A Survey of Some Network Reliability Analysis and Synthesis Results

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The purpose of this article is to introduce several results concerning the analysis and synthesis of reliable or invulnerable networks. First, the notion of signed reliability domination of systems is described and some applications to reliability analysis are reviewed. Then the analysis problem is considered and a brief summary of the difficulty of calculating various reliability measures is presented. Some relevant concepts in the synthesis of a most reliable network are studied. The article concludes with an introduction to a non-probabilistic approach to evaluate the vulnerability of a network. **NETWORKS, Vol. 54(2), 99–107 2009**

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1. INTRODUCTION

There have been numerous outstanding advances in reliability theory which use graph-theoretic ideas. An important problem in reliability theory is to determine the reliability of a system from the reliability of its components. Historically, network reliability has been concerned with the problem of determining the probability that there is a path of operational elements from a specified node to another node in the network. Various reliability measures have been defined in the literature and a number of graph-theoretic parameters have been used to derive formulas for calculating network reliability. For example, one of the most commonly used performance measures is the K-terminal reliability of a network. Suppose $G = (V, E)$ is an undirected graph and $K \subseteq V$ is a specified subset of V. Given that the elements (nodes or edges) of G may fail with known probabilities, the *K-terminal reliability* $R_K(G)$ of G is the probability that there is some connected subgraph H in G such that all elements of H are operational and H contains all nodes of K.

As a result of work on network reliability analysis, a graph invariant called *reliability domination* was introduced in [46]. This important concept has since been explored and applied by several researchers in reliability theory [2, 16, 29, 30, 41, 44, 45, 47]. The original definition of the reliability domination of a graph involves the concept of *formations* of a graph. Let $G = (V, E)$ be a graph and let $K \subseteq V$ be a specified subset of V. A *K-tree* is a tree of G which contains K and has all its degree-one nodes in K. A *formation* of G is a set of K-trees of G whose union yields the edge set of G. A formation is said to be *odd* if the number of K-trees in the formation is odd, and is said to be *even* otherwise. The *signed reliability domination* $d_K(G)$ is the number of odd minus the number of even formations of G. Domination possesses many interesting properties and plays a key role in network reliability.

In Section 2 of this survey, we first describe the notion of signed reliability domination of systems and review some of its applications to reliability analysis. We present several results showing how this invariant is related to other graph-theoretic properties.

The results are stated as theorems whose proofs are found elsewhere. Unless defined otherwise, graph-theoretic terminology used here follows Harary [28]. One exception is that we allow multiple edges and self-loops in a graph, so that by a graph we mean what Harary calls a pseudograph. Furthermore, in what follows, by domination of a graph or a system we mean its reliability domination.

The remainder of the article is organized as follows. Section 3 considers the analysis problem and gives a brief summary of the difficulty in calculating various reliability measures. In Section 4, we introduce some concepts for the synthesis of a *most reliable* network. In Section 5, we conclude with an introduction to a non-probabilistic approach to evaluating the ability of a network to resist damage.

2. COHERENT SYSTEMS AND DOMINATION

Let E be a finite set and let $\mathcal{P}(E)$ be the power set of E. A nonempty subset $\mathcal{C} \subseteq \mathcal{P}(E)$ is called a *clutter* on E if for any two elements $C_1 \in \mathcal{C}$ and $C_2 \in \mathcal{C}$, whenever $C_1 \subseteq C_2$, then $C_1 = C_2$. The pair $(E, \mathcal{C})$ will be referred to as a *system* and the system is *coherent* if each element of E is contained in some element of $\mathcal{C}$. A subset $A \subseteq E$ is called an *operating*

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state of the system $(E, \mathcal{C})$ if A contains an element of $\mathcal{C}$. Let $\Theta(E, \mathcal{C}) = \{A \subseteq E : C \subseteq A \text{ for some } C \in \mathcal{C}\}$ be the collection of all operating states of the system $(E, \mathcal{C})$. A *formation* $\mathcal{F}$ of a system $(E, \mathcal{C})$ is a subset of $\mathcal{C}$ with the property that $\bigcup_{T \in \mathcal{F}} T = E$. The formation $\mathcal{F}$ is odd or even depending on whether its cardinality is odd or even, respectively. The *signed domination* $d(E, \mathcal{C})$ of a system $(E, \mathcal{C})$ is defined to be the number of odd formations minus the number of even formations of $(E, \mathcal{C})$.

The notion of the signed domination was introduced in [46] in the context of reliability analysis of directed networks. Suppose $G = (V, E)$ is a digraph and $K \subseteq V$ is a specified subset of nodes of G such that a given source node $s \in K$. A subdigraph T is a rooted tree, rooted at s, if T has no directed cycles, the indegree of node s is 0 and the other nodes of T have indegree 1. A *K-tree*, rooted at s, is a rooted tree, rooted at s, such that (i) $s \in K$, (ii) every node of K is in the tree, and (iii) every node with outdegree 0 is a node of K. Clearly, a K-tree rooted at s of a digraph G constitutes a minimal subgraph G with the property that there is a directed path from s to each node of K. The subgraph is minimal in the sense that deletion of any edge from it results in the event that not all nodes in K can be reached from s. A digraph $G = (V, E)$ with $K \subseteq V$ and $s \in K$ is called a *K-digraph* if every edge of G lies in some K-tree, rooted at s, of G. Let $\mathcal{T}_K(G)$ be the collection of all the K-trees, rooted at s, of G. Clearly, $\mathcal{T}_K(G)$ constitutes a clutter on E. Furthermore, the system $(E, \mathcal{T}_K(G))$ is coherent if and only if G is a K-digraph. A *formation* $\mathcal{F}$ of G is a collection of K-trees, rooted at s, whose union constitutes the set of edges E of G. A formation $\mathcal{F}$ is odd or even depending on whether $\mathcal{F}$ contains an odd or even number of trees, respectively. The *signed domination* of a digraph $G = (V, E)$, with respect to a given subset $K \subseteq V$ and $s \in K$, is the number of odd minus the number of even formations of G. In this instance, we write $d_K(G)$ instead of $d(E, \mathcal{T}_K(G))$. The absolute value of $d_K(G)$ will be denoted by $D_K(G)$.

