APPM 4350 Project: Guitar Signal Analysis Report

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1 Introduction

The wave equation is one of the most prominent PDEs in physical problems. However, the derivation of the simple wave equation neglects several physical characteristics that make it a poor model in real world applications. In this lab we will investigate the the 1-D wave equation, in the context of comparing it to the spectral response of the strings on a guitar. In addition to modeling the response of the string using the classic wave equation, we will also compare three different modifications that take into account physical effects that were ignored in the derivation of the wave equation. By comparing these models to experimental data we will determine which is the most accurate.

2 Method and Procedure

For the experimental data of this lab, we collected the amplitude vs. time and frequency response (FFT) of each individual guitar string in addition to several chords, after tuning the guitar. This was done through use of the "Guitar Signal Analysis" LabVIEW program that was provided. Furthermore, background amplitude measurements were taken, in order to process it out of the guitar signals.
In our analysis, we will demonstrate the solution process to the given modified wave PDEs through separation of variables or through assumed solutions. Furthermore, we will provide insight for the physical characteristic that the equations accounts for.

3 Modeling Vibrations of a Stretched String

3.1 Undamped Wave Equation

The undamped wave equation, shown in 1, can be used to model vibrations of a stretched string.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},\ 0 < x < L, t > 0. \tag{1}$$

Here, u is the transverse displacement, which is a function of time t and distance x. Furthermore, c is the speed of a wave travelling along the string. In the case of a guitar string, $c^2 = \frac{T_0}{\rho_0(x)}$, where T_0 is the tension and $\rho_0(x)$ is the mass density of the string. Combined with Dirichlet boundary conditions and physical initial conditions, the undamped wave equation can be used to model the vibrations of a stretched string. In particular, solving this PDE via the method of separation of variables yields an explicit expression for $u(x,t)$.

Let 1 be subject to the following boundary conditions

$$\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0, \end{aligned} \tag{2}$$

and arbitrary initial conditions

$$\begin{aligned} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t}(x,0) &= g(x), \end{aligned} \tag{3}$$

corresponding to the initial position and the initial velocity of each segment of the string. Note that these are reasonable conditions dictated by the physical system. Notably, the ends of the guitar strings, $x = 0$ and $x = L$, are fixed and therefore their displacement is 0 as can be seen by the boundary conditions. In addition, the initial conditions are given by the arbitrary functions $f(x)$ and $g(x)$.

The method of separation of variables asserts that we look for solutions of the form

$$u(x,t) = \phi(x)h(t) \neq 0. \tag{4}$$

Combined with 4, this yields

$$\phi(x)\frac{d^2h}{dt^2} = c^2h(t)\frac{d^2\phi}{dx^2} \tag{5}$$

and subsequently

$$\frac{1}{c^2}\frac{1}{h(t)}\frac{d^2h}{dt^2} = \frac{1}{\phi(x)}\frac{d^2\phi}{dx^2} = -\lambda \tag{6}$$

This yields two ordinary differential equations:

$$\begin{aligned} \frac{d^2h}{dt^2} &= -\lambda c^2 h(t) \\ \frac{d^2\phi}{dx^2} &= -\lambda\phi(x). \end{aligned} \tag{7}$$

The boundary conditions given in 2 can now be expressed as

$$\phi(0) = \phi(L) = 0, \tag{8}$$

which, when combined with the ordinary differential equations in 7, yields the following boundary value problem

$$\begin{aligned} \frac{d^2\phi}{dx^2} &= -\lambda\phi(x) \\ \phi(0) &= 0 \\ \phi(L) &= 0. \end{aligned} \tag{9}$$

There are three possible cases for λ that could potentially yield solutions to the ODE subject to the prescribed boundary conditions.

1. $\underline{\lambda = q^2 < 0}$

   If λ is less than 0, then the general solution to 9 is

   $$\frac{d^2\phi}{dx^2} - q^2\phi = 0 \rightarrow \phi(x) = c_1e^{qx} + c_2e^{-qx}. \tag{10}$$

   If $\phi(0) = 0$, then

   $$\phi(0) = 0 = c_1 + c_2 \Rightarrow c_2 = -c_1 \Rightarrow \phi(x) = c_1(e^{qx} - e^{-qx}). \tag{11}$$

   If $\phi(L) = 0$, then

   $$\phi(L) = 0 = c_1(e^{qL} - e^{-qL}). \tag{12}$$

   Because the expression in the parentheses never equals 0, c_1 must be 0, which implies that $\phi(x) = 0$. Being that this is the trivial case, λ cannot be less than 0.

2. $\underline{\lambda = 0}$

   If λ is equal to 0, then the general solution to 9 is

$$\frac{d^2\phi}{dx^2} = 0 \to \phi(x) = c_1 x + c_2. \tag{13}$$

   If $\phi(0) = 0$, then

$$\phi(0) = 0 = c_2. \tag{14}$$

   If $\phi(0) = 0$, then

$$\phi(L) = 0 = c_1 L \Rightarrow c_1 = 0 \because L > 0. \tag{15}$$

   If $\lambda = 0$, $\phi(x)$ yields only the trivial solution, implying that λ cannot be equal to 0.

3. $\underline{\lambda = p^2 > 0}$

   If λ is greater than 0, then the general solution to 9 is

$$\frac{d^2\phi}{dx^2} + p^2\phi = 0 \to \phi(x) = A\cos(px) + B\sin(px). \tag{16}$$

   If $\phi(0) = 0$, then

$$\phi(0) = 0 = A. \tag{17}$$

   If $\phi(0) = 0$, then

$$\phi(L) = 0 = B\sin(pL). \tag{18}$$

   If B is equal to 0, then only the trivial solution exists. However, if $\sin(pL) = 0$, then a non-trivial solution exists. Specifically, $\sin(pL) = 0 \Rightarrow \lambda = (\frac{n\pi}{L})^2$ for $n \in \mathbb{N}^+$. These eigenvalues have the corresponding eigenfunction $\sin(\frac{n\pi x}{L})$. Therefore,

$$\phi_n(x) = c_n \sin(\frac{n\pi x}{L}),\ n \in \mathbb{N}^+. \tag{19}$$

The time-dependent ODE can be solved immediately now knowing that $\lambda > 0$.

$$\frac{d^2h}{dt^2} + \lambda c^2 h(t) = 0 \Rightarrow h(t) = c_1\cos(\sqrt{\lambda}ct) + c_2\sin(\sqrt{\lambda}ct). \tag{20}$$

Being that λ is a function of n, 20 can be expressed as

$$h_n(t) = c_{n1}\cos(\frac{n\pi ct}{L}) + c_{n2}\sin(\frac{n\pi ct}{L}),\ n \in \mathbb{N}^+. \tag{21}$$

Combining 4 and the principle of superposition, we see that

$$u(x,t) = \sum_{n=1}^{\infty}(A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}),\ 0 < x < L, t > 0. \tag{22}$$

