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Brief paper

Robust model predictive control of constrained linear systems with bounded disturbances[☆]

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Abstract

This paper provides a novel solution to the problem of robust model predictive control of constrained, linear, discrete-time systems in the presence of bounded disturbances. The optimal control problem that is solved online includes, uniquely, the initial state of the model employed in the problem as a decision variable. The associated value function is zero in a disturbance invariant set that serves as the 'origin' when bounded disturbances are present, and permits a strong stability result, namely robust exponential stability of the disturbance invariant set for the controlled system with bounded disturbances, to be obtained. The resultant online algorithm is a quadratic program of similar complexity to that required in conventional model predictive control.
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1. Introduction

Model predictive control is widely employed for the control of constrained systems and an extensive literature on the subject exists some of which is reviewed in Bemporad and Morari (1999); Allgöwer, Badgwell, Qin, Rawlings, and Wright (1999); Mayne, Rawlings, Rao, and Scokaert (2000). Several methods for achieving robustness have been considered. The simplest is to ignore the disturbance and rely on the inherent robustness of deterministic model predictive control applied to the nominal system (Scokaert & Rawlings, 1995; Marruedo, Álamo, & Camacho, 2002). *Open-loop* model predictive control that determines the current control action by solving on-line an optimal control problem in which the decision variable is a sequence

$\{u_0, u_1, \ldots, u_{N-1}\}$ of control *actions* was proposed in Zheng and Morari (1993); this method cannot contain the 'spread' of predicted trajectories resulting from disturbances. Hence *feedback* model predictive control in which the decision variable is a *policy* π, which is a sequence $\{\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$ of control *laws*, was advocated in, for example, Mayne (1995); Kothare, Balakrishnan, and Morari (1996); Lee and Yu (1997); Scokaert and Mayne (1998); De Nicolao, Magni, and Scattolini (2000); Magni, Nijmeijer, and van der Schaft (2001); Magni, De Nicolao, Scattolini, and Allgöwer (2003). Determination of a control policy is usually prohibitively difficult so research has concentrated on various simplifying approximations (Mayne, 1995; Kothare et al., 1996; Scokaert & Mayne, 1998; Park & Kwon, 1999; Kouvaritakis, Rossiter, & Schuurmans, 2000; Schuurmans & Rossiter, 2000; Lee & Kouvaritakis, 2000; Mayne & Langson, 2001; Chisci, Rossiter, & Zappa, 2001; Lee, Kouvaritakis, & Cannon, 2002; Langson, Chryssochoos, Raković, & Mayne, 2004). In many papers, following (Lee & Kouvaritakis, 2000), a sub-optimal control policy in which $\mu_i(x) = v_i + Kx$ is employed.

An important problem that arises in robust model predictive control when the disturbances are merely bounded is that asymptotic stability of the origin cannot be established; the best that can be achieved is robust asymptotic stability of a set Z. It is then useful to have a Lyapunov function that is zero in this set (since the value of the function cannot be guaranteed to decrease in this set); the set Z may be regarded as the 'origin' for the uncertain system. This requirement on the Lyapunov function has been addressed in various ways in the literature. In Scokaert and Mayne (1998) a terminal cost that is zero and a stage cost that is zero in Z and strictly positive outside was proposed; the value function for the resultant optimal control problem is zero in the set Z. A disadvantage of this method is the discontinuous nature of the stage cost. An improved proposal (Kerrigan & Maciejowski, 2003) uses a continuous stage cost and zero terminal cost but requires solution of a quadratic program to evaluate the stage cost. An alternative approach (Lee & Kouvaritakis, 2000) is to use a stage cost $|v_i|^2$ (when the suboptimal policy π in which $\mu_i(x) = v_i + Kx$ is employed) and a terminal cost of zero. With this approach, the stage cost does not involve the state, which is restrictive; attractivity, but not Lyapunov stability, of the set Z has been established.

