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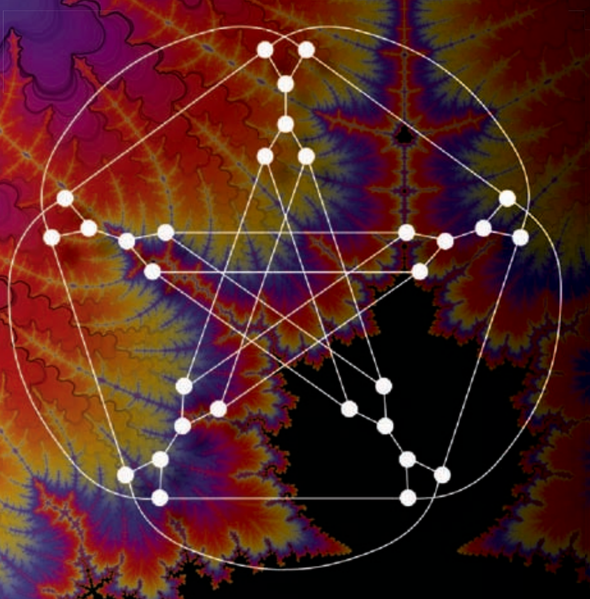

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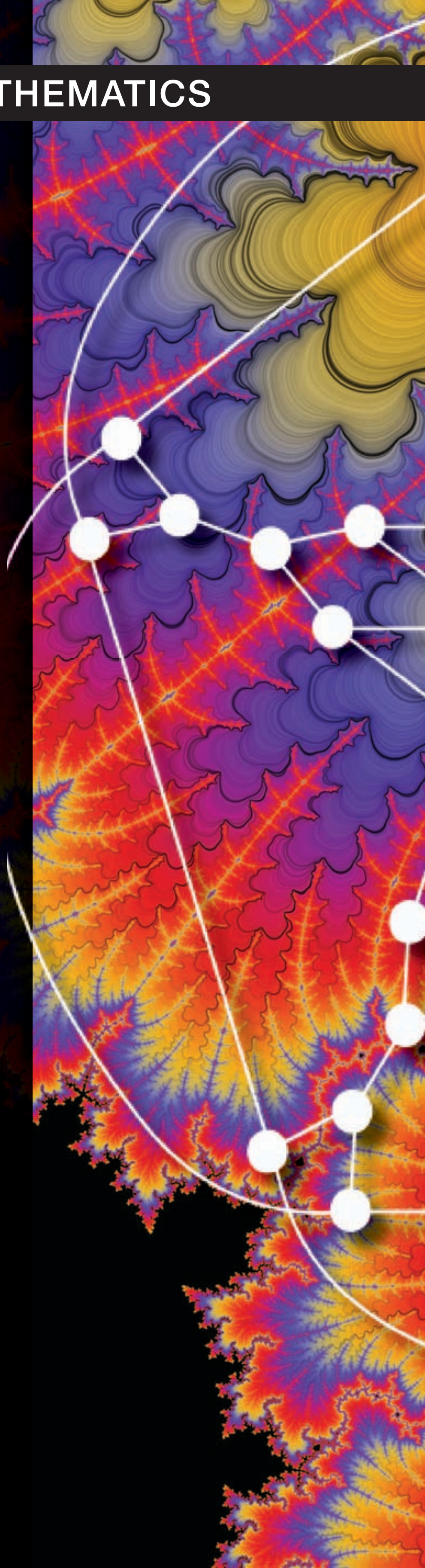

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Western Michigan University
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To

the memory of my mother and father.   G. C.

my mother and the memory of my father Stanley.   L. L.

my mother and the memory of my father.   P. Z.

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Preface to the Sixth Edition

Graph theory is an area of mathematics whose origin dates back to 1736 with the solution of the famous Königsberg Bridge Problem by the eminent Swiss mathematician Leonhard Euler. During the next several decades, topics in graph theory arose primarily through recreational mathematics. The development of graph theory received a substantial boost in 1852 when the young British mathematician Francis Guthrie introduced one of the best known problems in all of mathematics: the Four Color Problem. It wasn't until late in the 19th century, however, when graph theory became a theoretical area of mathematics through the research of the Danish mathematician Julius Petersen. Major progress in graph theory, however, didn't occur until World War II ended. Since then, though, the subject has developed into an area with a fascinating history, numerous interesting problems and applications in many diverse fields. It is the beauty of the subject, however, that has attracted so many to this field.

The goal of this sixth edition is, as with the previous editions, to describe much of the story that is graph theory – through its concepts, its theorems, its applications and its history. The audience for the sixth edition is beginning graduate students and advanced undergraduate students. The primary prerequisite required of students using this book is a knowledge of mathematical proofs. For some topics, an elementary knowledge of linear algebra and group theory is useful. For Chapter 21, an elementary knowledge of probability is needed. Proofs of some of the results that appear in this book have not been supplied because the techniques are beyond the scope of the book or are inordinately lengthy. Nevertheless, these results have been included due to their interest and since they provide a more complete description of what is known on a particular topic.

A one-semester course in graph theory using this textbook can be designed by selecting topics of greatest interest to the instructor and students. There is more than ample material available for a two-semester sequence in graph theory. Our goal has been to prepare a book that is interesting, carefully written, student-friendly and consisting of clear proofs. The sixth edition has been divided into shorter chapters as well as more sections and subsections to make reading and locating material easier for instructors and students. The following major additions have been made to the sixth edition:

- more than 160 new exercises
- several conjectures and open problems
- many new theorems and examples
- new material on graph decompositions
- a proof of the Perfect Graph Theorem

- material on Hamiltonian extension
- a new chapter on the probabilistic method in graph theory and random graphs.

At the end of the book is an index of mathematicians, an index of mathematical terms and an index of symbols. The references list research papers referred to in the book (indicating the page number(s) where the reference occurs) and some useful supplemental references. There is also a section giving hints and solutions to all odd-numbered exercises.

Over the years, there have been some changes in notation that a number of mathematicians now use. When certain notation appears to have been adopted by sufficiently many mathematicians working in graph theory so that this has become the norm, we have adhered to these changes. As with the fifth edition, the following notation is used in the sixth edition:

- a path is now expressed as $P = (v_1, v_2, \ldots, v_k)$ and a cycle as $C = (v_1, v_2, \ldots, v_k, v_1)$;
- the Cartesian product of two graphs G and H is expressed as $G \square H$, rather than the previous $G \times H$;
- the union of G and H is expressed by $G + H$, rather than $G \cup H$;
- the join of two graphs G and H is expressed as $G \vee H$, rather than $G + H$.

We are most grateful to Bob Ross, senior editor of CRC Press, who has been a constant source of support and assistance throughout the entire writing process.

Gary Chartrand, Linda Lesniak and Ping Zhang

Chapter 1

Introduction

The theory of graphs is one of the few fields of mathematics with a definite birth date.

It is the subject of graph theory of course that we are about to describe. The statement above was made in 1963 by the mathematician Oystein Ore who will be encountered in Chapter 6. While graph theory was probably Ore's major mathematical area of interest during the latter part of his career, he is also known for his work and interest in number theory (the study of integers) and the history of mathematics.

Although awareness of integers can be traced back for many centuries, geometry has an even longer history. Early geometry concerned distance, lengths, angles, areas and volumes, which were used for surveying, construction and astronomy. While geometry dealt with magnitudes, the German mathematician Gottfried Leibniz introduced another branch of geometry called the geometry of position. This branch of geometry did not deal with measurements and calculations, but rather with the determination of position and its properties. The famous mathematician Leonhard Euler said that it hadn't been determined what kinds of problems could be studied with the aid of the geometry of position but in 1736 he believed that he had found one, which led to the origin of graph theory. It is this event to which Oystein Ore was referring in his quote above. We will visit Euler again, in Chapter 5 as well as in Chapters 10 and 11.

1.1 Graphs

Graphs arise in many different settings. Let's look at three of these.

Example 1.1 Eight students $s_1, s_2, \ldots, s_8$ have been invited to a dinner. Each student knows only some of the other students. The students that each student knows are listed below.

$s_1: s_4, s_5, s_8$	$s_2: s_3, s_4, s_6, s_8$
$s_3: s_2, s_5, s_6, s_7, s_8$	$s_4: s_1, s_2, s_5, s_6, s_7$
$s_5: s_1, s_3, s_4, s_8$	$s_6: s_2, s_3, s_4, s_7$
$s_7: s_3, s_4, s_6$	$s_8: s_1, s_2, s_3, s_5$

In order to determine if these eight students can be seated at a round table where each student sits next to two students he or she knows, it is useful to represent this situation by the diagram shown in Figure 1.1. Each point or small circle in the diagram represents a student and two points are joined by a line segment if the two students know each other. This diagram is referred to as a *graph*.

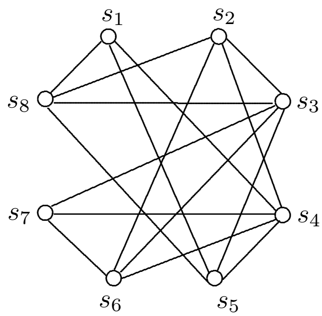


Figure 1.1: The diagram in Example 1.1

A related question is whether the students could be seated at a round table so that each student sits next to two students he or she does not know. ♦

Example 1.2 There are six special locations in a neighborhood park. Twelve trails are to be built between certain pairs of these locations, namely all pairs of locations except $\{a_1, a_2\}$, $\{b_1, b_2\}$, $\{c_1, c_2\}$ (see Figure 1.2(a)). A trail can be straight or curved. Can this be done without any trails crossing? This situation can be represented by the diagram with six points (each point representing a location), where two points are joined by a line segment or a curve if the two points represent locations to be joined by a trail (see Figure 1.2(b)). Once again, this diagram is a graph. ♦

Example 1.3 A chemical company is to ship eight chemicals (denoted by $c_1, c_2, \ldots, c_8$) to a chemistry department in a university. Because some pairs of chemicals should not be shipped in the same container, more than one container needs to be used for this shipment. Each chemical is listed below together with the chemicals that should not be placed in the same container as this chemical.

$c_1: c_2$	$c_2: c_1, c_8$	$c_3: c_5, c_6, c_7$	$c_4: c_5, c_7$
$c_5: c_3, c_4, c_8$	$c_6: c_3, c_7$	$c_7: c_3, c_4, c_6$	$c_8: c_2, c_5$

It would be useful to know the minimum number of containers needed to ship these eight chemicals. This situation can be represented by the diagram

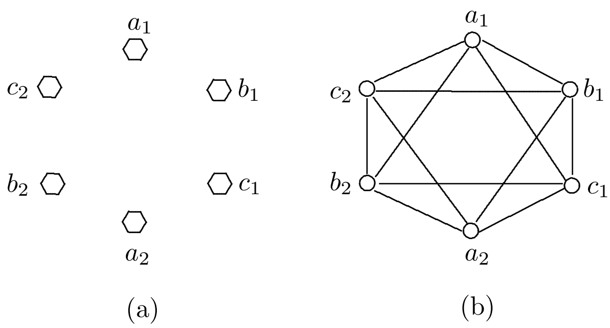


Figure 1.2: Constructing a graph in Example 1.2

in Figure 1.3, whose eight points represent the eight chemicals and where two points are joined by a line segment or curve if these chemicals cannot be shipped in the same container. Here too, this diagram is a graph. ♦

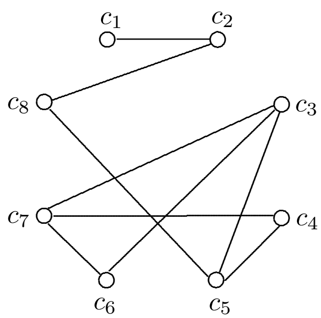


Figure 1.3: The graph in Example 1.3

We now give a formal definition of the term *graph.* A **graph** G is a finite nonempty set V of objects called **vertices** (the singular is **vertex**) together with a possibly empty set E of 2-element subsets of V called **edges**. Vertices are sometimes referred to as **points** or **nodes**, while edges are sometimes called **lines** or **links**. In fact, historically, graphs were referred to as *linkages* by some. Calling these structures graphs was evidently the idea of James Joseph Sylvester (1814–1897), a well-known British mathematician who became the first mathematics professor at Johns Hopkins University in Baltimore and who founded and became editor-in-chief of the first mathematics journal in the United States (the *American Journal of Mathematics*).

