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The average connectivity of a graph

Lowell W. Beineke[a], Ortrud R. Oellermann[b,1], Raymond E. Pippert[a,*]

[a] *Department of Mathematics, Indiana University Purdue University Fort Wayne, Fort Wayne, IN 46805-1499, USA*

[b] *The University of Winnipeg, Winnipeg, Man., Canada R3B 2E9*

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Abstract

In this paper, we consider the concept of the average connectivity of a graph, defining it to be the average, over all pairs of vertices, of the maximum number of internally disjoint paths connecting these vertices. Our main results are sharp bounds on the value of this parameter, and a construction of graphs whose average connectivity is the same as the connectivity. Along the way, we establish some new results on connectivity. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The best known and most useful of the measures of how well a graph is connected is the *connectivity*, defined to be the minimum number of vertices in a set whose deletion results in a disconnected or trivial graph (the latter applying only to complete graphs). This parameter has been extensively studied (see [6,7]). However, since its value is based on a worst-case situation, it does not always reflect what happens throughout the whole of the graph. Recent interest in the vulnerability and reliability of networks (transportation, communication, computer) has given rise to a host of other measures, some of which are more global in nature; see, for example, [1,7].

In this paper, we investigate the average connectivity, a new measure of global connectedness. Whereas other global parameters, such as toughness and integrity, are NP-hard computationally, the average connectivity can be computed in polynomial time

* Corresponding author.

E-mail addresses: beineke@ipfw.edu (L.W. Beineke), ortrud.oellermann@uwinnipeg.ca (O.R. Oellermann), pippert@ipfw.edu (R.E. Pippert).

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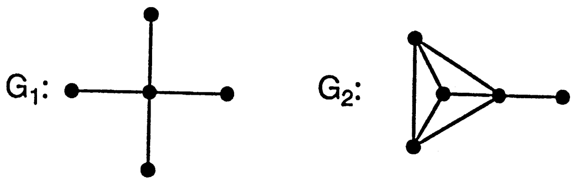

Fig. 1.

(see, for example, [2]), making it much more attractive for applications. Given that the average connectivity has these properties, it is surprising that such a natural measure has been overlooked thus far.

Other average parameters have been found to be more useful in some circumstances than the corresponding measures based on worst case situations. For example, the average distance between vertices in a graph was introduced as a tool in architecture and later turned out to be more valuable than the diameter when analyzing transportation networks. (For an excellent survey of this topic, see [8].)

Menger's classical theorem tells us that in a k-connected graph, every pair of vertices are joined by k internally disjoint paths. We use this idea in our definitions. Two vertices u and v in a graph G are said to be *k-connected* if there are k or more pairwise internally disjoint paths between them. The *(u,v)-connectivity* of G, denoted $\kappa_G(u,v)$, is defined to be the maximum value of k for which u and v are k-connected. It is a well-known fact that the connectivity $\kappa(G)$ equals $\min\{\kappa_G(u,v)\colon u,v \in V(G)\}$.

If the order of G is p, then the *average connectivity* of G, denoted $\bar{\kappa}(G)$, is defined to be

$$\bar{\kappa}(G) = \frac{\sum_{u,v} \kappa_G(u,v)}{\binom{p}{2}}.$$

(The expression $\sum_{u,v} \kappa_G(u,v)$ is sometimes referred to as the *total connectivity* of G.) In contrast to the connectivity, which gives the smallest number of vertices whose failure disconnects some pair of vertices, the average connectivity gives the expected number of vertices that must fail in order to disconnect an arbitrary pair of nonadjacent vertices.

As examples, we consider the two graphs in Fig. 1. Both of these graphs have connectivity 1, but the second would be a more reliable communication network than the first. This is reflected in the average connectivity since $\bar{\kappa}(G_1) = 1$ and $\bar{\kappa}(G_2) = 2.2$.

Clearly, for any graph $G, \bar{\kappa}(G) \geqslant \kappa(G)$. We also observe that

(i) $\bar{\kappa}(G) = 0$ if and only if G is a null graph (that is, has no edges);
(ii) if G is connected, $\bar{\kappa}(G) = 1$ if and only if G is a non-trivial tree;
(iii) if G has order p, then $\bar{\kappa}(G) \leqslant p - 1$, with equality if and only if G is complete.

The two graphs in Fig. 1 have the same number of vertices as well as the same connectivity, but not the same number of edges. One might expect that the difference

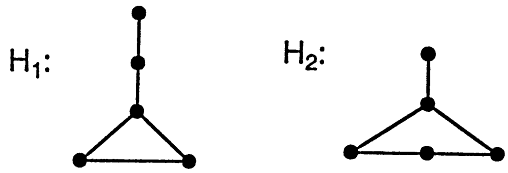

Fig. 2.

in the average connectivity is due primarily to the increased number of edges, but this is not the case. In Fig. 2 we show two graphs with the same numbers of vertices and edges (in fact, the same degrees of the vertices), but $\bar{\kappa}(H_1) = 1.3$ while $\bar{\kappa}(H_2) = 1.6$. It is not difficult to construct two graphs of the same order where the one with the fewer edges has the greater average connectivity.