The notions of K-trees, K-digraphs, formations, and the signed domination are applicable to undirected graphs as well. Suppose $G = (V, E)$ is an undirected graph and $K \subseteq V$. A *K-tree* of G is a tree of G containing all nodes of K such that every leaf of the tree belongs to K. The notions of K-graph, formation, and the signed domination are similarly defined.

The invariant $d_K(G)$ has been used in the following directed network reliability problem. Suppose we are given a directed network $G = (V, E)$ with $K \subseteq V$ and $s \in K$. The elements (edges or nodes) of G, at a given instant of time, are in one of two states, either failed or functioning. A node u is said to be able to *communicate* with another node v in G if there is a directed path of functioning elements from u to v. The reliability problem of interest here, known as the *source-to-K-terminal reliability problem*, is that of determining the probability that s can communicate with every node in K of G when the joint probability distribution for the states of the elements of G is known.

It is shown in [44] that the source-to-K-terminal reliability $R_K(G)$ of a digraph G can be written as $R_K(G) = \sum_H (-1)^{e(H)-n(H)+1} Pr(H)$, where the summation extends over the set of acyclic K-subgraphs H of G, $e(H)$ and $n(H)$ are the number of edges and nodes of H respectively, and $Pr(H)$ is the probability that all the elements of H are functioning. A digraph is said to be *acyclic* if it has no directed cycles and it is *cyclic* otherwise. This formula may be viewed as a simplification of the well-known inclusion–exclusion expression of probability. The simplification is substantial as it entails generation of only noncanceling terms of the inclusion–exclusion expression. The proof of the above formula involves two key results on the signed domination of digraphs. First, for any acyclic K-digraph H with $e(H)$ edges and $n(H)$ nodes, $d_K(H) = (-1)^{e(H)-n(H)+1}$. Secondly, $d_K(H) = 0$ if H is cyclic or not a K-digraph.

Another important application of $d_K(G)$ is in certain pivoting algorithms to compute the reliability of undirected graphs G. In this case the absolute value of $d_K(G)$ has been used as a measure of the computational complexity of pivoting algorithms [45]. The signed domination theorem, for undirected graphs, proved in [45] for this purpose states that $d_K(G)$ can be expressed in terms of the signed dominations of two smaller graphs: one obtained by deleting an edge x from G, and the other by contracting x. The graph obtained by deleting an edge x from G will be denoted by $G - x$. If x is an edge of G with endnodes u and v, then by the contraction of x, we mean deleting x and identifying the nodes u and v as a single node. The graph obtained by contracting x in G is denoted by $G|x$. Note that all other edges with endnodes u and v in G become self-loops in $G|x$. Each edge of G other than x can, in a natural way, be regarded as an edge of $G|x$. This identification, which we henceforth assume, constitutes a one-to-one correspondence between the edge sets of $G - x$ and $G|x$. If K is a subset of the node set of G, we shall denote by $K|x$ the corresponding subset of the node set of $G|x$. Specifically, the signed domination theorem for undirected graphs states that, if $G = (V, E)$ is an undirected graph, $K \subseteq V$ and x is an edge of G, then $d_K(G) = d_{K|x}(G|x) - d_K(G - x)$. It is of interest to note that this result does not hold, in general, for directed K-graphs.

This section summarizes well-known results pertaining to the domination of a system. Let $\mathcal{C}$ be a clutter on the set E, and let $\Theta(E, \mathcal{C})$ be the collection of all operating states S of the associated system. We can partition the set $\Theta(E, \mathcal{C})$ of operating states into two classes, depending upon the cardinality of the states, as follows: $\Theta_o(E, \mathcal{C}) = \{S \in \Theta(E, \mathcal{C}) : |S| \text{ is odd}\}$ and $\Theta_e(E, \mathcal{C}) = \{S \in \Theta(E, \mathcal{C}) : |S| \text{ is even}\}$.

The following theorem, first proved in [29], provides a characterization for the domination of a system.

Theorem 2.1. *For any system* $(E, \mathcal{C})$, $d(E, \mathcal{C}) = (-1)^{|E|}(|\Theta_e(E, \mathcal{C})| - |\Theta_o(E, \mathcal{C})|)$.

Theorem 2.1 holds for an arbitrary clutter and, in particular, reaffirms the fact that the domination of a noncoherent system is zero.