To indicate that a graph G has **vertex set** V and **edge set** E, we write $G = (V, E)$. To emphasize that V and E are the vertex set and edge set of a graph G, we often write V as $V(G)$ and E as $E(G)$. Each edge $\{u, v\}$ of G is usually denoted by uv or vu. If $e = uv$ is an edge of G, then e is said to **join** u and v.

As the examples described above indicate, a graph G can be represented by a diagram, where each vertex of G is represented by a point or small circle and an edge joining two vertices is represented by a line segment or curve joining the corresponding points in the diagram. It is customary to refer to such a diagram as the graph G itself. In addition, the points in the diagram are referred to as the vertices of G and the line segments are referred to as the edges of G. For example, the graph G with vertex set $V(G) = \{u, v, w, x, y\}$ and edge set $E(G) = \{uv, uy, vx, vy, wy, xy\}$ is shown in Figure 1.4. Even though the edges vx and wy cross in Figure 1.4, their point of intersection is not a vertex of G.

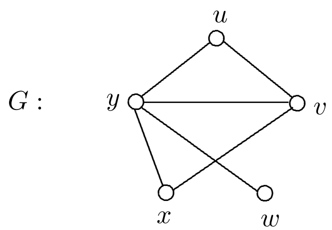


Figure 1.4: A graph

If uv is an edge of G, then u and v are **adjacent vertices**. Two adjacent vertices are referred to as **neighbors** of each other. The set of neighbors of a vertex v is called the **open neighborhood** of v (or simply the **neighborhood** of v) and is denoted by $N_G(v)$, or $N(v)$ if the graph G is understood. The set $N[v] = N(v) \cup \{v\}$ is called the **closed neighborhood** of v. If uv and vw are distinct edges in G, then uv and vw are **adjacent edges**. The vertex u and the edge uv are said to be **incident** with each other. Similarly, v and uv are incident.

For the graph G of Figure 1.4, the vertices u and v are therefore adjacent in G, while the vertices u and x are not adjacent. The edges uv and vx are adjacent in G, while the edges vx and wy are not adjacent. The vertex v is incident with the edge uv but is not incident with the edge wy.

The number of vertices in a graph G is the **order** of G and the number of edges is the **size** of G. The order of the graph G of Figure 1.4 is 5 and its size is 6. We typically use n and m for the order and size, respectively, of a graph. A graph of order 1 is called a **trivial graph**. A **nontrivial graph** therefore has two or more vertices. A graph of size 0 is called an **empty graph**. A **nonempty graph** then has one or more edges. In any empty graph, no two vertices are adjacent. At the other extreme is a **complete graph** in which every two distinct vertices are adjacent. The size of a complete graph of order n is $\binom{n}{2} = n(n-1)/2$. Therefore, for every graph G of order n and size m, it follows that $0 \le m \le \binom{n}{2}$. The complete graph of order n is denoted by K_n. The complete graphs K_n for $1 \le n \le 5$ are shown in Figure 1.5.

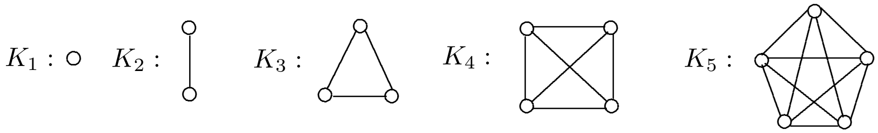


Figure 1.5: Some complete graphs

Two other classes of graphs that are often encountered are the paths and cycles. For an integer $n \geq 1$, the **path** P_n is a graph of order n and size $n-1$ whose vertices can be labeled by $v_1, v_2, \ldots, v_n$ and whose edges are $v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$. For an integer $n \geq 3$, the **cycle** C_n is a graph of order n and size n whose vertices can be labeled by $v_1, v_2, \ldots, v_n$ and whose edges are $v_1 v_n$ and $v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$. The cycle C_n is also referred to as an n**-cycle** and the **3-cycle** is also called a **triangle**. The paths and cycles of order 5 or less are shown in Figure 1.6. Observe that $P_1 = K_1$, $P_2 = K_2$ and $C_3 = K_3$.

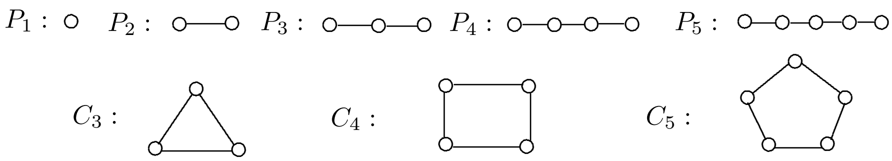


Figure 1.6: Paths and cycles of order 5 or less

1.2 The Degree of a Vertex

The **degree of a vertex** v in a graph G is the number of vertices in G that are adjacent to v. Thus, the degree of v is the number of vertices in its neighborhood $N(v)$. Equivalently, the degree of v is the number of edges incident with v. The degree of a vertex v is denoted by $\deg_G v$ or, more simply, by $\deg v$ if the graph G under discussion is clear. Hence, $\deg v = |N(v)|$. A vertex of degree 0 is referred to as an **isolated vertex** and a vertex of degree 1 is an **end-vertex** or a **leaf**. An edge incident with an end-vertex is called a **pendant edge**. The largest degree among the vertices of G is called the **maximum degree** of G and is denoted by $\Delta(G)$. The **minimum degree** of G is denoted by $\delta(G)$. (The symbols Δ and δ are the upper case and lower case Greek letter delta, respectively.) Thus, if v is a vertex of a graph G of order n, then

$$0 \leq \delta(G) \leq \deg v \leq \Delta(G) \leq n-1.$$

For the graph G of Figure 1.4,

$$\deg w = 1,\ \deg u = \deg x = 2,\ \deg v = 3 \text{ and } \deg y = 4.$$

Thus, $\delta(G) = 1$ and $\Delta(G) = 4$.

The First Theorem of Graph Theory

A well-known theorem in graph theory dealing with the sum of the degrees of the vertices of a graph was observed (indirectly) by Leonhard Euler in a 1736 paper [86] of his that is now considered the first paper ever written on graph theory – even though graphs were never mentioned in the paper. This observation is often referred to as the **First Theorem of Graph Theory**. Some have called Theorem 1.4 the **Handshaking Lemma**, although Euler never used this name.

Theorem 1.4 (**The First Theorem of Graph Theory**) *If G is a graph of size m, then*

$$\sum_{v \in V(G)} \deg v = 2m.$$

Proof. When summing the degrees of the vertices of G, each edge of G is counted twice, once for each of its two incident vertices. ■

The sum of the degrees of the vertices of the graph G of Figure 1.4 is 12, which is twice the size 6 of G, as expected from Theorem 1.4. The **average degree** of a graph G of order n and size m is

$$\frac{\sum_{v \in V(G)} \deg v}{n} = \frac{2m}{n}.$$

For example, the average degree of the graph G of Figure 1.4 (having order $n = 5$ and size $m = 6$) is $2m/n = 12/5$. Since the average degree of this graph is strictly between 2 and 3, it follows that G must have a vertex of degree 3 or more and a vertex of degree 2 or less. This graph actually has vertices of degrees 3 and 4 as well as vertices of degrees 1 and 2.

Even and Odd Vertices

A vertex in a graph G is **even** or **odd**, according to whether its degree in G is even or odd. Thus, the graph G of Figure 1.4 has three even vertices and two odd vertices. While a graph can have either an even or an odd number of even vertices, this is not the case for odd vertices.

Corollary 1.5 *Every graph has an even number of odd vertices.*

Proof. Suppose that G is a graph of size m. By Theorem 1.4,

$$\sum_{v \in V(G)} \deg v = 2m,$$

which, of course, is an even number. Since the sum of the degrees of the even vertices of G is even, the sum of the degrees of the odd vertices of G must be even as well, implying that G has an even number of odd vertices. ■

1.3 Isomorphic Graphs

There is only one graph of order 1, two graphs of order 2, four graphs of order 3 and eleven graphs of order 4. All 18 of these graphs are shown in Figure 1.7. This brings up the question of why every two graphs in Figure 1.7 are considered different. In fact, there is the related question of what it means for two graphs to be considered the same. The technical term for this is *isomorphic graphs* (graphs having the same structure).

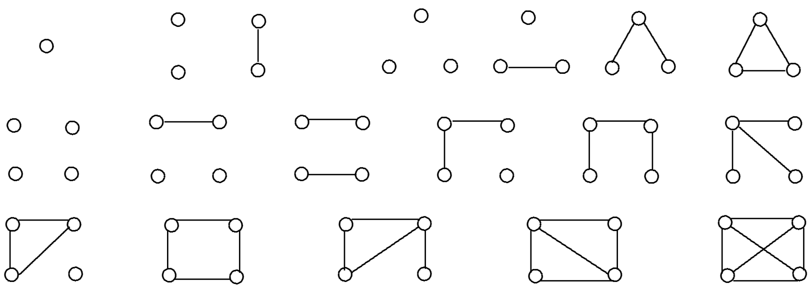


Figure 1.7: The (non-isomorphic) graphs of order 4 or less

Two graphs G and H are **isomorphic** if there exists a bijective function $\phi : V(G) \to V(H)$ such that two vertices u and v are adjacent in G if and only if $\phi(u)$ and $\phi(v)$ are adjacent in H. The function ϕ is called an **isomorphism** from G to H. If $\phi : V(G) \to V(H)$ is an isomorphism, then the inverse function $\phi^{-1} : V(H) \to V(G)$ is an isomorphism from H to G. If G and H are isomorphic, we write $G \cong H$. If there is no such function ϕ as described above, then G and H are **non-isomorphic graphs** and we write $G \ncong H$.

The graphs G and H of Figure 1.8 (both of order 7 and size 8) are isomorphic and the function $\phi : V(G) \to V(H)$ defined by

$$\phi(u_1) = v_4,\ \phi(u_2) = v_5,\ \phi(u_3) = v_1,$$

$$\phi(u_4) = v_6,\ \phi(u_5) = v_2,\ \phi(u_6) = v_3,\ \phi(u_7) = v_7$$

is an isomorphism, although there are three other isomorphisms. The graphs F_1 and F_2 of Figure 1.8 (both of order 7 and size 10) are not isomorphic however. An explanation of this will be given shortly.

The graphs G_1 and G_2 in Figure 1.9 are isomorphic, while G_1 and G_3 are not isomorphic. For example, the function $\phi : V(G_1) \to V(G_2)$ defined by

$$\phi(u_1) = v_1,\ \phi(u_2) = v_3,\ \phi(u_3) = v_5,$$

$$\phi(u_4) = v_2,\ \phi(u_5) = v_4,\ \phi(u_6) = v_6$$

is an isomorphism. The graph G_3 of Figure 1.9 contains three mutually adjacent vertices w_1, w_2, w_6. If G_1 and G_3 were isomorphic, then for an isomorphism

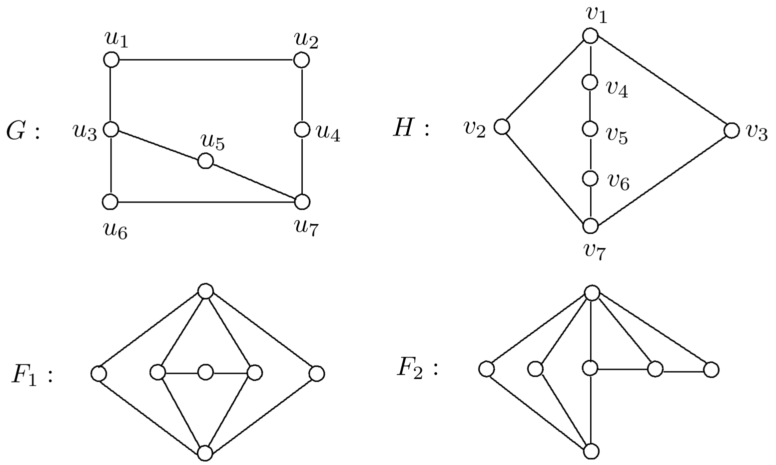


Figure 1.8: Isomorphic and non-isomorphic graphs

$\alpha : V(G_3) \to V(G_1)$, the vertices $\alpha(w_1), \alpha(w_2), \alpha(w_6)$ must also be mutually adjacent in G_1. Since G_1 does not contain three mutually adjacent vertices, there is no isomorphism from G_3 to G_1 and so $G_1 \not\cong G_3$. Furthermore, $G_2 \not\cong G_3$ as well.