Note that the initial conditions determine A_n and B_n. Specifically,

$$\begin{aligned} u(x,0) = f(x) &= \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \\ \frac{\partial u}{\partial t}(x,0) = g(x) &= \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L}\sin\frac{n\pi x}{L}, \end{aligned} \tag{23}$$

which implies that the coefficients can be found via the Fourier sine series of $f(x)$ and $g(x)$:

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \\ B_n\frac{n\pi c}{L} &= \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx. \end{aligned} \tag{24}$$

Here it is worth noting that this solution can be expressed in a form more suitable for interpretation in the context of musical stringed instruments such as a guitar. Note that 22 can be expressed as

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}(\sqrt{A_n^2+B_n^2}\sin(\omega t+\theta)),\ 0<x<L, t>0, \tag{25}$$

where

$$\begin{aligned} A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L} &= \sqrt{A_n^2+B_n^2}\sin(\omega t+\theta) \\ \theta &= \arctan\frac{A_n}{B_n} \\ \omega &= \frac{n\pi c}{L}. \end{aligned} \tag{26}$$

Furthermore, a relationship between ω and the wavenumber, $k = \frac{n\pi}{L}$, can be determined. If we assume that $u(x,t) = e^{ikx}e^{i\omega t}$, then the PDE stated in 1 yields

$$\begin{aligned} \frac{\partial^2}{\partial t^2}(e^{ikx}e^{i\omega t}) &= c^2\frac{\partial^2}{\partial x^2}(e^{ikx}e^{i\omega t}),\ 0<x<L, t>0 \\ -\omega^2 e^{ikx}e^{i\omega t} &= -k2c^2 e^{ikx}e^{i\omega t} \\ \omega &= kc \\ k &= \frac{\omega}{c}. \end{aligned} \tag{27}$$

Noting that $\omega = \frac{n\pi c}{L}$, we see that $k = \frac{n\pi}{L}$ as 26 predicted.

3.2 Damped Wave Equation

While 1 can be used to describe a vibrating string, it does not necessarily reflect reality as accurately as possible. After all, 1 is an idealized model in which there is no dampening of the string. Other models, such as 28, take into account the natural dampening observed in a guitar string:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} - \beta\frac{\partial u}{\partial t},\ 0<x<L,\ t>0, \tag{28}$$

where $\beta \geq 0$ is the dampening factor.

As in the undamped case, 28 is subject to the same Dirichlet boundary conditions

$$\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0, \end{aligned} \tag{29}$$

and arbitrary initial conditions

$$\begin{aligned} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t}(x,0) &= g(x), \end{aligned} \tag{30}$$

corresponding to the initial position and the initial velocity of each segment of the string respectively. The method of separation of variables can once again be utilized to find a solution to this PDE. Recall that this method asserts that we look for solutions of the form

$$u(x,t) = \phi(x)h(t) \neq 0. \tag{31}$$

Applying this solution to the PDE, we obtain

$$\frac{\partial^2}{\partial t^2}(\phi(x)h(t)) = c^2\frac{\partial^2}{\partial x^2}(\phi(x)h(t)) - \beta\frac{\partial}{\partial t}(\phi(x)h(t)) \tag{32}$$

and subsequently

$$\begin{aligned}\phi(x)\frac{\partial^2 h}{\partial t^2} &= c^2 h(t)\frac{\partial^2\phi}{\partial x^2} - \beta\phi(x)\frac{\partial h}{\partial t}\\ \phi(x)\frac{\partial^2 h}{\partial t^2} + \beta\phi(x)\frac{\partial h}{\partial t} &= c^2 h(t)\frac{\partial^2\phi}{\partial x^2}\\ \frac{1}{c^2h(t)}\frac{\partial^2 h}{\partial t^2} + \frac{\beta}{c^2h(t)}\frac{\partial h}{\partial t} &= \frac{1}{\phi(x)}\frac{\partial^2\phi}{\partial x^2} = -\lambda.\end{aligned} \tag{33}$$

This yields two ordinary differential equations:

$$\begin{aligned}\frac{1}{c^2h(t)}\frac{\partial^2 h}{\partial t^2} + \frac{\beta}{c^2h(t)}\frac{\partial h}{\partial t} &= -\lambda\\ \frac{1}{\phi(x)}\frac{\partial^2\phi}{\partial x^2} &= -\lambda.\end{aligned} \tag{34}$$

The boundary conditions given in 29 can now be expressed as

$$\phi(0) = \phi(L) = 0, \tag{35}$$

which, when combined with the ordinary differential equations in 33, yields the following boundary value problem

$$\begin{aligned}\frac{1}{\phi(x)}\frac{\partial^2\phi}{\partial x^2} &= -\lambda\\ \phi(0) &= 0\\ \phi(L) &= 0.\end{aligned} \tag{36}$$

There are three possible case for λ that could potentially yield solutions to the ODE subject to the the prescribed boundary conditions.

1. $\underline{\lambda = q^2 < 0}$

   If λ is less than 0, then the general solution to 9 is

   $$\frac{d^2\phi}{dx^2} - q^2\phi = 0 \rightarrow \phi(x) = c_1e^{qx} + c_2e^{-qx}. \tag{37}$$

   If $\phi(0) = 0$, then

   $$\phi(0) = 0 = c_1 + c_2 \Rightarrow c_2 = -c_1 \Rightarrow \phi(x) = c_1(e^{qx} - e^{-qx}). \tag{38}$$

   If $\phi(L) = 0$, then

   $$\phi(L) = 0 = c_1(e^{qL} - e^{-qL}). \tag{39}$$

   Because the expression in the parentheses never equals 0, c_1 must be 0, which implies that $\phi(x) = 0$. Being that this is the trivial case, λ cannot be less than 0.

2. $\underline{\lambda = 0}$

   If λ is equal to 0, then the general solution to 9 is

   $$\frac{d^2\phi}{dx^2} = 0 \rightarrow \phi(x) = c_1x + c_2. \tag{40}$$

If $\phi(0) = 0$, then
$$\phi(0) = 0 = c_2. \tag{41}$$

If $\phi(0) = 0$, then
$$\phi(L) = 0 = c_1 L \Rightarrow c_1 = 0 \because L > 0. \tag{42}$$

If $\lambda = 0$, $\phi(x)$ yields only the trivial solution, implying that λ cannot be equal to 0.