The primary goals of this paper are to derive bounds on the average connectivity of a graph (Section 2) and to study graphs for which the average connectivity equals the connectivity (Section 3).

2. Bounds

As noted above, the connectivity is a lower bound for the average connectivity, and in the next section, we investigate graphs in which this bound is sharp. Such graphs are "uniformly connected", in that all pairs of vertices have the same connectivity. In this section, we study upper bounds for the average connectivity.

Taking a cue from the relationship between the independence number and the average distance in a graph [3], we establish a relationship between the independence number and the average connectivity.

Theorem 2.1. *Let G be a graph of order p and independence number β. Then*

$$\bar{\kappa}(G) \leqslant \left[(p-1)\binom{p-\beta}{2} + (p-\beta)\binom{\beta}{2} + (p-\beta)^2\beta\right] \Big/ \binom{p}{2}.$$

Proof. Let G be a graph with p vertices and independence number β, and let S be a set of β independent vertices. The connectivity between any pair of vertices in G is at most $p-1$, so the contribution to the total connectivity of G of the pairs of vertices not in S is bounded by $(p-1)\binom{p-\beta}{2}$. On the other hand, if u or v (or both) is in S, then $\kappa_G(u,v) \leqslant p-\beta$, so such pairs contribute at most $(\binom{\beta}{2} + \beta(p-\beta))(p-\beta)$ to the total connectivity. Addition of these two contributions gives the desired result. □

It is easily checked that for given p and β with $\beta \leqslant p$, the join $K_{p-\beta} + \overline{K}_\beta$ attains the bound for the average connectivity given in Theorem 2.1. It can be seen from the proof that these are the only graphs that do.

Since the average connectivity is generally greater than the connectivity, it is natural to ask for the maximum average connectivity among graphs of order p and connectivity k. However, it is not difficult to see that the graph obtained from K_p by deleting $p-k-1$ edges at one vertex has connectivity k and average connectivity $p-4+(2k+4)/p$. Thus, without a restriction on the number of edges, the average connectivity can be quite large. Consequently, a more practical question is what is the largest possible average connectivity of a network with p vertices and q edges. To this end, we begin with a bound in terms of the degrees of a graph.

Theorem 2.2. *Let G be a graph with degree sequence $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p$. Then $\bar{\kappa}(G) \leqslant [\sum_{i=1}^{p}(i-1)d_i]/\binom{p}{2}$.*

Proof. Let $v_1, v_2, \ldots, v_p$ be the vertices of G and let $d_i = \deg v_i$, $1 \leqslant i \leqslant p$. Since $d_i \geqslant d_j$ if $i < j$, it follows that the connectivity between v_i and v_j is at most d_j. Hence, the total connectivity of G is bounded by $d_2 + 2d_3 + \cdots + (p-1)d_p$, and the theorem follows. □

Corollary 2.3. *If G is a graph with p vertices and q edges and degree sequence $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p$, then*

$$\bar{\kappa}(G) \leqslant \frac{2q}{p} - \sum_{i=1}^{p} \frac{(p-2i+1)}{p(p-1)} d_i.$$

Proof. This follows from Theorem 2.2 and the fact that $\sum_{i=1}^{p} d_i = 2q$. □

The next corollary is an important step in determining the greatest average connectivity in a graph with given numbers of vertices and edges.

We observe that its proof also implies that the average connectivity is bounded by the average degree just as the connectivity is bounded by the minimum degree.

Corollary 2.4. *Let G be a graph on p vertices and q edges with $q \geqslant p$, and let $r = 2q - p\lfloor 2q/p \rfloor$. Then*

$$\bar{\kappa}(G) \leqslant \frac{2q}{p} - \frac{r(p-r)}{p(p-1)}.$$

Proof. Let $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p$ be the degrees of a graph G having p vertices and q edges. Further, let $r = 2q - p\lfloor 2q/p \rfloor$ and $D = \sum_{i=1}^{p}(p-2i+1)d_i$. Note that for $j = p-i+1$ and $i \leqslant \lceil p/2 \rceil$, $(p-2i+1)d_i + (p-2j+1)d_j \geqslant 0$ since $d_i \geqslant d_j$. Consequently, $D \geqslant 0$. Moreover, D attains its minimum when the degrees are as nearly equal as possible. If $d = \lfloor 2q/p \rfloor$ (so $r = 2q - pd$), this occurs when $d_1 = \cdots = d_r = d+1$

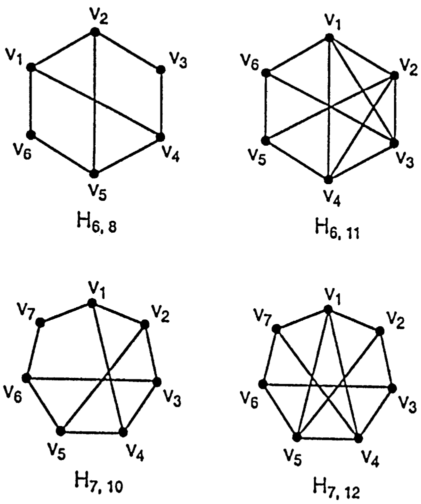

Fig. 3.