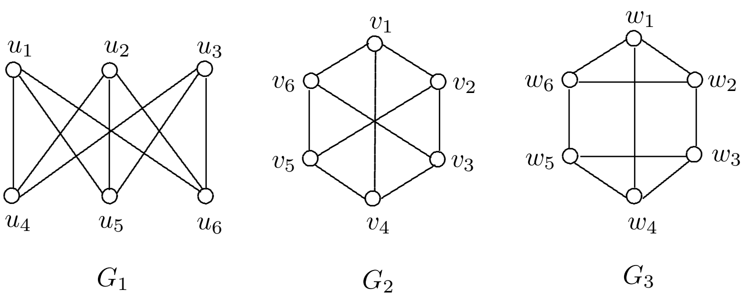


Figure 1.9: Isomorphic and non-isomorphic graphs

Suppose that two graphs G and H are isomorphic. Then there exists an isomorphism $\phi : V(G) \to V(H)$. Since ϕ is a bijective function, $|V(G)| = |V(H)|$. Furthermore, since two vertices u and v are adjacent in G if and only if $\phi(u)$ and $\phi(v)$ are adjacent in H, it follows that $|E(G)| = |E(H)|$. These facts are summarized below, together with another necessary condition for two graphs to be isomorphic.

Theorem 1.6 *If two graphs G and H are isomorphic, then they have the same order and the same size, and the degrees of the vertices of G are the same as the degrees of the vertices of H.*

Proof. We have already observed that isomorphic graphs have the same order and the same size. Let v be a vertex of G and suppose that $\deg v = k$. Then v is adjacent to k vertices, say $v_1, v_2, \ldots, v_k$. Suppose that v is not adjacent to $u_1, u_2, \ldots, u_\ell$. If $\phi : V(G) \to V(H)$ is an isomorphism, then $\phi(v)$ is adjacent to $\phi(v_1), \phi(v_2), \ldots, \phi(v_k)$ while $\phi(v)$ is not adjacent to $\phi(u_1), \phi(u_2), \ldots, \phi(u_\ell)$. Hence, $\phi(v)$ has degree k in H. ■

The proof of Theorem 1.6 also shows that every isomorphism from a graph G to a graph H maps every vertex of G to a vertex of the same degree in H.

It is therefore a consequence of Theorem 1.6 that if G and H are two graphs such that (1) the orders of G and H are different, or (2) the sizes of G and H are different or (3) the degrees of the vertices of G and those of the vertices of H are different, then G and H are not isomorphic. Since the graph F_2 in Figure 1.8 contains a vertex of degree 5 and no vertex of F_1 has degree 5, it follows by Theorem 1.6 that $F_1 \ncong F_2$.

The conditions described in Theorem 1.6 are strictly necessary for two graphs to be isomorphic – they are not sufficient. Indeed, the graphs G_1 and G_3 of Figure 1.9 have the same order, the same size and the degrees of the vertices of G_1 and G_3 are the same; yet G_1 and G_3 are not isomorphic.

Next, consider the four graphs H_1, H_2, H_3 and H_4 shown in Figure 1.10. The graphs H_1 and H_2 have order 7, size 7 and the degrees of the vertices of these two graphs are the same. Furthermore, each graph contains a single triangle. Nevertheless, $H_1 \ncong H_2$, for suppose that there is an isomorphism ϕ from H_1 to H_2. Since each graph has only one vertex of degree 3 and one vertex of degree 4, $\phi(u_1) = v_2$ and $\phi(y_1) = x_2$. Since v_1 is adjacent to u_1 and y_1, it follows that $\phi(v_1)$ is adjacent to v_2 and x_2. However, H_2 contains no vertex adjacent to v_2 and x_2. Thus, $H_1 \ncong H_2$.

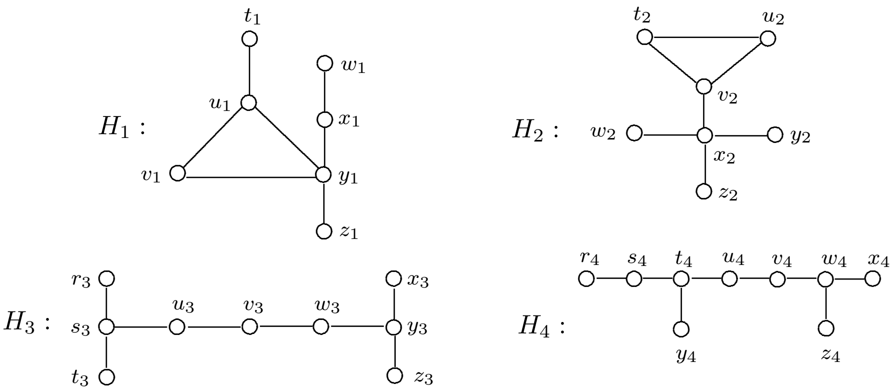


Figure 1.10: Non-isomorphic graphs

The graphs H_3 and H_4 of Figure 1.10 have order 9, size 8 and the degrees of the vertices of these two graphs are the same; yet these two graphs as well are *not* isomorphic. Suppose that there is an isomorphism ϕ from H_3 to H_4. Consider the vertex v_3. Since $\deg_{H_3} v_3 = 2$, it follows that $\phi(v_3)$ has degree 2 in H_4. Since v_3 is adjacent to u_3 and w_3, we must have $\phi(v_3)$ adjacent to $\phi(u_3)$ and $\phi(w_3)$. Since $\deg_{H_3} u_3 = \deg_{H_3} w_3 = 2$, it follows that $\phi(u_3)$ and $\phi(w_3)$ have degree 2 in H_4. But no vertex of degree 2 in H_4 is adjacent to two vertices of degree 2. Thus, $H_3 \not\cong H_4$.

Subgraphs

A graph H is a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, in which case we write $H \subseteq G$. If H is a subgraph of G, then G is a **supergraph** of H. If $V(H) = V(G)$, then H is a **spanning subgraph** of G. If H is a subgraph of a graph G where $H \not\cong G$, then H is a **proper subgraph** of G. Therefore, if H is a proper subgraph of G, then either $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$.

Figure 1.11 shows six graphs, namely G and the graphs G_i for $i = 1, 2, \ldots, 5$. All six of these graphs are proper subgraphs of G, except G itself and G_1. Although G is a subgraph of itself, it is not a proper subgraph of G. The graph G_1 contains the edge uz, which is not an edge of G and so G_1 is not even a subgraph of G. The graph G_3 is a spanning subgraph of G since $V(G_3) = V(G)$.

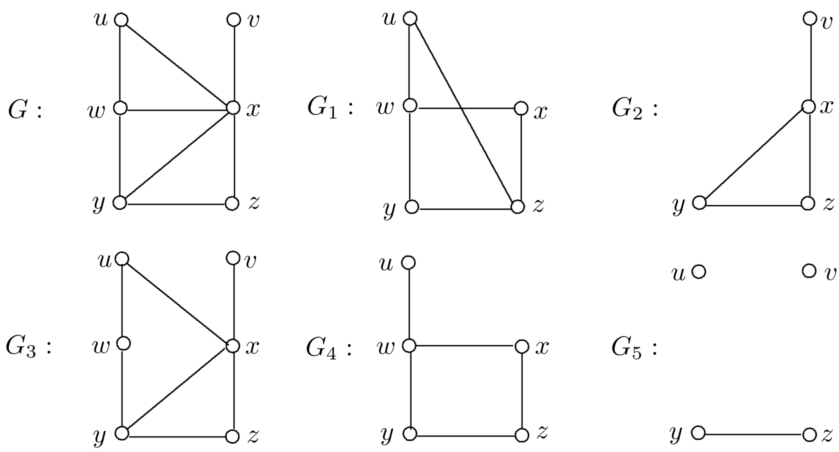


Figure 1.11: Graphs and subgraphs

Induced Subgraphs

For a nonempty subset S of $V(G)$, the **subgraph** $G[S]$ **of** G **induced by** S

has S as its vertex set and two vertices u and v are adjacent in $G[S]$ if and only if u and v are adjacent in G. A subgraph H of a graph G is called an **induced subgraph** if there is a nonempty subset S of $V(G)$ such that $H = G[S]$. Thus $G[V(G)] = G$. For a nonempty set X of edges of a graph G, the **subgraph** $G[X]$ **induced by** X has X as its edge set and a vertex v belongs to $G[X]$ if v is incident with at least one edge in X. A subgraph H of G is **edge-induced** if there is a nonempty subset X of $E(G)$ such that $H = G[X]$. Thus, $G[E(G)] = G$ if and only if G has no isolated vertices.

Once again, consider the graphs shown in Figure 1.11. Since $xy \in E(G)$ but $xy \notin E(G_4)$, the subgraph G_4 is not an induced subgraph of G. On the other hand, the subgraphs G_2 and G_5 are both induced subgraphs of G. Indeed, for $S_1 = \{v, x, y, z\}$ and $S_2 = \{u, v, y, z\}$, $G_2 = G[S_1]$ and $G_5 = G[S_2]$. The subgraph G_4 of G is edge-induced; in fact, $G_4 = G[X]$, where $X = \{uw, wx, wy, xz, yz\}$.

For a vertex v and an edge e in a nonempty graph $G = (V, E)$, the subgraph $G - v$, obtained by deleting v from G, is the induced subgraph $G[V - \{v\}]$ of G and the subgraph $G - e$, obtained by deleting e from G, is the spanning subgraph of G with edge set $E - \{e\}$. More generally, for a proper subset U of V, the graph $G - U$ is the induced subgraph $G[V - U]$ of G. For a subset X of E, the graph $G - X$ is the spanning subgraph of G with edge set $E - X$. If u and v are distinct nonadjacent vertices of G, then $G + uv$ is the graph with $V(G + uv) = V(G)$ and $E(G + uv) = E(G) \cup \{uv\}$. Thus, G is a spanning subgraph of $G + uv$. For the graph G of Figure 1.12, the set $U = \{t, x\}$ of vertices and the set $X = \{tw, ux, vx\}$ of edges, the subgraphs $G - u$, $G - wx$, $G - U$ and $G - X$ of G are also shown in that figure, as is the graph $G + uv$.

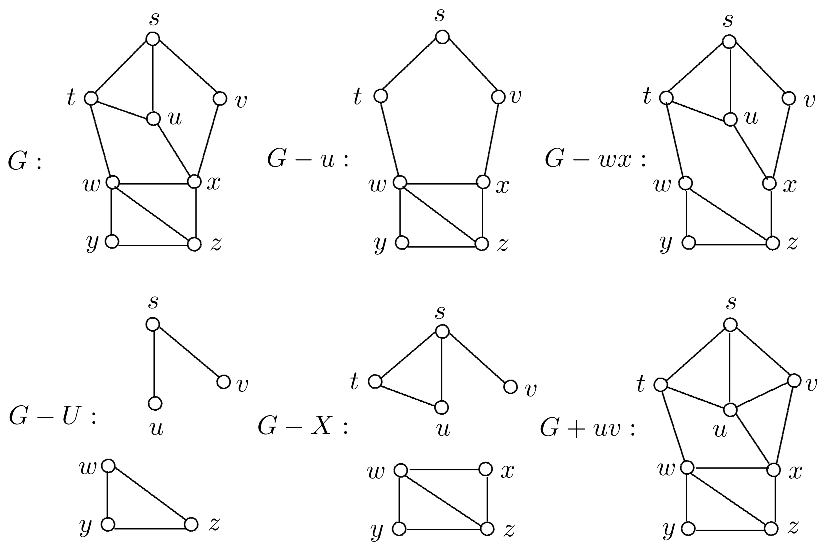


Figure 1.12: Deleting vertices and edges from and adding edges to a graph

If every two edges e_1 and e_2 of a graph G have the property that $G - e_1 \cong G - e_2$, then we write $G - e$ for the deletion of any edge from G. Hence, for $n \geq 2$, $K_n - e$ denotes the graph obtained by deleting any edge from K_n. If $G + uv \cong G + xy$ for any two pairs $\{u, v\}$ and $\{x, y\}$ of nonadjacent vertices of G, then we write $G + e$ for the addition of any edge to G. In particular, $C_4 + e = K_4 - e$.

