previous lemma gives one direction. Conversely, suppose $\mathbf{L}$ has the minimal join cover refinement property and $\mathrm{D}(\mathbf{L}) = \mathbf{L}$. Let $\mathbf{P}$ be order-isomorphic to $\mathbf{L}$, say $P = \{x_a : a \in L\}$, and let $h : \mathbf{FL}(\mathbf{P}) \twoheadrightarrow \mathbf{L}$ by extending the map $h(x_a) = a$. Then $\beta_0(a) = x_a$, and the minimal join cover refinement property allows us to define $\beta_n(a)$ for all n as in the statement of Theorem 2.4. By the extension of Lemma 2.7, h is lower bounded.

Similarly, we can mimic the proof of Theorem 5.7 to obtain the following.

THEOREM 5.32. *A lattice* $\mathbf{L}$ *is a retract of* $\mathbf{FL}(\mathbf{P})$ *for some ordered set* $\mathbf{P}$ *if and only if it satisfies the following conditions.*

(1) *Whitman's condition* (W).
(2) $L = \mathrm{D}(\mathbf{L}) = \mathrm{D}^d(\mathbf{L})$.
(3) $\mathbf{L}$ *has the minimal join cover refinement property and its dual.*

Our characterization of lower bounded homomorphic images of $\mathbf{FL}(X)$ is a little more artificial, but still easy enough.

THEOREM 5.33. *A lattice* $\mathbf{L}$ *is a lower bounded homomorphic image of a free lattice if and only if it satisfies the following conditions.*

(1) $L = \mathrm{D}(\mathbf{L}) = \mathrm{D}^d(\mathbf{L})$.
(2) $\mathbf{L}$ *has the minimal join cover refinement property.*
(3) *There exists a generating set* Y *for* $\mathbf{L}$ *such that* $\{y \in Y : y \geq a\}$ *is finite for every* $a \in L$.

[2]On the other hand, while it is possible to formulate analogues, the proper context for the results from Theorem 2.13 to Corollary 2.16 is finitely generated lattices.

The lattice of natural numbers ω, which is projective but not a lower bounded image of a free lattice, shows that we are not likely to do much better. It is also not hard to show that these conditions are independent.

COROLLARY 5.34. $\mathbf{FL}(\mathbf{P})$ *is a lower bounded image of a free lattice if and only if* $\{q \in P : q \geq p\}$ *is finite for every* $p \in P$.

The proof just uses the fact that the generators of $\mathbf{FL}(\mathbf{P})$ are meet prime.

3. Finite Sublattices of Free Lattices

The aim of this section is to prove that a finite semidistributive lattice satisfying Whitman's condition is projective, and thus can be embedded in a free lattice. It is useful to have a name for these lattices: by an *S–lattice* we shall mean a finite semidistributive lattice satisfying (W).

The proof is fairly long and complicated, and we have not found any significant improvements. Nonetheless, the idea behind it is fairly simple. The result was first conjectured by Bjarni Jónsson, who outlined a program for its proof in an unpublished set of seminar notes written in the 1960's. His first observation was that it is sufficient to prove that no S–lattice can contain a C–cycle. This leads us to a detailed study of C–cycles in S–lattices. Semidistributivity alone prevents us from having a C–cycle which is too short, or of too simple a form. On the other hand, longer or more complicated C–cycles lead to configurations which cannot exist in a finite lattice satisfying (W). Partial results along these lines were published by Jónsson and Nation in [**83**], and the program was carried through for planar lattices by Ivan Rival and Bill Sands [**115**]. Also, J. Ježek and V. Slavík proved Jónsson's conjecture for subdirectly irreducible S–lattices using different methods [**76**] (see Section 5 of this chapter). Finally, Nation completed Jónsson's program in 1980 [**104**], yielding the proof given here.

The reader may very well wonder at the motivation for some of the technical lemmas. In fact, Nation spent a fair amount of time in the 1970's writing computer programs to construct S–lattices generated by a C–cycle, which if they had ever terminated would have produced a counterexample. Invariably, these programs started constructing longer and longer chains. The argument that these chains become infinite was then formalized, and a different type of C–cycle was tried. Nation's programs were rather simple minded by today's standards, but certainly this qualifies as one of the first computer assisted proofs in lattice theory.

Configurations which make a lattice satisfying (W) infinite are now much better understood, thanks to the wonderful results of V. Slavík on (W)–covers of finite lattices [**123**]. However, for our purposes the simple configurations found by Nation's programs suffice. So let us begin!

LEMMA 5.35. *The following are equivalent for a finite lattice.*

(1) **L** *is projective.*
(2) **L** *is a sublattice of a free lattice.*
(3) **L** *is bounded and satisfies* (W)*.*
(4) **L** *is an S–lattice containing no* C*–cycle.*

Proof: By Corollaries 5.9 and 5.10, the first three conditions are equivalent for any finitely generated lattice. Moreover, they imply (W) and semidistributivity by Theorem 2.20 and its dual. On the other hand, a finite semidistributive lattice is bounded if and only if it contains no C–cycle by Lemma 2.62.

In the following we shall work with an S–lattice containing a C–cycle. Recall from Chapter II that a C–cycle is a finite sequence $a_0, \ldots, a_{n-1}$ $(n \geq 2)$ of join irreducible elements such that $a_i \; C \; a_{i+1}$ for all $i = 0, \ldots, n-1$, where the subscripts are computed modulo n. Here C is the union of two relations A, B that were defined by

$$\begin{aligned} a \; A \; b \quad & \text{if } b < a \text{ and } a \leq b \vee \kappa(b), \\ a \; B \; b \quad & \text{if } a \neq b,\ a \leq a_* \vee b \text{ and } a \not\leq a_* \vee b_*. \end{aligned}$$

Equivalently, these can be written as

$$\begin{aligned} a \; A \; b \quad & \text{if } b < a \leq \kappa(b)^*, \\ a \; B \; b \quad & \text{if } a \neq b,\ b \not\leq \kappa(a) \text{ and } b_* \leq \kappa(a). \end{aligned}$$

For technical reasons these relations are further subdivided:

$$\begin{aligned} a \; A_1 \; b \quad & \text{if } a \; A \; b \text{ and } a \wedge \kappa(b) > b_*, \\ a \; A_2 \; b \quad & \text{if } a \; A \; b \text{ and } a \wedge \kappa(b) = b_*, \\ a \; B_1 \; b \quad & \text{if } a \; B \; b \text{ and } b_* \not\leq a, \\ a \; B_2 \; b \quad & \text{if } a \; B \; b \text{ and } b_* < a. \end{aligned}$$

The remarks following the original definition of the relations A and B on page **??** show that the drawings of Figure 5.4 accurately represent these relations insofar as the joins and meets of the labeled elements are concerned. In particular, if $a \; B \; b$, then $a \leq \kappa(b)$ by Lemma 2.58.

Supposing there is a C–cycle in an S–lattice, there also exists a minimal C–cycle, i.e., one of minimal length. If p, q are two elements

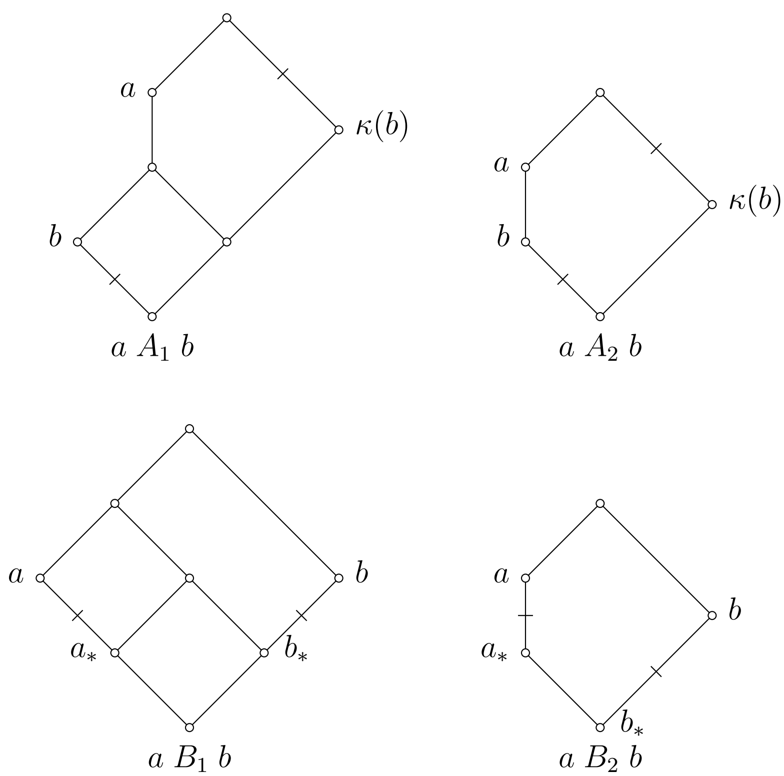


FIGURE 5.4

of a minimal C–cycle, then $p\ C\ q$ can only be true if q is the successor of p in the cycle. The following argument (which we shall refer to as the *free star principle*) will be useful several times: *If p, q are two distinct members of a minimal C–cycle and q is not the successor of p in the cycle, then $p\ B\ q$ does not hold; therefore $q \nleq \kappa(p)$ implies $q_* \nleq \kappa(p)$.*

Observe also that for any element p of a minimal C–cycle, p_{**} exists. Indeed, we have $q\ C\ p$ for some q, which implies that p_* is meet reducible (see Figure 5.4). Also, p_* is nonzero, since $p\ C\ r$ for some r. Consequently, p_* is join irreducible and p_{**} exists.

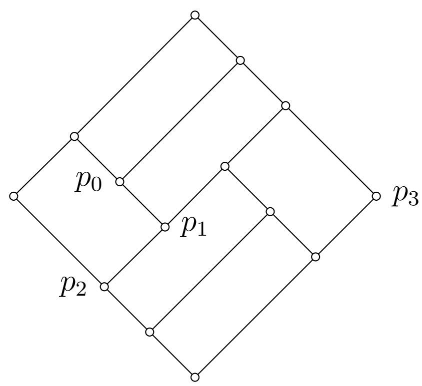


FIGURE 5.5

At this point it would seem appropriate to indicate to the reader our general plan for showing that no S–lattice can contain a C–cycle. First of all, C–cycles can exist in finite semidistributive lattices failing (W), as Figure 5.5 shows. This example, from Jónsson and Nation [**83**], has the cycle $p_0\ A\ p_1\ A\ p_2\ B\ p_3\ B\ p_0$. Thus our first order of business is to develop some configurations which cannot exist in a finite lattice satisfying (W). Now suppose that $p_0\ C\ p_1\ C\ \dots\ C\ p_n = p_0$ is a minimal C–cycle in an S–lattice. Whenever $p_i\ A\ p_{i+1}$, then $p_i > p_{i+1}$, so of course $p_{i*} > p_{i+1*}$. Some of our lemmas will state that under the right circumstances, if $p_i\ B\ p_{i+1}\ B\ \dots\ B\ p_j\ A\ p_{j+1}$, then $p_{i*} > p_{j+1*}$. If these circumstances always persisted, with appropriate indexing we would obtain $p_{0*} > p_{j+1*} > p_{k+1*} > \cdots > p_{0*}$, a contradiction. However, the proof will not be so easy. In most cases we can conclude at least that

$p_{i*} > p_{j+1**}$, whence $p_{i**} \geq p_{j+1**}$. In the remaining case, we find that $p_j\ A\ p_{j+1}$ is a single A_2 sandwiched between B's, and that this section of the cycle behaves enough like a sequence of all B's to enable us use our arguments at the next occurrence of an A in the cycle. Here we employ the notion of a B–type sequence, to be defined later. Our modified arguments allow us to obtain $p_{0**} \geq p_{j+1**} \geq p_{k+1**} \geq \cdots \geq p_{0**}$ (with appropriate indexing) for any minimal C–cycle in an S–lattice. Moreover, one of the inequalities will be strict, and thus lead to a contradiction, if our cycle contains any A_1 or any two consecutive A's. By the duality induced by Lemma 2.63, neither can our cycle contain two consecutive B's. Thus the A's and B's alternate, in which case we can show that one of the inequalities will be strict if any of the B's is a B_2. So we are left only cycles of the form $p_0\ B_1\ p_1\ A_2\ p_2\ B_1\ p_3 \ldots A_2\ p_n = p_0$ to consider. This type of a cycle is excluded by a separate argument, which will complete the proof.

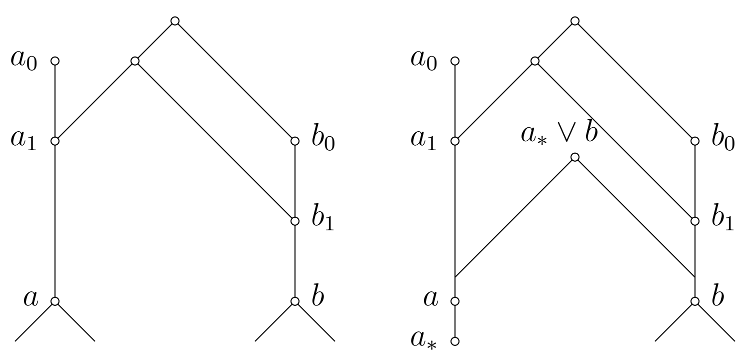


FIGURE 5.6

Let us begin with a pair of simple lemmas about consecutive A's or B's.

LEMMA 5.36. *If* $p_0\ A_2\ p_1\ A\ p_2$ *in a minimal* C*–cycle, then* $p_2 = p_{1*}$ *and* $p_1\ A_2\ p_2$.

Proof: By definition, $p_0 \; A_2 \; p_1$ means $p_0 \wedge \kappa(p_1) = p_{1*}$, so we can apply (W) to $p_0 \wedge \kappa(p_1) \leq p_2 \vee \kappa(p_2)$. We have $p_0 \nleq p_2 \vee \kappa(p_2)$, or else $p_0 \; A \; p_2$ and we would obtain a contradiction with the minimality of the C–cycle. If $\kappa(p_1) \leq p_2 \vee \kappa(p_2)$, then as we also have $p_1 \leq p_2 \vee \kappa(p_2)$ we get $p_0 \leq \kappa(p_1)^* \leq p_2 \vee \kappa(p_2)$, which we have just excluded. On the other hand, $p_{1*} \nleq \kappa(p_2)$, or else $p_2 \leq p_{1*} \leq \kappa(p_2)$. This leaves only the possibility $p_{1*} \leq p_2$. Then $p_{1*} = p_2$. Since $p_2 < p_1$ implies $p_1 \nleq \kappa(p_2)$, we have $p_1 \wedge \kappa(p_2) \leq p_{1*} = p_2$ and thus $p_1 \; A_2 \; p_2$.

LEMMA 5.37. *If $q_0 \; B \; q_1 \; B \; q_2$ in a minimal C–cycle, then $q_1 \; B_1 \; q_2$.*

Proof: Note that $q_0 \neq q_1$ by the definition of the relation B. We have $q_0 \leq q_{0*} \vee q_1 \leq q_{0*} \vee q_{1*} \vee q_2$. Since $q_{0*} \vee q_{1*} \leq \kappa(q_0)$, this implies $q_2 \nleq \kappa(q_0)$. By the free star principle $q_{2*} \nleq \kappa(q_0)$, and hence $q_{2*} \nleq q_{1*}$. Thus $q_1 \; B_1 \; q_2$.

Now it is time to consider some configurations excluded in finite lattices satisfying (W).

LEMMA 5.38. *A finite lattice* $\mathbf{L}$ *satisfying* (W) *cannot contain elements a, a_0, a_1, b, b_0 such that the following conditions hold*:

(1) *b is join reducible, and either a is join reducible or $a \leq a_* \vee b$,*
(2) *$a \leq a_1 \prec a_0$,*
(3) *$b \leq b_0$,*
(4) *$a \nleq b_0$,*
(5) *$b \nleq a_0$,*
(6) *$a_0 \nleq a_1 \vee b_0$,*
(7) *$b_0 \nleq a_1 \vee b$.*

Proof: Suppose that such a configuration exists in a finite lattice satisfying (W). The situation (depending on which one of the two cases in condition (1) takes place) is pictured in Figure 5.6. First observe that the element $a_1 = a_0 \wedge (a_1 \vee b_0)$ is meet reducible, and hence join irreducible, since a lattice satisfying (W) contains no doubly reducible elements. We claim that $a < a_1$. If a is join reducible, this is clear. If a is join irreducible and $a \leq a_* \vee b$, then $a_1 = a_0 \wedge (a_1 \vee b_0) \nleq a_{1*} \vee b$ by (W), while $a \leq a_* \vee b$, so that again $a \neq a_1$.

Let $a_2 = a_{1*}$ and $b_1 = b_0 \wedge (a_1 \vee b)$. We shall show that (1)–(7) hold with a_0, a_1, b_0 replaced by a_1, a_2, b_1. Obviously $a \leq a_2 \prec a_1$, $b \leq b_1$, $a \nleq b_1$ and $b \nleq a_1$. Also, by (W) we have $a_1 \nleq a_2 \vee b_1$, i.e., $a_0 \wedge (a_1 \vee b_0) \nleq a_2 \vee b_1$, because $a_0 \nleq a_2 \vee b_1$, $b_0 \nleq a_2 \vee b_1$ (since $a_2 \vee b_1 \leq a_1 \vee b_1 = a_1 \vee b$), $a_1 \nleq a_2$ and $a_1 \nleq b_1$. Finally, suppose $b_1 \leq a_2 \vee b$, i.e., $b_0 \wedge (a_1 \vee b) \leq a_2 \vee b$. Neither b_0 nor a_1 is below $a_2 \vee b$, and $b_1 \leq a_2$ fails because $b \nleq a_2$. We must therefore have

$b_0 \wedge (a_1 \vee b) \leq b$, and hence $b_0 \wedge (a_1 \vee b) = b$. But this too is excluded because $b < b_0$, $b < a_1 \vee b$, and b is meet irreducible.

Therefore by iterating this process we obtain an infinite descending chain $a_0 > a_1 > a_2 > \dots$, contradicting the finiteness of $\mathbf{L}$.

The next two lemmas are similar but easier.

LEMMA 5.39. *A finite lattice* $\mathbf{L}$ *satisfying* (W) *cannot contain elements* a, b, c *such that the following conditions hold*:

(1) $b \wedge (c \vee (a \wedge b)) \not\leq a$,
(2) $a \wedge (c \vee (a \wedge b)) \not\leq b$,
(3) $a \wedge b \not\leq c$,
(4) $a \not\leq b \vee c$,
(5) $b \not\leq a \vee c$.

Proof: Supposing that (1)–(5) hold, let $a_1 = a \wedge (b \vee c)$ and $b_1 = b \wedge (a \vee c)$. Then $a_1 < a$ and $b_1 < b$, and it is fairly simple to show that (1)–(5) hold with a and b replaced by a_1 and b_1. Thus we obtain two infinite descending chains contrary to the finiteness of $\mathbf{L}$.

The following lemma is motivated by the infinite chains in $\mathbf{FL}(\mathbf{2}+\mathbf{2})$, see Figure 5.3 on page 139.

LEMMA 5.40. *A finite lattice* $\mathbf{L}$ *satisfying* (W) *cannot contain elements* a, a_0, b, b_0 *such that the following conditions hold*:

(1) $a \leq a_0$,
(2) $b \leq b_0$,
(3) $a \not\leq b_0$,
(4) $b \not\leq a_0$,
(5) $a_0 \not\leq a \vee b$,
(6) $b_0 \not\leq a \vee b$,
(7) $a_0 \wedge b \not\leq a$,
(8) $a \wedge b_0 \not\leq b$.

Proof: Supposing that (1)–(8) hold, let $c = a_0 \wedge b_0$. Then $a \not\leq b \vee c$ since $b \vee c \leq b_0$, and similarly $b \not\leq a \vee c$. By (W), $c \not\leq a \vee b$. On the other hand, it is clear that $b \wedge c \not\leq a$ and $a \wedge c \not\leq b$. By the dual of Lemma 5.39 we can conclude that the configuration cannot occur in $\mathbf{L}$.

By a *B–type sequence* we shall mean a finite sequence $q_0, \dots, q_k$ of join irreducible elements of an S–lattice such that the following four conditions are satisfied:

(B1) $q_{i-1*} \not\leq q_i$ for $1 \leq i \leq k$,
(B2) $q_i \leq \kappa(q_j)$ for $0 \leq i < j \leq k$,
(B3) $q_{i*} \leq \kappa(q_{i-1})$ for $1 \leq i \leq k$,
(B4) $q_{j*} \not\leq \kappa(q_i)$ for $0 \leq i < i+2 \leq j \leq k$.

Note that (B2) implies $q_i \neq q_j$ for $i \neq j$.

LEMMA 5.41. *If $q_0, \dots, q_k$ is a subsequence of a minimal C–cycle such that $q_i \; B \; q_{i+1}$ for $0 \le i < k$, then $q_0, \dots, q_k$ is a B–type sequence.*

Proof: (B1) and (B3) are immediate from the relation $q_{i-1} \; B \; q_i$. (B2) follows from Lemmas 2.58 and 2.63, which give $q_i \le \kappa(q_{i+1}) < \kappa(q_{i+2}) < \cdots < \kappa(q_k)$. For (B4), first note that $q_i \; B \; q_{i+1} \; B \; q_{i+2}$ implies $q_{i+1} \le q_{i+1*} \vee q_{i+2}$, and $q_{i+1} \not\le \kappa(q_i)$ while $q_{i+1*} \le \kappa(q_i)$, so we must have $q_{i+2} \not\le \kappa(q_i)$. If $q_{i+2*} \le \kappa(q_i)$, we would have $q_i \; B \; q_{i+2}$, a contradiction to the minimality of the cycle. Thus $q_{i+2*} \not\le \kappa(q_i)$. For $j > i+2$, we have $q_{j*} \not\le \kappa(q_{i+1})$ by induction and $\kappa(q_i) \le \kappa(q_{i+1})$ since $\kappa(q_i) \; A^d \; \kappa(q_{i+1})$, so that $q_{j*} \not\le \kappa(q_i)$, as desired.

The difference between a B–type sequence and a sequence of B's is that in a B–type sequence we do not require $q_i \not\le \kappa(q_{i-1})$. This notion will play a crucial role in our proof.

By a subsequence we shall always mean a not necessarily consecutive subsequence. The next two technical lemmas put restrictions on subsequences of minimal C–cycles in an S–lattice (which we will eventually show don't exist at all).

LEMMA 5.42. *Let $q_0, \dots, q_k, p$ with $k \ge 0$ be a subsequence of a minimal C–cycle in an S–lattice such that $q_0, \dots, q_k$ is a B–type sequence and $q_k \; A \; p$. Denote by j the least index with $q_k \wedge \kappa(p) \le q_{j*}$. Then none of the following three cases can take place*:

(1) *$q_k \; A_1 \; p$ and $p_* \not\le q_{0*}$;*
(2) *$q_k \; A_2 \; p$, $p_{**} \not\le q_{0*}$, $\bigvee_{i<j} q_i \not\le \kappa(p)$ and $p \; B \; q_0$ does not hold;*
(3) *$q_k \; A_2 \; p$, $p_* \not\le q_{0*}$, $q_0 \le \kappa(p)$ and $\bigvee_{i<j} q_i \not\le \kappa(p)$.*

Proof: Suppose that one of the three cases takes place. Put $t = q_k \wedge \kappa(p)$, so that in cases (2) and (3) we have $t = p_*$. Since $t \le q_{k*}$ (which is a consequence of $q_k \; A \; p$), there exists a least number j with $t \le q_{j*}$. Since $p_* \le t$ and $p_* \not\le q_{0*}$ in all cases, we have $t \not\le q_{0*}$ and hence $j \ge 1$.

Define an element r as follows. Let $r = p_*$ in case (1), $r = p_{**}$ in case (2) and $r = 0$ in case (3). We will define five elements a, a_0, a_1, b, b_0 that will give us the configuration of Lemma 5.38:

$$a = p \vee t,$$

$$b = r \vee \bigvee_{i<j} q_{i*},$$

$$b_0 = r \vee \bigvee_{i<j-1} q_{i*} \vee q_{j-1}.$$

If $j = k$, let $a_0 = q_k$ and $a_1 = q_{k*}$. Otherwise $j < k$, and we take a_0 to be a minimal element in the interval q_{k*}/a with respect to the property $a_0 \nleq \kappa(q_{j-1})$. The existence of a_0 follows from $a \leq q_{k*}$ (a consequence of $q_k \mathrel{A} p$) and $q_{k*} \nleq \kappa(q_{j-1})$ (a consequence of (B4)). We have $p_* \leq t \leq q_{j*} \leq \kappa(q_{j-1})$ (the first inequality by $q_k \mathrel{A} p$ and the last by (B3)), whence by the free star principle $p \leq \kappa(q_{j-1})$. From this, $a \leq \kappa(q_{j-1})$ and $a_0 > a$. Clearly a_0 is join irreducible in the interval q_{k*}/a and we take a_1 to be the unique lower cover of a_0 in that interval. Note that in either case, whether $j = k$ or $j < k$, we have $a_1 \leq \kappa(q_{j-1})$.

Now we must prove that the seven conditions of Lemma 5.38 are satisfied.

In case (1) the element a is join reducible, as p, t are incomparable. In cases (2) and (3) we have $a = p$; since $\bigvee_{i<j} q_i \nleq \kappa(p)$ and $p \mathrel{B} q_0$ does not hold, by the free star principle $\bigvee_{i<j} q_{i*} \nleq \kappa(p)$. Hence in these cases $a = p \leq p_* \vee \bigvee_{i<j} q_{i*} = p_* \vee b = a_* \vee b$. In order to prove 5.38(1), it remains to show that b is join reducible. We have $b \neq r$, since $q_{j-1*} \leq b$ while $q_{j-1*} \nleq r$ (indeed, $r \leq p_* = t \leq q_{j*}$ and $q_{j-1*} \nleq q_{j*}$ by (B1)). Also, $b \neq q_{0*}$. In cases (1) and (2) this follows from $r \leq b$ while $r \nleq q_{0*}$; and if (3) is the case, then (as above) $\bigvee_{i<j} q_{i*} \nleq \kappa(p)$, while $q_{0*} \leq \kappa(p)$ by assumption. If $j = 1$, we are done. Otherwise, for $1 \leq i < j$ we have $q_{i-1*} \leq b$ while $q_{i-1*} \nleq q_{i*}$ by (B1), wherefore $b \neq q_{i*}$.

Conditions 5.38(2) and 5.38(3) are immediate. To prove 5.38(4), suppose that $a \leq b_0$. Then

$$t = q_k \wedge \kappa(p) \leq r \vee \bigvee_{i<j} q_{i*} \vee q_{j-1} = b_0$$

and we can apply (W) to this inequality. We have $q_k \nleq b_0$, since $b_0 \leq \kappa(q_k)$ by (B2) and the fact that $r \leq p_* \leq q_{k*}$. Also $\kappa(p) \nleq b_0$, as $q_k \mathrel{A} p$ yields $q_k \leq p \vee \kappa(p)$, while $p \vee b_0 \leq \kappa(q_k)$ as both terms are. We have $t \nleq r$. In case (1) this can be seen from $q_k \mathrel{A_1} p$ and in cases (2) and (3) it follows from $t = p_*$ while $r \leq p_{**}$. We have $t \nleq q_{i*}$ for $0 \leq i < j$ by the choice of j. This leaves only the possibility $t = q_{j-1}$, which however would imply $q_{j-1*} < t < q_j$, contrary to (B1).

To prove 5.38(5), suppose $q_{j-1*} \leq q_k$. If $j = k$, this contradicts (B1), so we may assume $j < k$. Now $q_{j-1*} \nleq \kappa(p)$, or else we would have $q_{j-1*} \leq q_k \wedge \kappa(p) = t \leq q_j$, contrary to (B1). Therefore $\kappa(p)^* = p \vee \kappa(p) = q_{j-1*} \vee \kappa(p)$, since $p \leq p_* \vee q_{j-1*} \leq q_{j-1*} \vee \kappa(p)$ and $q_{j-1*} \leq q_k \leq \kappa(p)^*$. Applying (SD$_\vee$) we get $\kappa(p)^* = (p \wedge q_{j-1*}) \vee \kappa(p)$. However, since $p_* \leq \kappa(p)$, we cannot have $p \wedge q_{j-1*} < p$; therefore $p \leq q_{j-1*}$. If $j = 1$, this contradicts our assumptions. So, $j > 1$. Now $q_{j-1*} < q_k$, because $t \leq q_k$ but $t \nleq q_{j-1*}$. Also, since $j < k$, (B4) gives

$q_{j-1} \leq q_{j-1*} \vee q_{k*} = q_{k*}$. Thus $p < q_{j-1} < q_k \leq \kappa(p)^*$, and $q_{j-1}\ A\ p$. This, however, contradicts the minimality of our original C–cycle.

Thus $q_{j-1*} \nleq q_k$. This means that 5.38(5) is satisfied, $b \nleq a_0$, or else we would have $q_{j-1*} \leq b \leq a_0 \leq q_k$.

In order to prove 5.38(6), suppose $a_0 \leq a_1 \vee b_0$. Then $a_0 \neq q_k$, since $b_0 \leq \kappa(q_k)$ by (B2). Thus $j < k$, so that $a_0 \nleq \kappa(q_{j-1})$ by the choice of a_0, whence $q_{j-1} \leq q_{j-1*} \vee a_0$. Since also $a_0 \leq a_1 \vee b_0$ and $r \leq p_* < a$, we compute

$$a_0 \vee b_0 = a_0 \vee \bigvee_{i<j} q_{i*} = a_1 \vee \bigvee_{i<j-1} q_{i*} \vee q_{j-1}.$$

Applying $(\mathrm{SD}_\vee)$, we obtain $a_0 \vee b_0 = a_1 \vee (a_0 \wedge q_{j-1}) \vee \bigvee_{i<j} q_{i*}$. However, we have proved $q_{j-1*} \nleq q_k$, so $q_{j-1} \nleq a_0$. Hence $a_0 \wedge q_{j-1} \leq q_{j-1*}$, and

$$a_0 \vee b_0 = a_1 \vee \bigvee_{i<j} q_{i*}.$$

But $a_1 \vee \bigvee_{i<j} q_{i*} \leq \kappa(q_{j-1})$ by the choice of a_0 and (B2), so $a_0 \vee b_0 \leq \kappa(q_{j-1})$, while $q_{j-1} \leq b_0 \leq a_0 \vee b_0$, a contradiction. Thus $a_0 \nleq a_1 \vee b_0$.

Finally, $b_0 \nleq a_1 \vee b$, for otherwise $q_{j-1} \leq b_0 \leq a_1 \vee b = a_1 \vee \bigvee_{i<j} q_{i*} \leq \kappa(q_{j-1})$ as above. Thus the last condition 5.38(7) is also satisfied, and the result follows from Lemma 5.38.

Lemma 5.43. *Let $q_0, \ldots, q_j, p$ with $j \geq 1$ be a subsequence of a minimal C–cycle in an S–lattice and suppose that there exists an element t in the lattice such that the following conditions are satisfied:*

(1) *$q_0, \ldots, q_j$ is a B–type sequence,*
(2) *$p_* \nleq q_{0*}$,*
(3) *$p_* \leq q_{j*}$,*
(4) *$p < t \leq \kappa(p)^*$,*
(5) *$t \leq \kappa(q_{j-1})$,*
(6) *$t \nleq p \vee \bigvee_{i<j-1} q_{i*} \vee q_{j-1}$,*
(7) *$p\ B\ q_0$ does not hold.*

Then $\bigvee_{i<j} q_{i*} \nleq \kappa(p)$.

Proof: Suppose that $\bigvee_{i<j} q_{i*} \leq \kappa(p)$, whence by the free star principle and (7) we also have $\bigvee_{i<j} q_i \leq \kappa(p)$. Let us apply Lemma 5.40 with

$$\begin{aligned} a &= p \vee \bigvee_{i<j} q_{i*}, \\ a_0 &= t \vee \bigvee_{i<j} q_{i*}, \\ b &= p_* \vee \bigvee_{i<j-1} q_{i*} \vee q_{j-1}, \\ b_0 &= \kappa(p). \end{aligned}$$

We need to check the eight conditions of Lemma 5.40.

Conditions 5.40(1)–5.40(3) are immediate consequences of our assumptions. Now by assumption (5) and (B2) we have $a_0 \leq \kappa(q_{j-1})$, whereas $q_{j-1} \leq b$, from which $b \not\leq a_0$ follows. By assumption (6), $t \not\leq a \vee b$ and hence $a_0 \not\leq a \vee b$. Thus 5.40(4) and 5.40(5) are also satisfied.

For 5.40(6), suppose $b_0 \leq a \vee b$. Then we would have $t \leq \kappa(p)^* = p \vee \kappa(p) \leq a \vee b$, contrary to assumption (6).

In order to prove 5.40(7), suppose $a_0 \wedge b \leq a$. Let us apply (W) to the inequality

$$a_0 \wedge b \leq p \vee \bigvee_{i<j} q_{i*} = a.$$

We have $a_0 \not\leq a$, since $a_0 \not\leq a \vee b$. Since $p < t \leq \kappa(q_{j-1})$ and $\bigvee_{i<j} q_{i*} \leq \kappa(q_{j-1})$ by (B2), we have $a \leq \kappa(q_{j-1})$ and consequently $b \not\leq a$. On the other hand, we cannot have $a_0 \wedge b \leq p$, for that would imply $a_0 \wedge b \leq p \wedge \kappa(p) = p_*$ and hence $q_{j-1*} \leq a_0 \wedge b \leq p_* \leq q_j$, contrary to (B1). Since $p_* \leq a_0 \wedge b$ and $p_* \not\leq q_{0*}$, we cannot have $a_0 \wedge b \leq q_{0*}$. For $1 \leq i < j$, though, $q_{i-1*} \leq a_0 \wedge b$ implies $a_0 \wedge b \not\leq q_{i*}$ by (B1). Therefore $a_0 \wedge b \not\leq a$.

It remains to prove 5.40(8). Suppose $a \wedge b_0 \leq b$ and let us apply (W) to

$$a \wedge b_0 \leq p_* \vee \bigvee_{i<j-1} q_{i*} \vee q_{j-1} = b.$$

Now $a \not\leq b$, since $p \leq a$ and $b \leq \kappa(p)$. By 5.40(6), $b_0 \not\leq b$. On the other hand, $q_{j-1*} \leq a \wedge b_0$ implies $a \wedge b_0 \not\leq p_*$, since $p_* \leq q_j$. Because $p_* \leq a \wedge b_0$ and $p_* \not\leq q_{0*}$, we have $a \wedge b_0 \not\leq q_{0*}$. For $1 \leq i \leq j-1$, $q_{i-1*} \leq a \wedge b_0$ implies $a \wedge b_0 \not\leq q_i$ by (B1) and thus $a \wedge b_0 \not\leq q_{i*}$. If $j = 1$, we must also note that $a \wedge b_0 \neq q_0$ since in this case $a \wedge b_0 \leq a \leq \kappa(q_0)$ as $p \leq q_{1*} \leq \kappa(q_0)$ by (B3).

The desired contradiction follows by Lemma 5.40.

Now we want to show that, with suitable hypotheses, a subsequence $q_0, \ldots, q_k, p$ of a minimal C–cycle has $p_{**} < q_{0*}$ (or better, $p_* < q_{0*}$). Lemmas 5.44, 5.45 and 5.46 do this.

LEMMA 5.44. *Let* $q_0, \ldots, q_k, p$ *with* $k \geq 0$ *be a subsequence of a minimal C–cycle in an S–lattice such that* $q_0, \ldots, q_k$ *is a B–type sequence and* $q_k \; A_1 \; p$. *Then* $p_* < q_{0*}$.

Proof: By Lemma 5.42(1), $p_* \leq q_{0*}$. Suppose $p_* = q_{0*}$. Then clearly $k > 0$ and we can apply Lemma 5.42(1) once more, this time to the sequence $q_1, \ldots, q_k, p$, to obtain $p_* \leq q_{1*}$. Since $p_* = q_{0*}$, we get $q_{0*} \leq q_{1*}$, which contradicts (B1).

LEMMA 5.45. *Let* $q_0, \ldots, q_k, p_1, p_2$ *with* $k \geq 0$ *be a subsequence of a minimal C–cycle in an S–lattice such that* $q_0, \ldots, q_k$ *is a B–type sequence and* $q_k \; A_2 \; p_1 \; A_2 \; p_2$. *Then* $p_{2*} < q_{0*}$.

Proof: We proceed by induction, with the case $k = 0$ following trivially from $q_0 \; A \; p_1 \; A \; p_2$. Thus we may assume $k > 0$ and $p_{2*} < q_{1*}$.

Suppose $p_{2*} \not< q_{0*}$. Now $q_{0*} \neq p_{2*}$, since $p_{2*} < q_1$; hence we have $p_{2*} \nleq q_{0*}$. (Note that this implies $p_2 \neq q_0$.) By Lemma 5.36, $p_2 = p_{1*}$ and so $p_{1**} \nleq q_{0*}$. Also, $p_1 \; B \; q_0$ does not hold as that would contradict the minimality of our cycle. By Lemma 5.42(2) we get $\bigvee_{i<j} q_i \leq \kappa(p_1)$, where $j \geq 1$ is minimal with respect to the property $q_k \wedge \kappa(p_1) = p_{1*} \leq q_{j*}$, i.e., $p_2 \leq q_{j*}$.

Now we can apply Lemma 5.43 to the sequence $q_0, \ldots, q_j, p_2$ with $t = p_1$. Conditions 5.43(1)–5.43(4) are easy to check. For 5.43(5), note that $p_{1*} \leq q_{j*} \leq \kappa(q_{j-1})$ by (B3), so by the free star principle $p_1 \leq \kappa(q_{j-1})$. By the above application of Lemma 5.42(2), $p_1 \nleq p_2 \vee \bigvee_{i<j} q_i$ and we obtain 5.43(6).

It remains to prove 5.43(7). Suppose, on the contrary, that $p_2 \; B \; q_0$. Then we can apply (W) to the inequality $p_2 = q_k \wedge \kappa(p_1) \leq p_{2*} \vee q_0$. Surely $p_2 \nleq p_{2*}$, and $p_2 \nleq q_0$ since $p_{2*} \nleq q_{0*}$. We have $q_k \nleq p_{2*} \vee q_0$ since by (B2), $p_{2*} \vee q_0 \leq \kappa(q_k)$. Likewise $\kappa(p_1) \nleq p_{2*} \vee q_0$, for otherwise $q_k \leq p_1 \vee \kappa(p_1) \leq p_1 \vee q_0 \leq \kappa(q_k)$ using $q_k \; A \; p_1$ and (B2), a contradiction. Thus 5.43(7) holds.

Lemma 5.43 then yields $\bigvee_{i<j} q_{i*} \nleq \kappa(p_2)$. That being the case, we may apply (W) to the inequality $p_2 = q_k \wedge \kappa(p_1) \leq p_{2*} \vee \bigvee_{i<j} q_{i*}$. Now, q_k is not below the right hand side since that is below $\kappa(q_k)$ by (B2). Similarly $\kappa(p_1)$ is not below the right hand side, for otherwise $q_k \leq \kappa(p_1)^* = p_1 \vee \kappa(p_1) \leq p_1 \vee \bigvee_{i<j} q_{i*} \leq \kappa(q_k)$, a contradiction. Of course $p_2 \nleq p_{2*}$, while $p_2 \nleq q_{i*}$ for $i < j$ by the choice of j.

The contradiction supplied by Lemma 5.43 means that our original assumption $p_{2*} \not< q_{0*}$ was wrong.

LEMMA 5.46. *Let $q_0, \dots, q_k, p$ with $k \geq 0$ be a subsequence of a minimal C–cycle in an S–lattice such that $q_0, \dots, q_k$ is a B–type sequence and $q_k\ A_2\ p$. If $p\ B\ q_0$ does not hold and $q_0 \nleq \kappa(p)$, then $p_{**} < q_{0*}$.*

Proof: The case $k = 0$ is trivial, so we may assume $k \geq 1$. Suppose $p_{**} \not< q_{0*}$. Since $q_0 \nleq \kappa(p)$ and $p\ B\ q_0$ does not hold, we have $q_{0*} \nleq \kappa(p)$, whence $q_{0*} \neq p_{**}$. Then $p_{**} \nleq q_{0*}$ and we can apply case (2) of Lemma 5.42 to obtain a contradiction.

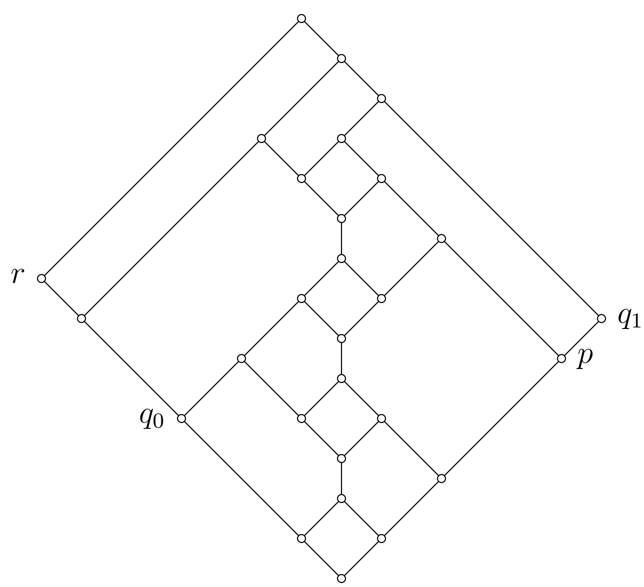


FIGURE 5.7

Now we must deal with the case $q_k\ A_2\ p\ B\ r$ with $q_0 \leq \kappa(p)$, where it will not be possible to conclude that $p_{**} < q_{0*}$ (see Figure 5.7). We shall show, however, that we can obtain a longer B–type sequence whenever $p_{**} \not< q_{0*}$. Our next four lemmas provide preliminaries for Lemma 5.51.

LEMMA 5.47. *Let $q_0, \dots, q_k, p$ with $k \geq 0$ be a subsequence of a minimal C–cycle in an S–lattice such that $q_0, \dots, q_k$ is a B–type sequence and $q_k \; A_2 \; p$. If $q_0 \leq \kappa(p)$ and $p_* \not< q_{0*}$, then $p_* \nleq q_{0*}$ and $\bigvee_{i<k} q_i \leq \kappa(p)$.*

Proof: Clearly, $k > 1$ because $q_0 \leq \kappa(p)$ while $q_k \; A_2 \; p$. First we shall prove $p_* \neq q_{0*}$, so that then $p_* \nleq q_{0*}$. For $k = 1$, this follows from (B1) as $p_* \leq q_{k*}$. If $k > 1$, then $q_k \nleq \kappa(q_0)$ by (B4), whence $p_* = q_{0*}$ would imply $q_0 \leq q_{0*} \vee q_{k*} = q_{k*}$. But this gives $q_0 \leq q_k \wedge \kappa(p) = p_* = q_{0*}$, a contradiction.

Denote by j the least index with $p_* \leq q_{j*}$, so that $j \geq 1$. Applying case (3) of Lemma 5.42, we obtain $\bigvee_{i<j} q_i \leq \kappa(p)$. If $j = k$, this is the desired result. So assume $j < k$ and $\bigvee_{i<k} q_i \nleq \kappa(p)$, i.e., $q_l \nleq \kappa(p)$ for some l with $j \leq l < k$. We shall apply Lemma 5.43 with $t = \kappa(p)^* \wedge \kappa(q_{j-1})$. Hypotheses 5.43(1)–5.43(3) and 5.43(5) hold immediately, while 5.43(7) is a consequence of $q_0 \leq \kappa(p)$. It remains to verify 5.43(4) and 5.43(6).

Since $p_* \leq q_{j*} \leq \kappa(q_{j-1})$, the free star principle yields $p \leq \kappa(q_{j-1})$. Hence $p \leq t \leq \kappa(p)^*$. We also have $q_{j-1*} \leq t$. Thus we cannot have $p = t$, for that would imply $q_{j-1*} \leq t \wedge \kappa(p) = p \wedge \kappa(p) = p_* \leq q_j$, contrary to (B1). This gives 5.43(4).

In order to prove 5.43(6), suppose

$$t = \kappa(p)^* \wedge \kappa(q_{j-1}) \leq p \vee \bigvee_{i<j-1} q_{i*} \vee q_{j-1}.$$

Let us apply (W) to this inequality, denoting the right hand side by u. We have already shown that $t > p$. Likewise, $p_* < t$ implies $t \nleq q_{i*}$ for $i \leq j-1$ by the choice of j, while $t \neq q_{j-1}$ because $t \leq \kappa(q_{j-1})$. If $\kappa(p)^*$ is below u, then the same holds for q_k, as $q_k \; A \; p$ implies $q_k \leq \kappa(p)^*$. But then $q_k \leq u \leq p \vee \bigvee_{i \leq j-1} q_i \leq \kappa(q_k)$ using (B2), a contradiction.

So we are left with the case when $\kappa(q_{j-1})$ is below u, which makes $u = \kappa(q_{j-1})^*$. This also implies that q_{j*} is below u. Recall that $j < k$, whence (B3) implies $q_{j*} = q_j \wedge (q_{j*} \vee q_{j+1*})$. Let us apply (W) to the inequality

$$q_{j*} = q_j \wedge (q_{j*} \vee q_{j+1*}) \leq p \vee \bigvee_{i<j-1} q_{i*} \vee q_{j-1}.$$

Suppose $q_{j*} \leq p$. Then since $p_* \leq q_{j*}$ we have either $q_{j*} = p$ or $q_{j*} = p_*$. Also this implies $j < k-1$, as by (B1) we cannot have $q_{k-1*} \leq p < q_k$. Hence, using (B4), $q_j \leq q_{j*} \vee q_{k*} \leq p \vee q_{k*} = q_{k*}$. If $q_{j*} = p$, this means $p < q_j < q_k \leq \kappa(p)^*$, so that $q_j \; A \; p$, contrary to the minimality of the C–cycle. If $q_{j*} = p_*$, then $q_j \leq \kappa(p)$ since $p_* \leq q_j$

and $p \not\leq q_j$, so we have $q_j \leq q_k \wedge \kappa(p) = p_* = q_{j*}$, a contradiction. Therefore $q_{j*} \not\leq p$.

We have $q_{j*} \not\leq q_{i*}$ for $i \leq j-1$ since j is minimal such that $p_* \leq q_{j*}$. Also, $q_{j*} \neq q_{j-1}$ by (B1).

Since $p_* \leq q_{j*} \leq \kappa(q_j)$, we get $p \leq \kappa(q_j)$ by the free star principle. As also $\bigvee_{i<j-1} q_{i*} \vee q_{j-1} \leq \kappa(q_j)$ by (B2), we have $q_j \not\leq u$.

In the last case we have $q_{j+1*} \leq u = \kappa(q_{j-1})^*$. By (B4) we know that $q_{j+1*} \not\leq \kappa(q_{j-1})$, so $u = q_{j-1} \vee \kappa(q_{j-1}) = q_{j+1*} \vee \kappa(q_{j-1})$. Applying ($\mathrm{SD}_\vee$) then yields $u = (q_{j-1} \wedge q_{j+1*}) \vee \kappa(q_{j-1})$, which can only happen if $q_{j-1} \leq q_{j+1*}$. But then we would have $q_{j-1} < q_{j+1} \leq \kappa(q_{j-1})^*$, so that $q_{j+1}\ A\ q_{j-1}$, contrary to the minimality of our cycle. We conclude that $t \not\leq u$.

All seven conditions of Lemma 5.43 have been verified. The lemma now yields $\bigvee_{i<j} q_i \not\leq \kappa(p)$, which contradicts the conclusion that we obtained earlier by applying Lemma 5.42. Therefore $\bigvee_{i<k} q_i \leq \kappa(p)$, as desired.

The next lemma has the same hypotheses and extends the conclusions of the preceding one.

LEMMA 5.48. *Let $q_0, \ldots, q_k, p$ with $k \geq 0$ be a subsequence of a minimal C–cycle in an S–lattice such that $q_0, \ldots, q_k$ is a B–type sequence and $q_k\ A_2\ p$. If $q_0 \leq \kappa(p)$ and $p_* \not< q_{0*}$, then $p = q_{k*}$.*

Proof: Suppose $p < q_{k*}$, and let us apply Lemma 5.43 with $t = q_{k*}$ and $j = k$. The hypotheses 5.43(1), 5.43(3)–5.43(5) and 5.43(7) follow immediately from our assumptions, and 5.43(2) is the easier of the two assertions of Lemma 5.47.

That leaves 5.43(6) to be checked. If it fails, then

$$q_{k*} = q_k \wedge \kappa(q_k) \leq p \vee \bigvee_{i<k-1} q_{i*} \vee q_{k-1},$$

in which case we can apply (W). Denote the right hand side by u.

Now $q_{k*} \not\leq p$ because $p < q_{k*}$.

For $i < k-1$, (B4) says that $q_{k*} \not\leq \kappa(q_i)$, whence $q_{k*} \not\leq q_{i*}$.

Suppose $q_{k*} \leq q_{k-1}$. Then $q_{k*} \leq q_{k-1*}$, as $q_{k*} \neq q_{k-1}$ by (B1). If $k = 1$, we obtain a contradiction from $p_* < q_{1*}$ and $p_* \not\leq q_{0*}$. If $k > 1$, $q_{k*} \leq q_{k-1*} \leq \kappa(q_{k-2})$ by (B3), while $q_{k*} \not\leq \kappa(q_{k-2})$ by (B4), so we get a contradiction again.

By (B2) and the fact that $p < q_k$, we have $u \leq \kappa(q_k)$, so $q_k \not\leq u$.

The last case is when $\kappa(q_k) \leq u$. We have just observed that $u \leq \kappa(q_k)$, so this implies $\kappa(q_k) = u$. If it happens that $\kappa(p) \wedge \bigvee_{i \leq k} q_i \not\leq \kappa(q_k)$, then $q_k \leq q_{k*} \sqcup (\kappa(p) \wedge \bigvee_{i \leq k} q_i)$. We can combine this with the conclusion $\bigvee_{i<k} q_i \leq \kappa(p)$ of Lemma 5.47 to obtain $\bigvee_{i \leq k} q_i = q_{k*} \vee$

$(\kappa(p) \wedge \bigvee_{i \le k} q_i)$. Applying $(\mathrm{SD}_\vee)$ to this yields $\bigvee_{i \le k} q_i = \bigvee_{i<k} q_i \vee q_{k*} \vee (q_k \wedge \kappa(p) \wedge \bigvee_{i \le k} q_i)$. But $q_k \wedge \kappa(p) = p_* < q_{k*}$ and we get $\bigvee_{i \le k} q_i = \bigvee_{i<k} q_i \vee q_{k*} \le \kappa(q_k)$, a contradiction. Thus instead we must have

$$\kappa(p) \wedge \bigvee_{i \le k} q_i \le p \vee \bigvee_{i<k-1} q_{i*} \vee q_{k-1} = u = \kappa(q_k).$$

Let us apply (W) to this inequality. We have $\kappa(p) \nleq \kappa(q_k)$, since $q_k \; A_2 \; p$ implies $\kappa(q_k) \; B^d \; \kappa(p)$, which makes these elements incomparable. Of course, $\bigvee_{i \le k} q_i \nleq \kappa(q_k)$. Then $\kappa(p) \wedge \bigvee_{i \le k} q_i$ must be below one of the joinands of u. Now $p_* \vee \bigvee_{i<k} q_i \le \kappa(p) \wedge \bigvee_{i \le k} q_i$ by Lemma 5.47, and so $p_* \vee \bigvee_{i<k} q_i$ is below one of the joinands of u. Since $q_{k-1} \nleq q_k$ by (B1), $p_* \vee \bigvee_{i<k} q_i \nleq p$. Since $p_* \nleq q_{0*}$ and $q_{i-1} \nleq q_{i*}$ for $i > 0$ by (B1) again, we have $p_* \vee \bigvee_{j<k} q_j \nleq q_{i*}$ for $0 \le i < k-1$. Then only $p_* \vee \bigvee_{i<k} q_i \le q_{k-1}$ remains possible. If $k = 1$, this says $p_* \le q_0$, which combined with the conclusion $p_* \nleq q_{0*}$ of Lemma 5.47 means $p_* = q_0$. But $p_* < q_k = q_1$, so this cannot be. If $k > 1$, a contradiction follows from $q_{k-2} \nleq q_{k-1}$.

We have verified all the seven hypotheses of Lemma 5.43. By that lemma we conclude that $\bigvee_{i<k} q_{i*} \nleq \kappa(p)$, which contradicts Lemma 5.47. Hence $p = q_{k*}$.

LEMMA 5.49. *If $p \; A_2 \; q \; B_1 \; r$ in an S–lattice, then $p \le q \vee r$.*

Proof: Put $a = q \vee r = q_* \vee r$, $b = \kappa(q)$ and $c = p$. By Lemma 5.39, one of the following five cases takes place:

(1) $b \wedge (c \vee (a \wedge b)) \le a$,
(2) $a \wedge (c \vee (a \wedge b)) \le b$,
(3) $a \wedge b \le c$,
(4) $a \le b \vee c$,
(5) $b \le a \vee c$.

Case (2) is not possible, because $q \le a \wedge c$ and $q \nleq b$.

Likewise, (3) cannot hold, because $r_* \le a \wedge b$ and $r_* \nleq c$. (If $r_* \le c$, then $r_* \le p \wedge \kappa(q) = q_*$, a contradiction.)

Case (4) is not possible, because $q \vee r \le p \vee \kappa(q) = q \vee \kappa(q)$ would imply $q \vee \kappa(q) = r \vee \kappa(q)$, since $q \le q_* \vee r \le \kappa(q) \vee r$. Applying $(\mathrm{SD}_\vee)$ yields $q \vee \kappa(q) = (q \wedge r) \vee \kappa(q) = \kappa(q)$ and hence $q \le \kappa(q)$, a contradiction.

If (5) holds, then $\kappa(q) \le q \vee r \vee p$, so that $q \vee r \vee p = \kappa(q) \vee q \vee r$, i.e., $r \vee p = r \vee \kappa(q)$; but then $r \vee p = r \vee (p \wedge \kappa(q)) = r \vee q_* = q \vee r$ and we get $p \le q \vee r$.

Finally, if (1) holds, then $\kappa(q) \wedge (p \vee r_*) \le b \wedge (c \vee (a \wedge b)) \le a = q \vee r$, since $r_* \le \kappa(q)$. Let us apply (W) to this inequality. Since

$r_* \leq \kappa(q) \wedge (p \vee r_*)$ but $r_* \nleq q$, we have $\kappa(q) \wedge (p \vee r_*) \nleq q$. Since $q_* \leq \kappa(q) \wedge (p \vee r_*)$ but $q_* \nleq r$, we have $\kappa(q) \wedge (p \vee r_*) \nleq r$. If $\kappa(q) \leq q \vee r$, then $b \leq a$ and the case (5) takes place. If $p \vee r_* \leq q \vee r$, we have $p \leq q \vee r$, as desired.

LEMMA 5.50. *Let $q_0, \ldots, q_k, p_0, p_1$ with $k \geq 1$ be a subsequence of a minimal C–cycle in an S–lattice such that $q_0, \ldots, q_k$ is a B–type sequence and $q_k\ A_2\ p_0\ B\ p_1$. If $q_0 \leq \kappa(p_0)$ and $p_{0*} \not< q_{0*}$, then $\bigvee_{i \leq k} q_i \leq p_0 \vee p_1$.*

Proof: Suppose $\bigvee_{i \leq k} q_i \nleq p_0 \vee p_1$. By Lemma 5.48 we have $p_0 = q_{k*}$.

First let us show that $q_k \leq p_0 \vee p_1$ implies $q_k \leq p_0 \vee p_{1*}$. Assume $q_k \leq p_0 \vee p_1$. Since $p_0 = q_{k*} \leq \kappa(q_k)$, this means that $p_1 \nleq \kappa(q_k)$, whence by the free star principle $p_{1*} \nleq \kappa(q_k)$. Therefore $q_k \leq q_{k*} \vee p_{1*} = p_0 \vee p_{1*}$.

Next we shall prove $p_{1*} \nleq p_{0*}$. Suppose $p_{1*} \leq p_{0*}$. It follows from $p_0 = q_{k*}$ and (B2) that $p_0 = q_k \wedge (p_0 \vee q_{k-1})$. Since $p_0\ B\ p_1$, we have $p_0 \leq p_{0*} \vee p_1$. Let us apply (W) to the inequality

$$p_0 = q_k \wedge (p_0 \vee q_{k-1}) \leq p_{0*} \vee p_1.$$

If $q_k \leq p_{0*} \vee p_1$, then $q_k \leq p_0 \vee p_{1*}$ by what we have proved above, so that $q_k \leq p_0 < q_k$, a contradiction. If $q_{k-1} \leq p_{0*} \vee p_1$, then since $p_{0*} < q_{k*} \leq \kappa(q_{k-1})$ by (B3), we must have $p_1 \nleq \kappa(q_{k-1})$. But then the free star principle yields $p_{1*} \nleq \kappa(q_{k-1})$, contrary to $p_{1*} \leq p_{0*} \leq \kappa(q_{k-1})$. Of course $p_0 \nleq p_{0*}$, and $p_0 \nleq p_1$ because $p_0\ B\ p_1$.

This means that $p_0\ B_1\ p_1$. By Lemma 5.49, $q_k \leq p_0 \vee p_1$. Our first observation above then implies $q_k \leq p_0 \vee p_{1*}$. Put

$$\begin{aligned} a &= q_k, \\ a_0 &= (\bigvee_{i \leq k} q_i) \wedge (p_0 \vee p_1), \\ b &= p_{0*} \vee p_{1*}, \\ b_0 &= \kappa(p_0) \wedge (p_0 \vee p_1). \end{aligned}$$

Clearly, $a \leq a_0$. To see that $a \neq a_0$, suppose otherwise, in which case we may apply (W) to the inequality

$$q_k = (\bigvee_{i \leq k} q_i) \wedge (p_0 \vee p_1) \leq p_0 \vee p_{1*}.$$

We are assuming that $\bigvee_{i \leq k} q_i \nleq p_0 \vee p_1$, and $p_1 \nleq p_0 \vee p_{1*}$ since $p_0\ B\ p_1$ implies $p_0 \vee p_{1*} \leq \kappa(p_1)$ by Lemma 2.58. On the other hand, $q_k \nleq p_0 = q_{k*}$ and $q_k \nleq p_{1*}$, for else $p_0 < q_k < p_1$. Thus the assumption $a = a_0$ leads to a (W)–failure. We conclude that $a < a_0$, and then we can choose an element a_1 such that $a \leq a_1 \prec a_0$. With this choice of the elements a, a_0, a_1, b, b_0 we shall check the seven conditions of Lemma 5.38.

Since $p_0 \; B_1 \; p_1$, the element b is join reducible (see Figure 5.4 on page 145). Since $q_k \le p_0 \vee p_{1*}$, we have $a \le a_* \vee b$. This proves 5.38(1), and 5.38(2) is satisfied by our choice of a_1. Condition 5.38(3) is clear, for by definition $p_0 \; B \; p_1$ implies $p_{1*} \le \kappa(p_0)$. Also 5.38(4) is easy to see: if $a \le b_0$, we would have $p_0 < q_k = a \le b_0 \le \kappa(p_0)$, a contradiction.