and $d_{r+1} = \cdots = d_p = d$. Hence,

$$D \geqslant \sum_{i=1}^{r} (p - 2i + 1)(d + 1) + \sum_{i=r+1}^{p} (p - 2i + 1)d$$

$$= \sum_{i=1}^{p} (p - 2i + 1)d + \sum_{i=1}^{r} (p - 2i + 1)$$

$$= r(p - r),$$

upon simplification. By Corollary 2.3, $\bar{\kappa}(G) \leqslant 2q/p - D/(p(p-1))$, and the result follows. □

We now show that the bound in Corollary 2.4 is sharp. That is, if p and q are positive integers satisfying $p \leqslant q \leqslant \binom{p}{2}$ and if $r = 2q - p\lfloor 2q/p \rfloor$, there is a graph $H_{p,q}$ having p vertices and q edges such that $\bar{\kappa}(H_{p,q}) = 2q/p - r(p-r)/(p(p-1))$. Our construction is based for the most part on the Harary graphs [4].

Let $d = \lfloor 2q/p \rfloor$ so that $r = 2q - dp$ and $0 \leqslant r < p$ as in the proof of the corollary. We consider four cases, determined by the parities of p and d. In each case, we start with the $\lfloor d/2 \rfloor$th power of a p-cycle $C_p = v_1 v_2 \cdots v_p v_1$, which we denote by $F_{p,q}$. For convenience, we show in Fig. 3 the four graphs $H_{6,8}, H_{6,11}, H_{7,10}$ and $H_{7,12}$, which correspond to the four cases (since, respectively, $d = 2$, 3, 2, and 3).

Case 1: p and d are both even. Then r is also even. Form $H_{p,q}$ by adding to $F_{p,q}$ the $r/2$ "diameter edges" $v_i v_{i+(p/2)}$ for $i = 1, 2, \ldots, r/2$.

Case 2: p is even and d is odd. Again r is even. Add to $F_{p,q}$ all $p/2$ of the edges $v_i v_{i+(p/2)}$ along with the $r/2$ edges $v_i v_{i+(p/2)-1}$ for $i = 1, 2, \ldots, r/2$.

Case 3: p is odd and d is even. Again r is even. Add to $F_{p,q}$ the $r/2$ edges $v_i v_{i+(p+1)/2}$ for $i = 1, 2, \ldots, r/2$.

Case 4: p and d are both odd. Note that $d \leqslant p-2$. First add the $(p+1)/2$ edges $v_i v_{i+(p-1)/2}$ for $i=1,2,\ldots,(p+1)/2$. (At this stage all vertices have degree d except $v_{(p+1)/2}$ which has degree $d+1$). Finally, add edges $v_i v_{i+(p+1)/2}$ for $i=1,2,\ldots,(r-1)/2$.

In each case, the graph $H_{p,q}$ has r vertices of degree $d+1$ and the rest of degree d.

By a tedious but straightforward argument it can be shown that $H_{p,q}$ has connectivity d. Furthermore, it can also be shown that if u and v are vertices of degree $d+1$, then $\kappa(u,v)=d+1$. Consequently,

$$\bar{\kappa}(H_{p,q}) = \left(d\binom{p}{2} + \binom{r}{2}\right)\bigg/\binom{p}{2} = \left\lfloor \frac{2q}{p} \right\rfloor + \frac{r(r-1)}{p(p-1)} = \frac{2q}{p} - \frac{r(p-r)}{p(p-1)}.$$

This establishes the claim that the bound in the corollary is sharp.

Consequently, we have determined, for $q \geqslant p$, the maximum average connectivity $\mu(p,q)$ among all graphs with p vertices and q edges. We now consider the cases with $q < p$, observing firstly that for all p,

$$\mu(p,0)=0, \quad \mu(p,1)=1\bigg/\binom{p}{2}, \quad \text{and} \quad \mu(p,2)=3\bigg/\binom{p}{2}.$$

For the other combinations of p and q, we have this lemma.

Lemma 2.5. *For* $3 \leqslant q < p$, $\mu(p,q)=2q(q-1)/p(p-1)$.

Proof. Assume $3 \leqslant q < p$. We first observe that $\bar{\kappa}(C_q \cup (p-q)K_1) = 2\binom{q}{2}/\binom{p}{2}$, so that $\mu(p,q) \geqslant 2q(q-1)/p(p-1)$.

For the reverse inequality, we note that it is sufficient to show that the total connectivity of any graph G with p vertices and q edges, when $p \geqslant 4$ and $3 \leqslant q < p$, is at most $q(q-1)$. For this, we use induction on p. The result clearly holds for $p=4$ since the only graphs of order 4 that have three edges are $K_{1,3}$, P_4, and $C_3 \cup K_1$, all of which have total connectivity 6. Let $p \geqslant 4$, assume that the result holds for graphs with p vertices, and let G be a graph with $p+1$ vertices and q' edges, $q' \leqslant p$. Let d be the minimum degree of a vertex in G. Note that $d<2$ since otherwise $q' \geqslant p+1$. Let v be a vertex of degree d in G. We consider the two possible values of d separately.