1.4 Regular Graphs

There are certain classes of graphs that occur so often that they deserve special mention and, in some cases, special notation. We describe some of the most prominent of these now.

A graph G is **regular** if the vertices of G have the same degree and is **regular of degree** r if this degree is r. Such graphs are also called r**-regular**. The complete graph of order n is therefore a regular graph of degree $n-1$ and every cycle is 2-regular. In Figure 1.13 are shown all (non-isomorphic) regular graphs of orders 4 and 5, including the cycles C_4 and C_5 and the complete graphs K_4 and K_5. Since no graph has an odd number of odd vertices, there is no 1-regular or 3-regular graph of order 5. Indeed, the pairs r, n of integers for which there exist r-regular graphs of order n are predictable.

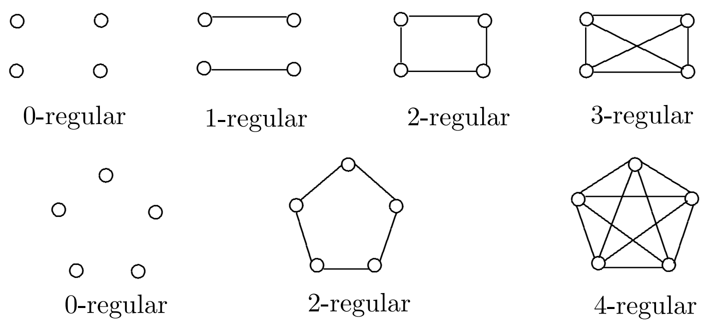


Figure 1.13: The regular graphs of orders 4 and 5

Theorem 1.7 *For integers r and n, there exists an r-regular graph of order n if and only if $0 \leq r \leq n-1$ and r and n are not both odd.*

Proof. That the conditions are necessary is an immediate consequence of Corollary 1.5 and the fact that $0 \leq \deg v \leq n-1$ for every vertex v in a graph of order n. For the converse, suppose that r and n are integers where $0 \leq r \leq n-1$ and r and n are not both odd. Assume first that r is even. If $r = 0$, then the graph of order n consisting of n isolated vertices is r-regular. So we may assume that $r = 2k$ for some integer $k \geq 1$ and $n \geq 2k+1$. Let G be the graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that v_i $(1 \leq i \leq n)$ is adjacent to $v_{i\pm1}, v_{i\pm2}, \ldots, v_{i\pm k}$ (subscripts expressed modulo n). The resulting graph G is then an r-regular graph of order n. If r is odd, then $n = 2\ell$ is even. Hence $r = 2k+1$ for some integer k with $0 \le k \le \ell - 1$. For the graph G in this case, the vertex v_i is adjacent to the $2k$ vertices described above and adjacent as well to $v_{i+\ell}$. Again, G is an r-regular graph of order n. ■

The 4-regular and 5-regular graphs of order 10 constructed in the proof of Theorem 1.7 are shown in Figure 1.14.

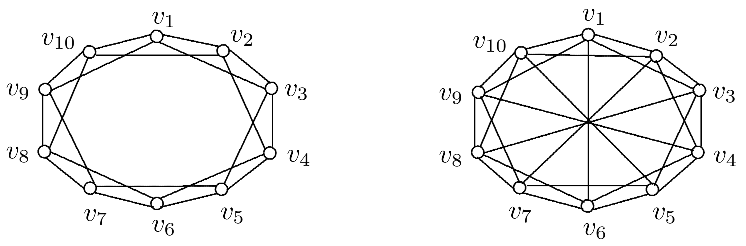


Figure 1.14: 4-regular and 5-regular graphs of order 10

The Petersen Graph

A 3-regular graph is also called a **cubic graph**. The graphs of Figure 1.9 are cubic as is the complete graph K_4. One of the best known cubic graphs is the **Petersen graph**, named for the Danish mathematician Julius Petersen whose 1891 research on regular graphs [186] is often credited as the beginning of the study of graphs as a theoretical subject. In fact, the Petersen graph is one of the best known graphs. Three different drawings of the Petersen graph are shown in Figure 1.15. We will have many occasions to encounter this graph.

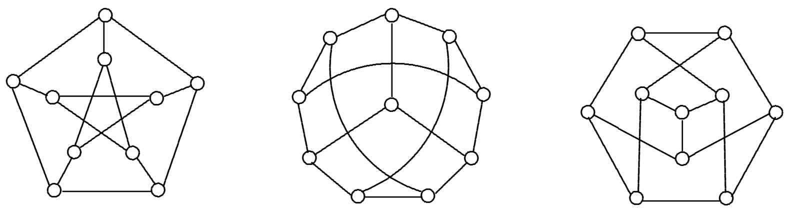


Figure 1.15: Three drawings of the Petersen graph

1.5 Bipartite Graphs

Another class of graphs that we often encounter are the bipartite graphs. A graph G is **bipartite** if $V(G)$ can be partitioned into two sets U and W (called

partite sets) so that every edge of G joins a vertex of U and a vertex of W. The graph G in Figure 1.16(a) is bipartite with partite sets $\{v_1, v_3, v_5, v_7\}$ and $\{v_2, v_4, v_6\}$. This graph is redrawn in Figure 1.16(b) to see more clearly that it is bipartite. If G is an r-regular bipartite graph, $r \geq 1$, with partite sets U and W, then $|U| = |W|$. This follows since the size of G is $r|U| = r|W|$. For two nonempty sets X and Y of vertices in a graph G, the set

$$[X, Y] = \{xy : x \in X, y \in Y\}$$

consists of those edges joining a vertex of X and a vertex of Y. Thus, if G is a bipartite graph with partite sets U and W, then $[U, W] = E(G)$.

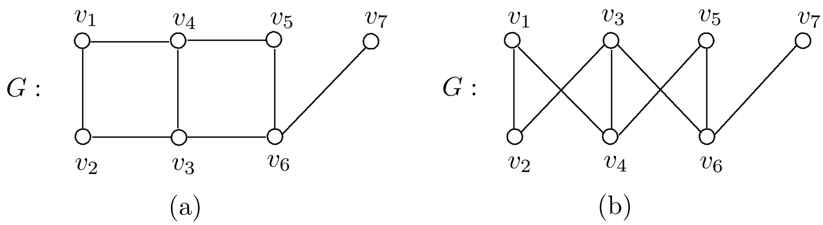


Figure 1.16: A bipartite graph

A graph G is a **complete bipartite graph** if $V(G)$ can be partitioned into two sets U and W (called **partite sets** again) so that uw is an edge of G if and only if $u \in U$ and $w \in W$. If $|U| = s$ and $|W| = t$, then this complete bipartite graph has order $s+t$ and size st and is denoted by $K_{s,t}$ (or $K_{t,s}$). The complete bipartite graph $K_{1,t}$ is called a **star**. The complete bipartite graphs $K_{1,3}$, $K_{2,2}$, $K_{2,3}$ and $K_{3,3}$ are shown in Figure 1.17. Observe that $K_{2,2} = C_4$. The star $K_{1,3}$ is sometimes referred to as a **claw**.

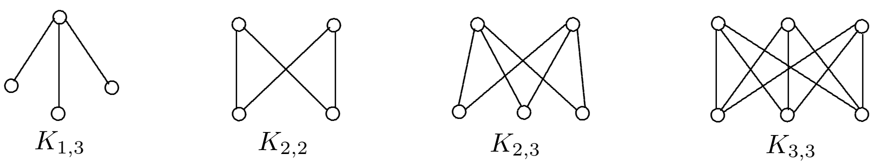


Figure 1.17: Complete bipartite graphs

Since the size of the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is $\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil = \lfloor n^2/4 \rfloor$, there are bipartite graphs of order n and size $\lfloor n^2/4 \rfloor$. No bipartite graph of order n can have a larger size however.

Theorem 1.8 *The size of every bipartite graph of order n is at most $\lfloor n^2/4 \rfloor$.*

Proof. Let G be bipartite graph of order n with partite sets U and W. Then $|U| = x$ and $|W| = n - x$ for some integer x with $1 \leq x \leq n-1$. Hence the size of G is at most $x(n - x)$.

Since $(n-2x)^2 \geq 0$, it follows that

$$n^2 \geq 4nx - 4x^2 = 4x(n-x)$$

and so $x(n-x) \leq n^2/4$. Since $x(n-x)$ is an integer, $x(n-x) \leq \lfloor n^2/4 \rfloor$. ■

No bipartite graph G can contain a triangle H, for otherwise, at least two vertices u and v of H must belong to the same partite set and so u and v are not adjacent. Hence if G is a graph of order $n \geq 3$ and size $m \leq \lfloor n^2/4 \rfloor$, then G need not contain a triangle. However, in 1907 the Dutch mathematician Willem Mantel [161] showed that any graph of order n with a larger size must contain a triangle.

Theorem 1.9 *Every graph of order $n \geq 3$ and size $m > \lfloor n^2/4 \rfloor$ contains a triangle.*

Proof. First, observe that the result is true for $n = 3$ and $n = 4$. Suppose, however, that the statement is false. Then there is a smallest integer $n \geq 5$ and a graph G of order n and size $m > \lfloor n^2/4 \rfloor$ not containing a triangle. Let uv be an edge of G. Since G contains no triangle, there is no vertex in G adjacent to both u and v. Hence, $(\deg u - 1) + (\deg v - 1) \leq n - 2$ and so $\deg u + \deg v \leq n$. Let $G' = G - u - v$. Since G' is a subgraph of G, it follows that G' also does not contain a triangle. Furthermore, G' has order $n-2$ and size

$$m' = m - (\deg u + \deg v) + 1 > \lfloor n^2/4 \rfloor - n + 1.$$

Thus,

$$m' > \frac{n^2}{4} - n + 1 = \frac{n^2 - 4n + 4}{4} = \frac{(n-2)^2}{4}.$$

From the defining property of the graph G, it follows that G' contains a triangle, producing a contradiction. ■

Therefore, from Theorems 1.8 and 1.9, it follows that every graph of order $n \geq 3$ and size $m > \lfloor n^2/4 \rfloor$ not only fails to be bipartite, it must, in fact, contain a triangle.

Complete Multipartite Graphs

Bipartite graphs belong to a more general class of graphs. For an integer $k \geq 1$, a graph G is a **k-partite graph** if $V(G)$ can be partitioned into k subsets $V_1, V_2, \ldots, V_k$ (again called **partite sets**) such that every edge of G joins vertices in two different partite sets. A 1-partite graph is then an empty graph and a 2-partite graph is bipartite. A **complete k-partite graph** G is a k-partite graph with the property that two vertices are adjacent in G if and only if the vertices belong to different partite sets. If $|V_i| = n_i$ for $1 \leq i \leq k$, then G is denoted by $K_{n_1,n_2,\ldots,n_k}$ (the order in which the numbers $n_1, n_2, \ldots, n_k$

are written is not important). If $n_i = 1$ for all i $(1 \leq i \leq k)$, then G is the complete graph K_k. A **complete multipartite graph** is a complete k-partite graph for some integer $k \geq 2$. Some complete multipartite graphs are shown in Figure 1.18

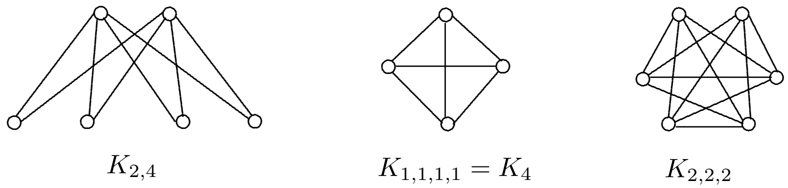


Figure 1.18: Some complete multipartite graphs

1.6 Operations on Graphs

There are many ways of producing a new graph from one or more given graphs. The most common of these is the complement of a graph.

