LINEAR SYSTEMS WITH OUTPUT CONSTRAINTS: THE THEORY AND APPLICATION OF MAXIMAL OUTPUT ADMISSIBLE SETS

Elmer G.Gilbert and Kok Tin Tan
Department of Aerospace Engineering, The University of Michigan
Ann Arbor, Michigan 48109-2140

Abstract

An initial state of an unforced linear system is output admissible with respect to a constraint set Y if the resulting output function satisfies the pointwise-in-time condition $y(t) \in Y, t \geq 0$. The set of all such initial states is the maximal output admissible set, O_∞. The properties of O_∞ and its characterization are investigated. In the discrete-time case it is generally possible to represent O_∞, or a close approximation of it, by a finite number of functional inequalities. Practical algorithms for generating the functions are described. O_∞ has important applications in the analysis and design of closed loop systems with state and control constraints. An example is given. The discrete-time results are used to obtain an alternative implementation of the error governor control scheme proposed by Kapasouris, Athans and Stein [6]. It works as well as their implementation, but reduces the computational load on the controller by several orders of magnitude.

1. Introduction

In this paper we are concerned with characterizing those initial states of an unforced linear system whose subsequent motion satisfies a specified, pointwise-in-time constraint. Such characterizations have important applications. Consider the following example. A linear, discrete-time, time-invariant plant is given together with a linear control law:

$$x(t+1) = Ax(t) + Bu(t), \; u(t) = Kx(t). \tag{1.1}$$

Physical constraints are imposed on both the state and control variables or linear combinations of them. If the constraints are violated serious consequences may ensue; for example, physical components may be damaged or saturation may cause a loss of closed-loop stability [1-3]. Constraints of this type may be summarized compactly by a single set inclusion:

$$Cx(t) + Du(t) \in Y. \tag{1.2}$$

Typically, the set Y is convex and contains the origin. For instance, it may be a polytope or a product set of balls associated with various norms. With (1.1) and (1.2) specified, it is desired to obtain a safe set of initial conditions, i.e., a set Z such that $x(0) \in Z$ implies (1.2) is satisfied for all integers $t \geq 0$.

The problem just cited, as well as others, can be stated precisely in a simple format. We are given a triple, $A \in \mathcal{R}^{n\times n}$, $C \in \mathcal{R}^{p\times n}$, and $Y \subset \mathcal{R}^p$, and are to determine if the motion of the discrete-time system

$$x(t+1) = Ax(t), \;\; x(t) \in \mathcal{R}^n, \;\; y(t) = Cx(t), \tag{1.3}$$

satisfies the output constraint

$$y(t) \in Y \tag{1.4}$$

for all $t \in \mathcal{I}^+$, where $\mathcal{I}^+$ is the set of non negative integers. Problem (1.1)-(1.2) fits this format by making the assignments $A + BK \rightarrow A$ and $C + DK \rightarrow C$.

To assist our subsequent discussions, we introduce some terminology. The *state constraint set* associated with A, C, Y is

$$X(C,Y) = \{x \in \mathcal{R}^n : Cx \in Y\}. \tag{1.5}$$

A set $Z \subset \mathcal{R}^n$ is *A-invariant* if $AZ \subset Z$; it is A, C, Y *output admissible* if $CA^tZ \subset Y$ for all $t \in \mathcal{I}^+$. In the future we may omit specific reference to A, C, and Y in these and other situations, when by context it causes no confusion. The same ideas and terminology extend to the following continuous-time system and output constraint:

$$\begin{aligned} \dot{x}(t) &= Ax(t), \;\; x(t) \in \mathcal{R}^n, \\ y(t) &= Cx(t) \in Y, \;\; \forall t \in \mathcal{R}^+, \end{aligned} \tag{1.6}$$

where $\mathcal{R}^+$ is the set of nonnegative reals. In the definition of output admissible it is required that $Ce^{At}Z \subset Y$ for all $t \in \mathcal{R}^+$.

If Z is output admissible and $x(0) \in Z, x(0)$ is a safe initial condition in the sense that (1.4) is satisfied for all $t \in \mathcal{I}^+$ (discrete time) or for all $t \in \mathcal{R}^+$ (continuous time). Output admissible sets have many important applications in the areas of stability analysis and controller design. They have appeared before in a variety of contexts, often without explicit mention. See, e.g., [3-13].

For discrete-time systems the following two facts are obvious: if Z is output admissible it must belong to X; if $Z \subset X$ and Z is A-invariant, Z is output admissible. The last observation has been used before to produce output admissible sets [3,9-13]. Assume, as in most of these papers, that A is asymptotically stable and X is a polyhedron containing the origin in its interior. Let $V(x) = x^T Px$ be a Lyapunov function for the system (1.3), generated by solving $A^TPA - P = -Q$, where $Q^T = Q > 0$. Then $P^T = P > 0$ and it is clear

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that [14] the set $W = \{x \in \mathcal{R}^n : V(x) \leq c\}, c > 0$, is A-invariant. If c is selected so that $W \subset X$, W is output admissible. Even if c is maximized subject to $W \subset X$, this approach is likely to be very conservative in the sense that much larger output admissible sets exist. The difficulty is that the ellipsoid W may not fit tightly into the polyhedron X [3]. This has led a number of authors [9-13] to consider conditions which imply that a specified polyhedral set W is A-invariant. The conditions have been applied, with special assumptions on C, D and Y, to closed-loop systems of the form (1.1), (1.2). This leads to techniques for finding K such that (1.1) is asymptotically stable and (1.2) is satisfied for all initial conditions in W. For the techniques to work the prespecified W must be sufficiently small and, as is the case with ellipsoidal W, much larger sets of safe initial conditions may exist. As a final point of interest, it is easy to give examples of output admissible sets which are not A-invariant; such sets have received little if any attention in the literature. While extensions of the preceding ideas to continuous-time systems are possible, things are more complicated. For example, it is no longer true that Z is output admissible whenever $Z \subset X$ and Z is A-invariant.

This paper moves in a direction which is different from the directions that are outlined in the preceding paragraph. Its objective is to investigate the properties and the determination of *maximal* output admissible sets. For the discrete-time and continuous-time systems these sets are defined, respectively, by

$$O_\infty(A, C, Y) = \{x \in \mathcal{R}^n : CA^t x \in Y \ \ \forall t \in \mathcal{I}^+\} \qquad (1.7)$$

and

$$O_\infty^c(A, C, Y) = \{x \in \mathcal{R}^n : Ce^{At} x \in Y \ \ \forall t \in \mathcal{R}^+\}. \qquad (1.8)$$

Our emphasis is on the discrete-time case, because it is simpler and offers significant computational advantages. References to maximal output admissible sets have appeared previously in connection with several nonlinear feedback control schemes which take into account control and state constraints; see [15] for the discrete-time case and [6-8] for the continuous-time case.