The clutters associated with the success and failure of a specific element $x \in E$ are defined as follows: Let $\mathcal{C} - x =$

$\{C - \{x\} : C \in \mathcal{C}\}$ and $\mathcal{C}_{-x} = \{C \in \mathcal{C} : x \notin C\}$. Here $\mathcal{C}_{-x}$ is certainly a clutter but $\mathcal{C} - x$ may not be. Define $\mathcal{C}_{+x}$ to be the collection of elements of $\mathcal{C} - x$ obtained by discarding those which are proper supersets of other elements of $\mathcal{C} - x$. Clearly, if $x \in A$ then $A \in \Theta(E, \mathcal{C})$ if and only if $A - \{x\} \in \Theta(E - \{x\}, \mathcal{C} + x)$, while if $x \notin A$ then $A \in \Theta(E, \mathcal{C})$ if and only if $A \in \Theta(E - \{x\}, \mathcal{C}_{-x})$. Clutters $\mathcal{C}_{+x}$ and $\mathcal{C}_{-x}$ are called *minors* of $\mathcal{C}$ with respect to x.

A consequence of Theorem 2.1 is the following result which is well known in the reliability community (see Boesch et al. [16] for a simple proof).

Corollary 2.1 (The domination theorem). *If $(E, \mathcal{C})$ is a system, $x \in E$ and $\mathcal{C}_{+x}$ and $\mathcal{C}_{-x}$ are minors of $\mathcal{C}$ with respect to x, then $d(E, \mathcal{C}) = d(E - \{x\}, \mathcal{C}_{+x}) - d(E - \{x\}, \mathcal{C}_{-x})$.*

Corollary 2.1 was first established in [43] for the K-terminal domination of graphs where $K = V$. Later in [45] the result was extended to K-terminal domination of a graph for arbitrary K. Subsequently, Agrawal and Barlow [2] showed that the signed domination theorem holds for all coherent systems and this result was later extended to general clutters by Huseby [29, 30].

If the system $(E, \mathcal{C})$ represents an undirected graph $G = (V, E)$ such that the clutter $\mathcal{C}$ is the collection of the K-trees of G, $K \subseteq V$, then it is easy to see that Corollary 2.1 reduces to the following.

Corollary 2.2. *If $G = (V, E)$ is an undirected graph $K \subseteq V$ and $x \in E$, then $d_K(G) = d_{K|x}(G|x) - d_K(G - x)$.*

Using Corollary 2.1 it can easily be shown that for any connected graph $G = (V, E)$, $K \subseteq V$, the unsigned domination $D_K(G) = (-1)^{|E|-|V|+1} d_K(G)$. This result yields the following.

Corollary 2.3. *Let $G = (V, E)$ be an undirected graph, and let $K \subseteq V$ be a nonempty subset. Suppose $x \in E$ is an edge such that (i) x is not a self-loop, and (ii) if x has an endnode u in $V - K$, then $\text{degree}_G(u) \neq 1$. Then $D_K(G) = D_{K|x}(G|x) + D_K(G - x)$.*

Corollaries 2.2 and 2.3 were proved in [45] for undirected K-graphs. Because Corollary 2.3 does not hold for an arbitrary edge x, it is of interest to ask that if the equality $D_K(G) = D_{K|x}(G|x) + D_K(G - x)$ does not hold for some edge x of an undirected graph G, then what can we say about G and x? The following corollary answers this question.

Corollary 2.4. *Let $G = (V, E)$ be an undirected graph and let $K \subseteq V$. Then $D_K(G) \neq D_{K|x}(G|x) + D_K(G - x)$ if and only if (i) x is a self-loop and $G - x$ is a K-graph, or (ii) x is incident on a degree-one node $u \in V - K$ and $G - x$ is a K-graph.*

Our next corollary yields a characterization for the K-terminal domination of undirected graphs $G = (V, E)$, with $K \subseteq V$, in terms of certain spanning connected subgraphs of G. We denote by $c(G, K)$ the number of connected components of G which contain at least one node of K. Let $S_o(G, K)$ denote the number of spanning subgraphs S of G such that each S has an odd number of edges and $c(S, K) = 1$. Similarly, $S_e(G, K)$ is the number of spanning subgraphs with an even number of edges and $c(S, K) = 1$. The following is an immediate consequence of Theorem 2.1, and the facts that $|\Theta_o(E, \mathcal{T}_K(G))| = S_o(G, K)$ and $|\Theta_e(E, \mathcal{T}_K(G))| = S_e(G, K)$, where $\mathcal{T}_K(G)$ is the set of K-trees of G.

Corollary 2.5. *For any undirected graph $G = (V, E)$ with a specified subset $K \subseteq V$, $D_K(G) = |S_e(G, K) - S_o(G, K)|$.*

Corollary 2.5 specialized to the all-terminal domination yields the characterization of $D_V(G)$ in terms of spanning connected subgraphs of G. More specifically, $S_o(G, V)$ and $S_e(G, V)$ are the number of spanning connected subgraphs with an odd and even number of edges, respectively. We can also partition the spanning connected subgraphs of G into two classes based on the nullity of the subgraphs rather than the number of edges in the subgraph. Specifically, let $\Theta(G)$ denote the collection of spanning connected subgraphs of G. If $H \in \Theta(G)$, and $n(H)$, $e(H)$ denote the number of nodes and edges of H, respectively, then let $S_o(G) = |\{H \in \Theta(G) : (e(H) - n(H) + 1) \text{ is odd }\}|$, $S_e(G) = |\{H \in \Theta(G) : (e(H) - n(H) + 1) \text{ is even }\}|$. It is clear that, in general, $S_o(G, V) \neq S_o(G)$ and $S_e(G, V) \neq S_e(G)$. The following corollary provides another characterization of $D_V(G)$ in terms of $S_o(G)$ and $S_e(G)$. A proof of this corollary is immediate from the observation that either $S_o(G, V) = S_o(G)$ and $S_e(G, V) = S_e(G)$ or $S_o(G, V) = S_e(G)$ and $S_e(G, V) = S_o(G)$.