The Complement of a Graph

The **complement** $\overline{G}$ of a graph G is that graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G. Any isomorphism from a graph G to a graph H is also an isomorphism from $\overline{G}$ to $\overline{H}$. Consequently, $\overline{G} \cong \overline{H}$ if and only if $G \cong H$. If G is a graph of order n and size m, then $\overline{G}$ is a graph of order n and size $\binom{n}{2} - m$. A graph G and its complement are shown in Figure 1.19. The complement $\overline{K}_n$ of the complete graph K_n is the empty graph of order n.

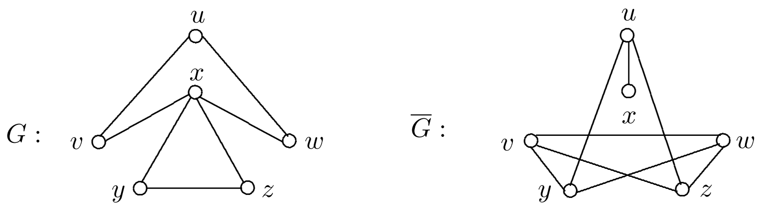


Figure 1.19: A graph and its complement

A graph G is **self-complementary** if G is isomorphic to $\overline{G}$. Certainly, if G is a self-complementary graph of order n, then its size is $m = \binom{n}{2}/2 = n(n-1)/4$. Since only one of n and $n-1$ is even, either $4 \mid n$ or $4 \mid (n-1)$; that is, if G is a self-complementary graph of order n, then either $n \equiv 0 \pmod 4$ or

$n \equiv 1 \pmod 4$. (See Exercise 29.) The self-complementary graphs of order 5 or less are shown in Figure 1.20.

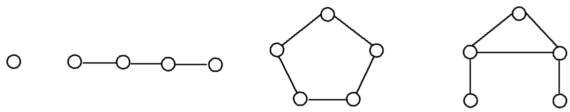


Figure 1.20: The self-complementary graphs of order 5 or less

The Union and Join of Graphs

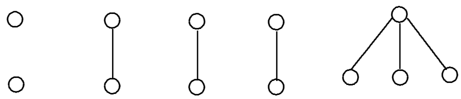
We next describe some common binary operations defined on graphs. This discussion introduces notation that will be especially useful in giving examples. Over the years, different authors have used different notation for the operations we are about to describe. In the following definitions, we assume that G_1 and G_2 are two graphs with disjoint vertex sets.

The **union** $G = G_1+G_2$ of G_1 and G_2 has vertex set $V(G) = V(G_1)\cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$. The union $G + G$ of two disjoint copies of G is denoted by $2G$. Indeed, if a graph G consists of k (≥ 2) disjoint copies of a graph H, then we write $G = kH$. The graph $2K_1 + 3K_2 + K_{1,3}$ is shown in Figure 1.21(a). The **join** $G = G_1 \vee G_2$ of G_1 and G_2 has vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set

$$E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}.$$

Using the join operation, we see that $\overline{K}_s \vee \overline{K}_t = K_{s,t}$. Another illustration is given in Figure 1.21(b).

$2K_1 + 3K_2 + K_{1,3}$:

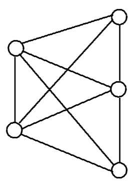


(a)

G_1 :

G_2 :

$G_1 \vee G_2$:



(b)

Figure 1.21: The union and join of graphs

The Cartesian Product of Graphs

The **Cartesian product** G of two graphs G_1 and G_2, commonly denoted by $G_1 \square G_2$ or $G_1 \times G_2$, has vertex set

$$V(G) = V(G_1) \times V(G_2),$$

where two distinct vertices (u, v) and (x, y) of $G_1 \square G_2$ are adjacent if either

$$(1)\ u = x \text{ and } vy \in E(G_2) \text{ or } (2)\ v = y \text{ and } ux \in E(G_1).$$

A convenient way of drawing $G_1 \square G_2$ is to first place a copy of G_2 at each vertex of G_1 (see Figure 1.22(b)) and then join corresponding vertices of G_2 in those copies of G_2 placed at adjacent vertices of G_1 (see Figure 1.22(c)). Equivalently, $G_1 \square G_2$ can be constructed by placing a copy of G_1 at each vertex of G_2 and adding the appropriate edges. As expected, $G_1 \square G_2 \cong G_2 \square G_1$ for all graphs G_1 and G_2.

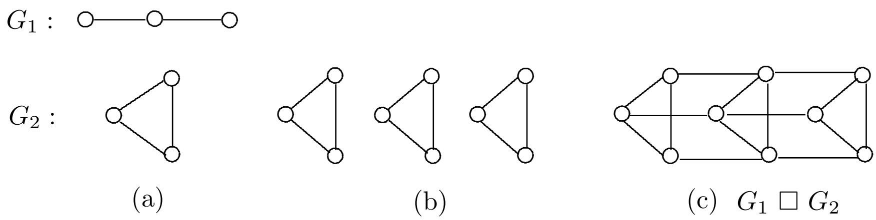


Figure 1.22: The Cartesian product of two graphs

Hypercubes

An important class of graphs is defined in terms of Cartesian products. The n**-cube** Q_n is K_2 if $n = 1$, while for $n \geq 2$, Q_n is defined recursively as the Cartesian product $Q_{n-1} \square K_2$ of Q_{n-1} and K_2. The n-cube can also be defined as that graph whose vertex set is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$ or $a_1 a_2 \cdots a_n$ where a_i is 0 or 1 for $1 \leq i \leq n$ (commonly called n**-bit strings**), such that two vertices are adjacent if and only if the corresponding ordered n-tuples differ at precisely one coordinate. The graph Q_n is an n-regular graph of order 2^n. The n-cubes for $n = 1, 2, 3$ are shown in Figure 1.23, where their vertices are labeled by n-bit strings. The graphs Q_n are also called **hypercubes**.

1.7 Degree Sequences

We saw in the First Theorem of Graph Theory (Theorem 1.4) that the sum of the degrees of the vertices of a graph G is twice the size of G and in Corollary 1.5

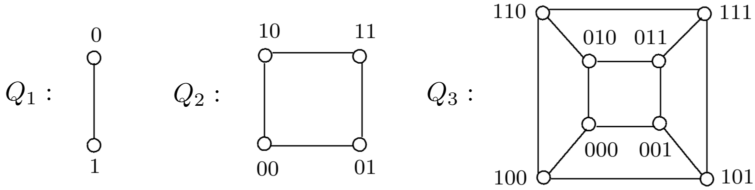


Figure 1.23: The n-cubes for $n = 1, 2, 3$

that G must have an even number of odd vertices. We have also described conditions under which a regular graph of order n can exist. We now consider the degrees of the vertices of a graph in more detail.

A sequence $d_1, d_2, \ldots, d_n$ of nonnegative integers is called a **degree sequence** of a graph G of order n if the vertices of G can be labeled $v_1, v_2, \ldots, v_n$ so that $\deg v_i = d_i$ for $1 \le i \le n$. For example, a degree sequence of the graph G of Figure 1.24 is $4, 3, 2, 2, 1$ (or $1, 2, 2, 3, 4$ or $2, 1, 4, 2, 3$, etc.). We commonly write the degree sequence of a graph as a nonincreasing sequence.

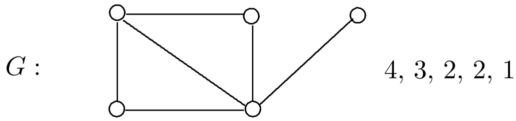


Figure 1.24: A degree sequence of a graph

A finite sequence s of nonnegative integers is a **graphical sequence** if s is a degree sequence of some graph. Thus, $4, 3, 2, 2, 1$ is graphical. There are some obvious necessary conditions for a sequence $s : d_1, d_2, \ldots, d_n$ of n nonnegative integers to be graphical. While the conditions that $d_i \le n - 1$ for all i $(1 \le i \le n)$ and $\sum_{i=1}^{n} \deg v_i$ is even are necessary for s to be graphical, they are not sufficient. For example, the sequence $3, 3, 3, 1$ satisfies both conditions but it is not graphical, for if three vertices of a graph of order 4 have degree 3 then the remaining vertex must have degree 3 as well.

It is not all that unusual for a graphical sequence to be the degree sequence of more than one graph. For example, the graphical sequence $3, 2, 2, 2, 1$ is the degree sequence of the two (non-isomorphic) graphs in Figure 1.25. On the other hand, each of the 18 graphs in Figure 1.7 has a degree sequence possessed by no other graph.

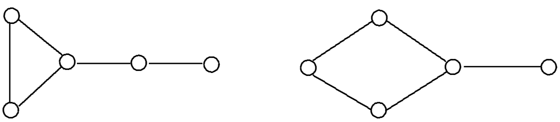


Figure 1.25: Two graphs with the same degree sequence

2-Switches

While two graphs with the same degree sequence need not be isomorphic, each can be obtained from the other by a sequence of edge shifts where at each step, two nonadjacent edges in some graph F are deleted and two nonadjacent edges in $\overline{F}$ are added to F such that the four edges involved are incident with the same four vertices.

Let H be a graph containing four distinct vertices u, v, w and x such that $uv, wx \in E(H)$ and $uw, vx \notin E(H)$. The process of deleting the edges uv and wx from H and adding uw and vx to H is referred to as a **2-switch** in H (see Figure 1.26, where a dashed line means no edge). This produces a new graph G having the same degree sequence as H. Of course, if G can be produced from H by a 2-switch, then H can be obtained from G by a 2-switch.

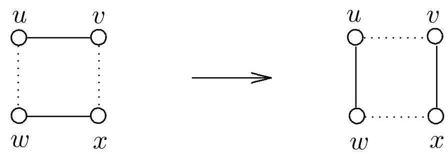


Figure 1.26: A 2-switch in a graph

Theorem 1.10 *Let $s : d_1, d_2, \ldots, d_n$ be a graphical sequence with $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n$ and let $\mathcal{G}_s$ be the set of all graphs F with degree sequence s such that $V(F) = \{v_1, v_2, \ldots, v_n\}$ where $\deg v_i = d_i$ for $1 \leq i \leq n$. Then every graph $H \in \mathcal{G}_s$ can be transformed into a graph $G \in \mathcal{G}_s$ by a sequence of 2-switches such that $N_G(v_1) = \{v_2, v_3, \ldots, v_{\Delta+1}\}$.*

Proof. Suppose that this statement is false. Let $W = \{v_2, v_3, \ldots, v_{\Delta+1}\}$. Among all graphs into which H can be transformed, let G be one for which the sum of the subscripts of the vertices in $N_G(v_1)$ is minimum. Since $N_G(v_1) \neq W$, the vertex v_1 is adjacent to a vertex v_k and is not adjacent to a vertex v_j with $j < k$ and so $d_j \geq d_k$. Consequently, there is a vertex v_ℓ such that $v_j v_\ell \in E(G)$ and $v_k v_\ell \notin E(G)$. Replacing the edges $v_1 v_k$ and $v_j v_\ell$ by $v_1 v_j$ and $v_k v_\ell$ is a 2-switch in G that produces a graph $G_1 \in \mathcal{G}_s$ for which the sum of the subscripts of the vertices in $N_{G_1}(v_1)$ is less than that of $N_G(v_1)$. Consequently, H can be transformed into G_1 and so this is a contradiction. ■

One consequence of Theorem 1.10 is that every two graphs with the same degree sequence are related in terms of 2-switches. The following theorem appeared in the book *Graphs and Hypergraphs* by Claude Berge [25].

