Mass-Radius Relation for Stable Netron Stars

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Neutron stars are compact remnants of collapsed massive stars ($M \geq M_{\odot}$). These stars are theorized to reach central densities exceeding that of the atomic nuclear density. In this paper we will model non-rotating neutron stars as pure neutron matter supported by the neutron degeneracy pressure to determine the mass-radius relationship for stable stars. We will compare this model to astrophysical measurements of neutron stars as well as the mass-radius relationship calculated using current tabulated dense matter equations of state (EoS).

I. INTRODUCTION

Because neutrons are so dense they are useful tools in understanding dense matter physics. Furthermore looking at the formation and evolution of these stars can provide us will a better understanding of star dynamics and black hole formations[8].

Basic early models of these stars assumed a composition of pure neutron matter from the core of a massive star after exhausting its fuel. This core was theorized to be supported from further collapse by the neutron degeneracy pressure, and if the star was still unstable and its gravitational force exceeded this pressure then it would collapse further into a black hole.

Recent physical measurements of neutron stars have shown that neutron stars exist at masses large than $0.7M_{\odot}$ which is beyond the mass supported by the neutron degeneracy pressure. Furthermore, theorized densities of such stars predict much more complicated internal structures of neutron stars as shown in Fig.1.

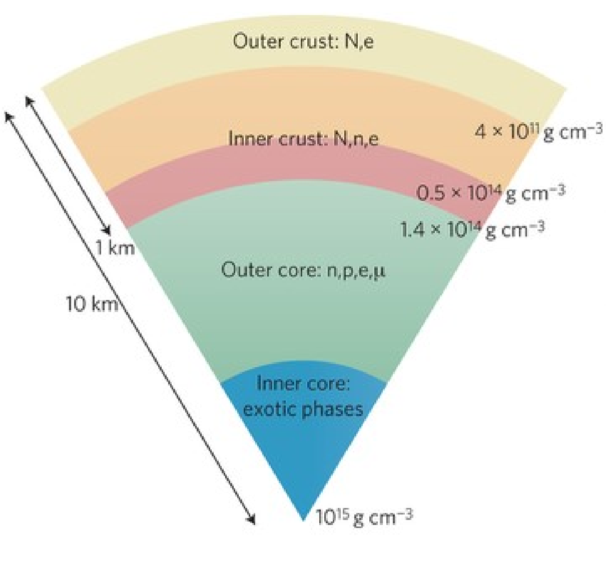

FIG. 1. Representation of theorised layer compositions of neutron stars[2].

Its theorized that just the first few layers of a neutron star are composed just nuclear matter (protons and neutrons) but as the density of the matter surpasses that of the atomic nuclear density more exotic phases of matter exist, the physics of which surpass our current understanding of dense matter. In order to model neutron stars, the equation of state (EoS) of these dense matters must be understood. It is in developing this EoS that neutron stars become a playground for dense matter physics, where physical observables such as mass and radius can validate or invalidate theorized equations of state for dense matter that cannot yet be measured in the laboratory[4].

II. NEUTRON STAR MODELING

Modeling neutron stars is a complicated task. Assuming the simplest case of symmetric non-rotating fluid, there are two hurdles to overcome to determine a mass-radius relationship: the relation for hydrostatic equilibrium in the general relativistic frame work, and a the equation of state for dense matter within the star.

A. Tolman-Oppenheimer-Volkov Equations

To model a neutron star we will consider a static, spherically symmetric, relativistic star in hydrostatic equilibrium. To do this we start with the static, spherically symmetric metric:

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega \tag{1}$$

The solution for a star will be the non-vacuum solutions to the full Einstein equation:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2}$$

Additionally, because we are modeling the star as a static perfect fluid the energy-momentum tensor is:

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$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} \tag{3}$$

$$= \begin{pmatrix} e^{2\alpha}\rho & & & \\ & e^{2\beta}p & & \\ & & r^2 p & \\ & & & r^2 p(\sin^2\theta) \end{pmatrix} \tag{4}$$

Thus we have three independent components to Eqn. 2 for the tt, rr, and $\theta\theta$ (same as $\phi\phi$ component due to symmetry):

$$\frac{1}{r^2}e^{-2\beta}(2r\partial_r\beta - 1 + e^{2\beta}) = 8\pi G\rho \tag{5}$$

$$\frac{1}{r^2}e^{-2\beta}(2r\partial_r\alpha + 1 - e^{2\beta}) = 8\pi G p \tag{6}$$

$$e^{-2\beta}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{1}{r}(\partial_r\alpha - \partial_r\beta)\right] = 8\pi G p \tag{7}$$

Replacing $\beta(r)$ with an expression for the mass:

$$m(r) = \frac{1}{2G}(r - re^{-2\beta})$$

This reduces the tt equation (Eqn. 5) to:

$$\frac{dm(r)}{dr} = 4\pi\rho(r)r^2 \tag{8}$$

and the rr equation (Eqn. 6) reduces to:

$$\frac{d\alpha}{dr} = \frac{Gm(r) + 4\pi G r^3 p}{r[r - 2Gm(r)]} \tag{9}$$

Using energy-momentum conservation, $\nabla_\mu T^{\mu\nu} = 0$, we see that the only nontrivial component is $\nu = r$ thus:

$$(\rho + p)\frac{d\alpha}{dr} = -\frac{dp}{dr}$$

Leading to (from Eqn. 9):

$$\frac{dP}{dr} = -\frac{(\rho + p)[Gm(r) + 4\pi G r^3 p]}{r[r - 2Gm(r)]} \tag{10}$$

Together Eqn. 8 and Eqn. 10 make the Tolman-Oppenheimer-Volkov (TOV) equations for hydrostatic equilibrium. These coupled equations along with the EoS will provide the mass-radius relationship for neutron stars[3].