3. $\underline{\lambda = p^2 > 0}$

If λ is greater than 0, then the general solution to 9 is
$$\frac{d^2\phi}{dx^2} + p^2\phi = 0 \to \phi(x) = A\cos(px) + B\sin(px). \tag{43}$$

If $\phi(0) = 0$, then
$$\phi(0) = 0 = A. \tag{44}$$

If $\phi(0) = 0$, then
$$\phi(L) = 0 = B\sin(pL). \tag{45}$$

If B is equal to 0, then only the trivial solution exists. However, if $\sin(pL) = 0$, then a non-trivial solution exists. Specifically, $\sin(pL) = 0 \Rightarrow \lambda = (\frac{n\pi}{L})^2$ for $n \in \mathbb{N}^+$. These eigenvalues have the corresponding eigenfunction $\sin(\frac{n\pi x}{L})$. Therefore,
$$\phi_n(x) = c_n \sin(\frac{n\pi x}{L}),\ n \in \mathbb{N}^+. \tag{46}$$

The time-dependent ODE can now be solved now knowing that $\lambda > 0$. However, it will be different than the time-dependent ODE obtained for the undamped wave equation.
$$\begin{gathered}\frac{\partial^2 h}{\partial t^2} + \beta\frac{\partial h}{\partial t} = -c^2\lambda h \to \frac{\partial^2 h}{\partial t^2} + \beta\frac{\partial h}{\partial t} + c^2\lambda h = 0 \\ \Rightarrow h(t) = e^{pt}[A\cos(qt) + B\sin(qt)],\end{gathered} \tag{47}$$

where $p = \frac{-\beta}{2}$ and $q_n = \pm\sqrt{(\frac{n\pi c}{L})^2 - \frac{\beta^2}{4}}$. Thus,
$$h_n(t) = e^{\frac{-\beta}{2}t}\left[A_n\cos\left(\sqrt{\left(\frac{n\pi c}{L}\right)^2 - \frac{\beta^2}{4}}t\right) + B_n\sin\left(\sqrt{\left(\frac{n\pi c}{L}\right)^2 - \frac{\beta^2}{4}}t\right)\right]. \tag{48}$$

Combining 31 and the principle of superposition, we see that
$$u(x,t) = e^{\frac{-\beta}{2}t}\sum_{n=1}^{\infty}\left[A_n\cos\left(\sqrt{\left(\frac{n\pi c}{L}\right)^2 - \frac{\beta^2}{4}}t\right) + B_n\sin\left(\sqrt{\left(\frac{n\pi c}{L}\right)^2 - \frac{\beta^2}{4}}t\right)\right]\sin\left(\frac{n\pi x}{L}\right), \tag{49}$$

where $0 < x < L$ and $t > 0$. Note that the initial conditions determine A_n and B_n. Specifically, A_n can be found via the initial position, $f(x)$, which yields
$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}. \tag{50}$$

Applying the Fourier sine series, we find that
$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx. \tag{51}$$

Determining B_n is more involved being that involves two functions dependent on time. Regardless, finding B_n follows a similar procedure.

$$\frac{\partial u}{\partial t}(x,0) = g(x) = \sum_{n=1}^{\infty}\left(B_n\sqrt{\left(\frac{n\pi c}{L}\right)^2 - \frac{\beta^2}{4}} - \frac{\beta}{2}A_n\right)\sin\left(\frac{n\pi x}{L}\right). \tag{52}$$

Though not shown here due to its dependence on $g(x)$, B_n can be isolated and determined after solving for A_n.

Finally, we can derive a relationship between ω and the wavenumber k. If we assume that $u(x,t) = e^{ikx}e^{i\omega t}$, then the PDE stated in 28 yields

$$\begin{aligned}\frac{\partial^2}{\partial t^2}(e^{ikx}e^{i\omega t}) &= c^2\frac{\partial^2}{\partial x^2}(e^{ikx}e^{i\omega t}) - \beta\frac{\partial}{\partial t}(e^{ikx}e^{i\omega t})\\ -\omega^2 e^{ikx}e^{i\omega t} &= -c^2k^2e^{ikx}e^{i\omega t} - i\omega\beta e^{ikx}e^{i\omega t}\\ -\omega^2 &= -c^2k^2 - i\omega\beta\\ \omega &= \frac{i\beta}{2} \pm \frac{\sqrt{4c^2k^2-\beta^2}}{2}.\end{aligned} \tag{53}$$

Note that this relationship gives us the expected ω predicted by 49 if $k = \frac{n\pi}{L}$.

3.3 Damped Frequency Wave Equation

Another damping scheme that we will investigate is given by the PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + \gamma\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\right)^2 u,\ \ 0 < x < L,\ \ t > 0 \tag{54}$$

for $c^2 > 0$ and $\gamma \geq 0$. Note that the γ damping term in this model will lead to altered frequencies. To solve this PDE, we will look for solutions in the form

$$u(x,t,k) \simeq A(k)e^{ikx}e^{-i\omega(k)t} \tag{55}$$

Plugging (55) into the PDE in (54) yields the relationship:

$$A(k)e^{ikx}e^{-i\omega(k)t}(\omega^2 - c^2k^2 + \gamma\omega k^2) = 0 \tag{56}$$

Solving (56) for the relation between frequency, ω, and wavenumber, k, using the quadratic equation gives

$$\omega(k) = \frac{-i\gamma k^2}{2} \pm ck\sqrt{1-(\frac{\gamma k}{2c})^2} \tag{57}$$

$$= \frac{-i\gamma k^2}{2} \pm ck\mu(k) \tag{58}$$

for

$$\mu(k) = \sqrt{1-(\frac{\gamma k}{2c})^2} \tag{59}$$

Thus the form of the solution in (54) becomes

$$u(x,t,k) = e^{-\frac{\gamma k^2 t}{2}}e^{ikx}[A(k)e^{-ick\mu(k)t} + B(k)e^{ick\mu(k)t}] \tag{60}$$

However, this expression is complex for almost all real-valued choices of x, t, k. We note, however, that if (60) is a solution to (54) then so is the complex conjugate. Taking the above solution and adding the complex conjugate yields

$$2u(x,t,k) = e^{-\frac{\gamma k^2 t}{2}}e^{ikx}[A(k)e^{-ick\mu(k)t}+B(k)e^{ck\mu(k)t}]+e^{-\frac{\gamma k^2 t}{2}}e^{-ikx}[\overline{A}(k)e^{ick\mu(k)t}+\overline{B}(k)e^{-ick\mu(k)t}] \tag{61}$$

Furthermore, A(k) might be complex, even for real k values, so we can write

$$\begin{aligned} A(k) &= |A(k)|\, e^{i\alpha(k)} \\ \overline{A}(k) &= |A(k)|\, e^{-i\alpha(k)} \end{aligned} \tag{62}$$

Similarly,

$$\begin{aligned} B(k) &= |B(k)|\, e^{i\beta(k)} \\ \overline{B}(k) &= |B(k)|\, e^{-i\beta(k)} \end{aligned} \tag{63}$$

With $\alpha(k)$ and $\beta(k)$ are real-valued. These together results in (61) becoming

$$\begin{aligned} u(x,t,k) =& e^{-\frac{\gamma k^2 t}{2}}\, |A(k)| \cos(k(x - c\mu(k)t) + \alpha(k)) + \\ & e^{-\frac{\gamma k^2 t}{2}}\, |B(k)| \cos(k(x + c\mu(k)t) + \beta(k)) \end{aligned} \tag{64}$$

We can now see that this yields exactly the generalized solution of D'Alembert's solution to the wave equation. This is even more evident for $\gamma = 0$ in which case (64) is exactly the familiar form of D'Alembert's formula. Using the generalized boundary and initial conditions $A(k)$, $B(k)$, $\alpha(k)$, and $\beta(k)$ can all be solved for in the typical manor.

