Cooperative Game Theory Approaches
for Network Partitioning

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Abstract. The paper is devoted to game-theoretic methods for community detection in networks. The traditional methods for detecting community structure are based on selecting denser subgraphs inside the network. Here we propose to use the methods of cooperative game theory that highlight not only the link density but also the mechanisms of cluster formation. Specifically, we suggest two approaches from cooperative game theory: the first approach is based on the Myerson value, whereas the second approach is based on hedonic games. Both approaches allow to detect clusters with various resolution. However, the tuning of the resolution parameter in the hedonic games approach is particularly intuitive. Furthermore, the modularity based approach and its generalizations can be viewed as particular cases of the hedonic games.

Keywords: Network partitioning, community detection, cooperative games, Myerson value, hedonic games.

1 Introduction

Community detection in networks or network partitioning is a very important topic which attracted the effort of many researchers. Let us just mention several main classes of methods for network partitioning. The first very large class is based on spectral elements of the network matrices such as adjacency matrix and Laplacian (see e.g., the survey [23] and references therein). The second class of methods, which is somehow related to the first class, is based on the use of random walks (see e.g., [1, 2, 6, 16, 18, 20] for the most representative works in this research direction.) The third class of approaches to network partitioning is based on methods from statistical physics [3, 21, 22]. The fourth class, which is probably most related to our approach, is based on the concept of modularity and its various generalizations [4, 9, 19, 24]. For a very thorough overview of the community detection methods we recommend the survey [7].

In essence, all the above methods (may be with some exception of the statistical physics methods), try to detect denser subgraphs inside the network and do not address the question: what are the natural forces and dynamics behind the

formation of network clusters. We feel that the game theory, and in particular, cooperative game theory is the right tool to explain the formation of network clusters.

In the present work, we explore two cooperative game theory approaches to explain possible mechanisms behind cluster formation. Our first approach is based on the Myerson value in cooperative game theory, which particularly emphasizes the value allocation in the context of games with interactions between players constrained by a network. The advantage of the Myerson value is in taking into account the impact of all coalitions. We use the efficient method developed in [15] and [14] based on characteristic functions to calculate quickly the Myerson value in the network. We would like to mention that in [14] a network centrality measure based on the Myerson value was proposed. It might be interesting to combine node ranking and clustering based on the same approach such as the Myerson value to analyze the network structure.

The second approach is based on hedonic games, which are games explaining the mechanism behind the formation of coalitions. Both our approaches allow to detect clusters with varying resolution and thus avoiding the problem of the resolution limit [8, 12]. The hedonic game approach is especially well suited to adjust the level of resolution as the limiting cases are given by the grand coalition and maximal clique decomposition, two very natural extreme cases of network partitioning. Furthermore, the modularity based approaches can be cast in the setting of hedonic games. We find that this gives one more, very interesting, interpretation of the modularity based methods.

Some hierarchical network partitioning methods based on tree hierarchy, such as [9], cannot produce a clustering on one resolution level with the number of clusters different from the predefined tree shape. Furthermore, the majority of clustering methods require the number of clusters as an input parameter. In contrast, in our approaches we specify the value of the resolution parameter and the method gives a natural number of clusters corresponding to the given resolution parameter.

In addition, our approach easily works with multi-graphs, where several edges (links) are possible between two nodes. A multi-edge has several natural interpretations in the context of social networks. A multi-edge can represent: a number of telephone calls; a number of exchanged messages; a number of common friends; or a number of co-occurrences in some social event.

The paper is structured as follows: in the following section we formally define network partitioning as a cooperative game. Then, in Section 3 we present our first approach based on the Myerson value. The second approach based on the hedonic games is presented in Section 4. In both Sections 3 and 4 we provide illustrative examples which explain the essence of the methods. Finally, Section 5 concludes the paper with directions for future research.