Corollary 2.6. *For any undirected graph $G = (V, E)$, $D_V(G) = S_e(G) - S_o(G)$.*

Since $D_V(G) > 0$ for any connected loopless graph G, the following is an immediate consequence of Corollary 2.6.

Corollary 2.7. *If G is a connected undirected graph without self-loops then $S_e(G) > S_o(G)$.*

Now the fact that $|\Theta(G)| = S_e(G) + S_o(G)$ implies that $|\Theta(G)|$ is odd if and only if $D_V(G)$ is odd. Indeed, the next theorem characterizes graphs G for which $D_V(G)$ is odd.

Theorem 2.2. *If $G = (V, E)$ is an undirected graph, then $D_V(G)$ is odd if and only if G is a connected bipartite graph.*

See [16] for a proof of Theorem 2.2. This theorem together with Turan's theorem on triangle-free graphs yields the following result.

Corollary 2.8. *The complete bipartite graph $K_{p,p}$ is the only graph among the simple graphs on $2p$ nodes and p^2 edges with an odd all-terminal domination. Likewise, $K_{p,p+l}$ is the* *only graph with $2p+1$ nodes and $p(p+1)$ edges having odd all-terminal domination.*

In the remainder of this section we deal with directed graphs. The nature of the invariant domination differs strikingly depending on whether G is a graph or a digraph. The signed domination $d_K(G)$ generally can assume any integer value and is never zero if G is an undirected K-graph. On the contrary, for digraphs $d_K(G)$ is 0, +1, or −1. Indeed, as mentioned earlier, a surprising fact is that $d_K(G) = \pm 1$ if and only if G is an acyclic K-digraph, and $d_K(G) = 0$ otherwise.

The next result (see [16] for a proof) on digraphs G relates $d_K(G)$ either to $d_{K|x}(G|x)$ or to $d_K(G-x)$, depending upon the nature of x.

Theorem 2.3. *Suppose $G = (V, E)$ is a digraph, $K \subseteq V$, $s \in K$, and $x = (s, u)$ is an edge of G. Then*

(i) *if indegree$(u) > 1$, then $d_K(G) = -d_K(G-x)$,*
(ii) *if indegree$(u) = 1$ and x lies on some K-tree rooted at s, then $d_K(G) = d_{K|x}(G|x)$.*

Note that $x = (s, u)$ being in some K-tree rooted at s is an essential condition for (ii) to be valid in the above theorem. To see this consider the digraph with two nodes and one edge $x = (s, u)$, and let $K = \{s\}$.

Theorem 2.3 affords a simple proof of a result first established for the case of $K = \{s, t\}$ by Satyanarayana and Prabhakar [46], and later extended to general K by Satyanarayana [44].

Corollary 2.9. *Suppose $G = (V, E)$ is a digraph, $K \subseteq V$ and $s \in K$. Then $d_K(G) = (-1)^{|E|-|V|+1}$ if G is an acyclic K-graph and $d_K(G) = 0$ otherwise.*

See [16] for a proof of Corollary 2.9.

Suppose $G = (V, E)$ is a K-digraph, $K \subseteq V$, and $s \in K$. Let $S^K(G)$ denote the collection of K-subgraphs of G. If $H \in S^K(G)$ and $n(H)$ and $e(H)$ denote the number of nodes and edges of H, respectively, then define

$S^K_o(G) = \{H \in S^K(G) : (e(H) - n(H) + 1) \text{ is odd}\}$,
$S^K_e(G) = \{H \in S^K(G) : (e(H) - n(H) + 1) \text{ is even}\}$.

Theorem 2.4. *If $G = (V, E)$ is an acyclic K-digraph, $K \subseteq V$ and $s \in K$, then $d_K(G) = (-1)^{|E|-|V|+1}(|S^K_e(G)| - |S^K_o(G)|)$ and $D_K(G) = |S^K_e(G)| - |S^K_o(G)|$.*

See [16] for a proof of the above theorem.

The all-terminal domination $D_V(G)$ of an undirected graph G is, by definition, related to the spanning trees of G because the K-trees of G in this case are the spanning trees. However, if $K \neq V$ not all K-trees are spanning; thus the connection between $D_K(G)$ and the spanning trees of G is not obvious. In fact, $D_K(G)$, for any arbitrary K, is equal to the number of spanning trees of a certain type. First we require some preliminaries. Suppose $G = (V, E)$ is an undirected graph and $<$ is a strict linear order on E. Let $T = (V, E')$ be a spanning tree of G and let $x \in E'$. Then the forest $T - x$ has exactly two connected components with node sets, say U and $V - U$. The collection of edges of G with one endnode in U and the other in $V - U$ is called the *fundamental cut* determined by x with respect to T. Likewise, if $x \in E - E'$ is an edge, then $T + x$ is unicyclic and the cycle in $T + x$ is called the *fundamental cycle* determined by x with respect to T. An edge $x \in E'$ is *internally active* in T if $x < y$ for all $y \in C - x$, where C is the fundamental cut determined by x with respect to T. Finally, an edge $x \in E - E'$ is *externally active* relative to T if $x < y$ for all $y \in C - x$, where C is the fundamental cycle determined by x with respect to T. The path of T obtained from the fundamental cycle determined by an externally active edge x is called a *broken cycle* of G [55].