3.4 String Stiffness

Another alternative to 1 for modeling a guitar string is 65 which takes into account the stiffness of the string.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - \alpha \frac{\partial^4 u}{\partial x^4},\ \ 0 < x < L,\ \ t > 0, \tag{65}$$

where $\alpha \geq 0$ is the stiffness of the string.

As in the case where stiffness is disregarded, 65 is subject to the same Dirichlet boundary conditions

$$\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0, \end{aligned} \tag{66}$$

and arbitrary initial conditions

$$\begin{aligned} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t}(x,0) &= g(x), \end{aligned} \tag{67}$$

corresponding to the initial position and the initial velocity of each segment of the string respectively. The method of separation of variables can once again be utilized to find a solution to this PDE. Recall that this method asserts that we look for solutions of the form

$$u(x,t) = \phi(x)h(t) \neq 0. \tag{68}$$

Applying this solution to the PDE yields

$$\phi(x)h''(t) = c^2 h(t)(\phi''(x) + \alpha\phi^{(4)}(x)). \tag{69}$$

Next, we divide through by $c^2\phi(x)h(t)$ to obtain

$$\frac{1}{c}\frac{h''}{h} = \frac{\phi''}{\phi} + \alpha\frac{\phi^{(4)}}{\phi} = -\lambda \tag{70}$$

where λ is the separation constant.

We first consider the spatially dependent ODE in ϕ. The boundary conditions given in 66 can now be expressed as

$$\phi(0) = \phi(L) = 0, \tag{71}$$

which, when combined with the ordinary differential equations in 71, yields the following boundary value problem

$$\begin{aligned} \phi'' + \alpha\phi^{(4)} &= -\lambda\phi \\ \phi(0) &= 0 \\ \phi(L) &= 0. \end{aligned} \tag{72}$$

There are three possible case for λ that could potentially yield solutions to the ODE subject to the the prescribed boundary conditions.

1. $\underline{\lambda < 0:}$

   If λ is less than 0, then we look for solutions of the form e^{rx} and obtain the following characteristic polynomial.

$$\alpha r^4 + r^2 + \lambda = 0 \longleftrightarrow r^2(r^2 + \frac{1+\lambda}{\alpha}) = 0, \alpha \gneq 0 \tag{73}$$

   This implies that either $r^2 = 0$ or $r^2 + \frac{1+\lambda}{\alpha} = 0$. The former would yield only a constant solution which does not solve the ODE in 72 for general λ and ϕ, so we consider the latter and compute that

$$r = \pm\sqrt{\frac{|\lambda| - 1}{\alpha}} \tag{74}$$

   which yields the general solution of

$$c_1 e^{\sqrt{\frac{|\lambda|-1}{\alpha}}x} + c_2 e^{-\sqrt{\frac{|\lambda|-1}{\alpha}}x}. \tag{75}$$

   Applying the boundary condition $\phi(0) = 0$, we discover

$$\phi(0) = 0 = c_1 + c_2 \longrightarrow c_2 = -c_1 \longrightarrow \phi(x) = c_1(e^{\sqrt{\frac{|\lambda|-1}{\alpha}}x} - e^{-\sqrt{\frac{|\lambda|-1}{\alpha}}x}). \tag{76}$$

   Applying the boundary condition $\phi(L) = 0$, we obtain

$$\phi(L) = 0 = c_1(e^{\sqrt{\frac{|\lambda|-1}{\alpha}}L} - e^{-\sqrt{\frac{|\lambda|-1}{\alpha}}L}). \tag{77}$$

   The expression in the parentheses can only be 0 if the exponents themselves are also 0. This is only possible if either $L = 0$ which does not match the restriction of a positive string length, $\alpha = \infty$ which is not true of general α, or $\lambda = -1$ which again only yields a constant solution that does not solve the ODE in 72. Therefore c_1 must be 0, which implies that $\phi(x) \equiv 0$. This is the trivial solution, so λ cannot be less than 0.

2. $\underline{\lambda = 0:}$

   If λ is equal to 0, then we look for solutions of the form e^{rx} and obtain the following characteristic polynomial.

$$\alpha r^4 + r^2 = 0 \longleftrightarrow r^2(r^2 + \frac{1}{\alpha}) = 0, \alpha \gneq 0 \tag{78}$$

   This implies that either $r^2 = 0$ or $r^2 + \frac{1}{\alpha} = 0$. The former would yield only a constant solution which which would become the trivial 0 once boundary conditions in 72 were applied, so we consider the latter and compute that

$$r = \pm i\frac{1}{\sqrt{\alpha}} \tag{79}$$

   which yields the general solution of

$$c_1 e^{i\frac{1}{\sqrt{\alpha}}x} + c_2 e^{-i\frac{1}{\sqrt{\alpha}}x}. \tag{80}$$

This can be written in terms of sines and cosines as

$$A \cos \frac{1}{\sqrt{\alpha}} x + B \sin \frac{1}{\sqrt{\alpha}} x. \tag{81}$$

Applying the boundary condition $\phi(0) = 0$, we discover

$$\phi(0) = 0 = A. \tag{82}$$

Applying the boundary condition $\phi(L) = 0$, we obtain

$$\phi(L) = 0 = B \sin \frac{1}{\sqrt{\alpha}} L \longrightarrow \frac{L}{\sqrt{\alpha}} = n\pi \longleftrightarrow \frac{1}{\sqrt{\alpha}} = \frac{n\pi}{L} \tag{83}$$

The final implication in 83 is not true for general α, therefore B must be 0, which creates the trivial solution for $\phi(x)$. Therefore λ cannot be equal to 0.

3. $\underline{\lambda > 0}$:

   If λ is greater than 0, then we look for solutions of the form e^{rx} and obtain the following characteristic polynomial.

$$\alpha r^4 + r^2 + \lambda = 0 \longleftrightarrow r^2(r^2 + \frac{1+\lambda}{\alpha}) = 0, \alpha \gtrneqq 0 \tag{84}$$

   This implies that either $r^2 = 0$ or $r^2 + \frac{1+\lambda}{\alpha} = 0$. The former would yield only a constant solution which does not solve the ODE in 72 for general λ and ϕ, so we consider the latter and compute that

$$r = \pm i \sqrt{\frac{\lambda+1}{\alpha}} \tag{85}$$

   which yields the general solution of

$$c_1 e^{i\sqrt{\frac{\lambda+1}{\alpha}}x} + c_2 e^{-i\sqrt{\frac{\lambda+1}{\alpha}}x}. \tag{86}$$

   This can be written in terms of sines and cosines as

$$A \cos \sqrt{\frac{\lambda+1}{\alpha}} x + B \sin \sqrt{\frac{\lambda+1}{\alpha}} x. \tag{87}$$

   Applying the boundary condition $\phi(0) = 0$, we discover

$$\phi(0) = 0 = A. \tag{88}$$

   Applying the boundary condition $\phi(L) = 0$, we obtain

$$\phi(L) = 0 = B \sin \sqrt{\frac{\lambda+1}{\alpha}} L \longrightarrow \sqrt{\frac{\lambda+1}{\alpha}} L = n\pi \longleftrightarrow \lambda = \alpha(\frac{n\pi}{L})^2 - 1. \tag{89}$$

