Variable Elimination Examples

Jesse Hoey
David R. Cheriton School of Computer Science
University of Waterloo
Waterloo, Ontario, CANADA, N2L3G1
jhoey@cs.uwaterloo.ca

If Jesse's alarm doesn't go off (A), Jesse probably won't get coffee (C); if Jesse doesn't get coffee, he's likely grumpy (G). If he is grumpy then it's possible that the lecture won't go smoothly L. If the lecture does not go smoothly then the students will likely be sad S.

$\mathbf{P(A = true)} = 0.3$

$\mathbf{P(C = true|A)} =$

A	$P(C = true\|A)$
t	0.8
f	0.15

$\mathbf{P(G = true|C)} =$

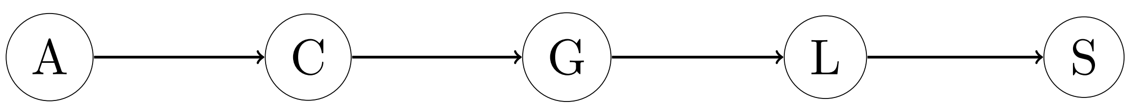
C	$P(G = true\|C)$
t	1.0
f	0.2

$\mathbf{P(L = true|G)} =$

G	$P(L = true\|G)$
t	0.7
f	0.2

$\mathbf{P(S = true|L)} =$

L	$P(S = true\|L)$
t	0.9
f	0.3



A=Jesse's alarm doesn't go off
C=Jesse doesn't get coffee
G=Jesse is grumpy
L=lecture doesn't go smoothly
S=students are sad

Query $P(S=true)$

Polytree (Fwd Inference):

$$P(S=true) = \sum_{A,C,G,L} P(S=true, A, C, G, L)$$ marginal sum over all descendents

$$= \sum_{A,C,G,L} P(A)P(C|A)P(G|C)P(L|G)P(S=true|L)$$ chain rule

$$= \sum_{A} P(A) \sum_{C} P(C|A) \sum_{G} P(G|C) \sum_{L} P(L|G)P(S=true|L)$$ distribute the sum

Variable Elimination: Query: S, Evidence: ϕ $A \to C \to G \to L \to S$

Ordering: L, G, C, A

$$P(S=true) = \sum_{A,C,G,L} f_0(A)f_1(C,A)f_2(G,C)f_3(L,G)f_4(L)$$

Factors

$f_0(A) = P(A) =$

A	$f_0(A)$
t	0.3
f	0.7

$f_1(C,A) = P(C|A) =$

A	C	$f_1(C,A)$
t	t	0.8
t	f	0.2
f	t	0.15
f	f	0.85

$f_2(G,C) = P(G|C) =$

C	G	$f_2(G,C)$
t	t	1.0
t	f	0.0
f	t	0.2
f	f	0.8

$f_3(L,G) = P(L|G) =$

G	L	$f_3(L,G)$
t	t	0.7
t	f	0.3
f	t	0.2
f	f	0.8

$f_4(L) = P(\underbrace{S=true}_{\text{Restricted}}|L) =$

L	$f_4(L)$
t	0.9
f	0.3

z_i	Pool	Eliminate	CPT	Meaning
L	$f_0(A), f_1(C,A), f_2(G,C), f_3(L,G), f_4(L)$	$f_5(G) = \sum_L f_3(L,G) f_4(L)$	G, $f_5(G)$ $= t$: $0.7*0.9+0.3*0.3 = 0.72$ f: $0.9*0.2+0.3*0.8=0.42$	$= \overbrace{\sum_A f_0(A) \sum_C f_1(C,A) \sum_G f_2(G,C)}^{\text{no } L} \overbrace{\sum_L f_3(L,G)f_4(L)}^{\text{contain } L}$ $= \sum_A f_0(A) \sum_C f_1(C,A) \sum_G f_2(G,C) f_5(G)$
G	$f_0(A), f_1(C,A), f_2(G,C), f_5(G)$	$f_6(C) = \sum_G f_2(G,C), f_5(G)$	C, $f_6(C)$ $= t$: $1.0*0.72+0*0.42=0.72$ f: $0.2*0.72+0.8*0.42=0.48$	$= \overbrace{\sum_A f_0(A) \sum_C f_1(C,A)}^{\text{no } G} \overbrace{\sum_G f_2(G,C)f_5(G)}^{\text{w/ } G}$ $= \sum_A f_0(A) \sum_C f_1(C,A) f_6(C)$
C	$f_0(A), f_1(C,A), f_6(C)$	$f_7(A) = \sum_C f_1(C,A) f_6(C)$	A, $f_7(A)$ $= t$: $0.8*0.72+0.2*0.48=0.672$ f: $0.15*0.72+0.85*0.48=0.516$	$= \boxed{\sum_A f_0(A)} \boxed{\sum_C f_1(C,A) f_6(C)}$ $= \sum_A f_0(A) f_7(A)$
A	$f_0(A)\ f_7(A)$	$f_8() = \sum_A f_0(A)\ f_7(A)$	$0.3*0.672+0.7*0.516 = 0.563$	