Note that if a spanning tree T has i internally active and j externally active edges, then $0 \leq i \leq |V| - 1$ and $0 \leq j \leq |E| - |V| + 1$. By $\mathcal{T}(G)$ we mean the set of all spanning trees of G, while $\mathcal{T}_{ij}(G)$ denotes the subcollection of $\mathcal{T}(G)$ of trees having i internally active and j externally active edges. Furthermore, let $t(G)$ and $t_{ij}(G)$ denote the cardinalities of $\mathcal{T}(G)$ and $\mathcal{T}_{ij}(G)$, respectively. The following is needed for our next definition.

Let $G = (V, E)$ be an undirected graph and let $K \subseteq V$ be a specified subset. Each spanning tree which is not a K-tree may be reduced to a K-tree by repeatedly pruning those leaves which are not nodes of K. Thus each spanning tree T of G contains exactly one K-tree T^K of G.

Let $\mathcal{T}_{*0}(G, K)$ consist of those trees $T \in \mathcal{T}(G)$ satisfying the following conditions:

(i) T has no externally active edges, and
(ii) if x is an internally active edge in T, then x is an edge of the unique K-tree T^K contained in T.

Finally, let $t_{*0}(G, K)$ denote the cardinality of $\mathcal{T}_{*0}(G, K)$.

Theorem 2.5. *Let $G = (V, E)$ be a connected undirected graph with a nonempty subset $K \subseteq V$, and let $<$ be a strict linear order on E. Then $D_K(G) = t_{*0}(G, K)$.*

A proof of Theorem 2.5 can be found in [16]. We can derive several corollaries from this theorem.

Corollary 2.10. *Let $G = (V, E)$ be an undirected graph with a strict linear order $<$ on E. If $x = \{u, v\}$ is the smallest edge of G, then $D_{\{u,v\}}(G) = t_{10}(G)$, where $t_{10}(G)$ is the number of spanning trees of G having exactly one internally active edge and zero externally active edges.*

Let $\mathcal{T}_0(G)$ be the collection of spanning trees having no externally active edges in a graph $G = (V, E)$ with respect to a strict linear order $<$ on E and let $t_0(G)$ be the cardinality of $\mathcal{T}_0(G)$. The following is an immediate consequence of Theorem 2.5.

Corollary 2.11. *For an undirected graph $G = (V, E)$, $D_V(G) = t_0(G)$.*

It is an obvious consequence of these results that the quantities $t_{*0}(G,K)$ and $t_0(G)$ are invariant with respect to the linear order $<$ of the edges of G. Indeed, Tutte [52], in his study of the chromatic polynomial of a graph, noted this fact for all parameters $t_{ij}(G)$. The value of the chromatic polynomial $P(G;\lambda)$ of a graph G gives the number of proper λ-colorings of G: that is, the number of ways of assigning colors to the nodes of G, using λ or fewer colors, so that no two adjacent nodes are assigned the same color. Tutte showed that for any connected graph G, evaluating $|(P(G;\lambda)/(1-\lambda))|$ when $\lambda = 1$ yields $t_{10}(G) = t_{01}(G)$. Hence, the next result follows directly from Corollary 2.10.

Corollary 2.12. *For any connected undirected graph $G = (V,E)$ and any edge $x = \{u,v\}$ such that $u \neq v$, evaluating $|(P(G;\lambda)/(1-\lambda))|$ when $\lambda = 1$ yields $D_{\{u,v\}}(G)$.*

An immediate consequence of this corollary is that $D_x(G) = D_y(G)$ for any pair of edges x and y of G. Moreover, Whitney [55] showed that $P(G;\lambda) = \sum_{i=1}^{|V|}(-1)^{|V|-i}m_i(G)\lambda^i$, where $m_i(G)$ is the number of spanning forests of G with i connected components and having no externally active edges. Note that an edge is externally active with respect to a given forest if and only if it is externally active with respect to some tree of the forest. Clearly when $i = 1$, then the spanning forests are the spanning trees of G, whence $m_i(G) = t_0(G)$. Therefore, the following corollary follows from Corollary 2.11.

Corollary 2.13. *If $G = (V,E)$ is an undirected graph, then evaluating $|(P(G;\lambda)/\lambda)|$ when $\lambda = 0$ yields $D_V(G)$.*

In [49] a polynomial $P(G,K;\lambda)$ in λ determined by the graph $G = (V,E)$ and $K \subseteq V$ was introduced. Like the classical chromatic polynomial $P(G;\lambda)$, this polynomial has integer coefficients that alternate in sign. Furthermore $P(G,K;\lambda) = P(G;\lambda)$ if K is the entire node set of G. The polynomial $P(G,K;\lambda)$ has several interesting properties, and in particular, it has been shown that evaluating $|(P(G,K;\lambda)/\lambda)|$ when $\lambda = 0$ yields $D_K(G)$.

