$$\mathbb{E}[V] = \sum_{x \in Dom(X)} P(x)V(x)$$

$$\mathbb{E}[V(Decision)] = \sum_{outcome} P(outcome|decision)\, V(outcome)$$

sum out what we can't control

Agents $\xrightarrow{\text{perform}}$ Actions $\xrightarrow{\text{results in}}$ Outcome

must. Even doing nothing is a choice

Usually not given. Only a chance of happening.

Lottery

Distribution over outcomes

$[p_1 : o_1, \dots, p_k : o_k]$

sum to 1

They **prefer** better outcomes

They are **rational**: do action w/ best outcome

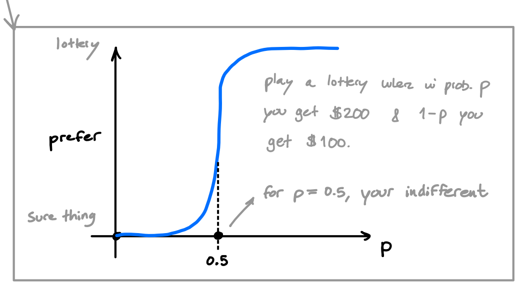
Preference

For any 2 outcomes o_1, o_2, either

Weak Preference	$o_1 \succeq o_2$	"at least as desireable"
Strong Preference	$o_1 \succ o_2$	"more desireable"
Indifferent	$o_1 \sim o_2$	"as desireable"

properties:

Complete	$o_1 \succeq o_2$ or $o_2 \succeq o_1$	
Transitive	$o_1 \succeq o_2 \wedge o_2 \succeq o_3 \Rightarrow o_1 \succeq o_3$	
Monotonic	$o_1 \succeq o_2$ & $p > q \Rightarrow [p : o_1, 1-p : o_2] \succeq [q : o_1, 1-q : o_2]$	prefers larger chance of better outcome.
Continuity	$o_1 \succ o_2 \wedge o_2 \succ o_3 \Rightarrow o_2 \sim \exists p, [1-p : o_1, p : o_3]$	



Given these properties we can define...

Utility Function:

Map preference to numerical values

Utility: Outcome $\rightarrow [0, 1]$

doesn't need to be in this range. But can always normalize

such that

① $o_1 \succeq o_2 \iff U(o_1) \geq U(o_2)$

Linear with Prob.s

② $U([p_1 : o_1, \dots, p_k : o_k]) = p_1 U(o_1) + \dots + p_k U(o_k)$

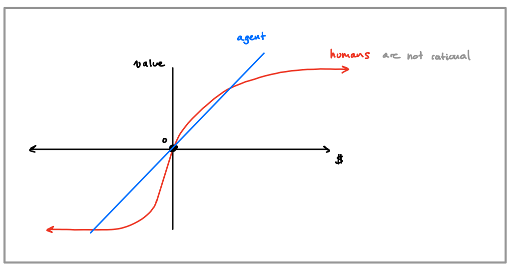
Rational Agents

Act to maximize expected utility.

a_1 results in, a_2 results in

$$U([p_1 : o_1, \dots, p_k : o_k]) \geq U([q_1 : o_1, \dots, q_k : o_k])$$

agent chooses a_1



Decision Theory

How to **tradeoff** preferences & probability of outcomes, between actions.

Agent $\longrightarrow$ action?

depends on

ability, beliefs — Std. Bayes Net; preferences (& tradeoffs)

Decision Network

specified by

decision Variables, Utility

Decision Variables

Agent $\xrightarrow{\text{chooses value}}$ $D = d_i$ $\xrightarrow{\text{leads to}}$ ω (outcome)

expected utility: $\mathcal{E}(u|D=d_i) = \sum P(\omega|D=d_i)\, u(\omega)$

Optimal Decision: $\arg\max_{d_i} \mathcal{E}(u|D=d_i)$

Decision Networks

For sequential decision problems

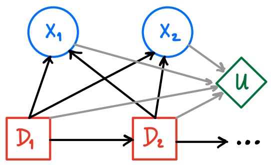
? $\longrightarrow$ (X) $P(X|?)$

? $\longrightarrow$ [D_i] Info Available when decision D is made (when agent has to act)