   Thus, the new basis solution for ϕ is

$$\phi_n(x) = C_n \sin \sqrt{\alpha(\frac{n\pi}{L})^2 - 1}\; x, \quad n \in \mathbb{Z}^+. \tag{90}$$

The time-dependent ODE can now be solved now knowing that $\lambda > 0$. It is identical to the time-dependent ODE obtained for the basic wave equation, so we can borrow the result from 20 and redisplay it below.

$$h(t) = c_1 \cos \sqrt{\lambda} ct + c_2 \sin \sqrt{\lambda} ct \tag{91}$$

Plugging in our obtained value of λ, we now have a basis solution for h.

$$h_n(t) = A_n \cos \sqrt{\alpha (\frac{n\pi}{L})^2 - 1}\ ct + B_n \sin \sqrt{\alpha (\frac{n\pi}{L})^2 - 1}\ ct, \quad n \in \mathbb{Z}^+. \tag{92}$$

Multiplying ϕ by h and absorbing constants yields a basis solution for u.

$$u_n(x,t) = \Big(a_n \cos \sqrt{\alpha (\frac{n\pi}{L})^2 - 1}\ ct + b_n \sin \sqrt{\alpha (\frac{n\pi}{L})^2 - 1}\ ct \Big) \sin \sqrt{\alpha (\frac{n\pi}{L})^2 - 1}\ x, \quad n \in \mathbb{Z}^+. \tag{93}$$

Applying the principle of superposition, we obtain a general solution for u.

$$u(x,t) = \sum_{n=1}^{\infty} \Big(a_n \cos \sqrt{\alpha (\frac{n\pi}{L})^2 - 1}\ ct + b_n \sin \sqrt{\alpha (\frac{n\pi}{L})^2 - 1}\ ct \Big) \sin \sqrt{\alpha (\frac{n\pi}{L})^2 - 1}\ x \tag{94}$$

where $0 < x < L$ and $t > 0$. Note that the initial conditions determine a_n and b_n, which can be computed similarly to how they can be computed in the basic wave equation.

Finally, we can derive a relationship between ω and the wavenumber k. If we assume that $u(x,t) = e^{ikx}e^{i\omega t}$, then the PDE stated in 65 yields

$$\begin{aligned} \frac{\partial^2}{\partial t^2}(e^{ikx}e^{i\omega t}) &= c^2 \frac{\partial^2}{\partial x^2}(e^{ikx}e^{i\omega t}) + \alpha \frac{\partial^4}{\partial x^4}(e^{ikx}e^{i\omega t}) \\ -\omega^2 e^{ikx}e^{i\omega t} &= -c^2k^2 e^{ikx}e^{i\omega t} + \alpha k^4 e^{ikx}e^{i\omega t} \\ -\omega^2 &= -c^2k^2 + \alpha k^4 \\ \omega &= \pm k\sqrt{c^2 - \alpha k^2}. \end{aligned} \tag{95}$$

4 Measured versus Modeled Data Analysis

4.1 Modeled String Vibrations

The four PDEs presented in section 3 aim to model the vibrations of a stretched string via different expressions of the intrinsic dampening. For instance, the first PDE is an idealistic model of a vibrating string whereas the second PDE models the dampening factor as a function of velocity. Due to the differences in the dampening models, each PDE will predict a slightly different frequency from the idealized frequency, which we define to be the frequency predicted by the idealized model.

The first, idealized PDE,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{96}$$

yields a frequency of

$$\omega = \frac{n\pi c}{L}. \tag{97}$$

The second PDE,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}, \tag{98}$$

yields a frequency of

$$\omega = \frac{i\beta}{2} \pm \frac{\sqrt{4c^2k^2 - \beta^2}}{2}. \tag{99}$$

The third PDE,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + \gamma \frac{\partial}{\partial t} \left(\frac{\partial}{\partial x} \right)^2 u, \tag{100}$$

yields a frequency of

$$\omega(k) = \frac{-i\gamma k^2}{2} \pm ck\sqrt{1 - (\frac{\gamma k}{2c})^2}. \tag{101}$$

The fourth PDE,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - \alpha \frac{\partial^4 u}{\partial x^4}, \tag{102}$$

yields a frequency of

$$\omega = \pm k\sqrt{c^2 - \alpha k^2} \tag{103}$$

Note that in the absence of a dampening factor, such as when $\beta = 0$ and $\gamma = 0$ in equations 67 and 69 respectively, the expressions for ω reduce down to the expression of ω found for the idealized PDE in Eq. 65. In addition, the dampening factors, β, γ, and α cause a slight reduction in the frequency from the idealized case.

In all four cases, though, the frequency is dependent upon the boundary conditions of the physical system. Recalling that the wavenumber $k = \frac{n\pi c}{L}$ is dependent upon L, the length of the string, we expect that variations in this length will directly affect the predicted and measured frequencies. Consider, for example, Eq. 66. By doubling the length of the string to $2L$ we halve the frequency thereby establishing a direct proportionality between frequency and the length of string.

This was also observed by measuring the frequency when holding the guitar strings down at a fret. Each fret essentially represents a different boundary condition. Rather than the boundary conditions being

$$\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0, \end{aligned} \tag{104}$$

the boundary conditions become

$$\begin{aligned} u(x_{fret_n},t) &= 0 \\ u(L,t) &= 0. \end{aligned} \tag{105}$$

Being that the strings were shorter, the measured frequencies were higher as the PDEs predicted.

4.2 Measured Data

Table 1 lists the frequency of the six basic notes on a six-string guitar that were measured in our experiment, and table 2 lists the measured string lengths for each fundamental note.

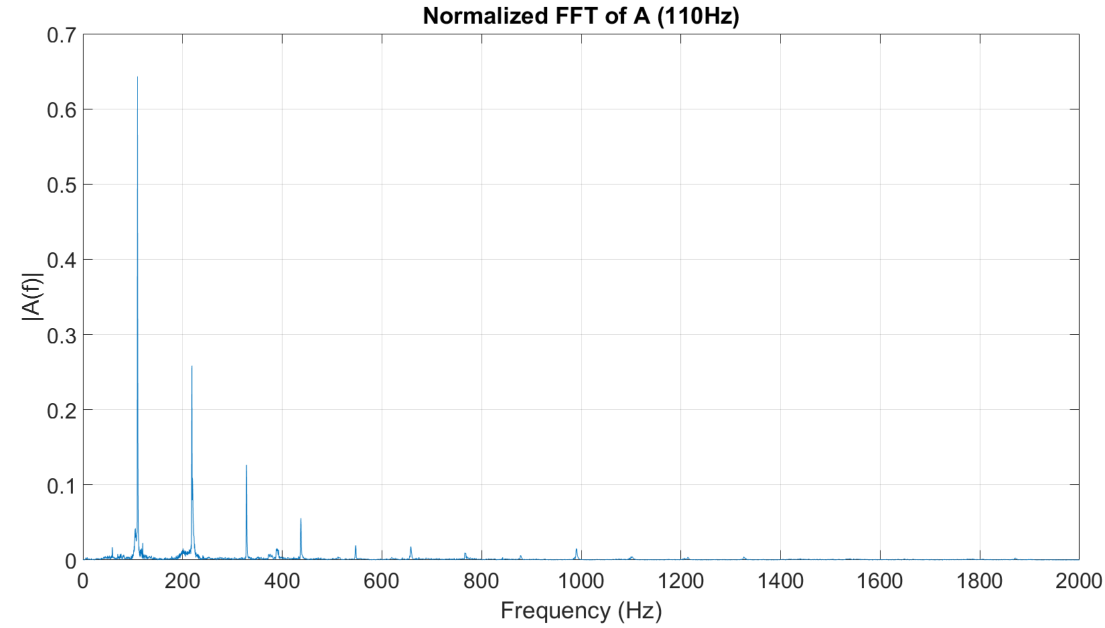
Note	Frequency
E	82 Hz
A	110 Hz
D	147 Hz
G	196 Hz
B	247 Hz
E	330 Hz