Case 1: $d=0$. If $q' < p$, then it follows from the induction hypothesis that the total connectivity of $G-v$, and hence that of G, is at most $q'(q'-1)$. If $q'=p$, the same conclusion follows from Corollary 2.4.

Case 2: $d=1$. In this case, $G-v$ has p vertices and $q'-1$ edges, so by the induction hypothesis, it has total connectivity at most $(q'-1)(q'-2)$. Since $\deg v=1, \kappa_G(v,w)$ is at most 1 for any other vertex w, and equals 1 for at most $q'-1$ vertices. Hence, G has total connectivity at most $(q'-1)(q'-2)+q'-1 < q'(q'-1)$.

The result now follows by induction. $\square$

Theorem 2.6. *The maximum average connectivity in a graph having p vertices and q edges is*, *with* $r=2q-p\lfloor 2q/p\rfloor$,

$$\mu(p,q)=\begin{cases} 0 & \text{if } q=0,\\ \dfrac{2}{p(p-1)} & \text{if } q=1,\\ \dfrac{6}{p(p-1)} & \text{if } q=2,\\ \dfrac{2q(q-1)}{p(p-1)} & \text{if } 3\leqslant q\leqslant p-1,\\ \dfrac{2q(p-1)-r(p-r)}{p(p-1)} & \text{if } q\geqslant p\geqslant 4.\end{cases}$$

Proof. This follows from Lemma 2.5, Corollary 2.4, and the comments following that corollary. □

3. Uniformly connected graphs

In this section, we consider graphs in which the connectivity between all pairs of vertices is the same, a desirable feature of a network. A graph G is therefore defined to be *uniformly k-connected* if $\kappa(u,v)=k$ for all pairs of vertices u and v; that is if $\bar{\kappa}(G)=\kappa(G)=k$. After showing that except for connectivities 0 and 1 all uniformly connected graphs are also both critically and minimally connected, we present some algorithms for constructing such graphs. It follows from our definition and Menger's theorem that a graph is uniformly k-connected if and only if it is k-connected and has no subgraph homeomorphic to $k_{1,1,k}$.

For low connectivity, it is thus easy to see that a graph is

(i) uniformly 0-connected if and only if it has no edges,
(ii) uniformly 1-connected if and only if it is a tree, and
(iii) uniformly 2-connected if and only if it is a cycle.

At the other extreme, a graph of order p is

(iv) uniformly $(p-1)$-connected if and only if it is complete, and
(v) uniformly $(p-2)$-connected if and only if it is the result of removing $\lfloor p/2\rfloor$ independent edges from K_p.

Clearly, any k-regular k-connected graph is uniformly k-connected, so for example the k-dimensional cube Q_k and the regular complete bipartite graph $K_{k,k}$ are both uniformly k-connected. Furthermore, it is straightforward to verify that the join K_1+G of K_1 with a k-regular k-connected graph G is uniformly $(k+1)$-connected; this is illustrated by the fact that wheels are uniformly 3-connected.

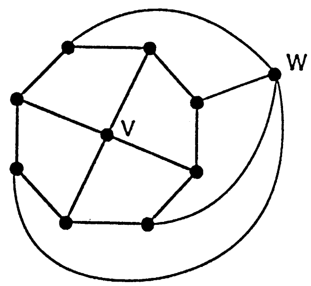

Fig. 4.

In the literature (see, for example [7]), a graph G having connectivity $k>0$ is called *critically k-connected* if $\kappa(G-v)<k$ for every vertex v, and *minimally k-connected* if $\kappa(G-e)<k$ for every edge e.

Theorem 3.1. *Let G be a uniformly k-connected graph.*

(a) *If $k \geqslant 1$, then G is minimally k-connected.*
(b) *If $k \geqslant 2$, then G is critically k-connected.*

Proof. Let G be uniformly k-connected.

(a) Assume $k \geqslant 1$, and let $e=vw$ be an edge of G. Suppose that $\kappa(G-e) \geqslant k$. Then in $G-e$, there are k internally disjoint v–w paths. These together with the path vw show that $\kappa_G(v,w) \geqslant k+1$. This contradicts the fact that G is uniformly k-connected, so G must be minimally k-connected.

(b) Assume $k \geqslant 2$, and let v be a vertex of G. Suppose that $\kappa(G-v) \geqslant k$, and let u and w be neighbors of v. Then in $G-v$, there are k internally disjoint u–w paths; these together with the path uvw show that $\kappa_G(u,w) \geqslant k+1$. This contradicts the fact that G is uniformly k-connected, so G must be critically k-connected also. □

We note that the converse of Theorem 3.1 is not true in that a graph can be both minimally and critically k-connected, but not uniformly k-connected. For example, the graph in Fig. 4 can be seen to be both minimally and critically 3-connected, but it is not uniformly 3-connected since it contains four internally disjoint v–w paths.

Mader [5] showed that in any minimally k-connected graph the vertices of degree greater than k induce an acyclic subgraph. Thus, it follows from Theorem 3.1(a) that the same must hold in every uniformly k-connected graph.