To prove 5.38(5), suppose $b \le a_0$. Then $p_{1*} \le \bigvee_{i \le k} q_i$. Using $q_k \le p_0 \vee p_{1*}$, we see that $\bigvee_{i \le k} q_i = \bigvee_{i<k} q_i \vee p_0 \vee p_{1*}$, whence $\bigvee_{i \le k} q_i = \bigvee_{i<k} q_i \vee p_0 \vee (q_k \wedge p_{1*})$ by (SD$_\vee$). However, $q_k \nleq p_{1*}$ since $p_0 < q_k$ and $p_0 \nleq p_{1*}$, whence $q_k \wedge p_{1*} \le q_{k*} = p_0$. Thus $\bigvee_{i \le k} q_i = \bigvee_{i<k} q_i \vee p_0 \le \kappa(q_k)$ using (B2), a contradiction. Hence $b \nleq a_0$.

Let us prove 5.38(6). If $a_0 \le a_1 \vee b_0$, then we may apply (W) to the inequality

$$a_0 = (\bigvee_{i \le k} q_i) \wedge (p_0 \vee p_1) \le a_1 \vee (\kappa(p_0) \wedge (p_0 \vee p_1)) = a_1 \vee b_0.$$

Of course, $a_0 \nleq a_1$, and since $a \le a_0$, we have $a_0 \nleq b_0$ by 5.38(4). Since $a_1 \vee b_0 \le p_0 \vee p_1$, while by assumption $\bigvee_{i \le k} q_i \nleq p_0 \vee p_1$, we have $\bigvee_{i \le k} q_i \nleq a_1 \vee b_0$. The last case is when $p_1 \le a_1 \vee b_0 \le a_0 \vee b_0$. Then, as $p_0 \le a_0$, it is easy to see that $p_0 \vee p_1 = a_0 \vee b_0$, i.e.,

$$p_0 \vee p_1 = (\bigvee_{i \le k} q_i \wedge (p_0 \vee p_1)) \vee (\kappa(p_0) \wedge (p_0 \vee p_1)).$$

Applying (SD$_\vee$) in its more general form, we obtain

$$p_0 \vee p_1 = p_0 \vee (p_1 \wedge \bigvee_{i \le k} q_i) \vee (p_1 \wedge \kappa(p_0)).$$

But $p_1 \nleq \bigvee_{i \le k} q_i$ by the argument used for 5.38(5) and $p_1 \nleq \kappa(p_0)$ by the definition of $p_0 \; B \; p_1$. Therefore $p_0 \vee p_1 \le p_0 \vee p_{1*} \le \kappa(p_1)$, a contradiction.

Now it remains to prove 5.38(7). If $b_0 \le a_1 \vee b$, we may apply (W) to the inequality

$$\kappa(p_0) \wedge (p_0 \vee p_1) \le a_1 \vee p_{1*},$$

since $p_{0*} < a \le a_1$. If $\kappa(p_0) \le a_1 \vee p_{1*}$, then using Lemma 5.47 we have $\bigvee_{i<k} q_i \le \kappa(p_0) \le a_1 \vee p_{1*} \le p_0 \vee p_1$. Since also $q_k \le p_0 \vee p_1$, this implies $\bigvee_{i \le k} q_i \le p_0 \vee p_1$, contrary to assumption. If $p_1 \le a_1 \vee p_{1*} = a_1 \vee b$, then again we would obtain $p_1 \le a_0 \vee b_0$, a possibility eliminated in the argument for 5.38(6). On the other hand, $b_0 \nleq a_1$, since $b \le b_0$ and $a_1 < a_0$, while $b \nleq a_0$ by 5.38(5). Finally, $b_0 \nleq p_{1*}$, because $p_{0*} \le b_0$ while $p_{0*} \nleq p_1$.

The contradiction furnished by Lemma 5.38 finishes the proof that $\bigvee_{i\le k} q_i \le p_0 \vee p_1$.

This brings us to the critical lemma about B–type sequences.

LEMMA 5.51. *Let*

$$r_{10}, \ldots, r_{1k_1}, r_{20}, \ldots, r_{2k_2}, \ldots, r_{n0}, \ldots, r_{nk_n}$$

with $n \ge 1$ *and* $k_i \ge 1$ *for all* i *be a subsequence of a minimal* C*–cycle in an* S*–lattice such that* $r_{i0}\ B\ \ldots\ B\ r_{ik_i}$ *for all* i *and* $r_{ik_i}\ A_2\ r_{i+1,0}$ *for* $i < n$*. If* $r_{10} \le \kappa(r_{i0})$ *and* $r_{i0*} \not< r_{10*}$ *for all* $i > 1$*, then the subsequence*

$$r_{10}, \ldots, r_{1,k_1-1}, r_{20}, \ldots, r_{2,k_2-1}, \ldots, r_{n0}, \ldots, r_{nk_n}$$

is a B*–type sequence.*

Proof: We proceed by induction on n. If $n = 1$, the assertion follows from Lemma 5.41. Let $n > 1$. Denote the subsequence in question by

$$q_0, \ldots, q_{s-1}, p_0, \ldots, p_m$$

where $q_{s-1} = r_{n-1,k_{n-1}-1}$ and $p_0 = r_{n0}$. Put $q_s = r_{n-1,k_{n-1}}$. Then $q_0, \ldots, q_s$ is a B–type sequence by induction and $p_0, \ldots, p_m$ is a B–type sequence by Lemma 5.41.

In order to verify (B1) for the whole subsequence, we need only to show that $q_{s-1*} \not\le p_0$. This follows from $p_0 < q_s$ and $q_{s-1*} \not\le q_s$.

Similarly, for (B2) we must prove that if $i < s$ and j is arbitrary, then $q_i \le \kappa(p_j)$. If $j = 0$, this follows from Lemma 5.47. If $j > 0$, then $p_{j-1}\ B\ p_j$ implies $\kappa(p_{j-1})\ A^d\ \kappa(p_j)$ and so $\kappa(p_{j-1}) \le \kappa(p_j)$; thus we get $q_i \le \kappa(p_0) \le \kappa(p_j)$.

Since $p_{0*} < q_{s*} \le \kappa(q_{s-1})$, the subsequence satisfies also (B3).

To obtain (B4), we must show that if $i < s$ and $\langle i, j\rangle \ne \langle s-1, 0\rangle$, then $p_{j*} \not\le \kappa(q_i)$. By virtue of the free star principle, we need only show that $p_j \not\le \kappa(q_i)$. By Lemma 5.48, $p_0 = q_{s*}$. If $j = 0$ and $i < s-1$ then, as $q_0, \ldots, q_s$ is a B–type sequence, $q_{s*} \not\le \kappa(q_i)$ and so we get $p_0 \not\le \kappa(q_i)$ in that case. Hence we can assume $j \ge 1$. Suppose that $p_j \le \kappa(q_i)$ for some $i < s$, and take the largest number i with this property.

By Lemma 5.50, $q_{s-1} \le p_0 \vee p_1$; since $p_0 = q_{s*} \le \kappa(q_{s-1})$, this implies $p_1 \not\le \kappa(q_{s-1})$. Thus we may assume that either $i < s-1$ or $j > 1$.

We have $q_i = r_{c,d}$ for some c, d with $c < n$ and $d \le k_c - 1$. If $d < k_c - 1$, then $q_{i+1} = r_{c,d+1}$, and $q_i\ B\ q_{i+1}$ implies $\kappa(q_i) \le \kappa(q_{i+1})$ by Lemma 2.63, so that $p_j \le \kappa(q_{i+1})$, contrary to the maximality of i. Hence $q_i = r_{c,k_c-1}$. Put $u = r_{c,k_c}$ and $v = r_{c+1,0}$. Then $q_0, \ldots, q_i, u, v$ is a subsequence of a minimal C–cycle, $q_0, \ldots, q_i, u$ is a B–type sequence by induction and $q_i\ B\ u\ A_2\ v$. By Lemma 5.48, $v = u_*$. Since $q_i\ B\ u$, we have $\kappa(q_i) \leq \kappa(u)$ and so $p_j \leq \kappa(u)$. Consequently, $u \nleq u_* \vee p_j = v \vee p_j$ and we get $u \wedge (v \vee p_j) = v$. On the other hand, $p_{j*} \nleq \kappa(v)$. If $i < s-1$, this follows from the maximality of i and the free star principle. If $i = s-1$, then $j > 1$ and $p_{j*} \nleq \kappa(v) = \kappa(p_0)$ follows from (B4) applied to $p_0, \ldots, p_m$. Thus $v \leq v_* \vee p_{j*}$. Combining these relations, we obtain the inequality

$$v = u \wedge (v \vee p_j) \leq v_* \vee p_{j*}$$

to which we may apply (W). Recall $u \nleq v \vee p_j$ from above, whence u is not below the right hand side. By property (B2), which was proved above, $v_* \leq \kappa(p_j)$, so that $v \vee p_j \nleq v_* \vee p_{j*}$. Of course $v \nleq v_*$. This leaves $v \leq p_{j*}$ as the only remaining possibility, which we now assume.

Let us apply Lemma 5.43 to the sequence $q_0, \ldots, q_i, u, v$ with the element $t = \kappa(v)^* \wedge (q_{i*} \vee p_j)$.

Condition 5.43(1) is clearly satisfied. For 5.43(2), note that $v_* \not< q_{0*}$ by one of our hypotheses, while equality is excluded by Lemma 5.47. Condition 5.43(3), $u_* \leq v_*$, is a consequence of $u\ A\ v$.

Condition 5.43(4) reads $v < \kappa(v)^* \wedge (q_{i*} \vee p_j) \leq \kappa(v)^*$. Because $v < p_j$, we have $v \leq \kappa(v)^* \wedge (q_{i*} \vee p_j)$. Moreover, the inequality is strict because $q_{i*} \leq \kappa(v)^* \wedge (q_{i*} \vee p_j)$ by Lemma 5.47, while $q_{i*} \nleq v = u_*$ by (B1).

Our assumption that $p_j \leq \kappa(q_i)$ gives condition 5.43(5).

If 5.43(6) fails, then we may apply (W) to the inequality

$$t = \kappa(v)^* \wedge (q_{i*} \vee p_j) \leq v \vee q_{0*} \vee \cdots \vee q_{i-1*} \vee q_i.$$

Denote the right hand side by w. Now $\kappa(v)^* \nleq w$, since $u \leq \kappa(v)^*$ while by induction, using (B2), $w \leq \kappa(u)$. Similarly, since $w \leq \kappa(p_j)$ by (B2) (which was proved above) and our assumption that $v \leq p_{j*}$, we have $q_{i*} \vee p_j \nleq w$. On the other hand, recall from 5.43(4) that $v < t$. Thus $t \nleq v$. Moreover, since $v = u_*$ is not below any of the elements $\kappa(q_0), \ldots, \kappa(q_{i-1})$ by (B4) applied to $q_0, \ldots, q_i, u$, we have $t \nleq q_{l*}$ for $0 \leq l \leq i-1$. Finally, suppose $t \leq q_i$, whence $v = u_* \leq q_i$. This makes $q_i\ B_2\ u$. Recall that $q_i = r_{c,k_c-1}$. If $k_c > 1$, then $r_{c,k_c-2}\ B\ q_i\ B_2\ u$ with $u_* < q_i$, so that $u_* \leq q_{i*} \leq \kappa(r_{c,k_c-2})$, contrary to (B4). Therefore $q_i = r_{c0}$. Then $c \neq 1$, for if $r_{10}\ B_2\ r_{11}\ A_2\ r_{20}$, we would have $r_{20*} < r_{11*} < r_{10*}$, contrary to one of our original hypotheses. However, $c > 1$ implies $r_{c0}\ B_1\ r_{c1}$, because we can use (B3) and (B4) inductively (as $c < n$) to obtain $r_{c0*} \leq \kappa(r_{c-1,k_{c-1}-1})$ and $r_{c1*} \nleq \kappa(r_{c-1,k_{c-1}-1})$, whence $r_{c1*} \nleq r_{c0*}$. Thus $t \nleq q_i$ and 5.43(6) holds.

For 5.43(7), $q_0 \leq \kappa(v)$ implies that $v\ B\ q_0$ does not hold.

Thus Lemma 5.43 applies to yield $q_{0*} \vee \cdots \vee q_{i*} \nleq \kappa(v)$. This, however, contradicts property (B2), which we have already proved.

As mentioned earlier, there is one exceptional type of cycle not covered by our general argument. We might as well dispose of this case first. The following lemma does most of the work.

Lemma 5.52. *Let $r_0, \ldots, r_5$ be a subsequence of a minimal C–cycle in an S–lattice such that $r_0\ A_2\ r_1\ B_1\ r_2\ A_2\ r_3\ B_1\ r_4\ A_2\ r_5$. Then*
(1) $r_0 \leq r_2 \vee r_4$, *and*
(2) $r_4 \nleq r_0 \vee r_2$.

Proof: By Lemma 5.49, $r_0 \leq r_1 \vee r_2$ and $r_2 \leq r_3 \vee r_4$. In particular $r_4 \nleq \kappa(r_2)$, since $r_3 \leq r_{2*} \leq \kappa(r_2)$. By the free star principle, $r_{4*} \nleq \kappa(r_2)$. Now $r_2 \neq r_{4*}$, since $r_3 \leq r_2$ and $r_3 \nleq r_{4*}$. Thus if $r_{4**} \leq \kappa(r_2)$, we would have $r_2\ B\ r_{4*}$. If $r_{4*} = r_5$, this immediately contradicts the minimality of the cycle; otherwise $r_{4*} > r_5$, in which case $r_{4*}\ A\ r_5$ (since $r_5 < r_{4*} < r_4 \leq \kappa(r_5)^*$), again shortening the cycle. Therefore $r_{4**} \nleq \kappa(r_2)$, i.e., $r_2 \leq r_{2*} \vee r_{4**}$. It follows in particular from this last statement that $r_{4**} \nleq r_{3*}$, since $r_{3*} \leq r_{2*}$.

We can now prove a strong version of (2). Note that $r_1 < r_0 \leq r_1 \vee r_2$ implies $r_0 \vee r_2 = r_1 \vee r_2$. Suppose $r_{4**} \leq r_0 \vee r_2$. Then we would have $r_1 \vee r_2 = r_1 \vee r_{2*} \vee r_{4**}$, whence $r_1 \vee r_2 = r_1 \vee r_{2*} \vee (r_2 \wedge r_{4**})$ by ($\mathrm{SD}_\vee$). Since $r_3 \leq r_2$ and $r_3 \nleq r_4$, we have $r_2 \nleq r_{4**}$, so that $r_2 \wedge r_{4**} \leq r_{2*}$. Thus $r_1 \vee r_2 = r_1 \vee r_{2*} \leq \kappa(r_2)$, a contradiction. Therefore $r_{4**} \nleq r_0 \vee r_2$.

Now suppose that (1) fails, i.e., $r_0 \nleq r_2 \vee r_4$. Then $(r_0 \vee r_2) \wedge (r_2 \vee r_4)$ is meet reducible, and hence join irreducible. Let us apply Lemma 5.38 with

$$
\begin{aligned}
a &= r_2, \\
a_0 &= (r_0 \vee r_2) \wedge (r_2 \vee r_4), \\
a_1 &= a_{0*}, \\
b &= r_{3*} \vee r_{4**}, \\
b_0 &= r_{3*} \vee r_{4*}.
\end{aligned}
$$

We must check the seven conditions 5.38(1)–5.38(7).

(1) $a = r_2 \leq r_{2*} \vee r_{4**} = a_* \vee b$ by the above remarks. Also $r_{3*} \nleq r_{4**}$ since $r_3\ B\ r_4$ while $r_{4**} \nleq r_{3*}$ was shown above, so b is join reducible.

(2) Clearly $a \leq a_0$; we need to show that $a < a_0$, i.e., $(r_0 \vee r_2) \wedge (r_2 \vee r_4) \neq r_2$. Otherwise, we could apply (W) to the inequality $r_2 = (r_0 \vee r_2) \wedge (r_2 \vee r_4) \leq r_{2*} \vee r_{4*}$. Now $r_0 \nleq r_2 \vee r_4$ by assumption, so the first meetand is not below the right hand side. On the other hand $r_2 \nleq r_{2*}$ and $r_2 \nleq r_{4*}$ (since $r_3 < r_2$ and $r_3\ B\ r_4$), which means that the second meetand must be below the right hand side, i.e., $r_4 \leq r_{2*} \vee r_{4*}$. But then, using $r_3 \leq r_{2*} < r_2 \leq r_3 \vee r_4$, we have $r_3 \vee r_4 = r_{2*} \vee r_{4*}$. Applying ($\mathrm{SD}_\vee$) we obtain $r_3 \vee r_4 = r_3 \vee r_{4*} \vee (r_{2*} \wedge r_4)$. However, $r_4 \nleq r_{2*}$

since $r_{4**} \nleq \kappa(r_2)$, so $r_{2*} \wedge r_4 \leq r_{4*}$. Thus $r_3 \vee r_4 = r_3 \vee r_{4*} \leq \kappa(r_4)$, a contradiction. Therefore $a < a_0$, whence $a \leq a_{0*} \prec a_0$, as desired.

(3) $b \leq b_0$ is clear.

(4) We have $r_3 < r_2 = a$, while $b_0 = r_{3*} \vee r_{4*} \leq \kappa(r_3)$ since $r_3 \; B \; r_4$. Hence $a \nleq b_0$.

(5) $b \nleq a_0$ follows from $r_{4**} \nleq r_0 \vee r_2$.

(6) If $a_0 \leq a_1 \vee b_0$, then we may apply (W) to the inequality

$$a_0 = (r_0 \vee r_2) \wedge (r_2 \vee r_4) \leq a_1 \vee r_{4*},$$

where we have used $r_{3*} < r_2 = a \leq a_1$. Now $r_0 \nleq a_1 \vee r_{4*}$ since $a_1 \vee r_{4*} \leq r_2 \vee r_4$. On the other hand $a_0 \nleq a_1$, and $a_0 \nleq r_{4*}$ since $r_2 \nleq r_{4*}$. Therefore we must have $r_4 \leq a_0 \vee r_{4*}$. But then, arguing as above, $r_3 \vee r_4 = a_0 \vee r_{4*}$, whence by $(\mathrm{SD}_\vee)$ $r_3 \vee r_4 = r_3 \vee r_{4*} \vee (r_4 \wedge a_0) = r_3 \vee r_{4*} \leq \kappa(r_4)$ because $r_4 \nleq a_0$, due to $a_0 \leq r_0 \vee r_2$. This is a contradiction, whereupon we conclude $a_0 \nleq a_1 \vee b_0$.

(7) If $b_0 \leq a_1 \vee b$, then $r_{4*} \leq a_1 \vee r_{4**}$ (where again we used $r_{3*} \leq a_1$). Now the last argument in (6) shows that $r_4 \nleq a_0 \vee r_{4*}$. Hence $r_4 \wedge (a_0 \vee r_{4*}) = r_{4*}$, so we may apply (W) to the inequality

$$r_{4*} = r_4 \wedge (a_0 \vee r_{4*}) \leq a_1 \vee r_{4**}.$$

Of course $r_4 \nleq a_1 \vee r_{4**}$ since $r_4 \nleq a_0 \vee r_{4*}$, while $a_0 \nleq a_1 \vee r_{4**}$ by (6). On the other hand, $r_{4*} \nleq a_1$ because $r_{4**} \nleq r_0 \vee r_2$, and obviously $r_{4*} \nleq r_{4**}$. Thus the assumption $b_0 \leq a_1 \vee b$ leads to a (W)–failure, whence $b_0 \nleq a_1 \vee b$.

By Lemma 5.38, this configuration cannot exist in an S–lattice. Therefore $r_0 \leq r_2 \vee r_4$.

Now we can easily eliminate minimal cycles which alternate B_1's and A_2's.

LEMMA 5.53. *An S–lattice cannot contain a minimal C–cycle of the form*

$$q_1 \; B_1 \; p_1 \; A_2 \; q_2 \; B_1 \; p_2 \; A_2 \ldots q_n \; B_1 \; p_n \; A_2 \; q_1.$$

Proof: Suppose we have such a cycle. Clearly, $n > 1$. Suppose $n = 2$. By Lemma 5.49, $p_1 \leq q_2 \vee p_2$ and $p_2 \leq q_1 \vee p_1$. Hence $p_1 \vee p_2 = q_1 \vee p_1 = q_2 \vee p_2$. By $(\mathrm{SD}_\vee)$ we get $p_1 \vee p_2 = q_1 \vee q_2 \vee (p_1 \wedge p_2)$. Now $p_1 \nleq p_2$ since $q_2 \leq p_1$ and $q_2 \nleq p_2$, so $p_1 \wedge p_2 \leq p_{1*}$, wherefore $p_1 \vee p_2 \leq q_1 \vee p_{1*} \leq \kappa(p_1)$, a contradiction. Hence $n > 2$.

By repeated application of Lemma 5.52(1),

$$p_1 \leq p_2 \vee p_3 \leq (p_3 \vee p_4) \vee p_3 = p_3 \vee p_4 \leq \cdots \leq p_{n-1} \vee p_n.$$

However, by Lemma 5.52(2), $p_1 \nleq p_{n-1} \vee p_n$.

With this exceptional case out of the way, we can proceed with the main argument.

LEMMA 5.54. *An S–lattice contains no C–cycle.*

Proof: A C–cycle cannot contain only A's, since $p\ A\ q$ implies $p > q$. Also, a C–cycle cannot contain only B's, as then by Lemma 2.63 the dual lattice would contain a cycle of A's. So, a minimal C–cycle must be, after choosing an appropriate starting element, of the form

$$\begin{aligned} q_{10}\ B^{k_1}\ q_{1k_1} = p_{10}\ A^{m_1}\ p_{1m_1} = q_{20}\ B^{k_2}\ q_{2k_2} = p_{20}\ A^{m_2}\ p_{2m_2} = \dots \\ = q_{n0}\ B^{k_n}\ q_{nk_n} = p_{n0}\ A^{m_n}\ p_{nm_n} = q_{10} \end{aligned}$$

with $n \geq 1$ and $k_i, m_i \geq 1$ for all i, where the notation $q_{10}\ B^{k_1}\ q_{1k_1}$ means that $q_{10}\ B\ q_{11}\ B\ \dots\ B\ q_{1k_1}$, etc. Moreover, a proper choice of q_{10} can guarantee that

(1) if the cycle contains an A_1, then $p_{n0}\ A_1\ p_{n1}$ (take Lemma 5.36 into account);
(2) if the cycle contains no A_1 but contains a consecutive pair of A_2's, then $p_{n0}\ A_2\ p_{n1}\ A_2\ p_{n2}$;
(3) if the cycle contains no A_1 and no consecutive pair of A_2's but contains a B_2, then $q_{n0}\ B_2\ q_{n1}$ (use Lemma 5.37).

Denote by i the largest number with $q_{i0**} \leq q_{10**}$. Let us prove by induction *for any $j = i, i+1, \dots, n$ that*

$$q_{i0}, \dots, q_{i,k_i-1}, q_{i+1,0}, \dots, q_{i+1,k_{i+i}-1}, \dots, q_{j0}, \dots, q_{jk_j}$$

is a B–type sequence, $m_j = 1$, $p_{j0}\ A_2\ p_{j1}$, and if $j < n$ then $q_{i0} \leq \kappa(p_{j1})$.

The induction assumption (saying that these statements are true for all $j' = i, \dots, j-1$) together with the maximality of i (according to which $q_{j'0**} \nleq q_{10**}$ for $j' > i$) ensure that the assumptions of Lemma 5.51, applied to the segment of the minimal C–cycle starting at q_{i0} and ending at q_{jk_j}, are satisfied. By that lemma, our subsequence is a B–type sequence. If $p_{j0}\ A_1\ p_{j1}$, we can apply Lemma 5.44 to this subsequence with $q_{jk_j}\ A_1\ p_{j1}$, and obtain $p_{j1*} < q_{i0*}$. Then $p_{jm_j**} \leq p_{j1**} < q_{i0**}$ and we get a contradiction with the maximality of i, since p_{jm_j} is either $q_{i+1,0}$ or q_{10}. Therefore $p_{j0}\ A_2\ p_{j1}$. If $p_{j0}\ A_2\ p_{j1}\ A_2\ p_{j2}$, we can similarly apply Lemma 5.45 to obtain a contradiction. By Lemma 5.36, we cannot have $p_{j0}\ A_2\ p_{j1}\ A_1\ p_{j2}$, so $m_j = 1$. If $j < n$ and $q_{i0} \nleq \kappa(p_{j1})$, we apply Lemma 5.46 in the same way to obtain $p_{j1**} < q_{i0*}$, i.e., $q_{j+1,0**} \leq q_{i0**}$, a contradiction again with the maximality of i.

In particular, $m_n = 1$ and $p_{n0}\ A_2\ p_{n1}$. By conditions (1) and (2), the minimal cycle contains no A_1 and $m_1 = \dots = m_n = 1$. Since these considerations also apply to the dual cycle, Lemma 2.63 gives us $k_1 = \dots = k_n = 1$. Thus we may simplify notation by relabeling the minimal C–cycle as

$$q_1\ B\ p_1\ A_2\ q_2\ B\ p_2\ A_2 \dots q_n\ B\ p_n\ A_2\ q_1.$$

Now suppose that one of the B's is B_2. By (3), $q_n\ B_2\ p_n$. The number i was chosen to be the largest with respect to $q_{i**} \le q_{1**}$. We have $i < n$, since otherwise $q_{n**} \le q_{1**} < p_{n**}$, which is not possible when $q_n\ B_2\ p_n$. Above we have proved that $q_i, q_{i+1}, \dots, q_{n-1}, p_{n-1}$ is a B–type sequence. If $q_i \nleq \kappa(q_n)$, we could apply Lemma 5.46 to the sequence $q_i, q_{i+1}, \dots, q_{n-1}, p_{n-1}, q_n$ and obtain $q_{n**} \le q_{i**}$, a contradiction with the maximality of i. Hence $q_i \le \kappa(q_n)$, and then Lemma 5.50 can be applied to $q_i, q_{i+1}, \dots, q_{n-1}, p_{n-1}, q_n, p_n$, yielding $p_{n-1} \le q_n \vee p_n$. But we can also show $p_{n-1} \nleq q_n \vee p_n$. For $q_n \le p_{n-1*} \le \kappa(p_{n-1})$, whence also $p_{n*} < q_{n*} \le \kappa(p_{n-1})$. By the free star principle, $p_n \le \kappa(p_{n-1})$ and so $q_n \vee p_n \le \kappa(p_{n-1})$.

Therefore the minimal C–cycle contains no B_2's and can be written as

$$q_1\ B_1\ p_1\ A_2\ q_2\ B_1\ p_2\ A_2 \dots q_n\ B_1\ p_n\ A_2\ q_1.$$

However, this is exactly the kind of cycle eliminated by Lemma 5.53, and the proof is complete.

Combining Lemmas 5.35 and 5.54, we obtain the desired result.

Theorem 5.55. *A finite lattice is a sublattice of a free lattice if and only if it is semidistributive and satisfies* (W)*.*

4. Related Topics

Based on Theorem 5.55, we can also characterize finite sublattices of free lattices in terms of forbidden sublattices. We begin with an important result due to B. Davey, W. Poguntke and I. Rival [**21**].

Theorem 5.56. *A finite lattice is semidistributive if and only if it contains no sublattice isomorphic to one of the six lattices* $\mathbf{L}_1$*,* $\mathbf{L}_2$*,* $\mathbf{L}_3$*,* $\mathbf{L}_4$*,* $\mathbf{L}_5$ *or* $\mathbf{L}_6$ *of* Figure 5.8*.*

Proof: It is easily verified that $\mathbf{L}_1$–$\mathbf{L}_4$ fail both semidistributive laws, while $\mathbf{L}_5$ fails ($\mathrm{SD}_\vee$) and $\mathbf{L}_6$ fails ($\mathrm{SD}_\wedge$). Conversely, let $\mathbf{L}$ be a finite lattice failing ($\mathrm{SD}_\vee$), and let us show that one of $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \mathbf{L}_4$ or $\mathbf{L}_5$ is a sublattice of $\mathbf{L}$. Clearly, we may assume that every proper subinterval of $\mathbf{L}$ satisfies ($\mathrm{SD}_\vee$).

Since $\mathbf{L}$ fails ($\mathrm{SD}_\vee$), there exists a triple $a, b, c \in L$ such that $a \vee b = a \vee c > a \vee (b \wedge c)$. Moreover, as $\mathbf{L}$ is finite, we can assume that a, b, c is a minimal triple with this property, in the sense that if a', b', c' is a failure with $\{a', b', c'\} \ll \{a, b, c\}$, then $\{a, b, c\} = \{a', b', c'\}$. As every proper subinterval of $\mathbf{L}$ satisfies ($\mathrm{SD}_\vee$), we have $a \vee b = a \vee c = 1$.

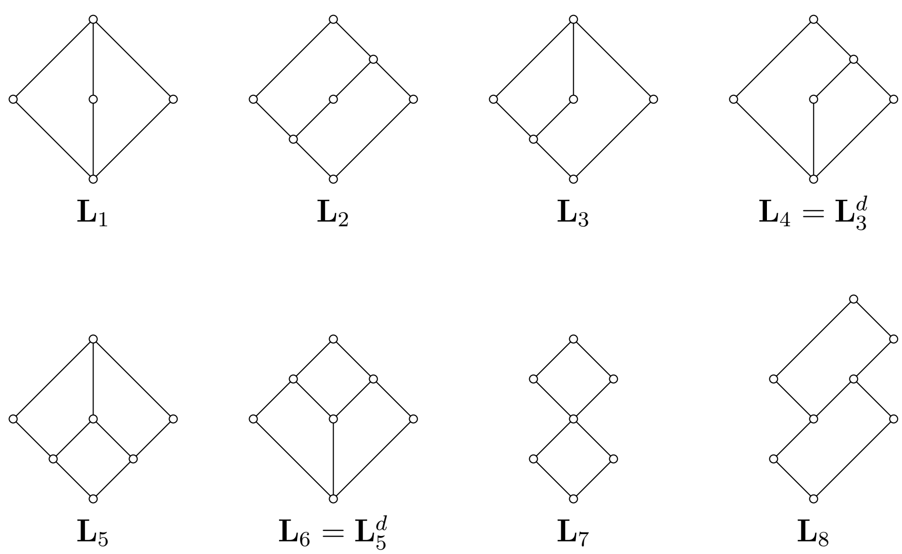


FIGURE 5.8

Likewise $b \wedge c = 0$, for otherwise the failure $a \vee (b \wedge c), b, c$ occurs in a proper subinterval.

Claim 1. *If* $0 < s < b$, *then* $b \vee c = s \vee c$. For assume that $b \vee c > s \vee c$, i.e., $b \nleq s \vee c$. If $a \vee (b \wedge (s \vee c)) < 1$, then the triple $a \vee (b \wedge (s \vee c)), b, s \vee c$ violates $(\mathrm{SD}_\vee)$ in the proper subinterval $1/s$ of $\mathbf{L}$. Thus $a \vee (b \wedge (s \vee c)) = 1$. But now the triple $a, b \wedge (s \vee c), c$ contradicts the minimality of a, b, c.

Claim 2. *If* $b < s < b \vee c$, *then* $s \wedge c = 0$. Indeed, if $s \wedge c > 0$, then $b \vee c = b \vee (s \wedge c)$ by the symmetric form of Claim 1, while in fact $b \vee (s \wedge c) \leq s < b \vee c$.

Consider first the case when $b \vee c = 1$. If $a \wedge b > 0$ and $a \wedge c > 0$, then we can apply Claim 1 to see that $\{1, b, c, (a \wedge b) \vee (a \wedge c), a \wedge b, a \wedge c, 0\}$ is a sublattice of $\mathbf{L}$ isomorphic to $\mathbf{L}_5$. If $a \wedge b > 0$ but $a \wedge c = 0$, then similarly $\{1, a, b, c, a \wedge b, 0\}$ is a sublattice isomorphic to $\mathbf{L}_3$. By symmetry we conclude that $a \wedge b = a \wedge c = 0$, so that $\{1, a, b, c, 0\}$ is a sublattice isomorphic to $\mathbf{L}_1$.

Next consider the case when $b \vee c < 1$. Then $a \wedge (b \vee c) > 0$, or else $\{1, a, b, c, b \vee c, 0\}$ would be a sublattice isomorphic to $\mathbf{L}_4$. Also, if $a \wedge (b \vee c) = a \wedge b$, then $\{1, b \vee c, a, b, c, a \wedge b, 0\}$ would be a sublattice isomorphic to $\mathbf{L}_2$, since then $a \wedge c = a \wedge (b \vee c) \wedge c = a \wedge b \wedge c = 0$ and, by Claim 1, $b \vee c = (a \wedge b) \vee c$. Hence, by symmetry, $a \wedge (b \vee c) > a \wedge b$ and $a \wedge (b \vee c) > a \wedge c$. If $b \vee (a \wedge (b \vee c)) = b \vee c = c \vee (a \wedge (b \vee c))$, then the triple $a \wedge (b \vee c), b, c$ would violate the minimality of a, b, c. If $b \vee (a \wedge (b \vee c)) < b \vee c = c \vee (a \wedge (b \vee c))$, then $\{1, b \vee c, a, b \vee (a \wedge (b \vee c)), c, a \wedge (b \vee c), 0\}$ would be a sublattice isomorphic to $\mathbf{L}_2$, since then $c \wedge (b \vee (a \wedge (b \vee c))) = 0$ by Claim 2 and hence $a \wedge c = a \wedge (b \vee c) \wedge c \leq (b \vee (a \wedge (b \vee c))) \wedge c = 0$. Therefore, by symmetry, $b \vee (a \wedge (b \vee c)) < b \vee c$ and $c \vee (a \wedge (b \vee c)) < b \vee c$. Let $e = b \vee (a \wedge (b \vee c))$ and $f = c \vee (a \wedge (b \vee c))$. Since the proper subinterval $1/a \wedge (b \vee c)$ of $\mathbf{L}$ satisfies $(\mathrm{SD}_\vee)$ and $a \vee e = a \vee f = 1$, it follows that $a \vee (e \wedge f) = 1$ and hence $a \wedge (b \vee c) < e \wedge f$. By Claim 2, $e \wedge c = 0$ and thus $e \wedge f \wedge c = 0$ and $a \wedge c = a \wedge (b \vee c) \wedge c \leq e \wedge f \wedge c = 0$. We see that $\{1, a, f, e \wedge f, c, a \wedge (b \vee c), 0\}$ is a sublattice isomorphic to $\mathbf{L}_2$.

Thus a finite lattice failing $(\mathrm{SD}_\vee)$ contains one of $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_3$, $\mathbf{L}_4$ or $\mathbf{L}_5$. Dually, a finite lattice failing $(\mathrm{SD}_\wedge)$ contains one of $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_3$, $\mathbf{L}_4$ or $\mathbf{L}_6$.

This argument was extended in Jónsson and Rival [**84**] to give a version which includes infinite lattices.

THEOREM 5.57. *A lattice* $\mathbf{L}$ *is semidistributive if and only if the variety* $\boldsymbol{V}(\mathbf{L})$ *contains none of the lattices* $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_3$, $\mathbf{L}_4$, $\mathbf{L}_5$ *or* $\mathbf{L}_6$ *of* Figure 5.8.

Both the preceding results have analogues for $(\mathrm{SD}_\vee)$ and $(\mathrm{SD}_\wedge)$ separately. This is implicit in the proof of Theorem 5.56, and the proof of Theorem 5.57 given in [**84**] requires only a minor modification found in Nation [**106**].

Now, following Antonius and Rival [**6**], we add Whitman's condition.

THEOREM 5.58. *A finite lattice is a sublattice of a free lattice if and only if it contains no sublattice isomorphic to one of the eight lattices* $\mathbf{L}_1, \ldots, \mathbf{L}_8$ *of* Figure 5.8.

Proof: With respect to Theorem 5.55 and Theorem 5.56, it is sufficient to show that if **L** is a finite semidistributive lattice not satisfying (W), then **L** contains a sublattice isomorphic to either $\mathbf{L}_7$ or $\mathbf{L}_8$. If $\mathbf{L}_7$ is not a sublattice, then **L** contains no doubly reducible elements. Suppose that this is the case and let $a, b, c, d \in L$ be such that $a \wedge b \leq c \vee d$ but $a \nleq c \vee d$, $b \nleq c \vee d$, $a \wedge b \nleq c$ and $a \wedge b \nleq d$. Clearly, a and b may be so chosen that $a \wedge (c \vee d) \prec a$ and $b \wedge (c \vee d) \prec b$. If $a \vee c \vee d$ and $b \vee c \vee d$ both cover $c \vee d$, then $a \vee c \vee d = b \vee c \vee d$, since $c \vee d$ is meet irreducible. This, however, violates $(\mathrm{SD}_\vee)$. Hence we may assume that there exists an element $e \in L$ such that $c \vee d < e < a \vee c \vee d$; since $a \wedge (c \vee d) \prec a$, we have $a \wedge (c \vee d) = a \wedge e$.

We can now choose $c', d' \in L$ such that $c \leq c' \prec c' \vee (a \wedge e)$ and $d \leq d' \prec d' \vee (a \wedge e)$. As above, if $a \wedge e$ covers both $c' \wedge a \wedge e$ and $d' \wedge a \wedge e$, then $c' \wedge a \wedge e = d' \wedge a \wedge e$. Therefore, by $(\mathrm{SD}_\wedge)$, we may assume that there exists an element $f \in L$ such that $c' \wedge a \wedge e < f < a \wedge e$, and we have $c' \vee f = c' \vee (a \wedge e)$. Now one can see that the elements $c' \wedge f, c', f, a \wedge e, c' \vee f, a, (a \vee c' \vee f) \wedge e, a \vee c' \vee f$ form a sublattice isomorphic to $\mathbf{L}_8$.

The following result can be found in Jónsson and Kiefer [**82**] (cf. Jónsson [**79**], where part of the argument is credited to a private communication from Dilworth).

THEOREM 5.59. *A meet semidistributive lattice without infinite chains is finite*; *if it is of length n, then it has at most 2^n elements.*

Proof: Suppose that **L** is an infinite meet semidistributive lattice without infinite chains. Since **L** satisfies the minimal condition, there exists a minimal element $a \in L$ with the property that the interval $a/0$ is infinite. Since **L** satisfies also the maximal condition, there exists a maximal element b with the property that $b \leq a$ and the interval a/b is infinite. Then a/b is an infinite interval, every proper subinterval of which is finite.

Therefore we can assume that any proper subinterval of **L** is finite. Take an atom c of **L** and take a maximal element d with the property $c \wedge d = 0$. Since **L** is meet semidistributive, d is the unique largest element with the property $c \nleq d$, i.e., $d = \kappa_{\mathbf{L}}(c)$. The lattice **L** is the disjoint union of $1/c$ and $d/0$. However, both these intervals are finite and so **L** is finite.

The second assertion will be proved by induction on n. The case $n = 1$ being trivial, let **L** be a finite meet semidistributive lattice of length $n \geq 2$. Take an atom c. As above, we can find an element d such that **L** is the disjoint union of $1/c$ and $d/0$. The lattices $1/c$ and $d/0$

are both of length at most $n-1$, so by induction they are of cardinality $\leq 2^{n-1}$ each. Then **L** has at most $2 \cdot 2^{n-1} = 2^n$ elements.

EXAMPLE 5.60. For any positive integer n denote by f_n the largest possible cardinality of a lattice of length n which is embeddable in a free lattice. By Theorem 5.59 we have $f_n \leq 2^n$ for any n. The following construction will enable us to find also a lower bound for the number f_n.

The *parallel sum* of two given lattices **L**, **K** is defined to be the lattice which is the disjoint union $L \cup K \cup \{0, 1\}$ and in which the lattices **L**, **K** are two disjoint intervals with no relations between elements of L and elements of K. It is easy to verify that the parallel sum of two finite sublattices of a free lattice is again a finite sublattice of a free lattice (using Theorem 5.55).

If **L** is of length n and cardinality f_n, then the parallel sum of **L** with itself is of length $n+2$ and cardinality $2f_n + 2$. This gives us $f_{n+2} \geq 2f_n + 2$ for any n. Moreover, $f_1 = 2$ and $f_2 = 4$. By induction it is now possible to prove $f_n \geq \sqrt{2^n}$ and we conclude that

$$\sqrt{2^n} \leq f_n \leq 2^n$$

for all n.

5. Finite Subdirectly Irreducible Sublattices of Free Lattices

By Jónsson's Lemma, for any finite subdirectly irreducible lattice **L**, the varieties $\mathcal{V}$ with $\mathbf{L} \notin \mathcal{V}$ form an ideal in the lattice of lattice varieties. If in addition **L** is bounded, then it is a splitting lattice (Corollary 2.76), with a conjugate variety $\mathcal{C}_{\mathbf{L}}$. Note that $\mathcal{C}_{\mathbf{L}}$ is the class of all lattices **K** such that $\mathbf{L} \notin \boldsymbol{HS}(\mathbf{K})$.

If **L** is finite, subdirectly irreducible, bounded and satisfies (W), then it is projective, and we can go one better. In that case, $\mathcal{C}_{\mathbf{L}}$ is the class of all lattices **K** such that **L** cannot be embedded into **K**. We can formalize this as follows.

THEOREM 5.61. *The following three conditions are equivalent for a lattice* **L**.

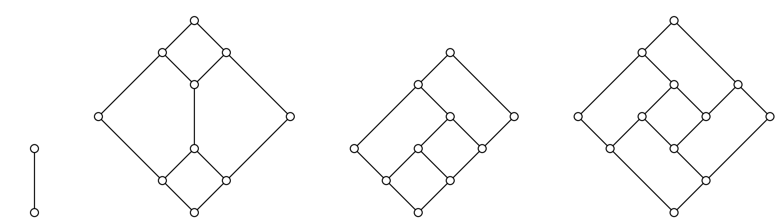
(1) **L** *is a finite, subdirectly irreducible sublattice of a free lattice.*
(2) **L** *is a projective and subdirectly irreducible lattice.*
(3) *The class of lattices containing no sublattice isomorphic to* **L** *is a variety.*

Proof: We know that finite sublattices of a free lattice are projective by Corollary 5.11, so (1) implies (2). If **L** satisfies (2), then by Theorem 5.1 the class $\mathcal{C}_{\mathbf{L}}$ of lattices containing no sublattice isomorphic to **L** is closed under homomorphic images. Now $\mathcal{C}_{\mathbf{L}}$ is clearly closed under sublattices, and the fact that $\mathcal{C}_{\mathbf{L}}$ is closed under direct products follows easily from $\mathbf{L}$ being subdirectly irreducible. Thus (2) implies (3).

So assume that $\mathcal{C}_{\mathbf{L}}$, defined as above, is a variety. Then $\mathcal{C}_{\mathbf{L}}$ is not the variety of all lattices, since $\mathbf{L}$ does not belong to $\mathcal{C}_{\mathbf{L}}$. By Corollary 2.78, the variety of all lattices is generated by the class of finite lattices, and hence there must exist a finite lattice not belonging to $\mathcal{C}_{\mathbf{L}}$. Consequently, this finite lattice contains a copy of $\mathbf{L}$, which forces $\mathbf{L}$ to be finite. Similarly, the variety of all lattices is generated by free lattices, so that a free lattice does not belong to $\mathcal{C}_{\mathbf{L}}$ and $\mathbf{L}$ is a sublattice of a free lattice. Since any lattice of smaller cardinality than $\mathbf{L}$ belongs to $\mathcal{C}_{\mathbf{L}}$, the lattice $\mathbf{L}$ cannot be a subdirect product of smaller lattices, and thus it must be subdirectly irreducible. Hence (3) implies (1).

Lattices which satisfy the equivalent conditions of Theorem 5.61 are called *primitive.* The aim of this section is to present a large collection of examples of primitive lattices.

EXAMPLE 5.62. The four lattices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$ pictured in Figure 5.9 are primitive.



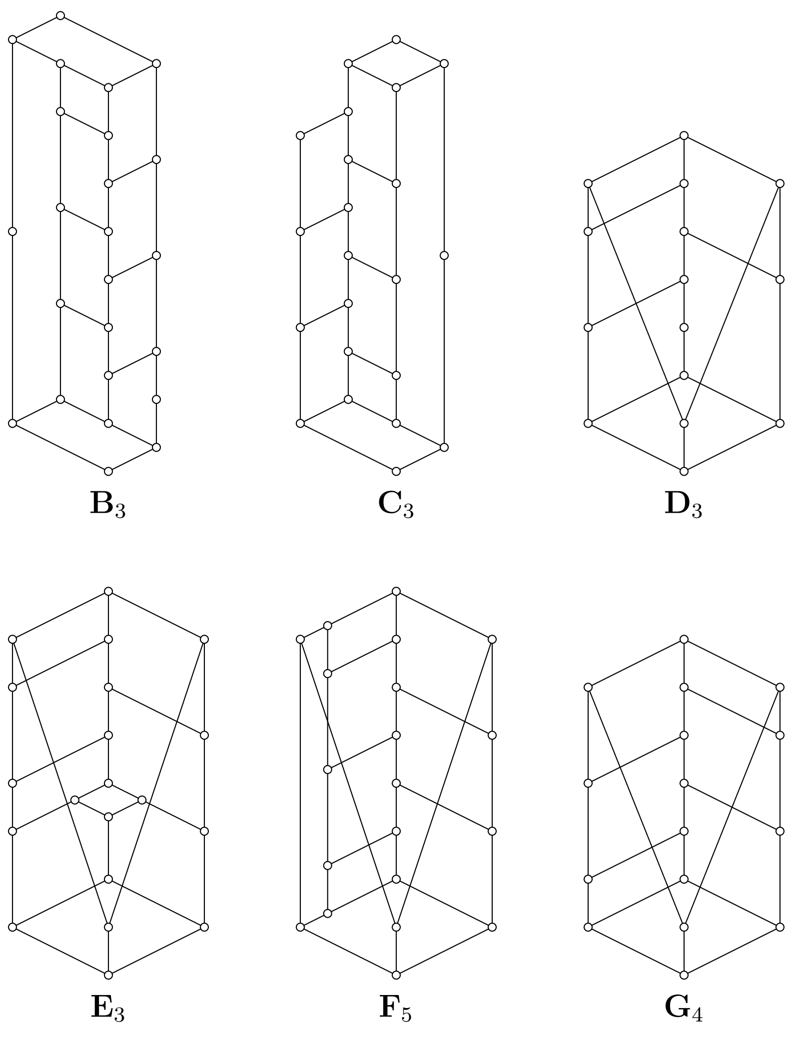
$\mathbf{A}_1$ $\mathbf{A}_2$ $\mathbf{A}_3$ $\mathbf{A}_4$

FIGURE 5.9

EXAMPLE 5.63. The six lattices $\mathbf{B}_3, \mathbf{C}_3, \mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_5, \mathbf{G}_4$ pictured in Figure 5.10 are primitive.

EXAMPLE 5.64. In fact, each of the six lattices from the preceding example is a typical representative of an infinite collection of primitive lattices. We have primitive lattices $\mathbf{B}_n$ $(n \geq 1)$, $\mathbf{C}_n$ $(n \geq 1)$, $\mathbf{D}_n$ $(n \geq 0)$, $\mathbf{E}_n$ $(n \geq 0)$, $\mathbf{F}_n$ $(n \geq 2)$ and $\mathbf{G}_n$ $(n \geq 2)$ with cardinalities given as follows:

$$|B_n| = 12 + 4n, \qquad |C_n| = 10 + 4n, \qquad |D_n| = 9 + 2n,$$
$$|E_n| = 14 + 2n, \qquad |F_n| = 10 + 2n, \qquad |G_n| = 8 + 2n.$$



$\mathbf{B}_3$ $\mathbf{C}_3$ $\mathbf{D}_3$

$\mathbf{E}_3$ $\mathbf{F}_5$ $\mathbf{G}_4$

FIGURE 5.10

We hope that this information, together with that each member of any of these six infinite collections is to be constructed similarly as its pictured representative, is sufficient for the reader to be able to imagine the precise definition which can be found in Ježek and Slavík [**76**].

EXAMPLE 5.65. There are also five general constructions, by means of which it is possible to obtain a larger primitive lattice from a given one.

For a finite lattice $\mathbf{L}$ denote by $\mathbf{R}(\mathbf{L})$ the parallel sum of $\mathbf{L}$ with the one-element lattice. This new lattice $\mathbf{R}(\mathbf{L})$ is primitive whenever $\mathbf{L}$ is.

An element a of a finite lattice $\mathbf{L}$ is said to be perfect if $0_{\mathbf{L}} < a < 1_{\mathbf{L}}$, $L - (a/0_{\mathbf{L}})$ is an interval and $a \leq x \vee y < 1_{\mathbf{L}}$ implies that either $a \leq x$ or $a \leq y$. Let $\mathbf{P}(\mathbf{L}, a)$ be the lattice such that $P(\mathbf{L}, a) = L \cup \{i, c\}$ for two elements i and c, $\mathbf{L}$ is an ideal of $\mathbf{P}(\mathbf{L}, a)$, i is the new largest element and $1_{\mathbf{L}} \wedge c = a$. If $\mathbf{L}$ is a primitive lattice and a is a perfect element which is not a coatom of $\mathbf{L}$ (that is, a is not covered by $1_{\mathbf{L}}$), then $\mathbf{P}(\mathbf{L}, a)$ is a primitive lattice.

Dually one can introduce coperfect elements and define $\mathbf{P}^*(\mathbf{L}, a)$. If $\mathbf{L}$ is a primitive lattice and a is a coperfect element which is not an atom of $\mathbf{L}$, then $\mathbf{P}^*(\mathbf{L}, a)$ is a primitive lattice.

When $\mathbf{P}(\mathbf{L}, a)$ has been constructed as before with universe $L \cup \{i, c\}$, we define $\mathbf{Q}(\mathbf{L}, a) = \mathbf{P}^*(\mathbf{P}(\mathbf{L}, a), c)$. If $\mathbf{L}$ is a primitive lattice and a is a perfect coatom of $\mathbf{L}$, then $\mathbf{Q}(\mathbf{L}, a)$ is a primitive lattice.

Finally, if $\mathbf{L}$ is primitive and a is a coperfect atom of $\mathbf{L}$, then the dually defined lattice $\mathbf{Q}^*(\mathbf{L}, a)$ is again primitive.

These five statements are not difficult to prove.

In [**76**] it is proved that the class of primitive lattices is just the smallest class containing all the lattices in Examples 5.62 and 5.64 and their duals and closed under the five constructions $\mathbf{R}$, $\mathbf{P}$, $\mathbf{P}^*$, $\mathbf{Q}$ and $\mathbf{Q}^*$. The proof will be omitted.

6. Summary

In Chapter VI we will begin to investigate the fine structure of free lattices. So this is an appropriate place for us to summarize what we know about the algebraic properties of free lattices and related types of lattices. Before doing so, however, let us describe an interesting strengthening of semidistributivity which holds in free lattices.

In [**114**], J. Reinhold observed that all sublattices of a free lattice share a property, called staircase distributivity, stronger than the meet semidistributivity. He also showed that free lattices have an even stronger property, called $*$–distributivity, which implies the upper continuity of a lattice. These properties originated in M. Erné's work on lattices of equational theories [**43**].

For an element a and an n–tuple $\langle b_1, \ldots, b_n \rangle$, $n \geq 1$ of a lattice $\mathbf{L}$, we define an element $a * \langle b_1, \ldots, b_n \rangle$ of $\mathbf{L}$ by induction on n as follows:

$$\begin{aligned} &a * \langle b_1 \rangle = a \wedge b_1, \\ &a * \langle b_1, \ldots, b_n \rangle = a \wedge (b_n \vee (a * \langle b_1, \ldots, b_{n-1} \rangle)) \quad \text{for } n \geq 2. \end{aligned}$$

Expanding the definition, this can be written as

$$a * \langle b_1, \ldots, b_n \rangle = a \wedge (b_n \vee (a \wedge (b_{n-1} \vee \ldots (a \wedge (b_2 \vee (a \wedge b_1))) \ldots))).$$

One can easily see that if $\mathbf{s}$ is a connected subsequence of $\mathbf{t}$, then $a * \mathbf{s} \leq a * \mathbf{t}$. (It is enough to verify this in two special cases, if $\mathbf{s}$ is either a beginning or an end of $\mathbf{t}$.) If B is a set of elements, then $a * B$ stands for $\bigcup_{n=1}^{\infty} \{a * \mathbf{s} : \mathbf{s} \in B^n\}$. For convenience, let $\overline{B} = \bigcup_{n=1}^{\infty} B^n$.

A lattice $\mathbf{L}$ is said to be $*$–*distributive* if for any element $a \in L$ and any subset B of L such that the join $\bigvee B$ exists, $\bigvee (a * B)$ also exists and

$$\bigvee (a * B) = a \wedge \bigvee B.$$

If the same is required to hold for all finite subsets B of L only, the lattice is said to be *staircase distributive.*

THEOREM 5.66. *If a lattice* $\mathbf{L}$ *satisfies* $\mathrm{D}^d(\mathbf{L}) = \mathbf{L}$ *and the dual of the minimal join cover refinement property, then* $\mathbf{L}$ *is* $*$–*distributive.*

Proof: First note that, assuming $\bigvee B$ exists, $a \wedge \bigvee B \geq a * \mathbf{s}$ for all $\mathbf{s} \in \overline{B}$ by induction on the length of $\mathbf{s}$. Thus it will suffice to prove that for all $w \in L$, $w \geq a * \mathbf{s}$ for all $\mathbf{s} \in \overline{B}$ implies $w \geq a \wedge \bigvee B$. This we will do by induction on the D^d-rank $\rho^d(w)$.

By definition, $w \in \mathrm{D}_0^d(\mathbf{L})$ if and only if w is meet prime. So if $w \in \mathrm{D}^d(\mathbf{L})$ and $w \geq a * \mathbf{s}$ for all $\mathbf{s} \in \overline{B}$, then in particular $w \geq a * b = a \wedge b$ for all $b \in B$, and hence either $w \geq a$ or $w \geq b$ for all $b \in B$, i.e., $w \geq a$ or $w \geq \bigvee B$. Thus $w \geq a \wedge \bigvee B$.

Now let $w \in \mathrm{D}_{k+1}^d(\mathbf{L})$, and assume $w \geq a * \mathbf{s}$ for all $\mathbf{s} \in \overline{B}$. Since the dual of the minimal join cover refinement property holds, there are finitely many meet covers $E_1, \ldots, E_k$ such that whenever $w \geq \bigwedge T$ nontrivially, then $E_i \gg T$ for some i. Because $w \in \mathrm{D}_{k+1}^d(\mathbf{L})$, we can assume that $E_i \subseteq \mathrm{D}_k^d(\mathbf{L})$ for all i.

If $w \geq a$ or $w \geq \bigvee B$, then we get $w \geq a \wedge \bigvee B$ as desired. So assume that $w \ngeq a$ and that there exists $c \in B$ with $w \ngeq c$. Then $a * \langle \mathbf{s}, c \rangle = a \wedge (c \vee (a * \mathbf{s}))$ is a nontrivial meet cover of w for any $\mathbf{s} \in \overline{B}$, and hence $E_i \gg \{a, c \vee (a * \mathbf{s})\}$ for some i. We want to show there is one i which works for all such sequences. For this it suffices to observe that if E_j fails to meet refine $\{a, c \vee (a * \mathbf{s})\}$, then E_j fail to meet refine $\{a, c \vee (a * \mathbf{t})\}$ whenever $\mathbf{s}$ is a connected subsequence of $\mathbf{t}$, since $a * \mathbf{t} \geq a * \mathbf{s}$ in that case. Hence there exists an i such that $E_i \gg \{a, c \vee (a * \mathbf{s})\}$ for all $\mathbf{s} \in \overline{B}$. We may suppose that E_1 has this property.

Consequently, for each $u \in E_1$, either $u \geq a$ or $u \geq a * \mathbf{s}$ for all $\mathbf{s} \in \overline{B}$. By induction, if $u \geq a * \mathbf{s}$ for all such $\mathbf{s}$, then $u \geq a \wedge \bigvee B$. Therefore $w \geq \bigwedge E_1 \geq a \wedge \bigvee B$, as desired.

Corollary 5.67. *Every free lattice* $\mathbf{FL}(X)$ *is both* $*$*–distributive and dually* $*$*–distributive.*

A $*$–distributive lattice is upper continuous. Indeed, let $\{c_i : i \in I\}$ be a chain with $c = \bigvee c_i$, and a be an element. Then $a \wedge c$ is clearly an upper bound for $\{a \wedge c_i : i \in I\}$, and if d is any other upper bound then, as it is easy to check, d is an upper bound for the set $a * \{c_i : i \in I\}$ which joins to $a \wedge c$.

Also, a staircase distributive lattice is meet semidistributive. If $a \wedge b_1 = a \wedge b_2 = c$, then $a \wedge (b_1 \vee b_2) = \bigvee (a * \{b_1, b_2\})$ where $a * \{b_1, b_2\}$ consists of the element c only, so that $a \wedge (b_1 \vee b_2) = c$.

It follows from Corollary 5.67 that each sublattice of a free lattice is staircase and dually staircase distributive. However, as noted in [**114**], the combination of this property with Whitman's condition (W) are not sufficient to characterize sublattices of free lattices.

This brings us to an important open problem.

Problem 5.68. *Which lattices* (*and in particular which countable lattices*) *are sublattices of a free lattice?*

A characterization is known for distributive sublattices of free lattices, due to Galvin and Jónsson [**65**]. A distributive lattice can be embedded into a free lattice if and only if it is isomorphic to the union of a countable chain of sublattices, each of which is either a one-element lattice, or an eight-element Boolean algebra, or a direct product of a two-element chain and a countable chain.

The next theorem gives the main connections between the various lattice-theoretic properties we have considered.

Theorem 5.69. *The following implications hold for an arbitrary lattice* $\mathbf{L}$.

(1) *If* $\mathbf{L}$ *satisfies* (W) *and is generated by a set of join prime elements, then* $\mathrm{D}(\mathbf{L}) = L$ *and* $\mathbf{L}$ *has the minimal join cover refinement property.*

(2) *If* $\mathrm{D}(\mathbf{L}) = L$*, then* $\mathbf{L}$ *satisfies* $(\mathrm{SD}_\vee)$.
(3) *If* $\mathbf{L}$ *has the minimal join cover refinement property, then it is lower continuous.*
(4) *If* $\mathbf{L}$ *is finitely separable, then it can be order-embedded into a free lattice, and hence it inherits the countable chain condition, and has order dimension at most* γ *whenever* γ *is an infinite cardinal with* $|L| \leq 2^\gamma$.
(5) *If* $\mathbf{L}$ *has the minimal join cover refinement property and* $\mathrm{D}(\mathbf{L}) = L$*, then it is dually* $*$*–distributive.*

Part (1) is found in Theorem 2.10; (2) is Corollary 2.23; (3) is Theorem 2.25; (4) combines Lemma 5.6 and Theorems 1.27 and 5.27; while (5) is the dual of Theorem 5.66.

