He is co-author of three books: *Principles of Circuit Synthesis*, 1959; *Theory of Linear Active Networks*, 1967; and *Basic Circuit Theory*, 1969.

Dr. Kuh was President of the IEEE Circuits and Systems Society in 1972. Since January 1, 1976, he has been a member of the Board of Directors of IEEE. At present, he serves on the Visiting Committee of the General Motors Institute. He is a member of the NSF Advisory Panel on Electrical Sciences and Analysis. He also serves as a member of the Evaluation Panel for the Institute for Applied Technology of the National Bureau of Standard. Among the various honors and awards he has received are as follows: NSF Senior Postdoctoral Fellow (1962), Miller Research Professorship, Berkeley (1965), National Electronics Conference Award (1966), Distinguished Alumni Award, University of Michigan (1970), IEEE Guillemin–Cauer Award (1973), and membership in the National Academy of Engineering (1975).

Isao Shirakawa (S'62–M'68) was born in Toyama Prefecture, Japan, on Sept. 12, 1939. He received the B. E., M. E., and Ph. D. degrees in electronics engineering from the University of Osaka, Osaka, Japan, in 1963, 1965, and 1968, respectively.

In 1968 he joined the Department of Electronics Engineering, University of Osaka, where he is now an Associate Professor. His main research interests are in the areas of network theory and graph theory.

Dr. Shirakawa is a member of the Institute of Electronics and Communication Engineers of Japan.

Line Removal Algorithms for Graphs and Their Degree Lists

FRANK BOESCH, MEMBER IEEE, AND FRANK HARARY, SENIOR MEMBER IEEE

***Abstract*—An important and basic characterization of a graph is the sequence or list of degrees of that graph. Problems regarding the construction of graphs with specified degrees occur in chemistry and in the design of reliable networks. A list of nonnegative integers is called *graphical* if there is a graph (called a *realization*) with the given list as its degree list. The usual algorithms for determining whether a given list is graphical are derived from the effect on a graphical list of the removal of a point from a graph. After reviewing such an algorithm by Havel–Hakimi and its generalization by Wang and Kleitman, we develop a corresponding algorithm based on the removal of a line from a graph.**

We conclude by reviewing and providing simple proofs of algorithms for a list to be multigraphical due to Hakimi and Butler. The conditions relating a graphical or multigraphical list to the point and line connectivity of their realizations, due to Edmonds, Wang and Kleitman, Boesch and McHugh, and Hakimi, are presented along with new and simple proofs of the multigraph case.

Introduction

A DEGREE LIST of a graph is a sequence of the degrees of its points; thus any permutation of a degree list yields a degree list of the same graph. A basic question regarding degree lists is to determine the properties that characterize them. Such properties are of interest for a variety of reasons, an important application being the construction of minimum cost networks with prescribed reliabilities [3], [6]. Thus we consider the problem of deciding when a list of nonnegative integers is a degree list of some graph called a *realization* of the list. When such a realization exists, the list is said to be *graphical*. We follow the terminology and notation of [7]. An existential answer to this question, in the form of a set of necessary and sufficient inequalities, was given by Erdös and Gallai [5]. An alternative answer in the form of an algorithm was given independently by Havel [11] and Hakimi [9]. In this latter case, the algorithm not only answers the question as to the existence of some graph, but it also constructs a realization. Both of these results can be found in [7]. For convenience we state here, as Theorem 1, the result on which the constructive algorithm is based. Clearly the degree of each point of a graph of order p is at most $p-1$.

Theorem 1 (Havel–Hakimi)

Let $\pi=[d_1,d_2,\cdots,d_p]$ be a list with $p-1 \geqslant d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p$. Then π is graphical if and only if the following list π' (obtained from π by removing d_1 and subtracting 1 from the next d_1 terms) is graphical:

$$\pi' = \left[d_2-1, d_3-1, \cdots, d_{d_1+1}-1, d_{d_1+2}, \cdots, d_p \right].$$

Manuscript received August 13, 1975; revised April 21, 1976. This work was supported in part by the Air Force Office of Scientific Research under Grant 73-2502.

F. Boesch is with Bell Telephone Laboratories, Holmdel, NJ 07733.
F. Harary is with the University of Michigan, Ann Arbor, MI 48109.