B. Equation of State

To solve the TOV equations we need an EoS that relates pressure and density of the matter composing the neutron star. Determining such a relationship is difficult as the composition of neutron stars are not fully understood as first principles quantum chromodynamics (QCD) are not yet able to describe interactions at the densities theorized inside these stars. There are, however, a large sample of currently proposed equations of state that span wide density and pressure ranges that are predicted from several physical assumptions. These equations of state are often numerically tabulated but are not a unified analytical relationship. Tabulated EoSs can be validated through experimental measurements of dense matter dynamics and astrophysical observations but are subject to the interpolation methods used and the assumption used for several physical parameters[1].

To develop an analytic EoS we can use the first order approximation that our neutron star is completely composed of neutron matter and model the star as a Fermi gas. Here, the energy density for the gas (where $k_F c$ is the Fermi energy) is[8]:

$$\epsilon = \frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_F}(k^2c^2 + m_N^2c^4)^{1/2}k^2 dk \tag{11}$$

$$= \epsilon_0 \int_0^{k_F/m_N c}(u^2 + 1)^{1/2}u^2 du \tag{12}$$

$$= \frac{\epsilon_0}{8}[(2x^3 + x)(1 + x^2)^{1/2} - \sinh^{-1}(x)] \tag{13}$$

$$\epsilon_0 = \frac{m_N^4 c^5}{\pi\hbar},\ x = \frac{k_F}{m_N c}$$

Where (for nucleon per electron A/Z and mass density ρ):

$$k_F = \hbar\left(\frac{3\pi^2\rho Z}{m_N A}\right)^{1/3} \tag{14}$$

For our cold Fermi-gas:

$$p = -\frac{U}{\partial V}\bigg|_{T=0} = n^2\frac{d(\frac{\epsilon}{n}}{dn} = n\frac{d\epsilon}{dn} - \epsilon \tag{15}$$

Which yields a pressure of:

$$p(k_F) = \frac{8\pi}{3(2\pi\hbar)^2}\int_0^{k_F}(k^2c^2 + m_N^2C^4)^{1/2}k^4 dk \tag{16}$$

$$= \frac{\epsilon_0}{24}[(2x^3 - 3x)(1 + x^2) + 3\sinh^{-1}(x)] \tag{17}$$

In the non-relativistic limit for $k_F \ll m_N$ this reduces to:

$$p = \kappa\epsilon^{5/3} \tag{18}$$

$$\kappa = \frac{\hbar^2}{15\pi^2 m_N}\left(\frac{3\pi^2 Z}{m_N c^2 A}\right)^{5/3}$$

Similarly in the relativistic limit:

$$p = \kappa\epsilon,\ \kappa = 1/3$$

Here we see the familiar form of the equation of state used in astrophysics called Polytropes where the pressure depends the density to some power. Here the EoS takes the form $p = \kappa\epsilon^{(n+1)/n}$ where K is a proportionality constant and n is the Polytropic Index. While this derivation is for pure non-interacting neutron gas this can be used to fit more complicated models such as non-symmetric proton and neutron matter or neutron interactions[6].

III. MASS-RADIUS RELATION

To determine the mass-radius relation for neutron stars we developed a numerical solver for the coupled TOV equations described by Eqn. 8 and Eqn. 10. Our chosen EoS is a polytrope described by:

$$p = \kappa_0 \rho^{\gamma},\ \gamma = 2.1,\ \kappa_0 = 3.54810^4\ \frac{fm^3}{MeV} \tag{19}$$

Where this is the best fit polytrope for an interacting, pure neutron star[6].

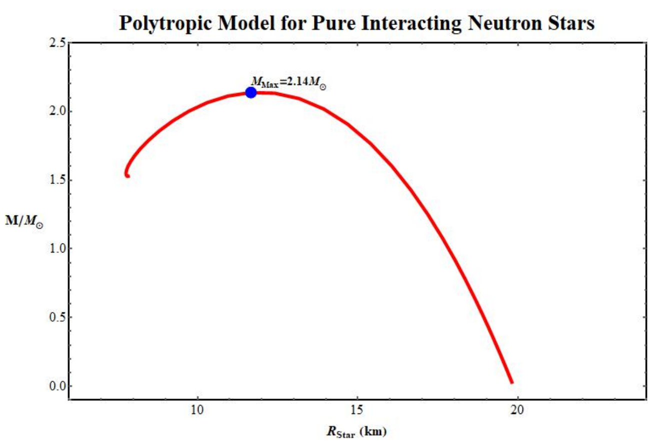

FIG. 2. Mass-radii relation from numerical solutions to the TOV equations using a polytropic EoS model for pure interacting neutron stars. Predicted maximum star mass of $2.14M_{\odot}$.