2 Network partitioning as a cooperative game

Let $g = (N, E)$ denote an undirected multi-graph consisting of the set of nodes N and the set of edges E. We denote a link between node i and node j as ij. The interpretation is that if $ij \in E$, then the nodes $i \in N$ and $j \in N$ have a connection in network g, while $ij \notin E$, then nodes i and j are not directly connected. Since we generally consider a multi-graph, there could be several edges between a pair of nodes. Multiple edges can be interpreted for instance as a number of telephone calls or as a number of message exchanges in the context of social networks.

We view the nodes of the network as players in a cooperative game. Let $N(g) = \{i : \exists j \text{ such that } ij \in g\}$. For a graph g, a sequence of different nodes $\{i_1, i_2, \ldots, i_k\}$, $k \geq 2$, is a path connecting i_1 and i_k if for all $h = 1, \ldots, k-1$, $i_h i_{h+1} \in g$. The length of the path l is the number of links in the path, i.e. $l = k - 1$. The length of the shortest path connecting i and j is distance between i and j. Graph g on the set N is connected graph if for any two nodes i and j there exists a path in g connecting i and j.

We refer to a subset of nodes $S \subset N$ as a coalition. The coalition S is connected if any two nodes in S are connected by a path which consists of nodes from S. The graph g' is a component of g, if for all $i \in N(g')$ and $j \in N(g')$, there exists a path in g' connecting i and j, and for any $i \in N(g')$ and $j \in N(g)$, $ij \in g$ implies that $ij \in g'$. Let $N|g$ is the set of all components in g and let $g|S$ is the subgraph with the nodes in S.

Let $g - ij$ denote the graph obtained by deleting link ij from the graph g and $g + ij$ denote the graph obtained by adding link ij to the graph g.

The result of community detection is a partition of the network (N, E) into subsets (coalitions) $\{S_1, ..., S_K\}$ such that $S_k \cap S_l = \emptyset, \forall k, l$ and $S_1 \cup ... \cup S_K = N$. This partition is *internally stable* or *Nash stable* if for any player from coalition S_k it is not profitable to join another (possibly empty) coalition S_l. We also say that the partition is *externally stable* if for any player $i \in S_l$ for whom it is benefitial to join a coalition S_k there exists a player $j \in S_k$ for whom it is not profitable to include there player i. The payoff definition and distribution will be discussed in the following two sections.

3 Myerson cooperative game approach

In general, cooperative game of n players is a pair $< N, v >$ where $N = \{1, 2, \ldots, n\}$ is the set of players and $v: 2^N \to R$ is a map prescribing for a coalition $S \in 2^N$ some value $v(S)$ such that $v(\emptyset) = 0$. This function $v(S)$ is the total utility that members of S can jointly attain. Such a function is called the characteristic function of cooperative game [13].

Characteristic function (payoff of coalition S) can be determined in different ways. Here we use the approach of [10, 11, 14, 15], which is based on discounting directed paths. The payoff to an individual player is called an imputation. The imputation in this cooperative game will be Myerson value [14, 15, 17].

Let $< N, v >$ be a cooperative game with partial cooperation presented by graph g and characteristic function v. An allocation rule Y describes how the value associated with the network is distributed to the individual players. Denote by $Y_i(v, g)$ the value allocated to player i from graph g under the characteristic function v.

Myerson proposed in [17] the allocation rule

$$Y(v, g) = (Y_1(v, g), \ldots, Y_n(v, g)),$$

which is uniquely determined by the following two axioms:

A1. If S is a component of g then the members of the coalition S ought to allocate to themselves the total value $v(S)$ available to them, i.e $\forall S \in N|g$

$$\sum_{i \in S} Y_i(v, g) = v(S). \tag{1}$$

A2. $\forall g$, $\forall ij \in g$ both players i and j obtain equal payoffs after adding or deleting a link ij,

$$Y_i(v, g) - Y_i(v, g - ij) = Y_j(v, g) - Y_j(v, g - ij). \tag{2}$$

Let us determine the characteristic function by the following way

$$v_g(S) = \sum_{K \in S|g} v(K).$$

Then the Myerson value can be calculated by the formula

$$Y_i(v, g) = \sum_{S \subset N \setminus \{i\}} (v_g(S \cup i) - v_g(S)) \frac{s!(n-s-1)!}{n!}, \tag{3}$$

where $s = |S|$ and $n = |N|$.