? $\longrightarrow$ ◇ u Utility depends on

$$\mathcal{E}(u|D) = \sum_{X_1 \dots X_n} P(X_1,\dots,X_n|D)\, U(X_1,\dots,X_n, D)$$

$$\prod_{j=1}^{n} P(X_j|D)$$



Optimal Decision:

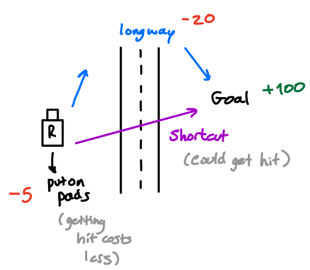
① Factor for each ____ and for ____

② Multiply, sum out : $\sum$ ____ ____ $\longrightarrow$ factor for $\mathcal{E}(u|D)$

③ Choose d_i with max value in that factor

Same as BN-VE but:
- ordering is fixed
- Utility factor
- Maximization step.

Delivery Robot



Decision Var's :

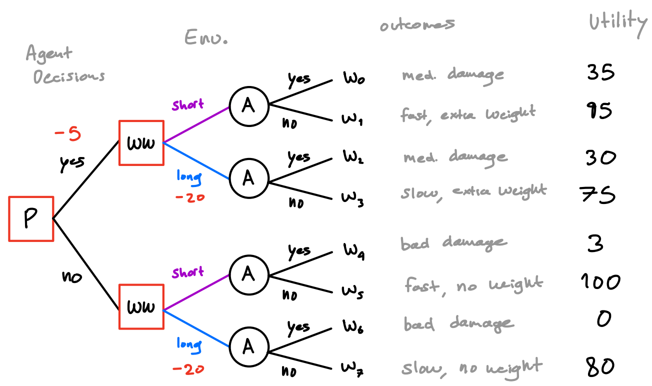
① Put on pads (P) : yes/no

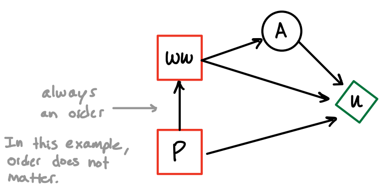
② Which way (ww) : long/short

Environment Var's :

① Accident (A) : yes/no

ww	A	P(A|ww)
long	yes	0.01
long	no	0.99
Short	yes	0.2
Short	no	0.8



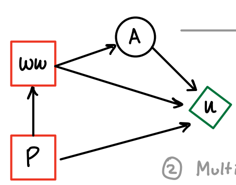


$$\mathcal{E}[u|P,ww] = \sum_{a \in A} P(A=a|P,ww)\, u(P,a,ww)$$

w.w	wp	$\mathcal{E}[u]$
S	T	83
ℓ	T	74.6
S	F	80.6
ℓ	F	79.2

As factors:

① Create Factors



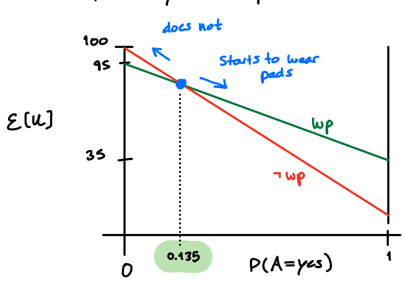
$f_1(A, ww) = P(A|ww)$

$f_2(A, ww, p) = P(u|A, ww, p)$

② Multiply & Sum Out

$$f_3(u, ww, p) = \sum_A f_1(A, ww)\, f_2(A, ww, p)$$
$$= \mathcal{E}(u|ww, p)$$

What probability of A, p would make the robot put on the pads?