We now proceed with the construction of uniformly k-connected graphs. We begin with a preliminary construction and a result on the connectivity of graphs resulting from this construction.

Let H and H' be disjoint graphs having, respectively, sets F and F' of t independent edges each. Let $F=\{f_1,f_2,\dots,f_t\}$ and $F'=\{f'_1,f'_2,\dots,f'_t\}$ with $f_i=v_iw_i$ and $f'_i=v'_iw'_i$ for $i=1,2,\dots,t$. Further, let D be the set of $2t$ edges $v_iv'_i$ and $w_iw'_i$ for

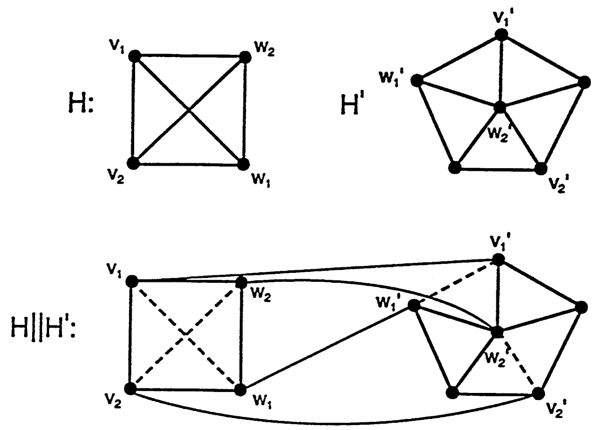

Fig. 5.

$i=1,2,\ldots,t$. The graph $G=((H\cup H')-(F\cup F'))\cup D$ is called a *t-amalgam* of H and H'. We denote it by $(H,F)\|(H',F')$, or simply by $H\|H'$ when notation for the sets F and F' is not important. An illustration for the case $t=2$ is given in Fig. 5.

Lemma 3.2. *Let H and H' be disjoint graphs with connectivities k and k' respectively. Assume that $k'\geqslant k\geqslant 2$, and let t be a positive integer with $k'\geqslant 2t$. Assume further that the edge-independence number of H and H' is at least t. If G is a t-amalgam of H and H', then any two vertices of H are k-connected in G.*

Proof. We adopt the notation of the lemma and the definition, and let x and y be two vertices in H.

Case 1: x and y are not adjacent in G. Suppose that $\kappa_G(x,y)=s<k$, and let S be an (x,y)-separating set of cardinality s. Further, let $R=S\cap V(H)$ and $r=|R|$, and define R' and r' similarly.

We first show that $H'-F'-R'$ is connected. Since x and y are in different components of $G-S$, they must be in different components of $H-F-R$, and hence $t+r\geqslant k$. Also, since $r+r'<k$, $r'<t$; and since $2t\leqslant k'$ by hypothesis, $r'+t<k'$. Consequently, $H'-F'-R'$ is connected. (We use this fact later.)

Now let C_x and C_y be the components of $H-F-R$ containing x and y, respectively. (As noted above, they are different components.) Let A be the set of edges of F having one end in C_x and the other in another component of $H-F-R$. Further, let X be the set of vertices in C_x that are on the edges in A, and let X' be the set of their neighbors in H'. In a like manner, define sets B,Y, and Y' for C_y. (See Fig. 6.).

Let $a=|A|$, whence $|X|=|X'|=a$. Observe that $H-A-R$ is disconnected, so $a+r\geqslant k$, the connectivity of H. Hence $r'<a$ (since $r+r'<k$), so there is a vertex v' in $X'-R'$. Similarly, there is a vertex w' in $Y'-R'$. Let v and w be the corresponding vertices in X and Y. Then, an x–v path in C_x, followed in succession by the edge vv', a v'–w' path in $H'-F'-R'$ (which graph we showed earlier to be connected), the

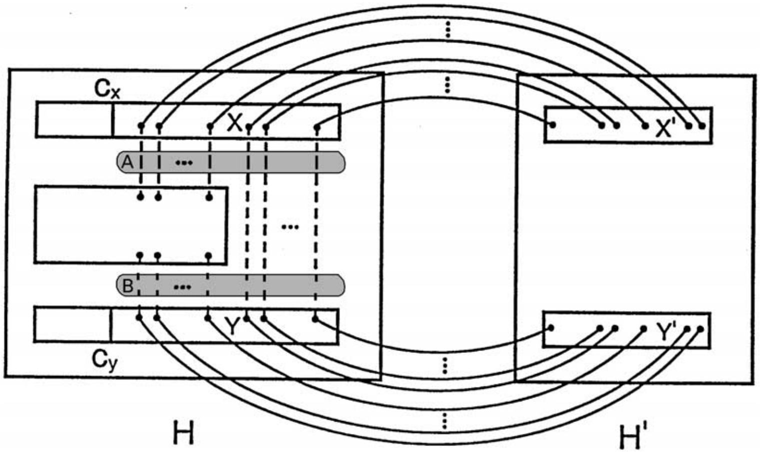

Fig. 6.