In this paper, we adopt a novel approach to obtain a suitable Lyapunov function that has value zero in the set Z. The novelty is due to the fact that, in the proposed procedure, the *initial state* of the nominal (or reference) model employed in the optimal control problem solved online is a decision variable in addition to the usual control sequence. The incorporation of the initial state yields several advantages: firstly, the value function is zero in Z, facilitating the proof of both robust attractivity and robust stability of Z; secondly, the effective terminal set for the 'tube' of trajectories emanating from the initial state in the uncertain case is not restricted, as in prior literature, to be Z which is analogous to a terminal set $\{0\}$ in the deterministic case, a restrictive requirement; thirdly it is possible to establish that the set Z is robustly *exponentially* stable for the controlled system.

Some preliminary results are presented in Sections 2 and 3. The new model predictive controller for bounded disturbances is presented in Section 4 where robust exponential stability of the set Z is proved. Some conclusions are drawn in Section 5. An extension of these results to the case when the disturbance is exponentially decaying can be found in Mayne, Seron, and Raković (2003).

Notation: Given two sets A, B, then $A \oplus B \triangleq \{a + b \mid a \in A, \ b \in B\}$ (set addition) and $A \ominus B \triangleq \{a \mid a \oplus B \subseteq A\}$ (set subtraction). A set Z is disturbance (or robustly positive) invariant for the discrete-time system $x^+ = f(x, w)$ where $w \in W$ if $f(x, w) \in Z$ for all $x \in Z$, $w \in W$. The distance of a point z from a set X is denoted by $d(z, X) \triangleq \inf\{|z - x| \mid x \in X\}$. The origin is exponentially stable (Lyapunov stable and exponentially attractive) for $x^+ = f(x)$ with a region of attraction X_N if there exists a $c > 0$ and a $\gamma \in (0, 1)$ such that any solution $x(\cdot)$ of $x^+ = f(x)$ with initial state $x(0) \in X_N$ satisfies $|x(i)| \leqslant c\gamma^i |x(0)|$ for all $i \geqslant 0$. A set Z is

robustly exponentially stable for $x^+ = f(x, w)$, $w \in W$, with a region of attraction X_N if there exists a $c > 0$ and a $\gamma \in (0, 1)$ such that any solution $x(\cdot)$ of $x^+ = f(x, w)$ with initial state $x(0) \in X_N$, and admissible disturbance sequence $w(\cdot)$ ($w(i) \in W$ for all $i \geqslant 0$) satisfies $d(x(i), Z) \leqslant c\gamma^i d(x(0), Z)$ for all $i \geqslant 0$.

2. The uncertain system

The discrete-time system to be controlled is described by

$$x^+ = Ax + Bu + w, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the control, and $w \in \mathbb{R}^n$ is a bounded disturbance; x^+ denotes the successor state. The system is subject to state and control constraints

$$u \in \mathbb{U}, \quad x \in \mathbb{X}, \tag{2}$$

where $\mathbb{U} \subset \mathbb{R}^m$ is compact, $\mathbb{X} \subset \mathbb{R}^n$ is closed, and each set contains the origin in its interior. The disturbance w is assumed to be bounded, i.e.

$$w \in W, \tag{3}$$

where W is compact and contains the origin (but may not have an interior). Let $\mathbf{u}$ denote the control *sequence* $\{u(0), u(1), \dots, u(N - 1)\}$ and $\mathbf{w}$ the disturbance sequence $\{w(0), w(1), \dots, w(N - 1)\}$. The solution of (1) at time i when the initial state is x (at time 0), and the control and disturbance sequences are, respectively, $\mathbf{u}$ and $\mathbf{w}$ is $\phi(i; x, \mathbf{u}, \mathbf{w})$. Let the *nominal (reference)* system corresponding to (1) be defined by

$$x^+ = Ax + Bu \tag{4}$$

and let $\bar{\phi}(i; x, \mathbf{u})$ denote the solution of (4) at time i when the initial state is x and the control sequence is $\mathbf{u}$.