The results of this paper are organized in the following way. Section 2 deals with basic issues: Y-dependent properties of O_∞, the determination of O_∞ by a finite number of operations, the characterization of Y and O_∞ by functional inequalities, simplifications which occur when C, A is unobservable and/or A is unstable. A condition for finite determinability, which is given in Section 2, leads to algorithmic procedures for the computation of O_∞. These procedures are presented along with some simple examples in Section 3. In Section 4 it is shown that O_∞ is finitely determined if: A is asymptotically stable, the pair C, A is observable, Y contains the origin in its interior and is bounded. If A is only Lyapunov stable, it often follows that O_∞ is not finitely determined. However, if the only characteristic roots of A which have unit magnitude are at $\lambda = 1$, an approximation of O_∞ is finitely determined. These matters are taken up in Section 5. In Section 6 we introduce a modification of the error governor control scheme of [6] which is based on O_∞ rather than O_∞^c. An example application shows that it is simple, fast and effective. Section 7 contains final remarks.

The following notations and terminology will be used. The identity matrix in $\mathcal{R}^{n \times n}$ is I_n. Let $\alpha \in \mathcal{R}, r \in \mathcal{R}^+, x \in \mathcal{R}^n, A \in \mathcal{R}^{m \times n}$, and $Z, Z_1, Z_2 \subset \mathcal{R}^n$. Then: x^i is the ith component of x, $||x|| = \sqrt{(x^T x)}$, $||A|| = \max\{||Ax|| : ||x|| = 1\}$, $S(r) = \{x : ||x|| \leq r\}$, $\alpha Z = \{\alpha x : x \in Z\}$, $-Z = (-1)Z$, $Z_1 + Z_2 = \{x_1 + x_2 : x_1 \in Z_1, x_2 \in Z_2\}$. Z is *symmetric* if $Z = -Z$. The boundary and interior of Z are denoted, respectively, by *bd* Z and *int* Z. The product set of Z with itself k times is Z^k.

We assume hereafter that $0 \in Y$. This assumption is satisfied in any reasonable application and has nice consequences. In particular, O_∞ and O_∞^c are nonempty and contain the origin.

2. Basic Results

Imposing special conditions on Y often imposes corresponding conditions on O_∞. Some implications of this type are summarized in the following theorem.

Theorem 2.1. (i) Each of the following properties of Y are inherited by O_∞: closure, convexity, symmetry. (ii) Suppose C, A is observable and Y is bounded. Then, O_∞ is bounded. (iii) Suppose A is Lyapunov stable (the characteristic roots of A satisfy the following conditions: $|\lambda_i(A)| \leq 1, i = 1, \ldots, n$, and $|\lambda_i(A)| = 1$ implies $\lambda_i(A)$ is simple) and $0 \in int\, Y$. Then, $0 \in int\, O_\infty$.

Proof: The results in (i) are easily verified from the definition of O_∞. To prove (ii) define $H^T = [C^T \ A^T C^T \ \ldots \ (A^T)^{n-1} C^T] \in \mathcal{R}^{n \times np}$. The observability assumption implies that H has rank n. Thus, if $Hx = z$ has a solution, it is given by $x = H^\dagger z$ where $H^\dagger = (H^T H)^{-1} H^T$ is the pseudo inverse of H. Define

$$O_t(A, C, Y) = \{x \in \mathcal{R}^n : CA^k x \in Y \text{ for } k = 0, \ldots, t\}. \qquad (2.1)$$

Clearly, $O_\infty \subset O_{n-1} = \{x : Hx \in Y^n\} = H^\dagger Y^n$. Since Y is bounded, so is $H^\dagger Y^n$. Thus, (ii) follows. The assumption of Lyapunov stability in (iii) implies that there exists a constant, $\gamma_1 > 0$, such that for all $x \in \mathcal{R}^n$ and $t \in \mathcal{I}^+, ||CA^t x|| \leq \gamma_1 ||x||$. Choose $\gamma_2 > 0$ so that $S(\gamma_2) \subset Y$. Then, $\gamma_1 ||x|| \leq \gamma_2$ implies $CA^t x \in Y$ for all $t \in \mathcal{I}^+$. Hence, $S(\gamma_2/\gamma_1) \subset O_\infty$ and the proof is complete. □

Obviously, the set $O_t(A, C, Y)$ satisfies the following condition:

$$O_\infty \subset O_{t_2} \subset O_{t_1} \ \ \forall t_1, t_2 \in \mathcal{I}^+ \text{ such that } t_1 \leq t_2. \qquad (2.2)$$

We say O_∞ is *finitely determined* if for some $t \in \mathcal{I}^+, O_\infty = O_t$. Let t^* be the smallest element in $\mathcal{I}^+$ such that $O_\infty = O_{t^*}$. We call t^* the *output admissibility index*. From (2.2) it follows that $O_\infty = O_t$ for all $t \geq t^*$.

Theorem 2.2. O_∞ is finitely determined if and only if $O_t = O_{t+1}$ for some $t \in \mathcal{I}^+$.

Proof: If O_∞ is finitely determined the equality holds for any $t \geq t^*$. From $O_t = O_{t+1}$ it is easily confirmed that $x \in O_t$ implies that $Ax \in O_t$; thus, O_t is A-invariant. Also, $O_t \subset X$. Hence, O_t is output admissible and $O_t \subset O_\infty$. Property (2.2) completes the proof. □

Finitely determined, maximal output admissible sets generally have a simpler structure than those which are not.

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Moreover, they can be obtained by finite recursive procedures. These matters, and conditions which imply finite determinability, are discussed in subsequent sections.

Next consider the situation where Y is defined by functional inequalities:

$$Y = \{y \in \mathcal{R}^p : f_i(y) \le 0, i = 1, \ldots, s\}. \tag{2.3}$$

Under what circumstances is O_∞ defined in a similar way?

Theorem 2.3. Suppose A is Lyapunov stable and that for $i = 1, \ldots, s$ the functions $f_i : \mathcal{R}^p \to \mathcal{R}$ are continuous and satisfy $f_i(0) \le 0$. Then: (i) the functions $g_i : \mathcal{R}^n \to \mathcal{R}$ given by

$$g_i(x) = \sup\{f_i(CA^t x) : t \in \mathcal{I}^+\} \tag{2.4}$$

are defined, (ii) $0 \in O_\infty$ and (iii)

$$O_\infty = \{x \in \mathcal{R}^n : g_i(x) \le 0, i = 1, \ldots, s\}. \tag{2.5}$$

Proof: Let $x \in \mathcal{R}^n$. Using the notation defined in the proof of Theorem 2.1, it follows that for all $t \in \mathcal{I}^+$, $CA^t x \in S(\gamma_1 \|x\|)$. Thus, by the compactness of S and the continuity of the f_i, the sequence $f_i(CA^t x), t \in \mathcal{I}^+$, is bounded from above. Therefore, the supremum in (2.4) is defined for all $x \in \mathcal{R}^n$. Results (ii) and (iii) are a direct consequence of $f_i(0) \le 0$ and the definition of O_∞. □

When Y is given by (2.3) and O_∞ is finitely determined

$$O_\infty = \{x \in \mathcal{R}^n : f_i(CA^t x) \le 0, \; i \in \{1, \ldots, s\}, t \in \{0, \ldots, t^*\}\}. \tag{2.6}$$

Not all of of the $s \cdot (t^* + 1)$ inequalities in (2.6) may be active in the sense that there exists an $x \in O_\infty$ such that $f_i(CA^t x) = 0$. The active inequalities have a special structure which is given in the following theorem.