Another interesting connection between $D_K(G)$ and the number of certain orientations of G was shown in [47]. An *orientation* of an undirected graph is an assignment of direction to each edge of the graph. Let $G = (V,E)$ be a connected undirected graph, and suppose $K \subseteq V$. A *rooted orientation*, with respect to K and the root $s \in K$, of G is an orientation of G such that exactly one node of the orientation, namely s, has indegree 0 and every node of outdegree 0 belongs to K. An orientation is *acyclic* if it has no directed cycles and is cyclic otherwise. The result proved in [47] asserts that if $N_K(G,s)$ is the number of rooted acyclic orientations of G, with respect to K and the root $s \in K$, then $D_K(G) = N_K(G,s)$ for any $s \in K$. An immediate consequence of this result is the fact that, if $i \in K$ and $j \in K$ are two nodes of G, then $N_K(G,i) = N_K(G,j)$ and hence the number of rooted acyclic orientations of a graph, with respect to a given K, is independent of the root selected from K.

3. COMPUTATIONAL COMPLEXITY OF NETWORK RELIABILITY

Valiant [53] showed that computing the K-terminal reliability $R_K(G)$ in general is NP-hard. Provan [41] proved that even for planar graphs the computation of $R_K(G)$ is NP-hard. These results motivated the search for classes of graphs G which admit polynomial time algorithms for the computation of $R_K(G)$; for example see [3,19,40].

A special case of the K-terminal problem is the following K-terminal node connectedness problem. In this model, edges do not fail but the nodes that are not in a specified subset K fail independently with known probabilities. The *K-terminal node connectedness reliability* of a graph G is then the probability that the surviving nodes of G induce a subgraph in which all nodes of K lie in a single component. AboElFotoh and Colbourn [1] showed that this problem also is NP-hard for general graphs and it remains so even for chordal graphs and comparability graphs.

Another network reliability analysis measure is the residual node connectedness reliability of a graph G. Again in this model, edges are perfectly reliable and the nodes fail independently of each other. The network is considered to be in an operational state if the surviving nodes induce a connected subgraph of G. The *residual node connectedness reliability* of a graph G, denoted $R(G)$, is the probability that the graph induced by the surviving nodes is connected. We first note that this problem is not a special case of the K-terminal reliability problems. The K-terminal models described earlier constitute coherent systems. The residual node connectedness model is not coherent since a supergraph of a connected graph may be disconnected.

Computing $R(G)$ is NP-hard and remains so even for split graphs as well as planar and bipartite graphs [51]. Efficient algorithms for computing $R(G)$ for various restricted classes of networks have been discovered in [20]. These restricted cases include graphs which are complements of planar graphs, trees, series-parallel graphs, partial k-trees, directed path graphs, and permutation graphs.

4. SYNTHESIS OF RELIABLE NETWORKS

In the most general sense, a network design or synthesis problem requires the construction of a network that meets certain predetermined specifications. Very often such a synthesis problem takes the form of a constrained optimization problem, which may appear to be innocuous at first sight. For example, the determination of an undirected graph with a specified number of nodes and edges having the maximum possible node connectivity was posed by Berge [12]. This problem was solved by Harary [27]. On the other hand, the determination of a undirected graph with a specified number of nodes n and edges e having the maximum number of spanning trees has evaded a complete solution as of the writing of this chapter. The problem has been solved for graphs having n nodes and a specified range of edges e. These include the case where the number of edges e is close to $n(n-1)/2$, e

is close to $n-1$, or e is equal to the number of edges in a regular or almost regular complete multipartite graph. Also for the range $n(n-1)/2 \geq e \geq n(n-1)/2-n$, the graphs with maximum number of spanning trees have been characterized independently in [32] and [38]. Furthermore, for the range $n(n-1)/2-n \geq e \geq n(n-1)/2-3n/2$ of graphs, as well as for regular and almost regular complete multipartite graphs, the problem has been solved in [39].

In this section we are concerned with the determination of a graph $G=(V,E)$ having the maximum possible all-terminal reliability $R_V(G)$ over all graphs having n nodes and e edges, a problem for which some formulations do not even admit a solution. More specifically, consider the all-terminal reliability function $R_V(G,p)$ on a graph G in which the nodes do not fail and all edges operate with the same operating probability p but independently of each other. Let s_k be the number of spanning connected subgraphs of G having exactly k edges. Then $R(G,p)=R_V(G,p)=\sum_{k=0}^{e} s_k p^k(1-p)^{e-k}$.

We see that the reliability depends on the value of p as well as the structure of the system. Thus, even though the restriction of the problem to a fixed number of nodes and edges constrains the collection of candidate systems to be finite, there may be no one system that maximizes the function for *all values* of p. On the other hand, if we fix p to begin with, then an optimal solution is sure to exist. In this section, we shall be exclusively concerned with the case of the all-terminal reliability model for an undirected simple graph and those node-edge pairs for which a "uniformly best" topology can be found, i.e., one for which the reliability is maximum for all values of p. A more complete survey may be found in [36].

