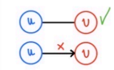
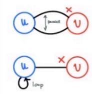
Graph

$G = (V, E)$

E: {edges}

pairs of vertices : (u,v)
- unordered
- distinct

V: {vertices}

Undirected	Edges are unordered pairs of vertices uv = vu	
Not a multigraph	Edges are pairs of distinct vertices • no **loops** (vertices are distinct) • no **parallel edges** (pairs are distinct)	

∴ a graph is a <u>Structure of Sets</u>

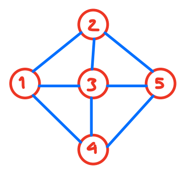
A <u>drawing</u> is not the <u>graph</u> itself, it is a specific way to represent the graph

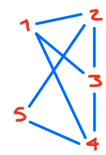
<u>Structure</u> (only 1)

$V(G) = \{1, 2, 3, 4, 5\}$

$E(G) = \{12, 13, 14, 23, 25, 34, 35, 45\}$

<u>Drawings</u> (many)



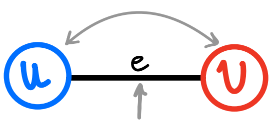


"Same" graph = Same Structure

Notation

u & v

1) adjacent
2) neighbours $\quad N(v) = \{\text{neighbours of } v\}$
3) ends of e



e is incident w/ u & v

Degree

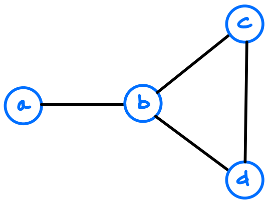
$\deg(v) = $ # edges <u>incident</u> w/ v $= |N(v)|$

Average Degree

(density of a graph)

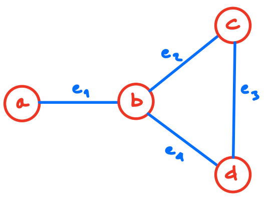
Ways to Specify a graph

1 Adjacency Matrix



$$
\begin{array}{c|cccc|}
 & a & b & c & d \\
\hline
a & 0 & 1 & 0 & 0 \\
b & 1 & 0 & 1 & 1 \\
c & 0 & 1 & 0 & 1 \\
d & 0 & 1 & 1 & 0 \\
\hline
\end{array}
$$

1 Incidence Matrix



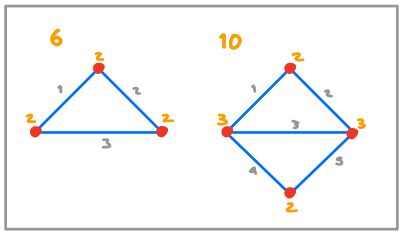
$$
\begin{array}{c|cccc|c}
 & e_1 & e_2 & e_3 & e_4 & \\
\hline
a & 1 & 0 & 0 & 0 & 1 \\
b & 1 & 1 & 0 & 1 & 3 \\
c & 0 & 1 & 1 & 0 & 2 \\
d & 0 & 0 & 1 & 1 & 2 \\
\hline
 & 2 & 2 & 2 & 2 &
\end{array}
$$

degree of v

Handshake Lemma

$$\sum_{v \in V} \deg(v) = 2|E|$$

why? every e has 2 ends
(contributes 2 to the total degree)



Corollary 1

$\sum$(vertices w/ odd degree) = even

Proof

$$2|E| = \sum \deg(v)$$
$$= \sum \deg(v_{even}) + \sum \deg(v_{odd})$$
$$\text{even} = \text{even} + \text{even}$$

Corollary 2

$$\text{avg } \deg(v) = \frac{2|E|}{|V|}$$

Isomorphism (vertices relabled)

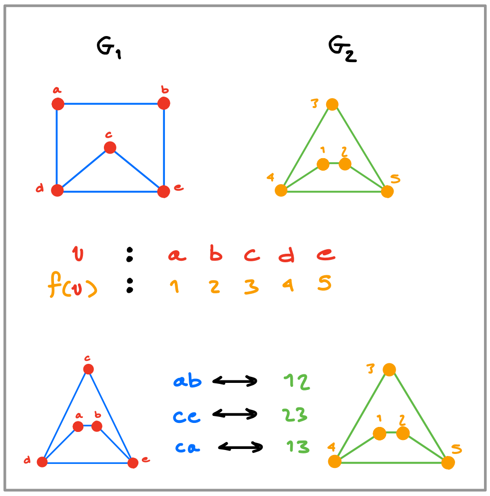
G_1 & G_2 are isomorphic :