Table 1: Notes on a basic six-string guitar.

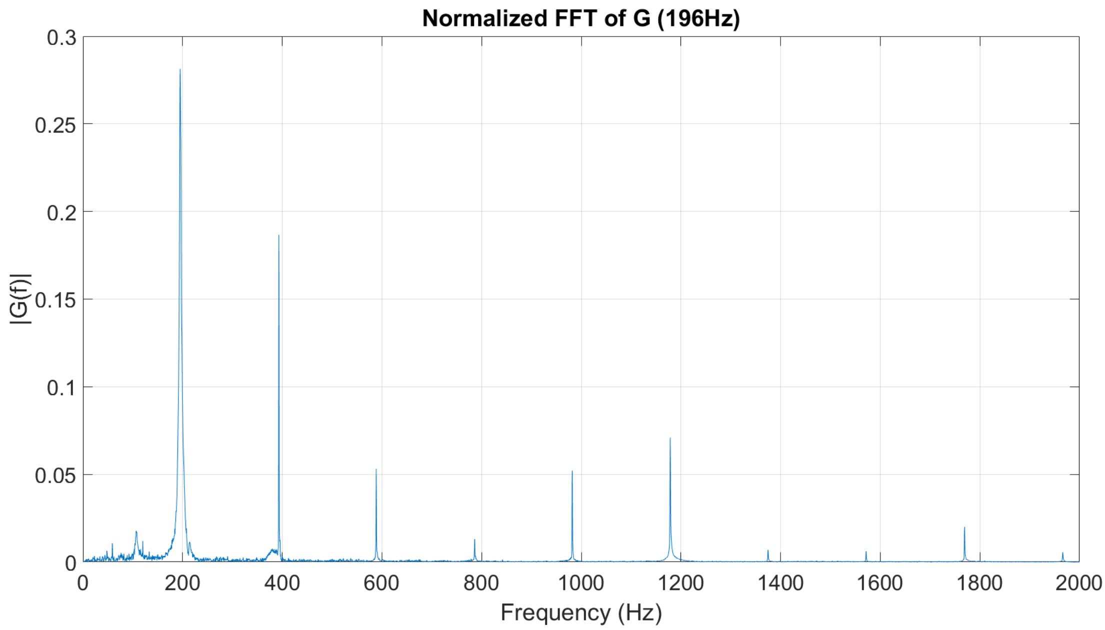
Note	String Length
E	65.30 cm
A	65.25 cm
D	65.20 cm
G	65.10 cm
B	65.20 cm
E	65.00 cm

Table 2: Measured string length of a basic six-string guitar.

Figure 1 depicts the normalized Fast Fourier transforms of the second and fourth vibrating strings of the six-string guitar. Each of these plots reveal clear harmonics that are evenly spaced at integer multiples of the fundamental frequency. For instance, Fig. 1a reveals the fundamental frequency of 110Hz (Note A) and harmonics at roughly 220Hz and 440Hz.



(a) Normalized FFT of A (100Hz)



(b) Normalized FFT of G (196Hz)

Figure 1: Normalized FFTs of A (100Hz) and G (196Hz).

Due to measurement error, though, the experimental data reveals that the frequencies are not necessarily centered on their expected values. Table 3 summarizes the experimental fundamental frequencies as reported by MATLAB.

Note	Measured Fundamental Frequency
E	82 Hz
A	109 Hz
D	147 Hz
G	195 Hz
B	248 Hz
E	329 Hz

Table 3: Fundamental frequencies of experimental data.

This information allows us to calculate a value for the velocity c in the wave equation. Specifically, we have a value for the length, L, and a value for the measured frequency, which is related to the angular frequency $\omega = 2\pi f$. Combined with the equation $\omega = \frac{n\pi c}{L}$, the velocity can be calculated. Table 4 summarizes the velocity of the waves on each of the six guitar strings. It is important to note, though, that $L = \frac{\lambda}{2}$.

Note	c (m/s)
E	207.09
A	284.49
D	383.38
G	507.78
B	646.78
E	855.40

Table 4: Calculated wave velocity, c on a basic six-string guitar.

4.3 Multiple Strings

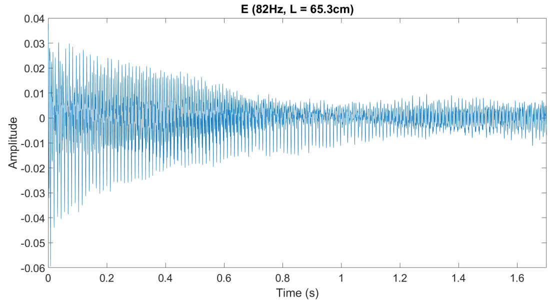
From the data of multiple strings being strummed at once, a clear peak in the FFT was visible at each string's frequency.

A common chord is the C major chord, which consists of a C (261.63Hz), a E (329.63Hz), and a G (392.00Hz). The harmonic for each of the notes in the chord is an integer multiple of its fundamental frequency. An octave higher corresponds to a frequency doubling, so those integers which are powers of two have the effect of raising the entire chord up one or more octaves.

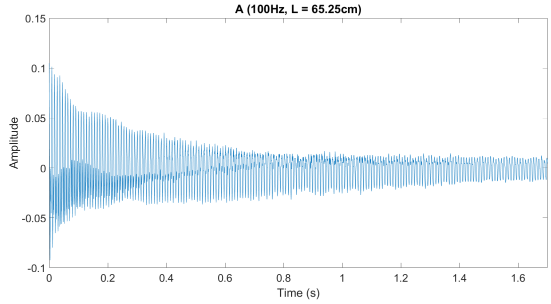
5 Time Scale of Decay Due to Dampening

Over time, the vibrations diminish until the string reaches its steady-state. This is due to the dampening of the strings and can be observed in a plot of amplitude versus time as depicted in Fig. 2. These figures suggest that the time scale of decay due to dampening is dependent upon the note played, which in turn corresponds to different guitar strings. That is, low E appears to decay quicker than A, and A appears to decay quicker than B. We can quantify this observation by extracting the envelope of the measured amplitude of each note and determining the amount of time required for the envelope to decay to the noise floor.