Suppose $K \in \mathbb{R}^{m \times n}$ is such that $A_K \triangleq A + BK$ is stable. Let Z be a disturbance invariant set for the controlled uncertain system $x^+ = A_K x + w$ satisfying, therefore

$$A_K Z \oplus W \subseteq Z, \tag{5}$$

where $\oplus$ denotes Minkowski set addition. To reduce conservativeness, it is desirable that Z be as small as possible. The minimal disturbance invariant set (Kolmanovsky & Gilbert, 1998) is $\sum_{i=0}^{\infty} A_K^i W$ (with Σ denoting here set addition) but this set is not necessarily polytopic unless controller $u = Kx$ is deadbeat (Mayne & Schroeder, 1997) or if K is such that $A_K^s = \beta I$ for some finite integer s and $\beta \in (0, 1)$ (Kolmanovsky & Gilbert, 1998); however, a polytopic, disturbance invariant, outer approximation of the minimal disturbance set may be computed (Raković, Kerrigan, Kouramas, & Mayne, 2004). The set Z serves as the 'origin' for the perturbed system. We require the following elementary result (Mayne & Langson, 2001) which we use in Section 4 to define the new controller:

Proposition 1. *Suppose Z is disturbance invariant for $x^+ = A_K x + w$. If $x \in \bar{x} \oplus Z$ and $u = \bar{u} + K(x - \bar{x})$, then $x^+ \in \bar{x}^+ \oplus Z$ for all $w \in W$ where $x^+ = Ax + Bu + w$ and $\bar{x}^+ = A\bar{x} + B\bar{u}$.*

Proposition 1 states the feedback policy $u(i) = \bar{u}(i) + K(x(i) - \bar{x}(i))$ keeps the states $x(i)$ of the uncertain system $x^+ = Ax + Bu + w$ close to the states $\bar{x}(i)$ of the nominal system $\bar{x}^+ = A\bar{x} + B\bar{u}$ (for all $\bar{u}(\cdot)$, $x(i) \in \bar{x}(i) \oplus Z$ if $x(0) \in \bar{x}(0) \oplus Z$).

3. Preliminary results

The model predictive controller we propose uses the solution of an optimal control problem in which the initial state of the nominal (reference) model is a decision variable. The properties of the value function for this new problem are related to the properties of a conventional optimal control problem in which **u** is the sole decision variable. Hence, in this section we recall some relevant results for the conventional problem.

The conventional optimal control problem $\mathbb{P}_N(x)$, in which x is the current state, has no uncertainties and has constraints that are tighter than the original (defined in (2)) precisely in order to guarantee that the final controller for the uncertain system ensures satisfaction of the original constraints. Problem $\mathbb{P}_N(x)$ is defined by

$$V_N^0(x) = \min_{\mathbf{u}} \{V_N(x, \mathbf{u}) \mid \mathbf{u} \in \mathscr{U}_N(x)\}, \tag{6}$$

$$\mathbf{u}^0(x) = \arg\min_{\mathbf{u}} \{V_N(x, \mathbf{u}) \mid \mathbf{u} \in \mathscr{U}_N(x)\}, \tag{7}$$

where the cost function $V_N(\cdot)$ is defined by

$$V_N(x, \mathbf{u}) \triangleq \sum_{i=0}^{N-1} \ell(x(i), u(i)) + V_f(x(N)). \tag{8}$$

For each i, $x(i) = \bar{\phi}(i; x, \mathbf{u})$, and $\mathscr{U}_N(x)$ is the set of control sequences satisfying the tighter control, state and terminal constraints defined by

$$u(i) \in \bar{\mathbb{U}} \triangleq \mathbb{U} \ominus KZ, \quad i \in \mathscr{I}_{N-1}, \tag{9}$$

$$x(i) \in \bar{\mathbb{X}} \triangleq \mathbb{X} \ominus Z, \quad i \in \mathscr{I}_{N-1}, \tag{10}$$

$$x(N) \in X_f \subset \mathbb{X} \ominus Z, \tag{11}$$

where $\mathscr{I}_{N-1} \triangleq \{0, 1, 2, \ldots, N-1\}$, and X_f is the terminal constraint set for $\mathbb{P}_N(x)$; X_f is assumed to have an interior. As before, $x(i)$ denotes $\bar{\phi}(i; x, \mathbf{u})$. Hence

$$\mathscr{U}_N(x) = \{\mathbf{u} \mid u(i) \in \bar{\mathbb{U}}, \ \bar{\phi}(i; x, \mathbf{u}) \in \bar{\mathbb{X}}, \ \forall i \in \mathscr{I}_{N-1},$$
$$\bar{\phi}(N; x, \mathbf{u}) \in X_f\}. \tag{12}$$