Table 5.1 shows the properties that hold in various generalizations of free lattices. The symbol $\times$ indicates that the property holds. All unmarked entries are known to be false.

[3]If a is a completely join irreducible element of a finitely presented lattice, then there is a finite set of elements such that if $u \geq a_*$ but $u \ngeq a$ then u is below one of these elements; see [**53**].

TABLE 5.1

	$\mathbf{FL}(X)$ X a set	$\mathbf{FL}(\mathbf{P})$ $\mathbf{P}$ an ord. set	fin. pres. lattices	proj. lattices	sublat. of free lattices
has prime gen. set	×	×			
(W)	×	×		×	×
$\mathrm{D}(\mathbf{L}) = L$	×	×		×	
$(\mathrm{SD}_\vee)$	×	×		×	×
min. join cover refin. prop.	×	×	×	×	
continuous	×	×	×	×	
fin. sep.	×		×	×	
countable CC	×		×	×	×
small dim.	×		×	×	×
$\kappa_{\mathbf{L}}(a)$ exists, for a cji	×	×	[3]	×	
$*$–dist.	×	×		×	
staircase dist.	×	×		×	×
fin. gen. $\Rightarrow$ weakly atomic	×	×		×	×

CHAPTER VI

Totally Atomic Elements

Totally atomic elements play a surprisingly important role in determining the fine structure of the lattice $\mathbf{FL}(X)$. Therefore it behooves us to study them more carefully. In particular, in the first section of this chapter we will provide an explicit description of the totally atomic elements in $\mathbf{FL}(X)$. As a consequence, we will see that for any finite set X, the number of totally atomic elements of $\mathbf{FL}(X)$ is finite.

Repeatedly in the arguments in the next four chapters, we will encounter conditions on an element of a free lattice which force that element to be totally atomic. Some of these results are of independent interest, and these are included in Section 3. In subsequent chapters, more will follow.

Later we shall also need to know the canonical form of $\kappa(w)$ for a join irreducible totally atomic element w. The syntactic algorithm obtained in Chapter III can be, of course, applied to find $\kappa(w)$ in a form which is not canonical and then the general procedure, described in Chapter I, can be used to transform $\kappa(w)$ into canonical form. However, we need a more specific characterization.

Throughout this chapter, let X be a finite set with at least three elements.

1. Characterization

Recall that an element a of a lattice is said to be totally atomic if it is both lower and upper atomic, i.e., if for every $b > a$ there exists an element c with $b \geq c \succ a$, and the dual condition holds.

Let us start our investigation of totally atomic elements in $\mathbf{FL}(X)$ by a summary of everything that we know about these elements from Chapter III.

THEOREM 6.1. *The following are equivalent for an element $w \in$ $\mathbf{FL}(X)$:*

(1) *w is totally atomic*;
(2) *each subelement of w is totally atomic*;
(3) *the lattices $\mathbf{L}^\vee(w) = \mathrm{J}(w)^\vee$ and $\mathbf{L}^\wedge(w) = \mathrm{M}(w)^\wedge$ are both semidistributive*;

(4) *w is either* 0*, or* 1*, or a completely join irreducible element all of whose canonical meetands are completely meet irreducible, or* (*dually*) *a completely meet irreducible element all of whose canonical joinands are completely join irreducible.*

Proof: The equivalence of (1) with (2) and (4) follows from Corollary 3.27 and Corollary 3.8, respectively. For (3), first note that if $w = x \in X$ then $\mathbf{L}^{\vee}(x) \cong \mathbf{L}^{\wedge}(x) \cong \mathbf{2}$, and all four conditions hold. Suppose say $w = \bigwedge w_i$ canonically. Then by Theorem 3.23, w is completely join irreducible if and only if $\mathbf{L}^{\vee}(w)$ is (meet) semidistributive, and dually every w_i is completely meet irreducible if and only if every $\mathbf{L}^{\wedge}(w_i)$ is (join) semidistributive. But $\mathbf{L}^{\wedge}(w)$ is a subdirect product of the $\mathbf{L}^{\wedge}(w_i)$'s by the dual of Corollary 3.21, so the latter condition is equivalent to $\mathbf{L}^{\wedge}(w)$ being semidistributive. The dual argument applies if $w = \bigvee w_j$, and we conclude that (3) is equivalent to (4).

Now we could use Theorem 6.1 and the syntactic algorithms of Chapter III, Section 5 to determine all the totally atomic elements in $\mathbf{FL}(X)$ when X is small. This would yield the following results.

EXAMPLE 6.2. If $X = \{x, y, z\}$ is a three-element set, then $\mathbf{FL}(X)$ contains precisely 17 totally atomic elements. The join irreducible ones are the elements

$$\begin{aligned}
&x,\\
&x \wedge y,\\
&x \wedge y \wedge z,\\
&(x \vee z) \wedge (y \vee z)
\end{aligned}$$

and their images under automorphisms of $\mathbf{FL}(X)$, i.e., permutations of the variables.

If $X = \{x, y, z, u\}$ is a four-element set, then $\mathbf{FL}(X)$ contains precisely 94 totally atomic elements. The join irreducible ones are the ones listed above and the elements

$$\begin{aligned}
&x \wedge y \wedge z \wedge u,\\
&(x \vee z \vee u) \wedge (y \vee z \vee u),\\
&(x \vee u) \wedge (y \vee u) \wedge (z \vee u),\\
&((x \wedge u) \vee (z \wedge u)) \wedge ((y \wedge u) \vee (z \wedge u))
\end{aligned}$$

and again their images under automorphisms of $\mathbf{FL}(X)$. Of course, the duals of these elements give us the meet irreducible totally atomic elements. In Corollary 6.11 below we give an exact formula for the number of totally atomic elements in $\mathbf{FL}(n)$.

With these examples in hand, it is a relatively straightforward matter to formulate a proposed characterization of totally atomic elements, and then verify that it is correct. In this way, Freese and Nation [**62**] found the characterization which we will give below.

Recall that an endomorphism of $\mathbf{FL}(X)$ is uniquely determined by its action on the generators. For every element $u \in \mathbf{FL}(X)$ we define two endomorphisms, σ_u and μ_u, of $\mathbf{FL}(X)$ by

$$\sigma_u(x) = x \vee u,$$
$$\mu_u(x) = x \wedge u,$$

for $x \in X$. We shall also let σ_u and μ_u denote the endomorphisms of the term algebra defined in the same way.

Denote by G the smallest subset of $\mathbf{FL}(X)$ such that

(1) $X \subseteq G$, and
(2) if $w \in G$ and x is an element of X not occurring in the canonical representation of w, then $\sigma_x(w) \in G$ and $\mu_x(w) \in G$.

The aim of this section is to prove that G is just the set of totally atomic elements of $\mathbf{FL}(X)$.

Every meet reducible element of G can be written as

$$w = \mu_{p_{n+1}} \sigma_{s_n} \mu_{p_n} \cdots \sigma_{s_2} \mu_{p_2} \sigma_{s_1}(p_1) \tag{1}$$

where

$$p_i = y_{i1} \wedge \cdots \wedge y_{il_i},$$
$$s_i = z_{i1} \vee \cdots \vee z_{ik_i}.$$

The y_{ij}'s and z_{ij}'s are distinct generators. We allow $n = 0$, in which case we just have $w = p_1 = \bigwedge y_{1j}$, and we allow $l_{n+1} = 0$ (i.e., $p_{n+1} = 1$). Otherwise we require k_i, $l_i \geq 1$, and $l_1 > 1$ (so that w is meet reducible). To get a feel for this notation, the last three elements given in Example 6.2 can be written as

$$\sigma_{z \vee u}(x \wedge y) = (x \vee z \vee u) \wedge (y \vee z \vee u)$$
$$\sigma_u(x \wedge y \wedge z) = (x \vee u) \wedge (y \vee u) \wedge (z \vee u)$$
$$\mu_u \sigma_z(x \wedge y) = ((x \wedge u) \vee (z \wedge u)) \wedge ((y \wedge u) \vee (z \wedge u)).$$

Lemma 6.3. *Let u and v be two elements of $\mathbf{FL}(X)$ and x be a generator occurring in neither u nor v. Then $u \leq v$ iff $\sigma_x(u) \leq \sigma_x(v)$ iff $\mu_x(u) \leq \mu_x(v)$.*

Proof: Since σ_x is an endomorphism, $u \leq v$ implies $\sigma_x(u) \leq \sigma_x(v)$. Conversely, assume $\sigma_x(u) \leq \sigma_x(v)$ where x is a generator occurring in neither u nor v. Let ζ_x be the endomorphism which sends x to 0, and fixes the rest of the generators. Then $\zeta_x \sigma_x(y) = y$ for $y \in X - \{x\}$.

Hence $u = \zeta_x\sigma_x(u) \le \zeta_x\sigma_x(v) = v$. The equivalence with $\mu_x(u) \le \mu_x(v)$ follows by duality.

Lemma 6.4. *The term at the right side of* (1) *is in canonical form.*

Proof: If $n = 0$, then $p_1 = \bigwedge y_{1j}$ is in canonical form. Likewise, $\sigma_{s_1}(p_1)$ is in canonical form, with our assumption that the variables are distinct. From here, it is routine to apply the canonical form criteria of Theorem 1.18, using Lemma 6.3 and induction.

We need one technical lemma.

Lemma 6.5. *For an element w as in* (1) *we have*

$$w = \mu_{p_{n+1}}\sigma_{s_n}\cdots\mu_{p_2}\sigma_{s_1}(p_1) \le \mu_{p_{n+1}}\sigma_{s_n}\cdots\mu_{p_2}\sigma_{s_1}(0) \vee y_{ij}$$

for all i and j.

Proof: It is sufficient to prove

$$\mu_{p_{n+1}}\sigma_{s_n}\cdots\mu_{p_2}\sigma_{s_1}(y_{1k}) \le \mu_{p_{n+1}}\sigma_{s_n}\cdots\mu_{p_2}\sigma_{s_1}(0) \vee y_{ij}$$

where k is chosen so that if $i = 1$, then $k = j$ (and otherwise k is arbitrary). If $n = 0$, this is trivial. Let $n \ge 1$ and let us proceed by induction. We need to show that the elements $a = \mu_{p_{n+1}}\sigma_{s_n}\cdots\mu_{p_2}(y_{1k})$ and $b_r = \mu_{p_{n+1}}\sigma_{s_n}\cdots\mu_{p_2}(z_{1r})$ are all below $\mu_{p_{n+1}}\sigma_{s_n}\cdots\mu_{p_2}\sigma_{s_1}(0)\vee y_{ij}$. Since $\sigma_{s_1}(0) = s_1$, clearly the elements b_r are below $\mu_{p_{n+1}}\sigma_{s_n}\cdots\mu_{p_2}\sigma_{s_1}(0)$. Using induction, we have

$$\begin{aligned} a = \mu_{p_{n+1}}\sigma_{s_n}\cdots\sigma_{s_2}(y_{1k} \wedge p_2) &\le \mu_{p_{n+1}}\sigma_{s_n}\cdots\sigma_{s_2}(0) \vee y_{ij} \\ &\le \mu_{p_{n+1}}\sigma_{s_n}\cdots\sigma_{s_2}\mu_{p_2}\sigma_{s_1}(0) \vee y_{ij}, \end{aligned}$$

since $0 \le \mu_{p_2}\sigma_{s_1}(0)$.

Lemma 6.6. *Let $w \ne 0$ be a meet reducible element of G, expressed as in* (1)*. Then we have the following.*

(1) $\mathrm{J}(w) = \{w\} \cup \{\mu_{p_{n+1}}\sigma_{s_n}\mu_{p_n}\cdots\sigma_{s_t}\mu_{p_t}(x) : t = 2,\dots,n+1$ *and* $x = y_{ij}$ *or* z_{ij} *for some* $i < t$ *and some* $j\}$.
(2) *If* $u \in \mathrm{J}(w)$, *then* $u < w$ *if and only if* $u = \mu_{p_{n+1}}\sigma_{s_n}\cdots\mu_{p_t}(z_{t-1,j})$ *for some* $t = 2,\dots,n+1$ *and some* j.
(3) $w_\dagger = \mu_{p_{n+1}}\sigma_{s_n}\mu_{p_n}\cdots\sigma_{s_2}\mu_{p_2}\sigma_{s_1}(0)$.
(4) $\mathrm{K}(w) = \{\mu_{p_{n+1}}\sigma_{s_n}\mu_{p_n}\cdots\sigma_{s_t}\mu_{p_t}(z_{ij}) : t = 2,\dots,n+1$ *and* $i < t\}$.
(5) $\bigvee\mathrm{K}(w) = \mu_{p_{n+1}}(s_1 \vee\cdots\vee s_n)$ *and* $w \nleq \bigvee\mathrm{K}(w)$.

Proof: (1) If $w = p_1$, then $\mathrm{J}(w) = \{w\}$ which agrees with (1). So let $n > 0$, and denote by τ the endomorphism $\mu_{p_{n+1}}\sigma_{s_n}\cdots\mu_{p_2}$. Then

$$w = \tau\sigma_{s_1}(\bigwedge_i y_{1i}) = \tau(\bigwedge_i(y_{1i} \vee \bigvee_j z_{1j})) = \bigwedge_i(\tau(y_{1i}) \vee \bigvee_j \tau(z_{1j})),$$

so that the elements $\tau(y_{1i})$ and $\tau(z_{1j})$ are in $\mathrm{J}(w)$. The rest follows easily by induction.

(2) By (1) we have $u = \mu_{p_{n+1}}\sigma_{s_n}\cdots\mu_{p_t}(x)$ for some t and some x. By Lemma 6.3, $u < w$ if and only if

$$x < \sigma_{s_{t-1}}\mu_{p_{t-1}}\cdots\sigma_{s_2}\mu_{p_2}\sigma_{s_1}(p_1).$$

If $x = z_{t-1,j}$ for some j, then clearly this inequality holds. But if $x = z_{ij}$ for some $i < t-1$, or if $x = y_{ij}$ for some i, set $x = 1$ and all the other generators to 0. Then the right-hand side evaluates to 0, while $x = 1$, so the inequality fails.

(3) If $n = 1$ and $k_1 = 0$, then both sides are equal to 0. Now let $k_1 > 0$. By definition and (2), $w_\dagger$ is the join of the elements $\mu_{p_{n+1}}\sigma_{s_n}\cdots\mu_{p_t}(z_{t-1,j})$ with $t = 2, \ldots, n+1$. Now if $t > 2$ and $s' = \bigvee_{k\neq j} z_{t-1,k}$ (possibly zero), then

$$\begin{aligned}\mu_{p_{n+1}}\sigma_{s_n}\cdots\mu_{p_t}(z_{t-1,j}) &= \mu_{p_{n+1}}\sigma_{s_n}\cdots\mu_{p_t}\sigma_{z_{t-1,j}}(0)\\ &\leq \mu_{p_{n+1}}\sigma_{s_n}\cdots\mu_{p_t}\sigma_{z_{t-1,j}}(\sigma_{s'}\mu_{p_{t-1}}(z_{t-2,1}))\\ &= \mu_{p_{n+1}}\sigma_{s_n}\cdots\mu_{p_{t-1}}(z_{t-2,1}).\end{aligned}$$

Therefore

$$w_\dagger = \bigvee_j \mu_{p_{n+1}}\sigma_{s_n}\cdots\mu_{p_2}(z_{1j}) = \mu_{p_{n+1}}\sigma_{s_n}\cdots\mu_{p_2}\sigma_{s_1}(0)$$

as claimed.

(4) Let u be an element of $\mathrm{J}(w)$ other than w. We have $u = \mu_{p_{n+1}}\sigma_{s_n}\cdots\mu_{p_t}(x)$ for some t and some x as in (1). By definition, $u \in \mathrm{K}(w)$ is equivalent to $w \not\leq w_\dagger \vee u$. By (3) and Lemma 6.3, $w \leq w_\dagger \vee u$ is equivalent to

$$\sigma_{s_{t-1}}\mu_{p_{t-1}}\cdots\sigma_{s_1}(p_1) \leq \sigma_{s_{t-1}}\mu_{p_{t-1}}\cdots\sigma_{s_1}(0) \vee x.$$

If we set all $y_{ij} = 1$ and all $z_{ij} = 0$, then the left-hand side of this inequality evaluates to 1 and the right-hand side is x. So, the inequality can be true only if x is not one of the z_{ij}'s. Conversely, if x is not a z_{ij}, then x is one of the y_{ij}'s, and the inequality holds by Lemma 6.5.

(5) Each $\mu_{p_{n+1}}(z_{ij})$ belongs to $\mathrm{K}(w)$ and the element $a = \mu_{p_{n+1}}(s_1 \vee \cdots \vee s_n)$ is the join of these elements, so $a \leq \bigvee \mathrm{K}(w)$. On the other hand, each element of $\mathrm{K}(w)$ is clearly below a, so $\bigvee \mathrm{K}(w) \leq a$. Setting each $z_{ij} = 0$ and each $y_{ij} = 1$, we conclude that $w \not\leq \bigvee \mathrm{K}(w)$.

This gives us the first half of our characterization.

LEMMA 6.7. *Every element of G is totally atomic.*

Proof: By induction on the length of w. If w is either a generator or 0 or 1, then it is clear that w is totally atomic. Now let w be any other element of G. It is sufficient to consider the case when w, expressed as in (1), is meet reducible. Then $w \not\leq \bigvee \mathrm{K}(w)$ by Lemma 6.6. By induction and Theorem 3.30, w is lower atomic. By induction and Lemma 6.4, each canonical meetand $\mu_{p_{n+1}}\sigma_{s_n} \cdots \mu_{p_2}(y_{1j} \vee \bigvee_j z_{1j})$ of w is totally atomic, so that by the dual of Corollary 3.8, w is also upper atomic.

To prove the converse of Lemma 6.7, we require a couple of lemmas.

Lemma 6.8. *Let $u \in G$ be join reducible with $u = u_1 \vee \cdots \vee u_n$ canonically, and let $x \in X$ be such that $u_1 \not\leq x$ and $u_i \leq x$ for $2 \leq i \leq n$. Then $x \wedge u \in X^\wedge$.*

Proof: If u is a join of generators, so that $u_i \in X$ for all i, then $u_i \leq x$ for $2 \leq i \leq n$ implies $n = 2$ and $u_2 = x$, whence $u = u_1 \vee x$ and $x \wedge u = x$. Thus, since $u \in G$, we may assume $u = \rho_{y_k} \cdots \rho_{y_1}(x_1 \vee \cdots \vee x_n)$, where $x_1, \ldots, x_n, y_1, \ldots, y_k$ are pairwise different generators, $\rho_{y_1} = \mu_{y_1}$, and $\rho_{y_i} = \sigma_{y_i}$ or μ_{y_i} for $i = 2, \ldots, k$.

Suppose $\rho_{y_i} = \sigma_{y_i}$ for at least one i. Then choosing r maximal such that $\rho_{y_r} = \sigma_{y_r}$, we have

$$u = \mu_{y_k} \cdots \mu_{y_{r+1}}\sigma_{y_r} \cdots \mu_{y_1}(x_1 \vee \cdots \vee x_n).$$

Moreover, $x \notin \{y_k, y_{k-1}, \ldots, y_{r+1}\}$ as $u \not\leq x$. For all $i \geq 2$ we now have

$$x \geq u_i = \mu_{y_k} \cdots \mu_{y_{r+1}}\sigma_{y_r} \cdots \mu_{y_1}(x_i).$$

If we set $x = 0$ and $z = 1$ for all $z \in X - \{x\}$, then u_i evaluates to 1 (there are two cases, depending on whether or not $x = y_r$), and we obtain $0 \geq 1$, a contradiction.

Therefore $\rho_{y_i} = \mu_{y_i}$ for all i. We get $u_i = x_i \wedge y_1 \wedge \cdots \wedge y_k$. As $u_1 \not\leq x$ and $u_i \leq x$ for $2 \leq i \leq n$, this implies $n = 2$ and $x_2 = x$. Thus

$$x \wedge u = x \wedge [(x_1 \wedge y_1 \wedge \cdots \wedge y_k) \vee (x \wedge y_1 \wedge \cdots \wedge y_k)] = x \wedge y_1 \wedge \cdots \wedge y_k \in X^\wedge,$$

as desired.

Lemma 6.9. *Let $w = w_1 \wedge \cdots \wedge w_m$, where each w_i is a meet irreducible element of G. For each i let $w_i = \bigvee_j w_{ij}$ canonically (if $w_i \in X$, let $w_i = w_{i1}$), and assume that $w_{ij} \leq w$ for all $j > 1$. Then $w \in G$.*

Proof: We can assume that $m \geq 2$ and that $w = w_1 \wedge \cdots \wedge w_m$ is an irredundant meet. If $w_i \in X$ for some i, then we are done by Lemma 6.8, for then $w \in X^\wedge \subseteq G$. Thus we may assume

$$w_i = \rho^i_{y_{ik_i}} \cdots \rho^i_{y_{i1}}(x_{i1} \vee \cdots \vee x_{in_i}),$$

where x_{ij}, y_{il} are pairwise distinct elements of X, $n_i \geq 2$, $k_i \geq 0$, each $\rho^i_{y_{ij}}$ is either $\sigma_{y_{ij}}$ or $\mu_{y_{ij}}$, and $\rho^i_{y_{i1}} = \mu_{y_{i1}}$ if $k_i \geq 1$.

First, let us consider the case when $k_i = 0$ for all i. For every pair $i \neq i'$ we have $x_{i2} \vee \cdots \vee x_{in_i} \leq w \leq w_{i'} = x_{i'1} \vee x_{i'2} \vee \cdots \vee x_{i'n_{i'}}$ but $x_{i1} \nleq w_{i'}$ (as $w_i \nleq w_{i'}$), so $\{x_{i2}, \ldots, x_{in_i}\} = \{x_{i'2}, \ldots, x_{i'n_{i'}}\}$. Therefore $w = \sigma_{x_{12}} \cdots \sigma_{x_{1n_1}}(x_{11} \wedge \cdots \wedge x_{m1}) \in G$, as desired.

Thus we may assume that $k_1 \geq 1$, and fix $y_1 = y_{1k_1}$, so that $w_1 = \rho^1_{y_1} w'_1$.

Suppose that $\rho^1_{y_1} = \mu_{y_1}$ and $\rho^i_{y_{ik_i}} = \sigma_{y_2}$ for some i and some y_2. Then $w_1 \leq y_1$ and $w_2 \geq y_2$, whence $y_1 \neq y_2$. However, by hypothesis we have $w_{22} \leq w_1$, which implies $y_2 \leq w_{22} \leq w_1 \leq y_1$, a contradiction.

Now suppose that w_1 begins with σ_{y_1} and w_i with σ_{z_1}. Then, for some $r \geq 1$ and some generators $z_1, \ldots, z_{r+1}$, w_i begins with $\sigma_{z_1} \cdots \sigma_{z_r} \mu_{z_{r+1}}$. If $y_1 \notin \{z_1, \ldots, z_r\}$, set $y_1 = 1$ and $z = 0$ for $z \in X - \{y_1\}$; then w_{12} evaluates to 1, but w_i evaluates to 0, contrary to $w_{12} \leq w_i$. Thus y_1 is one of $z_1, \ldots, z_r$. Since the σ_z's commute, we may assume that $y_1 = z_1$.

A similar argument applies when both w_1 and w_i begin with μ. So now we know that for all i, either $k_i = 0$ or w_i begins with ρ_{y_1}; ρ is either σ or μ, the same for all i. If $k_i > 0$, denote by w'_i the rest of w_i, so that $w_i = \rho_{y_1} w'_i$.

If $\rho = \mu$, then $k_i > 0$ for all i. Indeed, $k_i = 0$ would imply $x_{i2} \leq w_1 \leq y_1$ and hence $w_1 = y_1$, a contradiction.

If $\rho = \sigma$ and $k_i = 0$, then it follows from $y_1 \leq w_{12} \leq w_i = x_{i1} \vee \cdots \vee x_{in_i}$ that $y_1 = x_{ij}$ for some j, and we can assume that $y_1 = x_{i1}$. In this case put $w'_i = x_{i2} \vee \cdots \vee x_{in_i}$.

In any case we have now $w_i = \rho_{y_1} w'_i$ for all i, and the elements w'_i are meet irreducible, belong to G and do not contain y_1. It follows easily from Lemma 6.3 that if $w'_i = \bigvee_j w'_{ij}$ canonically, then also $w_i = \bigvee_j \rho_{y_1}(w'_{ij})$ is a canonical representation. Moreover, all the canonical joinands of w'_i except for one are below the element $w' = w'_1 \wedge \cdots \wedge w'_m$, since the elements w_i have the same property. Therefore, by induction, $w' \in G$. But then $w = \rho_{y_1}(w') \in G$.

THEOREM 6.10. *An element of* $\mathbf{FL}(X)$ *is totally atomic if and only if it belongs to* G.

Proof: Let us prove by induction on the length of w that if w is a totally atomic element, then $w \in G$. It is sufficient to consider the case when $w = w_1 \wedge \cdots \wedge w_m$ canonically and $m > 1$. By Theorem 6.1, each w_i is totally atomic and thus belongs to G by induction. Moreover, since w is completely join irreducible, Theorem 3.4 applies to say that for every i, either w_i is a generator or all but one of the canonical joinands of w_i are below w. Now $w \in G$ by Lemma 6.9.

The converse was proved in Lemma 6.7.

COROLLARY 6.11. *The free lattice* $\mathbf{FL}(X)$ *contains only finitely many totally atomic elements. Their number is* $|X|$ *plus two times the number of finite sequences* $Y_0, Y_1, \ldots, Y_k$ *of pairwise disjoint nonempty subsets of* X *such that* $|Y_0| \geq 2$. *If* $n = |X|$, *then the number of totally atomic elements in* $\mathbf{FL}(X)$ *is*

$$n + 2\left[\sum_{m=2}^{n}\binom{n}{m}\sum_{s=2}^{m}\binom{m}{s}\sum_{k=0}^{m-s} k!\, S(m-s,k)\right] \tag{2}$$

where $S(r,k)$ *is the Stirling number of the second kind, i.e., the number of partitions of an* r *element set into* k *blocks;* $S(0,0) = 1$.

Proof: By the last theorem and the definition of G, the number of meet reducible totally atomic elements is the same as the number of finite sequences $Y_0, Y_1, \ldots, Y_k$ of pairwise disjoint nonempty subsets of X such that $|Y_0| \geq 2$. Thus to prove that (2) counts the number of totally atomic elements, we need to show the expression in the brackets counts such sequences. We first choose a subset of X of size m, $m \geq 2$, to be $\bigcup Y_i$. From these m elements we choose a subset of size s, $s \geq 2$, for Y_0. Next we distribute the $m-s$ remaining elements into $Y_1, \ldots, Y_k$. The number of ways of doing this is the number of functions from the $m-s$ elements *onto* $\{1, \ldots, k\}$. This is easily seen to be $k!\, S(m-s,k)$.

Using this formula we can see, for example, that $\mathbf{FL}(5)$ has 667 totally atomic elements and $\mathbf{FL}(10)$ has 125,499,820.

COROLLARY 6.12. *Let* $w = w_1 \wedge \cdots \wedge w_n$ *be a join irreducible totally atomic element of* $\mathbf{FL}(X)$. *For each permutation* σ *of* $\{1, \ldots, n\}$, *there is an automorphism of* $\mathbf{FL}(X)$ *which maps* w_i *to* $w_{\sigma(i)}$ *(and so fixes* w*). Consequently, if* $w_i \geq \kappa(w)$ *for some* i, *then* $\kappa(w) = w_*$.

Proof: By Theorem 6.10, w has the form given in equation (1), with $p_1 = y_1 \wedge \cdots \wedge y_n$, and so

$$w_i = \mu_{p_{n+1}}\sigma_{s_n}\mu_{p_n}\cdots\sigma_{s_2}\mu_{p_2}\sigma_{s_1}(y_i)$$

The automorphism which sends $y_i \mapsto y_{\sigma(i)}$ and fixes the other generators clearly has the required properties.

It is clear from the definition of κ that any automorphism which fixes w also fixes $\kappa(w)$. So if $\kappa(w) \leq w_i$ for some i, then $\kappa(w) \leq w_i$ for all i by the first part, and hence $\kappa(w) \leq w$. This implies $\kappa(w) = w_*$.

It can happen that $\kappa(w) = w_*$, but this is extremely rare. Theorem 8.6 shows that this occurs only at the very top and bottom of $\mathbf{FL}(X)$.

2. Canonical Form of Kappa of a Totally Atomic Element

As before, let

$$w = \mu_{p_{n+1}} \sigma_{s_n} \mu_{p_n} \cdots \sigma_{s_2} \mu_{p_2} \sigma_{s_1}(p_1)$$

where $s_i = z_{i1} \vee \cdots \vee z_{ik_i}$ and $p_i = y_{i1} \wedge \cdots \wedge y_{il_i}$ be a meet reducible, nonzero, totally atomic element. We know then that $\kappa(w)$ exists and can be computed using Theorem 3.32. However, the formula in 3.32 does not give a canonical form for $\kappa(w)$. The purpose of the present section is to find the canonical form.

For $t = 0, \dots, n$ put

$$\phi_t = \mu_{p_{n+1}} \sigma_{s_n} \mu_{p_n} \cdots \sigma_{s_{t+1}} \mu_{p_{t+1}},$$

so that $w = \phi_0(1)$ and, by Lemma 6.6, $\mathrm{J}(w) - \{w\}$ is the set of the elements $\phi_t(x)$ for $t = 1, \dots, n$ and $x = y_{ij}$ or z_{ij} with $i \leq t$.

By induction on the length of w we define an element $\overline{\kappa}(w)$ as follows. Set

$$\begin{aligned}
B_t &= \bigwedge_{j=1}^{t} \bigwedge_{c=1}^{l_j} \overline{\kappa}(\phi_t(y_{jc})), \\
A_{ti} &= B_t \wedge \overline{\kappa}(\phi_t(z_{ti})), \\
E &= \bigvee_{t=1}^{n} \bigvee_{i=1}^{k_t} A_{ti}, \\
\overline{\kappa}(w) &= \begin{cases} \bigvee(X - \{y_{11}, \dots, y_{1l_1}\}) & \text{for } n = 0, \\ z_{11} \vee E & \text{for } n = 1 \text{ and } k_1 = 1, \\ E & \text{otherwise.} \end{cases}
\end{aligned}$$

We are going to show that $\overline{\kappa}(w) = \kappa(w)$ and that the element $\overline{\kappa}(w)$, when expanded by recursion, is in canonical form. The proof will be by induction on the length of w. There is no difficulty to check the assertion in the simple cases, when $n = 0$, or $n = 1$ and $k_1 = 1$. In $\mathbf{FL}(4)$ these cases are exemplified by

$$\begin{aligned}
\kappa(x \wedge y) &= z \vee t \\
\kappa(\sigma_z(x \wedge y)) &= z \vee [(y \vee z \vee t) \wedge (x \vee z \vee t) \wedge (x \vee y \vee t)] \\
\kappa(\mu_t \sigma_z(x \wedge y)) &= z \vee [(y \vee z) \wedge (x \vee z) \wedge (x \vee y)]
\end{aligned}$$

and it is absolutely straightforward to apply Theorem 3.32 to calculate these. (The exceptional case $n = 1$, $k_1 = 1$ is due to the fact that $z_{11} \leq \kappa(w)$ whenever $n = 1$, but for $k_1 > 1$ it gets absorbed in the canonical form.) In the following lemmas we shall suppose that we have already proved $\kappa(v) = \overline{\kappa}(v)$ for any element $v \in \mathrm{J}(w)$. For the time being, these lemmas are true only under this assumption; but as we shall prove $\kappa(w) = \overline{\kappa}(w)$ in Lemma 6.19, in later sections we can legitimately make use of them in the absolute sense. Also note that a property of a totally atomic element formulated and proved in the following for the element w can later be applied without warning to any element $\phi_t(x)$.

LEMMA 6.13. *For any generator x we have*

$$x \leq \overline{\kappa}(w) \quad \textit{iff} \quad p_{n+1} \wedge x \leq \overline{\kappa}(w) \quad \textit{iff} \quad x \notin \{y_{11}, \ldots, y_{n+1,l_{n+1}}\}.$$

Proof: We proceed by induction on n. The claim is immediate if $n = 0$, and easy to prove for the exceptional case $n = 1$, $k_1 = 1$. So we may assume that $\overline{\kappa}(w)$ is given by $E = \bigvee_t \bigvee_i A_{ti}$.

Writing $\phi_t(y_{jc}) = \mu_{p_{n+1}}\sigma_{s_n}\mu_{p_n} \cdots \sigma_{s_{t+1}}(y_{jc} \wedge \bigwedge_d y_{t+1,d})$ and similarly for $\phi_t(z_{ti})$, we have inductively

$$x \leq \kappa(\phi_t(y_{jc})) \quad \text{iff} \quad p_{n+1} \wedge x \leq \kappa(\phi_t(y_{jc})) \quad \text{iff} \quad x \notin \{y_{jc}, y_{t+1,1}, \ldots, y_{n+1,l_{n+1}}\}$$
$$x \leq \kappa(\phi_t(z_{ti})) \quad \text{iff} \quad p_{n+1} \wedge x \leq \kappa(\phi_t(z_{ti})) \quad \text{iff} \quad x \notin \{z_{ti}, y_{t+1,1}, \ldots, y_{n+1,l_{n+1}}\}.$$

Hence, since $B_t = \bigwedge_j \bigwedge_c \overline{\kappa}(\phi_t(y_{jc}))$ and $A_{ti} = B_t \wedge \overline{\kappa}(\phi_t(z_{ti}))$, we get

$$x \leq B_t \quad \text{iff} \quad p_{n+1} \wedge x \leq B_t \quad \text{iff} \quad x \notin \{y_{11}, \ldots, y_{n+1,l_{n+1}}\}$$
$$x \leq A_{ti} \quad \text{iff} \quad p_{n+1} \wedge x \leq A_{ti} \quad \text{iff} \quad x \notin \{z_{ti}, y_{11}, \ldots, y_{n+1,l_{n+1}}\}.$$

Since x is join prime and $\overline{\kappa}(w) = \bigvee_t \bigvee_i A_{ti}$, we then obtain

$$x \leq \overline{\kappa}(w) \quad \text{iff} \quad p_{n+1} \wedge x \leq \overline{\kappa}(w) \quad \text{iff} \quad x \notin \{y_{11}, \ldots, y_{n+1,l_{n+1}}\}$$

as claimed.

LEMMA 6.14. $\overline{\kappa}(w) \leq \kappa(\phi_1(y_{1c}))$ *for* $1 \leq c \leq l_1$.

Proof: To simplify the notation take $c = 1$. Note $\phi_1(y_{11}) = \mu_{p_{n+1}} \cdots \sigma_{s_2}(p_2 \wedge y_{11})$, and by induction $\kappa(\phi_1(y_{11})) = \overline{\kappa}(\phi_1(y_{11}))$. Let us consider the most general case only, when $\kappa(\phi_1(y_{11}))$ is given by

$$\kappa(\phi_1(y_{11})) = \bigvee_{t=2}^{n} \bigvee_{i=1}^{k_t} A'_{ti},$$
$$A'_{ti} = B'_t \wedge \kappa(\phi_t(z_{ti})),$$
$$B'_t = \bigwedge_{j=2}^{t} \bigwedge_{d=1}^{l_j} \kappa(\phi_t(y_{jd})) \wedge \kappa(\phi_t(y_{11})).$$

Clearly, $B_t \leq B'_t$ for $t \geq 2$. Hence $A_{ti} \leq A'_{ti}$ for $t \geq 2$ and any i. It is also clear that $A_{1i} \leq B_1 \leq \kappa(\phi_1(y_{11}))$, and we get $\overline{\kappa}(w) \leq \kappa(\phi_1(y_{11}))$. In the exceptional cases the proof is similar or simpler.

LEMMA 6.15. *Let* $1 \le t < n$. *Then*

$$\begin{aligned} \kappa(\phi_t(x)) &\le \kappa(\phi_{t+1}(x)), \\ \kappa(\phi_t(z_{ti})) &\le \kappa(\phi_{t+1}(y_{t+1,c})) \qquad \textit{for any } c, \\ \kappa(\phi_t(y_{tc})) &\le \kappa(\phi_{t+1}(y_{t+1,d})) \qquad \textit{for any } d. \end{aligned}$$

Proof: All the three assertions are consequences of induction and Lemma 6.14, applied with w replaced by $\phi_t(x)$, $\phi_t(z_{ti})$ and $\phi_t(y_{tc})$, respectively.

LEMMA 6.16. *Let* $k_1 > 0$. *The element* $k^\dagger$*, used for the calculation of* $\kappa(w)$ *in* Theorem 3.32*, is given by* $k^\dagger = \bigwedge_{c=1}^{l_1} \kappa(\phi_1(y_{1c}))$. *Moreover, for each* $t = 1, \ldots, n$ *we have*

$$k^\dagger \le \bigwedge_{j=1}^{t} \bigwedge_{c=1}^{l_j} \kappa(\phi_t(y_{jc})).$$

Proof: The element $k^\dagger$ is the meet of the elements $\kappa(v)$ with $v \in \mathrm{J}(w) - \{w\}$ and $\kappa(v) \ge \bigvee \mathrm{K}(w)$. By Lemma 6.6(1), $v \in \mathrm{J}(w) - \{w\}$ if and only if $v = \phi_t(x)$ with $x = y_{ij}$ or z_{ij} for some $i \le t$. By Lemma 6.6(5), $\bigvee \mathrm{K}(w) = \mu_{p_{n+1}}(s_1 \vee \cdots \vee s_n) = \bigvee(p_{n+1} \wedge z_{ij})$. Applying Lemma 6.13 with w replaced by all the permissible values of v, we have $\kappa(v) \ge \bigvee \mathrm{K}(w)$ if and only if $v = \phi_t(y_{jc})$ for some t, j, c $(j \le t)$. The rest follows from Lemma 6.15.

LEMMA 6.17. *Let* w' *be an element obtained from* w *by replacing one of the generators* $y_{11}, \ldots, y_{1l_1}$ *with a new generator, not occurring in* w. *Then* $w \le \kappa(w')$.

Proof: First note that w' is also totally atomic. Suppose, contrary to the claim, that $w' \le w'_* \vee w$. Without loss of generality, the replaced generator is y_{11}. Setting $y_{11} = 0$ and keeping all the remaining generators unchanged, both w'_* and w evaluate to elements below w'_* while w' evaluates to itself, a contradiction.

LEMMA 6.18. *An element* $v \in \mathrm{J}(w) - \{w\}$ *satisfies* $w \not\le \kappa(v)$ *if and only if* $v = \phi_t(z_{ti})$ *for some* t *and* i.

Proof: If $k_1 = 0$, the assertion is empty and so true. Let $k_1 > 0$. We have proved $\phi_t(z_{ti}) \le w$ in Lemma 6.6. From this it follows that $w \not\le \kappa(\phi_t(z_{ti}))$.

Let us first prove that $w \le \kappa(\phi_1(y_{11}))$. Since $w = \phi_1(\sigma_{s_1}(\bigwedge y_{1i}))$, it is sufficient to show that $\phi_1(z_{11}), \ldots, \phi_1(z_{1k_1}), \phi_1(y_{12})$ are all below $\kappa(\phi_1(y_{11}))$. But these follow from Lemma 6.17.

Similarly, $w \leq \kappa(\phi_1(y_{1i}))$ for all i and also $w \leq \kappa(\phi_1(x))$ for any generator x not occurring in w. By Lemma 6.15 we obtain $w \leq \kappa(\phi_t(y_{jc}))$ for any $j \leq t$ and any c.

Now it remains to prove that $w \leq \kappa(\phi_t(z_{ji}))$ if $j < t$. As before, it is sufficient to show that $\phi_1(x) \leq \kappa(\phi_t(z_{ji}))$ for any $x = y_{11}, z_{11}, \ldots, z_{1k_1}$. If $j > 1$, we can use induction, replacing w by the shorter word $\phi_1(x)$. If $j = 1$, our previous argument shows that $\phi_1(x) \leq \kappa(\phi_2(z_{1i}))$ since z_{1i} is either x or does not occur in $\phi_1(x)$, and $\kappa(\phi_2(z_{1i})) \leq \kappa(\phi_t(z_{1i}))$ by Lemma 6.15.

Lemma 6.19. $\overline{\kappa}(w) = \kappa(w)$.

Proof: We may assume $k_1 > 0$. By Theorem 3.32 and Lemma 6.18,

$$\kappa(w) = \bigvee\{x \in X : w \not\leq w_\dagger \vee x\} \vee \bigvee_{t=1}^{n} \bigvee_{i=1}^{k_t} (k^\dagger \wedge \kappa(\phi_t(z_{ti}))).$$

By Lemma 6.5, recalling that $w_\dagger = \mu_{p_{n+1}}\sigma_{s_n}\mu_{p_n} \cdots \sigma_{s_2}\mu_{p_2}\sigma_{s_1}(0)$, a generator x for which $w \not\leq w_\dagger \vee x$ must be different from all y_{ij}, and all these generators are below $\overline{\kappa}(w)$ by Lemma 6.13. By Lemma 6.16 we have $k^\dagger \wedge \kappa(\phi_t(z_{ti})) \leq A_{ti}$, and so we get $\kappa(w) \leq \overline{\kappa}(w)$.

By Lemma 6.14, $\overline{\kappa}(w) \leq \kappa(\phi_1(y_{1c}))$ for all c, so that $\overline{\kappa}(w) \leq k^\dagger$ by Lemma 6.16. From this we get $\overline{\kappa}(w) \leq \kappa(w)$ by definition.

Lemma 6.20. *The element $\overline{\kappa}(w)$ is in canonical form.*

Proof: We shall prove the assertion by induction on the length of w. The cases $n = 0$ and $n = 1$, $k_1 = 1$ are easy, so we shall assume that $\overline{\kappa}(w)$ is given by $\overline{\kappa}(w) = E$. The elements $\overline{\kappa}(\phi_t(x))$ are in canonical form by induction. For any element u denote by $\mathbf{S}_($u$)$ the set of generators below u.

We shall first show that each A_{ti} is in canonical form. By Lemma 6.13, the sets $\mathbf{S}_((\overline{\kappa}(\phi_t(x)))$ for $x = y_{jc}, z_{ti}$ are pairwise incomparable, so the meetands of A_{ti} are also pairwise incomparable. If $t = n$, then A_{ti} is clearly in canonical form. If $t = n-1$ and $k_n = 1$, then each $\overline{\kappa}(\phi_t(x))$ has two joinands, one of them being z_{n1}. In all other cases the joinands of an arbitrary meetand $\overline{\kappa}(\phi_t(x))$ of A_{ti} can be denoted by $A'_{t'i'}$, for $t' = t+1, \ldots, n$ and $i' = 1, \ldots, k_{t'}$ and we have

$$\mathbf{S}_((A'_{t'i'}) = X - \{x, y_{t+1,1}, \ldots, y_{n+1,l_{n+1}}, z_{t',i'}\}.$$

We see that A_{ti} is not below any joinand of any of its meetands, as for any of these joinands u there is always a generator in $\mathbf{S}_((A_{ti})$ not below u. By Theorem 1.18 it follows that A_{ti} is in canonical form.

The sets $\mathbf{S}_((A_{ti})$ are all pairwise incomparable, and thus the elements A_{ti} themselves are pairwise incomparable. For $x = y_{jc}$ or z_{ti}

we have $\mathbf{S}_{(}\overline{\kappa}(\phi_t(x))) \not\subseteq \mathbf{S}_{(}\overline{\kappa}(w))$ and hence $\overline{\kappa}(\phi_t(x)) \not\leq \overline{\kappa}(w)$. Again by Theorem 1.18 we conclude that $\overline{\kappa}(w)$ is in canonical form.

We have thus proved the following result.

THEOREM 6.21. *Let w be a nonzero meet reducible totally atomic element, expressed as in* (1). *Then the canonical expression for $\kappa(w)$ is*

$$\kappa(w) = \begin{cases} \bigvee(X - \{y_{11}, \dots, y_{1l_1}\}) & \text{if } n = 0 \text{ and } \\ & \quad w = p_1 = y_{11} \wedge \cdots \wedge y_{1l_1}, \\ z_{11} \vee [\bigwedge_{c=1}^{l_1} \kappa(p_2 \wedge y_{1c}) \wedge \kappa(p_2 \wedge z_{11})] & \text{if } n = k_1 = 1,\ s_1 = z_{11} \\ & \quad \text{and } w = \mu_{p_2}\sigma_{s_1}(p_1), \\ \bigvee_{t=1}^{n} \bigvee_{i=1}^{k_t} [\bigwedge_{j=1}^{t} \bigwedge_{c=1}^{l_j} \kappa(\phi_t(y_{jc})) \wedge \kappa(\phi_t(z_{ti}))] & \text{otherwise.} \end{cases}$$

EXAMPLE 6.22. Despite its imposing appearance, this formula can actually be used to calculate $\kappa(w)$ by hand when n is small. For example, let $X = \{a, b, c, d, e\}$ and $w = \sigma_e\mu_d\sigma_c(a \wedge b)$. Then

$$\begin{aligned} \kappa(w) = (&\overline{a} \wedge \overline{b} \wedge \overline{d} \wedge \overline{e}) \vee \\ &[(e \vee (\overline{a} \wedge \overline{d} \wedge \overline{e})) \wedge (e \vee (\overline{c} \wedge \overline{d} \wedge \overline{e})) \vee (e \vee (\overline{b} \wedge \overline{d} \wedge \overline{e}))] \end{aligned}$$

where $\overline{a}$ is the join of the generators other than a, $\overline{b}$ is the join of the generators other than b, etc.

EXAMPLE 6.23. On the other hand, since the formula is recursive it is not hard to program for use on longer totally atomic elements. Suppose the first ten elements of X are $a, b, c, d, e, f, g, h, i, j$ and now let

$$w = \sigma_j\mu_i\sigma_h\mu_{f\wedge g}\sigma_{d\vee e}(a \wedge b \wedge c).$$

Then $\kappa(w)$ is given canonically by

$$
\begin{aligned}
&(((j\vee(\overline{a}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{f}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{g}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{h}\wedge\overline{i}\wedge\overline{j}))\vee(\overline{a}\wedge\overline{f}\wedge\overline{g}\wedge\overline{i}\wedge\overline{j}))\wedge\\
&\quad((j\vee(\overline{b}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{f}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{g}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{h}\wedge\overline{i}\wedge\overline{j}))\vee(\overline{b}\wedge\overline{f}\wedge\overline{g}\wedge\overline{i}\wedge\overline{j}))\wedge\\
&\quad((j\vee(\overline{c}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{f}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{g}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{h}\wedge\overline{i}\wedge\overline{j}))\vee(\overline{c}\wedge\overline{f}\wedge\overline{g}\wedge\overline{i}\wedge\overline{j}))\wedge\\
&\quad((j\vee(\overline{d}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{f}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{g}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{h}\wedge\overline{i}\wedge\overline{j}))\vee(\overline{d}\wedge\overline{f}\wedge\overline{g}\wedge\overline{i}\wedge\overline{j})))\\
&\vee(((j\vee(\overline{a}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{f}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{g}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{h}\wedge\overline{i}\wedge\overline{j}))\vee(\overline{a}\wedge\overline{f}\wedge\overline{g}\wedge\overline{i}\wedge\overline{j}))\wedge\\
&\quad((j\vee(\overline{b}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{f}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{g}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{h}\wedge\overline{i}\wedge\overline{j}))\vee(\overline{b}\wedge\overline{f}\wedge\overline{g}\wedge\overline{i}\wedge\overline{j}))\wedge\\
&\quad((j\vee(\overline{c}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{f}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{g}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{h}\wedge\overline{i}\wedge\overline{j}))\vee(\overline{c}\wedge\overline{f}\wedge\overline{g}\wedge\overline{i}\wedge\overline{j}))\wedge\\
&\quad((j\vee(\overline{e}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{f}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{g}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{h}\wedge\overline{i}\wedge\overline{j}))\vee(\overline{e}\wedge\overline{f}\wedge\overline{g}\wedge\overline{i}\wedge\overline{j})))\\
&\vee((j\vee(\overline{a}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{b}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{c}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{f}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{g}\wedge\overline{i}\wedge\overline{j}))\wedge(j\vee(\overline{h}\wedge\overline{i}\wedge\overline{j})))\\
&\vee(\overline{a}\wedge\overline{b}\wedge\overline{c}\wedge\overline{f}\wedge\overline{g}\wedge\overline{i}\wedge\overline{j})
\end{aligned}
$$

where again $\overline{a}$ is the join of the generators other than a, etc.

Remark 1. There is an alternative way to prove that an element u is equal to $\kappa(w)$, more sophisticated than the brute force approach that we have taken. Namely, it is not difficult to prove that if w is a completely join irreducible element, then $u = \kappa(w)$ if and only if the following six conditions are satisfied:

(1) u is meet irreducible,
(2) $\mathrm{M}(\kappa(w)) - \{\kappa(w)\} = \mathrm{M}(u) - \{u\}$,
(3) $\kappa(w)/0 \cap X \subseteq u/0 \cap X$,
(4) $\kappa(w)^{\dagger} \le u^{\dagger}$,
(5) $\kappa(w)\ B^d\ v$ implies $u \ge u^{\dagger} \wedge v$,
(6) $w \nleq u$.

Using the characterization of totally atomic elements and Lemma 2.63, then gives a different derivation of the formula in Theorem 6.21.

3. The Role of Totally Atomic Elements

Our first theorem shows why totally atomic elements will play an important role in our investigation of covering chains in the next chapter.

Theorem 6.24. *Let w be a completely meet irreducible element in a free lattice, and u be a lower cover of w. If w_1 is the unique canonical joinand of w such that $w_1 \nleq u$, then w_1 is totally atomic.*

Proof: The element w_1 exists and is completely join irreducible by Theorem 3.5. (If w is a generator, then $w_1 = w$.) On the other hand, every canonical meetand of w_1 belongs to $\mathrm{M}(w)$ and is completely meet irreducible by the dual of Theorem 3.26, which makes w_1 upper atomic by Corollary 3.8.

LEMMA 6.25. *Let w be a nonzero join irreducible totally atomic element. If $\kappa(w)$ is also totally atomic, then $w \in X^{\wedge}$. If a canonical joinand of $\kappa(w)$ is totally atomic, then either $w \in X^{\wedge}$ or $w = \mu_{y_1 \wedge \cdots \wedge y_l}\sigma_z(x_1 \wedge \cdots \wedge x_m)$ for pairwise distinct generators $x_1, \ldots, x_m, z, y_1, \ldots, y_l$ with $m \geq 2$ and $l \geq 0$, and the canonical joinand is the generator z.*

Proof: This follows easily from Theorems 6.10 and 6.21, using induction.

THEOREM 6.26. *Let w be a totally atomic join irreducible element. Then $\kappa(w)$ has a lower cover if and only if either $w \in X^{\wedge}$ or $w = \mu_{y_1 \wedge \cdots \wedge y_l}\sigma_z(x_1 \wedge \cdots \wedge x_m)$ for pairwise distinct generators $x_1, \ldots, x_m, z, y_1, \ldots, y_l$ with $m \geq 2$, $l \geq 0$.*

Proof: Let $u \prec \kappa(w)$. By Theorem 6.24, the unique canonical joinand v of $\kappa(w)$ such that $v \nleq u$ is totally atomic. We can then apply Lemma 6.25.

THEOREM 6.27. *Let w be a join irreducible totally atomic element. Then $\kappa(w)^* = w \vee \kappa(w)$ is not completely meet irreducible.*

Proof: The unique upper cover of $\kappa(w)$ is the element $w \vee \kappa(w)$, and clearly w is a canonical joinand of $w \vee \kappa(w)$. If $w \in X$, then $w \vee \kappa(w) = 1$, and w is not completely meet irreducible by convention. Suppose that $w \vee \kappa(w)$ is completely meet irreducible. By the dual of Theorem 3.4, there is a canonical meetand v of w such that $v \geq w \vee \kappa(w)$. By Corollary 6.12, this implies that $w \geq \kappa(w) = w_*$ and hence $w = w \vee \kappa(w) = \kappa(w)^*$. However, w is meet reducible and hence not completely meet irreducible.

COROLLARY 6.28. *Let w be a completely meet irreducible element. If $u \prec w$, then u is meet reducible.*

Proof: By Theorem 6.24 there is a unique canonical joinand w_1 of w such that $w_1 \nleq u$, and w_1 is totally atomic. If u is not meet reducible, then $u = \kappa(w_1)$ and hence, since w_1 is totally atomic, by Theorem 6.27 the unique upper cover w of u is not completely meet irreducible, contradicting our assumption. Hence u is meet reducible.

LEMMA 6.29. *Let w be a completely join irreducible element. Then no element of $\mathrm{J}(w) - \{w\}$ belongs to $\kappa(w)^*/w$. Moreover, if w is meet reducible, then no canonical meetand of w is below $\kappa(w)^*$.*

Proof: Let g be the standard epimorphism of $\mathbf{FL}(X)$ onto $(\mathrm{J}(w) - \{w\})^{\vee}$. If $u \in \mathrm{J}(w) - \{w\}$ and $u \leq \kappa(w)^* = w \vee \kappa(w)$, then $u = g(u) \leq g(w) \vee g(\kappa(w)) = g(\kappa(w)) \leq \kappa(w)$ and hence $u \nleq w$.

Now let $w \notin X$ and suppose that there exists a canonical meetand v of w such that $v \leq \kappa(w)^* = w \vee \kappa(w)$. If $v \in X$, we have either $v \leq w$ or $v \leq \kappa(w)$, a contradiction in both cases. If $v \nleq X$, then each canonical joinand of v belongs to $\mathrm{J}(w) - \{w\}$ and hence is not above w. Being below $\kappa(w)^*$, each canonical joinand of v must then be below $\kappa(w)$ and we get $w \leq v \leq \kappa(w)$, a contradiction.

Theorem 6.30. *Let w be a completely join irreducible element such that $\kappa(w)$ is not totally atomic. Then the interval $\kappa(w)/w_*$ contains no completely join irreducible element.*

Proof: Suppose u is a completely join irreducible element in $\kappa(w)/w_*$. Then, since $\kappa(w)$ is a canonical meetand of w_*, the dual of Theorem 3.28 implies that $\kappa(w)$ is lower atomic. Since it is completely meet irreducible, it is totally atomic.

The preceding theorem shows that some intervals in a free lattice contain no completely join irreducible element. On the other hand, Alan Day's Theorem 4.1 says that if $u \nleq v$ in a finitely generated free lattice, then there exists a completely join irreducible element q with $q \leq u$ and $v \leq \kappa(q)$ (and hence $q \nleq v$). By way of contrast, we want to conclude this chapter with a technical result which shows that there exist pairs u, v with $u > v$ such that every completely meet irreducible element below u is also below v. This result will be used in the characterization of semisingular elements in Chapter VIII.

Theorem 6.31. *Suppose q is a completely meet irreducible element of $\mathbf{FL}(X)$ and $q^* \succ q \succ p$. Let q_1 be the canonical joinand of q not below p (if $q \in X$ then $q_1 = q$) and suppose $\kappa^d(q) \leq \kappa(q_1)$. Then there is no completely meet irreducible element v such that*

$$\begin{aligned} &v \leq q^* \wedge \kappa(q_1) \\ &v \nleq p \ \sqcup \ \kappa^d(q). \end{aligned}$$

Proof: Let q, q_1 and p be as in the statement of the theorem. By Theorem 6.24, q_1 is totally atomic. Thus $\kappa(q_1)$ is defined, and $p = q \wedge \kappa(q_1)$. Let $t = \kappa^d(q)$ and suppose that v is a completely meet irreducible element of minimal complexity satisfying

$$\begin{aligned} &v \leq q^* \wedge \kappa(q_1) \\ &v \nleq p \ \sqcup \ t. \end{aligned}$$

Then, by the dual of Theorem 6.30, t is totally atomic. Since $\kappa(t) = q$ has a lower cover, Corollary 6.26 applies, and $q_1 \in X$. Now if $t \in X$, then q is a coatom of $\mathbf{FL}(X)$, and $q^* \wedge \kappa(q_1) = p \vee t$ is another coatom. The theorem is trivial in this case, so we will assume $t \notin X$. The general situation is represented in Figure 6.1, where dashed lines indicate covers.

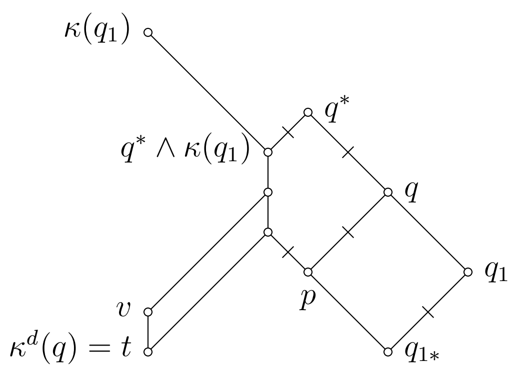


FIGURE 6.1

Next note that $v \notin X$, because $v \leq q^* = t \sqcup q$, $v \not\leq t$, and $v \leq q$ would imply $v \leq q \wedge \kappa(q_1) = p$, a contradiction. Thus v is a proper join, $v = \bigvee_i v_i$. For all i, $v_i \leq q^* \wedge \kappa(q_1)$, and there exists i_0 such that $v_{i_0} \not\leq p \sqcup t$. In fact, $v_{i_0} \geq t$, because $v_{i_0} \leq q^*$ and $v_{i_0} \not\leq q$ (as $v_{i_0} \leq \kappa(q_1)$).

As before $v_{i_0} \notin X$, so it is a proper meet, and we have

$$v_{i_0} = \bigwedge_j v_{i_0 j} \leq q^* = q \sqcup t.$$

Applying (W), we find that $v_{i_0 j_0} \leq q^*$ for some j_0. Since $v_{i_0 j_0} \in \mathrm{M}(v)$, $v_{i_0 j_0}$ is completely meet irreducible. By the minimal complexity of v, since $v_{i_0 j_0}$ satisfies the rest of the conditions, we must have $v_{i_0 j_0} \not\leq \kappa(q_1)$. Since $q_1 \in X$, this is equivalent to $q_1 \leq v_{i_0 j_0}$. We can summarize these properties by saying that $w = v_{i_0 j_0}$ is a completely meet irreducible element with

$$q_1 \sqcup t \leq w \leq q \sqcup t.$$

We claim that this cannot happen in the circumstances of the theorem.