Answer

Already normalized since bayes rule not used

$\Rightarrow P(S=False) = 0.437$

Query $P(S=true)$ & $P(S=False)$ at the same time $A \to C \to G \to L \to S$

Factors

Same except for f_4

$f_4(S,L) = P(S|L) =$

S	L	$f_4(S,L)$
t	t	0.9
t	f	0.3
f	t	0.1
f	f	0.7

z_i	Pool	Eliminate	CPT	Meaning
L	$f_0(A), f_1(C,A), f_2(G,C), f_3(L,G), f_4(S,L)$	$f_5(G,S) = \sum_L f_3(L,G) f_4(S,L)$	. . .	. . .
G	$f_0(A), f_1(C,A), f_2(G,C), f_5(G,S)$	$f_6(C,S) = \sum_G f_2(G,C), f_5(G,S)$	. . .	. . .
C	$f_0(A), f_1(C,A), f_6(C,S)$	$f_7(A,S) = \sum_C f_1(C,A) f_6(C,S)$	. . .	. . .
A	$f_0(A)$ $f_7(A,S)$	$f_8(S) = \sum_A f_0(A)\ f_7(A,S)$	S, $f_8(S)$ t: $0.3*0.672+0.7*0.516 = 0.563$ f: $0.3*0.328+0.7*0.484=0.437$	

Answers Already Normalized

Query $P(\underbrace{S=true}_{\text{Query}}|\underbrace{A=true}_{\text{Evidence}})$ $A \to C \to G \to L \to S$

Ordering: L, G, C

$$P(S=true|A=true) \propto \sum_{C,G,L} f_0(C)f_1(G,C)f_2(L,G)f_3(L)$$

$$= \sum_C f_0(C) \sum_G f_1(G,C) \sum_L f_2(L,G)f_3(L)$$

$$= \sum_C f_0(C) \sum_G f_1(G,C) f_4(G)$$

Factors:

$f_0(C) = P(C|A=true) =$

C	$f_0(C)$
t	0.8
f	0.2

$f_1(G,C) = P(G|C) =$

C	G	$f_1(G,C)$
t	t	1.0
t	f	0.0
f	t	0.2
f	f	0.8

$f_2(L,G) = P(L|G) =$

G	L	$f_2(L,G)$
t	t	0.7
t	f	0.3
f	t	0.2
f	f	0.8

$f_3(L) = P(S=true|L) =$

L	$f_3(L)$
t	0.9
f	0.3

z_i	Pool	Eliminate	CPT	Meaning
L	$f_0(C), f_1(G,C)$ $f_2(L,G)\ f_3(L)$	$f_4(G) = \sum_L f_2(L,G)\ f_3(L)$	G, $f_4(G)$ t: $0.7*0.9+0.3*0.3=0.72$ f: $0.2*0.9+0.8*0.3=0.42$	. . .
G	$f_0(C), f_1(G,C)$ $f_4(G)$	$f_5(C) = \sum_G f_1(G,C)\ f_4(G)$	C, $f_5(C)$ t: $1.0*0.72+0*0.42=0.72$ f: $0.2*0.72+0.8*0.42=0.48$	. . .
C	$f_0(C), f_5(C)$	$f_6() = \sum_C f_0(C)\ f_5(C)$	$= 0.8*0.72+0.2*0.48 = 0.672$	. . .

Already normalized (Bayes Rule not used)

$\therefore P(S=False) = 0.328$

Query | Evidence

Query $P(C{=}true \mid S{=}true)$

$A \rightarrow C \rightarrow G \rightarrow L \rightarrow S$

$$P(C{=}true \mid S{=}true) = \frac{P(C{=}t \mid S{=}t)}{P(S{=}t)} \leftarrow \sum_c P(C{=}c \mid S{=}t)$$

Ordering: L, G, A

Factors:

$f_0(A) = P(A) =$

A	$f_0(A)$
t	0.3
f	0.7

$f_1(A) = P(C = true \mid A) =$

A	$f_1(A)$
t	0.8
f	0.15

$f_2(G) = P(G \mid C = true) =$

G	$f_2(G)$
t	1.0
f	0.0

$f_3(L, G) = P(L \mid G) =$

G	L	$f_3(L,G)$
t	t	0.7
t	f	0.3
f	t	0.2
f	f	0.8

$f_4(L) = P(S = true \mid L) =$

L	$f_4(L)$
t	0.9
f	0.3

z_i	Pool	Eliminate	CPT
L	$f_0(A)$ $f_1(A)$ $f_2(G)$ $f_3(L,G)$ $f_4(L)$	$f_5(G) = \sum_L f_3(L,G) f_4(L)$	G / $f_5(G)$: $= t$: $0.7*0.9+0.3*0.3=0.72$; f: $0.9*0.2+0.3*0.8=0.42$
G	$f_0(A)$ $f_1(A)$ $f_2(G)$ $f_5(G)$	$f_6() = \sum_G f_2(G) f_5(G)$	$= 1.0*0.72+0.0*0.42=0.72$
A	$f_0(A)$ $f_1(A)$ $f_6()$	$f_7() = \sum_A f_0(A) f_1(A)$	$= 0.3*0.8+0.7*0.15=0.345$

Remaining Factors

$f_6() \times f_7() = (0.72) \times (0.345) =$ 0.2484

unnormalized

$$\frac{0.2484}{0.2484 + 0.3144} = 0.44$$

$P(C{=}False \mid S{=}true)$:

Ordering: L, G, A

Factors:

Same except ...

$f_1(A) = P(C = false \mid A) =$

C	$f_1(A)$
t	0.2
f	0.85

$f_2(G) = P(G \mid C = false) =$

G	$f_2(G)$
t	0.2
f	0.8

z_i	Pool	Eliminate	CPT
L	...	...	...
G	...	...	...
A	...	...	...

result: 0.3144