Numerically solving the TOV equations with this polytropic EoS yields the above relation for the mass and radii of neutron stars. To a first order this depiction roughly agrees with true masses and radii of neutron stars ($M \sim 1 - 3M_{\odot}$ and $R \sim 10km$). Stable stars will exist to the right of the maximum[?] shown in our model ($R \gtrsim 11.7\ km$).

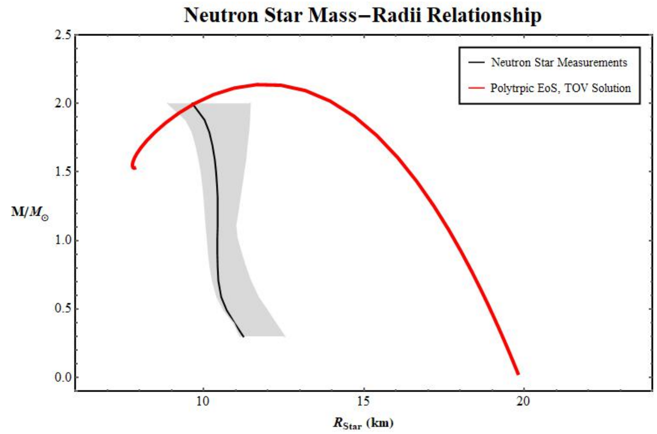

FIG. 3. Numerical polytropic model compared to neutron star mass and radius measurements[7]. Shaded gray is the minimum and maximum bounds on the measured radii with the most likely measurement as the blue line[9].

From Fig.3 we can see that our model agrees closely with the range of observed masses for neutron stars but the range of the star radius only agrees in order of magnitude. Here, the radii are greatly over estimated from those actually observed due to our model assumptions. We modeled the neutron star EoS for that of pure neutron matter, whereas true neutron stars are composed of more than neutrons and would likely have a more layered composition as shown in Fig.1. To represent neutron star models for a wider range of physical parameters than our polytropic model, we selected a few tabulated equations of state to generate mass-radii relationships for neutron stars (hard EoS - MS1, moderate EoS - H4, and soft EoS - SQM3)[5]. These equations of state have varying parameters corresponding to layer densities and compositions. The mass-radii relationships for these equations of state were generated by first linearly interpolating the tabulated EoS over our density pressure space and then once again numerically solving the TOV equations.

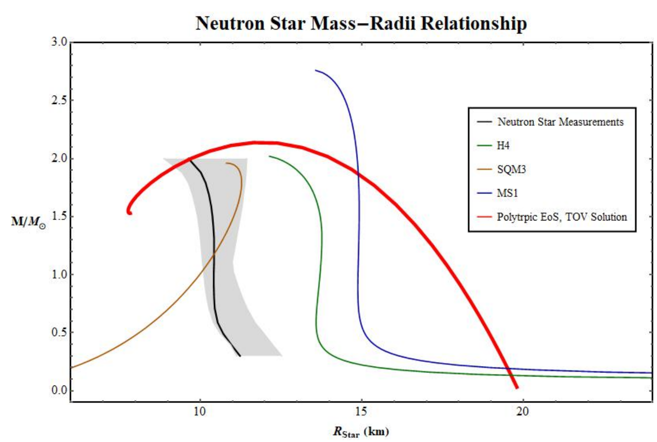

FIG. 4. Comparison of our modeled mass radius relationship for our polytropic EoS to that for several tabulated equations of state.

Here we can see that these tabulated equations of state more closely match the observed data with regards to the stars radius, however, they still vary from one another and the observed data. We can clearly see from Fig.4 that the mass-radii relationship models are highly dependent on the choice of an EoS and that there is still a large discrepancy among equations of state, illustrating the difficulty of modeling these stars.

IV. SUMMARY

Under the assumptions of non-rotating symmetric neutron stars, we found that the governing equations of hydrostatic equilibrium are the TOV equations for pressure and mass. Furthermore, under our assumption of pure interacting neutron matter composition we developed an EoS for neutron stars that can be fitted to a polytrope. Numerically solving the TOV equations with the use of this EoS led to a mass-radius relationship for neutron stars. Comparing this to actual astrophysical measurements of neutron stars we showed that to an order of magnitude our mass and radii agreed with that seen in nature. However, the characteristic curve governing this relationship is vastly different between our model fit and actual measurements. This is likely due to the many assumptions we made in our model, in particular the assumption of pure neutron matter. To further our investigation we used current tabulated equations of state, which implement a vast range of physical properties, to numerically solve the TOV equations. This mass-radius relationships more accurately modeled the characteristic curve of the actual data but there is still much left to be desired for fitting the true star properties.

Developing an accurate EoS is vital the modeling of neutron stars. However, the measurements from these stars can provide great insight into unanswered questions of dense matter physics and general relativity[8].

V. REFERENCES

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