A graph having a specified number of nodes and edges for which the reliability is maximum for all candidate graphs and all values of p is referred to as a *uniformly most-reliable* graph. Now it is clear that if a graph G_u exists which maximizes each of the coefficients s_k over all graphs having the same number of nodes and edges, then G_u is uniformly most-reliable. It is not known whether this condition is necessary. Nevertheless most approaches taken thus far have been concerned with the optimization of the individual coefficients. The following theorem plays an important role in the application of this approach and has a straightforward calculus proof.

Theorem 4.1. *Let G and H be two undirected simple graphs both having n nodes and e edges and let $s_k(G)$, $s_k(H)$ denote the number of spanning connected subgraphs of G and H, respectively, with exactly k edges.*

(i) *If there exists an integer $0 \leq k \leq e-1$ such that $s_i(G)=s_i(H)$ for $i=0,1,\ldots,k$ and $s_{k+1}(G)>s_{k+1}(H)$, then there exists a $\rho>0$ such that for all $0<p<\rho$ we have $R(G,p)>R(H,p)$.*
(ii) *If there exists an integer $0 \leq k \leq e$ such that $s_i(G)=s_i(H)$ for $i=e,e-1,\ldots,e-k$ and $s_{e-k-1}(G)>s_{e-k-1}(H)$, then there exists a $\rho<1$ such that for all $\rho<p<1$ we have $R(G,p)>R(H,p)$.*

We use the following notation for the invariants *minimum degree*, *edge connectivity*, and *node connectivity* of a graph G. By $\delta(G)$ we mean the minimum degree of a node in G. By $\lambda(G)$ we mean the edge connectivity of G, i.e., the least number of edges whose deletion disconnects G. The node connectivity κ is the least number of nodes whose deletion disconnects the graph G or reduces it to a 1-node graph.

An important consequence of Theorem 4.1 is the following corollary [8].

Corollary 4.1. *If G is uniformly most-reliable then*

(i) *G has the maximum number of spanning trees among all simple graphs having n nodes and e edges, and*
(ii) *G is max $-\ \lambda$, i.e., has the maximum possible value of λ among all simple graphs having n nodes and e edges, namely $\lambda(G)=\lfloor 2e/n \rfloor$, and the minimum number of cutsets of size λ among all such max-λ graphs.*

Since $\delta=\lambda$ in a max-λ graph, the number of cutsets of size $\lambda=\lfloor 2e/n \rfloor$ in a max-λ graph is at least as large as the number of minimum degree nodes. Therefore the number of such cutsets is at least as large as $(1+\lfloor 2e/n \rfloor)n-2e$, the minimum number of minimum degree nodes over all simple graphs having n nodes and e edges and $\lambda=\lfloor 2e/n \rfloor$. Thus if G has $\lambda=\lfloor 2e/n \rfloor$, $(1+\lfloor 2e/n \rfloor)n-2e$ nodes of degree λ and has the property that every cutset of size λ is an incident set, then G satisfies (ii) of Corollary 4.1. A max-λ graph is *super* $-\ \lambda$ if every cutset of size λ is the incident set of a minimum degree node. Bauer et al. [7,8] have shown that for $e \geq \lfloor 6n/5 \rfloor$ super-λ graphs exist and for each such n and e one exists having precisely $(1+\lfloor 2e/n \rfloor)n-2e$ nodes of minimum degree. The same authors have also completely described the structure of a max-λ graph having the minimum number of cutsets of size λ when $n \leq e < \lfloor 6n/5 \rfloor$ noting that they are not super-λ.

It is to be noted that uniformly most-reliable graphs do not always exist. This negative result was independently discovered by Kelmans [33] and Myrvold et al. [37]. They exhibited infinite families of counterexamples. Indeed, if $e=n(n-1)/2-(n+2)/2$ for $n \geq 6$ even or $e=n(n-1)/2-(n+5)/2$ for $n \geq 7$ odd, then there exists a graph, in each case, that maximizes $R(G,p)$ for p close to 1 but do not have the maximum number of spanning trees. Thus it follows by Theorem 4.1 and Corollary 4.1 that a uniformly most-reliable graph does not exist in these cases.

Some researchers have made use of the partial progress made on the spanning tree problem in determining (n,e) pairs for which a uniformly most-reliable graph exists. For example, it was independently discovered by Kelmans and Chelnokov [34] and Shier [50] that when $e \leq n(n-1)/2-\lfloor n/2 \rfloor$ the graph obtained by removing a matching from the complete graph has the maximum number of spanning trees over all graphs having the same number of nodes and edges. Using a reliability increasing operation known as a swing surgery, originally discovered by Kelmans [33], these graphs have been shown to be uniformly most-reliable [48].

If $n(n-1)/2 - \lfloor n/2 \rfloor > e \geq n(n-1)/2 - (n-2)$, and in special cases when $e = n(n-1)/2 - (n-1)$ or $e = n(n-1)/2 - n$, graphs having the maximum number of spanning trees have been identified among almost regular graphs in [25, 38], but no progress has been made to date on the uniformly most-reliable problem for these (n, e) pairs.

As for sparse cases, i.e., those with $n-1 \leq e \leq n+3$, the graphs having the largest number of spanning trees have been classified and also shown to be uniformly most-reliable [15, 54]. They are respectively trees, cycles, three multiple edges subdivided as evenly as possible, a particular subdivision of K_4 and a particular subdivision of $K_{3,3}$. Finally, in the case where e is the number of edges in a regular or almost regular (i.e., the degrees differ by at most one) complete multipartite graph, those graphs maximize the number of spanning trees [17, 42]. It is not known if they are uniformly most-reliable.