Let us determine the characteristic function which is determined by the scheme proposed by Jackson [11]: every direct connection gives to coalition S the impact r, where $0 \leq r \leq 1$. Moreover, players obtain an impact from indirect connections. Each path of length 2 gives to coalition S the impact r^2, a path of length 3 gives to coalition the impact r^3, etc. So, for any coalition S we obtain

$$v(S) = a_1 r + a_2 r^2 + \cdots + a_k r^k + \cdots + a_L r^L = \sum_{k=1}^{L} a_k r^k, \tag{4}$$

where L is a maximal distance between two nodes in the coalition; a_k is the number of paths of length k in this coalition. Set

$$v(i) = 0, \ \forall i \in N.$$

In [15] it was proven that the Myerson value can be found by the following simple procedure of allocation the general gain $v(N)$ to each player $i \in N$:

Stage 1. Two direct connected players together obtain r. Individually, they would receive nothing. So, each of them receives at least $r/2$. If player i has some direct connections then she receives the value $r/2$ times the number of paths of length 1 which contain the node i.

Stage 2. Three connected players obtain r^2, so each of them must receive $r^2/3$, and so on.

Arguing this way, we obtain the allocation rule of the following form:

$$Y_i(v, g) = \frac{a_1^i}{2}r + \frac{a_2^i}{3}r^2 + \cdots + \frac{a_L^i}{L+1}r^L = \sum_{k=1}^{L} \frac{a_k^i}{k+1}r^k, \qquad (5)$$

where a_k^i is the number of all paths of length k which contain the node i.

Example 1. Consider network of six nodes presented in Fig. 1. Below we show how to calculate characteristic function for different coalitions.
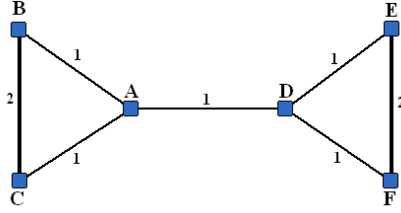

Fig. 1. Network of six nodes.

For the network $N = \{A, B, C, D, E, F\}$ we find $L = 3$, $a_1 = 9$, $a_2 = 4$, $a_3 = 4$. Consequently, the value of grand-coalition is

$$v(N) = 9r + 4r^2 + 4r^3.$$

For coalition $S = \{A, B, C, D\}$ we have $L = 2$, $a_1 = 5$, $a_2 = 2$ and we obtain

$$v(S) = 5r + 2r^2.$$

This way we can calculate the values of characteristic function for all coalitions $S \subset N$. After that we can find the Myerson vector.

Example 1 (ctnd). Let us calculate the Myerson value for player A in Example 1 using the allocation rule (5). Mark all paths which contain node A. The paths of length 1 are: {A,B}, {A,C}, {A,D}, hence $a_1^A = 3$. The paths of length 2 are: {B,A,C}, {B,A,D}, {C,A,D}, {A,D,E}, {A,D,F}, so $a_2^A = 5$. The paths of length 3: {B,A,D,E}, {B,A,D,F}, {C,A,D,E}, {C,A,D,F}, so $a_3^A = 4$. Consequently,

$$Y_A = \frac{3}{2}r + \frac{5}{3}r^2 + r^3.$$

Thus, we can propose the following algorithm for network partitioning based on the Myerson value: Start with any partition of the network $N = \{S_1, \ldots, S_K\}$. Consider a coalition S_l and a player $i \in S_l$. In cooperative game with partial cooperation presented by the graph $g|S_l$ we find the Myerson value for player i, $Y_i(g|S_l)$. That is reward of player i in coalition S_l. Suppose that player i decides to join the coalition S_k. In the new cooperative game with partial cooperation presented by the graph $g|S_k \cup i$ we find the Myerson value $Y_i(g|S_k \cup i)$. So, if for the player $i \in S_l : Y_i(g|S_l) \geq Y_i(g|S_k \cup i)$ then player i has no incentive to join to new coalition S_k, otherwise the player changes the coalition. The partition $N = \{S_1, \ldots, S_K\}$ is the Nash stable if for any player there is no incentive to move from her coalition. Notice that for unweighted graphs the definition of the Myerson value implies that for any coalition it is always beneficial to accept a new player (of course, for the player herself it might not be profitable to join that coalition), the Nash stability (internal stability) in this game coincides with the external stability.