The domain of the value function $V_N^0(\cdot)$ is $\bar{X}_N$ defined by

$$\bar{X}_N \triangleq \{x \mid \mathscr{U}_N(x) \neq \emptyset\}. \tag{13}$$

It is assumed that W is small enough to ensure that $Z \subset$ interior($\mathbb{X}$) and $KZ \subset$ interior($\mathbb{U}$) (the minimal Z is proportional to W). It is also assumed (in (8)) that

$$\ell(x, u) \triangleq (1/2)[x'Qx + u'Ru], \quad V_f(x) \triangleq (1/2)x'Px, \tag{14}$$

where Q, R and P are positive definite, and that the terminal cost $V_f(\cdot)$ and the terminal constraint set X_f satisfy the usual axioms (Mayne et al., 2000, p. 797), namely:

A1 : $A_K X_f \subset X_f$, $X_f \subset \mathbb{X} \ominus Z$, $KX_f \subset \mathbb{U} \ominus KZ$,
A2 : $V_f(A_K x) + \ell(x, Kx) \leqslant V_f(x) \ \forall x \in X_f$.

The solution of $\mathbb{P}_N(x)$ yields the optimal control sequence $\mathbf{u}^0(x) \triangleq \{u_0^0(x), u_1^0(x), \ldots, u_{N-1}^0(x)\}$ and the associated optimal state sequence $\mathbf{x}^0(x) \triangleq \{x_0^0(x), x_1^0(x), \ldots, x_N^0(x)\}$, $x_0^0(x) \triangleq x$, where, for each i, $x_i^0(x) \triangleq \bar{\phi}(i; x, \mathbf{u}^0(x))$. The (implicit) model predictive control law is, therefore,

$$\kappa_N^0(x) \triangleq u_0^0(x) \tag{15}$$

so that the nominal system, under model predictive control, satisfies

$$x^+ = Ax + B\kappa_N^0(x). \tag{16}$$

This control law ensures satisfaction (by the nominal system) of the tighter constraints for all initial states in $\bar{X}_N$. It also stabilizes the nominal system; this follows from the fact that, under the assumptions made above, the value function satisfies:

$$V_N^0(Ax + B\kappa_N^0(x)) \leqslant V_N^0(x) - \ell(x, \kappa_N^0(x)), \tag{17}$$

as shown in Mayne et al. (2000). It follows from (14), (17) and **A1**–**A2**, as shown in Mayne et al. (2000), that there exist constants $c_2 > c_1 > 0$ such that, with $x^+ = Ax + B\kappa_N^0(x)$,

$$c_1|x|^2 \leqslant V_N^0(x), \ \forall x \in \bar{X}_N, \tag{18}$$

$$V_N^0(x^+) \leqslant V_N^0(x) - c_1|x|^2, \ \forall x \in \bar{X}_N, \tag{19}$$

$$V_N^0(x) \leqslant c_2|x|^2, \ \forall x \in X_f. \tag{20}$$

If $\bar{X}_N$ is bounded, which we assume, the origin is *exponentially* stable for the controlled nominal system $x^+ = Ax + B\kappa_N^0(x)$ with a region of attraction $\bar{X}_N$. It is these properties that we exploit in the derivation of a new robust controller.

4. Robust model predictive controller

Given *any* $x \in \bar{X}_N \backslash Z$, it is not necessarily true that $V_N^0(Ax + B\kappa_N^0(x) + w)$ is smaller than $V_N^0(x)$ for all $w \in W$. Hence, to establish robust asymptotic stability of Z, we propose a novel optimization problem that is solved online; the new optimization problem involves, uniquely, the initial state. This is permissible because the initial state x_0 in the optimal control problem is now not the current state x of the plant, which cannot be instantaneously changed, but a parameter of the control law. The new optimal control problem

$\mathbb{P}_N^*(x)$, solved online, is a modification of $\mathbb{P}_N(x)$ defined in Section 3; the new problem $\mathbb{P}_N^*(x)$ is defined by

$$V_N^*(x) = \min_{x_0, \mathbf{u}} \{V_N(x_0, \mathbf{u}) \mid \mathbf{u} \in \mathscr{U}_N(x_0), x \in x_0 \oplus Z\}, \quad (21)$$

$$(x_0^*(x), \mathbf{u}^*(x)) = \arg \min_{x_0, \mathbf{u}} \{V_N(x_0, \mathbf{u}) \mid \mathbf{u} \in \mathscr{U}_N(x_0),$$
$$x \in x_0 \oplus Z\}. \quad (22)$$