Lemma 6.32. *Let q be a completely meet irreducible element which is not a coatom, and $t = \kappa^d(q) = \bigwedge t_i$ canonically. Then for every completely meet irreducible element $w \in q^*/t$ there exists an i such that $w^\dagger \wedge t_i = t$.*

Proof: Suppose not, and let w be a counterexample of minimal length. Thus $w^\dagger \wedge t_i > t$ for each i. If $u \in \mathrm{M}(w)$ with $w < u$, then $w^\dagger \leq u^\dagger$ and hence, by minimality, we have $u \not\leq q^*$. An easy application of (W) shows that

$$w^\dagger = \bigwedge{}^{u \in \mathrm{M}(w)}_{w<u} u \not\leq q \sqcup t = q^*.$$

By Lemma 3.28, $\mathrm{M}(t) \subseteq \mathrm{M}(w)$, so in particular each t_i is in $\mathrm{M}(w)$. Since t is a canonical joinands of q^*, $t_i \not\leq q^*$ for each i. Since $t < w^\dagger \wedge t_i$, it follows that $w^\dagger \wedge t_i \not\leq q$ and $w^\dagger \wedge t_i \not\leq t$. Now applying (W) and using these observations, it follows that $w^\dagger \wedge t_i \not\leq q \sqcup t = q^*$, and hence

$$w^\dagger \wedge t_i \not\leq w$$

for each i. It follows that $t_i \in \mathrm{K}'(w)$. On the other hand, $\bigwedge t_i = t \leq w$. By the dual of Theorem 3.30, this implies that w is not completely meet irreducible, a contradiction.

Now let us consider the various possibilities for t given by Theorem 6.26.

Case 1. Assume $t \in X^\wedge$ but, as above, $t \notin X$, so $t = x_1 \wedge \cdots \wedge x_m$ with $m > 1$. Then $q = \bigvee X - \{x_1, \ldots, x_m\} = z_1 \sqcup \ldots \sqcup z_n$ say, and we may suppose $q_1 = z_1$. By the lemma, we have $w^\dagger \wedge x_i = t$ for some i. But this implies $w^\dagger \leq \bigwedge_{j \neq i} x_j$, contrary to $z_1 = q_1 \leq w < w^\dagger$.

Case 2. Assume $t = \sigma_z(x_1 \wedge \cdots \wedge x_m)$. Then $q = \kappa(t) = z \sqcup (\kappa(z) \wedge \bigwedge \kappa(x_i))$, and $q_1 = z \leq t$, which contradicts our assumption that $\kappa^d(q) \leq \kappa(q_1)$. So this case does not occur.

Case 3. Assume $t = \mu_{y_1 \wedge \cdots \wedge y_l} \sigma_z(x_1 \wedge \cdots \wedge x_m)$ with $m > 1$ and $l \geq 1$. In this case $q = \kappa(t) = z \sqcup (\hat{z} \wedge \bigwedge \widehat{x_i})$, where $\hat{a} = \bigvee X - \{a, y_1, \ldots, y_l\}$ for $a \in X$, and we have $q_1 = z$. Let $p = y_1 \wedge \cdots \wedge y_l$. Again applying the lemma, we have $w^\dagger \wedge \mu_p \sigma_z(x_i) = t$ for some i, which implies that $w^\dagger \leq \bigwedge_{j \neq i} \mu_p \sigma_j(x_j) \leq p$. This contradicts the fact that $z = q_1 \leq w < w^\dagger$.

The simplest concrete example of Theorem 6.31 is obtained by taking $q = x$.

COROLLARY 6.33. *If X is finite, then there is no completely meet irreducible element v in $\mathbf{FL}(X)$ such that*

$$v \leq x^* \wedge \overline{x}$$

$$v \nleq x_* \sqcup \underline{x}$$

where $\overline{x} = \bigvee X - \{x\}$ *and* $\underline{x} = \bigwedge X - \{x\}$.

CHAPTER VII

Finite Intervals and Connected Components

A natural question which had been around for some time is whether there exist large finite intervals, or at least long chains of covers, in finitely generated free lattices. It turns out that the longest possible chain of covers in $\mathbf{FL}(3)$ has five elements and, in $\mathbf{FL}(X)$ with $|X| \geq 4$, only four elements. Moreover, in Section 1 we shall prove that for any X, each chain of covers not at the very bottom or the very top of $\mathbf{FL}(X)$ contains at most three elements.

Using this we will be able to show that finite intervals in $\mathbf{FL}(X)$ cannot be large. In Section 2 we will see that the only finite intervals are those at the bottom or the top, which can be described easily, and chains with at most three elements. We also show that if an interval contains a maximal three element chain, then the interval either is just this chain or has a very specific form resembling a pentagon. There is an easy criterion to determine which case holds.

The three element intervals will be discussed in Section 3. Although there are infinitely many in each $\mathbf{FL}(X)$ with X finite and $|X| \geq 3$, a detailed description of all such intervals can be given. This description is very useful as we shall see in the later chapters.

The last section characterizes the ordered sets that are connected components (with respect to the covering relation) of a free lattice.

Most of the results of the first three sections of this chapter were obtained by Freese and Nation in [**62**], and those of Section 4 by Freese in [**51**].

1. Chains of Covers

We begin with a simple consequence of Theorem 3.4 which will be used several times below.

Lemma 7.1. *Let X be a finite set and let Y be a nonempty, proper subset of X. In $\mathbf{FL}(X)$ let $a = \bigwedge Y$ and $b = \bigvee X - Y$. Suppose that a is a canonical joinand of a completely meet irreducible element v and b is a canonical meetand of a completely join irreducible element u. If $v^* \geq u_*$, then either $a \in X$ and v is in the connected component of 1, or $b \in X$ and u is in the connected component of 0.*

Proof: By Theorem 3.4, all but at most one of the elements of Y are above v^* and all but at most one of the elements of $X-Y$ are below u_*. Since generators are incomparable, this implies that either Y or $X-Y$ has only one element. In the former case $a \in X$ and v^* contains a and all but at most one member of $X-Y$. So it is either a coatom or 1 and thus v is in the connected component of 1. A similar argument applies if $b \in X$.

Our description of the connected component of 0 (see Section 7 of Chapter III) shows that if X is a finite set with at least three elements, then $\mathbf{FL}(X)$ contains a chain of three covers, and if $|X| = 3$, it contains a chain of four covers. In more detail, if $X = \{x_1, \ldots, x_n\}$ and $n = |X| \geq 4$, then all the chains of three covers beginning with 0 are of the form

$$0 \prec \underline{x}_i \prec \underline{x}_i \vee \underline{x}_j \prec \bigwedge_{k \neq i,j} (\underline{x}_i \vee \underline{x}_j \vee \underline{x}_k) \tag{1}$$

for $i \neq j \in \{1, \ldots, n\}$, where $\underline{x}_i = \bigwedge_{l \neq i} x_l$. If $n = 3$, then the chains of four covers beginning with 0 are

$$\begin{aligned} 0 \prec x_i \wedge x_j &\prec (x_i \wedge x_j) \vee (x_i \wedge x_k) \\ &\prec x_i \wedge ((x_i \wedge x_j) \vee (x_i \wedge x_k) \vee (x_j \wedge x_k)) \\ &\prec (x_i \wedge x_j) \vee (x_i \wedge x_k) \vee (x_j \wedge x_k) \end{aligned} \tag{2}$$

for a permutation i, j, k of 1, 2, 3. Of course, there are also the dual chains of three or four covers at the top of $\mathbf{FL}(X)$, ending with 1. Moreover, none of these chains is contained in a longer chain of covers.

The purpose of this section is to prove that these are the longest possible chains of covers in a free lattice.

THEOREM 7.2. *In a free lattice* $\mathbf{FL}(X)$*, each chain of at least three covers is either a subchain of one of the chains* (1) (*in the case* $|X| \geq 4$) *or* (2) (*in the case* $|X| = 3$) *or a subchain of the dual of one of these chains.*

Proof: Clearly, it is sufficient to prove the theorem for chains of precisely three covers. Let

$$t \prec u \prec v \prec w \tag{3}$$

be a chain of three covers in $\mathbf{FL}(X)$. Then X is a finite set (there are no covers in $\mathbf{FL}(X)$ when X is infinite) and $|X| \geq 3$. By our description of the chains of covers in the connected component of 0 and of 1, it suffices to show that the chain of covers (3) is contained in one of these connected components. Since the connected component of a generator does not contain long chains of covers (see Example 3.43), none of the elements t, u, v, w belongs to X.

Let us consider first the case when v is completely meet irreducible and u is completely join irreducible. By Theorem 6.24, the unique canonical joinand v_1 of v, such that $v_1 \not\leq u$, is totally atomic. Dually, the unique canonical meetand u_1 of u, such that $u_1 \not\geq v$, is totally atomic. The element v_{1*} is below u, because otherwise we would have $v = u \vee v_{1*}$, which is not possible when v_1 is a canonical joinand of v (see Theorem 1.20). But then, $v_{1*} \leq u_* = t$. Similarly, $u_1^* \geq w$. The situation is diagrammed in Figure 7.1.

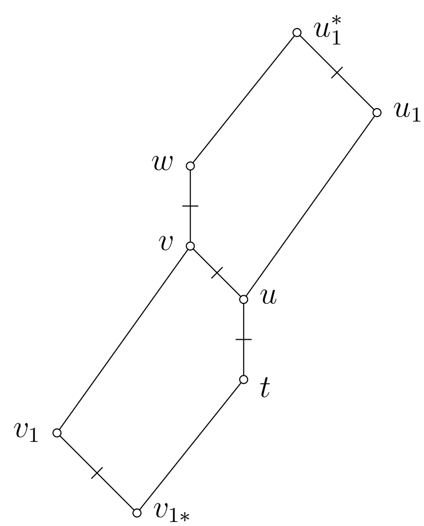


FIGURE 7.1

It is also clear that u_1 is the largest element above v_{1*} but not above v_1, and so $u_1 = \kappa(v_1)$ by definition. But since u_1 is totally atomic, Lemma 6.25 tells us that $v_1 \in X^\wedge$. Thus we have $v_1 = \bigwedge Y$ and $u_1 = \bigvee(X - Y)$ for a nonempty proper subset Y of X. It now follows from Lemma 7.1 that our chain (3) is in the connected component of 0 or 1, as desired.

So far, we have proved the theorem under the assumption that v is completely meet irreducible and u is completely join irreducible.

It is clear that each of the elements u and v is either completely meet or completely join irreducible. By Corollary 6.28, if v is completely meet irreducible, then u is meet reducible and hence completely join irreducible. Dually (and conversely), if u is completely join irreducible, then v is completely meet irreducible. Thus for the rest of the proof we may assume that u is completely meet irreducible and v is completely join irreducible.

We have $\kappa(v) = v_* = u$ and $\kappa^d(u) = u^* = v$.[1] By Theorem 6.24 and its dual, the unique canonical joinand u_1 of u such that $u_1 \not\leq t$ is totally atomic and similarly the unique canonical meetand v_1 of v such that $v_1 \not\geq w$ is totally atomic. Clearly, $w = v \vee \kappa^d(v_1) = u \vee \kappa^d(v_1)$ and $t = u \wedge \kappa(u_1) = v \wedge \kappa(u_1)$. The situation is diagrammed in Figure 7.2.

By Corollary 6.28, w is a proper join and t is a proper meet. Thus

$$\kappa^d(v_1) < w, \qquad t < \kappa(u_1). \tag{4}$$

If $v = v_1 \wedge \cdots \wedge v_m$ canonically, then applying (W) to $v = \bigwedge v_i \leq w = \kappa^d(v_1) \vee u$ yields that $w = v_i$ for some i. So $m = 2$ and $v_2 = w$ and $v = v_1 \wedge w$ canonically. Since w is a canonical meetand of v which is completely join irreducible, w is lower atomic. Dually t is upper atomic.

Suppose that t is a canonical joinand of w. Then, by Theorem 3.5, t is completely join irreducible since w is lower atomic. But then

$$t_* \prec t \prec u \prec v$$

with t a proper meet (by (4)) and u a proper join. We have already handled this case.

Thus by duality we may assume that t is not a canonical joinand of w and that w is not a canonical meetand of t. Now

$$w = \kappa^d(v_1) \vee u = \kappa^d(v_1) \vee u_1 \vee t.$$

---

[1]In Chapter VIII we will define a completely join irreducible element v to be *singular* if $\kappa(v) = v_*$, and characterize these elements in Theorem 8.6. Since that characterization depends on the result we are proving now, we cannot use it here.

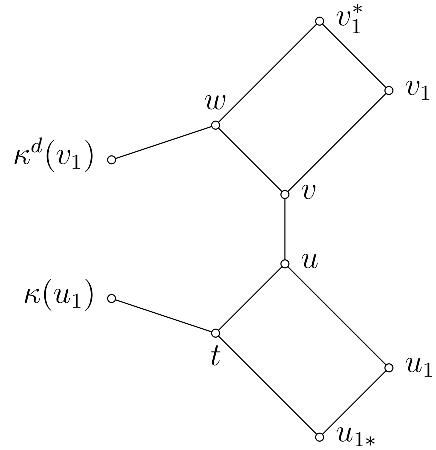


FIGURE 7.2

If $s = \bigvee\{w_i : w_i \leq t\}$, where $w = \bigvee w_i$ canonically, then $s < t$ since $t \neq w_i$ for all i and t is join irreducible. By the refinement property of canonical joins, Theorem 1.19, $w = \kappa^d(v_1) \vee u_1 \vee s$. Dually there is an element $r > w$ such that $t = \kappa(u_1) \wedge v_1 \wedge r$. Applying (W) to $t = \kappa(u_1) \wedge v_1 \wedge r \leq w = \kappa^d(v_1) \vee u_1 \vee s$ yields that either $\kappa(u_1) \leq w$ or $t \leq \kappa^d(v_1)$.

By duality we may assume that $\kappa(u_1) \leq w$. Since $w \geq u_1$, $\kappa(u_1) < w$. We claim that $\kappa(u_1) \prec w$. Since u is meet irreducible the interval w/u must be just the three element chain. Thus if $\kappa(u_1) \leq a < w$ then $a \wedge u = t$ since $t \leq \kappa(u_1)$. But then $a \wedge u_1 = u_{1*}$ which implies $a = \kappa(u_1)$ and thus $\kappa(u_1)^* = w$, as claimed.

We may assume that $\kappa(u_1)$ is not a generator since otherwise our chain would lie in the connected component of a generator, which is not possible. Since $\kappa^d(v_1) \vee v = w$ but $\kappa^d(v_1)_* \leq v$, $\kappa^d(v_1)$ is a canonical joinand of w. So by Theorem 3.34 $\kappa^d(v_1)$ is also a canonical joinand of $\kappa(u_1)$. In particular, $\kappa^d(v_1) \le \kappa(u_1) \le v_1^*$ and so, by the dual of Theorem 6.30, $\kappa^d(v_1)$ is totally atomic. By Lemma 6.25, there is a proper, nonempty subset Y of X such that $\kappa^d(v_1) = \bigwedge Y$ and $v_1 = \bigvee X - Y$. Now we are done by applying Lemma 7.1.

2. Finite Intervals

In this section we will describe all the finite intervals in a free lattice. We begin with a technical result which is of interest in itself.

THEOREM 7.3. *Any element of* $\mathbf{FL}(X)$ *other than* 1 *has at most one meet irreducible lower cover.*

Proof: The upper cover x^* of a generator has its other lower cover $x^* \wedge \overline{x}$ meet reducible, so we are really concerned with the possibility of two join reducible lower covers. Let w be an element with two distinct join reducible lower covers u and v. By Theorem 3.5(1), the unique canonical joinand v_1 of w such that $v_1 \not\le u$ is completely join irreducible, and we have $v_1 < v$ and $u = \kappa(v_1)$. Similarly, the unique canonical joinand u_1 of w such that $u_1 \not\le v$ is completely join irreducible, $u_1 < u$ and $v = \kappa(u_1)$. Since $w = u^* = v^*$, two applications of the dual of Theorem 3.34 tell us that u_1 is a canonical joinand of u and v_1 is a canonical joinand of v. Now by Theorem 3.5(2), u has a lower cover t such that $u_1 \not\le t$, *viz.,* $t = u \wedge \kappa(u_1) = u \wedge v$. Symmetrically, $v \succ u \wedge v = t$. Now u_1 is the unique canonical joinand of u such that $u_1 \not\le t$. By Theorem 6.24, u_1 is totally atomic. Similarly, v_1 is totally atomic. Since u_1 is totally atomic and the element $\kappa(u_1) = v$ has a lower cover, it follows from Theorem 6.25 that the totally atomic canonical joinand v_1 of $\kappa(u_1)$ is a generator. Similarly, u_1 is a generator and the elements u and v are both coatoms. We get $w = 1$.

THEOREM 7.4. *Let* w/u *be a finite interval of* $\mathbf{FL}(X)$ *not contained in the connected component of either* 0 *or* 1*. Then* w/u *is a chain of at most three elements.*

Proof: By Theorem 7.2, w/u is a lattice of height at most 2. Since $\mathbf{FL}(X)$ is semidistributive, the five-element lattice of height 2, $\mathbf{M}_3$, is not a sublattice of $\mathbf{FL}(X)$. Hence to prove the theorem it remains only to derive a contradiction from the assumption that w/u is a four-element interval, $w/u = \{u, v, t, w\}$, $u \prec v \prec w$ and $u \prec t \prec w$. By our description of the connected component of a generator in Example 3.43, none of the four elements is a generator, and without loss of generality we can assume that v is meet reducible. By the dual of Theorem 7.3, t is join reducible. Clearly, $t = \kappa(v)$. Let v_1 be the canonical meetand of v not above w. It is easy to see that $\kappa^d(v_1) < t$ and $\kappa^d(v_1) \vee u = t$. Because the lower cover of $\kappa^d(v_1)$ lies below u, it follows that $\kappa^d(v_1)$ is a canonical joinand of t. In fact, $\kappa^d(v_1)$ is the only canonical joinand of t not below u. By Theorem 6.24, $\kappa^d(v_1)$ is totally atomic. Similarly, v_1 is totally atomic. By the dual of Theorem 6.25 we get $v_1 \in X^\vee$. So there exists a nonempty proper subset Y of X such that $v_1 = \bigvee Y$ and $\kappa^d(v_1) = \bigwedge(X - Y)$. Now the Theorem follows from Lemma 7.1.

It follows from this theorem that the finite intervals of free lattices are the finite intervals that occur in the connected component of 0 or 1, since the three element chain is an interval in these components. Looking at the intervals that do occur, we obtain the following corollary.

THEOREM 7.5. *If* $\mathbf{L}$ *is a finite interval in a free lattice, then either* $\mathbf{L}$ *is the one element lattice or it is isomorphic to one of the lattices of* Figure 7.3 *or the dual of one of these lattices. Moreover, we can say the following about each of these intervals.*

(1) *The first lattice is the connected component of* 0 *in* $\mathbf{FL}(3)$ *and does not occur as an interval in any other free lattice. Nor does it occur anywhere else in* $\mathbf{FL}(3)$.
(2) *The second lattice only occurs as an interval in the connected component of* 0 *or of* 1 *in* $\mathbf{FL}(3)$.
(3) *The third lattice occurs as an interval in* $\mathbf{FL}(n)$ *for each* $n \geq 3$, *but its least element must be the* 0 *of* $\mathbf{FL}(n)$.
(4) *The fourth lattice occurs as an interval in* $\mathbf{FL}(n)$, *for* $n \geq 3$, *only when it is a subinterval of an interval isomorphic to the third lattice or its dual.*
(5) *The three element chain occurs as an interval infinitely often in each* $\mathbf{FL}(n)$, *with* $n \geq 3$, *but there are infinite intervals in* $\mathbf{FL}(n)$ *which do not contain it as a subinterval.*
(6) *The two element chain is a subinterval of every nontrivial interval.*

Proof: All parts of this theorem, except (5) and (6), follow from Theorem 7.4 and the description of the connected components of 0 given in Examples 3.44 and 3.45. Of course (6) is Day's Theorem. Three element intervals will be studied more thoroughly in the next section. There it will be shown that finitely generated free lattices contain infinitely many such intervals, see Corollary 7.14. By Theorem 6.24, if an interval a/b of a free lattice contains a three element subinterval, then (depending on whether the middle element is join or meet irreducible) there is a join irreducible totally atomic element q with $q \leq a$ and $b \leq \kappa(q)$, or the dual situation holds. If a/b and c/d are three

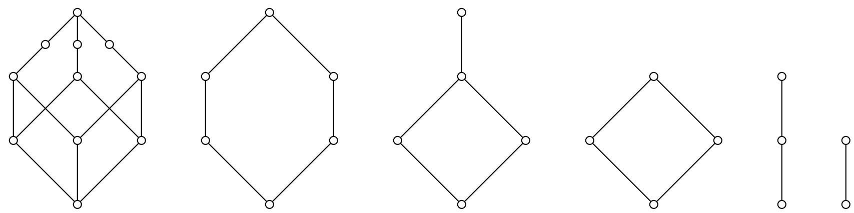


FIGURE 7.3. Finite Intervals

element intervals with $c \leq b$ and the middle elements of both are join reducible, then the totally atomic element associated with a/b must be distinct from the one associated with c/d. Using this and the fact that there are only finitely many totally atomic elements in any free lattice (Corollary 6.11), it is not hard to construct an infinite interval with no three element subinterval. We leave this to the reader. In fact more is true: *every infinite interval in a free lattice contains an infinite subinterval with no three element covering chain.* The proof is easier if you use Theorem 9.12, which shows that every infinite interval of a free lattice contains an element with no upper and no lower cover.

The next theorem gives a useful description of the intervals w/u when $u \prec v \prec w$ for some v, even when w/u is infinite. It also gives a test to decide if w/u is a three element interval.

THEOREM 7.6. *Let u, v, w be three elements of $\mathbf{FL}(X)$ such that $u \prec v \prec w$.*

(1) *Either $w/u = \{u, v, w\}$ or there exist two elements r, s such that $u \prec r \leq s \prec w$ and the interval w/u is the disjoint union $\{u, v, w\} \cup (s/r)$. Moreover, in the latter case, s/r is an infinite interval unless v is an atom or a coatom, in which case $r = s$.*

Suppose v is join irreducible, and let v_1 be the unique canonical meetand of v not above w.

(2) *We have $w/u = \{u, v, w\}$ if and only if $\kappa^d(v_1) \nleq \kappa(v)$.*
(3) *If $w/u \neq \{u, v, w\}$, then $s = w \wedge \kappa(v)$ and $r = u \vee \kappa^d(v_1)$.*

Proof: Suppose $w/u \neq \{u, v, w\}$. We may assume by duality that v is join irreducible. By the dual of Theorem 6.24, the unique canonical meetand v_1 of v such that $v_1 \ngeq w$ is totally atomic. (If $v \in X$, then $v_1 = v$.) Put $s = w \wedge \kappa(v)$ and $r = u \vee \kappa^d(v_1)$. It is sufficient to prove that if t is an element such that $u < t < w$ and $t \neq v$, then $r \leq t \leq s$.

If t is as given, then $t \not\geq v$ since $v \prec w$. Combined with $t > u = v_*$, this means that $t \leq \kappa(v)$. But also $t < w$, and we get $t \leq s$. On the other hand, since $v = w \wedge v_1$ and $t \nleq v$, we have $t \nleq v_1$; but $t < w \leq v_1^*$, so that $t \geq \kappa^d(v_1)$ and then clearly $t \geq r$. Note that this implies $\kappa^d(v_1) \leq t \leq \kappa(v)$. If s/r is finite, then it follows from what we have just proved that w/u is finite. It now follows from Theorem 7.5 that w/u must be isomorphic to $\mathbf{2} \times \mathbf{2}$ and that v is either an atom or a coatom.

This proves statements (1) and (3) and one direction of (2). To see the other direction, suppose $w/u = \{u, v, w\}$. Then $w \wedge \kappa(v) = u$. Now it is easy to see that $\kappa^d(v_1) \leq w$, so that if $\kappa^d(v_1) \leq \kappa(v)$ then $\kappa^d(v_1) \leq w \wedge \kappa(v) = u \leq v_1$, a contradiction. Thus $\kappa^d(v_1) \nleq \kappa(v)$, as desired.

In Chapter VIII we will also need the following consequence of the previous theorem and Theorem 6.31.

Corollary 7.7. *Suppose q is a completely meet irreducible element of $\mathbf{FL}(X)$ and $q^* \succ q \succ p$. Let q_1 be the canonical joinand of q not below p, and suppose $\kappa^d(q) \leq \kappa(q_1)$. If q is not a coatom, then the only completely meet irreducible element of q^*/p is q.*

Proof: By Theorem 7.6,

$$q^*/p = \{q^*, q, p\} \cup \left(q^* \wedge \kappa(q_1)/p \ \sqcup \ \kappa^d(q)\right).$$

The statement then follows from Theorem 6.31 since neither p nor q^* is completely meet irreducible by Theorem 6.27.

Lemma 7.22 below will give additional information about the intervals described in Theorem 7.6. It will show that, in the case $w/u \neq \{u, v, w\}$, the element r is join reducible, has no upper cover, and has no lower cover except w. The dual statements hold for s.

3. Three Element Intervals

In this section we characterize three element intervals in free lattices. Although this characterization is somewhat technical, it will turn out to be quite useful. We have divided this characterization into two parts. Theorem 7.9 gives a method for constructing three element intervals. Theorem 7.10 shows that every three element interval can be constructed in this way (or dually). It also gives some further properties of these intervals. At the end of the section we give several examples of three element intervals.

Let us start with a lemma which will prove to be useful in both directions.

Lemma 7.8. *If a canonical joinand of a meet irreducible totally atomic element $q \in \mathbf{FL}(X)$ is below $\kappa^d(q)$, then $q < \kappa^d(q) = q^*$.*

Proof: This follows directly from the dual of Corollary 6.12.

Our method for constructing three element intervals is given by the next result.

Theorem 7.9. *Let q be a meet irreducible totally atomic element of $\mathbf{FL}(X)$, $q = q_1 \vee \cdots \vee q_m$ canonically. Let $p_1, \dots, p_k$ be a finite collection ($k \geq 1$) of completely join irreducible elements satisfying the following three conditions*:

(1) $\kappa^d(q) \vee p_1 \vee \cdots \vee p_k$ *is in canonical form,*
(2) $q_2 \vee \cdots \vee q_m \leq p_1 \vee \cdots \vee p_k < q$,
(3) *if* $k = 1$, *then* $q \wedge (\kappa^d(q) \vee p_1) > p_1$.

Then the element

$$w = q \wedge (\kappa^d(q) \vee p_1 \vee \cdots \vee p_k)$$

is in canonical form, is completely join irreducible, and is the middle element of the three element interval

$$\kappa^d(q) \vee p_1 \vee \cdots \vee p_k / w_*.$$

Proof: It follows from (1) and (2) that $q \not\leq \kappa^d(q)$, so we can use Lemma 7.8 to see that $q_i \not\leq \kappa^d(q)$ for all i. Now by the dual of Theorem 3.34 it follows that the canonical expression for q^* is $q^* = \kappa^d(q) \vee q_1 \vee \cdots \vee q_m$.

If $q \leq \kappa^d(q) \vee p_1 \vee \cdots \vee p_k$, then $\kappa^d(q) \vee p_1 \vee \cdots \vee p_k = q^*$ by (2) and so, because $q^* = \kappa^d(q) \vee q_1 \vee \cdots \vee q_m$ canonically, $q_1 \leq p_i$ for some i. Using (2), this would imply $q \leq p_1 \vee \cdots \vee p_k < q$, a contradiction. Hence q is not below $\kappa^d(q) \vee p_1 \vee \cdots \vee p_k$. As the opposite inequality clearly fails, we see that the two elements are incomparable. In order to show that the expression for w is canonical, it remains by Theorem 1.18 to observe that $w \not\leq \kappa^d(q)$ (this follows from $q_i \not\leq \kappa^d(q)$), $w \not\leq p_i$ (because $p_1 \vee \cdots \vee p_k \leq w$ and w is meet reducible, so $p_1 \vee \cdots \vee p_k = w$ would imply $k = 1$ and $w = p_1$, contrary to (3)), and $w \not\leq q_j$ for all j if $m > 1$, i.e., if $q \notin X$ (because for $j > 1$ we have $q_j \leq p_1 \vee \cdots \vee p_k < w$, and $w \leq q_1$ would imply $q_2 < q_1$).

Since every element of $\mathrm{J}(w) - \{w\} = \bigcup \mathrm{J}(q_i) \cup \bigcup \mathrm{J}(p_j) \cup \mathrm{J}(\kappa^d(q))$ has a lower cover, in order to prove that w is completely join irreducible it is sufficient by Theorem 3.30 to show that $w \not\leq \bigvee \mathrm{K}(w)$. Suppose that $w \leq \bigvee \mathrm{K}(w)$, i.e., $w \leq t_1 \vee \cdots \vee t_r$ for some elements $t_i \in \mathrm{J}(w)$ such that $w \not\leq w_\dagger \vee t_i$. Clearly, $w_\dagger \geq p_1 \vee \cdots \vee p_k$. Since $\kappa^d(q)$ belongs to $\mathrm{J}(w)$ and $\kappa^d(q)_* = q \wedge \kappa^d(q) \leq w$, we have $\kappa^d(q)_\dagger \leq w_\dagger$. For each i we have $w_\dagger \vee t_i \geq w_\dagger \geq p_1 \vee \cdots \vee p_k$ and $w_\dagger \vee t_i \not\geq w$, so that $w_\dagger \vee t_i \not\geq \kappa^d(q)$, which can be also written as $\kappa^d(q)_\dagger \vee w_\dagger \vee t_i \not\geq \kappa^d(q)$. By Lemma 3.29 this implies $\kappa^d(q)_* \vee w_\dagger \vee t_i \not\geq \kappa^d(q)$, i.e., $\kappa^d(q)_* \vee w_\dagger \vee t_i \leq q$, and we get $t_i \leq q$. Hence $t_1 \vee \cdots \vee t_r \leq q$, which makes $\kappa^d(q) \not\leq t_1 \vee \cdots \vee t_r$. Applying (W) to the inequality

$$w = q \wedge (\kappa^d(q) \vee p_1 \vee \cdots \vee p_k) \leq t_1 \vee \cdots \vee t_r$$

we obtain $q = t_1 \vee \cdots \vee t_r$. Since q is canonically $q_1 \vee \cdots \vee q_m$, it follows that $q_1 \leq t_i$ for some i. But then $w \leq q = q_1 \vee \cdots \vee q_m \leq q_1 \vee p_1 \vee \cdots \vee p_k \leq t_i \vee w_\dagger$, a contradiction.

We have proved that w is completely join irreducible. By the dual of Theorem 3.5, the element $\kappa^d(q) \vee w = \kappa^d(q) \vee p_1 \vee \cdots \vee p_k$ is an upper cover of w. If the interval $\kappa^d(q) \vee p_1 \vee \cdots \vee p_k / w_*$ is not a three element chain, then $\kappa^d(q) \leq \kappa(w)$ by Theorem 7.6 and thus $w \leq \kappa^d(q) \vee p_1 \vee \cdots \vee p_k \leq \kappa^d(q) \vee w_* \leq \kappa(w)$, a contradiction. Hence the interval is a three element chain.

Now let us show that every three element interval arises in the manner just described, or dually.

THEOREM 7.10. *Let a join irreducible element w be the middle element of a three element interval $w_* \prec w \prec u$ in $\mathbf{FL}(X)$. Denote by q the canonical meetand of w not above u. Let $q = q_1 \vee \cdots \vee q_m$ canonically with $q_1 \not\leq w$, and let $p_1, \ldots, p_k$ be the canonical joinands of u that lie below w. Then q is totally atomic, $p_1, \ldots, p_k$ are completely join irreducible, and the three conditions* (1), (2) *and* (3) *of* Theorem *7.9 are satisfied. Moreover, the following statements hold.*

(4) $\{q_2, \ldots, q_m\} \ll \{p_1, \ldots, p_k\}$.
(5) $w = q \wedge u = q \wedge (\kappa^d(q) \vee p_1 \vee \cdots \vee p_k)$ *canonically.*
(6) $\kappa(w) < q$.

Proof: The situation is diagrammed in Figure 7.4. The element q is totally atomic by the dual of Theorem 6.24. Clearly, $\kappa^d(q) \leq u$. Since $\kappa^d(q) \not\leq w$ and the interval u/w_* has only three elements, we have $\kappa^d(q) \vee w_* = u$. The canonical join representation of u refines this, so $u = \kappa^d(q) \vee p_1 \vee \cdots \vee p_k$ and hence $w = q \wedge u = q \wedge (\kappa^d(q) \vee p_1 \vee \cdots \vee p_k)$. Since $\kappa^d(q)_* \vee p_1 \vee \cdots \vee p_k \leq w_* < u$, we see, by Theorem 1.20, that $\kappa^d(q)$ is a canonical joinand of u, and it follows that (1) holds.

Let us apply (W) to

$$w = q \wedge w_2 \wedge \cdots \wedge w_n \leq \kappa^d(q) \vee w_* = u,$$

where $q \wedge w_2 \wedge \cdots \wedge w_n$ is the canonical expression for w. Note $w \notin X$, so $q \neq w$ and hence $q \not\leq u$. If $w \leq \kappa^d(q)$, then $u = \kappa^d(q)$, which is impossible because u is join reducible by the dual of Corollary 6.28. The only remaining possibility is $w_i \leq u$ for some i. Since $w_2 \wedge \cdots \wedge w_n \geq u$,

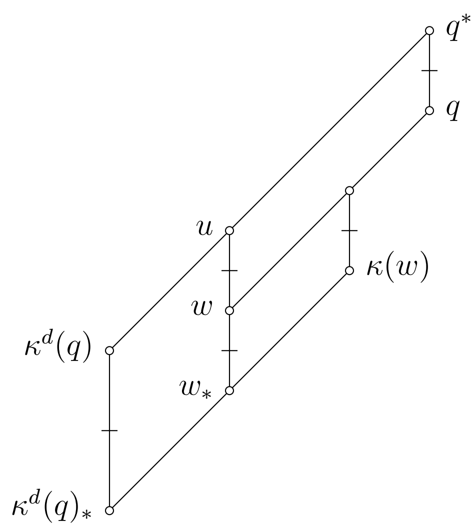


FIGURE 7.4

we get $n = 2$ and $w_2 = u$. Thus u is a canonical meetand of w and so the elements $p_1, \dots, p_k$ are completely join irreducible, because they belong to $\mathrm{J}(w)$ and w is completely join irreducible.

By Theorem 3.4, the elements $q_2, \dots, q_m$ are below w_*. By Lemma 7.8, $q_i \not\leq \kappa^d(q)$ for all i, and thus $q^* = \kappa^d(q) \vee q_1 \vee \cdots \vee q_m$ canonically by the dual of Theorem 3.34. Since $w_* \vee q_1 = q$, we have

$$q^* = u \vee q = u \vee q_1 = \kappa^d(q) \vee p_1 \vee \cdots \vee p_k \vee q_1$$

and consequently each canonical joinand of q^* is below one of the elements $\kappa^d(q),\ p_1, \dots, p_n, q_1$. Thus $\{q_2, \dots, q_m\} \ll \{p_1, \dots, p_k\}$ so that (4) holds and, in particular, $q_2 \vee \cdots \vee q_m \leq p_1 \vee \cdots \vee p_k$. The rest of condition (2) follows from $p_1 \vee \cdots \vee p_k \leq w_* < q$.

If $k = 1$, then $w = q \wedge (\kappa^d(q) \vee p_1) > w_* \geq p_1$, showing that (3) holds.

It remains to show that $\kappa(w) < q$. Note that $q \geq w$ implies $q \neq \kappa(w)$. Clearly, $u \wedge \kappa(w) = w_*$. If q is a generator, then $u \wedge \kappa(w) \leq q$ implies $\kappa(w) \leq q$ as desired. So let $m \geq 2$. Let us apply (W) to the inequality

$$w_* = u \wedge \kappa(w) \leq q_1 \vee \cdots \vee q_m = q.$$

Now $u \nleq q$ and $\kappa(w) \leq q$ is the conclusion we want. If $w_* \leq q_1$, then $q_2 \leq w_* \leq q_1$, a contradiction. If $w_* \leq q_i$ for some $i > 1$, then $w_* = q_i$; but in that case q cannot be a canonical meetand of w_*, and so by Theorem 3.34 we get $q \geq \kappa(w)$.

Example 7.11. Let q be a meet irreducible totally atomic element of $\mathbf{FL}(X)$ such that q is not a generator and q is not the join of two atoms. Let $q = q_1 \vee \cdots \vee q_m$ canonically. It is not difficult to show that with the choice $\{p_1, \ldots, p_k\} = \{q_2, \ldots, q_m\}$, the conditions of Theorem 7.9 are always satisfied. Consequently, the element

$$q \wedge (\kappa^d(q) \vee q_2 \vee \cdots \vee q_m)$$

is the middle element of a three element interval in $\mathbf{FL}(X)$.

Example 7.12. A second interesting case arises when we choose $q = x \in X$ and $k = 1$. Then we are looking for a completely join irreducible element $p < x$ satisfying (1),(2) and (3) to obtain a middle element $w = x \wedge (\underline{x} \vee p)$ of a three element interval in $\mathbf{FL}(X)$. Condition (2) is empty and it is not hard to show that (3) will be satisfied if and only if x is not a canonical meetand of p.

We shall construct an infinite sequence of such p's in $\mathbf{FL}(x, y, z, t)$. Let

$$\begin{aligned}
&a_0 = z, \quad b_0 = t, \\
&a_{i+1} = z \wedge (b_i \vee (y \wedge z)), \quad b_{i+1} = t \wedge (a_i \vee (y \wedge t)), \\
&c_i = (a_i \vee (y \wedge t)) \wedge (b_i \vee (y \wedge z)), \\
&p_i = \mu_x c_i
\end{aligned}$$

where μ_x is the endomorphism of $\mathbf{FL}(x, y, z, t)$ sending each generator to its meet with x. An inductive argument shows that each element $\underline{x} \vee p_i = (y \wedge z \wedge t) \vee p_i$ is in canonical form and the algorithm described in Theorem 3.32 can be used to prove that the elements p_i are completely join irreducible.

We see that $\mathbf{FL}(4)$, and similarly $\mathbf{FL}(X)$ for any finite set X of at least four elements, contains infinitely many three element intervals.

Example 7.13. To see that $\mathbf{FL}(x, y, z)$ has infinitely many three element intervals, again let $q = x$ and define $a_0 = x \wedge y$, $b_0 = x \wedge z$ and

$$a_{i+1} = \begin{cases} x \wedge (z \vee a_i), & i \text{ even}, \\ x \wedge (y \vee a_i), & i \text{ odd}, \end{cases} \qquad b_{i+1} = \begin{cases} x \wedge (y \vee b_i), & i \text{ even}, \\ x \wedge (z \vee b_i), & i \text{ odd}. \end{cases}$$

As we have seen in Example 3.40, the (slim) elements a_i and b_i are all completely join irreducible. It is also easy to check that for any i, $(y \wedge z) \vee a_i \vee b_i$ is in canonical form. This verifies condition (1), while conditions (2) and (3) are both empty. By Theorem 7.9, for any i the element

$$x \wedge ((y \wedge z) \vee a_i \vee b_i)$$

is the middle of a three element interval in $\mathbf{FL}(x, y, z)$.

Corollary 7.14. *For any finite set X of at least three elements, the free lattice $\mathbf{FL}(X)$ contains infinitely many three element intervals.*

4. Connected Components

As we saw in Chapter III, the connected component of a generator x_i in $\mathbf{FL}(x_1, \ldots, x_n)$ with $3 \leq n < \omega$ is the pentagon diagrammed in Figure 7.5. Notice that each of the covers of the pentagon is actually a cover in $\mathbf{FL}(X)$, except that $x_i^* \wedge \overline{x}_i$ does not cover $x_{i*} \vee \underline{x}_i$, where again $\overline{x}_i = \bigvee_{j \neq i} x_j$ and $\underline{x}_i = \bigwedge_{j \neq i} x_j$. We indicate this in Figure 7.5 with a dotted line.

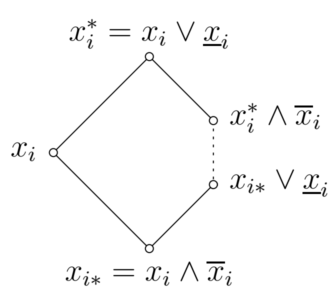


Figure 7.5

This example shows that, in describing connected components, we should not only indicate the order among the elements, but also which covers in the induced partial order of the connected component are actually covers in the free lattice. We can formalize this by using the notion of a *labeled ordered set*, by which we mean an ordered set, each covering pair of which is labeled with either a 1 (indicating a covering in the free lattice) or a 0 (indicating a noncover in the free lattice).[2] In diagrams, a covering labeled 0 will be indicated with a dotted line, while a covering labeled 1 is indicated with a solid line.

By an interpretation of a labeled ordered set P in $\mathbf{FL}(X)$ we shall mean an order isomorphism f of P onto a connected component of $\mathbf{FL}(X)$ such that for any pair a, b of elements of P, $f(a)$ is covered by $f(b)$ in $\mathbf{FL}(X)$ if and only if a, b is a covering pair labeled 1 in P. The aim of this section is to describe all the labeled ordered sets that have an interpretation in $\mathbf{FL}(X)$ for some finite set X.

The pentagon, with all covers labeled 1 except for the atom-to-coatom cover (as in Figure 7.5) will be denoted by $\overline{\mathrm{N}}_5$.

For a positive integer k, we let $\overline{\mathrm{N}}_5(k)$ denote the labeled ordered set consisting of k copies of $\overline{\mathrm{N}}_5$, with all the least elements identified and also all the middle elements identified. (For this purpose, the middle element of a pentagon is the only element which is both an atom and a coatom.) For example, $\overline{\mathrm{N}}_5(3)$ is pictured in Figure 7.6.

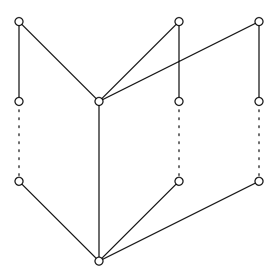


FIGURE 7.6. $\overline{\mathrm{N}}_5(3)$

Next let us give several examples of labeled ordered sets which are connected components in free lattices and elements that realize them. Then we shall prove that every connected component is isomorphic, or dually isomorphic, to one of these. These examples were found with the aid of our programs described in Chapter XI.

[2]A better term might be a *covers labeled ordered set* since combinatorists use the term labeled ordered set to mean an ordered set whose elements are labeled.

EXAMPLE 7.15. For each positive integer k there exists a finite set X such that $\overline{\mathrm{N}}_5(k)$ has an interpretation in $\mathbf{FL}(X)$. If $X = \{x_1, \dots, x_n\}$ and $1 \leq k \leq n-2$, then the connected component of the element $a = x_1 \wedge \cdots \wedge x_k$ is isomorphic to $\overline{\mathrm{N}}_5(k)$. If we let $\overline{a} = x_{k+1} \vee \cdots \vee x_n$ and $\underline{x}_i = \bigwedge_{j \neq i} x_j$, then the connected component of a consists of k pentagons:

$$a \wedge \overline{a} \prec a \prec a \vee \underline{x}_i,$$
$$a \wedge \overline{a} \prec (a \wedge \overline{a}) \vee \underline{x}_i < (a \vee \underline{x}_i) \wedge \overline{a} \prec a \vee \underline{x}_i$$

where i ranges over $1, \dots, k$. The proof is a straightforward application of the techniques developed in Chapter III. This proof, as well as those of the other examples below, also follows from the lemmas we shall prove below.

For any pair of nonnegative integers m and k, we shall let $\mathrm{C}(m, k)$ denote the ordered set consisting of m chains of length 2 and of k chains of length 1, with their greatest elements identified. The ordered set is considered to be a labeled ordered set with all labels set to 1. For example, $\mathrm{C}(2, 1)$ is pictured in Figure 7.7. We take $\mathrm{C}(0, 0)$ to be the one element ordered set.

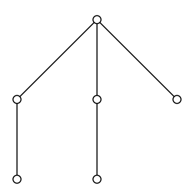


FIGURE 7.7. $\mathrm{C}(2, 1)$

EXAMPLE 7.16. As we have seen in Example 3.39, the element

$$((x \wedge (y \vee z)) \vee (y \wedge z)) \wedge ((y \wedge (x \vee z)) \vee (x \wedge z))$$

has no lower and no upper cover in $\mathbf{FL}(x, y, z)$. Thus the connected component of this element is the one-element set $\mathrm{C}(0, 0)$.

EXAMPLE 7.17. Let $m \geq 3$. If $X = \{x_1, \dots, x_n\}$ where $n \geq m$ and $n \geq 4$, then the connected component of the join $a = \underline{x}_1 \vee \cdots \vee \underline{x}_m$ of m distinct atoms $\underline{x}_i = \bigwedge_{j \neq i} x_j$ is isomorphic to $\mathrm{C}(m, 0)$. The connected component is the union of m chains

$$a \wedge \bigvee{}_{j \leq m}^{j \neq i} (x_i \wedge x_j) \prec a \wedge x_i \prec a$$

where i runs over $1, \dots, m$. This is illustrated for $m = 3$ and $n = 4$ in Figure 3.5 on page 117.

EXAMPLE 7.18. Let $m \geq 1$ and $k \geq 1$. In order to represent $\mathrm{C}(m,k)$, let $X = \{x_1, \dots, x_r, y_1, \dots, y_k\}$ where $r \geq m$ and $r \geq 4$. Put $\hat{x}_i = \bigwedge_{l \neq i} x_l$, $u_i = \bigvee_{l \neq i} x_l$, $v_j = \bigvee_{l \neq j} y_l$, $u = x_1 \vee \cdots \vee x_r$ and $v = y_1 \vee \cdots \vee y_k$. The connected component C of the element $a = \hat{x}_1 \vee \cdots \vee \hat{x}_m \vee y_1 \cdots \vee y_k$ is isomorphic to $\mathrm{C}(m,k)$. If $m = k = 1$, C is the union of the chains

$$
\begin{aligned}
&a \wedge (y_1 \vee ((x_1 \vee y_1) \wedge u \wedge (u_1 \vee v))) \prec a \wedge (x_1 \vee y_1) \prec a,\\
&a \wedge u \prec a.
\end{aligned}
$$

In all other cases, C is the union of the m chains

$$a\wedge[\bigvee_{p=1}^{k}((x_i\vee v)\wedge(u\vee v_p)\wedge(u_i\vee v))\vee\bigvee{}_{q\leq m}^{bq\neq i}((x_i\vee v)\wedge(x_q\vee v))] \prec a\wedge(x_i\vee v) \prec a$$

for $i = 1, \dots, m$ and of the k chains

$$a \wedge (u \vee v_j) \prec a$$

for $j = 1, \dots, k$.

EXAMPLE 7.19. By Theorem 3.30 and the dual of Corollary 3.7, $(x \vee (y \wedge z)) \wedge (y \vee (x \wedge z))$ is completely join irreducible but has no upper cover. Moreover, its lower cover has no lower cover of its own. Thus $\mathrm{C}(0,1)$, which is the two element chain, can be interpreted in $\mathbf{FL}(x,y,z)$ by

$$(x\vee(y\wedge z))\wedge(y\vee(x\wedge z))\wedge(z\vee((x\wedge(y\vee z))\vee(y\wedge(x\vee z)))) \prec (x\vee(y\wedge z))\wedge(y\vee(x\wedge z)).$$

EXAMPLE 7.20. $\mathrm{C}(0,2)$ can be interpreted in $\mathbf{FL}(x,y,z,u,v,w)$ by the connected component of the element

$$a = ((x \vee (y \wedge z)) \wedge (y \vee (x \wedge z))) \vee ((u \vee (v \wedge w)) \wedge (v \vee (u \wedge w))).$$

The two lower covers of a are

$$
\begin{aligned}
&a \wedge [z \vee ((y\mathbf{w}_z\mathbf{w}_u\mathbf{w}_v\mathbf{w}_w) \wedge (x\mathbf{w}_u\mathbf{w}_v\mathbf{w}_w)) \vee ((x\mathbf{w}_z\mathbf{w}_u\mathbf{w}_v\mathbf{w}_w) \wedge (y\mathbf{w}_u\mathbf{w}_v\mathbf{w}_w))],\\
&a \wedge [w \vee ((x\mathbf{w}_y\mathbf{w}_z\mathbf{w}_v\mathbf{w}_w) \wedge (x\mathbf{w}_y\mathbf{w}_z\mathbf{w}_u)) \vee ((x\mathbf{w}_y\mathbf{w}_z\mathbf{w}_u\mathbf{w}_w) \wedge (x\mathbf{w}_y\mathbf{w}_z\mathbf{w}_v))].
\end{aligned}
$$

EXAMPLE 7.21. For $\mathrm{C}(0,k)$ with $k \geq 3$, let $X = \{x_1, \dots, x_k, y\}$ and put $\hat{x}_i = \bigwedge_{j \neq i} x_j$. The connected component of the element $a = \hat{x}_1 \vee \cdots \vee \hat{x}_k$ is isomorphic to $\mathrm{C}(0,k)$. The lower covers of a are the elements $a \wedge (x_i \vee y)$ for $i = 1, \dots, k$.

Now that we have obtained a number of examples of interpretable labeled ordered sets, we shall prove that these and their duals are the only ones.

Lemma 7.22. *Let a, b, c, d, e be five distinct elements of* $\mathbf{FL}(X)$ *such that $d \prec b \prec e$, and $d \prec a < c \prec e$ (so that $d = b \wedge c$ and $e = a \vee b$), and such that the connected component of these five elements contains neither 0 nor 1. Then a is a proper join, has no upper cover, and has no lower cover except d. Similarly, c is a proper meet, has no lower cover, and has no upper cover except e.*

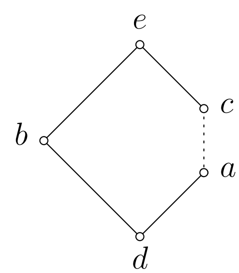


Figure 7.8

Proof: The situation is diagrammed in Figure 7.8. It is easy to see that the theorem is true for the connected component of a generator, so we shall suppose that these elements are not in such a connected component.

Suppose that there is a lower cover f of a such that $f \neq d$. Since b is not in the connected component of either 0 or 1, it follows from Theorem 7.2 that d has no lower cover. In particular, $f \wedge d$ is not a lower cover of d and thus there is an element u with $f \wedge d < u < d$. Then $u \vee f = a > b \wedge c$, violating (W). Thus a has no lower cover except d and, similarly, c has no upper cover except e.

Suppose that a is a proper meet. Then a is completely join irreducible with the lower cover $a_* = d$. Let a_1 be a canonical meetand of a such that $a_1 \not\geq c$, and let d_1 be the canonical meetand of d such that $d_1 \not\geq b$. Clearly, $d_1 \geq c$ as the canonical meet representation of d upper refines $\{b, c\}$. We have $b \wedge d_1 = d = a_* < a_1$. If $b \leq a_1$, then $c \leq a \vee b \leq a_1$, a contradiction. If $d_1 \leq a_1$, then $c \leq d_1 \leq a_1$, a contradiction again. Thus neither b nor d_1 is below a_1. In particular, a_1 is not a generator (as generators are meet prime). Let

$a_1 = a_{11} \vee a_{12} \vee \cdots \vee a_{1k}$ canonically. By Theorem 3.4 we may assume that $a_{12} \vee \cdots \vee a_{1k} \leq a_* = d$. Let us apply (W) to

$$a_* = d = b \wedge d_1 \leq a_1 = a_{11} \vee a_{12} \vee \cdots \vee a_{1k}.$$

We know already that $b \not\leq a_1$ and $d_1 \not\leq a_1$. If $d \leq a_{11}$, then $a_{12} \leq a_{11}$, a contradiction. So we must have $d \leq a_{1j}$ for some $j \geq 2$, and consequently $k = 2$ and $d = a_{12}$. By Theorem 3.26, the element $d = a_{12} \in \mathrm{J}(a)$ has a lower cover and we have a covering chain $d_* \prec d \prec b \prec e$, contradicting Theorem 7.2.

Thus a is not a proper meet. Since a is not a generator, it is a proper join. By duality, c is a proper meet.

By duality we may assume that b is a proper meet. Hence, b is completely join irreducible with the lower cover $b_* = d$. It remains to prove that c has no lower cover and a has no upper cover.

Suppose that c has a lower cover. Then c is completely join irreducible, and $c > a$ implies $c_* \geq a$. If there is an element v with $c_* < v < e$ and $v \neq c$, then $e = b \vee v = c \vee v$ (because b and c are lower covers of e) and hence $e = (b \wedge c) \vee v = v$ by semidistributivity, a contradiction. Thus the interval e/c_* is a three element chain.

By the dual of Theorem 6.24, the unique canonical meetand q of c such that $q \not\geq e$ is totally atomic. We have $q^* = q \vee e = q \vee a \vee b = q \vee b$ while $b_* = d \leq q$, so that $\kappa^d(q) = b$. The element q is not a generator, as b is not an atom by our assumption that b does not belong to the connected component of 0.

It follows easily from $e = b \vee a$, $b_* < a$ and $b \prec e$ that the canonical expression for e is $e = b \vee e_1$ for a completely join irreducible element $e_1 \leq a$ with $e_{1*} \leq d$. By Theorem 7.10 we have $q = q_1 \vee \cdots \vee q_m$ canonically with $q_2 \vee \cdots \vee q_m \leq e_1$. If $q_i < e_1$ for some i, then $q_i \leq e_{1*} \leq d < \kappa^d(q)$, so that a canonical joinand of q is below $\kappa^d(q)$. But then, as q is totally atomic, the dual of Corollary 6.12 implies that $b = \kappa^d(q) = q^*$, a contradiction. Hence $m = 2$ and $q_2 = e_1$.

The unique canonical meetand b_1 of b not above e is totally atomic by the dual of Theorem 6.24, and it is easy to see that $b_1 = \kappa(e_1) = \kappa(q_2)$. Since q_2 is totally atomic, Lemma 6.25 shows that the element $e_1 = q_2$ is a meet of generators, $e_1 = x_1 \wedge \cdots \wedge x_k$ and $b_1 = y_1 \vee \cdots \vee y_l$ where $X = \{x_1, \dots, x_k, y_1, \dots, y_l\}$. By Theorem 3.4 we may assume that $x_2, \dots, x_k \geq q^*$ and $y_2, \dots, y_l \leq b_*$. Since $b_* < q^*$ and any two distinct generators are incomparable, we get a contradiction unless either $k = 1$ or $l = 1$.

If $k = 1$, then q is the join of two generators, so that $b = \kappa^d(q)$ is a meet of generators and b_1 is a generator; but q_2 is also a generator, which contradicts $b_1 = \kappa(q_2)$.

If $l = 1$, then q_2 is an atom, q is the join of two atoms and c belongs to the connected component of 0, another contradiction.

This proves that the element c has no lower cover. Next suppose that a has an upper cover $a^* < c$. As before, a^*/d is a three element chain. Let q be the canonical joinand of a not below d, and let r be the canonical meetand of b not above e. By Theorem 6.24, both q and r are totally atomic. Easily, $r = \kappa(q)$. It follows by Lemma 6.25 that q is a meet of generators, $q = x_1 \wedge \cdots \wedge x_m$ and $r = y_1 \vee \cdots \vee y_k$ where $X = \{x_1, \ldots, x_m, y_1, \ldots, y_k\}$. By Theorem 3.4 all but one of the generators x_i are above a, and all but one of the generators y_j are below $b_* = d$. As before, this implies either $m = 1$ or $k = 1$. If $m = 1$, then a is above all but one generator, so that a is in the connected component of 1, a contradiction. If $k = 1$, then d is below all but one generator, so that d is in the connected component of 0, again a contradiction.

Lemma 7.23. *Let a, b, c, d, e be as in* Lemma *7.22. Moreover, let f be an element such that $d \prec f$, $f \neq a$ and $f \neq b$. Then the connected component of b contains a copy of $\overline{\mathrm{N}}_5(2)$ with the elements labeled as in* Figure 7.9.

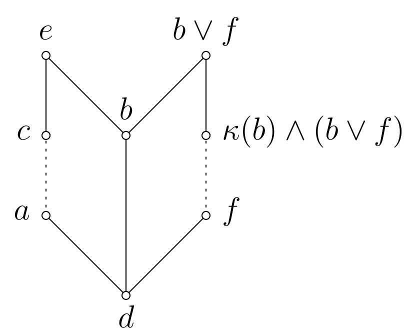


Figure 7.9

Proof: We have $d = a \wedge b = a \wedge f$, so by semidistributivity $d = a \wedge (b \vee f)$. Consequently $b \vee f \ngeq e$ and so, since $e \succ b$, we get

$e \wedge (b \vee f) = b$. We see that b is completely join irreducible and $b_* = d$. Denote by d_1 the canonical meetand of d not above f. It is clear that $d_1 \geq e$. Using the fact that $f \succ d$ and Theorem 1.20, it is not hard to show that every element above d is either above f or below d_1. From this it follows that $d_1 \wedge (b \vee f) \prec b \vee f$.

Suppose that $d_1 \wedge (b \vee f) > b$, and let b_1 be a canonical meetand of b not above $d_1 \wedge (b \vee f)$. Clearly $b_1 \ngeq f$. Since $d \prec f$, we get $b_1 \wedge f = d$ and hence $b_1 \leq d_1$. Because $f \leq \kappa(b)$ and $f \nleq b_1$, we have $b_1 \ngeq \kappa(b)$. By Theorem 3.34, this makes b_1 a canonical meetand of $b_* = d$, so that $b_1 = d_1$. However, this contradicts $b_1 \ngeq d_1 \wedge (b \vee f)$.

Thus $d_1 \wedge (b \vee f) = b$ and so $b \prec b \vee f$, since an element with $b < v < f \vee b$ would be above d, but neither above f nor below d_1. Now that we know $d \prec b \prec b \vee f$ and that $b \vee f/d$ is not a three element interval, we can use Theorem 7.6 to obtain the rest of the $\overline{\mathrm{N}}_5(2)$.

Lemma 7.24. *Let a, b, c, d, e be five elements as in* Lemma *7.22, and let g be an upper cover of b such that $g \neq e$. Put $f = d \vee \kappa^d(b_1)$, where b_1 is the canonical meetand of b not above g. Then $d \prec f$, $f \neq a$, $f \neq b$ and the connected component of b contains a configuration of* Figure 7.9 *with $g = b \vee f$.*

Proof: If g/d were a three element chain, then g would be a canonical meetand of b by Theorem 7.10. Since $d \prec b \prec e$ and $g \ngeq e$, the element g would then be totally atomic by Theorem 6.24. We would obtain a covering chain $d \prec b \prec g \prec g^*$ of length 3, a contradiction. So g/d is not a three element chain, and we can again apply Theorem 7.6 to obtain the rest of the $\overline{\mathrm{N}}_5(2)$.

Lemma 7.25. *Let C be a connected component of $\mathbf{FL}(X)$ containing neither* 0 *nor* 1*. If C contains a copy of $\overline{\mathrm{N}}_5$, then C is either $\overline{\mathrm{N}}_5(k)$ or the dual of $\overline{\mathrm{N}}_5(k)$ for some positive integer k.*

Proof: Let C contain elements a, b, c, d, e as in Lemma 7.22. By duality, it is sufficient to assume that b is a proper meet and to prove in that case that C is $\overline{\mathrm{N}}_5(k)$. By Lemma 7.23 and Lemma 7.24, there is a one-to-one correspondence between upper covers of b and the upper covers of d different from b. Since C contains no covering chain of length three, the element d has no lower cover and none of the upper covers of b has an upper cover. By the dual of Lemma 7.23, if an upper cover of b had more than two lower covers, then b would be a proper join, which is not the case. Finally, Lemma 7.22 can be used to show that the remaining elements, such as a and c, have only the one indicated upper or lower cover.