① bijection

$$f : V(G_1) \longrightarrow V(G_2)$$

② preserves <u>adjacency structure</u>

u, v are adjacent $\iff$ $f(u), f(v)$ are adjacent



How to | Show that 2 graphs are <u>not</u> isomorphic

find an <u>adjacency property</u> that does not hold (isomorphism preserves adjacency)

- degree sequence
- neighbours
- # vertices, #edges
- cycles
- Subgraphs
- planarity
- bipartness

<u>Note</u> : Isomorphism is an Equivalence Relation

1) G_1 is isomorphic to G_1 (reflexive)

2) G_1 is isomorphic to G_2 => G_2 is isomorphic to G_1

3) G_1 is isomorphic to G_2 $\wedge$ G_2 is isomorphic to G_3 => G_1 is isomorphic to G_3

Automorphism (different drawing, Same Structure & vertices)

① $f : V(G) \longrightarrow V(G)$

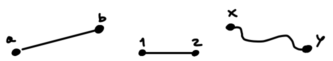
② u, v are adjacent $\iff$ $f(u), f(v)$ are adjacent

Isomorphism Class

All G's isomorphic to each other $G_1 \rightleftharpoons G_2 \rightleftharpoons G_3 \rightleftharpoons G_4 \cdots$

- each one is "equal"
- each represents the whole class

Class : "graph w/ 2 vertices & 1 edge"



There is only "one" such graph

one such class

Common Classes of graphs

Complete

every pair of vertices are adjacent

K_n :

- $p = n$
- $q = \binom{p}{2}$ (total # pairs)
- $(n-1)$ – regular

Regular

K-regular

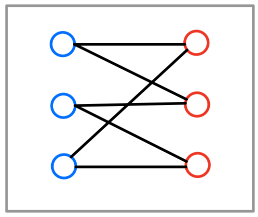
- every v has degree K
- # edges = $\frac{nK}{2}$

$\Sigma \deg = 2q$
$nK = 2q$
$q = \frac{nK}{2}$

Bipartite

Has a bipartition of V

1) partition (A, B)
2) each e has 1 end in A & 1 end in B

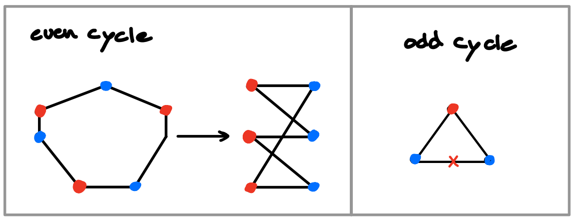


Properties

① odd cycle $\Rightarrow$ not bipartite

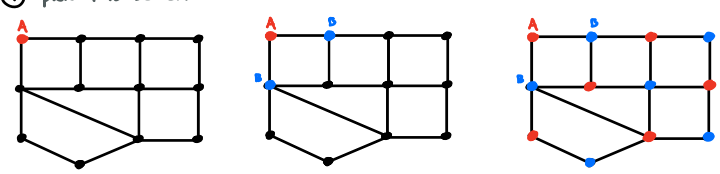
② Bipartite $\Leftrightarrow$ every subgraph is bipartite

} Bipartite $\Leftrightarrow$ no odd cycles



How to	Show that G is bipartite

① pick 1 to be $\in A$ ② neighbours must be in B ③ Repeat



How to	Show that G isn't bipartite

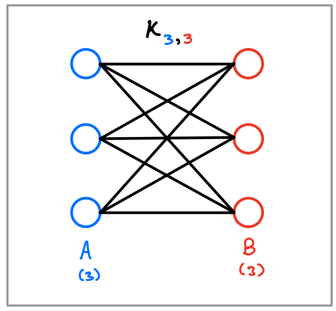
Find an odd cycle

Complete-Bipartite

$K_{m,n}$ $\qquad |A| = m \quad |B| = n$

- has bipartition (A,B) with max edges
- $\#$ edges $=$ total $\#$ pairs $= mn$

pair any $v \in A$ (m-choices) $\qquad$ with all $n \in B$ (n choices)