The modified list π' results from removing a point and its incident lines from some realization of π. Likewise a realization of π is obtained from a realization of π' by the addition of a point and certain incident lines. Clearly the sufficiency is the easier part of the theorem as it is obvious that if π' is graphical, so is π. However the proof of necessity involves certain degree-invariant line interchanges in a realization of π. Such a line interchange always involves replacing two lines tu and vw by tw and uv; these are called *surgeries* in point set topology and that term is used here. Recently it was noted by Wang and Kleitman [12] that these surgeries could be used to derive a generalization of Theorem 1. We state here as a lemma the general property of these surgeries which emerges in the derivations of these degree list algorithms.

Lemma

If $\pi=[d_1,d_2,\cdots,d_p]$ with $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p$ is a graphical list and v_i denotes the point of degree d_i in any realization, then for any j with $d_j>0$, there exists a realization having v_j adjacent to the first d_j points other than itself in the sequence of points $[v_1,v_2,\cdots,v_p]$.

The following necessary conditions of Wang and Kleitman follow readily from this lemma; the proof of sufficiency is again trivial.

Theorem 2 (Wang–Kleitman)

Let $\pi=[d_1,d_2,\cdots,d_p]$ be a list with $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p$ and let j be an integer $(1 \leqslant j \leqslant p)$ with $d_j<p$. Define $\bar{\pi}_j$ as the list which is obtained from π by removing the term d_j, and let $\bar{\pi}_j'$ be obtained from $\bar{\pi}_j$ by subtracting one from thre first d_j terms of $\bar{\pi}_j$. Then π is graphical if and only if $\bar{\pi}_j'$ is graphical.

Specifically, the resulting sequence is

$$\bar{\pi}_j'=\left[d_1-1,\cdots,d_{d_j}-1,d_{d_j+1},\cdots,\right.$$
$$\left. d_{j-1},d_{j+1},\cdots,d_p\right], \qquad \text{if } j>d_j$$

and

$$\bar{\pi}_j'=\left[d_1-1,\cdots,d_{j-1}-1,d_{j+1}-1,\cdots,\right.$$
$$\left. d_{d_j+1}-1,d_{d_j+2},\cdots,d_p\right], \qquad \text{if } j \leqslant d_j.$$

Notice that Theorem 2 provides for a general class of algorithms that test whether or not a given list is graphical and provide realizations of graphical lists.[1] These algorithms can aptly be named *point removal algorithms*, of which Theorem 1 specifies the most efficient for settling (in terms of having the fewest steps) whether a given list is graphical.

We are concerned here with discovering line removal algorithms for degree lists. Based on the knowledge of the point removal case, one might conjecture that a list $\pi=[d_1,d_2,\cdots,d_p]$ with $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p$ is graphical if and only if the p element list π' is graphical, where

$$\pi'=\left[d_1-1,d_2-1,d_3,\cdots,d_p\right].$$

It is easily verified by considering the example $\pi=[3,3,1,1]$ that this conjecture is not correct as π' is graphical but π is not. Interestingly enough it is the sufficiency which now provides the difficulty; in fact the necessity part of the conjecture is indeed correct as can be seen from the lemma.

We now give, as Theorem 3, a correct line removal algorithm.

Theorem 3

Let $\pi=[d_1,d_2,\cdots,d_p]$ be a list with $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p$ and j an integer $(1 \leqslant j \leqslant p)$ with $p>d_j>0$. Let π' be obtained from π by subtracting one from the jth term and the d_jth term not counting the jth so that

$$\pi'=\left[d_1\cdots,d_{d_j}-1,d_{d_j+1},\cdots,d_j-1,d_{j+1},\cdots,d_p\right], \quad \text{if } j>d_j$$

or

$$\pi'=\left[d_1,\cdots,d_j-1,d_{j+1},\cdots,\right.$$
$$\left. d_{d_j+1}-1,d_{d_j+2},\cdots,d_p\right], \qquad \text{if } j \leqslant d_j.$$

Then π is graphical if and only if π' is graphical.