Now we turn our attention to some other potential configurations of covers.

LEMMA 7.26. *Let a, b, c, d be four elements of* $\mathbf{FL}(X)$ *such that* $a \prec b$, $c \prec d$, $a < d$, $a \nleq c$ *and* $b \nleq d$. *Then either a has a lower cover or d has an upper cover.*

Proof: We have $a \wedge c < a$ and $d < b \vee d$. If neither of these is a cover, choose elements u and v with $a \wedge c < u < a$ and $d < v < b \vee d$. Then $b \wedge v = a < d = c \vee u$, violating (W).

LEMMA 7.27. *Let C be a connected component of* $\mathbf{FL}(X)$ *containing neither* 0 *nor* 1.

(1) *There is no quadruple a, b, c, $d \in C$ with $a \prec b \prec d$, $a \prec c \prec d$ and $b \neq c$ (see* Figure 7.10(1)*).*
(2) *If there is a quadruple a, b, c, $d \in C$ with $a \prec b \prec c$, $b \prec d$ and $c \neq d$ (see* Figure 7.10(2)*), then C contains* $\overline{\mathrm{N}}_5$.
(3) *If there is a quadruple a, b, c, $d \in C$ with $a \prec b$, $c \prec b$, $c \prec d$, $a \neq c$ and $b \neq d$ (see* Figure 7.10(3)*), then C contains* $\overline{\mathrm{N}}_5$.
(4) *If there is a quadruple a, b, c, $d \in C$ with $a \prec b \prec c$, $a < d \prec c$ and $b \neq d$ (see* Figure 7.10(4)*), then C contains* $\overline{\mathrm{N}}_5$.
(5) *If there is a quintuple a, b, c, $d, e \in C$ with $a \prec b$, $d \prec c \prec b$, $d \prec e$, $a \neq c$ and $c \neq e$ (see* Figure 7.10(5)*), then C contains* $\overline{\mathrm{N}}_5$.

Proof: Note that according to Theorem 7.2, there are no chains of length 3 in C.

(1) If $e \in d/a$ and is distinct from a, b, c, and d, then, by the coverings, $e \vee b = e \vee c = d$. By semidistributivity, this implies $d = e \vee (b \wedge c) = e \vee a = e$, a contradiction. Thus the interval d/a has only four elements. But by Theorem 7.5(3) and (4), this interval only occurs in the connected component of 0 and 1.

(2) Let a, b, c, $d \in C$, $a \prec b \prec c$, $b \prec d$ and $c \neq d$. If C does not contain $\overline{\mathrm{N}}_5$, then by (1) and Theorem 7.6, c/a is a three element interval. By Theorem 7.10, c is a canonical meetand of b. But then, since $b \prec d$, we have $c \prec c \vee d$. The chain $a \prec b \prec c \prec c \vee d$ gives a contradiction.

(3) Let a, b, c, $d \in C$, $a \prec b$, $c \prec b$, $c \prec d$, $a \neq c$ and $b \neq d$. By Lemma 7.26 either c has a lower cover or b has an upper cover, which means that there is either a configuration of (2) or a dual configuration.

(4) This follows from (1) and Theorem 7.6.

(5) Let a, b, c, $d, e \in C$, $a \prec b$, $d \prec c \prec b$, $d \prec e$, $a \neq c$ and $c \neq e$. If $d \leq a$, we can use (4). If $e \leq b$, the dual of (4) applies. If neither

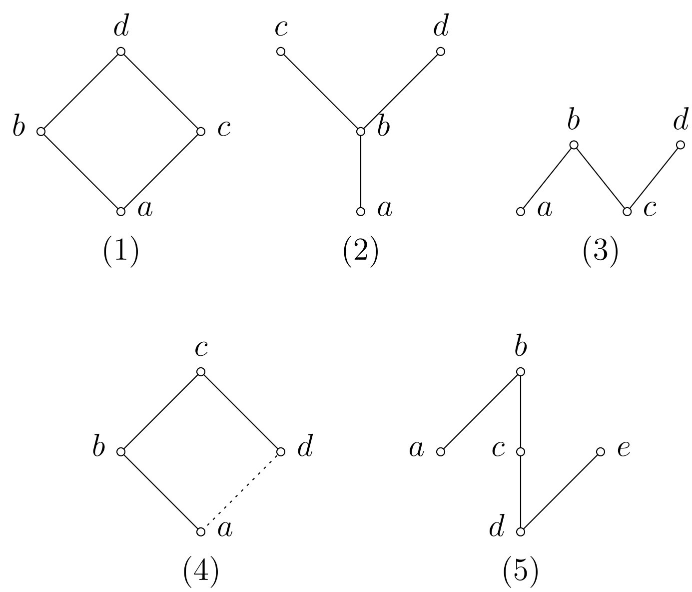


FIGURE 7.10

$d \leq a$ nor $e \leq b$, then by Lemma 7.26 either d has a lower cover or b has an upper cover. But then there is a chain of three covers in C.

LEMMA 7.28. *Let C be a connected component of $\mathbf{FL}(X)$ containing neither 0 nor 1, and not containing $\overline{\mathrm{N}}_5$. Then C is isomorphic either to $\mathrm{C}(m, k)$ or to the dual of $\mathrm{C}(m, k)$ for a pair of nonnegative integers m, k.*

Proof: Lemma 7.27 and the absence of chains of length 3 eliminate all the other possibilities.

We still have to show that C$(1, 0)$ and C$(2, 0)$ do not occur.

LEMMA 7.29. *Let $a \prec b \prec c$ be a three element interval in* $\mathbf{FL}(X)$ *with the middle element b join irreducible. Then c has a lower cover other than b.*

Proof: By Theorem 7.10, c is a canonical meetand of b, which is completely join irreducible. Thus all the canonical joinands of c are completely join irreducible, whence c is lower atomic by Corollary 3.8. Clearly c is not a generator, so it is join reducible and has a lower cover other than b.

LEMMA 7.30. C$(1, 0)$*, the three element chain, is never a connected component of* $\mathbf{FL}(X)$.

Proof: If $a \prec b \prec c$ is a connected component, then c/a is a three element interval by Theorem 7.6 and then by Lemma 7.29 and its dual, either a has an upper cover other than b, or c has a lower cover other than b.

LEMMA 7.31. C$(2, 0)$ *is never a connected component of* $\mathbf{FL}(X)$.

Proof: Suppose that there is a connected component of five distinct elements a, b, c, d, e in $\mathbf{FL}(X)$ such that $a \prec b \prec c$ and $d \prec e \prec c$. As usual, by Theorem 7.6 c/a and c/d are three element intervals. By the dual of Lemma 7.29, the elements b and e are both join irreducible; not being generators, they are proper meets. By Theorem 7.10 there are totally atomic elements q and r with $b = q \wedge c$ and $e = r \wedge c$ canonically. Moreover, according to that theorem both $\kappa^d(q)$ and $\kappa^d(r)$ are canonical joinands of c, and all the canonical joinands of c are completely join irreducible. Since c has precisely two lower covers, by Corollary 3.6 it has precisely two completely join irreducible canonical joinands, and we get $c = \kappa^d(q) \vee \kappa^d(r)$ canonically.

Let $q = q_1 \vee \cdots \vee q_m$ and $r = r_1 \vee \cdots \vee r_k$ $(m, k \geq 1)$ canonically, with $q_1 \nleq c$ and $r_1 \nleq c$. By Theorem 7.10, we have $q_2 \vee \cdots \vee q_m \leq \kappa^d(r)$; in the notation of 7.10, $q_2 \vee \cdots \vee q_m \leq p_1$ with p_1 being the canonical joinand of c which lies below b, so $p_1 = \kappa^d(r)$. As $\kappa^d(r)$ is join irreducible, this inequality is strict unless $m = 2$ and $q_2 = \kappa^d(r)$.

Suppose that $m = 2$ and $q_2 = \kappa^d(r)$. Since both q_2 and r are totally atomic, the dual of Lemma 6.25 implies r is a join of generators, $r = y_1 \vee \cdots \vee y_k$ and $q_2 = \kappa^d(r) = x_1 \wedge \cdots \wedge x_l$, where $X = \{x_1, \ldots, x_l, y_1, \ldots, y_k\}$. Since totally atomic elements must have the form of equation (1) of Chapter VI, q_1 is also a meet of l generators, all but one of which belong to $\{x_1, \ldots, x_l\}$. We can assume that $q_1 = y_1 \wedge x_2 \wedge \cdots \wedge x_l$, so that $q = \mu_{x_2 \wedge \cdots \wedge x_l}(y_1 \vee x_1)$. By Theorem 6.21,

$$\kappa^d(q) = \begin{cases} y_2 \wedge \cdots \wedge y_k & \text{if } l = 1, \\ x_2 \wedge (\underline{x}_1 \vee \underline{x}_2 \vee \underline{y}_1) & \text{if } l = 2, \\ \bigwedge_{i=2}^{l} (\underline{x}_1 \vee \underline{x}_i \vee \underline{y}_1) & \text{if } l > 2 \end{cases} \tag{5}$$

where, for any generator x, $\underline{x}$ denotes the atom not below x. If $k > 1$, then Theorem 7.10, this time applied to $\{d, e, c\}$, gives $r_2 \vee \cdots \vee r_k = y_2 \vee \cdots \vee y_k \leq \kappa^d(q)$. But, by (5), this is only possible if $l = 1$ and $k = 2$. In this case $|X| = 3$, and both q and r are coatoms. Then c is is the join of two generators and so also a coatom, and so its connected component is that of 1, which is certainly not $\mathrm{C}(2, 0)$. Hence $k = 1$. The elements q_1 and q_2 are now both atoms, so that their join q, and then also the element b (which is below q), belong to the connected component of 0. Again, the connected component of 0 is certainly not $\mathrm{C}(2, 0)$. We have obtained a contradiction.

Thus we may assume that $q_2 \vee \cdots \vee q_m < \kappa^d(r)$. Since the lower cover of $\kappa^d(r)$ is below r, this implies $q_2 \vee \cdots \vee q_m \leq r$. Similarly, $r_2 \vee \cdots \vee r_k \leq q$. Since q, r are incomparable, we get $q_1 \nleq r$ and $r_1 \nleq q$.

These last observations allow us to apply Lemma 6.9 to show that the element $p = q \wedge r$ is totally atomic. Moreover, it is easy to use Whitman's criteria for canonical form (Theorem 1.18) to see that the canonical form for p is $q \wedge r$. By the description of totally atomic elements given in Section 1 of Chapter VI, it follows that either q and r are generators, or $q_i = r_j$ for some i and j. If q and r are both generators, then $c = \kappa^d(q) \vee \kappa^d(r)$ belongs to the connected component of 0, a contradiction. If $q_i = r_j$, then clearly i, $j > 1$ and we have $q_i = r_j \leq \kappa^d(q)$. But since q is totally atomic, Corollary 6.12 then gives $\kappa^d(q) = q^*$, which is impossible because $b < q^*$ and $\kappa^d(q) < c$.

Combining the lemmas and examples from above, we can formulate the main result of this section.

THEOREM 7.32. *A labeled ordered set P is representable as a connected component of a free lattice if and only if one of the following four cases, or one of their duals, occurs*:

(1) *P is the four element Boolean algebra, which occurs only as the (unique) connected component of* $\mathbf{FL}(2)$.
(2) *P is the connected component of* 0 *in* $\mathbf{FL}(X)$ *with X finite and $|X| \geq 3$, and this representation of P is the only one. P has been described in* Example *3.44 for $|X| = 3$, and in* Example *3.45 for $|X| > 3$.*
(3) *P is $\overline{\mathrm{N}}_5(k)$ for some positive integer k.*

(4) *P is* $\mathrm{C}(m, k)$ *for a pair of nonnegative integers m and k such that $\langle m, k\rangle \neq \langle 1, 0\rangle$ and $\langle m, k\rangle \neq \langle 2, 0\rangle$.*

COROLLARY 7.33. *The connected component of a totally atomic element q not belonging to the connected component of either* 0 *or* 1 *is always either* $\overline{\mathrm{N}}_5(k)$, *if q is completely join irreducible with k canonical meetands, or the dual of* $\overline{\mathrm{N}}_5(k)$, *when q is completely meet irreducible with k canonical joinands.*

Proof: This result is clear if q is a generator. Otherwise its connected component must contain the configuration diagrammed in Figure 7.10(2), or its dual, and the result follows from Lemma 7.27(2).

CHAPTER VIII

Singular and Semisingular Elements

Recall that by Theorem 3.34, if w is completely join irreducible then all of the canonical meetands of w are canonical meetands of w_*, except those that lie above $\kappa(w)$. An element w for which some canonical meetand is above $\kappa(w)$ is called semisingular. More precisely, a completely join irreducible element w of a free lattice is a *semisingular element* if $\kappa(w) \le w_i$ for one of its canonical meetands w_i. Whenever w is semisingular, we will use the convention that $\kappa(w) \le w_1$. A completely join irreducible element is a *singular element* if $w_* = \kappa(w)$ (so that $\kappa(w) \le w_i$ for *every* i). In this case the covering $w_* \prec w$ is known as a *singular cover*. Thus a singular cover is one in which the top is a meet and the bottom is a join. For example, the meet of two coatoms is a singular element in a free lattice.

In this chapter we characterize singular and semisingular elements. It turns out that these characterizations are extremely useful, and their applications will be investigated in the next chapter.

If X is infinite, $\mathbf{FL}(X)$ trivially has no completely join irreducible elements, and hence no semisingular elements; thus in this chapter we assume X is finite.

1. Semisingular Elements

Whenever $w = \bigwedge w_i$ is completely join irreducible, then by Theorem 3.4, for each i there is a unique j such that $w_{ij} \nleq w_*$. Throughout this chapter we will denote this element by w_{i1}. (If $w_i \in X$, then we take $w_{i1} = w_i$.) Note $w_{i1} \nleq \kappa(w)$ (else $w \le w_i \le \kappa(w)$), and by Theorem 3.33, $w_{ij} < w_*$ for all $j > 1$.

The main result of this section is the following characterization of semisingular elements.

THEOREM 8.1. *A completely join irreducible element $w \in \mathbf{FL}(X)$ is semisingular if and only if $w_* = \kappa(w)$ (i.e., w is singular) or w is the middle element of a three element interval.*

Obviously if w is singular it is semisingular and, by Theorem 7.10(6), the middle element of a three element interval is semisingular. Thus it remains to prove the converse.

We begin with some general facts about completely join irreducible elements which are of independent interest.

Lemma 8.2. *Let* $w = \bigwedge_k x_k \wedge \bigwedge_i (\bigvee_j w_{ij})$ *(canonically) be a completely join irreducible element in a free lattice. Assume the indexing is such that* $w_{i1} \not\leq w$ *and* $w_{ij} < w$ *for* $j > 1$*. Then for each* i*,*

$$w_{i1*} \leq \kappa(w) \prec \kappa(w)^* \leq \kappa(w_{i1}).$$

Proof: In Chapter II (page **??**) we defined the relations A and B on completely join irreducible elements as follows:

$$\begin{aligned} &a \; A \; b \quad \text{if } b < a \text{ and } a \leq b \vee \kappa(b), \\ &a \; B \; b \quad \text{if } a \neq b,\ a \leq a_* \vee b \text{ and } a \not\leq a_* \vee b_*. \end{aligned}$$

We also noted there that if $a \; B \; b$, then $b_* \leq \kappa(a)$. Now by Theorem 2.60 and Lemma 3.11, $w \; B \; w_{i1}$ and thus $w_{i1*} \leq \kappa(w)$. By Lemma 2.63, $\kappa(w) \; A^d \; \kappa(w_{i1})$ which implies $\kappa(w) < \kappa(w_{i1})$ and the lemma follows.

We now begin the proof of Theorem 8.1. So assume that w is semisingular and that $\kappa(w) \leq w_1$. Let $v = \kappa(w)$. We claim that it is enough to show that either $w_* = v$ or w_1 is totally atomic. For if w_1 is totally atomic, then $\kappa^d(w_1) \not\leq \kappa(w)$, for otherwise $\kappa^d(w_1) \leq w_1$; thus in this case $w \sqcup \kappa^d(w_1) \succ w \succ w_*$ by Theorem 3.5, and $w \sqcup \kappa^d(w_1)/w_*$ is a three element interval by Theorem 7.6(2). Since w is completely join irreducible, w_1 is lower atomic by Corollary 3.27. Thus *it suffices to show that either* $w_* = v$ *or* w_1 *is completely meet irreducible.*

If w_1 is a generator, then clearly it is completely meet irreducible. Thus we assume w_1 is not a generator. Note that v is not a generator since otherwise $w = \kappa^d(v)$ would be an atom, but atoms are not semisingular. Thus $v = \bigvee v_r$ is a proper join.

Lemma 8.3. *Let* v*,* w*,* v_r*,* w_{ij} *be as given above.*

(1) $v \sqcup w_{11} = w_1$.
(2) $\{w_{1j} : j > 1\} \ll \{v_1, v_2, \dots\}$.

Proof: The first statement follows from the fact that $w_{1j} \leq w_* \leq v$ for $j > 1$, and the second follows from the first and canonical form, Theorem 1.19.

If $w_{11} \leq v^*$, then $w_1 \leq v^*$, which contradicts Lemma 6.29. Thus, by Lemma 8.2, we have $w_{11*} = w_{11} \wedge v^*$. Applying (W) to

$$w_{11*} = w_{11} \wedge v^* \leq \bigvee v_r = v$$

yields that $w_{11*} \leq v_r$ for some r. We may take $r = 1$, so that

$$w_{11*} \leq v_1. \tag{1}$$

If v_1 is a generator, then $v \wedge w_{11} = w_{11*} \leq v_1$ implies $w_{11} \leq v_1 < v$, a contradiction. So we may assume v_1 is not a generator. Then, according to our numbering convention, v_{11} is the only canonical meetand of v_1 not above v, and we have

$$\bigwedge v_{1s} = v_1 \leq w_1 = \bigvee w_{1j}. \tag{2}$$

Applying (W) to (2) leads to four cases. The first is easy.

Case i. $v_{1s} \leq w_1$ **for some** $s > 1$**.** By the dual of Theorem 3.4 and our numbering convention, $v^* \leq v_{1t}$ for all t except $t = 1$. Thus in this case, $v^* \leq v_{1s} \leq w_1$ which implies $w \leq v_{1s}$. As v_{1s} is completely meet irreducible by Theorem 3.26 and $w \leq v_{1s} \leq w_1$, the dual of Lemma 3.28 yields that w_1 is completely meet irreducible, as desired. Note that the reduction above shows that w is the middle of a three element interval in this case.

Case ii. $v_{11} \leq w_1$**.** By the dual of Lemma 8.2, $w = \kappa^d(v) \leq v_{11}^*$. Thus

$$\bigwedge w_k = w \leq v_{11}^* \leq v \;\sqcup\; v_{11} \leq w_1$$

as $v \not\leq v_{11}$ by our convention. Moreover, $v_{11} \not\geq \kappa^d(v) = w$. Hence by (W), $w_i \leq v \;\sqcup\; v_{11} \leq w_1$ for some i. This implies $i = 1$, and thus $v \vee v_{11} = w_1$ and

$$v_{11}^* = w_1. \tag{3}$$

Since $w_{11} \not\leq v$, we have by canonical form (Theorem 1.19) that

$$w_{11} \leq v_{11}.$$

These facts are depicted in Figure 8.1. Lines with cross marks indicate coverings.

Since $w_1 = v_{11}^* = v \vee v_{11}$, $\kappa^d(v_{11}) \leq v$, and since v is a join, we get $\kappa^d(v_{11}) < v$. Since $w_{11} \not\leq v$, we have $w_{11} \not\leq \kappa^d(v_{11})$. The dual of Theorem 3.34 implies that w_{11}, being a canonical joinand of v_{11}^* and not below $\kappa^d(v_{11})$, is a canonical joinand of v_{11} and so, by the dual of Theorem 3.26, w_{11} is upper atomic. Of course, it is also completely join irreducible since it is in $\mathrm{J}(w)$; see Theorem 3.26. Thus w_{11} is totally atomic. Also

$$v_1 \prec v_1 \;\sqcup\; w_{11}.$$

For if $v_1 < u \leq v_1 \;\sqcup\; w_{11}$, then $u \not\leq v$ as $v \wedge u \leq v \wedge v_{11} = v_1$ since $v_{1j} > v$ for $j > 1$. Hence $w \leq v^* \leq v \sqcup u$, and the usual (W) argument gives $v \;\sqcup\; u = w_1$, which, by canonical form (Theorem 1.19), implies $w_{11} \leq u$ and so $u = v_1 \;\sqcup\; w_{11}$.

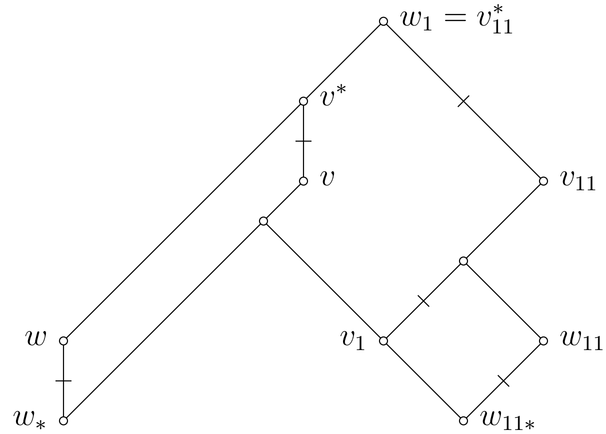


FIGURE 8.1

Now we show that

$$v_{11} = v_1 \sqcup w_{11}. \tag{4}$$

As previously observed,

$$v^* \wedge v_{11} = v_1, \tag{5}$$

so

$$w \wedge v_{11} \leq v_1. \tag{6}$$

As noted above, $\kappa^d(v_{11}) \leq v$. Now it follows from (5) that $\kappa^d(v_{11})_* \leq v_1$. By the dual of Theorem 3.34, every canonical joinand of v_{11} is either strictly below $\kappa^d(v_{11})$ (and hence below v_1) or is a canonical joinand of w_1. But every canonical joinand of w_1 except w_{11} is below w_*. Hence by (6), all the joinands of v_{11}, except w_{11}, are below v_1. Hence (4) holds and thus

$$v_1 \prec v_{11}.$$

Now we can apply Theorem 6.31. Let $p = v_1$ and $q = v_{11}$. Then $p \prec q \prec q^* = w_1$, and $q_1 = w_{11}$ is the canonical joinand of q not below p. Since $\kappa^d(v_{11}) \leq v \leq \kappa(w_{11})$ by the dual of Lemma 8.2, $\kappa^d(q) \leq \kappa(q_1)$. By Lemma 7.22 and the easy fact the q is not in the connected component of 0 or 1, $v_1 \sqcup \kappa^d(v_{11}) \neq v$. Thus v is a completely meet irreducible element with $v \leq q^* \wedge \kappa(q_1)$ and $v > p \sqcup \kappa^d(q)$, which contradicts Theorem 6.31.

This finishes case **ii**. Note that this case does not occur at all.

Case iii. $v_1 \leq w_{11}$**.** Then $w_{11*} \leq v_1 \leq w_{11}$ by (1), and since $w_{11} \not\leq v$ (else $w_1 \leq w_* \sqcup w_{11} \leq v$), we must have $v_1 = w_{11*}$.

Note also that $v_{11}^* \geq \kappa^d(v) = w$ by the dual of Lemma 8.2, while $w_{11} \wedge v = w_{11*} = v_1$ implies (using Theorem 3.4 and canonical form) that $v_{11} \geq w_{11}$. Thus $v_{11}^* \geq w \sqcup w_{11} = w_1$. The situation is depicted in Figure 8.2.

Suppose w_{11} is a generator, say $w_{11} = x$. Then $v_1 = v \wedge v_{11} \leq x$ implies that $v_{11} = x$ (as $v \not\leq x \leq v_{11}$). This yields $x_* = v_1 < v < w_1 \leq v_{11}^* = x^*$, while Corollary 7.7 shows that there are no completely meet irreducible elements besides x in the interval x^*/x_*. Therefore w_{11} is not a generator.

Now by Theorem 3.34, v_{11} is a canonical meetand of w_{11} (since $w_{11} \wedge v = v_1$ implies $v \leq \kappa(w_{11})$ and $v_{11} \not\geq v$) as well as of $v_1 = w_{11*}$. This makes it both lower and upper atomic by Theorem 3.26, i.e., v_{11} is totally atomic. On the other hand the dual of Lemma 8.2 gives $w = \kappa^d(v) > \kappa^d(v_{11})$. Thus $\kappa^d(v_{11}) \leq \kappa^d(v)_* = w_* \leq v$, so we have $\kappa^d(v_{11}) \leq v \leq w_1 \leq v_{11}^*$. The dual of Theorem 6.30 then yields that $t = \kappa^d(v_{11})$ is also totally atomic. By Lemma 6.25, this can only happen when t is a meet of generators. Thus there exists subsets $Y, Z \subseteq X$ such that

$$t = \bigwedge Z, \qquad v_{11} = \bigvee Y, \qquad X = Y \mathbin{\dot\cup} Z.$$

Since v_{11} is a canonical meetand of w_{11} which is completely join irreducible, by Theorem 3.4 there exists $y \in Y$ such that $\bigvee Y - \{y\} \leq w_{11*} = v_1$. On the other hand, Lemma 6.32 implies that for some t_i, i.e., for some $z \in Z$, $v \wedge z = t$. This implies that $v \leq \bigwedge Z - \{z\}$. As $v_1 < v$ and hence $\bigvee Y - \{y\} \leq \bigwedge Z - \{z\}$, we must have that either Y or Z is a singleton.

If $Y = \{y\}$, then $v_{11} = y$ and $t = \kappa^d(y)$ is an atom. But we also have $v_1 \leq y \wedge \bigwedge Z - \{z\}$, which is another atom. But $w_{11} \succ v_1$, while the covers of atoms in $\mathbf{FL}(X)$ are join reducible. Hence $|Y| > 1$.

If $Z = \{z\}$, then $t = z$ and $v \geq z \sqcup \bigvee Y - \{y\}$, which makes v a coatom of $\mathbf{FL}(X)$. Now $v < v^* \leq w_1$, which would imply $w_1 = 1$. But

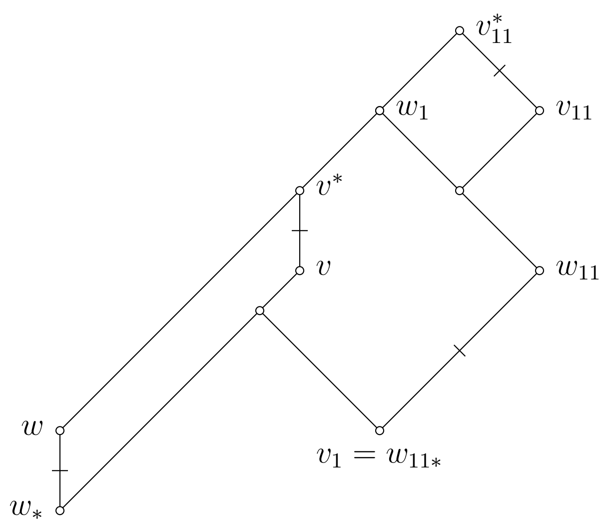


FIGURE 8.2

1 is never a canonical meetand, so this is impossible. Hence this case also does not occur at all.

Case iv. $v_1 \leq w_{1j}$ **for some** $j > 1$**.** By Theorem 3.4, we have $w_{1j} \leq w_*$. Hence by (1),

$$w_{11*} \leq v_1 \leq w_{1j} \leq w_*$$

and, since $w_{1j} \not\geq w_{11}$, we have $w_{11*} \leq w_{1j} \leq \kappa(w_{11})$. Now, by Theorem 6.30, this implies that $\kappa(w_{11})$ is totally atomic. Moreover, Lemma 8.3(2) gives $v_1 = w_{1j}$ and hence this element is totally atomic as well.

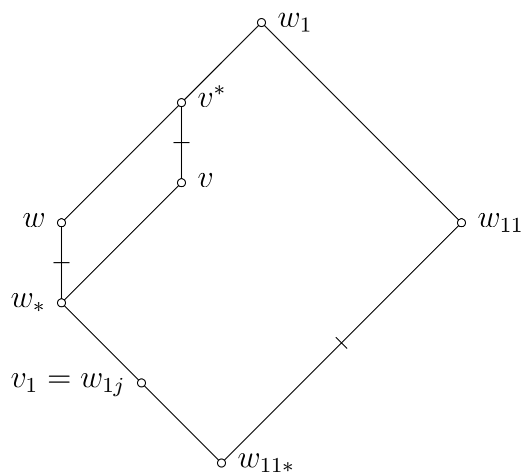


FIGURE 8.3

Let $v_2 = v \wedge \kappa(v_1)$ (momentarily we will show that this element actually is a canonical joinand of v). Then $v_2 \prec v \prec v^*$ by Theorem 3.5. If $\kappa(v_1) \geq \kappa^d(v) = w$, then $v_1 \leq \kappa(v_1)$, a contradiction. Thus, by Theorem 7.6(2), v^*/v_2 is a three element interval and, by Theorem 7.10, $v = v_1 \sqcup v_2$ canonically. Moreover, by the definition of v_2,

$$w_{1j*} = v_{1*} \leq v_2.$$

Since $\kappa(w_{11})$ is totally atomic, the dual of Theorem 6.27 implies w_{11*} is not completely join irreducible. Thus we cannot have $v_1 = w_{1j} = w_{11*}$. Hence

$$v_1 = w_{1j} > w_{11*}.$$

Thus $v_2 \geq v_{1*} \geq w_{11*}$, and both of the following hold:

$$w_{11*} \leq v_1 \qquad w_{11*} \leq v_2.$$

So we can apply all of the previous arguments (from the very beginning) with v_2 in place of v_1.

As before, v_2 is not a generator and thus

$$v_2 = \bigwedge v_{2s} \le \bigvee w_{1j} = w_1.$$

This leads to the same four cases that we are considering for v_1. If $v_{2s} \le w_1$ for some $s > 1$, then, w_1 is totally atomic by case **i**, as desired. The case $v_{21} \le w_1$ cannot occur by case **ii**. Hence $v_2 \le w_{1k}$ for some k. The case $k = 1$ is ruled out by case **iii**. Thus we may assume $k > 1$. But then $v_2 \le w_{1k} \le w_*$ and so $\kappa(w) = v = v_1 \sqcup v_2 \le w_*$. Thus $\kappa(w) = w_*$, as desired.

This completes the proof of Theorem 8.1.

Examples 7.11, 7.12 and 7.13 show us how to construct infinitely many three element intervals, and hence infinitely many semisingular elements, in each $\mathbf{FL}(X)$ with $|X| \ge 3$. For a concrete example, take $w = (x \sqcup y) \wedge (x \sqcup (z \wedge t))$ in $\mathbf{FL}(\{x, y, z, t\})$. Then $\kappa(w) = x \sqcup [(x \sqcup y) \wedge (y \sqcup z \sqcup t) \wedge (x \sqcup z \sqcup t)]$, and $w_* = (x \sqcup (z \wedge t)) \wedge \kappa(w)$ canonically, since $\kappa(w) \le x \sqcup y$.

Theorem 8.1 has two important corollaries. First, as we observed in the proof, only two of the four cases can actually occur. In case **i**, w is the middle of a three element interval. Case **iv** either reduces to case **i** or else gives $\kappa(w) = w_*$, in which event $v = \kappa(w)$ is the middle element of a three element interval $v_2 \prec v \prec w$. Thus we obtain the following corollary.

Corollary 8.4. *If w is a semisingular element of a free lattice, then either w is the middle element of a three element interval, or $w_* = \kappa(w)$ and w is the top element of a three element interval.*

This next corollary will turn out to be extremely useful.

Corollary 8.5. *Let w be a completely join irreducible element of $\mathbf{FL}(X)$ with $\kappa(w) \ne w_*$. If u is an element such that $u \wedge \kappa(w) = w_*$, then either $u = w_*$, or $u = w$, or $u \succ w$. Moreover, if $u \succ w$ then w is semisingular.*

Proof: By Theorem 3.34, if w is not semisingular, then each canonical meetand, w_i, of w is a canonical meetand of w_*. So if $w_* = u \wedge \kappa(w)$ and w is not semisingular, i.e., $w_i \not\ge \kappa(w)$ for all i, then by Theorem 1.19 we have $w_i \ge u$ for all i. Thus $u \le w$, which implies u is either w or w_*.

In the other case w is semisingular and thus, by Theorem 8.1, w is the middle element of a three element element. By Theorem 7.10, the canonical form of w_* is $w_* = \kappa(w) \wedge r$ for some $r \succ w$. The corollary follows easily from this.

2. Singular Elements

Recall that a singular element is a completely join irreducible element w with $\kappa(w) = w_*$. We have seen in Chapter III that these exist in the connected components of 0 and 1; in this section we will show that these are the only examples. This result first appeared in Freese, Ježek, Nation and Slavík [**58**], with a slightly more complicated proof.

Theorem 8.6. *Suppose that w is a singular element, and let $v = \kappa(w) = w_*$. Then either w is the meet of two coatoms or, dually, v is the join of two atoms.*

Proof: A quick look at the description of the connected components of 0 and 1 given in Examples 3.44 and 3.45, shows that the singular elements contained in one of these components are as described in the theorem. Thus it suffices to show that every singular element lies in one of these components. Moreover, we may assume that w is the middle element of a three element interval. For otherwise, it is the top element of a three element interval by Corollary 8.4, and so the dual of v is singular, completely join irreducible, and the middle element of a three element interval. So if we show that the dual of v is in the connected component of 0 or of 1, then w will be in the other connected component.

Thus we assume that our singular element w is join irreducible and the middle element of a three element interval $u \succ w \succ v$. By Theorem 7.2, four element covering chains are always in the connected component of 0 or 1, and so we may assume that u has no upper cover and that v has no lower cover.

Let u_1 be the canonical joinand of u not below v, and let w_1 be the canonical meetand of w not above u, as in Figure 8.4. Note that $u_1 = \kappa^d(w_1)$ by Theorem 3.5.

Let $u = u_1 \vee \cdots \vee u_k$ and $v = v_1 \vee \cdots \vee v_m$ canonically. Since v is completely meet irreducible, we get $u = u_1 \vee v = u_1 \vee \bigvee v_i$, and so by Theorem 1.19

$$\{u_2, \ldots, u_k\} \ll \{v_1, \ldots, v_m\}. \tag{7}$$

If $u_1 \geq v$, then $u_1 = u$, which means that u is a generator or a proper meet, both of which contradict Theorem 7.10. Thus there is an i such that $v_i \nleq u_1$. We take $i = 1$. Now apply (W) to the inequality

$$v_1 = \bigwedge v_{1j} \leq \bigvee u_i = u.$$

By assumption $v_1 \leq u_1$ cannot occur. If $v_1 \leq u_i$ for some $i > 1$, then by (7), $v_1 = u_i$. By Theorem 7.10, u is a canonical meetand of w,

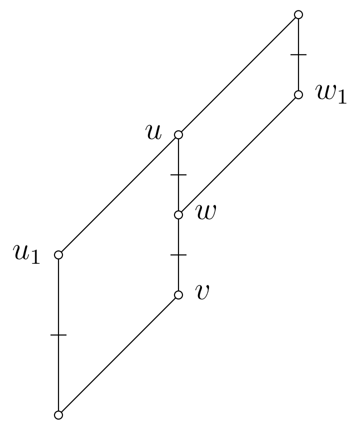


FIGURE 8.4

and hence $v_1 = u_i \in \mathrm{J}(w)$. Thus v_1 is completely join irreducible by Theorem 3.26. Then $v \succ v \wedge \kappa(v_1)$ by Theorem 3.5, contrary to our assumption that v has no lower cover.

Suppose that $v_{1j} \leq u$ for some $j > 1$. Then $v < v_{1j}$ by Theorem 3.4 (by our convention, v_{11} is the unique canonical meetand of v_1 not above v). Since v_{1j} is meet irreducible, $v_{1j} \neq w$. Hence $v_{1j} = u$. But $v_{1j} \in \mathrm{M}(v)$ and so is completely meet irreducible by the dual of Theorem 3.5, which implies u has an upper cover, contrary to our assumption.

Thus we must have $v_{11} \leq u$. By the dual of Lemma 8.2, $v_{11}^* \geq \kappa^d(v) = w$, so $w \leq v_{11}^* \leq u$. Since $v_{11}^* = w$ would imply that $w = v_{11} \vee v$ is a proper join, we must have $v_{11}^* = u$. Letting $u_{11} \wedge \cdots \wedge u_{1r}$ be the canonical form of u_1 and applying (W) to the inequality

$$u_1 = u_{11} \wedge \cdots \wedge u_{1r} \leq u = v_{11} \vee w$$

easily gives that $u_1 \leq v_{11}$, since $u_{ij} \nleq u$ is one of the criteria for canonical form. Now $\kappa^d(v_{11}) \leq v$ since $v \leq v_{11}^*$ and $v \nleq v_{11}$. Hence u_1, being a canonical joinand of $v_{11}^* = u$ and not equal to $\kappa^d(v_{11})$ (as

$u_1 \nleq v$), is a canonical joinand of v_{11} by Theorem 3.34. Thus, by Theorem 3.5, we have $u \succ v_{11} \succ v_{11} \wedge \kappa(u_1)$. Theorem 6.24 then shows that u_1 is totally atomic. By the same theorem, w_1 is totally atomic. Recall that $\kappa(u_1) = w_1$, and hence by Lemma 6.25 u_1 is a meet of variables and w_1 is a join of variables. Since u_1 is a canonical joinand of v_{11} and w_1 is a canonical meetand of w, we can apply Lemma 7.1 to conclude that w is in the connected component of either 0 or 1, as desired.

As an application of the results of this chapter we will show that no interval of a free lattice can be an infinite chain. This result turned out to be much more difficult than it might sound. In the next chapter we will discuss the history of this result and use Corollary 8.5 to prove a much stronger theorem, due to S. Tschantz, that every infinite interval of a free lattice contains a copy of $\mathbf{FL}(\omega)$; but the present argument is so easy we include it here.

COROLLARY 8.7. *No interval of a free lattice can be isomorphic to an infinite chain.*

Proof: Suppose $a > b$ in $\mathbf{FL}(X)$ and the interval a/b is an infinite chain. By Day's Theorem and the fact (from Chapter VII) that there are no long chains of covers in free lattices, there are elements c and d such that $a > c \succ d > b$ with a/c and d/b infinite. By the description of the connected component of a generator given in Example 3.43, a/b contains no generator. If c is join reducible, let c_1 be a canonical joinand of c not below d, and if d is meet reducible, let d_1 be a canonical meetand of d not above c. It is not possible for both of these to hold because otherwise $a \wedge d_1 = d \leq c = b \vee c_1$, since a/b is a chain, which violates (W). Thus by duality we may assume that c is join irreducible and so $c_* = d$. If d is join reducible, then c is a singular element. But then Theorem 8.6 would imply that either a/c or d/b is finite, a contradiction. Thus d is meet reducible and so $c_* = d = a \wedge d_1$. Since a does not cover or equal c, this violates Corollary 8.5.

CHAPTER IX

Tschantz's Theorem and Maximal Chains

In this chapter we give some applications of the characterization of semisingular elements in Chapter VIII. First we prove Tschantz's beautiful result that every infinite interval of a free lattice contains a copy of $\mathbf{FL}(\omega)$. Using Tschantz's Theorem, we will show that every infinite interval of a free lattice contains a *coverless element*, i.e., an element with no upper and no lower cover. Finally we shall show that if $a > b$ in a free lattice and a/b is neither atomic nor dually atomic, then there is a maximal chain in a/b without any covers. This is a little surprising in light of Day's Theorem.

1. Tschantz's Theorem

Tschantz's Theorem has an interesting history. After the first and third authors had characterized all finite intervals of free lattices [**62**], they tried to show that every infinite interval of a free lattice would contain a copy of $\mathbf{FL}(\omega)$. By Theorem 1.28 and Corollary 1.13, in order to do this one would simply have to show any such interval contains a join and meet irredundant set of three elements. This seemed like it should be easy. We were quite surprised when we could not do this and, in fact, were not every able to show the existence of two incomparable elements. That is, we could not rule out the possibility of an infinite interval in a free lattice which is a chain! We were able to derive some unlikely consequences of such an interval, but none produced a contradiction. We showed this problem to Tom Harrison who derived much stronger (and more unlikely) consequences, but still without reaching a contradiction. Harrison showed it to Steve Tschantz, who was able to show that such an interval could not exist, and, in fact, that every infinite interval contains $\mathbf{FL}(\omega)$.

Interestingly, most of Tschantz's proof is devoted to showing that there is no infinite interval which is a chain. We can simplify this part of the proof considerably by using the characterization of semisingular elements from the last chapter. For the rest we follow his original proof from [**131**].

We begin with a definition. Let u and $v \in \mathbf{FL}(X)$ and let

$$\mathrm{Q}(u, v) = \{q \in \mathbf{FL}(X) : q \text{ is completely join irreducible, } q \leq u \text{ and } v \leq \kappa(q)\}.$$

Clearly if $u' \leq u$ and $v' \geq v$, then $\mathbf{Q}_{(}u', v') \subseteq \mathbf{Q}_{(}u, v)$. This fact will be used repeatedly in this section. The strong version of Day's Theorem (Theorem 4.1) can be expressed in terms of $\mathbf{Q}_{(}u, v)$ as follows.

THEOREM 9.1. *If $u \not\leq v$, then $\mathbf{Q}_{(}u, v)$ is nonempty, and conversely.*

LEMMA 9.2. *If q and r are completely join irreducible elements, $q \leq r$, and $\kappa(q) \geq \kappa(r)$, then $q = r$.*

Proof: If $q < r$, then $q \leq r_* \leq \kappa(r) \leq \kappa(q)$, a contradiction.

LEMMA 9.3. *$\mathbf{Q}_{(}u, v) = \{q\}$ if and only if $q \in \mathbf{Q}_{(}u, v)$, $u \wedge \kappa(q) \leq v$, and $u \leq v \vee q$.*

Proof: Suppose that $\mathbf{Q}_{(}u, v) = \{q\}$. If $u \wedge \kappa(q) \not\leq v$, then there is a $q' \in \mathbf{Q}_{(}u \wedge \kappa(q), v) \subseteq \mathbf{Q}_{(}u, v)$. Since $q' \leq \kappa(q)$, $q' \neq q$. Similarly, if $u \not\leq v \vee q$, we can find another element in $\mathbf{Q}_{(}u, v \ \sqcup\ q)$, which is contained in $\mathbf{Q}_{(}u, v)$.

Conversely, assume $q \in \mathbf{Q}_{(}u, v)$, $u \wedge \kappa(q) \leq v$ and $u \leq v \vee q$. If q' is also in $\mathbf{Q}_{(}u, v)$, then

$$q' \leq u \leq v \vee q \leq \kappa(q) \vee q = \kappa(q)^*$$

so either $q \leq q'$ or $q' \leq \kappa(q)$ by the dual of Theorem 3.1. In the latter case, $q' \leq u \wedge \kappa(q) \leq v \leq \kappa(q')$, which is not possible. Thus $q \leq q'$. Similarly,

$$\kappa(q') \geq v \geq u \wedge \kappa(q) \geq q \wedge \kappa(q) = q_*$$

and $q \leq \kappa(q')$ would imply $\kappa(q') \geq v \vee q \geq u \geq q'$. Thus $\kappa(q') \leq \kappa(q)$. By Lemma 9.2 we conclude that $q = q'$.

Let

$$K = \{q \in \mathbf{FL}(X) : q \text{ is completely join irreducible and } q \text{ or } \kappa(q) \text{ is totally atomic}\}.$$

The case when $\mathrm{Q}(u, v)$ is finite but has more than one element involves totally atomic elements, just as do chains of covers of length at least two. Theorem 9.5 will show that, if $\mathrm{Q}(u, v) \cap K = \emptyset$ and $|\mathrm{Q}(u, v)| > 1$, then $\mathrm{Q}(u, v)$ is infinite. First we need a lemma.

LEMMA 9.4. *If $\mathrm{Q}(u, v) \cap K = \emptyset$ and $q \in \mathrm{Q}(u, v)$, then $|\mathbf{Q}_{(}u \wedge \kappa(q), v)| \neq 1$ and $|\mathbf{Q}_{(}u, v \vee q)| \neq 1$.*

Proof: Suppose $\mathbf{Q}_{(}u \wedge \kappa(q), v) = \{q'\}$. Then Lemma 9.3 gives $u \wedge \kappa(q) \leq v \vee q'$. We will apply (W) to this inequality, using repeatedly the inclusions implied by $q \in \mathrm{Q}(u, v)$ and $q' \in \mathbf{Q}_{(}u \wedge \kappa(q), v)$.

If $u \leq v \vee q'$, then $q \leq u \leq v \vee q' \leq \kappa(q)$, a contradiction.

If $\kappa(q) \leq v \vee q'$, then

$$q' \leq \kappa(q) \leq v \vee q' \leq \kappa(q') \vee q' = \kappa(q')^*.$$

But, since q' is a canonical joinand of $\kappa(q')^*$ and $\kappa(q)$ is completely meet irreducible, this implies by Lemma 3.28 that q' is totally atomic, which would imply $q' \in \mathbf{Q}_{(}u, v) \cap K$.

If $u \wedge \kappa(q) \leq v$, then $q' \leq u \wedge \kappa(q) \leq v \leq \kappa(q')$, a contradiction. If $u \wedge \kappa(q) \leq q'$, then $q' = u \wedge \kappa(q)$. Thus

$$q_* = q \wedge \kappa(q) \leq u \wedge \kappa(q) = q' \leq \kappa(q)$$

which would imply $\kappa(q)$ is totally atomic, and hence $q \in \mathrm{Q}(u, v) \cap K$.

Since none of these inclusions can hold, by (W) we conclude that $|\mathbf{Q}_{(}u \wedge \kappa(q), v)| \neq 1$. A similar argument shows that $|\mathbf{Q}_{(}u, v \vee q)| \neq 1$.

THEOREM 9.5. *If* $\mathrm{Q}(u, v) \cap K = \emptyset$ *and* $|\,\mathrm{Q}(u, v)| > 1$, *then* $\mathrm{Q}(u, v)$ *is infinite.*

Proof: We use induction to construct $q_0, q_1, q_2 \ldots \in \mathrm{Q}(u, v)$ and chains $u_0 \geq u_1 \geq u_2 \cdots$ and $v_0 \leq v_1 \leq v_2 \cdots$ such that the q_i's are distinct, $q_i \in \mathbf{Q}_{(}u_i, v_i) \subseteq \mathrm{Q}(u, v)$, and $|\mathbf{Q}_{(}u_i, v_i)| > 1$. Let $u_0 = u$, $v_0 = v$ and choose $q_0 \in \mathrm{Q}(u, v)$ arbitrarily. Assume u_i, v_i, and q_i have been chosen for $i < n$ satisfying the inductive conditions. Since $|\mathbf{Q}_{(}u_{n-1}, v_{n-1})| > 1$, either $u_{n-1} \wedge \kappa(q_{n-1}) \nleq v_{n-1}$ or $u_{n-1} \nleq v_{n-1} \vee q_{n-1}$ by Lemma 9.3. In the former case let $u_n = u_{n-1} \wedge \kappa(q_{n-1})$ and $v_n = v_{n-1}$ and in the latter case let $u_n = u_{n-1}$ and $v_n = v_{n-1} \vee q_{n-1}$. Since, in either case $u_n \nleq v_n$, we can choose a $q_n \in \mathbf{Q}_{(}u_n, v_n)$. By Lemma 9.4, $|\mathbf{Q}_{(}u_n, v_n)| > 1$. To see that the q_n's are distinct, let $i < j$. If $u_{i+1} = u_i \wedge \kappa(q_i)$, then $q_j \leq u_j \leq u_{i+1} \leq \kappa(q_i)$, which implies $q_i \neq q_j$. In the other case, $v_{i+1} = v_i \vee q_i$, so $q_i \leq v_{i+1} \leq v_j \leq \kappa(q_j)$. Again this implies $q_i \neq q_j$.

The next few results show how to avoid totally atomic and certain other elements.

LEMMA 9.6. *If* $\mathrm{Q}(u, v)$ *is infinite and* S *is a finite set of completely join irreducible elements in* $\mathbf{FL}(X)$, *then, for some* $u' \leq u$ *and* $v' \geq v$, $\mathbf{Q}_{(}u', v')$ *is infinite and* $\mathbf{Q}_{(}u', v') \cap S = \emptyset$. *Moreover,* u' *and* v' *can be taken so that*

$$u' = u \wedge \bigwedge \kappa(p_i) \quad \textit{and} \quad v' = v \vee \bigvee r_i$$

for some (possible empty) sequences p_i, $r_i \in \mathrm{Q}(u, v)$ *with* $p_j \leq \kappa(p_i)$ *and* $\kappa(r_j) \geq r_i$ *whenever* $i < j$.

Proof: Induct on $|\,\mathrm{Q}(u,v)\cap S|$. If $\mathrm{Q}(u,v)\cap S=\emptyset$, then we can choose $u'=u$ and $v'=v$. Otherwise let $s\in \mathrm{Q}(u,v)\cap S$. If there are infinitely many $q\in \mathrm{Q}(u,v)$ with $q\le \kappa(s)$, let $u''=u\wedge\kappa(s)$ and $v''=v$. Then $|\mathbf{Q}_(u'',v'')\cap S|<|\,\mathrm{Q}(u,v)\cap S|$, and the desired u' and v' can be found easily by induction. Similarly, if there are infinitely many $q\in \mathrm{Q}(u,v)$ with $s\le\kappa(q)$, then we can let $u''=u$ and $v''=v\vee s$ and complete the argument again by induction.

Consequently, if we let

$$Q=\{q\in \mathrm{Q}(u,v) : q\ne s,\ q\not\le\kappa(s),\text{ and }\kappa(q)\not\ge s\},$$

then we may assume that Q is infinite. Also, we may assume there are only finitely many $r\in \mathrm{Q}(u,v)$ with $\kappa(r)\ge\kappa(s)^*$, since there are only finitely many with $s\le\kappa(r)$. For each such r we define

$$A_r=\{q\in Q:\kappa(q)\ge r\}.$$

Let q be an arbitrary element of Q. As $q\not\le\kappa(s)$ and $\kappa(q)\not\le s$, by Theorem 9.1 there are completely join irreducible elements r and r' with

$$r\le q,\quad \kappa(s)\le\kappa(r),\quad r'\le s,\quad\text{and}\quad \kappa(q)\le\kappa(r').$$

Now if $r=q$, then, by Lemma 9.2, $r'\le s$ and $\kappa(s)\le\kappa(r)=\kappa(q)\le\kappa(r')$ imply $r'=s$. Then $\kappa(s)\le\kappa(r)=\kappa(q)\le\kappa(r')=\kappa(s)$, which implies $\kappa(q)=\kappa(s)$ and hence $q=s$, a contradiction. Thus $r\ne q$ and so

$$r\le q_*\le\kappa(q). \tag{1}$$

If $\kappa(s)=\kappa(r)$, then $r'\le s=r\le q$ and $\kappa(q)\le\kappa(r')$ imply $r'=q$. Then $r'\le s=r\le q=r'$ implies $s=q$, again a contradiction. Thus $\kappa(s)\ne\kappa(r)$, and so $\kappa(s)^*\le\kappa(r)$. This together with (1) shows each $q\in Q$ is in some A_r. Thus some A_r must be infinite. Choose such an r and let $u''=u$ and $v''=v\vee r$. Now $s\notin\mathbf{Q}_(u'',v'')$ since otherwise $r\le v\vee r\le\kappa(s)$, which implies $r\le\kappa(s)<\kappa(s)^*\le\kappa(r)$. On the other hand $\mathbf{Q}_(u'',v'')$ is infinite because $A_r\subseteq\mathbf{Q}_(u'',v'')$. Again the argument can be completed easily by induction.

This lemma has a corollary that for $u\not\le v$, there are $u'\le u$ and $v'\ge v$ such that $\mathbf{Q}_(u',v')\cap\ K=\emptyset$. We require the following stronger result.

Corollary 9.7. *If $\mathrm{Q}(u,v)$ is infinite, then there exist $u'\le u$ and $v'\ge v$ such that $\mathbf{Q}_(u',v')$ is infinite, $\mathbf{Q}_(u',v')\cap K=\emptyset$, and for all $q\in\mathbf{Q}_(u',v')$, $u'\not\le\kappa(q)^*$ and $v'\not\ge q_*$.*

Proof: As remarked above, we may assume that $\mathbf{Q}_(u,v)\cap\ K=\emptyset$. Let $S=\mathrm{J}(u)\cup\kappa^d(\mathrm{M}(v))$, and apply Lemma 9.6 to obtain $u'=u\wedge\bigwedge_{i=1}^m\kappa(p_i)$

and $v' = v \vee \bigvee_{i=1}^{n} r_i$ such that $\mathbf{Q}_(u', v') \cap S = \emptyset$ and the p_i's and r_i's satisfy the conditions of that theorem. Suppose that $q_* \leq v'$ for some $q \in \mathbf{Q}_(u', v')$. Then we can apply (W) to

$$q \wedge \kappa(q) = q_* \leq v' = v \vee \bigvee r_i.$$

If $q_* \leq v$, then $q_* \leq v \leq \kappa(q)$ since $q \in \mathbf{Q}_(u, v)$, and thus by the dual of (1) of Lemma 3.28, $\kappa(q) \in \mathrm{M}(v)$. Hence $q \in S$, a contradiction. If $q_* \leq r_i \leq \kappa(q)$, then by the dual of part (2) of Lemma 3.28, either $\kappa(q)$ is a generator or $\mathrm{J}(\kappa(q)) \subseteq \mathrm{J}(r_i)$, which, in either case, implies that $\kappa(q)$ is lower atomic since r_i is (as $r_i \in \mathrm{Q}(u, v)$). Hence $\kappa(q)$ is totally atomic, which implies $q \in K$, a contradiction.

Of course $q \leq v'$ is impossible since $v' \leq \kappa(q)$. If $\kappa(q) \leq v'$, then $v' = \kappa(q)$. If the number of r_i's $n = 0$, then $v = v' = \kappa(q)$ which implies $q \in S$. Thus $n > 0$. Now r_n is a canonical joinand of v' since $v \vee \bigvee_{i=1}^{n-1} r_i \vee r_{n*} \leq \kappa(r_n)$ but $v' \nleq \kappa(r_n)$ because $r_n \leq v'$. But since $\kappa(q)$ is completely meet irreducible and hence upper atomic, r_n is upper atomic and hence it is totally atomic. Thus $r_n \in \mathrm{Q}(u, v) \cap K$, a contradiction. Thus $q_* \nleq v'$, and a similar argument proves $u' \nleq \kappa(q)^*$.

This brings us to the critical step which produces a pair of incomparable elements in $\mathrm{Q}(u, v)$.

THEOREM 9.8. *If $\mathrm{Q}(u, v)$ is infinite, then there exist q, $q' \in \mathrm{Q}(u, v)$ such that*

$$q \leq \kappa(q') \quad \textit{and} \quad q' \leq \kappa(q).$$

Proof: By Corollary 9.7, we may assume that $\mathrm{Q}(u, v) \cap K = \emptyset$ and that for all $q \in \mathrm{Q}(u, v)$,

$$u \nleq \kappa(q)^* \quad \text{and} \quad q_* \nleq v. \tag{2}$$

Pick $q \in \mathrm{Q}(u, v)$. If $u \wedge \kappa(q) \nleq v \vee q$, then $\mathbf{Q}_(u \wedge \kappa(q), v \vee q)$ is nonempty by Theorem 9.1, so there is a q' with $q' \leq u \wedge \kappa(q)$ and $v \vee q \leq \kappa(q')$. Clearly these q and q' satisfy the conclusion of the theorem. Thus we may assume that $u \wedge \kappa(q) \leq v \vee q$ and we can apply (W). By (2), $u \leq v \vee q \leq \kappa(q)^*$ is not possible. If $u \wedge \kappa(q) \leq v$ then $q_* \leq v$, which also contradicts (2). If $u \wedge \kappa(q) \leq q$ then

$$q_* \leq u \wedge \kappa(q) \leq q \wedge \kappa(q) = q_*.$$

Thus in this case $u \wedge \kappa(q) = q_*$. Similarly, $\kappa(q) \leq v \vee q$ implies that $\kappa(q)^* = v \vee q$. So one of these must hold and so by duality we may assume that $q_* = u \wedge \kappa(q)$.

If $\kappa(q) = q_*$ then, by Theorem 8.6, either q is the meet of two coatoms or $\kappa(q)$ is the join of two atoms. In the first case q is totally atomic and in the second case $\kappa(q)$ is. So if either case occurs, $\mathrm{Q}(u, v) \cap$

$K \neq \emptyset$, a contradiction. Now we apply Corollary 8.5. Since $u = q$ cannot occur by (2), we conclude that $u \succ q$ and that q is semisingular. By Theorem 8.1, q is the middle element of a three element interval u/q_*. Let q_1 be the canonical meetand of q not above u. Then, by Theorem 7.10(6), $q_1 \geq \kappa(q)$ and, by the dual of Theorem 6.24, q_1 is totally atomic. If we let $r = \kappa^d(q_1)$, then $r \leq u$ and $\kappa(r) = q_1 \geq \kappa(q) \geq v$. Thus $r \in \mathrm{Q}(u, v) \cap K$. This contradiction completes the proof.

Given the two elements of the last theorem, it is not hard to get infinitely many such elements.

THEOREM 9.9. *If* $\mathrm{Q}(u, v)$ *is infinite, then there is an infinite sequence* $q_1, q_2, \ldots \in \mathrm{Q}(u, v)$ *such that* $q_i \leq \kappa(q_j)$ *for all* $i \neq j$.

Proof: Using Corollary 9.7, choose $u_1 \leq u$ and $v_1 \geq v$ so that $\mathbf{Q}_{(}u_1, v_1)$ is infinite, disjoint from K, and contains no q with $q_* \leq v_1$ or $u_1 \leq \kappa(q)^*$. We will inductively construct the q_i's and two auxiliary sequences determined from u_1, v_1, and the q_i's by $u_{i+1} = u_i \wedge \kappa(q_i)$ and $v_{i+1} = v_i \vee q_i$. First we prove by induction that this can be done so that $\mathbf{Q}_{(}u_n, v_n)$ is infinite. This is clearly true for $n = 1$. Assume $q_1, \ldots, q_{n-1}$ are defined and $\mathbf{Q}_{(}u_n, v_n)$ is infinite.