n-cube

$V(G) = \{$ length-n binary strings $\}$

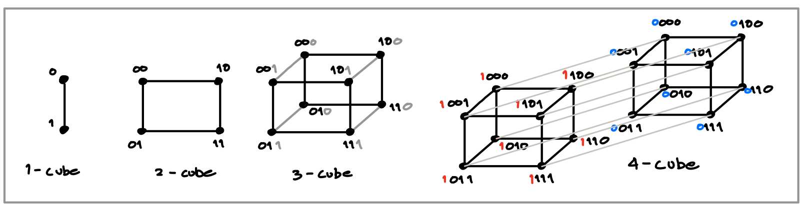
$E(G) = \{$ strings that differ in 1 bit $\}$

- $\#$ vertices $= 2^n$
- $\#$ edges $= n\,2^{n-1}$

$2^n \cdot n = 2q$ (handshake)

$q = n\,2^{n-1}$

- n-regular
- bipartite



Proof n-cube is bipartite

(A, B)

$\{$odd $\#$ zeros$\}$ $\quad$ $\{$even $\#$ zeros$\}$

① $(A\ B)$ partitions V

② Let $e = st$

we get from $s \to t$ by changing 1 bit

- $0 \to 1$: $\#0$'s $= \#0$'s -1 $\to$ parity of $\#0$'s flips
- $1 \to 0$: $\#0$'s $= \#0$'s $+1$ $\to$ parity of $\#0$'s flips

$\therefore$ parity of s, t is different

$\therefore$ for any edge, e: 1 end in A, 1 end in B

① + ② $\Rightarrow$ G is bipartite

Sequences

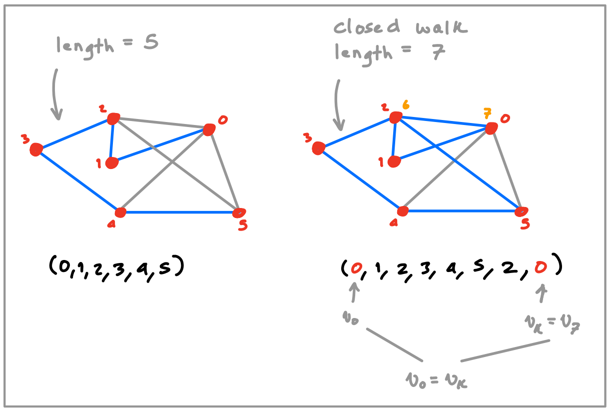
Sequence of alternating vertices & edges :

$$(v_0 \xrightarrow{v_0 v_1} , v_1 \xrightarrow{v_1 v_2} , v_2 \rightarrow , \ldots \rightarrow , v_{n-1} \xrightarrow{v_{n-1} v_n} , v_n)$$

Length : # edges crossed = k

Walk

- vertices/edges can be reused
- Closed : $v_0 = v_k$



Path

- vertices/edges can't be reused

Note: trivial walk/path

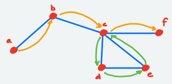
- length = 0 (single vertex)

Theorem

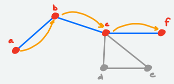
uv – walk $\longrightarrow$ $\exists$: uv – path

Proof

Idea: if you can get from $u - v$ by repeating a place you went, then you can get there without repeating.



remove parts
betwen repitition



by induction on $\boxed{P}$ ⟵ we choose this because it goes down as we remove parts of the walk & we want to reduce # repeated vertices

Strong or Weak induction ?