Theorem 2.4. Suppose O_∞ is given by (2.6). Then there exists a nonempty set of integers $S^* \subset \{1, \ldots, s\}$ and indices $t_i, i \in S^*$, such that: (i) $t^* = \max\{t_i^* : i \in S^*\}$, (ii)

$$O_\infty = \{x \in \mathcal{R}^n : f_i(CA^t x) \le 0, i \in S^*, t \in \{0, \ldots, t_i^*\}\}, \tag{2.7}$$

(iii) for all $i \in S^*$ and $t \in \{0, \ldots, t_i^*\}$ there exists an $x \in O_\infty$ such that $f_i(CA^t x) = 0$.

Proof: Let $S^* = \{i : \exists\, t \in \{0, \ldots, t^*\}$ and $x \in O_\infty$ such that $f_i(CA^t x) = 0\}$. If $i \notin S^*$ it follows that f_i is inactive in (2.6), i.e., $f_i(CA^t x) < 0$ for all $t \in \{0, \ldots, t^*\}$ and $x \in O_\infty$. For $i \in S^*$, define $t_i^* = \max\{t \in \{0, \ldots, t^*\} : \exists\, x \in O_\infty$ such that $f_i(CA^t x) = 0\}$. Then it is obvious that (2.7) holds. Result (i) follows from (2.7) and the definition of t^*. Clearly, there exists $x \in O_\infty$ such that $f_i(CA^{t_i^*} x) = 0$. This implies $f_i(CA^{t_i^* - t} x(t)) = 0, t = 0, \ldots, t_i^*$, where $x(t) = A^t x$. By (1.7), $x(t) \in O_\infty$ for all $t \in \mathcal{I}^+$. Hence, result (iii) is true. □

Remark 2.1. Suppose there exists an $i \in \{1, \ldots, s\}$ such that $f_i(Cx) < 0$ for all $x \in X(C, Y)$. Then it is obvious that $i \notin S^*$. Even when such trivial situations are excluded, it is possible that $S^* \ne \{1, \ldots, s\}$. Consider, for instance, the example: $n = p = 1, A = [-0.8], C = [1], f_1 = y - 1, f_2 = -y - 2$. Then $S^* = \{1\}$ and $t_1^* = t^* = 1$.

Remark 2.2. Result (iii) states that all the inequalities in (2.7) are active. This does not exclude the possibility that some of the inequalities may be redundant in the sense that they can be removed from (2.7) without destroying the characterization of O_∞. For example, redundant inequalities in (2.3) lead to redundant inequalities in (2.7). If such redundant inequalities are excluded, it is still possible to give examples where there are redundant inequalities in (2.7). However, these examples appear to be structurally unstable, i.e., small changes in A, C or the f_i eliminate redundant inequalities. Thus, from a practical point of views it appears that (2.7) offers the most economical representation of O_∞.

Remark 2.3. Because of Theorem 2.4, (2.4) and (2.5) can be replaced by

$$g_i(x) = \max\{f_i(CA^t x) : t \in \{0, \ldots, t_i^*\}\} \tag{2.8}$$

and

$$O_\infty = \{x \in \mathcal{R}^n : g_i(x) \le 0, i \in S^*\}. \tag{2.9}$$

These simplification in (2.4) and (2.5) are valuable in control applications where it is necessary to have an efficient computational procedure for testing whether or not $x \in O_\infty$.

A change in state coordinates for the system (1.3) produces a simple transformation of O_∞. To see this let $U \in \mathcal{R}^{n \times n}$ be the nonsingular matrix which describes the change in coordinates. Then it can be verified from (1.7) that:

$$O_\infty(A, C, Y) = U O_\infty(\tilde{A}, \tilde{C}, Y) \tag{2.10}$$

where $\tilde{C} = CU, \tilde{A} = U^{-1} A U$. Because of this relationship, we shall feel free in our subsequent derivations to express the pair C, A in whatever coordinate system seems most natural. This path is acceptable for both theoretical developments and the computation of O_∞.

The most obvious application of this idea is to unobservable systems. Choose the coordinate system in the usual way [16] so that C, A has the form:

$$C = [C_1 \;\; 0], \;\; A = \begin{bmatrix} A_1 & 0 \\ A_3 & A_2 \end{bmatrix}, \tag{2.11}$$

where $C_1 \in \mathcal{R}^{p \times m}, A_1 \in \mathcal{R}^{m \times m}, A_2 \in \mathcal{R}^{(n-m) \times (n-m)}, A_3 \in \mathcal{R}^{(n-m) \times m}$ and the pair C_1, A_1 is observable. The application of (1.7) then shows that

$$O_\infty(A, C, Y) = O_\infty(A_1, C_1, Y) \times \mathcal{R}^{n-m}. \tag{2.12}$$

Thus, O_∞ is a cylinder set whose determination depends only on the triple A_1, C_1, Y. Stated in a coordinate-free way, O_∞ is a cylinder set which has infinite extent in those directions which belong to the unobservable subspace.

Finally, we consider systems (1.3) which are observable but have *divergent* motions. Define $L = \{x \in \mathcal{R}^n :$ the sequence $A^t x, \; t \in \mathcal{I}^+$, is bounded$\}$. Clearly, L is a linear space. It is determined by the direct sum of the invariant subspaces of A which are associated with those eigenvalues that satisfy $|\lambda_i(A)| < 1$, together with the span of the eigenvectors of A which correspond to $|\lambda_i(A)| = 1$. We won't dwell on the details, but it is possible to determine a basis for L numerically from the real Schur form [17] of A. Let $m =$ dimension of L. Choose a basis for $\mathcal{R}^n$ in the following way: let the first m basis vectors be a basis for L, let the remaining $n - m$ basis vectors be a basis for the orthogonal complement of L. In this basis it is easy to see that the pair C, A has the form

$$C = [C_1 \;\; C_2], \; A = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}, \tag{2.13}$$

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where: $A_1 \in \mathcal{R}^{m\times m}$ is Lyapunov stable and $CA^t x$ diverges on $\mathcal{I}^+$ unless the last $n-m$ components of x are zero. Now assume Y is *bounded*. Then it is obvious that

$$O_\infty(A,C,Y) = O_\infty(A_1,C_1,Y)\times\{0\}. \qquad (2.14)$$

A coordinate-free interpretation of (2.14) is that $O_\infty \subset L$.

Remark 2.4. With respect to the determination of O_∞ the consequences of the preceding two paragraphs are clear. It is possible to successively eliminate from the representation of C, A the "unobservable" and "divergent motion" subspaces. Hence there is no loss of generality in restricting our attention to systems which are observable and (if Y is bounded) Lyapunov stable. This observation is important because several of our key results depend on the assumptions of observability and Lyapunov stability.

