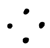
Handshake Lemma

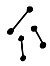
$$\sum_{v \in V} \deg(v) = 2|E|$$

Corollary 1

Σ(vertices w/ odd degree) = even



0-regular



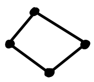
1-regular

Regular $\frac{pk}{2}$

Cycles

2-regular

"cycles"



Complete $\binom{p}{2}$

(p−1) − regular

n-cube $n2^{n-1}$

n-regular

bipartite

no odd cycles

Bipartite

Complete-Bipartite $|A||B|$

odd cycle => not bipartite
Bipartite ⟺ every subgraph is bipartite

} Bipartite ⟺ no odd cycles

Sequences

Theorem

uv − walk ⟶ ∃ : uv − path

Corollary

∃ : uv-path ∧ ∃ : vw-path => ∃ : uw-path

Subgraphs

Theorem

$\deg(v) \geq 2 \ \forall v \in V \Rightarrow G$ has a cycle

if any vertice is a leaf, can't guarantee there's a cycle

Connected / Disconnected

Theorem

Pick any u

connected $\iff \forall v \in V : \exists\ uv$-path

Theorem

disconnected $\iff \exists X :$ ① $C.I(X) = \emptyset$

② X is non-empty

③ X is a proper subset

$\left.\begin{array}{l} X = V \\ X = \emptyset \end{array}\right\}$ the cut Induced $(X) = \emptyset$

but G is not necessarily disconnected

connected : 1 component

Component : Induces empty cut
: connected

Eulerian Circuits

E.P $\iff$ at most 2 vertices of odd degree

must start & end with odd vertices

E.C $\iff$ all even degrees

Given: connected

Bridges

e is a **bridge** : $G \setminus e$ has more components

Lemma

$e = uv$ is a bridge in H (component) $\Rightarrow$ ① $H \setminus e$ has exactly 2 components

② u & v are in different components

Theorem

$e =$ bridge $\Leftrightarrow$ $e \notin$ cycle

Tree $\Rightarrow$ no cycles

Trees

Lemma

Tree $\Leftrightarrow$ every edge is a bridge

Theorem

$q = p - 1$ # edges in a tree

Theorem

$q = p - k$ # edges in a Forest

k: #components

Theorem

$\forall T$ with $p_T \geq 2$: # leaves ≥ 2

no cycles $\Rightarrow$ no odd cycles

Lemma

A tree is bipartite

Theorem

2 distinct path between 2 vertices => cycle

- no cycles $\not\Rightarrow$ Unique paths
- no cycles + connected => Unique paths

Lemma

tree : unique path between any 2 vertices

Span Trees

Theorem

Connected $\Longleftrightarrow$ has Span Tree

Corollary

$\left.\begin{array}{l} ① \text{ Connected} \\ ② \ q = p-1 \end{array}\right\} \Rightarrow$ Tree

Corollary

$\left.\begin{array}{l} ① \text{ Connected} \\ ② \text{ no cycles} \end{array}\right\} \Rightarrow$ Tree

$\left.\begin{array}{l} ① \text{ connected} \\ ② \ q = p-1 \\ ③ \text{ no cycles} \end{array}\right\}$ 2/3 hold => G is a tree

Corollary

$\left.\begin{array}{l} ① \ p-1 \text{ edges} \\ ② \text{ no cycles} \end{array}\right\} \Rightarrow$ Tree

Forest : $q = p - k$

Every Span tree has the same # edges	$q_T = \overset{\text{const}}{p_G} - 1$

If a span-tree doesn't have e : must not be a bridge
must be in a cycle

Corollary : Method 1

① Add : $e \notin E(T) \Rightarrow T + e$ has 1 cycle (C)

② Remove : $e' \in C \Rightarrow T + e - e'$: new span tree of G

Corollary : Method 2

① Remove : $e \in E(T) \Rightarrow T - e$ has 2 components $(\overbrace{C_1, C_2}^{\text{trees}})$

② Add : $e' \in C.I(V(C_1))$ or $C.I(V(C_2)) \Rightarrow T - e + e'$: new span tree of G