In problem $\mathbb{P}_N^*(x)$, the function $V_N(x, \mathbf{u})$ and the constraint set $\mathscr{U}_N(x)$ are defined as in Section 3; the only difference between $\mathbb{P}_N(x)$ and $\mathbb{P}_N^*(x)$ is that the initial state x_0 of the model employed in the latter problem is a decision variable that can be varied provided it satisfies the constraint

$$x \in x_0 \oplus Z,$$

where x is the state of the system being controlled. It is easily shown that $\mathbb{P}_N^*(x)$, like $\mathbb{P}_N(x)$, is a quadratic program. The solution of $\mathbb{P}_N^*(x)$ yields the optimal control sequence $\mathbf{u}^*(x) \triangleq \{u_0^*(x), u_1^*(x), \ldots, u_{N-1}^*(x)\}$ and the associated optimal state sequence $\mathbf{x}^*(x) \triangleq \{x_0^*(x), x_1^*(x), \ldots, x_N^*(x)\}$ where, for each $i > 0$, $x_i^*(x) \triangleq \bar{\phi}(i; x_0^*(x), \mathbf{u}^*(x))$ (the optimal initial state $x_0^*(x)$ is not necessarily equal to the current state x of the system being controlled as it is in Section 3). A pair $(x_0, \mathbf{u})$ is feasible for $\mathbb{P}_N^*(x)$ if $x \in x_0 \oplus Z$ and $\mathbf{u} \in \mathscr{U}_N(x_0)$ (i.e. $\mathbf{u}$ satisfies (9)–(11)). Motivated by the discussion following Proposition 1, we propose that control $u_0^*(x) + K(x - x_0^*(x))$ be applied to the system if its current state is x. The implicit model predictive control law $\kappa_N^*(\cdot)$, yielded by the solution of $\mathbb{P}_N^*(x)$, is, therefore

$$\kappa_N^*(x) \triangleq u_0^*(x) + K(x - x_0^*(x)). \quad (23)$$

The control applied to the system when its state is x is $\kappa_N^*(x)$ and is not necessarily equal to $u_0^*(x)$, the first element in the sequence $\mathbf{u}^*(x)$, as in conventional model predictive control. Let x be an arbitrary state in

$$X_N \triangleq \{x \mid \exists x_0 \text{ such that } x \in x_0 \oplus Z, \ \mathscr{U}_N(x_0) \neq \emptyset\},$$

the domain of the value function $V_N^*(\cdot)$. The optimal control and state trajectories $\mathbf{u}^*(x)$ and $\mathbf{x}^*(x)$ (for problem $\mathbb{P}_N^*(x)$) satisfy $u_i^*(x) \in \mathbb{U} \ominus KZ$ and $x_i^*(x) \in \mathbb{X} \ominus Z$ for all $i \in \mathscr{I}_{N-1}$; in addition, $x_N^*(x) \in X_f \subset \mathbb{X} \ominus Z$. An important consequence of the definition of the optimal control problem $\mathbb{P}_N^*(x)$ is the following result:

Proposition 2. (i) *The domain X_N of $V_N^*(\cdot)$ satisfies $X_N = \bar{X}_N \oplus Z \subseteq \mathbb{X}$, where $\bar{X}_N$ is the domain of $V_N^0(\cdot)$.* (ii) *$V_N^*(x) = V_N^0(x_0^*(x))$ and $\mathbf{u}^*(x) = \mathbf{u}^0(x_0^*(x))$ for all $x \in X_N$.* (iii) *For all $x \in Z$, $V_N^*(x) = 0$, $x_0^*(x) = 0$, $\mathbf{u}^*(x) = \{0, 0, \ldots, 0\}$, $\mathbf{x}^*(x) = \{0, 0, \ldots, 0\}$ and $\kappa_N^*(x) = Kx$.*