3. The Algorithmic Determination of O_∞

Theorem 2.2 suggests the following conceptual algorithm for determining t^* and O_∞.

Algorithm 3.1. Step 1: Set $t=0$. Step 2: If $O_{t+1}=O_t$, stop. Set $t^*=t$ and $O_\infty = O_t$. If $O_{t+1}\neq O_t$, continue. Step 3: Replace t by $t+1$ and return to Step 2. □

Clearly, the algorithm will produce t^* and O_∞ if and only if O_∞ is finitely determined. There appears to be no finite algorithmic procedure for showing that O_∞ is not finitely determined. Fortunately, as subsequent developments show, it is often possible to resolve the issue of finite determination by other means. Algorithm 3.1 is not practical because it does not describe how the test $O_{t+1}=O_t$ is implemented. The difficulty can be overcome if Y is defined by (2.3) and the hypotheses of Theorem 2.3 are satisfied, for then Step 2 leads to a set of mathematical programming problems.

Algorithm 3.2. Step 1: Set $t=0$. Step 2: Solve the following optimization problems for $i=1,\ldots,s$:

$$\text{maximize } J_i(x) = f_i(CA^{t+1}x) \qquad (3.1)$$

subject to the constraints

$$f_j(CA^k x)\le 0,\ \ j=1,\ldots,s,\ k=0,\ldots,t. \qquad (3.2)$$

Let J_i^* be the maximum value of $J_i(x)$. If $J_i^*\le 0$ for $i=1,\ldots,s$, stop. Set $t^*=t$ and define O_∞ by (2.6). Otherwise, continue. Step 3: Replace t by $t+1$ and return to Step 2.□

* **Remark 3.1.** After algorithm 3.2 has terminated, it is possible to obtain S^* and the indices t_i^* by solving a sequence of mathematical programming problems. Let $J_{it}^* = \max f_i(CA^t x)$ subject to the constraints $f_j(CA^k x)\le 0, j=1,\ldots,s,\ k=0,\ldots,t^*$. For each $i\in\{1,\ldots,s\}$: compute J_{it}^* for $t=t^*, t^*-1,\ldots$ until $J_{it}^*=0$ or $J_{it}^*<0$ for $t=t^*,\ldots,0$. If $J_{it}^*=0, i\in S^*$ and $t_i^*=t$. If $J_{it}^*<0$ for $t=t^*,\ldots,0,\ i\notin S^*$.

The success of Algorithm 3.2 depends on the existence of effective algorithms for solving the rather large mathematical programming problems which arise. This presents some difficulty because global optima are needed. Even when the $f_i,\ i=1,\ldots,s$, are convex, the difficulty remains because the programming problems require the *maximum* of a convex function subject to convex constraints. When Y is a polyhedron, the difficulty disappear. Then, the programming problems are linear and efficient computer codes for obtaining the global maxima abound.

Using the above ideas it is possible to characterize O_∞ for complex, finitely determined systems. In this section, we are content to apply them to some simple examples. Our purpose is to illustrate how various A, C, and Y affect both the determination of O_∞ and the properties of O_∞.

The data for the examples, along with t^* and the characteristic roots of A are summarized in Table 1. Examples 3.3-3.10 are particularly simple and it is possible to obtain t^* and O_∞ by hand calculations. In Example 3.1, Algorithm 3.2 was implemented by linear programming. Example 3.2 requires the maximization of quadratic functions subject to quadratic inequality constraints; these nonlinear programming problems were solved numerically using the optimization program VMCON [18]. Figure 1 shows O_∞ for Examples 3.1 and 3.2. In each example, O_∞ is determined by functional inequalities of the form $f_1(CA^t x)\le 0$. The dots on the boundary of O_∞ show where two inequalities are simultaneously active; the numbers adjacent to the intervening arcs show the value of t for which the corresponding inequality is active. Note that A is asymptotically stable in Examples 3.1-3.6 and only Lyapunov stable in Examples 3.7-3.10.

Example 3.1 is taken from [11]. The double shaded region shows the A-invariant subset of X obtained in that paper. The maximal output admissible set, O_∞, is obviously much larger. Example 3.2 illustrates the fact that Y need not be a polyhedron as in the prior literature devoted to the determination of A-invariant subsets of X. The boundary segments of O_∞ are sections taken from three ellipses. Examples 3.3-3.6 show that O_∞ may or may not be finitely determined when Y is unbounded or contains the origin in its boundary. In Example 3.7, A has its characteristic roots on the unit circle. Because CA^t is periodic in t with period $t=4$, it is easy to see that O_∞ is finitely determined. In Example 3.8, the characteristic roots are again on the unit circle but CA^t never repeats itself for $t\in\mathcal{I}^+$. In fact, for $t\in\mathcal{I}^+$, $||CA^t||=1$ and CA^t takes on essentially all directions. Thus, O_∞ is not finitely determined and $O_\infty = S(1)$. In Examples 3.9 and 3.10, A has a characteristic root at $\lambda=1$. Again, both possibilities with respect to finite determination exist. In Example 3.10 it is easy to see that $O_\infty = \{x : |[1\ 1]x|\le 1, |[1\ 0]x|\le 1\}$, a very simple set despite the fact that O_∞ is not finitely determined.

4. Conditions for Finite Determination of O_∞

It is desirable to have simple conditions which assure the finite determination of O_∞. Our main result in this direction is the following theorem.

Theorem 4.1. Suppose the following assumptions hold: (i) A is asymptotically stable ($|\lambda_i(A)|<1, i=1,\ldots,n$), (ii) the pair C, A is observable, (iii) Y is bounded, (iv) $0\in$ *int* Y. Then, O_∞ is finitely determined.

Proof: It is apparent from (ii) and (iii) and the proof of Theorem 2.1, that there exists an $r>0$ such that $O_t\subset S(r), t\in\mathcal{I}^+, t\ge n-1$. Moreover, by (i), $CA^t\to 0$ as $t\to+\infty$. Thus, by (iv), there exists a $k\in\mathcal{I}^+, k\ge n-1$, such that $CA^{k+1}S(r)\subset Y$. Hence, $CA^{k+1}O_k\subset Y$. This result and (2.1) imply that $O_{k+1}=O_k$ and by Theorem 2.2 the proof is complete. □

Remark 4.1. The k in the proof is an upper bound for t^*. There is a simple test for obtaining it. Choose γ so that

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$S(\gamma) \subset Y$. Then, $||CA^k|| \leq r^{-1}\gamma$ implies $CA^{k+1}S(r) \subset Y$ and $t^* \leq k$.

While the conditions in the theorem are sufficient for finite determinability, Examples 3.3, 3.5, 3.7 and 3.9 show that they are not necessary. On the other hand, Examples 3.4, 3.6, 3.8 and 3.10 illustrate difficulties in sharpening the sufficient conditions. As has been noted in Remark 2.2, assumption (ii) is not really a limitation. It is possible, by making additional assumptions on C and A to eliminate assumptions (iii) and (iv). We will not pursue the investigation of such assumptions here. Instead, because it has greater practical interest, we will investigate the relaxation of (i).

