

Training and Workshop on Python Programming and Numerical Methods

An Introduction to Numerical Linear Algebra and Differential Equations

RHUDAINA MOHAMMAD

Numerical Analysis & Scientific Computing Group

Institute of Mathematics, UP Diliman

rzmohammad@up.edu.ph



– LUNCH –
12:00 – 13:00

Hands-on Activity

2. Generate a set of $n = 100$ data points (x_i, y_i) for $i = 1, \dots, n$ with

$$\frac{1}{2} = x_1 < x_2 < \dots < x_{n-1} < x_n = 1$$

$$y_i = ae^{b(x_i+1)} + \delta_i, \quad i = 1, 2, \dots, n$$

where $a = b = 2$. Assume that δ_i 's are independent and normally distributed with mean 0 and variance 1.

Although the data points exhibit a nonlinear relationship, we can transform the problem into a linear form by taking the natural logarithm as follows.

$$\ln y = \ln a + b(x + 1)$$



Hands-on Activity

This leads to the following linear system

$$\begin{bmatrix} 1 & x_1 + 1 \\ 1 & x_2 + 1 \\ \vdots & \\ 1 & x_n + 1 \end{bmatrix} \begin{bmatrix} \ln a \\ b \end{bmatrix} = \begin{bmatrix} \ln y_1 \\ \ln y_2 \\ \vdots \\ \ln y_n \end{bmatrix}$$

Determine the best exponential curve that fits the given data points by solving the corresponding linear least squares problem via singular value decomposition. In one figure, plot the true and fitted nonlinear curves with the given data points. Discuss your results.



MORE ON SVD

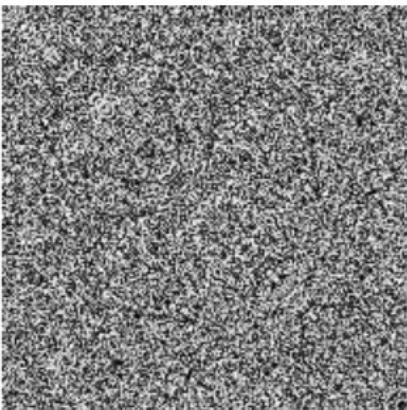
Dimensionality Reduction

High dimensionality is a common challenge in processing data from complex systems.

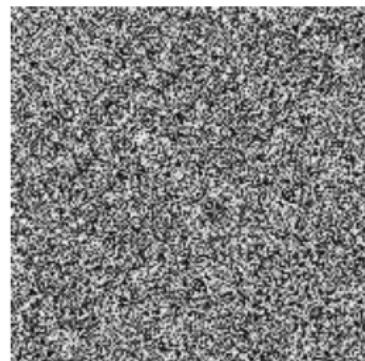
Dimensionality Reduction



$$A \in \mathbb{R}^{192 \times 168}$$



$$AA^T \in \mathbb{R}^{192 \times 192}$$



$$A^T A \in \mathbb{R}^{168 \times 168}$$

- ▶ Both correlation matrices AA^T and $A^T A$ are symmetric.
 - AA^T is formed by taking inner product of rows of A
 - $A^T A$ is formed by taking inner product of columns of A



Dimensionality Reduction

- ▶ Note that

$$A = U\Sigma V^T = U \begin{bmatrix} \hat{\Sigma} \\ 0 \end{bmatrix} V^T$$

Then,

$$AA^T = U \begin{bmatrix} \hat{\Sigma} \\ 0 \end{bmatrix} \underbrace{V^T V}_I \begin{bmatrix} \hat{\Sigma} & 0 \end{bmatrix} U^T = U \begin{bmatrix} \hat{\Sigma} & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$A^T A = V \begin{bmatrix} \hat{\Sigma} & 0 \end{bmatrix} \underbrace{U^T U}_I \begin{bmatrix} \hat{\Sigma} \\ 0 \end{bmatrix} V^T = V \hat{\Sigma}^2 V^T$$

- ▶ To get the singular values, it is better to compute the **eigendecomposition of $A^T A$** , which is smaller and more manageable than AA^T .
- ▶ This gives V and $\hat{\Sigma}$. From $\hat{\Sigma}$, we only keep p nonzero singular values in Σ_p ; and their corresponding columns V_p of V . From these matrices, we can construct $U_p = A V_p \Sigma_p^{-1}$



Dimensionality Reduction

- ▶ Since all nonzeros entries of Σ are in the diagonal, then we can write

$$A = U\Sigma V^T = \sum_{i=1}^m \sigma_i u_i v_i^T = \underbrace{\sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_m u_m v_m^T}_{\text{sum of rank-1 matrices}}$$

where singular values σ_i is the i th diagonal entry of Σ such that

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m \geq 0$$

Here, u_i and v_i are the i th columns of singular vectors U and V , respectively.

- ▶ For many systems, singular values decrease rapidly, and so, we can get a good approximation of A by truncating at some rank p :

$$A \approx A_p = \sum_{i=1}^p \sigma_i u_i v_i^T = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_p u_p v_p^T$$



Truncated SVD

$$A_{n \times m} = \underbrace{\begin{matrix} U_p \\ n \times p \end{matrix} * \begin{matrix} \Sigma_p \\ p \times p \end{matrix} * \begin{matrix} V_p^T \\ r \times m \end{matrix}}_{U \in \mathbb{R}^{n \times n}} \underbrace{\begin{matrix} 0 & * \\ * & 0 \end{matrix}}_{\Sigma \in \mathbb{R}^{n \times m}} \underbrace{\begin{matrix} * \\ 0 \end{matrix}}_{V \in \mathbb{R}^{m \times m}}$$

$$\approx \begin{matrix} U_p \\ n \times p \end{matrix} \quad \begin{matrix} \Sigma_p \\ p \times p \end{matrix} \quad \begin{matrix} V_p^T \\ p \times m \end{matrix}$$



Image Compression via Truncated SVD

Consider a grayscale image with n pixels in the vertical direction and m pixels in the horizontal direction

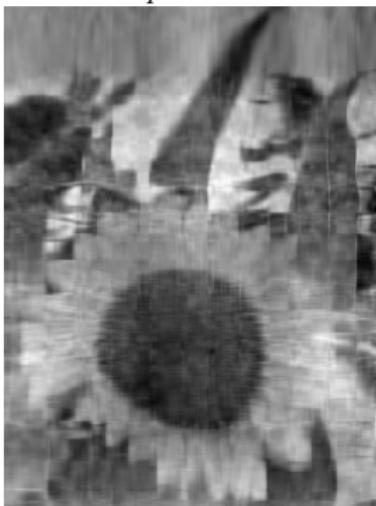
- Let A be an $n \times m$ matrix representation of a 8-bit grayscale image
- pixel values denote light intensities from 0 (black) to 255 (white)

(UP Diliman, 2021)



$A \in \mathbb{R}^{4000 \times 3000}$

$p = 20$



about 50% image
variance

$p = 200$



almost 80% image
variance



Hands-on Activity

Take a picture with a new friend from this workshop. Convert your colored digital image to blue B, green G, and red R intensity matrices. For each intensity matrix (channel), take its singular value decomposition and plot the singular values and corresponding image variance. Determine the lowest possible p so that the truncated SVDs yield at least 80% image variance in all color channels. Plot the corresponding compressed colored image.



– BREAK –
15:00 – 15:30