Theorem 1.11 *If G and H are two graphs with the same degree sequence, then H can be transformed into G by a (possibly empty) sequence of 2-switches.*

Proof. We proceed by induction on the order n of G and H. If $n \leq 4$, then the result is immediate. For a given integer $n \geq 5$, assume that every two graphs of order $n-1$ with the same degree sequence can be transformed into each other by a sequence of 2-switches. Let $s : d_1, d_2, \ldots, d_n$ be a graphical sequence with $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n$ and let $\mathcal{G}_s$ be the set of all graphs F with degree sequence s such that $V(F) = \{v_1, v_2, \ldots, v_n\}$ where $\deg v_i = d_i$ for $1 \leq i \leq n$. Let $W = \{v_2, v_3, \ldots, v_{\Delta+1}\}$. Let G and H be two graphs in $\mathcal{G}_s$. By Theorem 1.10, G can be transformed into a graph $G_1 \in \mathcal{G}_s$ such that $N_{G_1}(v_1) = W$ and H can be transformed into a graph $H_1 \in \mathcal{G}_s$ such that $N_{H_1}(v_1) = W$. Since $G_1 - v_1$ and $H_1 - v_1$ are two graphs of order $n-1$ with the same degree sequence, it follows by the induction hypothesis that $G_1 - v_1$ can be transformed into $H_1 - v_1$ by a sequence of 2-switches. Hence G_1 can be transformed into H_1 by a sequence of 2-switches and so G can be transformed into H by a sequence of 2-switches. ■

The Havel–Hakimi Theorem

There are necessary and sufficient conditions for a finite sequence of nonnegative integers to be graphical. One of these is due to Václav Havel [124] and S. Louis Hakimi [117] and is a consequence of Theorem 1.10. This result is often referred to as the Havel-Hakimi Theorem, despite the fact that Havel and Hakimi gave independent proofs and wrote separate papers containing this theorem.

Theorem 1.12 **(Havel–Hakimi Theorem)** *A sequence $s : d_1, d_2, \ldots, d_n$ of nonnegative integers with $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n$ and $\Delta \geq 1$ is graphical if and only if the sequence*

$$s_1 : d_2 - 1, d_3 - 1, \ldots, d_{\Delta+1} - 1, d_{\Delta+2}, \ldots, d_n$$

is graphical.

Proof. First, assume that s_1 is graphical. Then there exists a graph G_1 of order $n-1$ such that s_1 is a degree sequence of G_1. Thus, the vertices of G_1 can be labeled as $v_2, v_3, \ldots, v_n$ so that

$$\deg_{G_1} v_i = \begin{cases} d_i - 1 & \text{if } 2 \leq i \leq \Delta + 1 \\ d_i & \text{if } \Delta + 2 \leq i \leq n. \end{cases}$$

A new graph G can now be constructed by adding a new vertex v_1 to G_1 together with the Δ edges $v_1 v_i$ for $2 \leq i \leq \Delta + 1$. Since $\deg_G v_i = d_i$ for $1 \leq i \leq n$, it follows that $s : \Delta = d_1, d_2, \ldots, d_n$ is a degree sequence of G and so s is graphical.

Conversely, let s be a graphical sequence. By Theorem 1.10, there exists a graph G of order n having degree sequence s with $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $\deg v_i = d_i$ for all i $(1 \leq i \leq n)$ where $N_G(v_1) = \{v_2, v_3, \ldots, v_{\Delta+1}\}$. Then $G - v_1$ has degree sequence s_1. ■

Theorem 1.12 actually provides us with an algorithm for determining whether a given finite sequence of nonnegative integers is graphical. If, upon repeated application of Theorem 1.12, we arrive at a sequence every term of which is 0, then the original sequence is graphical. On the other hand, if we arrive at a sequence containing a negative integer, then the given sequence is not graphical.

We now illustrate Theorem 1.12 with the sequence

$$s : 5, 3, 3, 3, 3, 2, 2, 2, 1, 1, 1.$$

After one application of Theorem 1.12 (deleting 5 from s and subtracting 1 from the next five terms), we obtain

$$s'_1 : 2, 2, 2, 2, 1, 2, 2, 1, 1, 1.$$

Reordering this sequence, we have

$$s_1 : 2, 2, 2, 2, 2, 2, 1, 1, 1, 1.$$

Continuing in this manner, we get

$$\begin{aligned}
&s'_2 : 1, 1, 2, 2, 2, 1, 1, 1, 1\\
&s_2 : 2, 2, 2, 1, 1, 1, 1, 1, 1\\
&s'_3 = s_3 : 1, 1, 1, 1, 1, 1, 1, 1\\
&s'_4 : 0, 1, 1, 1, 1, 1, 1\\
&s_4 : 1, 1, 1, 1, 1, 1, 0\\
&s'_5 : 0, 1, 1, 1, 1, 0\\
&s_5 : 1, 1, 1, 1, 0, 0\\
&s'_6 : 0, 1, 1, 0, 0\\
&s_6 : 1, 1, 0, 0, 0\\
&s'_7 = s_7 : 0, 0, 0, 0.
\end{aligned}$$

Therefore, s is graphical. Of course, if we observe that some sequence prior to s_7 is graphical, then we can conclude by Theorem 1.12 that s is graphical. For example, the sequence s_3 is clearly graphical since it is the degree sequence of the graph $G_3 = 4K_2$ in Figure 1.27. By Theorem 1.12, each of the sequences s_2, s_1 and s is also graphical. To construct a graph with degree sequence s_2, we proceed in reverse from $s'_3 = s_3$ to s_2, observing that a vertex should be added to G_3 so that it is adjacent to two vertices of degree 1. We thus obtain a graph G_2 with degree sequence s_2 (or s'_2). Proceeding from s'_2 to s_1, we again add a new vertex joining it to two vertices of degree 1 in G_2. This gives a graph G_1 with degree sequence s_1 (or s'_1). Finally, we obtain a graph G with degree sequence s by considering s'_1; that is, we add a new vertex to G_1 joining it to vertices of degrees $2, 2, 2, 2, 1$. This procedure is illustrated in Figure 1.27.

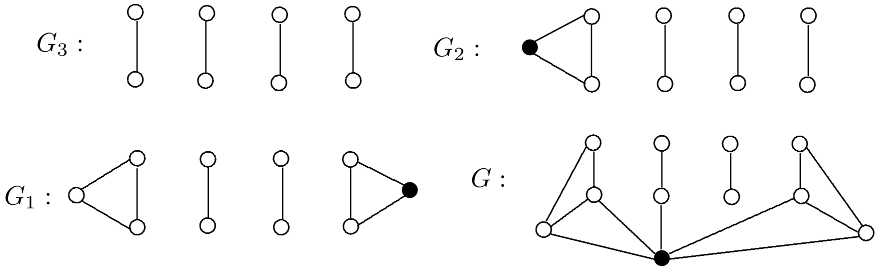


Figure 1.27: Construction of a graph G with a given degree sequence

It should be pointed out that the graph G in Figure 1.27 is not the only graph with degree sequence s. However, there are graphs that cannot be produced by the method used to construct the graph G in Figure 1.27. For example, the graph H of Figure 1.28 is such a graph.

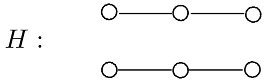


Figure 1.28: A graph that cannot be constructed by the method following Theorem 1.12

The Erdős–Gallai Theorem

Suppose that $s : d_1, d_2, \ldots, d_n$ is a graphical sequence with $d_1 \geq d_2 \geq \cdots \geq d_n$. Then there exists a graph G of order n with $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $\deg v_i = d_i$ for all i $(1 \leq i \leq n)$. Of course, the sum $\sum_{i=1}^{n} d_i$ is even. Let k be an integer with $1 \leq k \leq n-1$. Suppose that $V_1 = \{v_1, v_2, \ldots, v_k\}$ and $V_2 = \{v_{k+1}, v_{k+2}, \ldots, v_n\}$. Now the sum $\sum_{i=1}^{k} d_i$ counts every edge in $G[V_1]$ twice and counts each edge in $[V_1, V_2]$ once. The size of $G[V_1]$ is at most $\binom{k}{2} = k(k-1)/2$, while for each i $(k+1 \leq i \leq n)$ the number of edges joining v_i and V_1 is at most $\min\{k, d_i\}$. Thus,

$$\sum_{i=1}^{k} d_i \leq 2\binom{k}{2} + \sum_{i=k+1}^{n} \min\{k, d_i\} = k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}. \tag{1.1}$$

Hence, for every graphical sequence s, we must have both that $\sum_{i=1}^{n} d_i$ is even and (1.1) is satisfied for every integer k with $1 \leq k \leq n-1$. These conditions are not only necessary for a sequence of nonnegative integers to be graphical, they are sufficient as well. Since the proof of the sufficiency is technical, we omit this. This result was established by Paul Erdős and Tibor Gallai [81], who were introduced to graph theory as youngsters by Dénes König, author of the first book on graph theory. In fact, Gallai was König's only doctoral student.

Theorem 1.13 (**Erdős–Gallai Theorem**) *A sequence* $s : d_1, d_2, \ldots, d_n$ $(n \geq 2)$ *of nonnegative integers with* $d_1 \geq d_2 \geq \cdots \geq d_n$ *is graphical if and only if* $\sum_{i=1}^{n} d_i$ *is even and for each integer* k *with* $1 \leq k \leq n-1$,

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}.$$

Irregular Graphs

According to Theorem 1.7, there is an r-regular graph of order n if and only if $0 \leq r \leq n-1$ and rn is even. At the other extreme are nontrivial graphs, no two vertices of which have the same degree. A nontrivial graph G is **irregular** if $\deg u \neq \deg v$ for every two vertices u and v of G. Actually, no graph has this property.

Theorem 1.14 *No graph is irregular.*

Proof. Assume, to the contrary, that there exists an irregular graph G of order $n \geq 2$. Since the degree of every vertex of G is one of the n integers $0, 1, \ldots, n-1$, each of these integers is the degree of exactly one vertex. Thus, we may assume that $V(G) = \{v_1, v_2, \ldots, v_n\}$ where $\deg v_i = i-1$ for $1 \leq i \leq n$. Since $\deg v_1 = 0$, the vertex v_1 is isolated in G and since $\deg v_n = n-1$, it follows that v_n is adjacent to v_1. This is a contradiction. ■

By Theorem 1.14, for each integer $n \geq 2$, there is no graph of order n whose n vertices have distinct degrees. Since the degrees of the vertices of the graph G of Figure 1.24 are $4, 3, 2, 2, 1$, it is possible for $n-1$ vertices of a graph of order n to have distinct degrees. A graph G of order $n \geq 2$ is **nearly irregular** if exactly two vertices of G have the same degree.

Theorem 1.15 *For every integer* $n \geq 2$*, there are exactly two nearly irregular graphs of order* n.

Proof. First, observe that if G is a nearly irregular graph of order n, then G cannot contain a vertex of degree 0 *and* a vertex of degree $n-1$. Thus, either each vertex of G has one of the degrees $1, 2, \ldots, n-1$ or each vertex of G has one of the degrees $0, 1, \ldots, n-2$. Furthermore, if G is nearly irregular, then so is $\overline{G}$.

We show by induction that for each integer $n \geq 2$ there are exactly two nearly irregular graphs of order n. Since K_2 and $\overline{K_2}$ are nearly irregular, the result holds for $n = 2$. Assume for an integer $n \geq 3$ that there are exactly two nearly irregular graphs of order $n-1$. Necessarily, one of these graphs is a graph F with $\Delta(F) = n-2$ and $\delta(F) = 1$ while the other is $\overline{F}$ where $\Delta(\overline{F}) = n-3$ and $\delta(\overline{F}) = 0$. Then $H = F + K_1$ and $\overline{H}$ are nearly irregular graphs of order

n, where $\Delta(H) = n-2$ and $\delta(H) = 0$. We claim that these are the only nearly irregular graphs of order n. Assume, to the contrary, that there is a third graph G of order n that is nearly irregular. Then either $\Delta(G) = n-1$ or $\delta(G) = 0$, say the latter. Then $G = G_1 + K_1$, where G_1 is the nearly irregular graph of order $n-1$ with $\Delta(G_1) = n-2$. However then, $G_1 \cong H$, which is a contradiction. ∎

It follows by Theorem 1.15 then that for each integer $n \geq 2$, there exist exactly two distinct graphical sequences of length n having exactly two equal terms.