$\therefore$ If G is connected w' edge $\boxed{e}$

bridge $\rightarrow$ every span T has e

non bridge $\rightarrow$ $\exists$: span tree with e

Bipartite

Theorem Bipartite Characterization

bipartite $\iff$ no odd cycles

ie: component

bipartite $\iff$ every Subgraph is bipartite

Planarity

Lemma

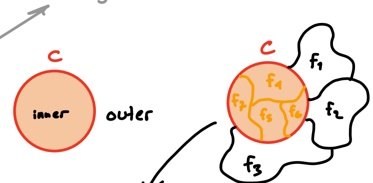
bridge $\iff$ Same face on both sides

non-bridge $\iff$ different faces on either side

Theorem

cycle seperates plane into 2 regions : ① inner
② outer

region $\not\Rightarrow$ face



$\sum_{f \in \text{inner}} \deg(f) = 2q$

C + interior edges

① every interior edge contributes 2 to the sum

② every edge in C contributes 1 to the sum

Lemma Handshake Lemma for faces

$$\sum_{f \in F} \deg(f) = 2|E|$$

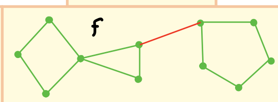
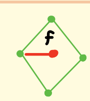
disconnected or connected

applies to all faces in a region !!!

could be the whole graph or could be inner region of a cycle

Sum of degrees

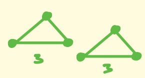
degree of single face

	Unbounded	bounded
degree(f) = # non bridges + 2(#bridges)		

face on one side

face on both sides

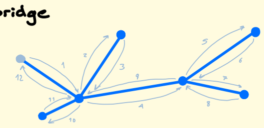
if boundary is disconnected, sum the lengths of each walk



$\deg(f_3) = 6$

Note Planar Embedding of a Tree:

- every edge a bridge
- 1 face



$|V| = 7$
$|E| = 6$

degree $= 2|E| = 2|V| - 2 = 12$

1 graph => many planar embeddings (different faces / degrees)

Theorem Euler's Formula

① connected
② Planar: s faces

$\left.\right\} \Rightarrow p - q + s = 2$

Structural → same, independent on drawing

drawing

Corollary

All planar embeddings have the same # faces: $2+m-n$

specific imbedding

Lemma

degree (every-face) $\geq d \Rightarrow q \leq \frac{d(p-2)}{d-2}$

given a planar graph

Lemma

G contains a cycle => every face boundary contains a cycle

Theorem to prove non-planar

$p \geq 3$: $q \geq 3p-6 \Rightarrow$ not planar

cycles have length ≥ 3

Corollary

$K_{3,3}$ is not planar

less edges allowed

Theorem prove non-planar (bipartite version)

$p \geq 3$ bipartite : $q \geq 2p-4 \Rightarrow$ non-planar

cycles have length ≥ 4

Corollary

K_5 is not planar

Theorem

planar $\Longleftrightarrow$ doesn't contain K_5 or $K_{3,3}$

planar $\Longleftrightarrow$ all subgraphs are planar

Theorem

planar $\Longleftrightarrow$ not an edge subdivision of K_5 or $K_{3,3}$

planar $\Longleftrightarrow$ all edge subdivisions are planar

Colouring

K-colourable $\not\Rightarrow$ (K-1)-colourable
K-colourable $\Rightarrow$ (K+1)-colourable
p vertices $\Rightarrow$ p-colourable

Extreme cases: minimum colours

min: 2

Theorem

Bipartite $\Leftrightarrow$ 2-colourable

only even cycles

min: P

Theorem

Complete $\Leftrightarrow$ p-colourable & not K-colourable for $K < p$

Colouring | Planar

Corollary

planar $\Rightarrow$ has a vertex of degree ≤ 5

Theorem * no matter the graph

planar $\Rightarrow$ is 6-colourable
planar $\Rightarrow$ is 5-colourable
planar $\Rightarrow$ is 4-colourable