$35p + (1-p)95 = 3p + (1-p)100$

accident

no accident

$p = 0.135$

Sequential Decisions $D_1, \dots D_n$

Each D_i has an information set parents(D_i) whose values will be know when its time to make D_i

$D_1 \rightarrow D_2 \rightarrow \dots$ parents(D_i) $\rightarrow D_i$

Actions depend on previous observations

Policy Sequence $\delta_1, \delta_2, \dots, \delta_n$

$$\underbrace{\delta_i(o)}_{\text{dom[parents}(D_i)]} = \underbrace{d_i}_{\text{dom}[D_i]}$$

expected Utility

$$\boxed{\mathcal{E}(u \mid \delta) = \sum_{\omega \models \delta} P(\omega)\, u(\omega)}$$

all worlds possible with this policy

Algorithm: Find δ^* (optimal policy)

Init:

Factors: each CPT, Utility Function

remaining D_i's $\leftarrow$ $\boxed{D_1, D_2 \dots D_n}$ (all of them)

Repeat:

$D_x \cdots D_y \cdots D_z$ D_i

ALL random var's that aren't parent of remaining

V.E Algorithm (1 by 1 in any order)

$\rightarrow f_{k+1}$

Pick D_i w/ no children. $\leftarrow$ i.e. In reverse order.

Eliminate by maximizing.

$f_{k+2} = \max_{D_i} f_{k+1}$ $\mathcal{E}(u \mid D_i)$

$\delta_i^* = \arg\max_{D_i} f_{k+1}$ (optimal for D_i) $\rightarrow \delta_1, \delta_2, \dots, \delta_i$

Note:

Pool does not reset.

Add f_{k+1} to pool.

$\downarrow \emptyset$

End:

① Eliminate remaining variables $\rightarrow \dfrac{\mathcal{E}(u \mid \delta^*)}{P(\text{Evidence})}$

② Divide by P(Evidence) (if any)

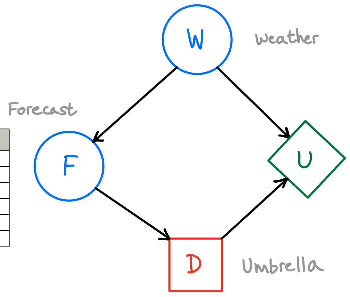
Example – Umbrella

W	P(W)
clear	0.7
rain	0.3

$f_1(W)$

W	F	P(F|W)
clear	sun	0.7
clear	cloud	0.2
clear	rain	0.1
rain	sun	0.15
rain	cloud	0.25
rain	rain	0.6

$f_2(W,F)$



W	D	U
clear	yes	20
clear	no	100
rain	yes	70
rain	no	0

$f_0(W,D)$

$$\mathcal{E}(U|D) = \sum_{W,F} P(W,F)\,U(W,D) = \sum_{W,F} f_0(W,D)\,f_1(W)\,f_2(W,F)$$

① **Eliminate W (VE)**

$$f_3(F,D) = \sum_{W} f_0(W,D)\,f_1(W)\,f_2(W,F) =$$

F	D	f_3
sun	yes	12.95
sun	no	49
cloud	yes	8.05
cloud	no	14
rain	yes	14
rain	no	7

② **Max out D**

$$f_4(F) = \max_{D} f_3(F,D) =$$

F	D	f_4
sun	yes	12.95
sun	no	49
cloud	yes	8.05
cloud	no	14
rain	yes	14
rain	no	7

$$\therefore\ \delta_D(F) =$$

F	δ(F)
sun	no
cloud	no
rain	yes

Sum-out F :

$$f_5(\,) = \sum_{F} f_4(F) = 49 + 14 + 14 = 77$$

(expected) value of optimal δ

This time, we observe F = cloud

policy is D = no. But $\mathcal{E}(D = no) \neq 14$

because this value considers F = cloud being unlikely

$$\mathcal{E}(U|D) = \sum_{W} P(W, F = cloud)\,U(W,D)$$

$f_2 \leftarrow f_2(W, F = cloud) =$

W	F	f_2
clear	sun	0.7
clear	cloud	0.2
clear	rain	0.1
rain	sun	0.15
rain	cloud	0.25
rain	rain	0.6