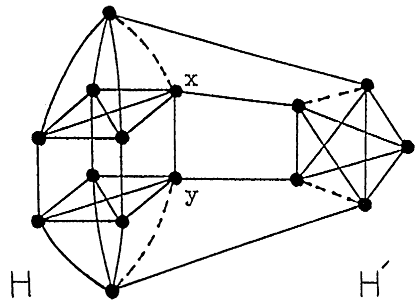

Fig. 7.

edge $w'w$, and a w–y path in C_y, is an x–y path in $G - S$. This is a contradiction to S being an (x, y)-separating set. Hence, $\kappa_G(x, y) \geqslant k$ if x and y are not adjacent.

Case 2: x and y are adjacent in G. Let $H_0 = H - xy$ and let $G_0 = G - xy$ (so G_0 is a t-amalgam of H_0 and H'). Then by Case 1, there are at least $k - 1$ internally disjoint x–y paths in G_0. These together with the edge xy show that $\kappa_G(x, y) \geqslant k$ in this case also. □

We observe that it is possible to have $\kappa_G(x, y) > \kappa_H(x, y)$ in an amalgam such as this, so the lemma cannot be extended to say that $\kappa_G(x, y) = \kappa_H(x, y)$. Fig. 7 shows such an example, with $t = 2$, $\kappa_H(x, y) = 4$ and $\kappa_G(x, y) = 5$.

Our next result is on the connectivity of pairs of vertices in certain graphs obtained from amalgams.

Lemma 3.3. *Let H and H' be disjoint k-connected graphs with, respectively, sets of independent edges F and F' of cardinality t and with sets of vertices $U = \{u_1, u_2, \ldots, u_s\}$ and $U' = \{u'_1, u'_2, \ldots, u'_s\}$ not incident with F or F'. Let G be obtained from the* *amalgam* $(H,F)\|(H',F')$ *by adding the edges* u_iu_i' *for* $i=1,2,\ldots,s$. *If* $s+2t=l$ *and* $l\leqslant k$, *then for any vertices* x *in* H *and* x' *in* H', $\kappa_G(x,x')=l$.

Proof. Clearly, for any such vertices x and x', $\kappa_G(x,x')\leqslant l$. To show the reverse inequality, we let S be the set of l vertices that are in U or are on the edges in F. Let J be obtained from H by adding a vertex z adjacent to all of the vertices in S. Since $l\leqslant k$, J is clearly l-connected. Hence, there exist l internally disjoint x–z paths in J, none of which contains an edge in F. Consequently, $H-F$ contains a set of l paths that start at x and are otherwise disjoint, and are such that each ends in a different vertex of S. There is a corresponding set of l paths in $H'-F'$ that start in U' or at the vertices on the edges in F' and end at x'. Together with the l edges between $H-F$ and $H'-F'$, these paths form a set of l internally disjoint x–x' paths in G, which proves the result. □

Corollary 3.4. *Let* $k\geqslant 2$, *let* H *and* H' *be disjoint* $(k-1)$*-regular uniformly* $(k-1)$*-connected graphs of the same order*, *and let* M *be the edges of a perfect matching between their vertex sets. Then* $G=H\cup H'\cup M$ *is a* k*-regular uniformly* k*-connected graph.*

Proof. Since G is k-regular, we need only show that $\kappa_G(u,v)\geqslant k$ for all pairs of vertices u and v. If u and v are in the same uniformly $(k-1)$-connected graph, say H, then there are $k-1$ internally disjoint u–v paths in H and there is obviously another using the matching edges of M at u and v, together with a path in H'. If u and v are in different graphs, the result follows from Lemma 3.3 (with $t=0$). □

Before making our next definition—the one on which all of our constructions are based—we observe that if $\kappa(G)\geqslant 2t$, then G contains a set of t independent edges. (This is easily proved by induction on t; later, in Lemma 3.7, we establish a much stronger result.) It follows as a consequence of this observation that one can form a $\lfloor k/2\rfloor$-amalgam from any pair of k-connected graphs.

A *k-linkage* of two graphs H and H' of connectivity k is defined as follows: if k is even, say $k=2t$, a k-linkage is just a t-amalgam of H and H'; if k is odd, say $k=2t+1$, it is a t-amalgam of H and H' with an edge ww' added, where w is a vertex of maximum degree in H not on an edge in the independent set used in forming the amalgam and w' is a corresponding vertex in H'. Note that in a graph having only one vertex w of maximum degree, this constraint precludes the choice of any edge incident with w as an element of the independent set of edges used in forming the amalgam. We shall extend the notation $\|$ that we introduced for amalgams to include linkages.

After we give one more definition, we will be able to give constructions for forming larger uniformly k-connected graphs from smaller ones. We say that a graph is a *k-optimal building block* if it is uniformly k-connected and has at most one vertex whose degree exceeds k. For clarity, we begin with a construction that uses just two building blocks.