A fair amount of work has been done on determining graphs which maximize the reliability for p close to 1. As already noted, such graphs must be max-λ, and must have the minimum number of cutsets of size λ. Indeed, these graphs maximize the coefficients $s_{e-\lfloor 2e/n \rfloor}, s_{e-\lfloor 2e/n \rfloor+1}, \ldots, s_e$. In an attempt to isolate those graphs that, for example, maximize $s_{e-\lfloor 2e/n \rfloor-1}$, some researchers have concentrated on a deeper study of super-λ graphs, for if $e \geq \lfloor 6n/5 \rfloor$, as previously noted, a max-λ graph which minimizes the number of cutsets of size λ must be a super-λ graph.

Towards this end Esfahanian and Hakimi [24] proposed the concept of the *restricted edge connectivity* λ', i.e., the minimum size of a cutset which does not isolate a single node. It can be shown that a graph is super-λ if and only if $\lambda(G) = \delta(G)$ and $\delta(G) < \lambda'(G)$ so that $\lambda'(G)$ quantifies the super-λ property. That is, $\lambda'(G)$ equals the minimum degree of an edge in G, defined to be the sum of the degrees of the endnodes minus 2. These authors have shown the optimal super-λ graphs to be most-reliable among the super-λ graphs for p sufficiently close to 1.

Recently, Meng and Ji [35] have advanced these considerations to the study of graphs that maximize the minimum size of a cutset that produces only components of order at least three. Work done to date regarding these issues has concentrated on a study of circulants. We hasten to note that, even in cases where a uniformly most-reliable graph may not exist, the study of those graphs that optimize the reliability for p close to 1 is of practical significance so a continuation of this study is justified from a pragmatic point of view.

As a final note, we comment on the extension of the problem to the class of all multigraphs. In the sparse cases previously discussed it was shown that the graphs determined to be uniformly most-reliable among all simple n-node e-edge graphs remain so when all multigraphs having n nodes and e edges are considered [26]. On the other hand, the simpler problem of whether the complete graph with a matching removed has the maximum number of spanning trees among all multigraphs remains unresolved at this time. Finally, a surprising negative result was reported in [36]; for certain n, e, and large p, a multigraph optimizes the reliability over the class of all planar multigraphs.

5. NETWORK VULNERABILITY MEASURES

There are two entirely different approaches to measuring the susceptibility of a network to damage. The use of probabilistic models is the standard approach in reliability theory. A different approach uses graph invariants as deterministic measures. We have suggested that these methods be called vulnerability theory, and other authors also have used this term. We note, in passing, that it might be preferable to call this approach invulnerability theory to emphasize its analogy to reliability theory.

This approach is fairly well known in the graph theory literature. In fact, there is a section describing the use of connectivity as a measure of a network's susceptibility to damage in Berge's book [12], the second book ever written on graph theory. Here we give a few examples and refer the reader to survey papers such as [6, 13, 14] for more details.

Having agreed on some reasonable parameter that can be used as a measure of vulnerability, one can then define an optimization problem that relates to the design of invulnerable networks. For example, if the edge connectivity λ is used as the measure of vulnerability, then an appropriate design question would be to create graphs that have the maximum possible value for λ, given the number of nodes n and the number of edges e. There are many solutions to this particular optimization problem; for example, an edge-disjoint union of hamiltonian cycles clearly has maximum λ. As in the case of designing graphs that have optimal reliability properties, we call problems that optimize a vulnerability parameter over some class of graphs *synthesis problems*. In the previous section we discussed Harary's solution to the problem of maximizing the node connectivity κ for given e and n which showed that this maximum value is $\lfloor 2e/n \rfloor$. However since $\kappa \leq \lambda \leq \lfloor 2e/n \rfloor$, it follows that maximizing κ for given e and n implies that λ is also maximized. It can be easily shown that the converse is not true.

There are many other graph invariants that are reasonable vulnerability measures: see [6, 13, 14]. For one possible example consider the *toughness* τ of a graph $G = (V, E)$, which is defined as $\tau = \min_{W \subseteq V} \left\{ \frac{|W|}{k(G-W)} : k(G-W) \geq 2 \right\}$, where $k(H)$ is the number of components of graph H. The toughness of K_n is defined to be infinity. The toughness invariant was introduced by Chvatal [18] in the study of hamiltonian cycles. Initial partial results on the synthesis problem were independently obtained by Doty [21] and Jackson and Katerinis [31]; further progress has been made by Doty and Ferland [22, 23]. Recently, Bauer et al. have published a comprehensive survey on toughness [9].

The last vulnerability measure we mention here is the *integrity* i of a graph $G = (V, E)$, defined as $i = \min_{W \subseteq V} \{|W| + m(G-W)\}$, where $m(H)$ is the maximum number of nodes among all components of a graph H. Note that in contrast to toughness, W is not required to be a disconnecting set. However, it is easily shown that for any connected graph, other than the complete graph, the minimizing set W for determining i is in fact a disconnecting set. Integrity was introduced as a vulnerability measure by Bagga et al. [4]; also see their related work [5]. The synthesis problem for integrity was solved by Beineke et al. [11]. There are also many vulnerability results given in the surveys by Barefoot et al. [6] and by Beineke et al. [10].

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