↳ We look at the 1^{st} & last time a v appears. There could be duplicates within this walk. So use strong induction

Proof: by S.I on K, on the statement : " uv-walk $\Rightarrow$ uv-path"

Let W = uv-walk

Base: $K = 0$ W has 0 repeated vertices $\Rightarrow$ W = path

Inductive: $K \geq 1$

I.H : W has $< K$ repeated vertices $\Rightarrow$ uv-path

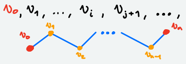
Consider W :

- uv-walk : $\overset{u}{v_0}, v_1, v_2, \dots \overset{v}{v_t}$
- K – repeated vertices

- w = repeated vertex
- i, j be smallest/largest index where w is repeated

Consider W' :

- remove part between v_i & v_j $\Rightarrow$ $W' = v_0, v_1, \dots, v_i, v_{j+1}, \dots, v_t$
- length $-(K-2)$ uv walk



I.H

Since W' is a uv walk with $< K$ vertices $\Rightarrow$ $\exists$ uv-path

Proof Alternate

Among all uv-walks, let $W = v_0, \dots, v_K$ be the shortest in length

Case 1: no v is repeated

$\Rightarrow$ W is a path ✓

Case 2: v is repeated

$\therefore$ $\exists$ $i < j$ s.t $v_i = v_j$

Then $v_0, \dots, v_i, v_{j+1}, v_K$ is a uv-walk, shorter than W

Contradiction.

$\therefore$ a uv-path exists

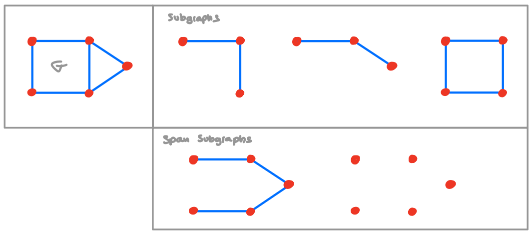
Corollary

$\exists$: uv-path $\wedge$ $\exists$: vw-path $\Rightarrow$ $\exists$: uw-path

Subgraphs

Subgraph

$V(H) \subseteq V(G)$



Spanning Subgraph

$V(H) = V(G)$

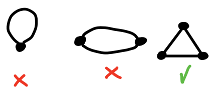
$\boxed{G \setminus e}$ Subgraph; without e

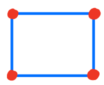
$\boxed{G + e}$ Subgraph; adds e

} Spanning

Cycles

- $p = q$ = length
- min length = 3



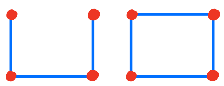


Note:

can be <u>represented</u> as a closed walk, starting with any vertex : $(v_0, \dots, v_k, v_0)$

remove an edge => it's a "path"

P = "uv-path" + uv => Cycle



*path cycle

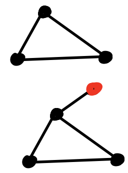
Hamilton Cycle

Spanning Subgraph + Cycle

Theorem

$\deg(v) \geq 2 \quad \forall v \in V \implies G$ has a cycle

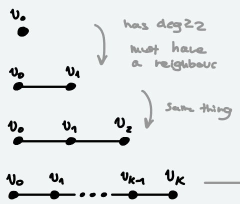
converse not true



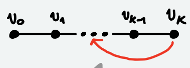
$\deg(v) \geq 2 \quad \forall v \in V$

$\deg(v) \ngeq 2 \quad \forall v \in V$

Proof (longest path argument)

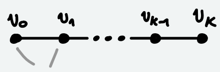


at this point, we've ran out of vertices in G.
but v_k must have deg ≥ 2



v_k must attach to some other vertice in G
This is a cycle

Let $P = v_0, \dots, v_k$ be the longest path in G

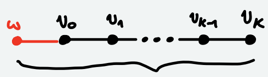


v_0 has neighbour v_1

but $\deg(v_0) \geq 2$

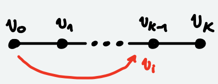
So v_0 must have another neighbour, ω

<u>Case 1</u>:



longer path
Contradiction!