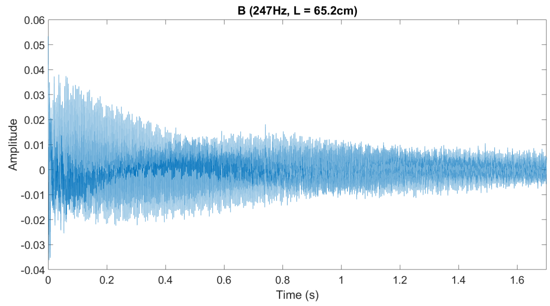
Extracting the envelope is accomplished by taking the magnitude of the Hilbert transform of each signal, as shown in Fig. 3 where the extracted envelope, depicted in red, is overlaid upon the original signal.



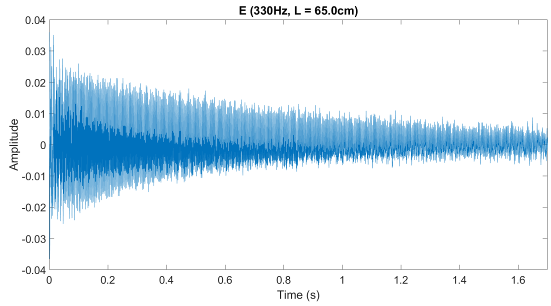
(a) Amplitude v. Time of Low E



(b) Amplitude v. Time of A

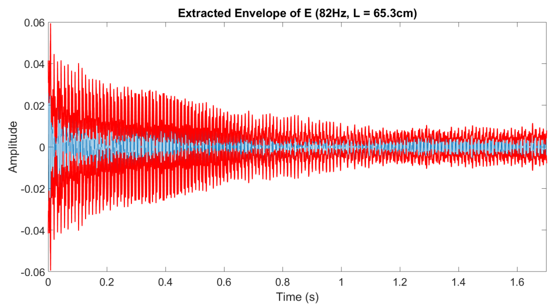


(c) Amplitude v. Time of B

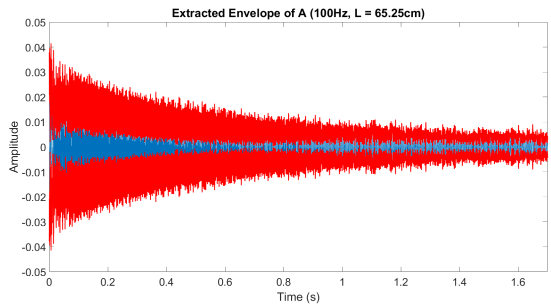


(d) Amplitude v. Time of High E

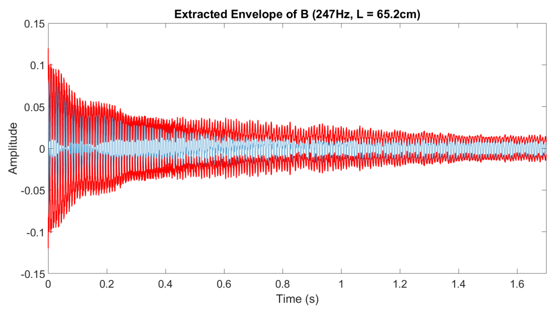
Figure 2: Amplitude versus Time Plots of Low E (82Hz), A (100Hz), B (247Hz), and High E (330Hz).



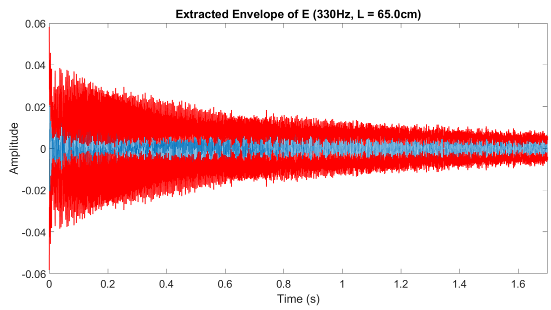
(a) Extracted Envelope of Low E



(b) Extracted Envelope of A



(c) Extracted Envelope of B



(d) Extracted Envelope of High E

Figure 3: Extracted Envelopes of Low E (82Hz), A (100Hz), B (247Hz), and High E (330Hz). The envelope is highlighted in red and the original signal is depicted in blue.

Focusing upon the extracted envelope of E (82Hz) shown in Fig. 4, we see that the amplitude reaches the noise floor in approximately 1.1s.
Likewise, A (100Hz) fully decays in approximately 1.35s, B (247Hz) fully decays in approximately 1.5s, and E (330Hz) fully decays in approximately 1.7s. This suggests that the time scale of decay is dependent upon which string is measured. Most notably, thinner strings that correspond to higher frequency notes appear to take longer to fully decay.

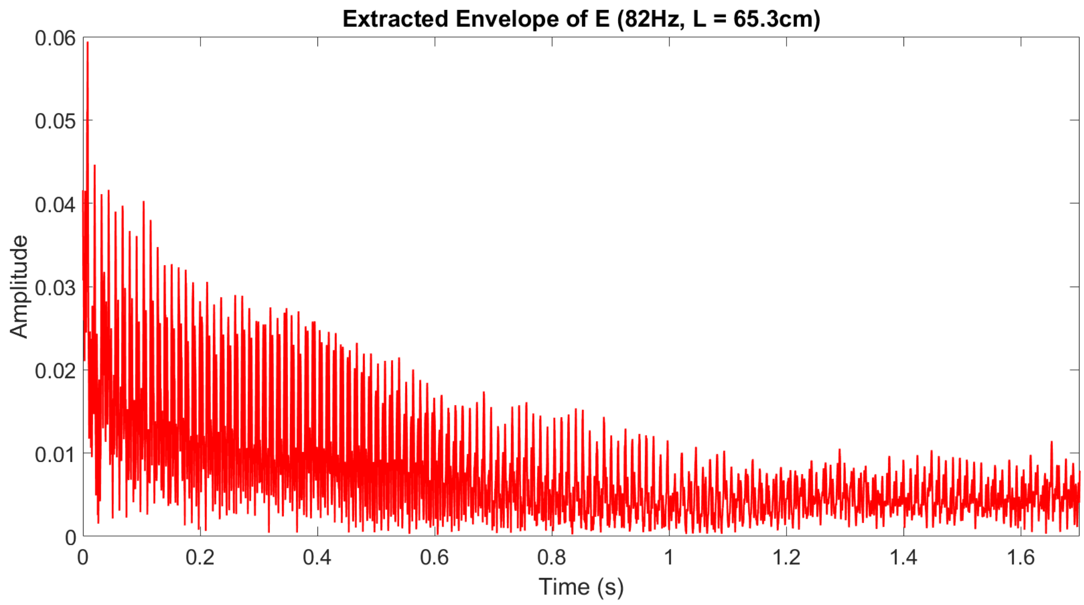


Figure 4: Extracted Envelope of Low E (82Hz).