Proof. (i) It follows from the definitions of X_N and $\bar{X}_N$ that $X_N = \{x \mid \exists x_0 \in \bar{X}_N \text{ such that } x \in x_0 \oplus Z\} = \bar{X}_N \oplus Z$. (ii) Follows from the definitions of $V_N^*(\cdot)$ and $V_N^0(\cdot)$. (iii) Let x be an arbitrary point in Z. Since $x \in 0 \oplus Z$, it follows from

the definition of $\mathbb{P}_N^*(x)$ that the pair $(0, \mathbf{0})$ (where $\mathbf{0}$ is the zero control sequence $\{0, 0, \ldots 0\}$) is feasible for $\mathbb{P}_N^*(x)$. Hence $V_N^*(x) \leqslant V_N(0, \mathbf{0}) = 0$ which establishes the desired result. $\square$

The next result shows that the model predictive control law $\kappa_N^*(\cdot)$, unlike the nominal control law, $\kappa_N^0(\cdot)$, reduces the Lyapunov function at all $x \in X_N$ despite the additive bounded disturbance w.

Proposition 3. *Suppose $x \in X_N$ so that $(x_0^*(x), \mathbf{u}^*(x))$, where $\mathbf{u}^*(x) = \{u_0^*(x), u_1^*(x), \ldots, u_{N-1}^*(x)\}$, exists and is feasible for $\mathbb{P}_N^*(x)$; $\mathbf{x}^*(x) = \{x_0^*(x), x_1^*(x), \ldots, x_N^*(x)\}$ is the associated state trajectory (for the nominal system). Then, for all $x^+ \in Ax + B\kappa_N^*(x) \oplus W$, $(x_1^*(x), \tilde{\mathbf{u}}(x))$ with $\tilde{\mathbf{u}}(x)$ defined by*

$$\tilde{\mathbf{u}}(x) \triangleq \{u_1^*(x), u_2^*(x), \ldots, u_{N-1}^*(x), Kx_N^*(x)\},$$

is feasible for $\mathbb{P}_N^*(x^+)$, and

$$V_N^*(x^+) \leqslant V_N^*(x) - \ell(x_0^*(x), \kappa_N^0(x_0^*(x))). \quad (24)$$

Proof. The state sequences associated with $\mathbf{u}^*(x)$ and $\tilde{\mathbf{u}}(x)$ are, respectively, $\mathbf{x}^*(x)$ and $\tilde{\mathbf{x}}(x)$ where

$$\tilde{\mathbf{x}}(x) = \{x_1^*(x), x_2^*(x), \ldots, x_N^*(x), A_K x_N^*(x)\}.$$

Because $x \in x_0^*(x) \oplus Z$, it follows, from Proposition 1 and (23), that $x^+ \in x_1^*(x) \oplus Z$. Since $(x^*(x), \mathbf{u}^*(x))$ is feasible for $\mathbb{P}_N^*(x)$, constraints (9)–(11) are satisfied by $\mathbf{u}^*(x)$ and $\mathbf{x}^*(x)$. Hence constraint (9) is satisfied by the first $N - 1$ elements of $\tilde{\mathbf{u}}(x)$ and constraint (10) is satisfied by the first N elements of $\tilde{\mathbf{x}}(x)$. Since $x_N^*(x) \in X_f$, it follows from **A1** that $Kx_N^*(x) \in \mathbb{U} \ominus KZ$ and $A_K x_N^*(x) \in X_f$. Hence the last element $Kx_N^*(x)$ of $\tilde{\mathbf{u}}(x)$ satisfies constraint (9) and the last element $A_K x_N^*(x)$ of $\tilde{\mathbf{x}}(x)$ satisfies constraint (11). Thus $\tilde{\mathbf{u}}(x) \in \mathscr{U}_N(x_1^*(x))$. Since $x^+ \in x_1^*(x) \oplus Z$ and $\tilde{\mathbf{u}}(x) \in \mathscr{U}_N(x_1^*(x))$, the pair $(x_1^*(x), \tilde{\mathbf{u}}(x))$ is feasible for $\mathbb{P}_N^*(x^+)$ and $x^+ \in X_N$. To prove (24), note that $x^+ \in x_1^*(x) \oplus Z$ so that $(x_1^*(x), \mathbf{u}^0(x_1^*(x)))$ is feasible for $\mathbb{P}_N^*(x^+)$. Hence $V_N^*(x^+) \leqslant V_N^0(x_1^*(x))$. But, from (17), $V_N^0(x_1^*(x)) \leqslant V_N^0(x_0^*(x)) - \ell(x_0^*(x), \kappa_N^0(x_0^*(x)))$ since $x_1^*(x)$ is the state of the nominal system at time 1 if at time 0 its state is $x_0^*(x)$ and its control is $\kappa_N^0(x_0^*(x)) = u_0^0(x_0^*(x))$; since $V_N^*(x) = V_N^0(x_0^*(x))$, (24) follows. $\square$