<u>Case 2</u>:



$\omega = v_i$ for some $i \geq 2$

P is still the longest path ✓
∴ G has a cycle

Connected

$\forall (u, v), \ \exists \ uv$ - path (check $\binom{|V|}{2}$ paths)

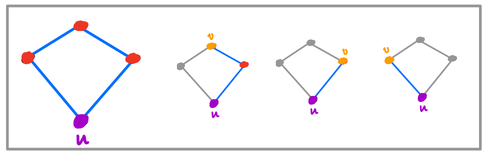
How to | Show that G is connected

(check $|V| - 1$ paths)

Theorem

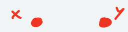
Pick any vertex u :

$\forall v \in V : \ \exists \ uv$-path $\Rightarrow G$ is connected



Proof

To show connectedness : $\exists \ xy$ path $\forall (x, y)$



<u>Assume</u>: $\forall v \in V : \exists \ uv$-path

↳ ux path exists

↳ uy path exists

xu-path $\wedge$ uy-path $\Rightarrow$ xy path

$\therefore \ \forall (x, y) : xy$ path exist

$\therefore \ G$ is connected

How to	Show that G is disconnected

A pair (u,v) has no path between them (difficult)

we make it easier to show disconnectedness with **components/cuts**

Component

Maximally connected subgraph

- Subgraphs that produce empty cuts
- not a proper-subgraph of another connected subgraph

$\boxed{G \text{ is connected} \iff G \text{ has 1 component}}$ → component induces an empty cut

Cuts

Let $X \subseteq V$

Cut induced by X = $\{e \in E : \text{exactly 1 end of } e \text{ is in } X\}$

Theorem

disconnected $\iff \exists X$: ① $C.I(X) = \phi$
$\subseteq V$
② X is non-empty
③ X is a proper subset

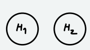
$X = V$, $X = \phi$ } the cut Induced $(X) = \phi$
↓
but G is not necessarily disconnected

Proof

=>

<u>Assume</u>: [G is disconnected]

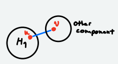
↳ G contains at least 2 components : H_1, H_2



↳ $V(H_1)$ is a non-empty proper subset of $V(G)$

<u>Assume</u>: b.w.o.c: $C.I(V(H_1)) \neq \phi$

↳ $\exists e = uv$ where $u \in V(H_1)$ & $v \notin V(H_1)$



↳ ∴ H_1 & [other component] are connected

↳ ∴ H_1 is not a maximally connected Subgraph (not a component)

contradiction!

↳ $C.I(V(H_1)) = \phi$

<u>Conclusion</u>: $\exists X : C.I(X) = \phi$

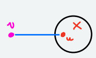
<=

<u>Assume</u>: $\exists X$: ① $C.I(X) = \phi$
② X is non-empty
③ X is a proper subset

∴ For some v :

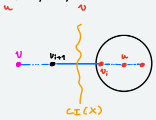
<u>Assume</u>: G is connected

$\exists$ uv-path where $u \in X$



$v_0, v_1, \ldots, v_k$
$u \qquad\qquad v$

Let i be the index where: $v_i \in X$, $v_{i+1} \notin X$



∴ $e = v_i v_{i+1}$ is in the $C.I(X)$

↳ $C.I(X) \neq \phi$ (contradiction)

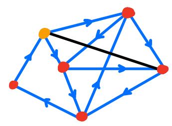
↳ no uv-path exists

↳ G is disconnected

Eulerian Circuits

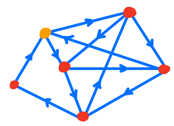
Euler Path

- uses every edge exactly once



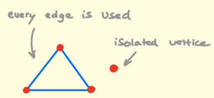
Eulerian Circuit

- closed euler path

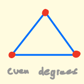


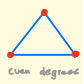
Note

E.C $\not\Rightarrow$ G is connected



all even degrees $\not\Rightarrow$ E.C





no E.C

Theorem

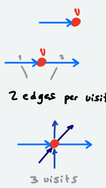
<u>Given:</u> G is connected

E.C $\iff$ all even degrees

Proof

① $\boxed{\text{E.C} \Rightarrow \text{even degrees}}$

Why?



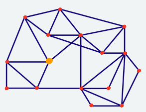
Everytime we visit a v

we leave using a different edge

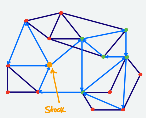
So $\deg(v) = \underbrace{2 \cdot \#\text{visits}}_{\text{even}}$

② $\boxed{\text{even degrees} \Rightarrow \text{E.C}}$

Why?