1.8 Multigraphs

There are occasions when a graph is not the appropriate structure to model a particular situation. For example, suppose that we are considering various locations in a certain community and there are roads between some pairs of locations that do not pass through any other location. Although this situation may be represented by a graph, there may be some characteristics in this network of roads that are not captured by a graph. For example, suppose that there are pairs of locations connected by two or more roads (not passing through any other location) and this information is important to us.

In the definition of a graph G, every two distinct vertices are joined by either one edge or no edge of G. There will be occasions when we will want to permit more than one edge to join two vertices. A **multigraph** is a nonempty set of vertices, every two of which are joined by a finite number of edges. Hence a multigraph H may be expressed as $H = (V, E)$, where E is a multiset of 2-element subsets of V. Two or more edges that join the same pair of distinct vertices are called **parallel edges**. The **underlying graph** of a multigraph H is that graph G for which $V(G) = V(H)$ and $uv \in E(G)$ if u and v are joined by at least one edge in H.

An edge joining a vertex to itself is called a **loop**. Structures that permit both parallel edges and loops (including parallel loops) are sometimes called **pseudographs**. For emphasis then, every two vertices of a graph are joined by at most one edge and loops are not permitted. In a multigraph, every two vertices are permitted to be joined by more than one edge but this is not required. Also, no multigraph contains a loop. In a pseudograph, every two vertices are permitted to be joined by more than one edge and loops are permitted. However, parallel edges and loops are not required in pseudographs. There are authors who refer to multigraphs or pseudographs as graphs and those who refer to what we call graphs as **simple graphs**. Consequently, when reading any material written on graph theory, it is essential that there is a clear understanding of how the term *graph* is being used. According to the terminology introduced here then, every multigraph is a pseudograph and every graph is both a multigraph and a pseudograph.

In Figure 1.29, H_1 and H_4 are multigraphs while H_2 and H_3 are pseudographs. Of course, H_1 and H_4 are also pseudographs while H_4 is the only graph in Figure 1.29. For a vertex v in a multigraph G, the **degree** $\deg v$ of v in G is the number of edges of G incident with v. In a pseudograph, each loop at a vertex contributes 2 to its degree. For the pseudograph H_3 of Figure 1.29, $\deg u = 5$ and $\deg v = 2$.

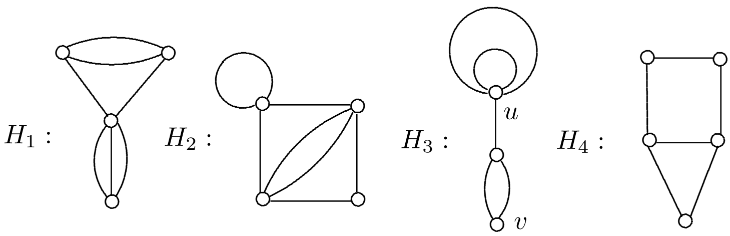


Figure 1.29: Multigraphs and pseudographs

The degree sequence of the multigraph G_1 in Figure 1.30 is $5, 4, 3$ and that of the multigraph G_2 is $4, 3, 2, 1$. That is, G_1 and G_2 are irregular multigraphs.

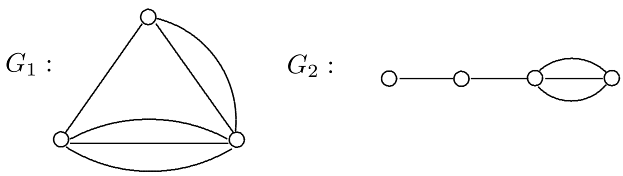


Figure 1.30: Two irregular multigraphs

The multigraphs G_1 and G_2 in Figure 1.30 illustrate the following result.

Theorem 1.16 *For every connected graph G of order at least 3, there exists an irregular multigraph whose underlying graph is G.*

Proof. Let $E(G) = \{e_1, e_2, \ldots, e_m\}$, where $e_i = u_iv_i$ for $i = 1, 2, \ldots, m$. Replacing e_i by 2^{i-1} parallel edges joining u_i and v_i produces a multigraph H. Since every two vertices of G have distinct sets of edges incident with them and every positive integer has a unique base 2 representation, their degrees in H are distinct and so H is irregular. ■

When describing walks in multigraphs or in pseudographs, it is often necessary to list edges in the sequence as well as vertices in order to specify the edges being used in the walk. For example,

$$W = (u, e_1, u, e_3, v, e_6, w, e_6, v, e_7, w)$$

is a $u - w$ walk in the pseudograph G of Figure 1.31.

$G:$

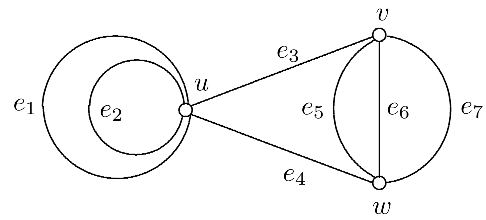


Figure 1.31: Walks in a pseudograph

Exercises for Chapter 1

Section 1.1. Graphs

1. An electronics company keeps on hand wire segments of a fixed length and of different colors for various purposes. Each wire is either colored blue (b), green (g), purple (p), red (r), silver (s), white (w) or yellow (y). The company has many wire segments of each color. All of the wire segments have been randomly stored in a large barrel. Eight handfuls of wires are removed from the barrel and each collection of wires is placed in a box. The boxes are denoted by B_i $(1 \le i \le 8)$. The colors of the wire segments in each box are:

$$\begin{array}{llll} B_1 = \{b, r\} & B_2 = \{p, r, s, w\} & B_3 = \{p, w, y\} & B_4 = \{g, r, y\} \\ B_5 = \{g\} & B_6 = \{b, g, y\} & B_7 = \{g, p, s, w, y\} & B_8 = \{s, w, y\}. \end{array}$$

   We are interested in those pairs of boxes containing at least one wire segment of the same color. Model this situation by a graph.

2. A graph $G = (V, E)$ of order 8 has the power set of the set $S = \{1, 2, 3\}$ as its vertex set, that is, V is the set of all subsets of S. Two vertices A and B of V are adjacent if $A \cap B = \emptyset$. Draw the graph G, determine the degree of each vertex of G and determine the size of G.

Section 1.2. The Degree of a Vertex

3. A graph G of order 26 and size 58 has 5 vertices of degree 4, 6 vertices of degree 5 and 7 vertices of degree 6. The remaining vertices of G all have the same degree. What is this degree?

4. A graph G has order $n = 3k + 3$ for some positive integer k. Every vertex of G has degree $k + 1$, $k + 2$ or $k + 3$. Prove that G has at least $k + 3$ vertices of degree $k + 1$ or at least $k + 1$ vertices of degree $k + 2$ or at least $k + 2$ vertices of degree $k + 3$.

5. The degree of every vertex of a graph G is one of three consecutive integers. For each degree x, the graph G contains exactly x vertices of degree x. Prove that for every graph G with this property, two-thirds of the vertices of G have odd degree.

6. Show for every positive integer k that there exists a graph G of order $2k$ containing two vertices of degree i for each $i = 1, 2, \ldots, k$.

Section 1.3. Isomorphic Graphs

7. Consider the pairs G_1, G_2 and H_1, H_2 of graphs in Figure 1.32.

(a) Determine whether $G_1 \cong G_2$.

(b) Determine whether $H_1 \cong H_2$.

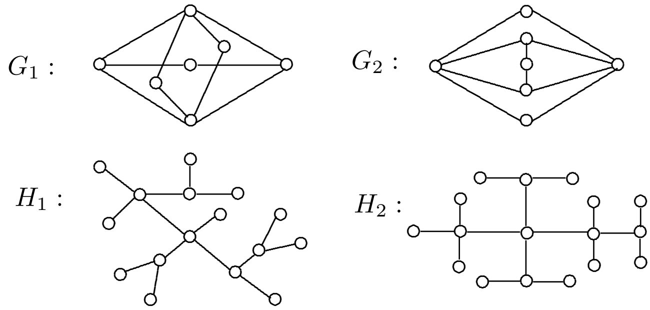


Figure 1.32: The graphs G_1, G_2, H_1, H_2 in Exercise 7

8. (a) Determine all non-isomorphic graphs of order 5.

   (b) Determine the minimum size of a graph G of order 5 such that every graph of order 5 and size 5 is isomorphic to some subgraph of G.

9. (a) Let G and H be two isomorphic graphs where one or more vertices of G (and of H) have degree r. Let S be the set of vertices of degree r in G and T be the set of vertices of degree r in H. Prove that $G[S] \cong H[T]$.

   (b) Use the result in (a) to show that the graphs G and H in Figure 1.33 are not isomorphic.

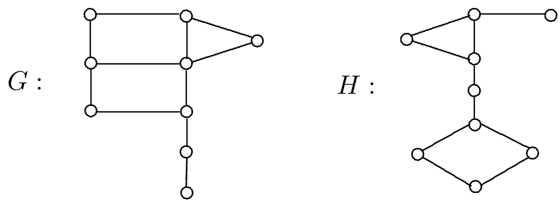


Figure 1.33: The graphs G and H in Exercise 9

10. (a) Give an example of three graphs of size 3, no two of which are isomorphic, such that in each graph, every two edge-induced subgraphs of the same size are isomorphic.

    (b) Give an example of two graphs H and G of the same order and two spanning subgraphs F_1 and F_2 of G such that $F_i \cong H$ for $i = 1, 2$ and $G - E(F_1) \not\cong G - E(F_2)$.

Section 1.4. Regular Graphs

11. Show that if G is a nonregular graph of order n and size $rn/2$ for some integer r with $1 \leq r \leq n-2$, then $\Delta(G) - \delta(G) \geq 2$.

12. For each integer $k \geq 2$, give an example of k non-isomorphic regular graphs, all of the same order and same size.

13. Give an example of a nonregular graph G containing an edge e and a vertex u such that $G - e$ and $G - u$ are both regular.

14. (a) Give an example of two non-isomorphic regular graphs G_1 and G_2 of the same order and same size such that (1) for every two vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, $G_1 - v_1 \not\cong G_2 - v_2$ and (2) there exist 2-element subsets $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$ such that $G_1 - S_1 \cong G_2 - S_2$.

    (b) Give an example of two non-isomorphic regular graphs H_1 and H_2 of the same order and same size such that (1) for every 2-element subsets $S_1 \subseteq V(H_1)$ and $S_2 \subseteq V(H_2)$, $H_1 - S_1 \not\cong H_2 - S_2$ and there exist 3-element subsets $S_1' \subseteq V(H_1)$ and $S_2' \subseteq V(H_2)$ such that $H_1 - S_1' \cong H_2 - S_2'$.

15. Prove for every graph G and every integer $r \geq \Delta(G)$ that there exists an r-regular graph containing G as an induced subgraph.

16. Let G be a graph of order n all of whose vertices have degree r, where r is a positive integer, except for exactly one vertex of each of the degrees $r-1, r-2, \ldots, r-j$, where $1 < j < r$. Show, in fact, that there exists an r-regular graph of order $2n$ containing G as an induced subgraph.

17. Let $S = \{1, 2, 3, 4, 5\}$. The vertex set of a graph G is the set of 2-element subsets of S. Two vertices of G are adjacent if the vertices are disjoint. What familiar graph is G?

18. For positive integers k and n with $n > 2k$, the graph $G_{n,k}$ is that graph whose vertices are the k-element subsets of an n-element set $S = \{1, 2, \ldots, n\}$ and where two vertices (k-element subsets) A and B are adjacent if A and B are disjoint. The graph $G_{n,k}$ is called the **Kneser graph**.

    (a) Determine the graphs $G_{6,1}$ and $G_{5,2}$.

    (b) Show that $G_{n,k}$ is an r-regular graph for some integer r.

Section 1.5 Bipartite Graphs

19. A bipartite graph G of order n has partite sets U and W where $|U| = 10$. Every vertex of U has degree 6. In W, there are four vertices of degree 2 and three vertices of degree 4. All other vertices of G have degree 8. What is n?

20. Show for each integer $n \geq 2$ that there is exactly one bipartite graph of order n having size $\lfloor n^2/4 \rfloor$.