The main result of this section follows.

Theorem 1. *The set Z is robustly exponentially stable for the controlled uncertain system $x^+ = Ax + B\kappa_N^*(x) + w$ where $w \in W$. The region of attraction is X_N.*

Proof. From Proposition 2, $V_N^*(x) = V_N^0(x_0^*(x))$ for all $x \in X_N$. Hence, using (18)–(20) and (24), we obtain:

$$V_N^*(x) = V_N^0(x_0^*(x)) \geqslant c_1 |x_0^*(x)|^2, \ \forall \ x \in X_N, \quad (25)$$

$$V_N^*(x^+) - V_N^*(x) \leqslant -\ell(x_0^*(x), \kappa_N^0(x_0^*(x))) \leqslant -c_1 |x_0^*(x)|^2,$$
$$\forall x \in X_N, \ \forall x^+ \in (Ax + B\kappa_N^*(x)) \oplus W, \qquad (26)$$

$$V_N^*(x) = V_N^0(x_0^*(x)) \leqslant c_2 |x_0^*(x)|^2, \ \forall x \in X_f \oplus Z. \qquad (27)$$

where $c_2 > c_1$. Let $x(i)$ denote the solution of $x^+ = Ax + B\kappa_N^*(x) + w$ given its initial state is $x(0) = x$ and disturbance is $w(\cdot)$; from Proposition 3, $x(i) \in X_N$ for all i. For all $\alpha \geqslant 0$, let $S_\alpha \triangleq \{x \mid V_N^*(x) \leqslant \alpha\}$. Then $S_0 = Z$ and there exists an $\alpha > 0$ such that $S_\alpha \subset X_f \oplus Z$. From (25)–(27), $V_N^*(x(i)) \leqslant \gamma^i V_N^*(x(0))$, $\gamma \triangleq (1 - c_1/c_2) \in (0, 1)$ and $|x_0^*(x(i))| \leqslant c\delta^i |x_0^*(x(0))|$, $\delta \triangleq \sqrt{\gamma}$, for all $x(0) = x \in S_\alpha$, some $c < \infty$. Since $\bar{X}_N$ and, hence, X_N are bounded, it follows from (26) that there exists a finite integer I such that $x(i) \in S_\alpha$ for all $i \geqslant I$, all $x \in X_N$, and every admissible disturbance $w(\cdot)$. Hence there exists a $d > c$ such that $|x_0^*(x(i))| \leqslant d\delta^i |x_0^*(x(0))|$ for all $x(0) = x \in X_N$ and every admissible disturbance $w(\cdot)$. Since $x(i) \in x_0^*(x(i)) \oplus Z$, the set Z is robustly exponentially stable for the controlled uncertain system with a region of attraction X_N.

4.1. Illustrative example

The numerical example is control of the constrained, sampled double integrator defined by

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u + w.$$

The state constraints are $x \in \mathbb{X} \triangleq \{x \mid [0 \ 1]x \leqslant 2\}$; the control constraint is $u \in \mathbb{U} \triangleq \{u \mid |u| \leqslant 1\}$ and the disturbance is bounded: $w \in W \triangleq \{w \mid \|w\|_\infty \leqslant 0.1\}$. The cost function is defined by (8) and (14) with $Q = I$, $R = 0.01$; the terminal cost $V_f(x)$ is the value function $(1/2)x'Px$ for the optimal unconstrained problem for the nominal system so that