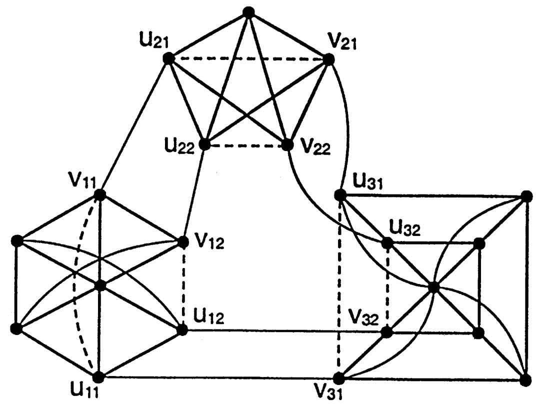

Fig. 8.

Basic Construction. *Let H and H' be k-optimal building blocks. Form a k-linkage $G = H \| H'$.*

Theorem 3.5. *The graph G obtained by the Basic Construction is uniformly k-connected.*

Proof. Let G be a k-linkage of two k-optimal building blocks H and H' and let u and v be two vertices in G. If u and v are in the same building block, then by Lemma 3.2, $\kappa_G(u,v) \geqslant k$. The reverse inequality holds since at least one of the two vertices has degree k. On the other hand, if u and v are not in the same building block, that $\kappa_G(u,v) = k$ follows from Lemma 3.3. □

Observe that by iterating this construction arbitrarily large uniformly k-connected graphs can be formed, starting with just one regular uniformly k-connected graph (such as K_{k+1} or $K_{k,k}$). However, other constructions are possible, and the next one that we give connects building blocks in a cyclic manner.

Cyclic Construction. *Let $k = 2t$, t even, and let $H_1, H_2, \ldots, H_n$ be k-optimal building blocks. Let $F_i = \{u_{i,1}v_{i,1}, \ldots, u_{i,t}v_{i,t}\}$ be a set of t independent edges in H_i for $i = 1, 2, \ldots, n$. Further, for $i = 1, 2, \ldots, n$, let M_i be the set of edges $\{v_{i,j}u_{i+1,j}$: $j = 1, 2, \ldots, t\}$ (with subscripts modulo n). Form $G_n = [(H_1 \cup H_2 \cup \cdots \cup H_n) - (F_1 \cup F_2 \cup \cdots \cup F_n)] \cup (M_1 \cup M_2 \cup \cdots \cup M_n)$.*

An example for $n = 3$ and $k = 4$ is given in Fig. 8

Theorem 3.6. *The graph G_n obtained by the Cyclic Construction is uniformly k-connected.*

Proof. We proceed by induction on n. To this end, we observe that the graph G_2 is a graph obtained by the Basic Construction, so the result holds for $n = 2$. For $n > 2$, G_n is the amalgam $(G_{n-1}, M_{n-1}) \| (H_n, F_n)$ (where G_{n-1} is the result of the Cyclic

Construction applied to $H_1, H_2, \ldots, H_{n-1}$ with independent sets $F_1, F_2, \ldots, F_{n-1}$, and M_{n-1} is the matching $\{v_{n-1,j}u_{1,j}: j = 1, 2, \ldots, t\}$). By the cyclic nature of the construction, a similar amalgam exists for each H_i.

As our induction hypothesis, we assume that G_{n-1} is uniformly k-connected. Now let x and y be two vertices in G_n. If they are in the same building block, then (as before) at most one of them has degree greater than k, so $\kappa_{G_n}(x, y) \leqslant k$; and by Lemma 3.2, they are k-connected in G_n. Thus $\kappa_{G_n}(x, y) = k$. On the other hand, if they are in different building blocks, we may assume without loss of generality that y is in H_n. By the induction hypothesis, G_{n-1} is uniformly k-connected, so by Lemma 3.3, $\kappa_{G_n}(x, y) = k$. This shows that G_n is uniformly k-connected, which completes the induction. □

For more general constructions, we require another result and an additional definition.

Lemma 3.7. *Every k-connected graph other than K_2 contains a pair of disjoint sets of $\lceil k/2 \rceil$ independent edges.*

Proof. Let G be a k-connected graph. We consider three cases, depending on the diameter of G.

Case 1: $\operatorname{diam} G \geqslant 3$. Let u and v be two vertices at distance at least 3. Then there exist k internally disjoint u–v paths. If k is even, the set of second edges of these paths is a collection of k independent edges, while if k is odd, the first and last edges of one path together with the second edges of the others form a set of $k+1$ independent edges. In either case, we easily get a pair of disjoint sets of $\lceil (k/2) \rceil$ independent edges.

Case 2: $\operatorname{diam} G = 1$. Then G is complete and has order at least $k+1$, so for $k > 1$, it clearly has disjoint sets of $\lceil k/2 \rceil$ independent edges.