Example 1 (ctnd). Let us clarify this approach on the network

$$N = \{A, B, C, D, E, F\}$$

presented in Fig. 1. Natural way of partition here is $\{S_1 = (A, B, C), S_2 = (D, E, F)\}$. Let us determine under which condition this structure will present the stable partition.

Suppose that characteristic function is determined by (4). For coalition S_1 the payoff $v(S_1) = 4r$. The payoff of player A is $Y_A(g|S_1) = r$. Imagine that player A decides to join the coalition S_2.

Coalition $S_2 \cup A$ has payoff $v(S_2 \cup A) = 5r + 2r^2$. The imputation in this coalition is $Y_A(g|S_2 \cup A) = r/2 + 2r^2/3, Y_D(g|S_2 \cup A) = 3r/2 + 2r^2/3, Y_E(g|S_2 \cup A) = Y_F(g|S_2 \cup A) = 3r/2 + r^2/3$. We see that for player A it is profitable to join this new coalition if $r/2 + 2r^2/3 > r$, or $r > 3/4$. Otherwise, the coalitional structure is stable.

Thus, for the network in Fig. 1 the Myerson value approach will give the partition $\{S_1 = (A, B, C), S_2 = (D, E, F)\}$ if $r < 3/4$ and, otherwise, it leads to the grand coalition. This example already gives a feeling that the parameter r can be used to tune the resolution of network partitioning. Such tuning will be even more natural in the ensuing approach.

4   Hedonic coalition game approach

There is another game-theoretic approach for the partitioning of a society into coalitions based on the ground-breaking work [5]. We apply the framework of Hedonic games [5] to network partitioning problem, particularly, specifying the preference function.

Assume that the set of players $N = \{1, \ldots, n\}$ is divided into K coalitions: $\Pi = \{S_1, \ldots, S_K\}$. Let $S_\Pi(i)$ denote the coalition $S_k \in \Pi$ such that $i \in S_k$. A player i preferences are represented by a complete, reflexive and transitive

binary relation $\succeq_i$ over the set $\{S \subset N : i \in S\}$. The preferences are additively separable [5] if there exists a value function $v_i : N \to \mathbb{R}$ such that $v_i(i) = 0$ and

$$S_1 \succeq_i S_2 \Leftrightarrow \sum_{j \in S_1} v_i(j) \geq \sum_{j \in S_2} v_i(j).$$

The preferences $\{v_i, i \in N\}$ are symmetric, if $v_i(j) = v_j(i) = v_{ij} = v_{ji}$ for all $i, j \in N$. The symmetry property defines a very important class of Hedonic games.

As in the previous section, the network partition Π is *Nash stable*, if $S_\Pi(i) \succeq_i S_k \cup \{i\}$ for all $i \in N, S_k \in \Pi \cup \{\emptyset\}$. In the Nash-stable partition, there is no player who wants to leave her coalition.

A potential of a coalition partition $\Pi = \{S_1, \ldots, S_K\}$ (see [5]) is

$$P(\Pi) = \sum_{k=1}^{K} P(S_k) = \sum_{k=1}^{K} \sum_{i,j \in S_k} v_{ij}. \tag{6}$$

Our method for detecting a stable community structure is based on the following better response type dynamics:

Start with any partition of the network $N = \{S_1, \ldots, S_K\}$. Choose any player i and any coalition S_k different from $S_\Pi(i)$. If $S_k \cup \{i\} \succeq_i S_\Pi(i)$, assign node i to the coalition S_k; otherwise, keep the partition unchanged and choose another pair of node-coalition, etc.