$=$

W	f_2
clear	0.2
rain	0.25

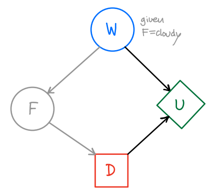
unnormalized

$f_1(W) \leftarrow P(W, F = cloudy)\,P(W) =$

W	f_1
clear	0.21
rain	0.075

(0.2) 0.7
(0.25) 0.3

unnormalized



① **Eliminate W (VE)**

$$f_7(D) = \sum_{W} f_0(W,D)\,f_6(W) =$$

D	f_7
yes	8.05
no	14

② **Max out D**

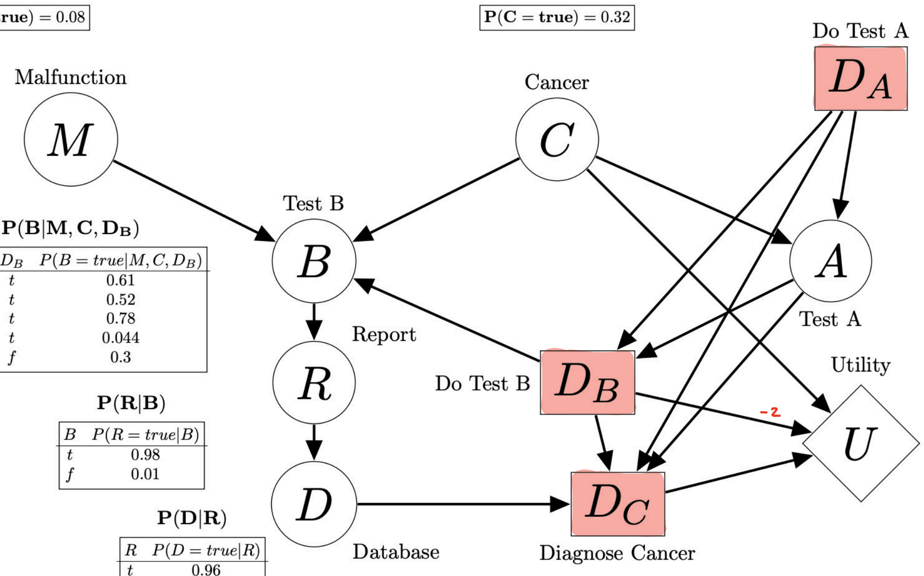
$$f_8(\,) = \max_{D} f_3(D) = 14 \qquad \therefore\ \delta_D = no$$

Divide by P(Evidence) to get expected value:

$$\sum_{W} P(W, F = cloudy)\,P(W) = \underbrace{(0.2)\,0.7}_{W=clear} + \underbrace{(0.25)\,0.3}_{W=rain} = 0.215 \longrightarrow \frac{14}{0.215} = 65$$

$\mathbf{P(M = true) = 0.08}$

$\mathbf{P(C = true) = 0.32}$



$\mathbf{P(B|M, C, D_B)}$

M	C	D_B	$P(B = true\|M, C, D_B)$
t	t	t	0.61
t	f	t	0.52
f	t	t	0.78
f	f	t	0.044
–	–	f	0.3

$\mathbf{P(R|B)}$

B	$P(R = true\|B)$
t	0.98
f	0.01

$\mathbf{P(D|R)}$

R	$P(D = true\|R)$
t	0.96
f	0.001

$\mathbf{P(A|C, D_A)}$

C	D_A	$P(A = true\|C, D_A)$
t	t	0.7
f	t	0.35
–	f	0.462

$\mathbf{U(C, D_B, D_C)}$

Max reward: 20

C	D_B	D_C	$U(U, D_B, D_C)$
t	t	t	18
t	t	f	0
t	f	t	20
t	f	f	2
f	t	t	2
f	t	f	18
f	f	t	4
f	f	f	20

TP
FN -18
TN
FP -16

<u>Init</u>: Factors:

$f_0(C) =$

C	f_0
t	0.32
f	0.68

$f_1(M) =$

M	F_1
t	0.08
f	0.92

$f_2(B,M,C,DB) =$

B	M	C	D_B	f_3
t	t	t	t	0.61
t	t	t	f	0.39
t	f	t	t	0.52
t	f	t	f	0.48
f	t	t	t	0.78
f	t	t	f	0.22
f	f	t	t	0.044
f	f	t	f	0.956
–	–	f	t	0.3
–	–	f	f	0.7