$$P = \begin{bmatrix} 2.0066 & 0.5099 \\ 0.5099 & 1.2682 \end{bmatrix}$$

and $u = Kx$ is the optimal unconstrained controller for (A, B, Q, R). The set Z is computed as a polytopic, disturbance invariant, outer invariant approximation of the minimal disturbance set (Raković et al., 2004) and the terminal constraint set X_f satisfies **A1**. The horizon length is $N = 9$. The sets X_9 and $\bar{X}_9$ (the domains of $V_N^*(\cdot)$ and $V_N^0(\cdot)$, respectively) are shown in Fig. 1. In Fig. 2, a state trajectory for initial condition $x_0 = (-5, -2)'$ is shown; the dash-dot line is the actual trajectory $\{x(i)\}$ for a sequence of extreme disturbances while the solid line is the sequence $\{x_0^*(x(i))\}$ of optimal initial states. The sets $x_0^*(x(i)) \oplus Z$ are shown shaded in Fig. 2. Also shown in Fig. 2 are the sets X_f and the set $X_f \oplus Z$ which is the effective terminal set for the 'tube' of trajectories; $X_f \oplus Z$ can be much larger than Z.

5. Conclusion

We have presented a robust model predictive controller for constrained linear systems with bounded disturbances. A
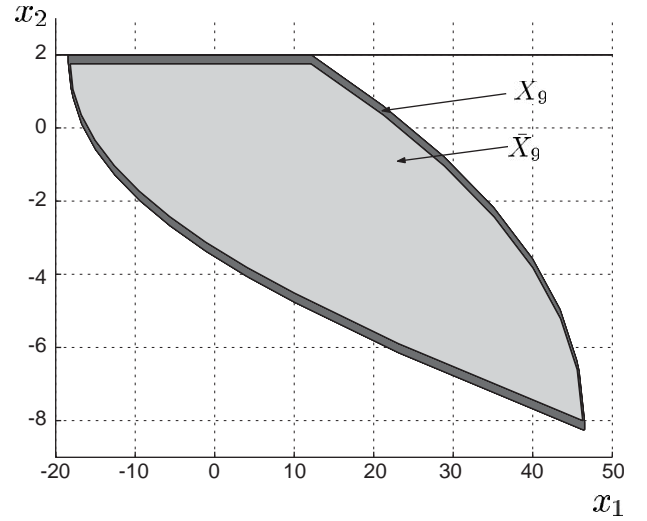

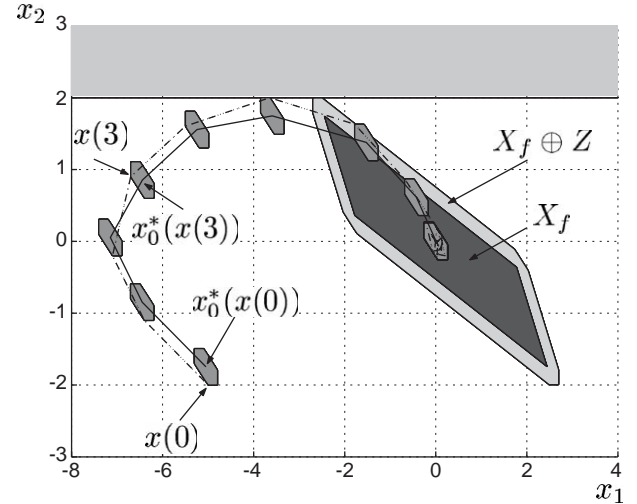
Fig. 1. Feasibility Sets for example of Section 4.1.

Fig. 2. Trajectories for example of Section 4.1.

novel feature of the robust model predictive controller is the fact that the decision variable in the (nominal) optimal control problem solved online incorporates the initial state of the model as well as the control sequence. The resultant value function is zero in the disturbance invariant set Z permitting robust exponential stability of Z to be established. The controller is relatively simple, merely requiring, in the optimal control problem solved online, optimization over the initial state x_0 of the model and a sequence of control actions subject to satisfaction of a tighter set of constraints than in the original problem; the optimal control problem is a standard quadratic program of approximately the same complexity as that required for conventional model predictive control (the dimension of the decision variable is increased by n). The results depend on linearity and cannot easily be extended to nonlinear systems.

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