By Theorem 9.8, there are q_n and $q' \in \mathbf{Q}_{(}u_n, v_n)$ with $q_n \leq \kappa(q')$ and $q' \leq \kappa(q_n)$. Note $q' \in \mathbf{Q}_{(}u_n \wedge \kappa(q_n), v_n \vee q_n) = \mathbf{Q}_{(}u_{n+1}, v_{n+1})$ and if this is the only element in $\mathbf{Q}_{(}u_{n+1}, v_{n+1})$ then, by Lemma 9.3,

$$q_* = q' \wedge \kappa(q') \leq u_{n+1} \wedge \kappa(q') \leq v_{n+1} = v_n \vee q_n = v_1 \vee q_1 \sqcup \ldots \sqcup q_n \leq \kappa(q').$$

We apply (W) to the inclusion $q' \wedge \kappa(q') \leq v_1 \vee \bigvee q_i$. Now $q'_* \leq v_1$ cannot occur by our choice of v_1. If $q'_* \leq q_i \leq \kappa(q')$ for some $i \leq n$, then $\kappa(q')$ is totally atomic by the dual of Lemma 3.28, which is not possible. Clearly $q' \leq v_n \vee q_n \leq \kappa(q')$ is impossible. So the only remaining case is $\kappa(q') \leq v_n \vee q_n \leq \kappa(q')$, so that $v_n \vee q_n = \kappa(q')$. Now, as before, q_n is a canonical joinand of $v_n \vee q_n = \kappa(q')$, since $v_n \vee q_{n*} \leq \kappa(q_n)$. Thus, since $\kappa(q')$ is upper atomic, q_n is, and so it is totally atomic, a contradiction. Thus $\mathbf{Q}_{(}u_{n+1}, v_{n+1})$ has more than one element, and so by Theorem 9.5 it is infinite.

Finally, by the definition of u_i and v_i, for $i < j$ we have $q_i \leq v_j \leq \kappa(q_j)$ and $q_j \leq u_j \leq \kappa(q_i)$.

We can now prove Tschantz's Theorem.

THEOREM 9.10. *If* u/v *is an infinite interval in* $\mathbf{FL}(X)$, *then* u/v *contains a sublattice isomorphic to* $\mathbf{FL}(\omega)$.

Proof: First we show that $\mathrm{Q}(u, v)$ is infinite. Since u/v is infinite, for each $n > 0$ there is a chain $v = v_0 < v_1 < \cdots < v_n = u$ of length n by Theorem 5.59. Clearly $\mathbf{Q}_{(}v_{i+1}, v_i) \subseteq \mathrm{Q}(u, v)$ and $\mathbf{Q}_{(}v_{i+1}, v_i) \cap$

$\mathbf{Q}_(v_{j+1}, v_j) = \emptyset$ for $i \neq j$. Since each $\mathbf{Q}_(v_{i+1}, v_i)$ must be nonempty by Theorem 9.1, this means $|\,\mathrm{Q}(u, v)| \geq n$. Since n was arbitrary, $\mathrm{Q}(u, v)$ must be infinite.

By Theorem 9.9 there are $q_1, q_2, \ldots, q_6 \in \mathrm{Q}(u, v)$ such that $q_i \leq \kappa(q_j)$ for $i \neq j$. For $\{i, j, k\} = \{1, 2, 3\}$ let $w_i = v \vee q_j \vee q_k \vee q_{i+3}$. Then $q_i \leq w_j \wedge w_k$ and $w_i \leq \kappa(q_i)$, so $w_j \wedge w_k \nleq w_i$. Moreover, $q_{i+3} \leq w_i$ and $w_j \vee w_k \leq \kappa(q_{i+3})$ so $w_i \nleq w_j \vee w_k$. Thus by Theorem 1.13, $\{w_1, w_2, w_3\}$ generate a sublattice isomorphic to $\mathbf{FL}(3)$. So by Theorem 1.28, u/v contains $\mathbf{FL}(\omega)$ as a sublattice.

Corollary 9.11. *If u/v is an infinite interval in $\mathbf{FL}(X)$ and $|Y| \leq |X| + \omega$, then u/v contains a sublattice isomorphic to $\mathbf{FL}(Y)$.*

Proof: This corollary follows from the last theorem if X is finite. Otherwise the sublattice generated by

$$S = \{(x \wedge u) \vee v : x \in X - \mathrm{var}(u) - \mathrm{var}(v)\}$$

is isomorphic to $\mathbf{FL}(S)$ and is contained in u/v. Since $|S| = |X|$, the result follows.

If we prove this, quote the ref.

2. Coverless Elements

In this section we use Tschantz's Theorem to prove a sort of complement to Day's Theorem. Recall that an element of $\mathbf{FL}(X)$ is said to be coverless if it has no lower and no upper covers. In Example 3.39 we gave an example of a coverless element, but now we can show that there are many such elements.

Theorem 9.12. *Every infinite interval of a free lattice contains a coverless element.*

Throughout this section $\sigma : \mathbf{FL}(X) \rightarrowtail \mathbf{FL}(Y)$ denotes a lattice embedding. Let

$$w = w_1 \sqcup \ldots \sqcup w_n \tag{3}$$

be the canonical form of an element of $\mathbf{FL}(X)$, and assume $n > 1$. The theorem will be proved with the aid of Theorem 9.10 and Theorem 3.4. In order to apply the latter theorem, we need to understand how σ affects the canonical form of w. While it is not true that every $\sigma(w_i)$ is a canonical joinand of $\sigma(w)$, we will show that this is true if w_i is not a generator.

Lemma 9.13. *If w_1 is a canonical joinand of w and $w_1 \notin X$, then $\sigma(w_1)$ is a canonical joinand of $\sigma(w)$.*

Proof: Let the canonical form of w be given by (3) and let $w_1 = w_{11} \wedge \cdots \wedge w_{1k}$ canonically. Since $w_1 \notin X$, we have $k > 1$.

Let $\sigma(w) = u_1 \sqcup \ldots \sqcup u_r$ canonically. Since σ is a lattice embedding, $\sigma(w) = \sigma(w_1) \sqcup \ldots \sqcup \sigma(w_n)$, but $\sigma(w) > \sigma(w_2) \sqcup \ldots \sqcup \sigma(w_n)$. It follows that $\{u_1, \ldots, u_r\} \ll \{\sigma(w_1), \ldots, \sigma(w_n)\}$ but $\{u_1, \ldots, u_r\} \not\ll \{\sigma(w_2), \ldots, \sigma(w_n)\}$. Hence for some i we have

$$u_i \leq \sigma(w_1) = \sigma(w_{11}) \wedge \cdots \wedge \sigma(w_{1k}) \leq \sigma(w) = u_1 \sqcup \ldots \sqcup u_r.$$

For all j, we have $w_{1j} \not\leq w$, by one of the basic properties of canonical forms. Thus, since σ is a lattice embedding, $\sigma(w_{1j}) \not\leq \sigma(w)$ for all j. Hence, by (W), $u_i \leq \sigma(w_1) \leq u_j$ for some j. This forces $i = j$ and $u_i = \sigma(w_1)$, proving the lemma.

LEMMA 9.14. *If $w \in \mathbf{FL}(X)$, $w = w_1 \wedge \cdots \wedge w_n$ canonically with $n > 1$, and $w_1 = w_{11} \sqcup w_{12}$ canonically where $w_{1j} \notin X$ and $w_{1j} \not\leq w$ for $j = 1, 2$, then $\sigma(w)$ is not completely join irreducible in $\mathbf{FL}(Y)$.*

Proof: Applying Lemma 9.13 to w_1, we see that both $\sigma(w_{11})$ and $\sigma(w_{12})$ are canonical joinands of $\sigma(w_1)$. Moreover, since σ is a lattice embedding, $\sigma(w_{1j}) \not\leq \sigma(w)$ for $j = 1, 2$. Now by Theorem 3.4, $\sigma(w)$ is not completely join irreducible.

Now we are ready to complete the proof of Theorem 9.12. Take an infinite interval in a free lattice $\mathbf{FL}(X)$. By Tschantz's Theorem, there is an embedding σ of $\mathbf{FL}(\omega)$ into this interval. Let w be given by

$$w = ((x_0 \sqcup x_1) \wedge [(x_2 \wedge x_3) \sqcup (x_4 \wedge x_5)]) \sqcup (x_6 \wedge [(x_7 \wedge x_8) \sqcup (x_9 \wedge x_{10})]).$$

By the dual of Lemma 9.14 (with $w_1 = (x_0 \sqcup x_1) \wedge [(x_2 \wedge x_3) \sqcup (x_4 \wedge x_5)]$) we have that $\sigma(w)$ has no upper cover. By the same lemma, neither of its joinands has a lower cover. Thus, by Theorem 3.5, $\sigma(w)$ has no lower cover. This proves Theorem 9.12.

3. Maximal Chains

In this section we characterize those intervals in free lattices which contain a dense maximal chain, i.e., one with no covers. It follows from Theorem 3.5 that an element of a free lattice can have no more upper covers than the number of its canonical meetands. Thus it can have only finitely many upper (and lower) covers. This implies that if an interval of a free lattice is atomic, then every maximal chain in that interval must contain an atom and thus cannot be dense. Of course, if it is dually atomic it also cannot have a dense chain. The next theorem proves a converse to this.

THEOREM 9.15. *Suppose that $c > d$ in a free lattice and that c/d is neither atomic nor dually atomic. Then there is a maximal chain C from c to d such that every element of $C - \{c, d\}$ is coverless.*

Proof: First we prove the theorem under the stronger hypothesis that c/d has no atom and no coatom and then we shall show how to derive the full result from this.

Let $a_i \succ b_i$, $i > 0$, be an enumeration of all the covers in c/d. Let $C_0 = \{c, d\}$. We build chains C_i's with the following properties.

(1) Each C_i is finite.
(2) Each element of $C_i - \{c, d\}$ is coverless.
(3) $C_i \subseteq C_j$ if $i \leq j$.
(4) C_i has an element which is incomparable with at least one of a_i and b_i.

Let $C = \bigcup C_i$. Clearly any maximal chain in c/d which contains C will have no cover.

Inductively, suppose that $C_0, \ldots, C_{i-1}$ have been constructed satisfying (1)–(4). Let $a = a_i$ and $b = b_i$. If C_{i-1} already has an element incomparable with either a or b, we let $C_i = C_{i-1}$. Otherwise, since C_{i-1} is finite, there are elements $e > f$ in C_{i-1} with $e \geq a \succ b \geq f$ and C_{i-1} has no element strictly between e and f. Since the elements of $C_{i-1} - \{c, d\}$ are coverless, and c has no lower cover above d, and d has no upper cover below c, we must have $e > a \succ b > f$.

Let us first consider the case when a is join reducible and b is meet reducible. Let q be the canonical joinand of a which is not below b. Then q is completely join irreducible and $\kappa(q) > b$.

If $q \sqcup f < a$ and the interval $a/q \sqcup f$ is infinite, we can choose a coverless element r in this interval by Theorem 9.12. In this case we let $C_i = C_{i-1} \cup \{r\}$. Clearly r is incomparable to b, as $q \leq r$ and $q \nleq b$ implies $r \nleq b$, while $a > r$ and $a \succ b$ implies $r \ngeq b$.

Now suppose that $q \sqcup f < a$, but $a/q \sqcup f$ is finite. Theorem 7.2 implies that any chain of covers containing a can be of length at most two, since the hypotheses of the theorem insure that a is not in the connected component of 0 or 1. We claim that

$$(q \sqcup f) \wedge \kappa(q) \prec q \sqcup f.$$

Indeed, $(q \sqcup f) \wedge \kappa(q)$ is above f and hence joins with q to $q \sqcup f$. Now the definition of κ shows that the above is a covering.

Thus we must have $a \succ q \sqcup f \succ (q \sqcup f) \wedge \kappa(q)$ and also $(q \sqcup f) \wedge \kappa(q) \leq a \wedge \kappa(q) = b$. Hence $a/(q \sqcup f) \wedge \kappa(q)$ is not a three element interval, and by Theorem 7.6 there exists an element v with $(q \sqcup f) \wedge \kappa(q) \prec v \leq b \prec a$. Moreover $v < b$, since otherwise $\mathbf{2} \times \mathbf{2}$ is contained in the connected component of a, and, by Theorem 7.32, this only occurs in the connected components of 0 and 1. This situation is diagrammed in Figure 9.1.

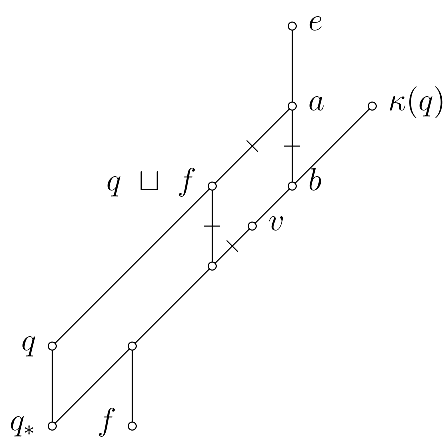


FIGURE 9.1

If $e \wedge \kappa(q)/b$ is infinite, we can again let $C_i = C_{i-1} \cup \{r\}$ for any coverless element r in this interval. If $e \wedge \kappa(q)/b$ is finite, then $b \prec e \wedge \kappa(q)$ by the dual of the arguments above. However, by Lemma 7.22, the only cover of b is a. If $e \wedge \kappa(q) = b$, then

$$e \wedge \kappa(q) = b \leq a = v \sqcup q$$

gives a violation of (W).

This leaves the possibility that $q \sqcup f = a$. Dually, we may assume that $\kappa(q) \wedge e = b$. But then

$$e \wedge \kappa(q) = b \leq a = f \sqcup q$$

is a clear violation of (W).

Now we must deal with the case when either a is join irreducible or b is meet irreducible. By duality we may assume the former, so that a is in fact completely join irreducible. If $e \wedge \kappa(a) = b = a_*$, then, by Corollary 8.5, e either covers a or is equal to a. But this contradicts the fact that e has no lower cover in e/f. If $e \wedge \kappa(a)/b$ is infinite we can simply augment C_{i-1} with a coverless element from this interval. Thus we may assume that $e \wedge \kappa(a)/b$ is finite. Arguments as above show that we have the situation diagrammed in Figure 9.2. However, this contradicts Lemma 7.22, which says that a must be a proper join in this situation. Thus the theorem is true in the case c/d has no atom and no coatom.

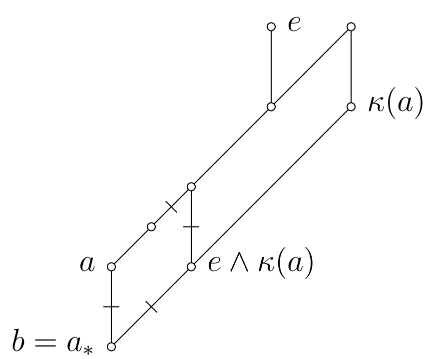


FIGURE 9.2

Now suppose that c/d is a nontrivial interval which is neither atomic nor dually atomic. Then, by definition, we can find e and f with $d < e < c$ and $c < f < c$ such that e/d has no atom and c/f has no coatom.

Since e/d is atomless, we can find a descending chain $e_0 = e > e_1 > e_2 > \cdots > d$ with $\bigwedge e_n = d$. Suppose that for some n, $e_n \vee f < c$. Then since e_n/d is infinite, it contains a coverless element v such that $d < v < e_n$. Since c/f has no coatom, $c/(e_n \vee f)$ also has no coatom. In particular it is infinite. Thus we can find a coverless element $u \in$

$c/(e_n \vee f)$ with $e_n \vee f < u < c$. Now all of the intervals v/d, u/v, and c/u are neither atomic nor dually atomic, and hence all contain dense maximal chains by what was proved above. The union of these chains is a dense maximal chain in c/d.

Thus we may assume that $e_n \vee f = c$ for all n. But then, since free lattices are continuous by Theorem 1.22,

$$f = f \vee d = f \vee \bigwedge e_n = \bigwedge (f \vee e_n) = c,$$

a contradiction.

CHAPTER X

Infinite Intervals

In this chapter we continue our study of the structure of infinite intervals in free lattices. By Corollary 8.7 no interval of a free lattice can be an infinite chain and, in fact, Tschantz's Theorem (Theorem 9.10) shows that every infinite interval contains $\mathbf{FL}(\omega)$. However these theorems do not rule out the possibility that there could be a triple of elements $a < c < b$ such that the intervals c/a and b/c are both infinite, and c is comparable with every element of b/a. In the first section of this chapter we shall prove that no such triple exists, confirming a conjecture formulated by Steve Tschantz in [**131**]. The proof turns out to be surprisingly difficult, and uses much of the machinery we have developed so far.

In Section 2 we prove a nice consequence of this result: *With the exception of some elements in the connected components of* 0 *and* 1*, each join irreducible element of* $\mathbf{FL}(X)$ *is a canonical joinand of some other element.* One might suspect that there is a simpler, more direct proof of this fact, but we were unable to find one.

In the third section we investigate maps from the sublattice $\mathbf{FL}(X)-Y$, for $Y \subseteq X$, onto $\mathbf{2}$. We show, for example, that every nongenerator of $\mathbf{FL}(X)$ is either below $\bigvee x_*$ or above $\bigwedge x^*$. This leads to the study of the relationship of elements and their duals, particularly, completely join irreducible elements such that $\kappa(a) = \mathbf{d}_a$.

Throughout this chapter we shall be working in a free lattice $\mathbf{FL}(X)$ with X finite and $|X| \geq 3$, unless otherwise specified. (Tschantz's conjecture and its corollary are trivially true when X is infinite.)

1. Tschantz Triples

By a *Tschantz triple* we mean a triple a, c, b of elements of $\mathbf{FL}(X)$ such that $a < c < b$, the intervals c/a and b/c are both infinite, and c is comparable with every element of b/a. By a *Tschantz element* we mean an element c for which there exists a Tschantz triple a, c, b.

Our aim is to show that these things do not exist.

THEOREM 10.1. *There is no Tschantz triple in* $\mathbf{FL}(X)$.

The proof will be divided into a series of lemmas and will occupy the rest of this section.

Lemma 10.2. *Let a, c, b be a Tschantz triple with the central element c join irreducible. Then c is a coverless element, and there exists an element $b' \leq b$ such that 0, c, b' is a Tschantz triple and $c = b' \wedge c_1$ for some canonical meetand c_1 of c.*

Proof: Since c/a and b/c are infinite, by Theorem 9.12 we can find coverless elements a' and b' with $a \leq a' < c < b' \leq b$. Clearly a', c, b' is also a Tschantz triple. By Theorem 9.15 there is a maximal chain from a' to b' consisting entirely of coverless elements. Of course, c must be in any maximal chain in b'/a', so c is coverless.

Let $c = c_1 \wedge \cdots \wedge c_k$ canonically. Since $c < b$, at least one canonical meetand, say c_1, is not above b. As c is coverless, we can find b'' with $c < b'' < b \wedge c_2 \wedge \cdots \wedge c_k$. Again it is obvious that a, c, b'' is a Tschantz triple, and $c_1 \wedge b'' = c$.

Moreover, we claim that 0, c, b'' is a Tschantz triple. Suppose $e \leq b''$ and that e is incomparable with c. Then the element $a \sqcup e$ is in b''/a and not below c, so

$$\bigwedge c_i = c \leq a \sqcup e.$$

By the choice of b'', no c_i is below b'', and hence no c_i is below $a \sqcup e$. As $c > a$ and $c \not\leq e$, this violates (W).

For the rest of this section we shall suppose that $c = c_1 \wedge \cdots \wedge c_k$ (canonically) is a meet reducible Tschantz element corresponding to a Tschantz triple 0, c, b with $b \wedge c_1 = c$ (so that $b \leq c_2 \wedge \cdots \wedge c_k$), and that no element shorter than c is a Tschantz element. With respect to Lemma 10.2, duality, and the easy fact that a generator cannot be a Tschantz element (since for any $x \in X$, the interval x^*/x_* contains elements incomparable with x; also generators are not coverless), in order to prove Theorem 10.1 it suffices to derive a contradiction from these assumptions.

The following lemma applies to Tschantz triples, but is more general because it does not require that the intervals are infinite. We will apply it to Tschantz triples with $p = c$, $q = c_1$ and $e = b$ to obtain Lemma 10.4. Another interesting special case is when e is an arbitrary completely join irreducible element, $p = e_*$ and $q = \kappa(e)$.

Lemma 10.3. *Let p be a meet reducible element, and let q be a canonical meetand of p. Suppose there is an element e such that $e > p$, $e \wedge q = p$ and every element below e is comparable with p. Then, for any element t of $\mathbf{FL}(X)$, one of the following holds*:

(1) $t \geq p$,

(2) $t \le q$,
(3) $t \le s$ *for a proper subelement* s *of* q *such that* $s \ngeq p$.

Proof: Replacing e by a smaller element if necessary, we can always ensure that every proper subelement of q above p is also above e. For let S denote the (finite) set of all proper subelements s of q with $s \ge p$. Let $e' = e \wedge \bigwedge S$. Then $e' > p$ because $e \wedge \bigwedge S = p$ would imply $q \ge s \ge p$ for some $s \in S$, since q is a canonical meetand of p, contrary to canonical form; see Theorem 1.17.

Suppose that there is an element t satisfying neither (1) nor (2) nor (3). We have $e \wedge t < p$, since $e \wedge t$ is comparable with p and $t \ngeq p$. Hence $e \wedge t < q$. Denote by s_0 the shortest subelement s of q such that $e \wedge t \le s$ and $s \ngeq e$. Since $s = q$ satisfies these properties, such an s_0 exists. By the requirement imposed on e above, either $s_0 = q$ or $s_0 \ngeq p$. Hence $t \nleq s_0$, since otherwise either (2) and (3) would hold.

If s_0 is a proper join, $s_0 = u_1 \vee \cdots \vee u_m$ canonically, then an application of (W) to $e \wedge t \le s_0$ yields $e \wedge t \le u_i$ for some i, since $e \nleq s_0$ and $t \nleq s_0$. But then u_i contradicts the minimality of s_0.

If s_0 is a proper meet, $s_0 = u_1 \wedge \cdots \wedge u_m$ canonically, then $s_0 \ngeq e$ implies $u_i \ngeq e$ for at least one i. But $e \wedge t \le u_i$ and u_i again contradicts the minimality of s_0.

Finally, if s_0 is a generator, then $e \wedge t \le s_0$ implies either $e \le s_0$ or $t \le s_0$, a contradiction.

Lemma 10.4. *Every element of* $\mathbf{FL}(X)$ *is either above* c *or below* c_1 *or below one of the (finitely many) completely meet irreducible subelements of* c_1 *that are not above* c.

Proof: This follows from Lemma 10.3 with the substitutions indicated above. Let s be a maximal subelement of c_1 not above c, and suppose $s \nleq c_1$. Then any element strictly above s is above c, since otherwise it would be below another subelement of c_1 not above c, contradicting the maximality of s. Thus s is completely meet irreducible with $s^* = s \vee c$.

Lemma 10.5. *Let* u *be an element incomparable with* c. *Then either* $u \le c_1$ *or* $u \vee c \ge c_i$ *for some* $i = 1, \ldots, k$.

Proof: Suppose that $u \nleq c_1$ and that c is a canonical joinand of $u \vee c$. By Lemma 10.2, c is not completely join irreducible and so, according to Theorem 3.5, there is no lower cover of $u \vee c$ above all the remaining canonical joinands of $u \vee c$. Consequently, there is an increasing sequence $d_0 < d_1 < d_2 < \cdots$ converging up to $u \vee c$, with $d_0 = u$. Since none of the elements d_i is either above c or below c_1, by Lemma 10.4 each of them is below one of the finitely many subelements

s of c_1 that are not above c. But then all the elements d_i are below one such element s and we get $u \vee c \leq s$, so that $c \leq s$, a contradiction.

This proves that if $u \nleq c_1$, then c is not a canonical joinand of $u \vee c$, so that $u \vee c = u \vee c'$ for some $c' < c$. By (W), it follows from $u \vee c' \geq c = c_1 \wedge \cdots \wedge c_k$ that $u \vee c' \geq c_i$ for some i.

The next few lemmas will analyze the canonical joinands of c_1 which lie below c, with the aim of showing that there is exactly one such and that it is totally atomic.

LEMMA 10.6. *The element c_1 has a canonical joinand below c. Moreover, every canonical joinand of c_1 below c is totally atomic.*

Proof: Because c is coverless, no canonical meetand c_i of c is completely meet irreducible; in particular, $c_1 \notin X$. Moreover, since generators are meet prime, $c \leq x \in X$ implies $c_i \leq x$ for some i. Thus, since b can be replaced with a smaller element if necessary, we can assume that $b \leq c_2 \wedge \cdots \wedge c_k$, that $b \leq x$ whenever $c \leq x \in X$, and that $b \leq c \vee u$ whenever $u \nleq c_1$ is a subelement of c.

Suppose that there is no canonical joinand of c_1 below c. Under this assumption, we will prove by induction that for any element $w \in \mathbf{FL}(X)$, either $w \geq b$ or $b \wedge w \leq c$. This will make b an upper cover of c, which is impossible. The reduction in the first paragraph (and the definition of a Tschantz triple) make this true when w is a generator. The case when w is a proper meet is straightforward. So let $w = w_1 \vee w_2$, where both w_1 and w_2 have the property, and suppose that $w \ngeq b$ and $b \wedge w \nleq c$, whence $b \wedge w > c$. Denote by $f : \mathbf{FL}(X) \to \mathbf{L}^{\vee}(c)$ the standard epimorphism. If $u \in \mathrm{J}(c)$ and $u \leq w$, then $u \leq c_1$ because otherwise, again using the reduction in the first paragraph, we would have $b \leq c \vee u \leq w$. Hence $f(w) \leq f(c_1) = c_1$, and then also $f(w_1) \leq c_1$. If we suppose that $w_1 \nleq c_1$, then w_1 is incomparable with c as $b \wedge w_1 \leq c$ by induction. Consequently, by Lemma 10.5, $w_1 \vee c \geq c_i$ for some i. This gives $c_i = f(c_i) \leq f(w_1 \vee c) = f(w_1) \vee f(c) \leq c_1$, so that $i = 1$ and $f(w_1) \vee c = c_1$. But c_1 has no canonical joinand below c, so $f(w_1) = c_1$ and hence $w_1 \geq c_1$. As $w_1 \neq c_1$, we have $w_1 > c_1$. Thus $b \wedge w_1 > c$ and (by induction) $w_1 \geq b$, which makes $w \geq b$, a contradiction. Hence $w_1 \leq c_1$. Similarly, $w_2 \leq c_1$. But then $w \leq c_1$, so that $b \wedge w \leq c$, a contradiction.

Thus there is at least one canonical joinand of c_1 below c. Now we want to show that any such joinand is totally atomic.

Let t be a canonical joinand of c_1 such that $t < c$. Then $c_1 = t \vee r$ where r is the join of the remaining canonical joinands of c_1. Suppose that t is not completely join irreducible, so that there is an increasing sequence $r = r_0 < r_1 < r_2 < \cdots$ converging up to c_1. For every subelement s of c_1 not above c there exists an i with $r_i \not\leq s$, since otherwise we would have $c_1 \leq s$ and consequently $c \leq s$. Because there are only finitely many subelements of c_1, it follows that there is an i such that $r_i \not\leq s$ for any subelement s of c_1 with $s \not\geq c$. If u is any element above r_i, then $u \not\leq s$ for any subelement s of c_1 with $s \not\geq c$ and so, by Lemma 10.4, either $u \leq c_1$ or $u \geq c$. If $u \geq c$, then $u \geq c \vee r = c_1$. We see that either r_i, c_1, 1 is a Tschantz triple, or else the interval $1/c_1$ is finite. But c_1 is not a Tschantz element, since it is shorter than c. Thus $1/c_1$ is finite. It is not hard to see that of the elements in the connected component of 1, only the coatoms can be canonical meetands of another element (cf. Theorem 10.20). Hence c_1 is a coatom and $t \in X$, contrary to our assumption that it was not completely join irreducible.

Thus t is completely join irreducible. Since $t < c$ and c is coverless, we can find an element v with $t < v < c$. By the dual of Day's theorem, there is a completely meet irreducible element q such that $q \geq v$ and $q \not\geq c$. As $b \wedge q$ must be comparable to c, we have

$$t < v \leq b \wedge q \leq c \leq c_1 = \bigvee c_{1i}.$$

Applying (W) to this inequality gives $t < q \leq c_1$. Hence t is upper atomic by Theorem 3.28, since t is a canonical joinand of c_1 and q is completely meet irreducible. Altogether, this makes t totally atomic.

Lemma 10.7. *Let t be a canonical joinand of c_1 below c. There exists a unique canonical meetand d of t not above b (if $t \in X$, let $d = t$). Every element of $\mathbf{FL}(X)$ above t is either above c or below c_1 or below d.*

Proof: Clearly, there exists a canonical meetand d of t not above c. Since $b \wedge d$ is comparable with c, we have $b \wedge d \leq c$ and hence $b \wedge d \leq c_1$. By (W), taking the canonical representation of c_1 into account, we see that $b \wedge d$ is below a canonical joinand of c_1, and consequently $b \wedge d = t$. Hence every other canonical meetand of t is above b. If $u \geq t$ and $u \not\geq c$, then $b \wedge u \leq c \leq c_1$, so that by (W), either $u \leq c_1$ or $b \wedge u$ is below a canonical joinand of c_1. The last possibility together with $t \leq b \wedge u$ implies $b \wedge u = t$, and consequently $u \leq d$.

Lemma 10.8. *There are at most two canonical joinands of c_1 below c. If c_1 has two distinct canonical joinands below c, then both are atoms.*

Proof: Let t_1 and t_2 be two distinct canonical joinands of c_1 below c. By Lemma 10.6 they are both totally atomic. Since $t_1 \vee t_{2*} \not\geq t_2$ by canonical form, $t_1 \leq \kappa(t_2)$. Similarly, $t_2 \leq \kappa(t_1)$.

Since $\kappa(t_2) \ngeq c$, by Lemma 10.7 the element $\kappa(t_2)$ is either below c_1 or below the unique canonical meetand d_1 of t_1 not above b. In the latter case we have $\kappa(t_2) = d_1$, because $\kappa(t_2)^*$ is above c, while d_1 is not. But then the element $\kappa(t_2)$ is a canonical meetand of a totally atomic element, so that it is totally atomic itself, and consequently by Lemma 6.25, t_2 is a meet of generators and $\kappa(t_2) \in X^\vee$.

Similarly, either $\kappa(t_1) \leq c_1$ or $\kappa(t_1) \in X^\vee$ is the unique canonical meetand d_2 of t_2 not above b.

Suppose that $\kappa(t_1) \leq c_1$. Since $\kappa(t_1)$ is above all the canonical joinands of c_1 except t_1, we have $\kappa(t_1)^* = c_1$. If also $\kappa(t_2) \leq c_1$, then similarly $c_1 = \kappa(t_2)^*$, so that c_1 has two distinct meet irreducible lower covers and thus $c_1 = 1$ by Theorem 7.3, a contradiction. Hence $\kappa(t_2) = d_1$, and t_2 is a meet of generators. Now t_2 is a canonical joinand of $c_1 = \kappa(t_1)^*$ not below t_1, so it is a canonical joinand of $\kappa(t_1)$ by Theorem 3.34. It follows from Lemma 6.25 that t_2 is a generator, and then the element $d_1 = \kappa(t_2)$ is a coatom. But d_1 is a canonical meetand of the totally atomic element t_1. It is then clear that t_1 is a meet of two coatoms, so that c is above a meet of two coatoms, a contradiction.

We get $\kappa(t_1) \nleq c_1$ and, similarly, $\kappa(t_2) \nleq c_1$. Consequently, $\kappa(t_1) = d_2$ and $\kappa(t_2) = d_1$. Hence t_1 determines $t_2 = b \wedge d_2$, and *vice versa*, which makes t_1 and t_2 the only two canonical joinands of c_1 below c. Moreover, $t_1 \in X^\wedge$ and $t_2 \in X^\wedge$; but d_1 is a canonical meetand of t_1, so d_1 is a generator and thus t_2 is an atom. Similarly, t_1 is an atom.

We can dispense now with the case when c_1 has two canonical joinands below c.

Lemma 10.9. *The element c_1 has exactly one canonical joinand below c.*

Proof: Otherwise, by Lemma 10.8, there are exactly two canonical joinands of c_1 below c, and they are of the form $\underline{x}_1$ and $\underline{x}_2$ where $x_1, x_2 \in X$ and $\underline{x} = \bigwedge(X - \{x\})$. By Lemma 10.7, both $\underline{x}_1$ and $\underline{x}_2$ have all but one canonical meetand above b, which makes $b \leq \bigwedge(X - \{x_1, x_2\})$.

If there is an element w such that $w \nleq c_1$ and $\underline{x}_1 \vee \underline{x}_2 \vee w \ngeq c$, then $b \wedge (\underline{x}_1 \vee \underline{x}_2 \vee w) \leq c \leq c_1$, from which it follows by (W) that $b \wedge (\underline{x}_1 \vee \underline{x}_2 \vee w)$ is below a canonical joinand of c_1. However, the element $b \wedge (\underline{x}_1 \vee \underline{x}_2 \vee w)$ is above two distinct canonical joinands of c_1, so this is impossible. Hence for every element w we have either $w \leq c_1$ or $\underline{x}_1 \vee \underline{x}_2 \vee w \geq c$.

Clearly, every canonical joinand of c_1 not below c is below $\kappa(\underline{x}_1) \wedge \kappa(\underline{x}_2) = x_1 \wedge x_2$, and hence $c_1 \leq \underline{x}_1 \vee \underline{x}_2 \vee (x_1 \wedge x_2)$. This implies $|X| > 3$.

Denote the elements of X by $x_1, \dots, x_n$. Let $j \in \{3, \dots, n\}$. With j fixed, let us define three infinite sequences r_i, s_i, t_i $(i = 0, 1, 2, \cdots)$ of elements of $\mathbf{FL}(X)$:

$$r_0 = \bigwedge_{l \neq 2,j} x_l, \qquad r_{i+1} = x_1 \wedge (\underline{x}_2 \vee s_i) \wedge (\underline{x}_j \vee t_i),$$

$$s_0 = \bigwedge_{l \neq 1,j} x_l, \qquad s_{i+1} = x_2 \wedge (\underline{x}_1 \vee r_i) \wedge (\underline{x}_j \vee t_i),$$

$$t_0 = \bigwedge_{l \neq 1,2} x_l, \qquad t_{i+1} = x_j \wedge (\underline{x}_1 \vee r_i) \wedge (\underline{x}_2 \vee s_i).$$

Each of the three sequences is decreasing: it is easy to check that $r_1 \leq r_0$, $s_1 \leq s_0$, $t_1 \leq t_0$, and the rest follows by simultaneous induction.

Easily by induction on i,

$$\underline{x}_2 \vee \underline{x}_j \leq r_i, \qquad \underline{x}_1 \vee \underline{x}_j \leq s_i, \qquad \underline{x}_1 \vee \underline{x}_2 \leq t_i.$$

Consequently,

$$\underline{x}_1 \vee \underline{x}_2 \vee \underline{x}_j \leq (\underline{x}_1 \vee r_i) \wedge (\underline{x}_2 \vee s_i) \wedge (\underline{x}_j \vee t_i)$$

for all i.

Let us prove by induction on the length of an arbitrary element w that if either $w \leq \underline{x}_1 \vee r_i$ for all i, or $w \leq \underline{x}_2 \vee s_i$ for all i, or $w \leq \underline{x}_j \vee t_i$ for all i, then $w \leq \underline{x}_1 \vee \underline{x}_2 \vee \underline{x}_j$. The claim is vacuously true for generators, and is clearly preserved under joins. So assume that $w = w_1 \wedge w_2$ and that both w_1 and w_2 satisfy the claim. Let $w \leq \underline{x}_1 \vee r_i$ for all i; the proof would be similar in the other two cases. If $w \leq \underline{x}_1$, then $w \leq \underline{x}_1 \vee \underline{x}_2 \vee \underline{x}_j$ is evident. If either w_1 or w_2 is below $\underline{x}_1 \vee r_i$ for all i, the same follows by induction. Thus we may assume that $w \not\leq \underline{x}_1$ and that there exists an i_0 such that $w_1 \not\leq \underline{x}_1 \vee r_i$ and $w_2 \not\leq \underline{x}_1 \vee r_i$ for all $i \geq i_0$. Applying (W) to $w_1 \wedge w_2 \leq \underline{x}_1 \vee r_i$, we get $w \leq r_i$ for all $i \geq i_0$, and so in fact for all i. Hence $w \leq r_0$ and $w \leq \underline{x}_2 \vee s_i$ for all i and $w \leq \underline{x}_j \vee t_i$ for all i. As before, we either get the desired conclusion from one of the latter two alternatives, or else we get $w \leq r_0 \wedge s_0 \wedge t_0 = 0$, and the claim holds trivially.

This shows that each of the three sequences $\underline{x}_1 \vee r_i$, $\underline{x}_2 \vee s_i$ and $\underline{x}_j \vee t_i$ converges down to $\underline{x}_1 \vee \underline{x}_2 \vee \underline{x}_j$.

Let us prove by induction on i that none of the three elements r_i, s_i, t_i is below $\underline{x}_1 \vee \underline{x}_2 \vee (x_1 \wedge x_2)$. This is easy to check for $i = 0$, so let $i > 0$. Suppose

$$r_i = x_1 \wedge (\underline{x}_2 \vee s_{i-1}) \wedge (\underline{x}_j \vee t_{i-1}) \leq \underline{x}_1 \vee \underline{x}_2 \vee (x_1 \wedge x_2).$$

Clearly x_1 is not below $\underline{x}_1 \vee \underline{x}_2 \vee (x_1 \wedge x_2)$, and by induction neither are s_{i-1} and t_{i-1}. On the other hand, $\underline{x}_2 \vee \underline{x}_j \leq r_i$, and this is enough to keep r_i from being below any of the elements $\underline{x}_1$, $\underline{x}_2$ and $x_1 \wedge x_2$. By (W) we get $r_i \nleq \underline{x}_1 \vee \underline{x}_2 \vee (x_1 \wedge x_2)$. The argument is similar (though not totally symmetric) for s_i and t_i in place of r_i.

We have seen above that $c_1 \leq \underline{x}_1 \vee \underline{x}_2 \vee (x_1 \wedge x_2)$, and we have just proved $r_i \nleq \underline{x}_1 \vee \underline{x}_2 \vee (x_1 \wedge x_2)$. Hence $r_i \nleq c_1$ for all i. By our earlier observation, this implies $\underline{x}_1 \vee \underline{x}_2 \vee r_i \geq c$, i.e., $\underline{x}_1 \vee r_i \geq c$ for all i. Since the sequence converges down to $\underline{x}_1 \vee \underline{x}_2 \vee \underline{x}_j$, we get $\underline{x}_1 \vee \underline{x}_2 \vee \underline{x}_j \geq c$.

Since $j > 2$ was arbitrary, this proves $c \leq \bigwedge_{j>2}(\underline{x}_1 \vee \underline{x}_2 \vee \underline{x}_j) = (\underline{x}_1 \vee \underline{x}_2)^*$, making the interval $c/0$ finite, a contradiction.

The unique canonical joinand of c_1 below c will be denoted by t in the rest of this section. By Lemma 10.6, the element t is totally atomic. Let us denote by $q_1, \dots, q_m$ all the remaining canonical joinands of c_1, i.e., those that are not below c. Put $q = q_1 \vee \dots \vee q_m$. As usual, $q \leq c_1 \wedge \kappa(t) \prec c_1$ by Theorem 3.5.

Should we replace q by r, q_j by r_j or c_{1j}?

Let us choose an element v with $t < v < c$. It follows from Lemma 10.7 that *every element above v is either above c or below c_1.* For if $w \leq d$, where d is the unique canonical meetand of t not above b, then $b \wedge w \leq b \wedge d = t$ and hence $w \ngeq v$.

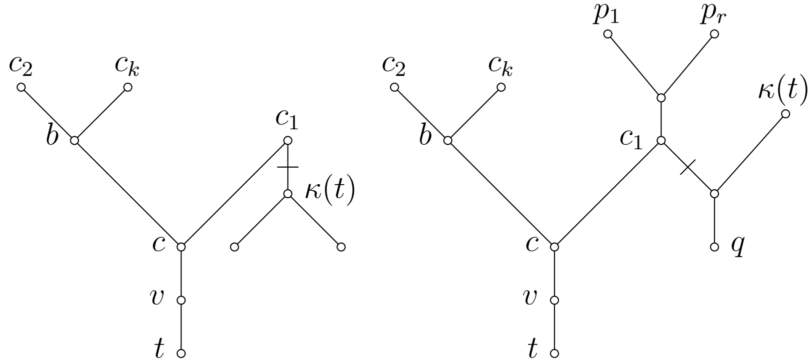


Figure 10.1

The argument will eventually split into two cases, depending on whether $\kappa(t) < c_1$ (and hence $\kappa(t)^* = c_1$) or $\kappa(t) \nleq c_1$. These situations are sketched in Figure 10.1. For the case when $\kappa(t) \nleq c_1$, we require another series of lemmas.

LEMMA 10.10. *Suppose $\kappa(t) \nleq c_1$. If u is an element such that $u \geq c$, $u \nleq c_1$ and $u \ngeq c_i$ for all i, then $u \wedge \kappa(t) \leq q_j$ for some j.*

Proof: If $v \vee (u \wedge \kappa(t)) \geq c$, then $v \vee (u \wedge \kappa(t)) \geq c_i$ for some i by (W), which implies $u \geq v \vee (u \wedge \kappa(t)) \geq c_i$, a contradiction. Thus $v \vee (u \wedge \kappa(t)) \leq c_1$ by the remark above, and hence $u \wedge \kappa(t) \leq c_1$. By (W), $u \wedge \kappa(t) \leq c_{1j}$ for a canonical joinand c_{1j} of c_1. If $u \wedge \kappa(t) \leq t$, then $u \wedge \kappa(t) = t_*$, whence $u \succ t$ by Corollary 8.5, which is impossible because $u \geq c$. Thus we must have $u \wedge \kappa(t) \leq q_j$ for some j.

LEMMA 10.11. *If $\kappa(t) \nleq c_1$ and $q < c_1 \wedge \kappa(t)$, then c_1 is lower atomic.*

Proof: Suppose, on the contrary, that there is an increasing infinite sequence $u_0 < u_1 < u_2 < \cdots$ converging up to c_1. Clearly, we can assume that $u_i \nleq \kappa(t)$ for almost all i; this together with $u_i \leq c_1 \leq \kappa(t)^*$ implies $u_i \geq t$. If $c \vee u_i = c_1$, then $q \leq u_i$ and hence $u_i \geq q \vee t = c_1$, a contradiction. Thus $c \vee u_i < c_1$ for all i, and hence we can assume that $u_i \geq c$, as the sequence $c \vee u_i$ could otherwise be considered instead of the sequence u_i. Since c is coverless, c_1 has no upper cover, and hence there exists an element $w > c_1$ such that $w \ngeq b$. Now c_1 is not a Tschantz element, as it is shorter than c which is one of minimal length, so the triple u_i, c_1, w is not a Tschantz triple. As the intervals w/c_1 and c_1/u_i are all infinite, this means that for each i there exists an element $s_i \in w/u_i$ incomparable with c_1. Since $s_i \leq w$ implies $s_i \ngeq b$, we also have $s_i \ngeq c_j$ for $j \geq 2$. Hence by Lemma 10.10, for every i there exists an index j_i such that $s_i \wedge \kappa(t) \leq q_{j_i}$. It follows that $u_i \wedge \kappa(t) \leq q_{j_i} \leq q$ for all i. However, the sequence $u_i \wedge \kappa(t)$ converges up to $c_1 \wedge \kappa(t)$ by continuity, while all its members are below q, contrary to our assumption that $q < c_1 \wedge \kappa(t)$.

LEMMA 10.12. *If $\kappa(t) \nleq c_1$, then $q = c_1 \wedge \kappa(t)$.*

Proof: Suppose, on the contrary, that $q < c_1 \wedge \kappa(t)$. Then c_1 is lower atomic by Lemma 10.11, and consequently the elements $q_1, \ldots, q_m$ are completely join irreducible. Choose an index $j \in \{1, \ldots, m\}$ such that q_{j*} is maximal among the elements $q_{1*}, \ldots, q_{m*}$. Now $q_j \leq q_{j*} \vee c$ would imply $c_1 = c \vee q_{j*} \vee \bigvee_{i \neq j} q_i$, which is not refined by $\{t, q_1, \ldots, q_m\}$. Hence $q_j \nleq q_{j*} \vee c$ and $c \leq \kappa(q_j)$; in fact, $q < \kappa(q_j)$ as c is coverless. Also $c_1 \wedge \kappa(q_j) \prec c_1 \leq \kappa(q_j)^*$, as usual.

If $\kappa(q_j) \leq c_1$, then $c_1 = \kappa(q_j)^*$. This implies that every canonical joinand of c_1 except q_j is a canonical joinand of $\kappa(q_j)$ by Theorem 3.34. In particular, t is a canonical joinand of $\kappa(q_j)$, so no canonical joinand of $\kappa(q_j)$ is above c. Also, no c_i is below $\kappa(q_j)$ since $\kappa(q_j) < c_1$. But then $c_1 \wedge \cdots \wedge c_k \leq \kappa(q_j)$, when expanded, gives a violation of (W).

This proves that $\kappa(q_j) \not\leq c_1$, i.e., $c_1 \wedge \kappa(q_j) < \kappa(q_j)$. Suppose that there is an element e with $c_1 \wedge \kappa(q_j) \prec e \leq \kappa(q_j)$. By Theorem 3.5, the unique canonical meetand w of $c_1 \wedge \kappa(q_j)$ not above e is completely meet irreducible. Clearly $w \geq c_1$, but $w \neq c_1$, since c_1 is not completely meet irreducible, so $w > c_1$. Now $e \wedge w \leq c_1$ when expanded again gives a violation of (W).

Thus there exists an infinite decreasing sequence $\kappa(q_j) \geq u_0 > u_1 > u_2 > \cdots$ converging down to $c_1 \wedge \kappa(q_j)$. Clearly, for almost all i, u_i is not above any of the elements $c_2, \ldots, c_k$ and u_i is incomparable with c_1. We can assume that this is true for all i. By Lemma 10.10 there exists an $r \in \{1, \ldots, m\}$ such that $u_i \wedge \kappa(t) \leq q_r$ for all i. (More precisely, the lemma says that for each i there is an r_i, but as there are only finitely many elements to choose from, it follows that there is one r good for all i.) Hence $\kappa(q_j) \wedge c_1 \wedge \kappa(t) \leq q_r$. We have $q_{j*} \leq \kappa(q_j) \wedge c_1 \wedge \kappa(t)$, so $q_{j*} \leq q_r$, and $q_{j*} \neq q_r$ as $q_r \not\leq q_j$. Hence $q_{j*} \leq q_{r*}$, and by the maximality of q_{j*} we get $q_{j*} = q_{r*}$. If $\kappa(q_j) \wedge c_1 \wedge \kappa(t) \leq q_j$, then $\kappa(q_j) \wedge c_1 \wedge \kappa(t) = q_{j*}$, and this meet is refined by the canonical expression of q_{j*} as the meet of $\kappa(q_j)$ and the canonical meetands of q_j not above $\kappa(q_j)$. It follows that each canonical meetand of q_j is either above $c_1 \wedge \kappa(t)$, or else above $\kappa(q_j)^*$ and hence also above c_1. This implies that $q_j \geq c_1 \wedge \kappa(t) > q$, a contradiction. Hence $\kappa(q_j) \wedge c_1 \wedge \kappa(t) \not\leq q_j$. But then $r \neq j$ and $\kappa(q_j) \wedge c_1 \wedge \kappa(t) = q_r$. Now $q_{j*} = q_{r*}$ implies by the dual of Theorem 7.3 that $q_{j*} = 0$, and by the maximality of q_{j*}, the elements $q_1, \ldots, q_m$ all are atoms. Since $\kappa(t)^* \geq c_1$, $\kappa(t)$ is a canonical meetand of $\kappa(q_j) \wedge c_1 \wedge \kappa(t) = q_r$, so that $\kappa(t) \in X$ and t is also an atom. Consequently, c_1 is a join of atoms. Since $t < c$, there is an atom $t' < c$ different from t; of course, $t' \neq q_i$ for all i because the q_i's are not below c. However $t' \leq c \leq c_1 = t \vee q_1 \vee \cdots \vee q_m$, which is impossible because atoms are join prime. This final contradiction shows that our initial assumption that $q < c_1 \wedge \kappa(t)$ was wrong.

LEMMA 10.13. *Suppose $\kappa(t) \not\leq c_1$. Then t and $c_1 \wedge \kappa(t)$ are the only two canonical joinands of c_1, the second being a lower cover.*

Proof: Since $q = q_1 \sqcup \ldots \sqcup q_m = c_1 \wedge \kappa(t)$ is a proper meet by the last lemma, we must have $m = 1$. Thus $c_1 = t \vee q$ canonically.

LEMMA 10.14. *Suppose $\kappa(t) \not\leq c_1$. Then the element $c_1 \wedge \kappa(t)$ is upper atomic.*

Proof: Denote by $p_1, \ldots, p_r$ the canonical meetands of $c_1 \wedge \kappa(t)$ that are above c_1; it follows from Lemma 10.13 that they are strictly above c_1 as $c_1 \wedge \kappa(t)$ is a subelement of c_1.

Suppose, contrary to the assertion, that there is an infinite decreasing sequence $u_0 > u_1 > u_2 > \cdots$ converging down to $c_1 \wedge \kappa(t)$. We have $u_i \not\geq c_1$ for almost all i, so we can assume that $u_i \not\geq c_1$ for all i. Since $u_i \geq t_*$, but $u_i \not\geq t$ as $u_i \vee t \geq (c_1 \wedge \kappa(t)) \vee t = c_1$, it follows that $u_i \leq \kappa(t)$. The sequence $v \vee u_i$ converges by continuity down to $v \vee (c_1 \wedge \kappa(t)) = c_1$. From this fact it follows that there is an index i such that $v \vee u_i$ is not above any of the elements $c_2, \ldots, c_k$, $p_1, \ldots, p_r$, $\kappa(t)$. Let us fix one such index i.

Since the element $c_1 \wedge \kappa(t)$ is shorter than c, being a canonical joinand of c_1, the triple 0, $c_1 \wedge \kappa(t)$, u_i is not a Tschantz triple. The intervals $c_1 \wedge \kappa(t)/0$ and $u_i/c_1 \wedge \kappa(t)$ are both infinite, so there exists an element $e < u_i$ incomparable with $c_1 \wedge \kappa(t)$. Note $e \not\leq c_1$ else $e \leq c_1 \wedge u_i \leq c_1 \wedge \kappa(t)$. Thus $v \vee e \not\leq c_1$, which implies $v \vee e \geq c = c_1 \wedge \cdots \wedge c_k$. We apply (W) to this inclusion. Now $v < c$, and $c \not\leq e$ because $t < c$ while $e \leq u_i \leq \kappa(t)$. Nor is $v \vee e$ above any of the elements $c_2, \ldots, c_k$, since $v \vee u_i$ is not above any of them. Hence

$$v \vee e \geq c_1 \geq c_1 \wedge \kappa(t) = p_1 \wedge \cdots \wedge p_r \wedge \kappa(t).$$

Now $v \vee e$ is not above any of the elements $p_1, \ldots, p_r$, $\kappa(t)$, since $v \vee u_i$ is not above them. Clearly $v \not\geq c_1 \wedge \kappa(t)$, and by construction $e \not\geq c_1 \wedge \kappa(t)$. This violation of (W) provides a final contradiction.

We need only a single simple lemma for the case $\kappa(t) < c_1$.

Lemma 10.15. *Let w be a nonzero, join irreducible, totally atomic element such that $\kappa(w)$ is semisingular. Then $w = (z \vee x_1) \wedge \cdots \wedge (z \vee x_m)$ for an integer $m \geq 2$ and pairwise distinct generators z, $x_1, \ldots, x_m$. Moreover, $\kappa(w) = z \vee d$ canonically for an upper atomic element d with the upper covers $z \vee d$, $x_1 \vee d, \ldots, x_m \vee d$.*

Proof: By semisingularity, $\kappa(w)$ has a totally atomic canonical joinand below w. By Lemma 6.25, this canonical joinand is a generator and w is either a meet of generators or an element of the form $w = \mu_{y_1 \wedge \cdots \wedge y_l} \sigma_z(x_1 \wedge \cdots \wedge x_m)$. If either $w \in X^\wedge$ or $l > 0$, then it is easy to check using Theorem 6.21 that $\kappa(w)$ is not semisingular. On the other hand, if $w = (z \vee x_1) \wedge \cdots \wedge (z \vee x_m)$, then $\kappa(w) = z \vee d$ where d is a meet of coatoms, $d = \overline{z} \wedge \overline{x}_1 \wedge \cdots \wedge \overline{x}_m$, and it is easy to check that this is semisingular with the desired upper covers.

Now for the last lemma.

Lemma 10.16. *There is an element b' with $c < b' \leq b$ such that every element above c is either above b' or below c_1.*

Since $b \wedge c_1 = c$, the element b' will be an upper cover of c, contrary to our earlier result that c must be coverless. Thus this lemma will complete the proof of Theorem 10.1.

Proof: In the case when $\kappa(t) \nleq c_1$, the element $c_1 \wedge \kappa(t)$ is the only canonical joinand of c_1 except t by Lemma 10.13. In that case, as before let $p_1, \ldots, p_r$ denote the canonical meetands of $c_1 \wedge \kappa(t)$ other than $\kappa(t)$; they are all strictly above c_1. Denote by c_1, $h_1, \ldots, h_r$ the upper covers of the element $c_1 \wedge \kappa(t)$, which is upper atomic by Lemma 10.14.

In either case, let S be the finite set consisting of the following elements:

(1) the generators not below c_1;
(2) the canonical meetands of any canonical joinand of c_1 which does not belong to X;
(3) the elements $p_1, \ldots, p_r$, $h_1, \ldots, h_r$, $\kappa(t)$ if $\kappa(t) \nleq c_1$.

Now every element $s \in S$ has the property that $s \nleq c_1$, and hence $v \vee s \geq c$. Define

$$b' = b \wedge \bigwedge_{s \in S} (a \vee s).$$

As $b' \geq c$, this element is below no canonical joinand of c_1. An easy application of (W) then yields $b' \nleq c_1$. Consequently, the element b' satisfies $c < b' \leq b$ and $v \vee s \geq b'$ for all $s \in S$.

We will prove by induction on the length of an arbitrary element $u \in \mathbf{FL}(X)$ that either $u \leq c_1$ or $v \vee u \geq b'$. This gives the claim of the lemma when $u \geq c$.

If $u \in X$ and $u \nleq c_1$, then $u \in S$ and consequently $v \vee u \geq b'$.

Let $u = u_1 \vee u_2$ where u_1 and u_2 are both shorter than u. If $u \nleq c_1$, then $u_i \nleq c_1$ for at least one $i \in \{1, 2\}$. By induction $v \vee u_i \geq b'$, and hence $v \vee u \geq v \vee u_i \geq b'$.

Let $u = u_1 \wedge u_2$ where u_1 and u_2 are both shorter than u, and assume $u \nleq c_1$. Then $u_1 \nleq c_1$ and $u_2 \nleq c_2$, so that by induction, $v \vee u_1 \geq b'$ and $v \vee u_2 \geq b'$. Since $u \nleq c_1$, we have $v \vee u \geq c$. If $u \geq c$, then $u_1 = v \vee u_1 \geq b'$ and $u_2 = v \vee u_2 \geq b'$, so that $u \geq b'$ and hence $v \vee u \geq b'$. So we may assume $u \ngeq c$ (and hence $u \ngeq c_1$). Then $v \vee u \geq c$ implies by (W) that $v \vee u \geq c_i$ for some i. If $i > 1$, then $v \vee u \geq c_i \geq b \geq b'$ and we are through. Thus we assume that $v \vee u \geq c_1$.

Consider first the case when $\kappa(t) \nleq c_1$, so that

$$v \vee u \geq c_1 \geq c_1 \wedge \kappa(t) = p_1 \wedge \cdots \wedge p_r \wedge \kappa(t).$$

Let us apply (W) to this inequality. If $v \vee u$ is above one of the elements $p_1, \ldots, p_r$ or $\kappa(t)$, then it is above b' because these elements belong to S.

Clearly, $v \ngeq c_1 \wedge \kappa(t)$. This leaves the possibility that $u \geq c_1 \wedge \kappa(t)$. Now $u \neq c_1 \wedge \kappa(t)$ since $u \nleq c_1$, and it is not above c_1 because it is not above c. Hence u is above one of the remaining upper covers $h_1, \ldots, h_r$ of $c_1 \wedge \kappa(t)$. Since $h_i \in S$ for all i, we get $v \vee u \geq v \vee h_i \geq b'$ for some i, as desired.

It remains to consider the case when $\kappa(t) \leq c_1$, i.e., $c_1 = \kappa(t)^*$. If there exists a canonical joinand r of c_1 such that $v \ngeq r$ and $u \ngeq r$, then (W) applied to $v \vee u \geq c_1 \geq r$ (expanding r) yields $v \vee u \geq s$ for some canonical meetand s of r. But then $s \in S$ and we get $v \vee u \geq v \vee s \geq b'$. So we may assume that $u \geq r$ for any canonical joinand r of c_1 not below v. If $\kappa(t)$ is not semisingular, then every canonical joinand of $\kappa(t)$ is a canonical joinand of $c_1 = \kappa(t)^*$, and on the other hand the only canonical joinand of c_1 below v is t, which is of course not a canonical joinand of $\kappa(t)$. So, if $\kappa(t)$ is not semisingular, u is above every canonical joinand of $\kappa(t)$, and thus $u \geq \kappa(t)$, which contradicts our assumption that u is incomparable with c_1. Hence $\kappa(t)$ must be semisingular, and thus by Lemma 10.15 we have $t = (z \vee x_1) \wedge \cdots \wedge (z \vee x_m)$ for pairwise distinct generators z, $x_1, \ldots, x_m$ with $m \geq 2$. Also, $\kappa(t) = z \vee d$ canonically for an upper atomic element d with upper covers $d \vee z$, $d \vee x_1, \ldots, d \vee x_m$. Now d is a canonical joinand of c_1 not below v, so we have $u \geq d$. Since $u \nleq c_1$, in fact $u > d$. Hence u is above an upper cover of d. It is not above $\kappa(t) = z \vee d$, so it is above one of the upper covers $d \vee x_1, \ldots, d \vee x_m$. In particular, d is above a generator x_i. Moreover, $x_i \nleq c_1$ because c_1/d is a three element interval, its middle element $\kappa(t)$ being semisingular. This makes $x_i \in S$ and $v \vee u \geq v \vee x_i \geq b'$.

We conclude that there are no Tschantz triples in a free lattice.

It is not hard to formulate problems related to Tschantz triples to which we do not know the answer:

Problem 10.17. *Does there exist a quadruple $a < c_1 \leq c_2 < b$ of elements of a free lattice $\mathbf{FL}(X)$ such that the intervals c_1/a and b/c_2 are both infinite and every element of b/a is comparable with either c_1 or c_2?*

A negative answer to this problem would be a strengthening of Theorem 10.1, which says that there is no such quadruple with $c_1 = c_2$. After the formulation of his conjecture in [**131**], S. Tschantz remarked that a stronger theorem may prove easier to establish inductively, and he suggested the following stronger version: No infinite interval u/v can be the union of finitely many intervals u/w_i and w_i/v if all the u/w_i and w_i/v are required to be infinite. However, this is not true. For example, if $|X| > 3$, then every element of $1/0$ is comparable with either one of the generators or with the meet of all coatoms, and the corresponding intervals are all infinite.