Case 3: $\operatorname{diam} G = 2$. Here we use induction on k. The result obviously holds for $k = 1$ and $k = 2$. We assume that it holds for r-connected graphs with $r < k$, and let G be a graph of connectivity k and diameter 2. Let u and v be vertices at distance 2. Then $G - \{u, v\}$ is $(k-2)$-connected, so it contains a disjoint pair of sets S_1 and S_2 of $\lceil (k-2)/2 \rceil$ independent edges each, either by the induction hypothesis or one of the previous cases. Since u and v have at least k common neighbors, it follows that some common neighbor x is not on an edge in S_1 or S_2. Consequently, $S_1 \cup \{ux\}$ and $S_2 \cup \{vx\}$ are disjoint sets of $\lceil k/2 \rceil$ independent edges. The result follows by induction. □

Motivated by Lemma 3.7 and our ultimate goal, we make one further definition. Let $k \geqslant 2$, let H be a k-optimal building block, and let $t = \lfloor k/2 \rfloor$. A *family of complementary t-sets* in H is defined as follows: If k is even, it is a collection of pairwise disjoint sets of t independent edges in H; if k is odd, it is a collection of such sets of t edges, each set augmented by the same vertex of maximum degree in H not on any of the edges in the sets.

Linear Construction. *Let $k \geqslant 3$, let $H_1, H_2, \ldots, H_n$ be k-optimal building blocks, and let $t = \lfloor k/2 \rfloor$. Let E_i and F_i be a pair of complementary t-sets in H_i. Define graphs*

$G_1, G_2, \ldots, G_n$ recursively as the following k-linkages:

$$G_1 = H_1, \text{ and for } i \geqslant 1, \quad G_{i+1} = (G_i, F_i) \parallel (H_{i+1}, E_{i+1}).$$

We note that Lemma 3.7 guarantees that the Linear Construction can be effected.

Theorem 3.8. *The graph G_n obtained by the Linear Construction is uniformly k-connected.*

Proof. The proof, by induction on n, follows along the lines of that of Theorem 3.5. □

Our next result shows that as long as there are enough pairwise disjoint sets of independent edges, the linkages can be made in a more general way.

Tree-like Construction. *Let $k \geqslant 3$ and $t = \lfloor k/2 \rfloor$. Further, let $H_1, H_2, \ldots, H_n$ be k-optimal building blocks, and assume that for $i = 1, 2, \ldots, n$, $\{E_{i,1}, E_{i,2}, \ldots, E_{i,m_i}\}$ is a family of complementary t-sets in H_i. Define graphs $G_1, G_2, \ldots, G_n$ recursively as follows*:

$G_1 = H_1$, and for $i \geqslant 1$, $G_{i+1} = (G_i, F_i) \parallel (H_{i+1}, E_{i+1,1})$, where F_i is any set $E_{h,j}$ with $h \leqslant i$ and $j \leqslant m_h$ that was not already used in constructing G_i.

As in the case of the Linear Construction, the constraint $k \geqslant 3$ ensures that the Tree-like Construction is not vacuous, but can actually be carried out since each of the building blocks contains at least two complementary t-sets. In many cases a building block contains more than two, allowing a tree-like structure to appear, as illustrated in Fig. 9.

Theorem 3.9. *The graph obtained by the Tree-like Construction is uniformly k-connected.*

Proof. The proof of this theorem also follows that of Theorem 3.5. □

We note that it is important that the graphs used in the constructions be building blocks. The example following Lemma 3.2 (see Fig. 7) shows that the local connectivity of two vertices of degree greater than k in the same subgraph can increase through a linkage, so each subgraph should have at most one vertex of degree greater than the connectivity. This is also the reason for the constraint on the augmenting vertex in the t-sets. Of course, this does not mean that uniformly k-connected graphs have only one such vertex, since some of our constructions yield uniformly k-connected graphs having arbitrarily many vertices of arbitrarily high degree.

While these constructions yield infinite families of uniformly k-connected graphs, they by no means exhaust the possibilities. Clearly, additional uniformly k-connected graphs can be constructed by judicious use of more than one of the construction

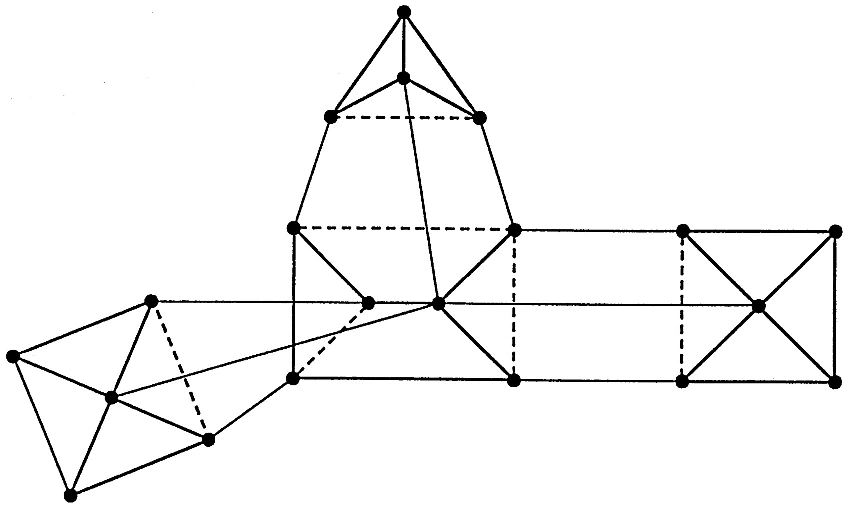

Fig. 9.

methods to obtain, say, graphs with tree-like structures appended to vertices of a cyclic structure.

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