Proof: The necessity of π' being graphical whenever π is graphical follows immediately from the lemma. For the sufficiency, we assume that d_i' denotes the ith term of π' after it has been rearranged into nonincreasing order and that v_i' is the corresponding point in a realization. Now in either of the two cases for π', the term d_j-1 can be identified with a d_k' for some $k \geqslant j$. The lemma then assures that there exists a realization of π' which has v_k' adjacent to the first d_j-1 points in the list $[d_1',d_2',\cdots,d_p']$. Now let v_r' denote the point of largest index r such that

$$d_r'=d_{d_j}-1, \qquad \text{if } j>d_j$$

or

$$d_r'=d_{d_j+1}-1, \qquad \text{if } j \leqslant d_j.$$

Note that $r \geqslant d_j$ for $j>d_j$ and $r \geqslant d_j+1$ for $j \leqslant d_j$. In either case it follows that in this realization v_k' will not be adjacent to v_r'. Hence the required line $v_k'v_r'$ can be added to obtain a realization of π. | |

We will now consider the case of pseudographical and multigraphical lists. Recall that a multigraph can contain more than one line joining the same pair of points but does not have any loops (self-loops), while a pseudograph is permitted to have loops as well as multiple lines, a loop adding two to the degree of its point. In these cases there are simple and explicit necessary and sufficient conditions for realizability. First we mention the easily verified result

[1]It can be shown that this class of algorithms does not generate all realizations.

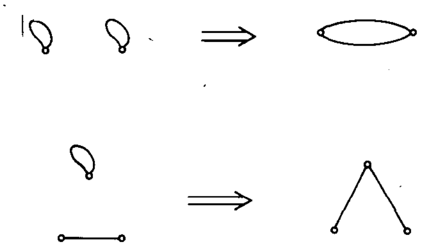

Fig. 1. Pseudograph to multigraph transformations.

that a list of nonnegative integers is pseudographical if and only if the sum of its terms is even. The conditions for multigraphical lists were given by Hakimi [9].

Theorem 4 (Hakimi)

A list $\pi=[d_1,d_2,\cdots,d_p]$ of nonnegative integers with $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p$ is multigraphical if and only if

$$\sum_{i=1}^{p} d_i \text{ is even}$$

and

$$\frac{1}{2}\sum_{i=1}^{p} d_i \geqslant d_1.$$

We mention that the proof of necessity in Theorem 4 is simple. A proof of sufficiency can be given by using the surgeries shown in Fig. 1 to delete the loops from any pseudograph realization of the list. The inequality condition ensures that all the loops can be removed from any pseudograph realization by these surgeries.

Another interesting fact about multigraphs is that the point removal algorithms for graphs are not applicable to multigraphs, as seen by the list $\pi=[3, 3, 1, 1]$. However the conjecture given above regarding a line removal algorithm is valid for multigraphs. Specifically the following line removal theorem was proven by Butler [2].

Theorem 5 (Butler)

Let $\pi=[d_1,d_2,\cdots,d_p]$ be a list with $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p$ and let j be an integer $(1 \leqslant j \leqslant p)$ with $d_j>0$. Then π is multigraphical if and only if the p element list π' is multigraphical where

$$\pi'=\left[d_1-1,d_2,\cdots,d_{j-1},d_j-1,d_{j+1},\cdots,d_p\right].$$

We comment that the proof of sufficiency is simple, and the proof of necessity uses the same surgeries as in the proof of the lemma. Theorem 5 can be used to prove Threorem 4 in a straightforward manner. However, we will show that it has a more important application in deriving connectivity properties of degree list realizations.

Consider the problem of determining the maximum line connectivity among all graphs which realize a given list. This problem was first solved by Edmonds [4] using a rather complicated surgery construction. More recently, Wang and Kleitman [13] have shown that a recursive application of Theorem 2 with $j=p$ automatically generates a graph of maximum line connectivity.

The related but easier problem of finding multigraphs with maximum line connectivity among all multigraph realizations of a given list can be handled using Edmonds' approach but not by the Wang and Kleitman algorithm. Butler [2], however has shown that a repeated application of Theorem 5 with $j=p$ will generate a multigraph of maximum line connectivity. We state as Theorem 6 Edmonds' result for this multigraph case and give a constructive proof that is simpler than either Butler's or Edmonds' proofs.

Theorem 6 (Edmonds)

The maximum line connectivity among all multigraph (pseudograph) realizations of a multigraphical (pseudographical) list $\pi=[d_1,d_2,\cdots,d_p]$ with $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p=n>1$ is n.