$f_3(R,B) =$

B	R	f_3
t	t	0.98
t	f	0.02
f	t	0.01
f	f	0.99

$f_4(D,R) =$

R	D	f_4
t	t	0.96
t	f	0.04
f	t	0.001
f	f	0.999

$f_5(A,C,D_A) =$

C	DA	A	f_5
t	t	t	0.7
t	t	f	0.3
f	t	t	0.35
f	t	f	0.65
–	f	t	0.462
–	f	f	0.538

$f_6(C,D_B,D_C) =$

C	D_B	D_C	U
t	t	t	18
t	t	f	0
t	f	t	20
t	f	f	2
f	t	t	2
f	t	f	18
f	f	t	4
f	f	f	20

---

① <u>Remaining</u>: D_A, D_B, D_C

No child in $\uparrow$: M, C, B, R

No children: D_C

<u>Variable Elim</u>:

$f_7(B,C,D_B) = \sum_M f_1(M) f_2(B,M,C,D_B)$

$f_8(R,C,D_B) = \sum_B f_3(R,B) f_7(B,C,D_B)$

$f_9(D,C,D_B) = \sum_R f_4(D,R) f_8(R,C,D_B)$

$f_{10}(A,D,D_A,D_B,D_C) = \sum_C f_0(C) f_5(A,C,D_A) f_6(C,D_B,D_C) f_9(D,C,D_B)$

Pool

$f_0(C)$ $\cancel{f_2(B,M,C,DB)}$ $f_4(D,R)$ $\cancel{f_1(M)}$ $f_3(R,B)$ $f_5(A,C,D_A)$ $f_6(C,D_B,D_C)$

$f_0(C)$ $\cancel{f_3(R,B)}$ $f_4(D,R)$ $f_5(A,C,D_A)$ $f_6(C,D_B,D_C)$ $\cancel{f_7(B,C,D_B)}$

$f_0(C)$ $\cancel{f_4(D,R)}$ $f_5(A,C,D_A)$ $f_6(C,D_B,D_C)$ $\cancel{f_8(R,C,D_B)}$

$\cancel{f_0(C)}$ $\cancel{f_5(A,C,D_A)}$ $\cancel{f_6(C,D_B,D_C)}$ $\cancel{f_9(D,C,D_B)}$

<u>Max Out</u>:

$f_{11}(A,D,D_A,D_B) = \max_{D_C} f_{10}(A,D,D_A,D_B,D_C)$ expected Utility

$\delta_C^* = \arg\max_{D_C} f_{10}(A,D,D_A,D_B,D_C)$ policy

② <u>Remaining</u>: D_A, D_B

No child in $\uparrow$: M, C, B, R, D

No children: D_B

<u>Variable Elim</u>: relevent: D

$f_{12}(A,D_A,D_B) = \sum_D f_{11}(A,D,D_A,D_B)$

Pool

$\cancel{f_{11}(A,D,D_A,D_B)}$

<u>Max Out</u>:

$f_{13}(A,D_A) = \max_{D_B} f_{12}(A,D_A,D_B)$

$\delta_B^* = \arg\max_{D_B} f_{10}(A,D,D_A,D_B,D_C)$

③ <u>Remaining</u>: D_A

No child in $\uparrow$: All

<u>Variable Elim</u>: relevent: A

$f_{14}(D_A) = \sum_A f_{13}(A,D_A)$

Pool

$\cancel{f_{13}(A,D_A)}$

<u>Max Out</u>:

$f_{15}(\,) = \max_{D_A} f_{14}(D_A)$

$\delta_B^* = \arg\max_{D_A} f_{14}(D_A)$

<u>Final Policy</u>

do test A
  if A= (+)
    do test B:
    diagnose based on D
  else:
    diagnose no cancer

Final Expected Value: 15.9