In particular, we consider systems where A is Lyapunov stable. By Remark 2.3 this represents no loss of generality when Y is bounded. The most common situation, and the one which we treat here, is where the only characteristic roots of A that are on the unit circle are at $\lambda = 1$. The Lyapunov stability of A implies that these roots are all simple. Thus, by doing a block diagonalization of A, where each block corresponds to a distinct eigenvalue of A [16], it is possible to find a choice of coordinates which puts A, and consequently C, into the form:

$$C = [C_L \ \ C_S], \ \ A = \begin{bmatrix} I_d & 0 \\ 0 & A_S \end{bmatrix}. \qquad (4.1)$$

Here: the partitioning of C and A is dimensionally consistent, d is the number of characteristic roots at $\lambda = 1$, and $A_S \in \mathcal{R}^{(n-d)\times(n-d)}$ is asymptotically stable. The representation (4.1) simplifies our developments.

Theorem 4.2. Suppose A, C has the form (4.1) and Y is closed. Define

$$\begin{aligned} \hat{C} &= \begin{bmatrix} C_L & 0 \\ C_L & C_S \end{bmatrix} \in \mathcal{R}^{2p\times n}, \qquad (4.2) \\ X_L &= \{x : [C_L \ \ 0]x \in Y\} \subset \mathcal{R}^n. \end{aligned}$$

Then: (i)$O_\infty \subset X_L$, (ii) $O_\infty(A, C, Y) = O_\infty(A, \hat{C}, Y \times Y)$

Proof: Result (i) implies (ii) because if it holds,the additional condition $[C_L \ \ 0]x \in Y$, which is contained in the augmented triple $A, \hat{C}, Y \times Y$, is satisfied automatically. Suppose, contrary to (i), there exists an $x \in O_\infty(A, C, Y)$ such that $C_L x \notin Y$. Since Y is closed, this implies the existence of an $r > 0$ such that for all $w \in S(r), C_L x + w \notin Y$. Because $x \in O_\infty(A, C, Y)$, it follows that $y(t) = CA^t x = [C_L \ \ 0]x + [0 \ \ C_S A_S^t]x \in Y$ for all $t \in \mathcal{I}^+$. From the asymptotic stability of A_S there exists a $k \in \mathcal{I}^+$ such that $||[0 \ \ C_S A_S^k]x|| < r$. Thus, $y(k) \notin Y$ and this contradiction completes the proof. □

Consider the application of the theorem to Example 3.10. The matrices C and A already have the form (4.1) with $d = 1$, so no transformation of coordinates is necessary. It is easily confirmed that while $O_\infty(A, C, Y)$ is not finitely determined, $O_\infty(A, \hat{C}, Y \times Y)$ is. Unfortunately, the trick of replacing A, C, Y by $A, \hat{C}, Y \times Y$ does not work generally. See Example 4.1 in Table 1. Again, this system satisfies (4.1) with $d = 1$. But now neither $O_\infty(A, C, Y)$ nor $O_\infty(A, \hat{C}, Y \times Y)$ is finitely determined.

To see why, examine Figure 2, which shows x^2-x^3 sections of $O_\infty(A, C, Y)$ taken for several values of $x^1 \in [0, 1]$. Because $O_\infty \subset \{x : |x^1| \leq 1\} = X_L$ and is symmetric, there is no need to consider values of x^1 outside $[0, 1]$. Each of the sections is a polygon. At $x^1 = 1$, the section of $O_\infty(A, C, Y)$ is a polygon with only 5 vertices; but as x^1 approaches 1 from below, the number of vertices in the polygon near the origin increases. A detailed analysis shows that if x^1 is sufficiently close to 1, the number of vertices may be arbitrarily large. Thus, Example 4.1 is vastly more complicated than Example 3.10. The characterization of $O_\infty(A, C, Y)$ requires the intersection of an infinite number of half spaces and there is no way that $O_\infty(A, C, Y)$ can be finitely determined.

The sections displayed in Figure 2 do show an interesting property. The number of vertices becomes very large only when x^1 is very close to 1. This suggests that $O_\infty(A, C, Y)$ can be approximated closely by $O_\infty(A, C, Y) \bigcap \{x : |x^1| \leq 1 - \epsilon\} = O_\infty(A, \hat{C}, Y(\epsilon) \times Y)$ where $Y(\epsilon) = [-1 + \epsilon, 1 - \epsilon]$ and $0 < \epsilon$ is small. As will now be seen, the approximation $O_\infty(A, \hat{C}, Y(\epsilon) \times Y)$ has the advantage that it is finitely determined.

5. The Approximation of O_∞ for Lyapunov Stable Systems

Assume that Y is given by (2.3). Then there is a natural choice for $Y(\epsilon)$:

$$Y(\epsilon) = \{y : f_i(y) \leq -\epsilon, i = 1, \ldots, s\}. \qquad (5.1)$$

Define

$$\epsilon_0 = -\max\{f_i(0) : i = 1, \ldots, s\}. \qquad (5.2)$$

Theorem 5.1. Suppose C, A and $\hat{C}$ are given by (4.1) and (4.2), C, A is observable and A_S is asymptotically stable. Assume: (i) the functions $f_i : \mathcal{R}^p \to \mathcal{R}, i = 1, \ldots, s$, are continuous, (ii) $\epsilon_0 > 0$, (iii) $Y = Y(0)$ is bounded. Then, for each $\epsilon \in (0, \epsilon_0], O_\infty(A, \hat{C}, Y(\epsilon) \times Y)$ is finitely determined. Furthermore,

$$O_\infty(A, C, Y(\epsilon)) \subset O_\infty(A, \hat{C}, Y(\epsilon) \times Y) \subset O_\infty(A, C, Y). \qquad (5.3)$$

Proof: Note that

$$\begin{aligned} O_t &= O_t(A, \hat{C}, Y(\epsilon) \times Y) \qquad (5.4) \\ &= \{x : f_i([C_L \ \ 0]x) \leq -\epsilon, \\ &\quad f_i([C_L \ \ 0]x + [0 \ \ C_S A_S^k]x) \leq 0, \\ &\quad i = 1, \ldots, s, k = 0, \ldots, t\}. \end{aligned}$$

Clearly, O_t is nonempty. As in the proof of Theorem 4.1 there exists an $r > 0$ such that $O_{n-1} \subset S(r)$. Suppose $x \in O_{n-1}$. Then $||x|| \leq r$ and $f_i([C_L \ \ 0]x) \leq -\epsilon$. Since $A_S^t \to 0$ as $t \to +\infty$ and f_i is continuous, there exists a $k \in \mathcal{I}^+$, which is independent of $x \in O_{n-1}$ and $i = 1, \ldots, s$, such that $f_i([C_L \ \ 0]x + [0 \ \ C_S A_S^{k+1}]x) \leq 0$. Thus, $\hat{C}A^{k+1}O_{n-1} \subset Y(\epsilon) \times Y$. Consequently, $\hat{C}A^{k+1}O_k \subset Y(\epsilon) \times Y$ and, by the reasoning used in the proof of Theorem 4.1, O_∞ is finitely determined. The right inclusion of (5.3) follows from part (ii) of Theorem 4.2 and $Y(\epsilon) \subset Y$. Again by Theorem 4.2, $O_\infty(A, C, Y(\epsilon)) = O_\infty(A, \hat{C}, Y(\epsilon) \times Y(\epsilon))$, and the left inclusion is obvious. □