contraction maintains planarity

G is planar $\Rightarrow$ G/e is planar

Colouring | Duals

Dual graph is planar → 4/5/6 colour theorems hold

G	G^*
v, $\deg(v) = k$	f, $\deg(f) = k$
f, $\deg(f) = k$	v, $\deg(v) = k$

G is connected $\Rightarrow (G^*)^* = G$

minimum # colours needed is not preserved

Matchings

$|M| \leq |C|$

$|M| = |C| \implies M$ is max

bipartite

$\boxed{\phi}$ smallest matching

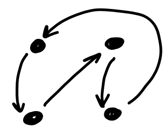
$\boxed{V}$ largest cover

Konig's Theorem

Augment Matchings

has an augmented path $\implies$ M isn't max

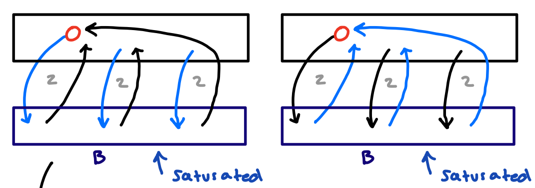
M is max $\implies$ no augmented path



Matchings | Bipartite

Augmented path $\iff$ From $a \in A$ unsaturated
To $b \in B$ unsaturated

why can't it start/end in A



bipartite $\implies$ even cycles

$\Downarrow$

2-regular Bipartite

$\Downarrow$

perfect matching

$\Downarrow$

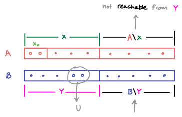
A is fully saturated

M_{max}

every vertex saturated

Find Largest Matching

not reachable from Y



not reachable from X

P = path starting at X_0

X_0 = { unsaturated in A }

X = { $v \in A$: reachable by any P }

Y = { $v \in B$: reachable by any P }

During:	After:
$\lvert C\rvert = Y \cup (A \setminus X)$	$\left.\begin{matrix}\lvert C\rvert \\ \lvert M\rvert\end{matrix}\right\} = Y \cup (A \setminus X)$
$\lvert M\rvert = \lvert C\rvert - U$	

Looking at matching's of bipartite graphs from a different angle

Hall's Theorem

A matching saturates $A \iff \forall D \subseteq A : |N(D)| \geq |D|$

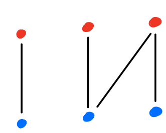
matching saturates $A \implies$ Max matching

① $|A| = |B|$
② Halls Condition
$\iff$ perfect matching

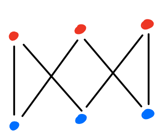
ie: k-regular

k regular $\implies$ perfect matching

no subset in A whose neighbour set is smaller than itself



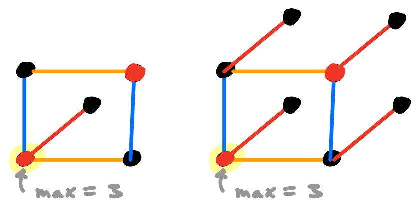
not regular



regular

Matchings | Regular

$$G: \boxed{\begin{array}{c}\text{bipartite} \\ \text{max degree } k\end{array}} \subseteq \boxed{\begin{array}{c}\text{bipartite} \\ k\text{-regular}\end{array}}$$



Bipartite

$$\boxed{|M| \geq \frac{q}{\text{max degree}}}$$

C = covers all edges

c_{min} = covers all edges <u>exactly once</u>

$$q \geq |C_{min}| (\text{max deg})$$

$$\frac{q}{\text{max deg}} = \text{greatest average degree}$$

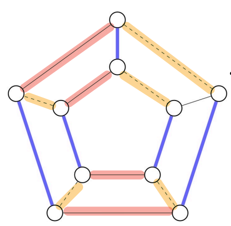
need at least $\frac{q}{d}$ vertices to cover all edges

$$|C_{min}| \geq \frac{q}{d}$$

Edge Colouring

Edge k-colouring

partitions the edges of G into k total matchings



3 matchings

Figure 8.11: A graph with an edge 3-colouring

bipartite :

$\boxed{\text{max degree} = k \implies \text{has edge-}k\text{-colouring}}$

$q \geq 1$ $\boxed{\implies \text{matching saturating each vertex of max degree}}$

regular: every vertex has max degree

$\boxed{\text{regular} \implies \text{perfect matching}}$