6 Dampened String Model

The principal goal of this project is to investigate a better model for the evolution of a string over time than that of the classic wave equation. From figure 5 below we can see the problem with using the simple wave equation when applying it to physical data.

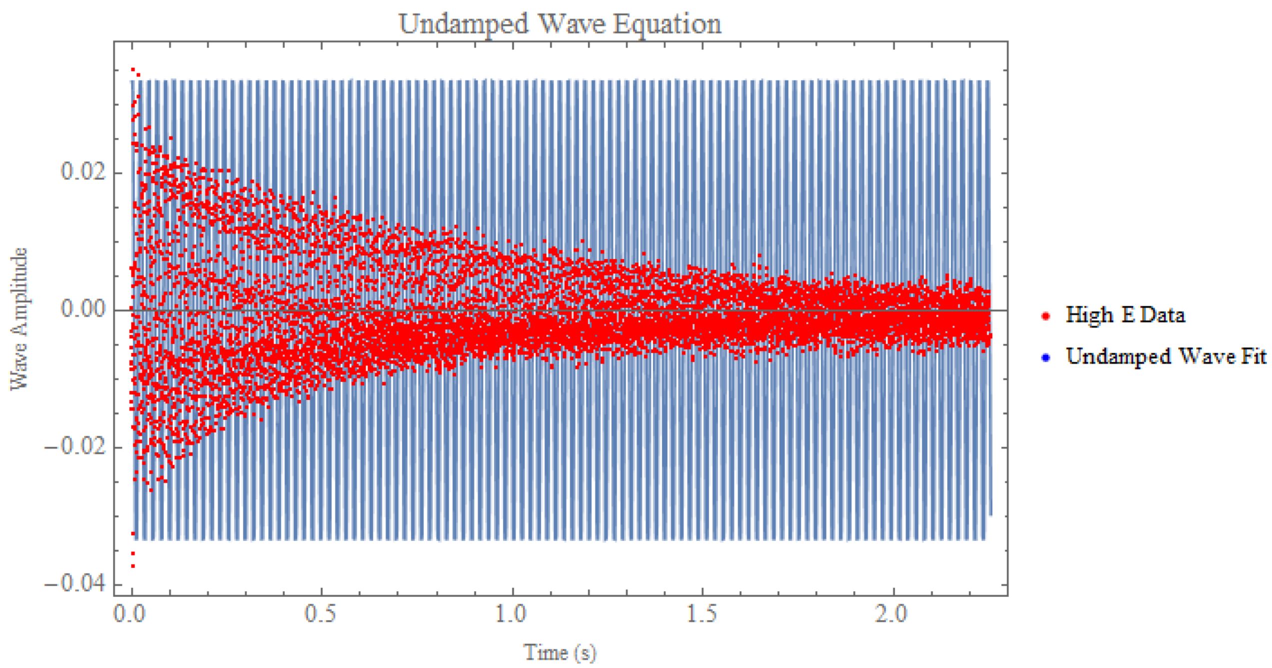


Figure 5: Best-fit model of high E string amplitude vs. time using the simple wave equation.

Here, the wave characteristics are determined by the initial conditions of the plucked string, and maintain those characteristics for all time. Clearly, however, due to dampening the actual amplitude of the wave decreases over time until it reaches the floor level. Using the models developed in this investigation, however, we can develop a much more realistic fit to our data. Take, for example, just the simple damped model described by (28) with solution (49). Once again taking the initial conditions to be a plucked string with no initial velocity and only the first ten terms of the Fourier

Series, we get the following fit

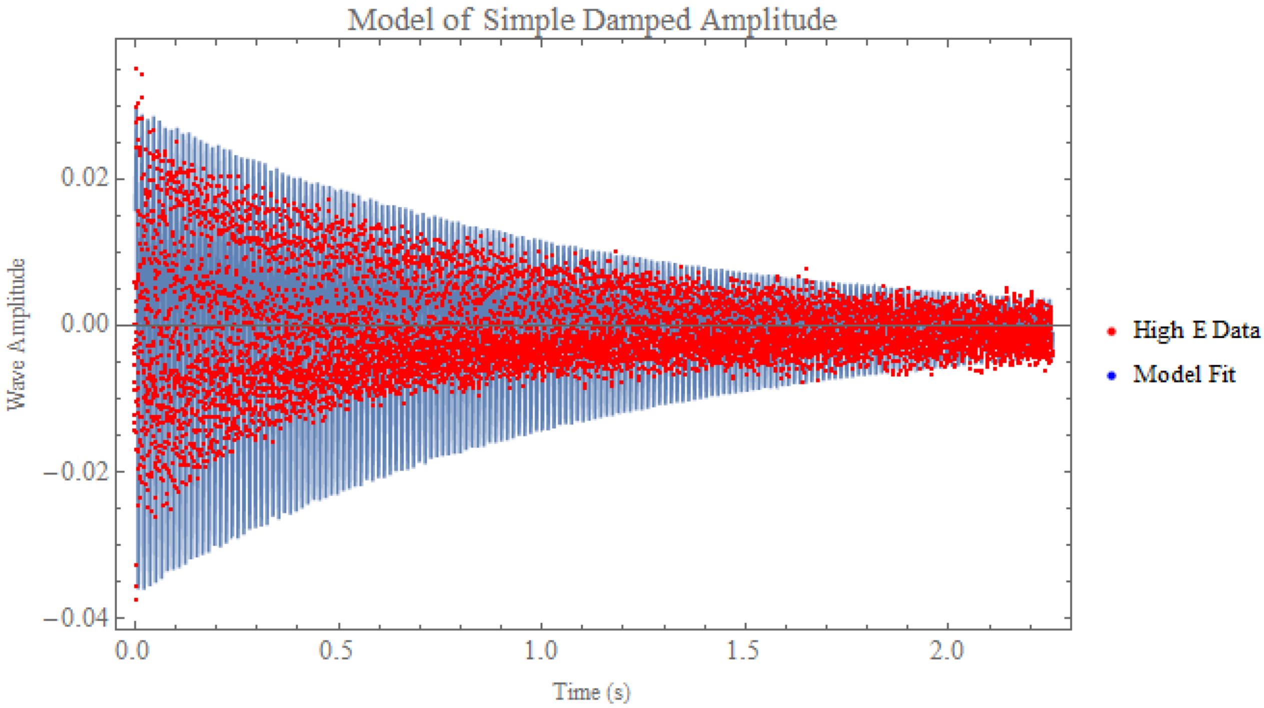


Figure 6: Best-fit model of high E string amplitude vs. time using the simple damped wave equation solution (49).

Clearly figure 6 is more realistic for our physical case. Furthermore, from this fitting processes we can determine the damping constant β, which was found to be $1.87\frac{1}{s}$ for the high E string. In addition, this fitting gives us the the experimental value of c in our best-fit, which was found to be $90.34\frac{m}{s}$. This leads to an experimentally determined principle frequency of $436Hz$ from the relationship in (91).