Since the game has the potential (6), the above algorithm is guaranteed to converge in a finite number of steps.

Proposition 1. *If players' preferences are additively separable and symmetric ($v_{ii} = 0, v_{ij} = v_{ji}$ for all $i, j \in N$), then the coalition partition Π giving a local maximum of the potential $P(\Pi)$ is the Nash-stable partition.*

One natural way to define a symmetric value function v with a parameter $\alpha \in [0, 1]$ is as follows:

$$v_{ij} = \begin{cases} 1 - \alpha, & (i,j) \in E, \\ -\alpha, & (i,j) \notin E, \\ 0, & i = j. \end{cases} \tag{7}$$

For any subgraph $(S, E|S)$, $S \subseteq N$, denote $n(S)$ as the number of nodes in S, and $m(S)$ as the number of edges in S. Then, for the value function (7), the potential (6) takes the form

$$P(\Pi) = \sum_{k=1}^{K} \left(m(S_k) - \frac{n(S_k)(n(S_k) - 1)\alpha}{2} \right). \tag{8}$$

We can characterize the limiting cases $\alpha \to 0$ and $\alpha \to 1$.

Proposition 2. *If $\alpha = 0$, the grand coalition partition $\Pi_N = \{N\}$ gives the maximum of the potential (8). Whereas if $\alpha \to 1$, some local maximum of (8)*

corresponds to a network decomposition into disjoint maximal cliques, given such decomposition exists. (A maximal clique is a clique which is not contained in another clique.)

Proof: It is immediate to check that for $\alpha = 0$ the grand coalition partition $\Pi_N = \{N\}$ gives the maximum of the potential (8), and $P(\Pi_N) = m(N)$.

For values of α close to 1, the partition into maximal cliques $\Pi = \{S_1, \ldots, S_K\}$ gives the maximum of (8). Indeed, assume that a player i from the clique $S_\Pi(i)$ of the size m_1 moves to a clique S_j of the size $m_2 < m_1$. The player $i \in S_\Pi(i)$ and S_j are connected by at most m_2 links. The impact on $P(\Pi)$ of this movement is not higher than

$$m_2(1 - \alpha) - (m_1 - 1)(1 - \alpha) \leq 0.$$

Now, suppose that player i from the clique $S_\Pi(i)$ moves to a clique S_j of the size $m_2 \geq m_1$. The player $i \in S_\Pi(i)$ is connected with the clique S_j by at most $m_2 - 1$ links. Otherwise, it contradicts the fact that Π is maximal clique cover and the clique S_j can be increased by adding of i. If i has an incentive to move from $S_\Pi(i)$ to the clique S_j, then for new partition the sum (8) would be not higher than for partition Π by

$$m_2 - 1 - m_2\alpha - (m_1 - 1)(1 - \alpha) = m_2 - m_1 - \alpha(m_2 - m_1 + 1).$$

For α close to 1, this impact is negative, so there is no incentive to join the coalition S_j.

The grand coalition and the maximal clique decomposition are two extreme partitions into communities. By varying the parameter α we can easily tune the resolution of the community detection algorithm.

Example 2. Consider graph $G = G_1 \cup G_2 \cup G_3 \cup G_4$, which consists of $n = 26$ nodes and $m = 78$ edges (see Fig. 2.) This graph includes 4 fully connected subgraphes: $(G_1, 8, 28)$ with 8 vertices connected by 28 links, $(G_2, 5, 10)$, $(G_3, 6, 15)$ and $(G_4, 7, 21)$. Subgraph G_1 is connected with G_2 by 1 edge, G_2 with G_3 by 2 edges, and G_3 with G_4 by 1 edge.

Firstly, find the potentials (8) for large-scale decompositions of G for any parameter $\alpha \in [0, 1]$. It is easy to check, that $P(G) = 78 - 325\alpha$, $P(\{G_1, G_2 \cup G_3 \cup G_4\}) = 77 - 181\alpha$, $P(\{G_1, G_2 \cup G_3, G_4\}) = 76 - 104\alpha$, $P(\{G_1, G_2, G_3, G_4\}) = 74 - 74\alpha$.