PROBLEM 10.18. *Describe all meet reducible elements a of $\mathbf{FL}(X)$ such that every element above a is comparable with a canonical meetand of a.*

Any element a with precisely two canonical meetands, one of them being an upper cover of a while the other one being $\kappa(w)$ for a completely join irreducible element w with $w_* < a$, has the desired property, so there is a large class of examples to this problem. Are there any other examples?

2. Join Irreducible Elements that are not Canonical Joinands

The canonical joinands of any element in a free lattice are of course join irreducible. A join irreducible element is its own (unique) canonical joinand, but we would expect that most join irreducible elements are also canonical joinands of some other element. We will verify this by characterizing, for a fixed free lattice $\mathbf{FL}(X)$ with $3 \le |X| < \infty$, those finitely many elements of $\mathrm{J}(\mathbf{FL}(X))$ which are not canonical joinands of some other element. We begin with a lemma of independent interest. Recall that the connected component of 0 is described in Examples 3.44 and 3.45.

LEMMA 10.19. *Let $a = a_1 \wedge \cdots \wedge a_k$ (canonically) be a meet reducible element of $\mathbf{FL}(X)$ such that the intervals a_i/a are all finite. Then a belongs to the connected component of* 1.

Proof: As it is easy to check, a does not belong to the connected component of 0. Suppose that it does not belong to the connected component of 1. Then, by Theorem 7.4, each of the intervals a_i/a is a chain of two or three elements.

Suppose that there is an element u with $a \prec u \prec a_i$ for some i. There is an index $j \neq i$ such that $u \not\leq a_j$, and there is an element v with $a \prec v \leq a_j$. Since $a_i/a = \{a_i, u, a\}$, we also have $v \not\leq a_i$. On the other hand, there is no element s with $a_i < s < a_i \vee v$, for otherwise $s \wedge v = a$ would contradict the canonical expression of a; see Theorem 1.19. Hence we have a chain of three covers $a \prec u \prec a_i \prec a_i \vee v$, which implies by Theorem 7.2 that these elements are in the connected component of either 0 or 1, a contradiction.

So, all of the elements a_i are upper covers of a, which makes $k = 2$. Arguing as above, we see that both a_1 and a_2 are covered by $a_1 \vee a_2$.

This gives another configuration which only exists in the connected components of 0 and 1, by Theorem 7.6.

THEOREM 10.20. *The following are the only examples of join irreducible elements of* $\mathbf{FL}(X)$ *that are not canonical joinands of any other element*:

(1) *the element* 0*,*
(2) *the meet of two distinct coatoms,*
(3) *the unique cover of the join of two distinct atoms.*

Proof: Of course 0 is not a canonical joinand of another element, and the meet of two coatoms is not a joinand of any of the three elements above it. If $w = \underline{x} \vee \underline{y}$ is the join of two atoms, then w^*/w is a singular cover. Hence for any element $u \not\leq w^*$ we have $w^* \vee u = w \vee u = \underline{x} \vee \underline{y} \vee u$, so w^* is not a canonical joinand. It is just as easy to check that the other join irreducible elements in the connected components of 0 and 1, *viz.*, the atoms and $\overline{x} \wedge \overline{y} \wedge \overline{z}$ if $|X| = 3$, are canonical joinands. It remains to show that any join irreducible element not in the top or bottom connected components is a canonical joinand.

Suppose that there is a join irreducible element $a \in \mathbf{FL}(X)$, not in the connected component of either 0 or 1, which is not a canonical joinand of another element. Clearly $a \notin X$, so we can write $a = a_1 \wedge \cdots \wedge a_k$ canonically with $k > 1$.

Let u be an arbitrary element incomparable with a. Since a is not a canonical joinand of $a \vee u$, there exists an element $v < a$ with $a \vee u = v \vee u$. Applying (W) to $v \vee u \geq a = \bigwedge a_i$, we see that $v \vee u$ is above at least one canonical meetand of a. Thus if u is an element incomparable with a, then $a \vee u \geq a_i$ for an $i \in \{1, \ldots, k\}$.

It follows that if $i \in \{1, \ldots, k\}$ and $a \leq w < a_i$, then there is no element below w incomparable with a. If, moreover, the interval w/a is infinite, this means that 0, a, w is a Tschantz triple, as $a/0$ is infinite because a is not in the connected component of 0. But there are no Tschantz triples in $\mathbf{FL}(X)$ by Theorem 10.1, and thus the interval w/a is finite for any element w with $a \leq w < a_i$. From this, and the fact that there are no long chains of covers in $\mathbf{FL}(X)$ (Theorem 7.2), it easily follows that every element of a_i/a belongs to the connected component of a. We have seen in Section 4 of Chapter VII that a connected component of any element is always finite. So, the intervals a_i/a are finite. Hence, by Lemma 10.19, a belongs to the connected component of 1, a contradiction.

3. Splittings of a Free Lattice

By the results of Chapter I, for each nonempty proper subset Y of X there is a decomposition of $\mathbf{FL}(X)$ into the disjoint union of a principal filter and a principal ideal,

$$\mathbf{FL}(X) = (1/a_Y) \cup (b_Y/0) \qquad \text{where } a_Y = \bigwedge Y \text{ and } b_Y = \bigvee (X - Y).$$

There is a similar, not so well known decomposition of $\mathbf{FL}(X)$ into the disjoint union of X, a principal filter and a principal ideal,

$$(1)\quad \mathbf{FL}(X) = X \cup (1/a) \cup (b/0) \qquad \text{where } a = \bigwedge_{x\in X} x^* \text{ and } b = \bigvee_{x\in X} x_*.$$

The result has a very simple proof: by induction on the length of the canonical form, an arbitrary element $u \in \mathbf{FL}(X)$ satisfies either $u \in X$ or $u \geq \bigwedge_{x\in X} x^*$ or $u \leq \bigvee_{x\in X} x_*$. A straightforward calculation (as in the proof of the next theorem) shows that these three possibilities are mutually exclusive. The aim of this section is to study decompositions of a similar kind and related questions.

Formula (1) gives a decomposition of the sublattice $\mathbf{FL}(X) - X$ into the disjoint union of a filter and an ideal. In the following theorem we shall describe all decompositions of $\mathbf{FL}(X) - Y$ into the disjoint union of a filter and an ideal, for Y a subset of X. Note that these decompositions correspond to homomorphisms of the sublattice $\mathbf{FL}(X) - Y$ onto $\mathbf{2}$.

THEOREM 10.21. *Let Y, Z_1, Z_2 be three pairwise disjoint subsets of X such that $X = Y \cup Z_1 \cup Z_2$, and such that if $Y = \emptyset$ then both Z_1 and Z_2 are nonempty. Put*

$$a = \bigwedge Z_1 \wedge \bigwedge_{x\in Y} x^*, \qquad b = \bigvee Z_2 \vee \bigvee_{x\in Y} x_*.$$

Then

$$\mathbf{FL}(X) = Y \cup (1/a) \cup (b/0)$$

is a decomposition of $\mathbf{FL}(X)$ into the disjoint union of Y, a principal filter and a principal ideal. Conversely, every decomposition of $\mathbf{FL}(X)$ into the disjoint union of a set of generators, an ideal and a filter is of this form.

Proof: Let us prove by induction on the length of an arbitrary element $u \in \mathbf{FL}(X)$ that either $u \in Y$ or $u \geq a$ or $u \leq b$. This is clear for $u \in X$. Otherwise, by duality it is sufficient to consider the case when u is a proper meet, $u = u_1 \wedge \cdots \wedge u_k$ canonically. If $u_i \leq b$ for some i, then clearly $u \leq b$. If $u_i \in Y$ for some i, then $u \leq u_{i*} \leq b$. If neither is the case, then by induction $u_i \geq a$ for all i, so that $u \geq a$.

We have proved $\mathbf{FL}(X) = Y \cup (1/a) \cup (b/0)$. If $y \in Y$, then $y \ngeq a$ as generators are meet prime. Similarly, $y \nleq b$, while $a \nleq b$ by (W). Hence Y is disjoint from both $1/a$ and $b/0$. The hypotheses of the theorem insure that the latter two are disjoint as well.

Now let Y be a subset of X and $\mathbf{FL}(X) = Y \cup J \cup I$ be a disjoint union, where J is a filter and I is an ideal of $\mathbf{FL}(X)$. Put $Z_1 = X \cap J$ and $Z_2 = X \cap I$. Clearly, $X = Y \cup Z_1 \cup Z_2$ is a disjoint union. For $x \in Y$ we have $x^* \in J$, since $x^* \in I$ would imply $x \in I$. So, the principal filter over the element $a = \bigwedge Z_1 \wedge \bigwedge_{x\in Y} x^*$ is contained in J and, similarly, the principal ideal over the element $b = \bigvee Z_2 \vee \bigvee_{x\in Y} x_*$ is contained in I. By the first part of the proof, $Y \cup (1/a) \cup (b/0)$ accounts for all the elements of $\mathbf{FL}(X)$. Hence J coincides with the principal filter and I with the principal ideal.

The element a in (1) has the following peculiar property:

$$a \text{ is completely join irreducible and } \kappa(a) = \mathbf{d}_a, \tag{2}$$

where $\mathbf{d}_a$ denotes the element dual to a.

In $\mathbf{FL}(x, y, z)$ there are at least four completely join irreducible elements a with $\kappa(a) = \mathbf{d}_a$ other than the one defined in (1), namely:

$$a_1 = (xy^* \vee xz^* \vee yx^* \vee zx^*) \wedge (xy^* \vee xz^* \vee yz^* \vee zx^*) \wedge (xy^* \vee xz^* \vee zx^* \vee zy^*),$$
$$a_2 = (xy^* \vee xz^* \vee yx^* \vee zy^*) \wedge (xz^* \vee yx^* \vee zx^* \vee zy^*) \wedge (xz^* \vee yz^* \vee zx^* \vee zy^*),$$
$$a_3 = (xy^* \vee yx^* \vee zx^*) \wedge (xy^* \vee zx^* \vee zy^*),$$
$$a_4 = (xz^* \vee yx^* \vee zx^*) \wedge (xz^* \vee yz^* \vee zx^*)$$

where xy^* stands for $x \wedge (y \vee (x \wedge z))$, etc. These elements were discovered with the aid of a computer program written in LISP.[1] It seems likely that these four elements and those obtained from them by an automorphism of $\mathbf{FL}(x, y, z)$ are not the only examples.

PROBLEM 10.22. *Which completely join irreducible elements $a \in \mathbf{FL}(X)$ satisfy $\kappa(a) = \mathbf{d}_a$? Are there infinitely many for fixed X?*

Observe that the orbits, under automorphisms of $\mathbf{FL}(x, y, z)$, of the four elements a_1, a_2, a_3, a_4 are of lengths 6, 2, 6 and 3, respectively.

PROBLEM 10.23. *Is the element a from* (1) *the only element of $\mathbf{FL}(X)$ which is invariant under the automorphisms of $\mathbf{FL}(X)$ and satisfies $\kappa(a) = \mathbf{d}_a$?*

By a *middle element* of $\mathbf{FL}(X)$ we mean an element which is incomparable with its dual. Note that the set of middle elements is a convex subset of $\mathbf{FL}(X)$. An element $a \in \mathbf{FL}(X)$ with $a \geq \mathbf{d}_a$ will be called an

[1] Computer algorithms for lattices are discussed at length in Chapter XI.

upper element, and an element a with $a \leq \mathbf{d}_a$ a *lower element*. These form an order filter and an order ideal, respectively.

THEOREM 10.24. *An element $a \in \mathbf{FL}(X)$ is a minimal middle element if and only if it is completely join irreducible and $\kappa(a) = \mathbf{d}_a$.*

Proof: Let a be a minimal middle element. Then a is not a generator. Suppose first that it is a proper join, $a = a_1 \vee \cdots \vee a_k$ canonically. Since all elements strictly below a are lower elements, and the join (if it exists) of a chain of lower elements is a lower element by an elementary argument, we get that a is lower atomic. Hence the elements $a_1, \ldots, a_k$ are all completely join irreducible and all the lower covers of a are lower elements, i.e., $a \wedge \kappa(a_i) \leq \mathbf{d}_a \vee \kappa^d(\mathbf{d}_{a_i})$ for all i. Since $a \nleq \mathbf{d}_a$, there is an index j with $a \nleq \mathbf{d}_{a_j}$. The situation is depicted in Figure 10.2, with $j = k = 2$. For any $i \neq j$, the fact that $a_j \leq \kappa(a_i)$ implies that $a \wedge \kappa(a_i) \leq \mathbf{d}_a \vee \kappa^d(\mathbf{d}_{a_i}) \leq \mathbf{d}_{a_j}$. Since $a \nleq \mathbf{d}_{a_j}$ and $a \succ a \wedge \kappa(a_i)$, this yields $\mathbf{d}_{a_j} \wedge a = a \wedge \kappa(a_i)$. As $\kappa(a_i)$ is a canonical meetand of this element, we get $\kappa(a_i) \geq \mathbf{d}_{a_j}$. By duality, $\kappa^d(\mathbf{d}_{a_i}) \leq a_j$. We have $a_{j*} \leq \mathbf{d}_a$, since $a_j \leq \mathbf{d}_{a_j}$ follows from the minimality of a and for $l \neq j$, $a_{j*} \leq a \wedge \kappa(a_j) \leq \mathbf{d}_a \vee \kappa^d(\mathbf{d}_{a_j}) \leq \mathbf{d}_{a_l}$ as above. So, from $\kappa^d(\mathbf{d}_{a_i}) \leq a_j$ we infer that $\kappa^d(\mathbf{d}_{a_i}) = a_j$, since a strict inequality would give $\kappa^d(\mathbf{d}_{a_i}) \leq a_{j*} \leq \mathbf{d}_a \leq \mathbf{d}_{a_i}$, a contradiction. Hence $\kappa^d(\mathbf{d}_{a_i}) = a_j$ for all $i \neq j$. In particular, $k = 2$. We can assume $i = 1$ and $j = 2$, as in Figure 10.2. Our equation now reads $\kappa^d(\mathbf{d}_{a_1}) = a_2$, which is equivalent to $\mathbf{d}_{a_1} = \kappa(a_2)$. Dualizing this we obtain $a_1 = \kappa^d(\mathbf{d}_{a_2})$ and hence $\mathbf{d}_{a_2} = \kappa(a_1)$. Hence $\mathbf{d}_a \vee \kappa^d(\mathbf{d}_{a_1}) \geq a \wedge \kappa(a_1)$ reads $\mathbf{d}_a \vee \ a_2 \geq a \wedge \mathbf{d}_{a_2}$. By (W) we get $a_2 \geq a \wedge \mathbf{d}_{a_2}$, since $\mathbf{d}_a \geq a \wedge \mathbf{d}_{a_2}$ would imply $a \leq \mathbf{d}_a \vee a_2 \leq \mathbf{d}_{a_2}$, contrary to the choice of $j = 2$, and the remaining two possibilities are by duality equivalent to these two. Since $a \succ a \wedge \mathbf{d}_{a_2} = a \wedge \kappa(a_1)$, this means that $a_2 = a \wedge \mathbf{d}_{a_2}$, and that $\mathbf{d}_{a_2}$ is a canonical meetand of a_2. However, this is impossible, because a canonical meetand is always shorter than the element itself.

Now let a be a proper meet, $a = a_1 \wedge \cdots \wedge a_k$ canonically. As before, the element a is lower atomic and hence completely join irreducible. By the minimality of a we have $a_* \leq \mathbf{d}_{(a_*)} = (\mathbf{d}_a)^*$. By (W) and duality, the inequality $a_* = a \wedge \kappa(a) \leq \mathbf{d}_a \vee \kappa^d(\mathbf{d}_a) = \mathbf{d}_a{}^*$ gives either $\mathbf{d}_a \geq a_*$ or $\kappa^d(\mathbf{d}_a) \geq a_*$. If $\mathbf{d}_a \geq a_*$, then by duality $a \leq (\mathbf{d}_a)^*$ and hence $\mathbf{d}_a = \kappa(a)$, which is just what we need to prove. Let $\mathbf{d}_a \ngeq a_*$, so that $\kappa^d(\mathbf{d}_a) \geq a_*$. The last inequality cannot be strict, because in that case we would again have $a_* \leq \mathbf{d}_a$. So, $\kappa^d(\mathbf{d}_a) = a_*$. By Theorem 3.34 it follows that $\kappa(a)$ is a canonical meetand of $\kappa^d(\mathbf{d}_a)$, which is clearly a contradiction.

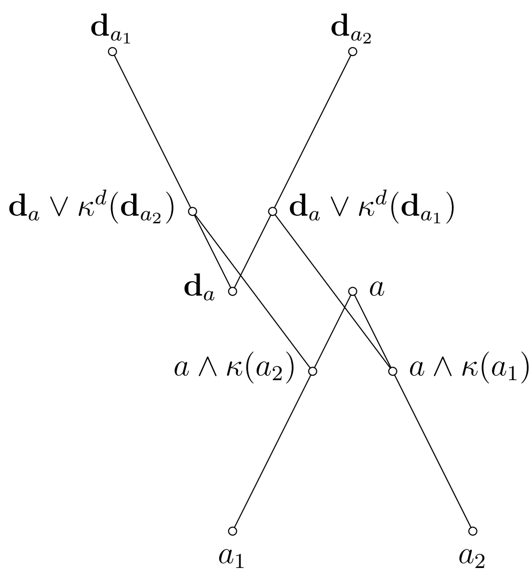


FIGURE 10.2

Conversely, if a is completely join irreducible and $\kappa(a) = \mathbf{d}_a$, then $a \nleq \mathbf{d}_a$ but $a_* \leq \kappa(a) = \mathbf{d}_a < (\mathbf{d}_a)^* = \mathbf{d}_{(a^*)}$, which makes a a minimal middle element.

PROBLEM 10.25. *Is every middle element of* $\mathbf{FL}(X)$ *above a minimal one*?

If a is an arbitrary element, then $a \vee \mathbf{d}_a$ is an upper element and $a \wedge \mathbf{d}_a$ is a lower element. Every generator is a minimal upper element and, at the same time, a maximal lower element.

THEOREM 10.26. *There are no minimal upper elements in* $\mathbf{FL}(X)$ *except for the generators.*

Proof: Suppose that a is a minimal upper element and $a \notin X$. Then both a and $\mathbf{d}_a$ are incomparable with all generators, for $a \not\geq x$ by minimality, and $a \leq x$ implies $\mathbf{d}_a \geq x \geq a \geq \mathbf{d}_a$ and hence $a \in X$.

Suppose first that a is a proper meet, so that then $\mathbf{d}_a$ is a proper join. If there is an element u with $\mathbf{d}_a < u < a$, then $u \vee \mathbf{d}_u$ is an upper element strictly below a (since a proper meet is join irreducible), a contradiction with the minimality of a. Hence $\mathbf{d}_a$ is a lower cover of a, or, stated in different terms, a a singular element. However, it follows from our characterization of singular elements in Theorem 8.6 that these do not satisfy $a_* = \mathbf{d}_a$.

So, a is a proper join. Let $a = a_1 \vee \cdots \vee a_k$ canonically. Since $a = a_1 \vee \cdots \vee a_k \geq \mathbf{d}_{a_1} \wedge \cdots \wedge \mathbf{d}_{a_k} = \mathbf{d}_a$, by (W) we have either $a \geq \mathbf{d}_{a_i}$ or $a_i \geq \mathbf{d}_a$ for some i. But these two conditions are equivalent, so we may assume both $a \geq \mathbf{d}_{a_1}$ and $a_1 \geq \mathbf{d}_a$; these inequalities are strict because a_1 is shorter than a. Since both a_1 and $\mathbf{d}_{a_1}$ are below a and $a_1 \vee \mathbf{d}_{a_1}$ is an upper element, by the minimality of a we get $a = a_1 \vee \mathbf{d}_{a_1}$. Hence $a_2 \vee \cdots \vee a_k \leq \mathbf{d}_{a_1}$ and $a_1 \leq \mathbf{d}_{a_2} \wedge \cdots \wedge \mathbf{d}_{a_k}$. If there is an element u with $\mathbf{d}_a < u < a_1$, then $\mathbf{d}_{a_1} < \mathbf{d}_u < a$, $u \vee \mathbf{d}_u$ is below a, and by the minimality of a we get $a = u \vee \mathbf{d}_u$, so either $a_1 \leq u$ or $a_1 \leq \mathbf{d}_u$. The former case is clearly out, and the latter gives $u < a_1 \leq \mathbf{d}_u < a$, making $\mathbf{d}_u$ an upper element strictly below a. This proves $\mathbf{d}_a \prec a_1$. Dually, $\mathbf{d}_{a_1} \prec a$. Hence a_1 is completely join irreducible and $\kappa(a_1) = \mathbf{d}_{a_1}$. To see this last part, note that $\mathbf{d}_{a_1}$ is completely meet irreducible and above $a_{1*} = \mathbf{d}_a$, but not above a_1 by the minimality of a, while $(\mathbf{d}_{a_1})^* = a$ is above a_1. Because $a_{1*} = \mathbf{d}_a = \mathbf{d}_{a_1} \wedge \cdots \wedge \mathbf{d}_{a_k}$ and $\kappa(a_1) = \mathbf{d}_{a_1}$, the elements $\mathbf{d}_{a_2}, \ldots, \mathbf{d}_{a_k}$ are canonical meetands of a_1. The situation is diagrammed (with $k = 3$ and a_1 not semisingular) in Figure 10.3.

First assume that a_1 is not semisingular, in which case $a_1 = \mathbf{d}_{a_2} \wedge \cdots \wedge \mathbf{d}_{a_k}$ canonically. Suppose that there exists a chain $u_0 > u_1 > u_2 > \cdots$ converging down to a_1. If $a \wedge u_i = a_1$ for some i, then a canonical meetand $\mathbf{d}_{a_r}$ of a_1 is above a and hence above $\mathbf{d}_{a_1}$, a contradiction. Hence $a \wedge u_i > a_1$ and we can assume that $a_1 < u_i < a$ for all i. By duality $\mathbf{d}_a < \mathbf{d}_{u_i} < \mathbf{d}_{a_1}$, and therefore $u_i \vee \mathbf{d}_{u_i} \leq a$. But $u_i \vee \mathbf{d}_{u_i}$ is an upper element, and consequently the minimality of a implies $u_i \vee \mathbf{d}_{u_i} = a$. So, each of the elements $a_1, \ldots, a_k$ is either below u_i or below $\mathbf{d}_{u_i}$. Not all of the elements $a_2, \ldots, a_k$ can be below $\mathbf{d}_{u_i}$, as

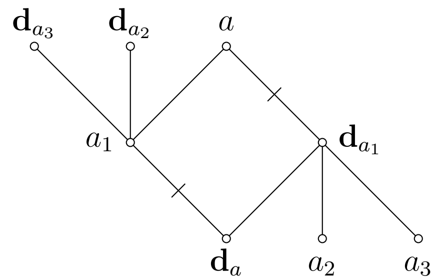


FIGURE 10.3

$\mathbf{d}_{u_i} < \mathbf{d}_{a_1} = a_2 \vee \cdots \vee a_k$. So there exists an $a_r \neq a_1$ such that $a_r \leq u_i$ for all i. But this implies $a_r \leq a_1$, a contradiction. We conclude that in this case a_1 is upper atomic, and hence totally atomic.

Now assume that a_1 is semisingular. Being a canonical joinand of a, it is not singular, and hence is the middle element of a three element interval. Then by Theorem 7.9 we know that $a_1 = u \wedge q$ canonically, where $u \succ a_1$ and $q \geq \kappa(a_1)^* = a$. The argument of the preceding paragraph then shows that there is no chain below a converging down to a_1, so that a_1 has an upper cover below a. Since a_1 has two canonical meetands and two upper covers, it is upper atomic, and hence again totally atomic.

Thus in either case a_1 is totally atomic. Dually, $\mathbf{d}_{a_1} = \kappa(a_1)$ is totally atomic. Hence a_1 is a meet of generators, $\mathbf{d}_{a_1}$ is a join of generators, and $a = {\mathbf{d}_{a_1}}^*$ is above a generator, a contradiction.

CHAPTER XI

Term Rewrite Systems and Varieties of Lattices

Term rewrite systems provide a strong form of the solution of the word problem for free algebras in certain varieties. For example, one can tell whether two lattice terms are equal in all distributive lattices by rewriting each as a join of meets of variables, applying the absorptive law, and then comparing their join normal forms. When properly formalized, this algorithm becomes an (associative and commutative) term rewrite system for distributive lattices.

This chapter contains two main results. The first is that the variety of all lattices has no such term rewrite system. The second generalizes the distributive result by exhibiting an infinite collection of lattice varieties, each generated by a finite lower bounded lattice, all of which have term rewrite systems. Along the way, we shall prove some results on varieties generated by finite lower bounded lattices which are of some interest in themselves.

The study of the equational theory of a class $\mathcal{K}$ of algebras and their free algebras $\mathbf{F}_{\mathcal{K}}(X)$ is greatly facilitated by a normal form for the terms over the language of $\mathcal{K}$. For terms u and v over some set of variables X, u is *equivalent to* v modulo $\mathcal{K}$ if the equation $u \approx v$ holds identically in $\mathcal{K}$ (i.e., for all substitutions of the variables into all algebras in $\mathcal{K}$). We write this $u \approx v \pmod{\mathcal{K}}$. By a *normal form* we mean an effective choice function from the equivalence classes of this relation. We will use the notation $\mathrm{nf}(w)$ for such a normal form function. Having a normal form is equivalent to the equational theory being decidable. Moreover, if this normal form can be computed efficiently, it is very helproofful for computer implementations of the free algebras in $\mathcal{K}$.

A term rewrite system, abbreviated TRS, constitutes a very specific method for transforming terms. A normal form TRS transforms terms into a unique normal form and, as such, is computationally useful. (The definitions will be given below.) Not every decidable equational theory has a normal form TRS. For example, it is easy to see that commutative groupoids have no such TRS. An *associative and commutative* TRS, denoted AC TRS, is one in which we are allowed to apply the associative and commutative laws, as well as the rewrite rules.

The class of all lattices, $\mathcal{L}$, has a very nice normal form, *viz.*, the Whitman canonical form, which is unique up to associativity and commutativity. In this chapter, we reserve the term *canonical form* for Whitman's normal form. The canonical form of a term is not only the shortest, but also lattice theoretically the best way to write it: if $w = w_1 \vee \cdots \vee w_n$ canonically, then $w_1, \ldots, w_n$ are the lowest possible elements of the free lattice that irredundantly join to w, see Theorem 1.19.

Our first objective is to prove that, despite this very nice canonical form, there is no finite, convergent AC TRS for lattice theory. The existence of such an AC TRS is raised as Problem 32 in Dershowitz, Jouannaud and Klop [**38**].

We would like to thank George McNulty for suggesting this problem to us and Stan Burris for several enlightening lectures on term rewrite systems. The results in the first three sections of this chapter appeared in a paper [**57**], and we would like to thank the referee of that paper for a simplification in the proof.

1. Term Rewrite Systems

Term rewrite systems were pioneered by Trevor Evans [**44**] who gave a convergent TRS for quasigroups. The subject was popularized with Knuth and Bendix [**91**] who gave methods which could sometimes convert equational axioms into a convergent TRS. They were able to use these methods to find a convergent TRS for groups. Since that time the subject has become popular, especially with computer scientists. Equational TRS's were introduced by Lankford and Ballantyne [**93**] and Peterson and Stickel [**111**]. A good general reference is Dershowitz and Jouannaud [**37**]; see also Jouannaud and Kirchner [**85**]. Ježek [**75**] considers TRS–like systems for groupoids, and Burris and Lawrence [**17**] consider AC TRS's for certain finite rings.

A set R of ordered equations is called a *term rewrite system* and abbreviated TRS. The equations are written with an arrow: $p \to q$. A *substitution* is simply an endomorphism of the term algebra. If $(p \to q) \in R$ and r is a term which has a subterm of the form $\sigma(p)$ for some substitution σ, then we can rewrite r by replacing (one occurrence of) $\sigma(p)$ by $\sigma(q)$. If t is the resulting term, then we write $r \to_R t$ and call this a *one step rewrite*. A term rewrite system R is finite if R is; it is *terminating* if there is no infinite sequence of (one step) rewrites. This means that if we start with any term and apply the rewrite rules repeatedly in any order, we will eventually reach a term which cannot be further rewritten. A terminating TRS is a *convergent* TRS if, for every term s, every sequence of rewrites starting with s terminates with the same term, which is then called the *normal form* of s. If R is a convergent TRS, we denote the normal form of a term w by $\mathrm{nf}_R(w)$ or $\mathrm{nf}(w)$, when R is understood. We say that an equational theory E has a convergent TRS provided there is a TRS such that $s \approx t$ is in E if and only if $\mathrm{nf}(s) = \mathrm{nf}(t)$.

Not every recursive equational theory has a finite, convergent TRS. It is easy to see that theories which contain the commutative law, $x \cdot y \approx y \cdot x$, do not have such a TRS. This defect can be corrected sometimes by an equational TRS. Let E_0 be a set of regular equations (an equation is *regular* if the set of variables occurring on the left side is the same as those on the right) and define $s \equiv t$ if $E_0 \models s \approx t$. An *equational* TRS is a pair $\langle E_0, R\rangle$ where R is a TRS. In such a system we allow sequences of rewrites of the form

$$s_0 \equiv s_1 \to_R s_2 \equiv s_3 \to_R \cdots . \tag{1}$$

A term u is *terminal* for an equational TRS if no rewrite rule applies to it nor to any $u' \equiv u$. A *normal form* equational TRS is one in which, for every term s, every sequence in the above form, with $s_0 = s$, eventually terminates, that is, ends in a terminal element and the $\equiv$–class of this element is unique. We let $\mathrm{nf}(s)$ denote some representative of this $\equiv$–class. It would make more sense to define $\mathrm{nf}(x)$ to be the equivalence class, but our definition is notationally easier. In this chapter we will be concerned with the case when E_0 consists of the associative and commutative laws (for meet and join). In this case the rewrite system is called an AC TRS.

The next lemma collects some basic facts about equational TRS's. We say that a term v is an E_0*–subterm* of u if v is a subterm of u' for some $u' \equiv u$. Since all the equations in E_0 are regular, we can talk about the variables occurring in $\mathrm{nf}(u)$ since all the members of an E_0–class depend on the same variables. (It is not hard to see that if we allowed E_0 to be nonregular, the equational term rewrite system it determined could not be convergent.)

LEMMA 11.1. *Suppose that $\langle E_0, R\rangle$ is a finite, normal form, equational TRS. Then the following hold.*

1. *If v is an E_0–subterm of u and* $\mathrm{nf}(u) \equiv u$, *then* $\mathrm{nf}(v) \equiv v$.
2. *If w is a term and σ is an automorphism of the term algebra, then* $\mathrm{nf}(\sigma(w)) \equiv \sigma(\mathrm{nf}(w))$.
3. *If $u = \sigma(v)$ for some endomorphism of the term algebra and* $\mathrm{nf}(u) \equiv u$, *then* $\mathrm{nf}(v) \equiv v$.
4. *The variables which occur in* $\mathrm{nf}(w)$ *all occur in w.*

Proof: (1) follows since none of the rewrite rules can apply to u. (2) is a direct consequence of the way rewrite rules are applied. For (3), first note that if v' is an E_0–subterm of v then $\sigma(v')$ is an E_0–subterm of u. So if $p \to q$ is a rewrite rule and $v' = \tau(p)$ for some substitution τ, then $p \to q$ would apply to u under the substitution $\sigma\tau$. But no rewrite rule can apply to u because it is in normal form.

If there was a variable occurring in $\mathrm{nf}(w)$ which did not occur in w, then there must be a rewrite rule of the form $u(x_1, \dots, x_n) \to v(x_1, \dots, x_n, y_1, \dots, y_k)$ with $k \geq 1$. But then, applying this rule under the substitution which maps y_1 to $u(x_1, \dots, x_n)$ and fixing the other variables, we obtain an infinite chain of rewrites:

$$u(\mathbf{x}) \to v(\mathbf{x}, u(\mathbf{x}), \mathbf{y}) \to v(\mathbf{x}, v(\mathbf{x}, u(\mathbf{x}), \mathbf{y}), \mathbf{y}) \to \cdots$$

where $\mathbf{x} = x_1, \dots, x_n$ and $\mathbf{y} = y_2, \dots, y_k$.

We will actually be using only a few basic facts about free lattices, and for reference we should list these explicitly.

First recall that, since both lattice operations are associative, we include in our definition of terms expressions which omit unnecessary parentheses. Thus

$$x \vee (y \vee z) \qquad x \vee y \vee z \qquad (x \vee y) \vee z \tag{2}$$

are all terms. If s and t are terms in a variable set X, then $s \leq t$ of course means $s^{\mathbf{FL}(X)} \leq t^{\mathbf{FL}(X)}$, and $s \approx t$ holds in lattice theory if and only if $s \leq t$ and $t \leq s$ both hold.

We need Whitman's solution to the word problem for free lattices, which is Theorem 1.11, and its consequence that every element of a free lattice is either meet irreducible or join irreducible. We will use the criterion for deciding if a term is in canonical form given in Theorem 1.18. Recall that this criterion provides a way to reduce a term to canonical form, and that each step of the process reduces the length of the term. Also we will use the fact of Theorem 1.19, that the canonical join representation of an element in a free lattice join refines ($\ll$) every other join representation of that element.

We will use the following lemma from Tschantz [**131**], which is really just the argument of Lemma 3.28.

Lemma 11.2. *Let u_1 be a canonical joinand of u. Then*

(1) *if $u_1 \leq s \wedge t \leq u$ then either $s \leq u$ or $t \leq u$ or $u_1 = s \wedge t$;*
(2) *if $u_1 \leq s \vee t \leq u$ then either $u_1 \leq s$ or $u_1 \leq t$;*
(3) *if $u_1 \leq x \leq u$, where $x \in X$, then $u_1 = x$.*

Finally, in $\mathbf{FL}(x_1, \ldots, x_n)$, let $\underline{x}_i = \bigwedge_{j \neq i} x_j$. It is not hard to show that $0 = \bigwedge x_k \prec \underline{x}_i \prec \underline{x}_i \vee \underline{x}_j$ for $i \neq j$; see Example 3.45. However, we only require the following easy fact.

LEMMA 11.3. *If $u > \underline{x}_i$ in $\mathbf{FL}(x_1, \ldots, x_n)$, then $u \geq \underline{x}_j$ for some $j \neq i$.*

Proof: Every element of $\mathbf{FL}(x_1, \ldots, x_n)$ is either below x_k or above $\underline{x}_k$. If $u > \underline{x}_i$, then $u \nleq x_j$ for some $j \neq i$, and thus $u \geq \underline{x}_j$.

2. No AC TRS for Lattice Theory

In this section we prove that there is no finite, convergent AC TRS for the equational theory of lattices. From now on we use $u \equiv v$ to mean that the lattice terms u and v are equivalent modulo AC. We use $u \approx v$ to mean that u is equivalent to v in lattice theory, i.e., they represent the same element of the free lattice. Of course, $u = v$ means that u and v are the same term.

Suppose there is a finite, convergent AC TRS for lattice theory. Let $\operatorname{nf}(t)$ denote its normal form. This normal form is really only defined up to equivalence modulo AC. We will assume that $\operatorname{nf}(w)$ chooses some element of this AC class which does not have any unnecessary parentheses. For example, the three terms in (2) are AC equivalent and, if they are in normal form (we will see that they are), then the value of $\operatorname{nf}(w)$ for any of them, must be $x \vee y \vee z$. This means that if $\operatorname{nf}(w) = u_1 \vee \cdots \vee u_k$ then each u_i is assumed to be either a formal meet or a variable. The expression 'w is in normal form' means $\operatorname{nf}(w) \equiv w$. Recall that the term *canonical form* refers to Whitman's canonical form. Also, as above, we say that v is an AC–*subterm* of u if v is a subterm of u' for some $u' \equiv u$.

LEMMA 11.4. *Let $t = t_1 \vee \cdots \vee t_n$ be in canonical form with $n > 1$. Then $\operatorname{nf}(t)$ is a formal join, say $\operatorname{nf}(t) = u = u_1 \vee \cdots \vee u_m$, and there is a map σ from $\{1, \ldots, n\}$ onto $\{1, \ldots, m\}$ such that $t_i \leq u_{\sigma(i)}$ for all i. In particular, $1 < m \leq n$. Moreover, $u_1, \ldots, u_m$ is an antichain.*

Proof: Since $n > 1$, $t \not\approx x$ for all variables and hence u cannot be a variable. If u is a formal meet $u = u_1 \wedge \cdots \wedge u_m$, then $u_i \approx u$ for some i since $u \approx t$ is meet irreducible. But then by Lemma 11.1, $u_i \equiv \operatorname{nf}(u_i) = \operatorname{nf}(u) \equiv u$, which is clearly false.

So suppose $u = u_1 \vee \cdots \vee u_m$. Then, by the refinement property of the Whitman canonical form (Theorem 1.19), for each i, there is a $\sigma(i)$ such that $t_i \leq u_{\sigma(i)}$. Clearly

$$u' = \bigvee_{j \in \operatorname{range} \sigma} u_j$$

is an AC–subterm of u and so is in normal form. It is also clear that $u' \approx u$ and hence they have the same normal form. Thus $u' \equiv \mathrm{nf}(u') = \mathrm{nf}(u) \equiv u$, which is not possible if u' is a proper subterm of u. This implies that σ must be onto.

A similar argument shows that the u_i's form an antichain.

LEMMA 11.5. *Suppose that w is a term in normal form, w is a formal meet, and x is a variable such that $x \vee w$ is in canonical form. Then $x \vee w$ is in normal form.*

Proof: By Lemma 11.4,

$$\mathrm{nf}(x \vee w) = s \vee t \tag{3}$$

for some s and t with $x \leq s < x \vee w$ and $w \leq t < x \vee w$. Moreover, each of s and t is either a generator or a formal meet. Also, (3) implies that $\mathrm{nf}(t) = t$. We claim that $w \equiv t$. If $w \approx t$ then $w \equiv t$ since both are in normal form. In the other case, $w < t$. By a repeated application of parts (1) and (2) of Lemma 11.2, t has a subterm $v \vee r$ with $r \approx w$ (and v possibly a join). Since r is a subterm of t, it is in normal form. Hence $r \equiv w$.

For any term u, let u' denote the image of u under the endomorphism which maps every variable to x. Of course, $u' \approx x$ for all u. Now the chain of rewrites that transforms $x \vee w$ to $s \vee t$ can be applied in the same way to $x \vee w'$, yielding $s' \vee t'$. But t' has $v' \vee r' \equiv v' \vee w'$ as a subterm. Since $\mathrm{nf}(v') = x$, some additional rewrites transform $v' \vee w'$ into $x \vee w'$. Thus the concatenation of these rewritings transforms $x \vee w'$ into a term having $x \vee w'$ as a subterm, and this clearly leads to an infinite chain of rewrites, a contradiction. Thus $t \equiv w$.

Consequently $\mathrm{nf}(x \vee w) \equiv s \vee w$. Unless $x \equiv s$, there is a chain of rewrites (of positive length) which transforms $x \vee w$ to $s \vee w$. When these rewrites are applied to $x \vee w'$, they yield $s' \vee w'$. Since $\mathrm{nf}(s') = x$, a further chain of rewrites transforms $s' \vee w'$ to $x \vee w'$. This again leads to an infinite chain of rewrites, and thus we must have $s \equiv x$.

The next lemma shows that, if a term $t = t_1 \vee \cdots \vee t_n$ has the property that the t_i's are the same except for a change of variables, then Lemma 11.4 can be strengthened.

LEMMA 11.6. *Suppose that $t = t_1 \vee \cdots \vee t_n$ canonically with $n > 1$ and that there is a subgroup $\mathbf{G} \cong \mathbf{S}_n$ of the automorphism group of the term algebra which acts faithfully on $\{t_1, \ldots, t_n\}$. Let $\mathrm{nf}(t) = r = r_1 \vee \cdots \vee r_m$. Then $m = n$ and, after reordering, $r_i \geq t_i$, for $i = 1, \ldots, n$, and $r_i \not\geq t_j$ for $i \neq j$.*

Proof: By part (2) of Lemma 11.1, $\sigma(r) \equiv r$ for each $\sigma \in G$. Thus

$$\sigma(r_1) \vee \cdots \vee \sigma(r_m) = \sigma(r) \equiv r = r_1 \vee \cdots \vee r_m$$

from which it follows that, for each i, $\sigma(r_i) \equiv r_j$, for some j. Suppose r_i is above exactly k of the t_j's. Since $\mathbf{S}_n$ acts transitively on the k element subsets of $\{t_1, \ldots, t_n\}$, each k–element subset has some r_i above it. By Lemma 11.4, $k < n$. Suppose $k > 1$. Then $n \geq 3$, and there are at least $\binom{n}{k}$ different r_i's. Since $\binom{n}{k} > n$ for $1 < k < n-1$, this contradicts Lemma 11.4 unless $k = n - 1$. But in this case the join of the r_i's is clearly redundant, since $n > 2$.

Lemma 11.7. *Let x, y, $z_1, \ldots, z_n$, and $e_1, \ldots, e_s$ be distinct variables, and let $\underline{z}_i = \bigwedge_{j \neq i} z_j$. If n, $s \geq 1$ and $k \geq 0$, then the following elements are all in normal form.*

$$z_1 \wedge \cdots \wedge z_n \tag{4}$$

$$\bigvee_{i=1}^{n} (x \wedge \underline{z}_i) \tag{5}$$

$$\bigvee_{i=1}^{n} (x \wedge z_i) \tag{6}$$

$$x \wedge \left[\bigvee_{i=1}^{k} (x \wedge z_i) \vee \bigvee_{j=1}^{s} e_j \right] \tag{7}$$

Proof: Straightforward applications of Theorem 1.18 show that each of these elements is in canonical form. By Lemma 11.6, $\mathrm{nf}(z_1 \wedge \cdots \wedge z_n) = r_1 \wedge \cdots \wedge r_n$. There is an obvious term algebra endomorphism mapping $z_1 \wedge \cdots \wedge z_n$ onto $r_1 \wedge \cdots \wedge r_n$ so, by part (3) of Lemma 11.1, $z_1 \wedge \cdots \wedge z_n$ is in normal form.

Let w be the element of (5), let $\mathrm{nf}(w) = r = r_1 \vee \cdots \vee r_m$, and let $a_i = x \wedge \underline{z}_i$. By part (4) of Lemma 11.1, the only variables that can occur in r are x, $z_1, \ldots, z_n$, and hence, r can be viewed as an element of $\mathbf{FL}(x, z_1, \ldots, z_n)$. By Lemma 11.6, $n = m$ and we may assume that $r_i \geq a_i$ and $r_i \not\geq a_j$ for all distinct i and j. But if $r_i > a_i$, then, by Lemma 11.3, $r_i \geq a_j$ for some $j \neq i$, a contradiction. Thus $a_i \approx r_i$. Since r_i is a subterm of r, $\mathrm{nf}(r_i) \equiv r_i$, and $\mathrm{nf}(a_i) \equiv a_i$ by the previous example. Thus $a_i \equiv r_i$ and so $\mathrm{nf}(w) = r_1 \vee \cdots \vee r_n \equiv a_1 \vee \cdots \vee a_n$, proving w is in normal form.

Since there is an obvious term algebra endomorphism mapping the element of (6) to the element of (5), the former element is in normal form by part (3) of Lemma 11.1.

Choose $n \geq k + s$. Then there is an endomorphism mapping the element

$$\bigvee_{i=1}^{k}(x \wedge z_i) \vee \bigvee_{j=1}^{s} e_j$$

to the element of (6), and hence the former is in normal form by Lemma 11.1 again. It now follows from the dual of Lemma 11.5 that the element of (7) is in normal form.

THEOREM 11.8. *There is no finite, convergent AC term rewrite system for the equational theory of lattices.*

Proof: Let w be the term given in (6), where n is large enough so that the length of w is greater than the length of the left hand side of all of the rewrite rules. Clearly $w \leq x$ and thus $x \wedge w \approx w$, so some rewrite rule must apply to $x \wedge w$. Since $\mathrm{nf}(w) \equiv w$, every proper AC–subterm of $x \wedge w$ is in normal form. Since the left hand side of a rewrite rule can never match a term in normal form, the left hand side of some rewrite rule must match $x \wedge w$.

What terms with length less than the length of $x \wedge w$ match $x \wedge w$? That is, for which terms t is there a term algebra endomorphism mapping t onto $x \wedge w$? (To avoid confusion, we will use letters at the beginning of the alphabet to denote variables.) Besides a and $a \wedge b$, the term

$$a \wedge [(c_1 \wedge d_1) \vee \cdots \vee (c_k \wedge d_k) \vee e_1 \vee \cdots \vee e_s]$$

matches $x \wedge w$ under an obvious substitution. (This term with $k = n$ and no e_j's also matches $x \wedge w$, but has the same length.) Some or all of the c_i's can be equal to each other or to a, but the other variables must be distinct. All of these terms have endomorphisms onto the term

$$a \wedge [(a \wedge d_1) \vee \cdots \vee (a \wedge d_k) \vee e_1 \vee \cdots \vee e_s]$$

which is in normal form by Lemma 11.7. Hence any term shorter than $x \wedge w$, which has an endomorphism onto $x \wedge w$, is in normal form and so cannot be the left hand side of a rewrite rule. Thus no rewrite rule applies to $x \wedge w$.

3. An Extension

By a terminating AC TRS, we mean one in which every sequence of rewritings ends in a *terminal* element in a finite number of steps. Recall that a term u is terminal if no rewrite rule applies to it nor to any $u' \equiv u$. Suppose we weaken this notion by defining u to be terminal if, for all $u' \equiv u$, if u' rewrites to v, then $v \equiv u$. (We would also have to modify the requirement on sequences as in (1) by insisting that, when $s_i \to_R s_{i+1}$, we have $s_i \not\equiv s_{i+1}$.) Does Theorem 11.8 still hold for such a system? In most parts of the proof we produced an infinite chain of rewrites with terms of increasing length. For example, the proof of Lemma 11.5 constructed certain subterms. A closer look at the proof shows that these are proper subterms and this implies that rewriting will produce longer and longer terms. Since AC equivalence classes are finite, such sequences cannot exist even with our modified definition of a terminal element. The proof that $x \equiv s$ showed that if this failed, we could rewrite $x \vee w'$ to $s' \vee w'$ and then rewrite the latter back to $x \vee w'$. Since x is AC equivalent only to itself, this also leads to an infinite chain of rewrites even under our modified definition of terminal element. However, part (3) of Lemma 11.1 is no longer valid. The only places where it is not obvious that the use of this cannot be avoided is in the proof that the terms given in (6) and (7) of Lemma 11.7 are in normal form. To see that they are, note by Lemma 11.6,

$$\mathrm{nf}((x \wedge z_1) \vee \cdots \vee (x \wedge z_n)) = r_1(x, z_1, \ldots, z_n) \vee \cdots \vee r_n(x, z_1, \ldots, z_n),$$

for some $r_1, \ldots, r_n$. This rewrite rule applies to the element of (5) to yield

$$(x \wedge \underline{z}_1) \vee \cdots \vee (x \wedge \underline{z}_n) \to r_1(x, \underline{z}_1, \ldots, \underline{z}_n) \vee \cdots \vee r_n(x, \underline{z}_1, \ldots, \underline{z}_n).$$

Since the left side is in normal form, this implies that (by our new rule for rewriting) these terms are AC equivalent, and so, after renumbering, $x \wedge \underline{z}_i \equiv r_i(x, \underline{z}_1, \ldots, \underline{z}_n)$. But it is easy to see that this implies that $r_i(x, z_1, \ldots, z_n) = x \wedge z_i$ or $z_i \wedge x$, which implies the element (6) is in normal form.

To see that the element of (7) is in normal form, it suffices, by the dual of Lemma 11.5, to show that $w = \bigvee_{i=1}^{k}(x \wedge z_i) \vee \bigvee_{j=1}^{s} e_j$ is in normal form. By Lemma 11.4, we may assume the normal form of this element is $r_1 \vee \cdots \vee r_m$, where each r_i is either a meet or a variable and $m \le k + s$. Choose $n > k + s$. Then there is an endomorphism σ mapping w to $\bigvee_{i=1}^{n}(x \wedge z_i)$ and this induces the rewrite:

$$\bigvee_{i=1}^{n}(x \wedge z_i) \to \sigma(r_1) \vee \cdots \vee \sigma(r_m).$$

Since the left side is in normal form, these two elements must be AC equivalent. If r_i is a meet then $\sigma(r_i)$ is also. Thus, since $n > m$, some of the r_i's must be variables. But, by Lemma 11.4, each r_i must satisfy either $x \wedge z_j \le r_i < w$ or $e_j \le r_i < w$, for some j. This implies that if r_i is a variable, it must be some e_j. If r_i is not a variable, then, since

$\sigma(r_i) = x \wedge z_j$, for some j, $r_i = x \wedge z_j$. It is not hard to see that this implies that w is in normal form.

Thus, even under this weaker set of rules for rewriting, there is no finite, convergent AC TRS for lattices.

As we mentioned earlier, Peterson and Stickel [**111**] have shown that the equational theory of distributive lattices does have a convergent AC TRS. However deciding if an equation is true in all distributive lattices is harder than deciding if it is true in all lattices. To make this more precise, we define the *term equivalence problem*, denoted TEP, for a class $\mathcal{K}$ of lattices. An instance of this problem is given two terms u and v (in the language of lattices) and asks if the equation $u \approx v$ holds in every lattice in $\mathcal{K}$. As we showed in Chapter XI, when $\mathcal{K}$ is the class of all lattices, the TEP is polynomial time. On the other hand, P. Bloniarz, H. B. Hunt, and D. Rosenkrantz have shown that the TEP for the class of distributive lattices is co–NP complete [**12**]. In fact, they show that the TEP for $\mathcal{K} = \{\mathbf{L}\}$ is co–NP complete for any finite, nontrivial lattice $\mathbf{L}$. On the other hand, the equational theory of the class of modular lattices was shown to be undecidable by Freese [**49**].

4. The Variety Generated by $\mathbf{L}^{\sqcup}(w)$

In this section we want to consider structural properties of lattices in the variety $\boldsymbol{V}(\mathbf{L})$ generated by a finite, lower bounded lattice. The definition of lower bounded homomorphism and lower bounded lattice and the basic results on these concepts which we require are found in Chapter II, Sections 1–6 and Chapter III, Sections 2–4. We are particularly interested in the case when $\mathbf{L}$ is also subdirectly irreducible; in that case, by Theorem 3.24, we know that $\mathbf{L}$ is isomorphic to $\mathbf{L}^{\sqcup}(w)$ for a (non-unique) join irreducible element $w \in \mathbf{FL}(\omega)$. In particular, we want to provide a syntactic answer to the question: *When is* $\mathbf{L}^{\sqcup}(u)$ *in* $\boldsymbol{V}(\mathbf{L}^{\sqcup}(v))$*?*

The variety generated by a finite lattice is, of course, locally finite. By Jónsson's Lemma [**80**], if $\mathbf{L}$ is finite then the subdirectly irreducible lattices in $\boldsymbol{V}(\mathbf{L})$ are contained in $\boldsymbol{HS}(\mathbf{L})$. If, in addition, $\mathbf{L}$ is lower bounded, then every finite lattice in $\boldsymbol{V}(\mathbf{L})$ is lower bounded by Theorem 2.17.

If $\mathbf{L}$ is a finite lower bounded lattice, and X is a finite set, then there is an efficient way to represent the relatively free lattice $\mathbf{F}_{\boldsymbol{V}(\mathbf{L})}(X)$. Note that $\mathbf{F}_{\boldsymbol{V}(\mathbf{L})}(X)$, being a finite sublattice of a direct power of $\mathbf{L}$, is itself a lower bounded lattice. Let f be the canonical epimorphism from $\mathbf{FL}(X)$ to $\mathbf{F}_{\boldsymbol{V}(\mathbf{L})}(X)$ with $f(x) = x$ for $x \in X$, and, as usual, let $\beta_f : \mathbf{F}_{\boldsymbol{V}(\mathbf{L})}(X) \to \mathbf{FL}(X)$ be the function which maps each element to its least preimage. By Corollary 3.14, $\mathbf{F}_{\boldsymbol{V}(\mathbf{L})}(X) \cong A_X(\mathbf{L})^{\sqcup}$ where

$$A_X(\mathbf{L}) = \{\beta_f(c) : c \in \mathrm{J}(\mathbf{F}_{\boldsymbol{V}(\mathbf{L})}(X))\}.$$

On the other hand, using Theorems 3.13 and 3.15, we have for $u \in \mathrm{J}(\mathbf{FL}(X))$,

$$\begin{aligned} u \in A_X(\mathbf{L}) \quad &\text{iff} \quad \mathrm{J}(u) \subseteq A_X(\mathbf{L}) \\ &\text{iff} \quad \mathbf{L}^{\sqcup}(u) \in \boldsymbol{H}(\mathbf{F}_{\boldsymbol{V}(\mathbf{L})}(X)) \\ &\text{iff} \quad \mathbf{L}^{\sqcup}(u) \in \boldsymbol{V}(\mathbf{L})\,. \end{aligned}$$

Combining these facts, we have the following representation for $\mathbf{F}_{\boldsymbol{V}(\mathbf{L})}(X)$.

THEOREM 11.9. *Let* $\mathbf{L}$ *be a finite lower bounded lattice and* X *a finite set. Then* $\mathbf{F}_{\boldsymbol{V}(\mathbf{L})}(X) \cong A_X(\mathbf{L})^{\sqcup}$ *where* $A_X(\mathbf{L}) = \{u \in J(\mathbf{FL}(X)) : \mathbf{L}^{\sqcup}(u) \in \boldsymbol{V}(\mathbf{L})\}$.

To construct $A_X(\mathbf{L})$, we need to find the elements $\beta_f(c)$ for $c \in \mathrm{J}(\mathbf{F}_{\boldsymbol{V}(\mathbf{L})}(X))$, where $f : \mathbf{FL}(X) \twoheadrightarrow \mathbf{F}_{\boldsymbol{V}(\mathbf{L})}(X)$ is the natural homomorphism. Of course, for such a c there is a homomorphism h from $\mathbf{F}_{\boldsymbol{V}(\mathbf{L})}(X)$ onto a subdirectly irreducible lattice $\mathbf{K}$ separating c and its lower cover c_* in $\mathbf{F}_{\boldsymbol{V}(\mathbf{L})}(X)$, and $\beta_f(c) = \beta_{hf}(h(c))$. Thus we can find the set $A_X(\mathbf{L})$ by looking at maps from $\mathbf{FL}(X)$ onto subdirectly irreducible lattices in $\boldsymbol{V}(\mathbf{L})$.

For example, let $\mathbf{L} = \mathbf{2}$. Now $\mathbf{2}$ is the only subdirectly irreducible in its variety, and if we take $X = \{x, y, z\}$, there are 6 homomorphisms from $\mathbf{FL}(X)$ onto $\mathbf{2}$, from which it follows that

$$A_X(\mathbf{2}) = \{x, y, z, x \wedge y, x \wedge z, y \wedge z\}. \tag{8}$$

The join closure of this set is the 18 element free distributive lattice on X.

If we take $\mathbf{L} = \mathbf{N}_5$, the only subdirectly irreducibles are $\mathbf{2}$ and $\mathbf{N}_5$. Again let $X = \{x, y, z\}$. Then there are 6 homomorphisms from $\mathbf{FL}(X)$ onto $\mathbf{N}_5$. Thus $A_X(\mathbf{N}_5)$ consists of the elements of the set in (8) together with $x \wedge (y \vee (x \wedge z))$ and its (six) automorphic images. So $A_X(\mathbf{N}_5)$ has 12 elements and its closure gives the 99 element lattice $\mathbf{F}_{\boldsymbol{V}(\mathbf{N}_5)}(3)$.

Let $\mathbf{L}$ be a finite lower bounded lattice. We define the $\boldsymbol{V}(\mathbf{L})$–*canonical form* of a lattice term t to be the Whitman canonical form of the evaluation of t in the relatively free lattice $A_X(\mathbf{L})^{\sqcup}$. Equivalently, if $t^{\mathbf{FL}(X)} = w$, the $\boldsymbol{V}(\mathbf{L})$–canonical form of t is the (Whitman) canonical form of the image of w under the standard epimorphism $h : \mathbf{FL}(X) \twoheadrightarrow A_X(\mathbf{L})^{\vee}$. Thus, the $\boldsymbol{V}(\mathbf{L})$–canonical form of t is the Whitman canonical form of

$$\bigvee\{v \in A_X(\mathbf{L}) : v \leq t^{\mathbf{FL}(X)}\}.$$

This will not in general be the shortest term equivalent to t modulo $\boldsymbol{V}(\mathbf{L})$; rather it will be the canonical form of the lowest element in $\mathbf{FL}(X)$ with this property.

If $\mathbf{A}$ is any finite algebra of finite type, then the word problem for $\boldsymbol{V}(\mathbf{A})$ is solvable. But if a variety has a normal form for terms and an efficient algorithm for finding them, then there is a practical way to do it, *viz.,* to determine if $u \approx v$ holds in $\mathbf{L}$, put both terms in normal form and see if they are equal. For finite lower bounded lattices, the $\boldsymbol{V}(\mathbf{L})$–canonical form serves this purpose. For example, the $\boldsymbol{V}(\mathbf{N}_5)$–canonical form of $x \wedge (y \vee z)$ is $[x \wedge ((x \wedge y) \vee z)] \vee [x \wedge (y \vee (x \wedge z))]$. This corresponds to the equation

$$x \wedge (y \vee z) \approx [x \wedge ((x \wedge y) \vee z)] \vee [x \wedge (y \vee (x \wedge z))]\,,$$

which holds in $\boldsymbol{V}(\mathbf{N}_5)$.

In the next section we will show that, for a specific finite lower bounded lattice $\mathbf{L}_2$, there is an AC TRS which rewrites an arbitrary lattice term to its $\boldsymbol{V}(\mathbf{L}_2)$–canonical form. Similar arguments extend this result to an infinite sequence of lattices $\mathbf{L}_i$ $(i \geq 2)$. The particular identity given above can also be used to show why the method for deriving AC TRS's in the next section does not apply to $\boldsymbol{V}(\mathbf{N}_5)$.

For reference, we formalize the idea behind these arguments.

Corollary 11.10. *Let* $\mathbf{L}$ *be a finite lower bounded lattice. For any* $w \in \mathbf{FL}(X)$*, the equation*

$$w \approx \bigvee\{v \in A_X(\mathbf{L}) : v \leq w\}$$

holds in $\boldsymbol{V}(\mathbf{L})$*.*

Now we turn to the main question for this section: *For join irreducible elements* u *and* v *of* $\mathbf{FL}(X)$*, when is* $\mathbf{L}^{\sqcup}(u)$ *in* $\boldsymbol{V}(\mathbf{L}^{\sqcup}(v))$*?* or equivalently, since $\mathbf{L}^{\vee}(u)$ is subdirectly irreducible, *when is* $\mathbf{L}^{\sqcup}(u) \in \boldsymbol{HS}(\mathbf{L}^{\sqcup}(v))$*?* We will consider X and v to be fixed, and seek criteria for u to have this property.