As Example 4.1 shows, the output admissibility index for $A, \hat{C}, Y(\epsilon) \times Y$ may increase without bound as $\epsilon \to 0$. The inclusions of (5.3) provide a way of judging whether or not for a given ϵ the approximation of $O_\infty(A, C, Y)$ by

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$O_\infty(A, \hat{C}, Y(\epsilon) \times Y)$ is sufficiently good. They state, in effect, that the approximation is no worse than what would happen if the constraint set Y were replaced by $Y(\epsilon)$.

In most applications, an acceptable choice for ϵ is clear. This is certainly true in Example 4.1. Table 2 shows how t^* changes with ϵ. For $\epsilon = 0.05$, there is a reasonable compromise between t^* and the accuracy of the approximation. Note that for small ϵ, t^* appears to increase logarithmically with respect to ϵ^{-1}. This result can be confirmed by a detailed analysis of the example.

6. An Application of O_∞ to the Design of a Nonlinear Controller

Maximal output admissible sets have important applications in system analysis and controller design. A simple example of an application to analysis is the regulator control system described in the first paragraph of Section 1. Suppose, for $i = 1, \ldots, p$, the actuator for the ith component of u saturates when $|u^i| > 1$ and it is desired to estimate the domain of attraction for the closed loop system in the presence of this nonlinearity. Let (1.2) take the form $u(t) \in Y = \{y : ||y||_\infty \le 1\}$. Then for any $x(0) \in O_\infty(A + BK, K, Y)$ saturation is avoided and the motion is described by the linear equations (1.1). Thus, if $A + BK$ is asymptotically stable , $x(t) \to 0$ and O_∞ is a subset of the domain of attraction.

Kapasouris, Athans and Stein [6-8] have more interesting applications. They allow dynamic compensators and exploit the properties of O_∞^c to obtain nonlinear controllers which avoid actuator saturation and give a much improved overall response to large inputs. Here, we present a discrete-time modification of their continuous-time, error governor scheme [6].

Our control system configuration is shown in Figure 3. To avoid any confusion, t represents continuous time and k represents discrete time. The error governor generates a scalar gain κ, multiplying the sampled error signal $e(k) = r(kT) - c(kT)$. We assume, as is in [6], that the plant is linear and asymptotically stable and that a linear, Lyapunov stable compensator has been designed by some methodology so that with $\kappa = 1$ good, linear-system, closed loop performance is obtained. If κ stays fixed at 1 and $||u(k)||_\infty > 1$, the resulting actuator saturation may cause a serious degradation in the closed loop response. Reset windup [6] is but one example of this sort of difficulty; see [1] for additional discussion. The idea of the error governor is to adjust κ downward if $\kappa = 1$ has the potential to create immediate or subsequent saturation. It is argued in [6] that such gain reductions will be relatively infrequent and that when they do occur that their effect on the response is much less damaging than actuator saturation.

In our implementation of the error governor, $\kappa(k) \in [0, 1]$ is adjusted according to the state of the discrete-time compensator so that for all $k \in \mathcal{I}^+, ||u(k)||_\infty \le 1$. Assume that the compensator and error governor are described by

$$x_c(k+1) = A_c x_c(k) + B_c \kappa(k) e(k),\ u(k) = C_c x_c(k). \quad (6.1)$$

Set $Y = \{y : ||y||_\infty \le 1\}$ and use Algorithm 3.2 to obtain a characterization of $O_\infty(A_c, C_c, Y)$ of the form (2.5)-(2.7). If A_c has characteristic roots at $\lambda = 1$ (assume there are no others on the unit circle), it may be necessary to replace $O_\infty(A_c, C_c, Y)$ by an approximation $O_\infty(A_c, \hat{C}_c, Y(\epsilon) \times Y)$ as described in Section 5. Set

$$\kappa(k) = \max\{\kappa \in [0, 1] : A_c x_c(k) + B_c \kappa e(k) \in O_\infty\}. \quad (6.2)$$

Then it is obvious that $x_c(k) \in O_\infty$ implies $x_c(k+1) \in O_\infty$. Thus, if $x_c(0) \in O_\infty$, which is a reasonable assumption for a compensator, $||u(k)||_\infty \le 1, k \in \mathcal{I}^+$. Moreover, it is certain that the maximization problem has a solution because O_∞ is closed and $A_c x_c(k) + B_c \kappa e(k) \in O_\infty$ for $\kappa = 0$. It should be noted that the on-line implementation of (6.2) is straightforward. Since O_∞ is defined by a system of linear inequalities, the upper limit imposed on κ by each inequality is given by a simple formula. The least of these limits is $\kappa(k)$. The real-time implementation of the overall control strategy requires that the computation of $\kappa(k)$ and $x_c(k+1)$ in (6.1) be done in less than T seconds.

The intuitive basis for the operation of the error governor is clear. If $x_c(k) \in int\ O_\infty$ and $e(k)$ is sufficiently small, $\kappa(k) = 1$. Consequently, when both $r(t)$ and $e(k)$ are reasonably small, the closed loop system satisfies the linear equations of motion. When $e(k)$ is large, or when $A_c x_c(k)$ is near the boundary of O_∞ and $B_c \kappa e(k)$ points toward the boundary, $\kappa(k) < 1$ and the compensator action is reduced to avoid saturation. As noted in [6], the basis for the error governor is entirely intuitive. There is no theory which actually proves that the error governor has better response characteristics. Because of the asymptotic stability of the plant it is easy to show that the closed loop system is bounded-input,bounded-output stable [6].

We have tested the modified error governor controller on the aircraft control problem described in [6]. There, state models are given for both the longitudinal motion of the aircraft (4th order) and the continuous-time compensator (8th order). There are two inputs to the aircraft: an elevator and a flaperon. The physical limits on both of the inputs are ± 25 degrees. Thus, saturation occurs when $||u||_\infty > 25$. The compensator in [6] was obtained by adding integral control and using the LQG/LTR methodology. Let A_a, B_a, C_a denote the system matrices for this continuous-time compensator. Our discrete-time compensator was derived from it by the usual zero-order hold approach: $A_c = e^{A_a T}, B_c = \int_0^T e^{A_a t} dt\, B_a, C_c = C_a$. The choice $T = 0.05$ provided a reasonable approximation to the continuous-time controller.