Other coalition partitions give smaller potentials: $P(\{G_1 \cup G_2, G_3 \cup G_4\}) = 76 - 156\alpha < 76 - 104\alpha$, $P(\{G_1 \cup G_2 \cup G_3, G_4\}) = 77 - 192\alpha < 77 - 181\alpha$, $P(\{G_1, G_2, G_3 \cup G_4\}) = 75 - 116\alpha < 76 - 104\alpha$, $P(\{G_1 \cup G_2, G_3, G_4\}) = 75 - 114\alpha < 76 - 104\alpha$.

We solve a sequence of linear inequalities in order to find maximum of the potential for all $\alpha \in [0, 1]$. The result is presented in the table below.
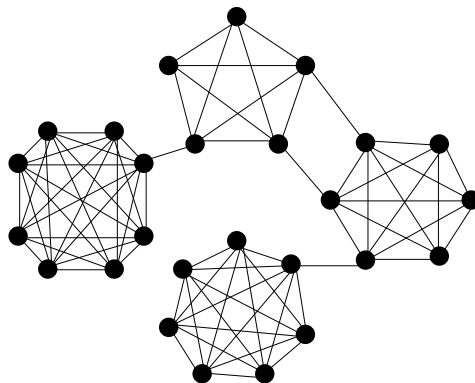
Fig. 2. Graph with four fully connected subgraphs.

Nash-stable coalition partitions in Example 2.

α	coalition partition	potential
$[0, 1/144]$	$G_1 \cup G_2 \cup G_3 \cup G_4$	$78 - 325\alpha$
$[1/144, 1/77]$	$G_1, G_2 \cup G_3 \cup G_4$	$77 - 181\alpha$
$[1/77, 1/15]$	$G_1, G_2 \cup G_3, G_4$	$76 - 104\alpha$
$[1/15, 1]$	G_1, G_2, G_3, G_4	$74 - 74\alpha$

Example 1 (ctnd). Note that for the unweighted version of the network example presented in Fig. 1, there are only two stable partitions: $\Pi = N$ for small values of $\alpha \leq 1/9$ and $\Pi = \{\{A, B, C\}, \{D, E, F\}\}$ for $\alpha > 1/9$.

Example 3. Consider the popular example of the social network from Zachary karate club (see Fig. 3). In his study [25], Zachary observed 34 members of a karate club over a period of two years. Due to a disagreement developed between the administrator of the club and the club's instructor there appeared two new clubs associated with the instructor (node 1) and administrator (node 34) of sizes 16 and 18, respectively.

The authors of [9] divide the network into two groups of roughly equal size using the hierarchical clustering tree. They show that this split corresponds almost perfectly with the actual division of the club members following the break-up. Only one node, node 3, is classified incorrectly.

Let us now apply the hedonic game approach to the karate club network. We start from the final partition $N = \{S_{15}, S_{19}\}$, which was obtained in [9]. We calculate the potential for grand-coalition $P(N) = 78 - 561\alpha$ and for partition $P(S_{15}, S_{19}) = 68 - 276\alpha$. From the equation $P(N) = P(S_{15}, S_{19})$ we obtain the cutoff point $\alpha = 2/57$. So, if $\alpha < 2/57$, $P(N)$ is larger than $P(S_{15}, S_{19})$, so partition $\{S_{15}, S_{19}\}$ is not Nash-stable. For $\alpha = 2/57$ the potential increases if the node 3 moves from S_{19} to S_{15}. For the new partition $P(S_{16}, S_{18}) = 68 - 273\alpha$. Comparing with potential of the grand coalition we obtain $\alpha = 5/144$. For $\alpha = 5/144$ the potential increases if the node 10 moves to S_{16}. Now $P(S_{17}, N \setminus S_{17}) =$
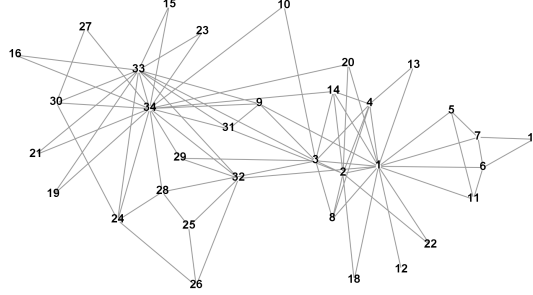
Fig. 3. Zachary karate club network.