21. Prove for a 3-partite graph of order $n = 3k$ and size m that $m \leq 3k^2$.

22. Let G be a nonempty graph with the property that whenever $uv \notin E(G)$ and $vw \notin E(G)$, then $uw \notin E(G)$. Prove that G is a complete multipartite graph.

Section 1.6. Operation on Graphs

23. Determine all bipartite graphs G such that $\overline{G}$ is bipartite.

24. Let G be a graph of odd order $n = 2k + 1 \geq 3$ for some positive integer k. Prove that if the vertices of G have exactly the same degrees as the vertices of $\overline{G}$, then G has an odd number of vertices of degree k.

25. (a) Show that there are exactly two 4-regular graphs G of order 7.

    (b) How many 6-regular graphs of order 9 are there?

26. Prove that there is no regular self-complementary graph of even order.

27. We have seen that C_5 is a self-complementary graph. Therefore, there is a regular self-complementary graph of order 5. Show that there is a regular self-complementary graph of order 5^n for every positive integer n.

28. Let G_1 and G_2 be self-complementary graphs, where G_2 has even order n. Let G be the graph obtained from G_1 and G_2 by joining each vertex of G_2 whose degree is less than $n/2$ to every vertex of G_1. Show that G is self-complementary.

29. Prove that there exists a self-complementary graph of order n for every positive integer n with $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$.

30. (a) Give an example of a graph G of order 6 and size 7 such that G is isomorphic to a subgraph H of $\overline{G}$.

    (b) Give an example of a graph G of order 7 and size 10 such that G is isomorphic to a subgraph H of $\overline{G}$.

31. Prove for every integer $n \geq 3$ that there exists a graph G of order n and size $\left\lfloor \binom{n}{2}/2 \right\rfloor$ that is isomorphic to a graph $H \subseteq \overline{G}$.

32. Let P be the Petersen graph. Show that $\overline{P}$ contains a subgraph H such that $H \cong P$.

33. For $i = 1, 2$, let u_i be a vertex in a graph G_i of order n_i and size m_i.

    (a) Determine the degree of u_1 in $G_1 + G_2$.

    (b) Determine the degree of u_1 in $G_1 \vee G_2$.

    (c) Determine the degree of (u_1, u_2) in $G_1 \,\square\, G_2$.

34. Determine the order and size of each of the graphs $P_3 \vee 2P_3$, $P_3 \,\square\, 2P_3$ and $Q_1 + Q_2 + Q_3$.

Section 1.7. Degree Sequences

35. Find a sequence of 2-switches that transforms the graph G of Figure 1.34 into the graph H.

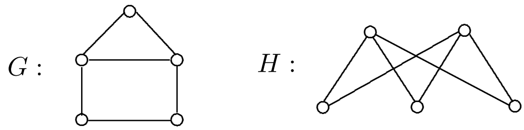


Figure 1.34: The graphs G and H in Exercise 35

36. For two pairs G_1, H_1 and G_2, H_2 of graphs shown in Figure 1.35, determine the minimum number of 2-switches required to transform

    (a) G_1 into H_1 and (b) G_2 into H_2.

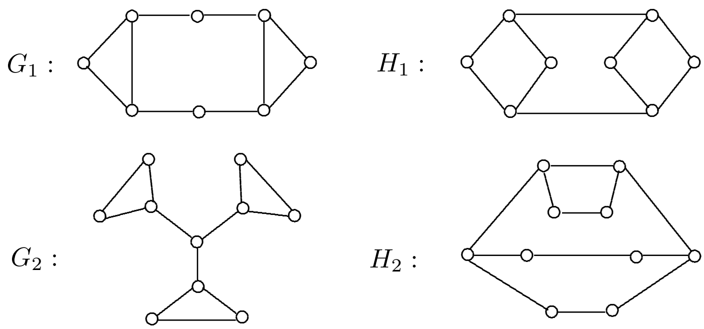


Figure 1.35: The graphs in Exercise 36

37. Let $s : 2, 2, 2, 2, 2, 2, 2, 2, 2, 2$ and let $\mathcal{G}_s$ be the set of all graphs with degree sequence s. Let G be a graph with $V(G) = \mathcal{G}_s$ where two vertices F and H in G are adjacent if F can be transformed into H by a single 2-switch. To which familiar graph is G isomorphic?

38. Give an example of a graphical sequence s (where $\mathcal{G}_s$ is the set of all graphs with degree sequence s) such that (1) the graph G has $V(G) = \mathcal{G}_s$, (2) two vertices F and H of G are adjacent if F can be transformed into H by a single 2-switch and (3) G contains a triangle.

39. Let $s : d_1, d_2, \ldots, d_n$ be a graphical sequence with $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n$. Show, for each integer k with $1 \leq k \leq n$, that there exists a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$ where $\deg v_i = d_i$ for $1 \leq i \leq n$ having the property that v_k is adjacent to either (1) the vertices of $\{v_1, v_2, \ldots, v_{d_k}\}$ if $k > d_k$ or (2) the vertices of $\{v_1, v_2, \ldots, v_{d_k+1}\} - \{v_k\}$ if $1 \leq k \leq d_k$.

40. Let G and H be two graphs that are neither empty nor complete. The graph H is said to be obtained from G by an **edge rotation** if G contains three vertices u, v, and w where $uv \in E(G)$ and $uw \notin E(G)$ and $H \cong G - uv + uw$.

    (a) Show that the graph G_2 of Figure 1.36 is obtained from G_1 by an edge rotation.

    (b) Show that G_3 of Figure 1.36 cannot be obtained from G_1 by an edge rotation.

    (c) Show that for every two nonempty, noncomplete graphs G and H of the same order and same size, there exists a sequence $G = G_0$, G_1, …, $G_k = H$ of graphs such that G_{i+1} is obtained from G_i by an edge rotation for $i = 0, 1, \ldots, k-1$.

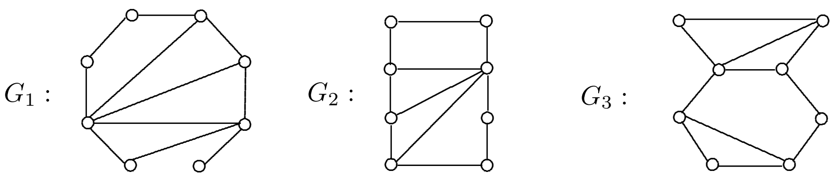


Figure 1.36: The graphs in Exercise 40

41. Determine whether the following sequences are graphical. If so, construct a graph with the appropriate degree sequence.

    (a) $4, 4, 3, 2, 1$

    (b) $3, 3, 2, 2, 2, 2, 1, 1$

    (c) $7, 7, 6, 5, 4, 4, 3, 2$

    (d) $7, 6, 6, 5, 4, 3, 2, 1$

    (e) $7, 4, 3, 3, 2, 2, 2, 1, 1, 1$.

42. Prove that a sequence $d_1, d_2, \ldots, d_n$ is graphical if and only if $n - d_1 - 1, n - d_2 - 1, \ldots, n - d_n - 1$ is graphical.

43. Prove that for every integer x with $0 \leq x \leq 5$, the sequence $x, 1, 2, 3, 5, 5$ is not graphical.

44. For which integers x $(0 \leq x \leq 7)$, if any, is the sequence $7, 6, 5, 4, 3, 2, 1, x$ graphical?

45. Use Theorem 1.13 to determine whether the sequence $s : 6, 6, 5, 4, 3, 2, 2$ is graphical.

46. We have seen that there is only one graphical sequence d_1, d_2, d_3, d_4, d_5 with $4 = d_1 \geq d_2 \geq d_3 \geq d_4 \geq d_5 = 1$ such that at least one term is 3 and at least one term is 2. How many graphical sequences d_1, d_2, d_3, d_4, d_5 are there with $4 = d_1 \geq d_2 \geq d_3 \geq d_4 \geq d_5 = 2$ such that at least one term is 3?

47. Show that for every finite set S of positive integers, there exists a positive integer k such that the sequence obtained by listing each element of S a total of k times is graphical. Find the minimum such k for $S = \{2, 6, 7\}$.

48. According to Theorem 1.15, for each integer $n \geq 2$, there exist exactly two distinct graphical sequences of length n having exactly two equal terms. What terms are equal for these two sequences?

49. Two finite sequences s_1 and s_2 of nonnegative integers are called **bigraphical** if there exists a bipartite graph G with partite sets V_1 and V_2 such that s_i lists the degrees of the vertices of G in V_i for $i = 1, 2$. Prove that the sequences $s_1 : a_1, a_2, \ldots, a_r$ and $s_2 : b_1, b_2, \ldots, b_t$ of nonnegative integers with $r \geq 2$, $a_1 \geq a_2 \geq \cdots \geq a_r$, $b_1 \geq b_2 \geq \cdots \geq b_t$, $0 < a_1 \leq t$ and $0 < b_1 \leq r$ are bigraphical if and only if the sequences $s_1' : a_2, a_3, \cdots, a_r$ and $s_2' : b_1 - 1, b_2 - 1, \ldots, b_{a_1} - 1, b_{a_1+1}, \ldots, b_t$ are bigraphical.

50. The graphs G and H of order 10 have vertex sets $V(G) = \{u_1, u_2, \ldots, u_{10}\}$ and $V(H) = \{v_1, v_2, \ldots, v_{10}\}$ and edge sets $E(G) = \{u_iu_j : i + j \geq 11\}$ and $E(H) = \{v_iv_j : i + j \geq 12\}$. How are G and H related?

51. (a) Let n be a given positive integer and let r and s be nonnegative integers such that $r + s = n$ and s is even. Give an example of a graph containing r even vertices and s odd vertices.

    (b) Determine the minimum size of a graph G containing r even vertices and s odd vertices and satisfying the properties in (a).

    (c) Determine the maximum size of a graph G containing r even vertices and s odd vertices and satisfying the properties in (a).

52. (a) Let G be a graph of order $n \geq 4$. Prove that if $\deg v \geq \frac{2n+1}{3}$ for every vertex v of G, then every edge of G belongs to a complete subgraph of order 4.

(b) Show that the result in (a) is best possible in general by showing that $\frac{2n+1}{3}$ cannot be replaced by $\frac{2n}{3}$.

Section 1.8. Multigraphs

53. We saw that the irregular multigraph G_2 in Figure 1.30 has degree sequence $4, 3, 2, 1$. Give an example of an irregular multigraph (if such a multigraph exists) having degree sequence

(a) $5, 4, 3, 2, 1$

(b) $6, 5, 4, 3, 2, 1$

(c) $7, 6, 5, 4, 3, 2, 1$.

54. Prove for every connected graph G of order $n = 3$ or $n = 4$ and size m that it is possible to label the edges of G by $e_1, e_2, \ldots, e_m$ and replace e_i by i parallel edges for each i $(1 \le i \le m)$ such that the degrees of the vertices of the resulting multigraph H are distinct.

55. Determine which of the following sequences are the degree sequences of a multigraph.

(a) $s_1 : 3, 2, 1$ (b) $s_2 : 5, 2, 1$

(c) $s_3 : 6, 4, 2$ (d) $s_4 : 3, 2, 2$

(e) $s_5 : 4, 4, 2, 2$ (f) $s_6 : 5, 3, 2, 1$

(g) $s_7 : 4, 4, 4, 4$ (h) $s_8 : 7, 5, 3, 1$.

56. Prove that a sequence $s : d_1, d_2, \ldots, d_n$ $(n \ge 1)$ of nonnegative integers with $d_1 \ge d_2 \ge \cdots \ge d_n$ is the degree sequence of a multigraph if and only if $\sum_{i=1}^{n} d_i$ is even and $d_1 \le \frac{1}{2} \sum_{i=1}^{n} d_i$.

57. Let G be a connected graph of order n where the vertices of G are labeled as $v_1, v_2, \ldots, v_n$ in some way. A multigraph H of size m with $V(H) = V(G)$ is obtained by replacing each edge v_iv_j of G by $\min\{i, j\}$ parallel edges.

(a) Find m if $G = K_5$.

(b) Find sharp upper and lower bounds for m if $G = C_5$.

(c) Find the minimum value of m if G is bipartite.

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