The compensator is Lyapunov stable with two characteristic roots of A_c on the unit circle at $\lambda = 1$. Thus, an approximation of $O_\infty(A_c, C_c, Y)$ was considered. The system equations already have the form (4.1), so Algorithm 3.2 was applied to the determination of $O_\infty = O_\infty(A_c, \hat{C}_c, Y(\epsilon) \times Y)$. The set Y was defined by (2.3) with $f_i(y) = 0.04|y_i| - 1$ and $s = 2$. Note that the Minkowski assumptions are satisfied and $\mu_Y(y) = 0.04||y||_\infty$. The required optimization problems were solved by linear programming for several values of ϵ. Table 3 summarizes the results. It appears that $O_\infty(A_c, \hat{C}_c, Y \times Y)$ is not finitely determined. However, $\epsilon = 0.004$ gives a very good approximation: by Theorem 5.1 it follows that the error of the approximation is at most 0.4%. The computational time required by Algorithm 3.2 is not large. For $\epsilon = 0.004$ and an Apollo DN 3500, it is 6.1 seconds; for $\epsilon = 0.00004$, it is 11.7 seconds.

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The actual characterization of $O_\infty(A_c, \hat{C}_c, Y(\epsilon) \times Y)$ is quite simple. For $\epsilon = 0.004$, it is given by the representation (2.7):

$$O_\infty = \{x_c : |x_c^7| \le 24.9, |x_c^8| \le 24.9, |C_{c1} A_c^t x_c| \le 25,$$
$$|C_{c2} A_c^k x_c| \le 25, t = 0, \ldots, 7, k = 0, \ldots, 9\}, \qquad (6.3)$$

where C_{c1} and C_{c2} are the rows of C_c. Suppose that the $C_{c1} A_c^t$ and $C_{c2} A_c^k$ are precomputed. Then the testing of $x \in O_\infty$ requires 18 inner product evaluations and 20 comparisons. The computational time is dominated by the 18 $\times$ 8=144 multiplies which are required. With appropriate precomputed data, the evaluation of (6.2) requires only a few more comparisons and multiplies.

The closed loop system was simulated with the step input $r_1(t) = r_2(t) = 10$. Figure 4 shows $\kappa(k)$. For $k \ge 15 (t \ge 0.75), \kappa(k) = 1$ and the system behaves as a linear system. Figure 5 shows the output response, $c_1(t)$ and $c_2(t)$, for three different situations: (i) $\kappa = 1$ and no saturation, (ii) $\kappa = 1$ and saturation, (iii) $\kappa(k)$ determined by the error governor ($\epsilon = 0.004$.) As expected in (i), the resulting linear sampled-data feedback system performs well and has essentially the same response as the corresponding continuous-time system in [6]. The response (ii) shows the effects of adding saturation to the same system. Both overshoot and settling time are badly degraded. With the error governor in place saturation does not occur and the response (iii) is much improved.

The results shown in Figures 4 and 5 are very close to those in [6]. Thus, our discrete-time modification appears to maintain the nice properties of the continuous-time mechanization in [6]. However, there is a substantial saving in the computational time required for the controller. On a Macintosh 512K the simulation of our implementation takes 13.1 seconds; the corresponding time reported in [8] is approximately 29,000 seconds. The reason for the large time is not clear from the discussion in [8], but it is certainly related to the complexity of testing whether or not a point x belongs to $O_\infty^c(A_a, C_a, Y)$. On an Apollo DN 3500 the simulation takes 1.24 seconds, which is about four times faster than real time. Hence, for systems of significant complexity, it appears that the implementation of practical, on-line controllers is feasible.

7. Conclusion

In this paper we have developed a general theory which pertains to the maximal output admissible sets O_∞. Our most important contributions concern the algorithmic characterization of O_∞. If Y is bounded, $0 \in int\ Y$ and A has no characteristic roots on the unit circle, O_∞ may be determined in a finite number of steps. In most cases the steps can be implemented by solving several easily formulated mathematical programming problems. When A has characteristic roots on the unit circle it may turn out that the computations are not finite. If this happens and the characteristic roots are all at $\lambda = 1$, there is a practical way out: by introducing slightly stronger constraints a close approximation of O_∞ may be obtained finitely.

Using representations of the form (2.7)-(2.9) it is possible to test numerically whether or not x belongs to O_∞. The computational effort is reasonable, even for systems of fairly high order, and the structure of computations is suitable for parallel processing. This situation makes it feasible to implement feedback control laws for linear plants with a variety of state and control constraints. The example in Section 6 effectively demonstrates this potential. The error governor control methodology of [6] is modified and applied to the 12th order aircraft control problem considered in [6]. The resulting nonlinear, discrete-time controller performs in the same effective way as the controller in [6] and it reduces computational load on the controller by several orders of magnitude. We have obtained similar improvements with modified versions of the reference governor controllers described in [7]. Additional details on these and other promising control schemes will be reported in the future.

While many of the properties of O_∞ carry over to O_∞^c, the characterization of O_∞^c is by no means as simple. A practical solution of this difficulty exists too. By taking $T > 0$ sufficiently small, it is possible to approximate as closely as desired $O_\infty^c(A, C, Y)$ by $O_\infty(e^{AT}, C, Y)$. Details will be published in the near future.

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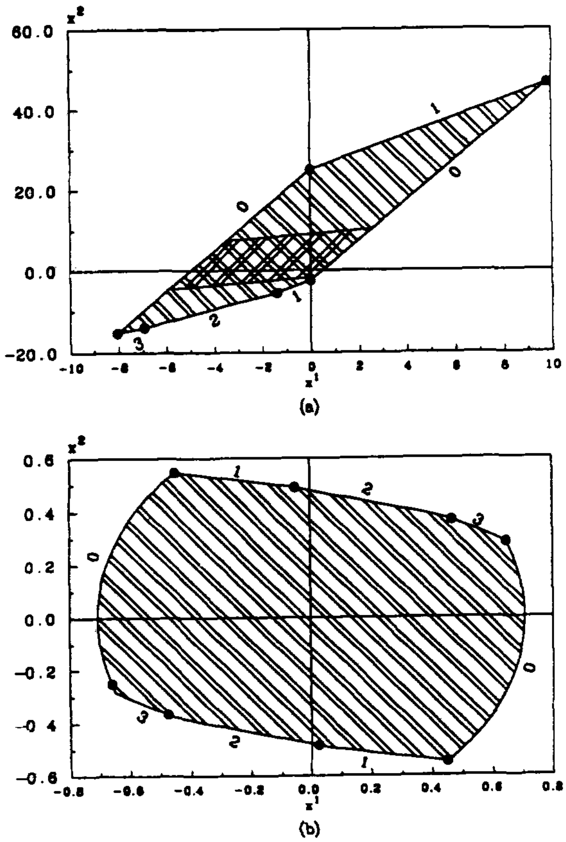

Figure 1: (a) O_∞ for Example 3.1. (b) O_∞ for Example 3.2.