Proof: First we give the proof for pseudographs. It is clear that any realization has line connectivity $\lambda \leqslant n$. We construct a realization with $\lambda=n$ as follows. First note that $C_p(k)$, defined as a p-point cycle with k-multiple lines whenever two points are adjacent, has $\lambda=2k$. If the consecutive points on the cycle are labeled $v_0, v_1,\cdots,v_{p-1}$ and we add to $C_p(k)$ a line from v_j to $v_{(j+\lfloor p/2\rfloor)\bmod p}$ for each $1 \leqslant j \leqslant \lceil p/2\rceil$ then the resulting multigraph has $\lambda=2k+1$.[2] We denote the result by $C_p(k)*\lceil p/2\rceil K_2$, since we are adding $\lceil p/2\rceil$ disjoint lines to $C_p(k)$ in a prescribed fashion.[3] Now by distinguishing even and odd n, it is easily seen by subtracting the degree list of $C_p(n/2)$ or $C_p(\lfloor n/2\rfloor)*\lceil p/2\rceil K_2$ from π that the resulting list is pseudographical.

To establish the result for multigraphs, let a pseudograph realization be constructed as above. Then it is clear that the elimination of the loops by the transformations shown in Fig. 1 does not decrease the line connectivity of a pseudograph. As noted previously however, the condition for a multigraph allows all loops to be removed by these transformations. Hence the result is also established for multigraphs. ‖

A collection of more difficult problems is presented by considering the maximum connectivity among all realizations. The pseudograph case is given in Boesch and McHugh [1]. The conditions are the same as those in Theorem 6. The construction varies slightly from Theorem 6 in that it uses the graphs C_p^k and $C_p^k*\lceil p/2\rceil K_2$, which we may be defined as follows. As per [7] let C_p denote a cycle on p points, and C_p^k denotes the graph which results from

[2]Herein $\lfloor x\rfloor$ denotes the greatest integer not exceeding x and $\lceil x\rceil$ the smallest integer not less than x. K_p denotes the complete graph on p points; thus K_2 is a line.

[3]We do not use union as this would imply by the notation of [7] that the $\lceil p/2\rceil$ lines are added arbitrarily.

adding to C_p a line between any pair of points of C_p that are distance at least 2 but not greater than k apart. The symbol $C_p^k * \lceil p/2 \rceil K_2$ is used here to denote the graph resulting from the addition of $\lceil p/2 \rceil$ disjoint lines to C_p^k in the same manner as defined for the multigraph case in Theorem 6. Both of these graphs were originally introduced (but not with this notation) by Harary [8] to solve the problem of finding the maximum connectivity among all graphs with a given number of points and lines. The most difficult of these degree list connectivity problems is the determination of the maximum connectivity among all graphs which realize a given graphical list. The solution to this problem was conjectured by Hakimi [10] and proven by Wang and Kleitman [12]. Rather than present the details here, we shall give a special case which was derived independently in both of these papers. This case is obtained by a slight variation of the construction in [8].

Theorem 7 (Wang–Kleitman and Hakimi)

A list of p integers $[k,n,\cdots,n]$ with $p-1 \geqslant k \geqslant n > 1$ and $k+(p-1)n$ even is realizable as a graph with connectivity n.

Now we use Theorems 5 and 7 to establish the conditions for a multigraph realization of a list to have maximum connectivity. This multigraph connectivity problem was first solved by Hakimi [10] without the aid of Theorem 5 as a preliminary. We show how the proof can be clarified by using these results.

Theorem 8 (Hakimi)

A multigraphical list $\pi=[d_1,d_2,\cdots,d_p]$ with $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p$ has a multigraph realization with connectivity $n (p-1 \geqslant n \geqslant 3)$ if and only if

i) $$d_p \geqslant n$$

and

ii) $$\frac{1}{2}\sum_{i=1}^{p} d_i \geqslant d_1 + \frac{(p-1)(n-1)}{2}.$$

Proof: The necessity of i) is straightforward. For ii), we need only note that the multigraph obtained by removing a point of degree d_1 is certainly $(n-1)$ connected. We then obtain ii) by invoking the well-known ([17, p. 44]) condition that $np \leqslant 2q$ for an n-connected multigraph with p points and q lines.

For sufficiency, we first observe that the case $n=p-1$ can be handled as follows. Subtract the degree list for K_p from π and consider the remainder. Conditions i) and ii) ensure that this remainder is multigraphic. Hence we can assume $n<p-1$. Now the repeated application of Theorem 5 (with $j=2$) shows that the problem can be reduced to finding an n-connected realization of the list $\hat{\pi}=[k,n,n,\cdots,n]$, where $k \geqslant n < p-1$. Now it is easily shown that if $\hat{\pi}$ also satisfies ii) then $k \leqslant p-1$. In this case, the desired result is obtained by invoking Theorem 7.