Homomorphic images were considered at length in Chapters II and III. In particular, Theorem 3.19 can be paraphrased as follows.

Theorem 11.11. *Let* $\mathbf{L}$ *be a finite lower bounded lattice. A lattice* $\mathbf{K}$ *is a homomorphic image of* $\mathbf{L}$ *if and only if* $\mathbf{K} \cong B^{\vee}$ *for some* J*–closed subset* $B \subseteq \mathrm{J}(\mathbf{L})$*.*

We will use the following characterization of sublattices.

Theorem 11.12. *Let* $\mathbf{L}$ *be a finite lower bounded lattice. A lattice* $\mathbf{S}$ *is isomorphic to a sublattice of* $\mathbf{L}$ *if and only if* $\mathbf{S} \cong A^{\vee}$*, where* $A =$

$\{\beta_h(c) : c \in L\} \cap \mathrm{J}(\mathbf{FL}(X))$ *for some finite set* X *and homomorphism* $h : \mathbf{FL}(X) \to \mathbf{L}$.

It should be emphasized that $\beta_h(a)$ is the least element $w \in \mathbf{FL}(X)$ such that $h(w) \geq a$. When h is not an epimorphism, this need not be a preimage of a.

Proof: If $h : \mathbf{FL}(X) \to \mathbf{L}$ is a homomorphism, then h is lower bounded and $\mathbf{S} = h(\mathbf{FL}(X))$ is a sublattice of $\mathbf{L}$ isomorphic to $A^{\vee}$, where A is as given in the statement of the theorem.

Conversely, assume $\varepsilon : \mathbf{S} \to \mathbf{L}$ is an embedding. Then, as in the proof of Corollary 2.15 (or just because $\mathbf{L}$ is finite), ε is lower bounded and for $s \in \mathbf{S}$, we have $s = \beta_\varepsilon \varepsilon(s)$. In particular, the range of β_ε is all of S. Also, $\mathbf{S}$ is a lower bounded lattice by Corollary 2.14. Thus there exist a finite set X and an epimorphism $f : \mathbf{FL}(X) \twoheadrightarrow \mathbf{S}$, so that $\mathbf{S} \cong B^{\vee}$ where $B = \{\beta_f(s) : s \in S\} \cap \mathrm{J}(\mathbf{FL}(X))$. Now let $h = \varepsilon f$. By Theorem 2.1, we have $\beta_h = \beta_f \beta_\varepsilon$. Since the range of β_ε is S, this implies that β_h and β_f have the same range. Hence we also have $B = \{\beta_h(c) : c \in L\} \cap \mathrm{J}(\mathbf{FL}(X))$, as desired.

Now we have both homomorphic images and sublattices characterized separately, and we can combine them in a rather straightforward way.

Theorem 11.13. *Let* $\mathbf{L}$ *be a finite lower bounded lattice. A lattice* $\mathbf{K}$ *is in* $\boldsymbol{HS}(\mathbf{L})$ *if and only if* $\mathbf{K} \cong B^{\vee}$ *for some* J*–closed subset* $B \subseteq \{\beta_h(p) : p \in \mathrm{J}(\mathbf{L})\}$ *for some finite set* X *and homomorphism* $h : \mathbf{FL}(X) \to \mathbf{L}$.

While the preceding characterization is nice, it is only good up to isomorphism, which is not sufficient for our purposes. Instead, we can use the following concrete version.

Corollary 11.14. *Let* $u, v \in \mathrm{J}(\mathbf{FL}(X))$. *Then* $\mathbf{L}^{\vee}(u) \in \boldsymbol{V}(\mathbf{L}^{\vee}(v))$ *if and only if* $u = \beta_h(p)$ *for some homomorphism* $h : \mathbf{FL}(X) \to \mathbf{L}^{\vee}(v)$ *and* $p \in \mathrm{J}(v)$.

Proof: Assume $h : \mathbf{FL}(X) \to \mathbf{L}^{\vee}(v)$ and $u = \beta_h(p)$, for some $p \in \mathrm{J}(v)$. Let $A = \{\beta_h(q) : q \in \mathrm{J}(v)\}$, so that $A^{\vee} \cong h(\mathbf{FL}(X))$; note the comment after Theorem 3.13. Since $u \in A$ and A is J–closed, we have $\mathrm{J}(u) \subseteq A$. Hence $\mathbf{L}^{\vee}(u)$ is a homomorphic image of the sublattice $h(\mathbf{FL}(X))$ of $\mathbf{L}^{\vee}(v)$.

Conversely, suppose $\mathbf{L}^{\vee}(u) \in \boldsymbol{HS}(\mathbf{L}^{\vee}(v))$. Then there exist a sublattice $\mathbf{S}$ of $\mathbf{L}^{\vee}(v)$ and homomorphisms g and ε such that $g : \mathbf{S} \twoheadrightarrow \mathbf{L}^{\vee}(u)$ is an epimorphism, and $\varepsilon : \mathbf{S} \to \mathbf{L}^{\vee}(v)$ is an embedding. Since $u \in \mathrm{J}(\mathbf{FL}(X))$, we have the standard epimorphism $f : \mathbf{FL}(X) \twoheadrightarrow \mathbf{L}^{\vee}(u)$.

Of course, $u = \beta_f(u)$. Because $\mathbf{FL}(X)$ is free and g is onto, there is a homomorphism $k : \mathbf{FL}(X) \to \mathbf{S}$ such that $f = gk$. Then we can define $h : \mathbf{FL}(X) \to \mathbf{L}^\vee(v)$ by $h = \varepsilon k$. Because the range of each of these homomorphisms is in $\boldsymbol{V}(\mathbf{L}^\vee(v))$, they are all lower bounded.

Now $\beta_f = \beta_k\beta_g$ by Theorem 2.1, whence $u = \beta_k(s_0)$ for $s_0 = \beta_g(u) \in S$. But $\beta_h = \beta_k\beta_\varepsilon$, and $\beta_\varepsilon\varepsilon(s) = s$ for $s \in S$, so $u = \beta_h\varepsilon(s_0)$. Finally, since $u \in \mathrm{J}(\mathbf{FL}(X))$ and β_h preserves joins, we can choose p join irreducible in $\mathbf{L}^\vee(v)$ with $u = \beta_h(p)$.

In order to convert Corollary 11.14 into a useful algorithm, let us go through the construction of β_h carefully for a homomorphism $h : \mathbf{FL}(X) \to \mathbf{L}^\vee(v)$. First recall that the domain of β_h is the ideal $h(1_{\mathbf{FL}(X)})/0$; if h is given, then of course we know $h(1_{\mathbf{FL}(X)})$. The map $\beta_0 : h(1_{\mathbf{FL}(X)})/0 \to X^\wedge$ is given by

$$(*) \qquad \beta_0(a) = \bigwedge\{x \in X : h(x) \geq a\}\,,$$

with the empty meet being assigned the value $1_{\mathbf{FL}(X)}$. Now, by Corollary 2.8, β_h can be constructed inductively using β_0 and the structure of $\mathbf{L}^\vee(v)$ only. For $a \in \mathrm{D}_0(\mathbf{L}^\vee(v))$, we have $\beta_h(a) = \beta_0(a)$. If $a \in \mathrm{D}_k(\mathbf{L}^\vee(v))$ and we know $\beta_h(b)$ for each $b \in \mathrm{D}_{k-1}(\mathbf{L}^\vee(v))$, then by Theorem 2.47,

$$\beta_h(a) = \beta_0(a) \wedge \bigwedge_{\bigvee B \leq h(1_{\mathbf{FL}(X)})}^{B \in \mathcal{M}^*(a)} \bigvee_{b \in B} \beta_h(b)$$

where $\mathcal{M}^*(a)$ denotes the set of minimal nontrivial join covers of a in $\mathbf{L}^\vee(v)$ whose join is minimal (as in Section 4 of Chapter II). Terms with $\bigvee B \not\leq h(1_{\mathbf{FL}(X)})$ are simply omitted.

Recall that it suffices to compute $\beta_h(p)$ for $p \in \mathrm{J}(\mathbf{L}^\vee(v)) = \mathrm{J}(v)$, as β_h preserves joins. Also note that if $p \in \mathrm{J}(v)$ and $p = \bigwedge x_i \wedge \bigwedge \bigvee p_{jk}$ canonically, then $\mathcal{M}^*(a)$ consists exactly of the join covers $\{p_{j1}, p_{j2}, \ldots\}$. This was proved for $\mathbf{FL}(X)$ in Lemma 3.11, and is clearly inherited by $\mathbf{L}^\vee(v)$. Thus, once we know $\beta_0 : \mathbf{L}^\vee(v) \to X^\wedge$, then we can compute β_h for $p \in \mathrm{J}(v)$ by

(†)

$$\beta_h(p) = \begin{cases} \beta_0(p) & \text{if } p \in X^\wedge \\ \beta_0(p) \wedge \bigwedge \bigvee \beta_h(p_{jk}) & \text{if } p = \bigwedge x_i \wedge \bigwedge \bigvee p_{jk} \text{ canonically} \end{cases}$$

again omitting the joins not in the domain of β_h.[1]

[1] The formula (†) and its variation (‡) below are the only places where v plays an explicit role. All the results in this section could be phrased with $\mathbf{L}^\vee(v)$ replaced wherever it occurs by an arbitrary finite, lower bounded lattice $\mathbf{L}$, as we have sometimes done. However, if we are thinking about algorithms, then it is useful to have (†).

The preceding discussion shows that we can compute β_h directly from β_0. The next part, concerning β_0, is quite general.

Let $h : \mathbf{FL}(X) \to \mathbf{L}$ be a homomorphism, where X and $\mathbf{L}$ are finite. Then $\beta_0 : \mathbf{L} \to X^{\wedge}$ defined by $(*)$ is a partial map with the property that for every $x \in X$, $\{a \in L : \beta_0(a) \leq x\}$ is an ideal, say $b_x/0$. The homomorphism h is then determined by $h(x) = b_x$. This determines $h(1_{\mathbf{FL}(X)}) = h(\bigvee X)$, and hence the domain of β_0 and β_h. Moreover, it suffices to know the values of β_0 on $\mathrm{J}(\mathbf{L})$. β_0 is subject to the restriction that, if $\beta_0(p_i) \leq x$ and $r \leq \bigvee p_i$, then $\beta_0(r) \leq x$. While our theorems were stated in terms of homomorphisms h and β_h, it is more efficient in practice to work with certain partial maps $\gamma : \mathrm{J}(\mathbf{L}) \to X^{\wedge}$.

Lemma 11.15. *Let X be finite, and let $\mathbf{L}$ be a finite lattice. A partial map $\gamma : \mathrm{J}(\mathbf{L}) \to X^{\wedge}$ is $\beta_0|_{\mathrm{J}(\mathbf{L})}$ for some homomorphism $h : \mathbf{FL}(X) \to \mathbf{L}$ if and only if it satisfies*

(1) *the domain of γ is $(c/0) \cap \mathrm{J}(\mathbf{L})$ where*

$$c = \bigvee\{p \in \mathrm{J}(\mathbf{L}) : \gamma(p) \text{ is defined and } \gamma(p) \neq 1_{\mathbf{FL}(X)}\};$$

(2) *for all $x \in X$, if $r \leq p_1 \vee \cdots \vee p_n$ and $\gamma(p_i) \leq x$ for all i, then $\gamma(r) \leq x$.*

The first condition is merely a technicality which will have no effect on our calculations. In the presence of (2), it merely keeps us from assigning the trivial value $1_{\mathbf{FL}(X)}$ where β_0 should be undefined. Note that (2) implies that γ is order preserving by taking $n = 1$.

Proof: The preceding discussion shows that the conditions are necessary. Conversely, given γ satisfying (1) and (2), define $h : X \to \mathbf{L}$ by $h(x) = \bigvee\{p \in \mathrm{J}(\mathbf{L}) : x \geq \gamma(p)\}$. Of course, this map h extends to a homomorphism from $\mathbf{FL}(X)$ to $\mathbf{L}$. Using condition (2) and the definition of h, we have $h(x) \geq p$ if and only if $x \geq \gamma(p)$. Since $\gamma(p) \in X^{\wedge}$, this implies $\gamma(p) = \bigwedge\{x \in X : h(x) \geq p\} = \beta_0(p)$ for all $p \in \mathrm{J}(\mathbf{L})$.

Now it is a straightforward matter to combine Theorem 11.15 with Corollary 11.14 and (†) to find the elements u such that $u = \beta_h(p)$ for some homomorphism $h : \mathbf{FL}(X) \to \mathbf{L}^{\vee}(v)$ and some $p \in \mathrm{J}(v)$. Given a partial map $\gamma : \mathrm{J}(\mathbf{L}^{\vee}(v)) \to X^{\wedge}$, define $\delta_\gamma : \mathrm{J}(\mathbf{L}^{\vee}(v)) \to \mathbf{FL}(X)$ by

$$(\ddagger)\quad \delta_\gamma(p) = \begin{cases} \gamma(p) & \text{if } p \in X^{\wedge} \\ \gamma(p) \wedge \bigwedge \bigvee \delta_\gamma(p_{jk}) & \text{if } p = \bigwedge x_i \wedge \bigwedge \bigvee p_{jk} \text{ canonically} \end{cases}$$

omitting the joins with undefined terms.

Corollary 11.16. *Let $u, v \in \mathrm{J}(\mathbf{FL}(X))$. Then $\mathbf{L}^{\vee}(u) \in \boldsymbol{V}(\mathbf{L}^{\vee}(v))$ if and only if $u = \delta_\gamma(p)$ for some $p \in \mathrm{J}(v)$ and partial map $\gamma : \mathrm{J}(v) \to X^{\wedge}$ satisfying conditions* (1) *and* (2) *of* Lemma *11.15.*

This yields an algorithm which can be reasonably straightforward to apply, as the following examples illustrate.

EXAMPLE 11.17. Let $X = \{x, y, z\}$ and let $v = x \wedge (y \vee z)$. Then $\mathrm{J}(v) = \{v, y, z\}$ and $\mathbf{L}^{\vee}(v)$ is diagrammed in Figure 3.2(1) on page 111. With the map

$$\begin{aligned} \gamma(y) &= y \\ \gamma(z) &= x \wedge z \\ \gamma(v) &= x \end{aligned}$$

we obtain $\delta_{\gamma}(x \wedge (y \vee z) = x \wedge (y \vee (x \wedge z))$. Thus $\mathbf{L}^{\vee}(x \wedge (y \vee (x \wedge z)))$, which is the pentagon diagrammed in Figure 3.2(2) on page 111, is a sublattice of $\mathbf{L}^{\vee}(v)$.

More generally, for this v, we obtain $\mathbf{L}^{\vee}(u) \in \boldsymbol{V}(\mathbf{L}^{\vee}(v))$ exactly when u has the form either p or $p \wedge (q \vee r)$ with p, q, $r \in X^{\wedge}$. Some of these will not be join irreducible, e.g., $1_{\mathbf{FL}(X)} \wedge (x \vee y) = x \vee y$. These we can discard, and we are left with (up to permutations of the variables) the terms:

$$x \qquad x \wedge y \qquad x \wedge (y \vee z) \qquad x \wedge (y \vee (x \wedge z))$$

as expected.

EXAMPLE 11.18. Let $|X| = 5$ and let $v = (x_2 \vee x_3) \wedge (x_4 \vee x_5)$. Then $\mathbf{L}^{\vee}(u) \in \boldsymbol{HS}(\mathbf{L}^{\vee}(v))$ when u has one of the forms

$$p \qquad p \wedge (q \vee r) \qquad (q \vee r) \wedge (s \vee t) \qquad p \wedge (q \vee r) \wedge (s \vee t)$$

with p, q, r, s and $t \in X^{\wedge}$. In particular, note that terms like $x_1 \wedge (x_2 \vee x_3) \wedge (x_4 \vee x_5)$ are included, which is perhaps surprising at first. There are roughly 2^{2^5} such terms, most of which are not distinct, so enumerating them is not necessarily something you want to do by hand.

For practical purposes, let us observe that we can use a slightly weakened version of condition (2) of Lemma 11.15 for determining which maps are β_h for some homomorphism h.

THEOREM 11.19. *Let X be finite, and let $\mathbf{L}$ be a finite lower bounded lattice. Let $\gamma : \mathrm{J}(\mathbf{L}) \to X^{\wedge}$ be a partial map which satisfies*

(1) *the domain of γ is $(c/0) \cap \mathrm{J}(\mathbf{L})$ where*

$$c = \bigvee \{p \in \mathrm{J}(\mathbf{L}) : \gamma(p) \text{ is defined and } \gamma(p) \neq 1_{\mathbf{FL}(X)}\};$$

(2′) *$p \leq q$ implies $\gamma(p) \leq \gamma(q)$.*

Let $h : \mathbf{FL}(X) \to \mathbf{L}$ be the homomorphism extending the map $h(x) = \bigvee\{p \in \mathrm{J}(\mathbf{L}) : x \geq \gamma(p)\}$. Define $\delta_\gamma : \mathrm{J}(\mathbf{L}) \to \mathbf{FL}(X)$ by $\delta_\gamma(p) = \gamma(p)$ if p is join prime, and inductively

$$\delta_\gamma(p) = \gamma(p) \wedge \bigwedge\nolimits_{\bigvee B \leq h(1_{\mathbf{FL}(X)})}^{B \in \mathcal{M}^*(p)} \bigvee \delta_\gamma(b).$$

Then $\delta_\gamma = \beta_h$.

Proof: We will show by induction on the D-rank of an element $p \in \mathrm{J}(\mathbf{L})$ that $\delta_\gamma(p) = \beta_h(p)$. If p is join prime, then this works exactly as in the proof of Lemma 11.15, as condition (2) does not come into play. If p is not join prime, then we use the fact that

$$\beta_h(p) = \beta_0(p) \wedge \bigwedge\nolimits_{\bigvee B \leq h(1_{\mathbf{FL}(X)})}^{bB \in \mathcal{M}^*(p)} \bigvee_{b \in B} \beta_h(b).$$

If $B \in \mathcal{M}^*(p)$, then each $b \in B$ has D-rank $\rho(b) < \rho(p)$, and hence $\beta_h(b) = \delta_\gamma(b)$. Clearly $\gamma(p) \geq \beta_0(p)$ (and this inequality could be strict), so, comparing the formulas for $\delta_\gamma(p)$ and $\beta_h(p)$, we need to show that $\delta_\gamma(p) \leq \beta_0(p)$, i.e., $\delta_\gamma(p) \leq x$ whenever $h(x) \geq p$. But if $h(x) = \bigvee\{q \in \mathrm{J}(\mathbf{L}) : x \geq \gamma(q)\} \geq p$, then either $p \leq q$ for some q with $\gamma(q) \leq x$, whence $\gamma(p) \leq x$ by (2′), or there is a nontrivial join cover C of p with $\gamma(c) \leq x$ for all $c \in C$. In the latter case, there exists $B \in \mathcal{M}^*(p)$ with $\bigvee B \leq \bigvee C$. Of course, $h(x) \geq \bigvee C \geq \bigvee B$, so by induction $\delta_\gamma(b) = \beta_h(b) \leq x$ for all $b \in B$, whence by the definition of $\delta_\gamma(p)$ we have $\delta_\gamma(p) \leq x$, as desired.

Finally, we should mention the preservation results from Nation [**108**].

THEOREM 11.20. *Let $\mathbf{L}$ be a finite lower bounded lattice, and let $\mathbf{K}$ be in $\boldsymbol{V}(\mathbf{L})$.*

(1) *If $\rho(u) \leq k$ for all $u \in \mathrm{J}(\mathbf{L})$, then $\rho(v) \leq k$ for all $v \in \mathrm{J}(\mathbf{K})$.*
(2) *If $|\mathcal{M}(u)| \leq m$ for all $u \in \mathrm{J}(\mathbf{L})$, then $|\mathcal{M}(v)| \leq m$ for all $v \in \mathrm{J}(\mathbf{K})$.*
(3) *If $|A| \leq n$ for each $A \in \mathcal{M}(u)$ and $u \in \mathrm{J}(\mathbf{L})$, then $|B| \leq n$ for each $B \in \mathcal{M}(v)$ and $v \in \mathrm{J}(\mathbf{K})$.*

The first property is a consequence of Theorem 2.41 (for homomorphic images), Corollary 2.15 (for sublattices), and the observation that finite direct products do not increase the D-rank of join irreducibles. It can be generalized in various ways; see [**108**] and [**30**]. Properties (2) and (3), which restrict the number of minimal nontrivial join covers and the number of elements in each, hold when $\mathbf{L}$ is any finite lattice.[2] These preservation properties are the motivation for some of the

[2]The analogues of (2) and (3) for $\mathcal{M}^*(u)$ are false. Recall from Lemma 3.11 that if $u = \bigwedge_i \bigvee_j u_{ij} \wedge \bigwedge_k x_k$ canonically, then $\mathcal{M}^*(u)$ contains precisely the join

rewrite rules in the next section, and are illustrated by the examples above. However, we do not need them directly, so we will omit the proof.

5. A Lattice Variety with AC TRS

It is elementary to show that the variety of distributive lattices has an AC TRS. Since the variety generated by a finite lattice is always finitely based (McKenzie [**98**], cf. Baker [**7**]), it is conceivable that every nontrivial finitely generated lattice variety has an AC TRS. In this section we will show that the lattice $\mathbf{L}_2 = \mathbf{L}^{\sqcup}(x \wedge (y \vee z))$ (see Figure 3.2(1) on page 111) generates a variety with a finite, convergent AC TRS. Then we will indicate the modifications necessary to produce an infinite collection of varieties with AC TRS's.

We start by applying the results of the previous section to $\boldsymbol{V}(\mathbf{L}_2)$.

THEOREM 11.21. *A lattice term w in the variables X is in $\boldsymbol{V}(\mathbf{L}_2)$–canonical form if and only if*

(1) *w is in Whitman canonical form,*
(2) *if $w \in X$ or if w is formally a meet, then either $w \in X^{\wedge}$, or it has the form $w = t \wedge (u \vee v)$ with t, u, $v \in X^{\wedge}$,*
(3) *if w is formally a join, $w = w_1 \vee \cdots \vee w_n$, then each w_i satisfies (2).*

Proof: By the results of the last sections, the $\boldsymbol{V}(\mathbf{L}_2)$–canonical form of w is an element of $A_X^{\vee}(\mathbf{L}_2)$ and, if w is join irreducible, it is in $A_X(\mathbf{L}_2)$. Also if u is join irreducible, $u \in A_X(\mathbf{L}_2)$ if and only if $\mathbf{L}^{\vee}(u) \in \boldsymbol{V}(\mathbf{L}_2)$. By Corollary 11.16, if $u \in \mathrm{J}(\mathbf{FL}(X))$, then $\mathbf{L}^{\vee}(u) \in \boldsymbol{V}(\mathbf{L}_2)$ if and only if $u \in X^{\wedge}$ or

$$\begin{aligned} u &= \delta_\gamma(x \wedge (y \vee z)) \\ &= \gamma(x \wedge (y \vee z)) \wedge (\gamma(y) \vee \gamma(z)) \end{aligned}$$

for some map $\gamma : \{y, z, x \wedge (y \vee z)\} \to X^{\wedge}$ such that, for all $t \in X$, $\gamma(x \wedge (y \vee z)) \leq t$ whenever $\gamma(y) \leq t$ and $\gamma(z) \leq t$. Theorem 11.19 and the fact that $\{y, z, x \wedge (y \vee z)\}$ is an antichain show that this last condition can be ignored; this is also easy to see directly. Thus a term is in $\boldsymbol{V}(\mathbf{L}_2)$–canonical form if and only if it is the Whitman canonical form of a join of such terms. Conditions (1)–(3) describe these terms.

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covers $U_i = \{u_{i1}, u_{i2}, \dots\}$. Straightforward applications of Corollary 11.16 show that $\mathbf{L}^{\vee}((x \vee y) \wedge (r \vee s) \wedge (t \vee u))$ is in $\boldsymbol{V}(\mathbf{L}^{\vee}((x \vee y) \wedge [z \vee ((r \vee s) \wedge (t \vee u))]))$, and $\mathbf{L}^{\vee}(x \wedge (y \vee s \vee t)))$ is in $\boldsymbol{V}(\mathbf{L}^{\vee}(x \wedge (y \vee (z \wedge (s \vee t)))))$, violating the extensions of (2) and (3), respectively.

Our rewrite rules will be designed so that at least one of them applies to any lattice term not in $\boldsymbol{V}(\mathbf{L}_2)$–canonical form. The first three rules are designed to bring an arbitrary term into the form specified by (2) and (3), without worrying about Whitman canonical form.

(9)
$$r \wedge (s \vee (t \wedge (u \vee v))) \to [r \wedge (s \vee (t \wedge u))] \vee [r \wedge (s \vee (t \wedge v))] \vee [r \wedge t \wedge (u \vee v)]$$

(10) $$r \wedge (s \vee t \vee u) \to [r \wedge (s \vee t)] \vee [r \wedge (s \vee u)] \vee [r \wedge (t \vee u)]$$

(11)
$$(s \vee t) \wedge (u \vee v) \to [s \wedge (u \vee v)] \vee [t \wedge (u \vee v)] \vee [u \wedge (s \vee t)] \vee [v \wedge (s \vee t)]$$

In each of these rules, the right hand side is the $\boldsymbol{V}(\mathbf{L}_2)$–canonical form of the left hand side. This is easily checked by showing that the right hand side is in $\boldsymbol{V}(\mathbf{L}_2)$–canonical form, and if a term w is below the left hand side and either $w \in X^\wedge$ or $w = w_0 \wedge (w_1 \vee w_2)$ with each $w_i \in X^\wedge$, then w is below one of the joinands of the right hand side. In particular, by Corollary 11.10, each of these rules corresponds to an equation valid in $\boldsymbol{V}(\mathbf{L}_2)$. The critical observation is that if w is a lattice term which does not satisfy conditions (2) and (3) of Theorem 11.21, then one of these rewrite rules applies to w.

All the rest of our rewrite rules are designed to put a term which does satisfy conditions (2) and (3) of Theorem 11.21 into Whitman canonical form. Each of the remaining rules corresponds to an equation true in all lattices, and hence can be applied without the assumption that (2) and (3) hold. However, our primary concern is that there be enough of them so that at least one applies whenever (2) and (3) hold, but (1) fails. Throughout we use the standard criteria for canonical form, Theorem 1.18.

A term $w \in X^\wedge$ can be put into Whitman canonical form using just the idempotent law

(12) $$x \wedge x \to x$$

to eliminate repeated variables.

Now consider words of the form $w = t \wedge (u \vee v)$ with each of these a meet of variables, write $t = \bigwedge T$, $u = \bigwedge U$ and $v = \bigwedge V$. Then, by a straightforward application of the algorithm of Theorem 1.18, w will be in canonical form if and only if

(a) t, u and v are in canonical form,
(b) $u \vee v$ is in canonical form, i.e., $U \not\subseteq V$ and $V \not\subseteq U$,
(c) for all i, $t_i \nleq u \vee v$, i.e., $u \neq t_i$ and $v \neq t_i$,
(d) for all i, $u \vee v \nleq t_i$, i.e., $T \cap U \cap V = \emptyset$,
(e) $u \ngeq t \wedge (u \vee v)$, i.e., there exists $u_j \notin T \cup V$,

(f) $v \not\geq t \wedge (u \vee v)$, i.e., there exists $v_k \notin T \cup U$.

Condition (a) can be achieved using (12). The following rewrite rules take care of the rest: (13) and (14) for (b), (15) for (c), (16) for (d), and (17) for (e) and (f).

$$x \vee x \to x \tag{13}$$

$$x \vee (x \wedge y) \to x \tag{14}$$

$$x \wedge (x \vee y) \to x \tag{15}$$

$$x \wedge ((x \wedge y) \vee (x \wedge z)) \to (x \wedge y) \vee (x \wedge z) \tag{16}$$

$$x \wedge ((x \wedge y) \vee (y \wedge z)) \to x \wedge y \tag{17}$$

Next we turn to getting a join of terms of the form $\bigwedge Y$ or $t \wedge (u \vee v)$ into canonical form. We can assume that each of these terms is itself in canonical form, for otherwise one of the above rewrite rules applies. We begin by finding rewrite rules to assure that we have an antichain.

For two meets of variables which are comparable, $\bigwedge Y \leq \bigwedge Z$, rules (13) and (14) suffice.

Next, $\bigwedge Y \leq t \wedge (u \vee v)$ if and only if $Y \supseteq T \cup U$ or $Y \supseteq T \cup V$. The following rules cover this case, including various degenerate forms.

$$[x_1 \wedge y_1 \wedge y_2 \wedge x_2] \vee [y_1 \wedge x_2 \wedge ((y_2 \wedge x_2) \vee z)] \to y_1 \wedge x_2 \wedge ((y_2 \wedge x_2) \vee z) \tag{18}$$

$$[y_1 \wedge y_2 \wedge x_2] \vee [y_1 \wedge x_2 \wedge ((y_2 \wedge x_2) \vee z)] \to y_1 \wedge x_2 \wedge ((y_2 \wedge x_2) \vee z) \tag{19}$$

$$[x_1 \wedge y_1 \wedge y_2] \vee [y_1 \wedge (y_2 \vee z)] \to y_1 \wedge (y_2 \vee z) \tag{20}$$

$$[y_1 \wedge y_2] \vee [y_1 \wedge (y_2 \vee z)] \to y_1 \wedge (y_2 \vee z) \tag{21}$$

$$[x_1 \wedge y_2 \wedge x_2] \vee [x_2 \wedge ((y_2 \wedge x_2) \vee z)] \to x_2 \wedge ((y_2 \wedge x_2) \vee z) \tag{22}$$

$$[y_2 \wedge x_2] \vee [x_2 \wedge ((y_2 \wedge x_2) \vee z)] \to x_2 \wedge ((y_2 \wedge x_2) \vee z) \tag{23}$$

Let us explain in some detail how (18)–(23) were obtained; the same general method will be used repeatedly in the remaining cases. We begin by observing that by symmetry we can assume $Y \supseteq T \cup U$. Then make a Venn diagram for the sets T, U, V and Y, remembering that $T \cap U \cap V = \emptyset$ and $Y \supseteq T \cup U$ (see Figure 11.1). Label each nonempty sector with a variable, as indicated. Since we are doing the case $y \leq t \wedge (u \vee v)$, our rule has the form $y \vee [t \wedge (u \vee v)] \to t \wedge (u \vee v)$. Substituting the meets of the appropriate variables gives us the most general rewrite rule for this case:

$$
\begin{aligned}
&[x_1 \wedge x_2 \wedge y_1 \wedge y_2 \wedge z' \wedge z'' \wedge z''']\\
&\qquad\qquad \vee [y_1 \wedge x_2 \wedge z''' \wedge ((y_2 \wedge x_2 \wedge z') \vee (z \wedge z' \wedge z'' \wedge z'''))]\\
&\qquad \to y_1 \wedge x_2 \wedge z''' \wedge ((y_2 \wedge x_2 \wedge z') \vee (z \wedge z' \wedge z'' \wedge z''')) \,.
\end{aligned}
$$

This would be used as a rewrite rule by mapping each variable to the meet of the generators in the appropriate set, e.g. $\sigma(y_1) = \bigwedge(T - U - V)$. As is often the case, we can simplify the general rule a little bit. Since y_1 and z never occur in the same meet of variables, and z''' occurs in a meet if and only if one of the two does, we can get rid of z''' and assign those variables to both y_1 and z. Thus $\sigma(y_1)$ now becomes $\bigwedge(T - U)$, etc. Similarly, we can eliminate z' and z''. This gives us (18).

Of course, we cannot apply (18) if the sector corresponding to one of the variables in the equation is empty. In that case, we would like to define σ of such variables to be 1, but that is not allowed as part of our rewrite procedure. Instead, we substitute 1 for every possible subset of the variables in (18). For those substitutions with the property that neither side evaluates to 1, we obtain a valid new rewrite rule. Some substitutions may yield trivial rules, covered by earlier rewrites; these can be omitted. The rest we keep, in this case giving us (19)–(23).

For this inclusion, we obtained only six rules. After the next case, we start getting large numbers of degenerate cases, far too many to write down. *From now on, we will write down only the most general rule. To obtain the complete list, substitute* 1 *for the variables in every combination, and keep those rules for which neither side evaluates to* 1*, and both sides are not the same.* We will make no attempt to count the number of nontrivial rules obtained.

Continuing, $t \wedge (u \vee v) \leq \bigwedge Y$ if and only if $Y \subseteq T \cup (U \cap V)$. If $Y \subseteq T$, then (14) suffices again. Otherwise we use one of the rules obtained from the general rule

$$
[x \wedge y_1 \wedge ((u' \wedge y_2) \vee (v' \wedge y_2))] \vee [y_1 \wedge y_2] \to y_1 \wedge y_2 \,. \tag{24}
$$

It is fairly easy to see that, up to symmetry in u and v, $p \wedge (q \vee r) \leq t \wedge (u \vee v)$ if and only if $p \wedge (q \vee r) \leq t$ and either $p \wedge (q \vee r) \leq u$ or $q \leq u$, $r \leq v$. Also, $p \wedge (q \vee r) \leq t$ if and only if for all i, $t_i \geq p$ or $t_i \geq q \vee r$ (and not both by canonical form). Likewise for $p \wedge (q \vee r) \leq u$.

First we do the former case: $p \wedge (q \vee r) \leq t$ and $p \wedge (q \vee r) \leq u$. This situation is covered by the bunch of rules given by the general rule

$$
[P \wedge (Q \vee R)] \vee [T \wedge (U \vee v)] \to T \wedge (U \vee v) \tag{25}
$$

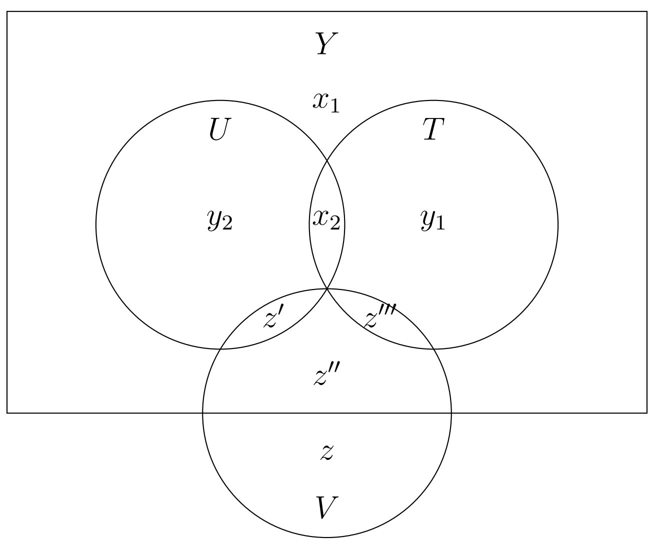

FIGURE 11.1

where

$$
\begin{aligned}
P &:= p' \wedge t_1 \wedge u_1 \wedge x_1 \\
Q &:= q' \wedge t_2 \wedge u_2 \wedge x_2 \\
R &:= r' \wedge t_2 \wedge u_2 \wedge x_2 \\
T &:= t_1 \wedge t_2 \wedge x_1 \wedge x_2 \\
U &:= u_1 \wedge u_2 \wedge x_1 \wedge x_2
\end{aligned}
$$

For the latter case: $p \wedge (q \vee r) \leq t$ and $q \leq u$ and $r \leq v$, we have a set of rules given by

$$[P \wedge (Q \vee R)] \vee [T \wedge (U \vee V)] \to T \wedge (U \vee V) \tag{26}$$

where

$$\begin{aligned}
P &:= p' \wedge t_1 \wedge x_2 \wedge x_5 \\
Q &:= q' \wedge u' \wedge t_2 \wedge x_1 \wedge x_2 \wedge x_3 \wedge x_4 \wedge x_6 \\
R &:= r' \wedge v' \wedge t_2 \wedge x_1 \wedge x_3 \wedge x_5 \wedge x_6 \wedge x_7 \\
T &:= t_1 \wedge t_2 \wedge x_2 \wedge x_3 \wedge x_5 \wedge x_6 \\
U &:= u' \wedge x_1 \wedge x_2 \wedge x_3 \wedge x_4 \\
V &:= v' \wedge x_1 \wedge x_5 \wedge x_6 \wedge x_7
\end{aligned}$$

The last condition for canonical form, Theorem 1.18, implies that if $w = \bigvee \bigwedge w_{ij}$ is in canonical form then $w_{ij} \nleq w$ for all i and j. Since a meet of generators is join prime and we have already covered the cases where the joinands do not form an antichain, this will not apply if $w_i = \bigwedge Y$, or if $w_i = t \wedge (u \vee v)$ and w_{ij} is a meetand of t. For the remaining cases, we have $w_i = t \wedge (u \vee v)$ and $w_{ij} = u \vee v$. As usual, we may assume that u and v are meets of generators and hence join prime (otherwise one of (9)–(11) applies). Then $w_{ij} \leq w$ implies that there exist k and l such that $u \leq w_k$ and $v \leq w_l$. This can occur in several ways, depending on the form of w_k and w_l.

The first case is covered by the rules derived from the general rule

$$[t \wedge ((p \wedge x_1) \vee (q \wedge x_2))] \vee p \vee q \to p \vee q\,. \tag{27}$$

The next case is covered by those derived from

$$\begin{aligned}
&[t \wedge ((p \wedge x_1) \vee (q \wedge r \wedge x_2 \wedge x_3))] \vee p \vee [q \wedge x_2 \wedge ((r \wedge x_2) \vee s)] \\
&\qquad \to p \vee [q \wedge x_2 \wedge ((r \wedge x_2) \vee s)]\,.
\end{aligned} \tag{28}$$

The last rewrite rules are derived from

$$\begin{aligned}
&\vee [p \wedge x_1 \wedge ((a \wedge x_1) \vee b)] \vee [q \wedge x_2 \wedge ((r \wedge x_2) \vee s)] \\
&\to [p \wedge x_1 \wedge ((a \wedge x_1) \vee b)] \vee [q \wedge x_2 \wedge ((r \wedge x_2) \vee s)]\,.
\end{aligned} \tag{29}$$

Now we have rewrite rules with the property that at least one of them applies to any term not in $\boldsymbol{V}(\mathbf{L}_2)$–canonical form. It remains to show that the rewriting process always terminates (necessarily then in the $\boldsymbol{V}(\mathbf{L}_2)$–canonical form). For this purpose we introduce a measure on lattice terms. This requires a construction which is fairly interesting in itself.

LEMMA 11.22. *Let $p > 1$ be an integer. There exists a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ which is associative, commutative, with $f(1,1) = p$ and for all n, $m \in \mathbb{N}$ satisfies $f(n+1, m) \geq pf(n, m)$.*

Proof: Our construction is motivated by the basic representation theorem of N. G. Alimov [**5**] (see Fuchs [**64**]) that every cancellative totally ordered semigroup containing no anomalous pair can be embedded into the additive group of real numbers. Given p, we will define a mapping $r : \mathbb{N} \to \mathbb{Q}$ such that

(1) r is strictly increasing,
(2) $r(1) = 1$ and $r(p) = 2$,
(3) for each i, j there exists $n \in \mathbb{N}$ such that $r(i) + r(j) = r(p^n)$.

It is then fairly straightforward to see that the function f defined by

$$f(i, j) = r^{-1}(r(i) + r(j))$$

has the desired properties.

The map r is defined inductively. Let $r(1) = 1$ and, as a convenience, $r(0) = 0$. Then define

$$r(p^i + j) = 1 + r(i) + \frac{j}{p^{i+1} - p^i}\,(r(i+1) - r(i))$$

for $0 \leq j < p^{i+1} - p^i$. Note that the formula also holds for $j = p^{i+1} - p^i$.

Now it is easy to show inductively that there are increasing, integer-valued functions $a(k)$ and $b(k)$ such that

(i) $r(k+1) - r(k) = \frac{1}{p^{a(k)}(p-1)^{b(k)}}$, and
(ii) $r(k) \in \frac{1}{p^{a(k)}(p-1)^{b(k)}}\,\mathbb{N}$.

Hence, any number which is greater than $r(k)$ and differs from it by a multiple of $\frac{1}{p^{a(k)}(p-1)^{b(k)}}$ is also in the range of r. Also, since $r(p^i) = 1 + r(i)$, we have $k = p^n$ for some n if and only if $r(k) - 1 \in \operatorname{range} r$. Combining these observations, we can conclude that $r(i) + r(j) = r(p^n)$ for some n, as desired.

Now we define an integer valued measure $|w|$ for lattice terms w as follows, using the above construction with $p = 5$. Let $|x| = 1$ for $x \in X$. Inductively, define $|u \vee v| = |u| + |v|$ and $|u \wedge v| = f(|u|, |v|)$.

LEMMA 11.23. *The measure $|w|$ has the following properties.*

(1) *$|w| \in \mathbb{N}$ for any lattice term w.*
(2) *$|w|$ respects associativity and commutativity, i.e., $|t \wedge (u \wedge v)| = |(t \wedge u) \wedge v|$ and $|u \wedge v| = |v \wedge u|$, and similarly for joins.*
(3) *If u is a proper subterm of v, then $|u| < |v|$.*
(4) *Let w' be obtained from w by replacing a subterm u by a term u' with $|u'| < |u|$. Then $|w'| < |w|$.*

These properties are straightforward to verify, using of course the associativity, commutativity, and monotonicity of f and addition.

LEMMA 11.24. *If $u \to v$ by application of any of the rules represented by* (9)–(29), *then* $|u| > |v|$.

Proof: First we note that since $|w|$ is monotonic in its subterms (property (4) of the Lemma 11.23), we can just consider the effect of applying a rewrite rule to an entire term. Secondly, rules (12)–(29) omit a superfluous term, and that definitely lowers the measure. It is fairly straightforward to then check that each of (9)–(11) replaces a term by at most four terms each with measure at most 1/5 that of the original, since if $|u| > |u'|$ then $|u \wedge v| = f(|u|, |v|) \geq 5f(|u'|, |v|) = |u' \wedge v|$.

Thus we have constructed a finite set of rewrite rules, and a measure $|u|$ on terms, with the following properties.

(1) Each rewrite rule corresponds to an equation valid in $\boldsymbol{V}(\mathbf{L}_2)$.
(2) At least one of the rules applies to any term not in $\boldsymbol{V}(\mathbf{L}_2)$–canonical form.
(3) If $u \to v$ by one of the rules, then $|v| < |u|$.

It follows that any sequence of applications of these rules must terminate in the $\boldsymbol{V}(\mathbf{L}_2)$–canonical form of a term. Hence we conclude that

THEOREM 11.25. *The rules represented by* (9)–(29) *provide a finite, convergent AC TRS for* $\boldsymbol{V}(\mathbf{L}_2)$.

COROLLARY 11.26. *The lattice equations corresponding to rules* (9)–(11),

$$\begin{aligned} r \wedge (s \vee (t \wedge (u \vee v))) &\approx [r \wedge (s \vee (t \wedge u))] \vee [r \wedge (s \vee (t \wedge v))] \vee [r \wedge t \wedge (u \vee v)] \\ r \wedge (s \vee t \vee u) &\approx [r \wedge (s \vee t)] \vee [r \wedge (s \vee u)] \vee [r \wedge (t \vee u)] \\ (s \vee t) \wedge (u \vee v) &\approx [s \wedge (u \vee v)] \vee [t \wedge (u \vee v)] \vee [u \wedge (s \vee t)] \vee [v \wedge (s \vee t)], \end{aligned}$$

together with the equational axioms for lattice theory, form an equational basis for the variety $\boldsymbol{V}(\mathbf{L}_2)$.

6. More Varieties with AC TRS

In the last section we constructed an AC TRS for $\boldsymbol{V}(\mathbf{L}_2)$. In this section we will see how to extend these arguments to obtain an infinite list of finitely generated lattice varieties which have a finite, convergent AC TRS, although to actually construct an AC TRS for one of these varieties would be prohibitively tedious. This raises an interesting question.

PROBLEM 11.27. *Does every finitely generated lattice variety have a finite, convergent AC term rewrite system? What about every variety generated by a finite lower (or upper) bounded lattice?*[3]

In particular, we do not know if $\boldsymbol{V}(\mathbf{N}_5)$ has an AC TRS.

Recall that $\mathbf{L}_2 = \mathbf{L}^\vee(x \wedge (y \vee z))$. Let $\mathbf{L}_n = \mathbf{L}^\vee(x \wedge (y_1 \vee \cdots \vee y_n))$. The modifications required in order to construct an AC TRS for $\boldsymbol{V}(\mathbf{L}_n)$ with $n \geq 3$ are as follows. First, using Corollary 11.10, we must replace (9) by

$$(9') \quad r \wedge (s \vee (t \wedge (u \vee v))) \to [r \wedge (s \vee (t \wedge u) \vee (t \wedge v))] \vee [r \wedge t \wedge (u \vee v)]$$

and replace (10) by the n+1–distributive law

$$(10') \qquad r \wedge (s_1 \vee \cdots \vee s_{n+1}) \to \bigvee_i r \wedge (\bigvee_{j \neq i} s_j)\,.$$

Rule (11) remains the same. The rules for canonical form must be revised to include terms which are meets of variables, or of the form $t \wedge (u_1 \vee \cdots \vee u_k)$ with $2 \leq k \leq n$, where each of these is a meet of variables. Again there are only finitely many cases to consider, and the general rule for each one can be derived using the procedure described on page 290. (The important observation is the conditions for canonical form can be written as set theoretic conditions on the sets of variables involved.) Finally, in view of (10′), for $n \geq 4$ we need to replace the number 5 by n+2 in the definition of $|w|$.

However we can do much better.

THEOREM 11.28. *Let m, $n_1, \dots, n_m$ be positive integers, and let x_{ij}, for $1 \leq i \leq m$ and $1 \leq j \leq n_i$, be distinct variables. Let*

$$w = \bigwedge_{i=1}^{m} \bigvee_{i=1}^{n_i} x_{ij}.$$

Then $\boldsymbol{V}(\mathbf{L}^\vee(w))$ has a finite, convergent AC TRS.

---

[3] The proof that there is no AC TRS for lattices only involved words in $X^{\wedge\vee\wedge}$, but with an arbitrary finite number of variables. It was crucial that these elements, and their subelements, have a canonical form in $\mathbf{FL}(X)$, but (W) was used only in a relatively weak way. It was required that these elements not be both meet and join reducible. (W) is also used implicitly in the proof of Lemma 11.2, but can be replaced there by an argument using upper and lower bounds, $\alpha(u)$ and $\beta(u)$, when these exist in relatively free lattices. Let $\mathcal{U}$ denote the variety generated by all the finite lattices $\mathbf{L}^\vee(w)$ and $\mathbf{L}^\wedge(w)$ where w appears explicitly in the proof of Theorem 11.8. This is a locally finite variety. By following the argument of Theorem 11.8 carefully, with only minor modifications, one can show that *if $\mathcal{V}$ is a variety with $\mathcal{U} \leq \mathcal{V} \leq \mathcal{L}$, then $\mathcal{V}$ has no finite, convergent AC TRS.*

We will only sketch the proof. For the remainder of this section, fix positive integers m, $n_1, \dots, n_m$ and let $w = \bigwedge_{i=1}^{m} \bigvee_{i=1}^{n_i} x_{ij}$ as in Theorem 11.28. Since we already know how to find an AC TRS for each $\boldsymbol{V}(\mathbf{L}_n)$, we may assume $m \geq 2$ and $n_i \geq 2$ for at least two i's, and hence without loss of generality for all i. (For if $v \in \mathrm{J}(\mathbf{FL}(X))$ and x is a variable not occurring in v, then $\mathbf{L}^\vee(x \wedge v) \cong \mathbf{L}^\vee(v)$.)

First we should apply Corollary 11.16.

LEMMA 11.29. *If X is any set and $u \in \mathrm{J}(\mathbf{FL}(X))$, then $\mathbf{L}^\vee(u) \in \boldsymbol{V}(\mathbf{L}^\vee(w))$ if and only if $u \in X^\wedge$ or*

$$u = u_0 \wedge \bigwedge_{i=1}^{q} \bigvee_{i=1}^{r_i} u_{ij}.$$

with u_0, $u_{ij} \in X^\wedge$, $q \leq m$ and $r_i \leq n_i$.

Implicit in the preceding lemma is the fact that if a lattice term u has the given form, then so does the Whitman canonical form of u. If w were allowed to have D-rank 2 or more, this would no longer be true, as is shown by the examples in the footnote to Theorem 11.20.

COROLLARY 11.30. *Let u be a lattice term in the variables X. Then u is in $\boldsymbol{V}(\mathbf{L}^\vee(w))$–canonical form if and only if*

(1) *u is in Whitman canonical form,*
(2) *if u is in X or is formally a meet, then either $u \in X^\wedge$, or*

$$u = u_0 \wedge \bigwedge_{i=1}^{q} \bigvee_{i=1}^{r_i} u_{ij}$$

with u_0, $u_{ij} \in X^\wedge$, $q \leq m$ and $r_i \leq n_i$,
(3) *if u is formally a join, $u = u_1 \vee \cdots \vee u_n$, then each u_i satisfies (2).*

Using the preceding corollary, we can find the $\boldsymbol{V}(\mathbf{L}^\vee(w))$–canonical form of any given lattice term. The first part of our term rewrite system consists of the rules $P \to Q$ where P is

$$r \wedge (s \vee (t \wedge (u \vee v))), \qquad \bigwedge_{i=1}^{m+1} (s_{i1} \vee s_{i2})$$

or, for some i with $1 \leq i \leq m$,

$$\left(\bigvee_{k=1}^{n_i+1} s_{ik}\right) \wedge \bigwedge_{j \neq i} \left(\bigvee_{p=1}^{n_j} s_{jp}\right)$$

or one of its degenerate forms where not all the s_{jp}'s occur (and it is still not in $\boldsymbol{V}(\mathbf{L}^{\vee}(w))$–canonical form), and Q is the $\boldsymbol{V}(\mathbf{L}^{\vee}(w))$–canonical form of P.

This procedure gives us the rewrite rule (9′) from above; for each i with $1 \le i \le m$, and each sequence r_j ($j \ne i$) with $0 \le r_j \le n_j$ such that the left hand side is not in $\boldsymbol{V}(\mathbf{L}^{\vee}(w))$–canonical form, the rule

$$(\bigvee_{k=1}^{n_i+1} s_{ik}) \wedge \bigwedge_{j\ne i}(\bigvee_{p=1}^{r_j} s_{jp}) \to \bigvee_{k=1}^{n_i+1} \big[(\bigvee_{t\ne k} s_{it}) \wedge \bigwedge_{j\ne i}(\bigvee_{p=1}^{r_j} s_{jp})\big]$$

$$\vee \bigvee_{n_q>n_i}^{q} \bigvee_{F\subseteq\{1,\dots,n_q\}}^{|F|=n_i} \big[(\bigvee_{f\in F} s_{qf}) \wedge (\bigvee_{k=1}^{n_i+1} s_{ik}) \wedge \bigwedge_{j\ne q,i}(\bigvee_{p=1}^{r_j} s_{jp})\big] \tag{10″}$$

and, if $n_i \ge 4$ for some i, the rule
(11″)

$$\bigwedge_{i=1}^{m+1} (s_{i1} \vee s_{i2})$$

$$\to \bigvee_{i=1}^{m+1} \bigvee_{j=1}^{2} (s_{ij} \wedge \bigwedge_{k\ne i}(s_{k1} \vee s_{k2}))$$

$$\vee \bigvee_{1\le q<r\le m+1} \Big[((s_{q1} \wedge s_{r1}) \vee (s_{q1} \wedge s_{r2})$$

$$\vee (s_{q2} \wedge s_{r1}) \vee (s_{q2} \wedge s_{r2})) \wedge \bigwedge_{k\ne q,r} (s_{k1} \vee s_{k2})\Big].$$

If $n_i < 4$ for all i, then the right hand side of (11″) will not be in $\boldsymbol{V}(\mathbf{L}^{\vee}(w))$–canonical form, and should be modified accordingly. (Actually, it would suffice to use (11″) in this case also, because the right hand side is below the left and above its $\boldsymbol{V}(\mathbf{L}^{\vee}(w))$–canonical form. Rule (10″) would then apply after any use of (11″).) The justification for each of these rules is provided by showing that if u has the form given by Corollary 11.30, and u is below the left hand side, then u is below the right hand side.

Now we must add rewrite rules to reduce a term of the form given by Corollary 11.30 to Whitman canonical form. As before, we only need finitely many of these, and in fact each one shortens a term by omitting a subterm. Again, these rules are derived by the algorithm described on page 290.

Define a measure $|t|$ on lattice terms as in Section 5, but using a value of the parameter p in the definition of f from Lemma 11.22 which is greater than the number of joinands on the right hand side of any of these rewrite rules. With a little care, one can then show that $|u| > |v|$ whenever $u \to v$ by one of these rewrite rules. In particular, for $(11'')$ we need to observe that

$$|(s_{q1} \vee s_{q2}) \wedge (s_{r1} \vee s_{r2})| > |(s_{q1} \wedge s_{r1}) \vee (s_{q1} \wedge s_{r2}) \vee (s_{q2} \wedge s_{r1}) \vee (s_{q2} \wedge s_{r2})|$$

whenever $p > 4$.

Thus, for any w as in Theorem 11.28, we can construct a finite set of rewrite rules, and a measure $|u|$ on terms, with the following properties.

(1) Each rewrite rule corresponds to an equation valid in $\boldsymbol{V}(\mathbf{L}^{\vee}(w))$.
(2) At least one of the rules applies to any term not in $\boldsymbol{V}(\mathbf{L}^{\vee}(w))$–canonical form.
(3) If $u \to v$ by one of the rules, then $|v| < |u|$.

It follows that any sequence of applications of these rules must terminate in the $\boldsymbol{V}(\mathbf{L}^{\vee}(w))$–canonical form of a term. Thus the rules form a finite, convergent AC TRS for $\boldsymbol{V}(\mathbf{L}^{\vee}(w))$.

This pretty much reaches the limit of the simple ideas developed in this section.

Open Problems

Below are the open problems which appeared in the text.

PROBLEM 1.25 (page 29). *Which unary polynomials on free lattices are fixed point free? For which unary polynomials f does $\bigvee f^i(a)$ exist for all a?*

PROBLEM 5.28 (page 141). *Characterize those ordered sets which can be embedded into a free lattice.*

PROBLEM 5.68 (page 175). *Which lattices (and in particular which countable lattices) are sublattices of a free lattice?*

PROBLEM 10.17 (page 261). *Does there exist a quadruple $a < c_1 \leq c_2 < b$ of elements of a free lattice $\mathbf{FL}(X)$ such that the intervals c_1/a and b/c_2 are both infinite and every element of b/a is comparable with either c_1 or c_2?*

PROBLEM 10.18 (page 262). *Describe all meet reducible elements a of $\mathbf{FL}(X)$ such that every element above a is comparable with a canonical meetand of a.*

PROBLEM 10.22 (page 265). *Which completely join irreducible elements $a \in \mathbf{FL}(X)$ satisfy $\kappa(a) = \mathbf{d}_a$? Are there infinitely many for fixed X?*

PROBLEM 10.23 (page 265). *Is the element a from* (1) *on page 264 the only element of $\mathbf{FL}(X)$ which is invariant under the automorphisms of $\mathbf{FL}(X)$ and satisfies $\kappa(a) = \mathbf{d}_a$?*

PROBLEM 10.25 (page 267). *Is every middle element of $\mathbf{FL}(X)$ above a minimal one?*

PROBLEM **??** (page **??**). *Can one decide if an ordered set of size n is a lattice in time faster than $O(n^{5/2})$? Can the various data structures mentioned in* Theorem **??** *be computed in time faster that $O(n^{5/2})$?*

PROBLEM **??** (page **??**). *Is there a polynomial time algorithm which decides if $w \in \mathbf{FL}(\mathbf{P})$ is completely join irreducible?*

Problem **??** (page **??**). *Is there a polynomial time algorithm which decides if a finitely presented lattice is finite*?

Problem **??** (page **??**). *Is there an algorithm to decide if a finitely presented lattice is weakly atomic*? *of finite width*?

Problem **??** (page **??**). *Is there a polynomial time algorithm which decides if a finitely presented lattice is projective*?

Problem 11.27 (page 296). *Does every finitely generated lattice variety have a finite, convergent AC term rewrite system*? *What about every variety generated by a finite lower* (*or upper*) *bounded lattice*?

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List of Symbols

Notation	Page(s)	Description
$A \subseteq B$, $A \subset B$	10	Set inclusion, proper set inclusion.
$a \vee b$, $\bigvee S$	10	Join, least upper bound.
$a \wedge b$, $\bigwedge S$	10	Meet, greatest lower bound.
$\mathbf{L} = \langle L, \sqcup, \wedge \rangle$	10	A lattice on the set L.
$\mathrm{J}(\mathbf{L})$	11	The join irreducible elements of L.
$\mathrm{M}(\mathbf{L})$	11	The meet irreducible elements of L.
$a \prec b$, $b \succ a$	11	a is covered by b, b covers a.
b/a	11	The interval from a to b.
$\mathbf{L}[C]$	12	$\mathbf{L}$ with a convex set doubled.
λ	12	Natural homomorphism from $\mathbf{L}[C]$ onto $\mathbf{L}$.
$\ker f$	73	The kernel of f.
$t^{\mathbf{L}}(a_1, \ldots, a_n)$, $t^{\mathbf{L}}$	13	Interpretation of a term t in $\mathbf{L}$.
$\mathbf{FL}(X)$, $\mathbf{FL}(n)$	14	Free lattice.
$\mathbf{F}_{\mathcal{V}}(X)$	14	Relatively free lattice in $\mathcal{V}$ over X.
$\mathbf{2}$	14	The two element lattice.
$\underline{x} = \bigwedge(X - \{x\})$	17	Atoms of $\mathbf{FL}(X)$.
$\overline{x} = \bigvee(X - \{x\})$	17	Coatoms of $\mathbf{FL}(X)$.
(W)	17	Whitman's condition.
(W+)	18	A modification of Whitman's condition.
$A \ll B$, $C \gg D$	20	Join and meet refinement.
$(\mathrm{SD}_\vee)$, $(\mathrm{SD}_\wedge)$	24	Join and meet semidistributivity.
$\boldsymbol{H}$, $\boldsymbol{S}$, $\boldsymbol{P}_u$, $\boldsymbol{V}$	33	Class closure operators.
$\beta(a)$, $\beta_h(a)$	35	The least preimage of a.
$\alpha(a)$, $\alpha_h(a)$	35	The largest preimage of a.
$A^\wedge$, $A^\vee$	37	Meet closure of A, join closure of A.
$\beta_k(a)$, $\beta_{k,h}(a)$	37	Step k in computing $\beta(a)$.
$\alpha_k(a)$, $\alpha_{k,h}(a)$	37	Step k in computing $\alpha(a)$.
$\mathcal{C}(a)$	37	Nontrivial join covers of a.
$\mathcal{M}(a)$	39	Minimal nontrivial join covers of a.
$\mathrm{D}_k(\mathbf{L})$	40	Elements such that $\beta(a) = \beta_k(a)$.

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