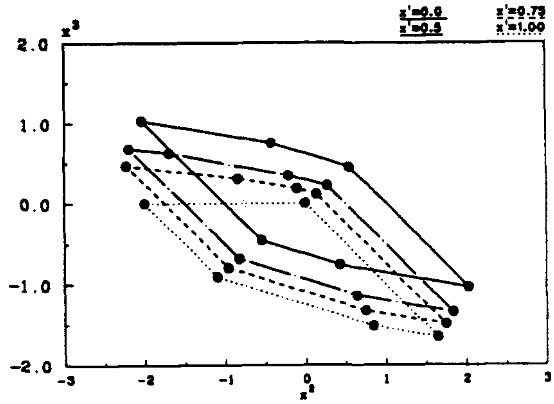

Figure 2: Sections of O_∞ for Example 4.1 - $x^1 = 0.0, 0.5, 0.75, 1.0$.

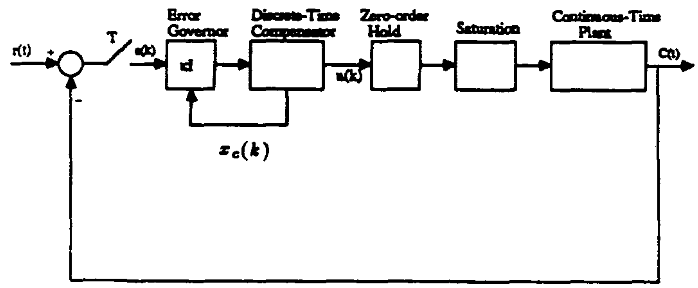

Figure 3: Implementation of the modified error governor.

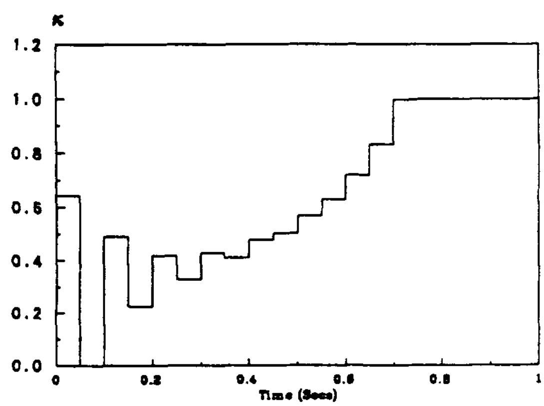

Figure 4: Gain produced by error governor for aircraft control system with step input $r_1 = r_2 = 10$.

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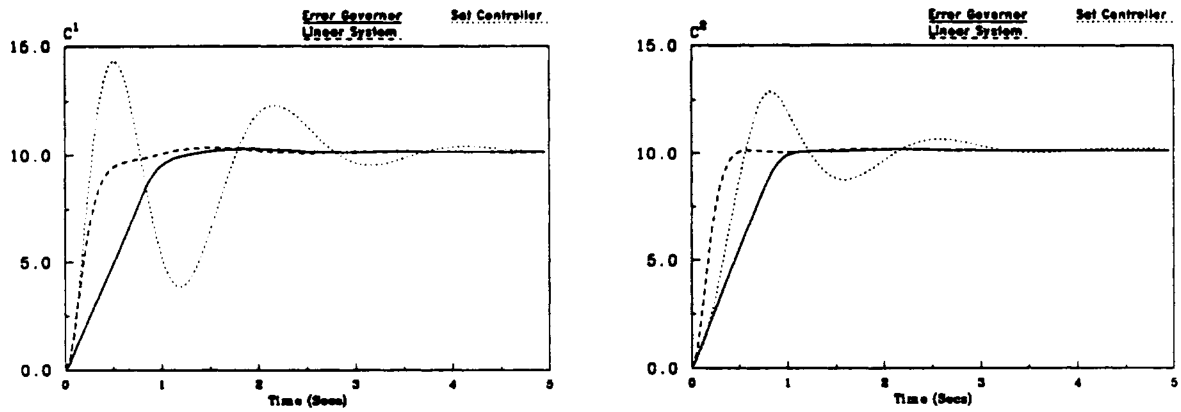

Figure 5: Output responses for aircraft control system with step input $r_1 = r_2 = 10$.

Example	A	C	Y	λ_i	t^*
3.1	$\begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.5 \end{bmatrix}$	$\begin{bmatrix} -1.0 & 0.2 \end{bmatrix}$	$[-0.5,\ 5.0]$	0.6, 0.3	3
3.2	$\begin{bmatrix} 0.6 & 1.0 \\ 0.0 & 0.76 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 1.0 \\ 1.0 & -1.0 \end{bmatrix}$	$S(1)$	0.76,0.6	3
3.3	$\begin{bmatrix} 0.5 & 1.0 \\ 0.0 & 0.3 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 0.0 \\ 1.0 & 1.0 \end{bmatrix}$	$[-1.0,\ +\infty) \times (-\infty,\ 1.0]$	0.5, 0.3	2
3.4	$\begin{bmatrix} 0.5 & 1.0 \\ 0.0 & 0.3 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 0.0 \end{bmatrix}$	$[-1.0,\ +\infty)$	0.5, 0.3	∞
3.5	$\begin{bmatrix} 0.6 & -0.1 \\ 0.1 & -0.3 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 1.0 \end{bmatrix}$	$[0.0,\ 1.0]$	0.58, −0.28	1
3.6	$\begin{bmatrix} 0.5 & 1.0 \\ 0.0 & 0.2 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 1.0 \end{bmatrix}$	$[0.0,\ 1.0]$	0.5, 0.2	∞
3.7	$\begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}$	$\begin{bmatrix} 1.0 & 1.0 \end{bmatrix}$	$[-1.0,\ 1.0]$	$e^{\pm j\frac{\pi}{4}}$	3
3.8	$\begin{bmatrix} \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 1.0 \end{bmatrix}$	$[-1.0,\ 1.0]$	$e^{\pm j1}$	∞
3.9	$\begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 0.5 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 1.0 \\ 1.0 & 0.0 \end{bmatrix}$	$[-1.0,\ 1.0]^2$	1.0, 0.5	0
3.10	$\begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 0.5 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 1.0 \end{bmatrix}$	$[-1.0,\ 1.0]$	1.0, 0.5	∞
4.1	$\begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 0.5 & 1.0 \\ 0.0 & 0.0 & 0.6 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 1.0 & 1.0 \end{bmatrix}$	$[-1.0,\ 1.0]$	1.0, 0.6, 0.5	∞

Table 1: Data for Examples.

ϵ	1.0	0.5	0.25	0.10	0.05	1×10^{-2}	1×10^{-3}	1×10^{-4}	1×10^{-5}	1×10^{-6}
t^*	2	3	3	4	5	7	10	13	16	19

Table 2: Example 4.1: t^* vs ϵ.

ϵ	4×10^{-2}	4×10^{-3}	4×10^{-4}	4×10^{-5}	4×10^{-6}	4×10^{-7}	4×10^{-8}	4×10^{-9}
t_1^*	5	7	9	11	13	14	16	17
t_2^*	8	9	11	12	14	15	17	19

Table 3: Aircraft Control System: t_1^* and t_2^* vs ϵ.

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