$68 - 272\alpha$ and the new cutoff point is $\alpha = 10/289$. Finally, in order to find the upper bound of the resolution parameter, we have to check that for any player there is no incentive to move from her coalition to the empty coalition.

Thus, for $1/16 \geq \alpha \geq 10/289$ the Nash-stable partition is

$$S_{17} = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 17, 18, 20, 22\} \cup \{N \setminus S_{17}\}.$$

Notice that in this new partition the node 3 belongs to the "right" coaltion.

Another natural approach to define a symmetric value function is, roughly speaking, to compare the network under investigation with the configuration random graph model. The configuration random graph model can be viewed as a null model for a network with no community structure. Namely, the following value function can be considered:

$$v_{ij} = \beta_{ij} \left(A_{ij} - \gamma \frac{d_i d_j}{2m} \right), \tag{9}$$

where A_{ij} is a number of links between nodes i and j, d_i and d_j are the degrees of the nodes i and j, respectively, $m = \frac{1}{2} \sum_{l \in N} d_l$ is the total number of links in the network, and $\beta_{ij} = \beta_{ji}$ and γ are some parameters.

Note that if $\beta_{ij} = \beta, \forall i, j \in N$ and $\gamma = 1$, the potential (8) coincides with the network modularity [9,19]. If $\beta_{ij} = \beta, \forall i, j \in N$ and $\gamma \neq 1$, we obtain the generalized modularity presented first in [22]. The introduction of the non-homogeneous weights was proposed in [24] with the following particularly interesting choice:

$$\beta_{ij} = \frac{2m}{d_i d_j}.$$

The introduction of the resolution parameter γ allows to obtain clustering with varying granularity and in particular this helps to overcome the resolution limit [8].

Thus, we have now a game-theoretic interpretation of the modularity function. Namely, the coalition partition $\Pi = \{S_1, \ldots, S_K\}$ which maximises the

modularity

$$P(\Pi) = \sum_{k=1}^{K} \sum_{i,j \in S_k, i \neq j} \left(A_{ij} - \frac{d_i d_j}{2m} \right) \tag{10}$$

gives the Nash-stable partition of the network in the Hedonic game with the value function defined by (9), where $\gamma = 1$ and $\beta_{ij} = \beta$.

Example 1 (ctnd). For the network example presented in Fig. 1 we calculate $P(N) = 3/2, P(\{B,C\} \cup \{A,D\} \cup \{E,F\}) = P(\{A,B,C,D\} \cup \{E,F\}) = 7/2$ and $P(\{A,B,C\} \cup \{D,E,F\}) = 5$. Thus, according to the value function (9) with $\gamma = 1$ and $\beta_{ij} = \beta$ (modularity value function), $\Pi = \{\{A,B,C\},\{D,E,F\}\}$ is the unique Nash-stable coalition.

Example 3 (ctnd). Numerical calculations show that the partition $S_{17} \cup \{N \setminus S_{17}\}$ gives the maximum of potential function (10). It means that this partition is Nash stable.

5   Conclusion and future research

We have presented two cooperative game theory based approaches for network partitioning. The first approach is based on the Myerson value for graph constrained cooperative game, whereas the second approach is based on hedonic games which explain coalition formation. We find the second approach especially interesting as it gives a very natural way to tune the clustering resolution and generalizes the modularity based approaches. Our near term research plans are to test our methods on more social networks and to develop efficient computational Monte Carlo type methods.

Acknowledgements

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