Hence the proof has been reduced to verifying that the process of obtaining $\hat{\pi}$ via the repeated application of Theorem 5 with $j=2$ preserves ii). The details are as follows. If π satisfies ii) and π', as defined in Theorem 5, is then substituted in ii) the only change is that the left-hand side is decreased by one and the right-hand side either remains the same or is also decreased by one. Thus the only case when π' does not automatically satisfy ii) is when

$$\frac{1}{2}\sum_{i=1}^{p} d_i = d_1 + \left\lceil \frac{(p-1)(n-1)}{2} \right\rceil$$

and

$$d_1 = d_3.$$

Thus

$$3d_1 + (p-3)d_p \leqslant \sum_{i=1}^{p} d_i \leqslant 2d_1 + 1 + (p-1)(n-1)$$

or

iii) $$d_1 + (p-3)d_p \leqslant 1 + (p-1)(n-1).$$

Now as $\pi \neq \hat{\pi}$, either $d_1 \geqslant d_p + 1$ or $d_1 = d_p > n$. In either of these cases iii) yields the contradiction $n \geqslant p-1$. Hence the proof is complete. | |

The exceptional cases $n=1$, 2 in Theorem 8 as well as $n=1$ in Theorem 6 are known and easily established. They are due to Hakimi [9] and can be summarized as follows.

Theorem 9 (Hakimi)

A multigraphical list $\pi=[d_1,d_2,\cdots,d_p]$ with $d_1 \geqslant d_2 \geqslant \cdots \geqslant_p (p \geqslant 3)$ has a 1-connected[4] multigraph realization if and only if

$$d_p \geqslant 1$$

and

$$\frac{1}{2}\sum_{i=1}^{p} d_i \geqslant p-1$$

and it has a 2-connected multigraph realization if and only if

$$d_p \geqslant 2$$

and

$$\frac{1}{2}\sum_{i=1}^{p} d_i \geqslant d_1 + (p-2).$$

[4]For $n=1$, n-line-connected and n-connected are equivalent and simply mean connected.

The proofs of necessity of these conditions is straightforward. For the sufficiency we shall comment that a simple proof of the case $n=1$ results from showing that π can be written as the sum of the degree list of a tree and the degree list of some multigraph by using Theorem 5; the theorem then follows from the well-known realizability conditions for a tree ([7, p. 62]). The sufficiency for $n=2$ follows by subtracting the degree list of C_p from π and noting that the result is multigraphic.

In closing we note again that the problem of determining if there is an n-connected realization of a given graphical list is the most difficult of the problems discussed herein. Finally we state as Theorem 10 the Hakimi conjecture which was proven by Wang and Kleitman.

Theorem 10 (Wang and Kleitman)

A graphical list $\pi=[d_1,d_2,\cdots,d_p]$ with $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p$ has an n-connected ($n \geqslant 2$) realization if and only if

i) $$d_p \geqslant n$$

and

ii) $$\frac{1}{2}\sum_{i=1}^{p} d_i - \sum_{i=1}^{n-1} d_i \geqslant p-n-\frac{(n-1)(n-2)}{2}.$$

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Frank Boesch (S'57–M'58) was born in New York, NY, on September 28, 1936. He received the B.S., M.S., and Ph.D. degrees, all in electrical engineering, from the Polytechnic Institute of Brooklyn, Brooklyn, NY, in 1957, 1960, and 1963, respectively.

Until September 1959, he was employed as a Member of the Technical Staff of Bell Telephone Laboratories, Murray Hill, NJ. From then until June, 1963, he was an Instructor in the Department of Electrical Engineering, Polytechnic Institute of Brooklyn. In 1963, he joined the Technical Staff of Bell Telephone Laboratories, Whippany, NJ. He is now with Bell Telephone Laboratories, Holmdel, NJ. He is also an Adjunct Professor of Electrical Engineering at the Polytechnic Institute of Brooklyn. During the 1968–1969 academic year, he served as a Visiting McKay Lecturer in the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley.

Frank Harary (SM'70) was born in New York, NY, on March 11, 1921. He received the B.A. degree from Brooklyn College, Brooklyn, NY, the M.A. and Ph.D. degrees from the University of California, Berkeley, CA, in 1941, 1945, and 1948, respectively.

Since 1948, he has been Professor of Mathematics, the University of Michigan, Ann Arbor, and since 1950, Faculty Associate at the Research Center for Group Dynamics of the Institute for Social Research at the same university.