① pick any vertex



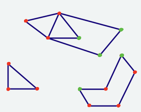
② Keep taking unused edges until you get stuck

claim: you get stuck only at the start vertex

when you visit a new v, there's an odd # of exits (1,3,...)
(at least 1 exit

for the start point. Every time we visit it again, it has even exits (0,2,...)

If we have used all edges, we're done. Otherwise:

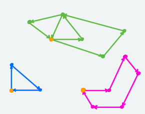


③ Remove all edges in the walk

claim: all v's still have even degree

* G : all even degrees
* Walk : all even degrees

} even # − even # = even #

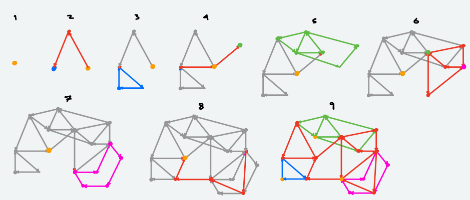


④ Repeat (1-3) for the components

we get a closed walk for each component

⑤ put them together

- go along the main walk
- when we hit a component, follow it's closed walk in a loop



Strong induction on #edges, m, on statement: "even degrees => E.c"

Base Case: m = 0 (trivial E.c)

Ind step: $m \geq 1$

$\boxed{\text{I.H}}$ "even degrees => E.c" for #edges < m

Let G be connected with all even degree vertices

↳ All degrees ≥ 2 $\overset{\text{theorem}}{\Rightarrow}$ G has a cycle (call it C)

Case 1: C has all edges of G => C "is a" E.c (we're done)

Case 2: otherwise:

G': remove edges in C from G

↳ both G & C have all even degrees $\overset{\text{claim}}{\Rightarrow}$ G' has all even degrees

For each component in G': #edges < m

I.H ↳ each component has a E.C

Since G is connected:

each component has vertex in common with C

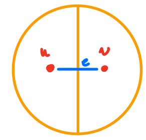
Obtain E.C of G by:

attaching E.C of a component when we first encounter a vertex in it while traversing the cycle C.

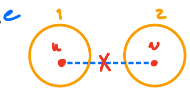
$\boxed{\text{Bridge}}$ e is a bridge : $G \setminus e$ has more components

Note : A component can have at most 1 bridge

H



$H \setminus e$



Lemma

Given : H is a component in G

$e = uv$ is a bridge in H $\Rightarrow$ ① $H \setminus e$ has exactly 2 components

② u & v are in different components

Proof

Assume : $e = uv$ is a bridge in H

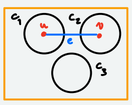
① " $H \setminus e$ has exactly 2 components "

Assume : # components in $H \setminus e$ ≥ 3 (contradiction)

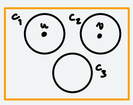
C_1 has u

C_2 has v

C_3 has neither



H



$H \setminus e$

components can't be empty

doesn't have u or v

$V(C_3)$ is a non-empty proper subset of $V(H)$

↳ $V(C_3)$ induces an empty cut in $H \setminus e$

add back e

Consider H :

Since e & C_3 : e is not in cut induced by $V(C)$

↳ cut-induced by $V(C_3)$ is empty in H as well

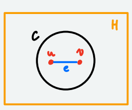
↳ H is disconnected

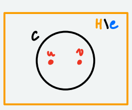
↳ contradiction (since H is a component)

$\therefore$ ① $H \setminus e$ has exactly 2 components

② " u & v are in different components in $H \setminus e$ "

Assume : u & v are in the same component (C) in $H \setminus e$





↳ $V(C)$ is a non-empty, proper subset of $V(G)$

↳ $V(C)$ induces the empty cut in $H \setminus e$

↳ $V(C)$ induces the empty cut in H

↳ H is disconnected

↳ contradiction (since H is a component)

$\therefore$ ② u & v are in different components in $H \setminus e$

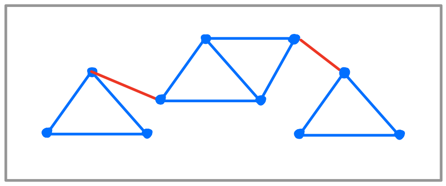
How to	Determine if e is a bridge

Theorem

e = bridge $\iff$ $e \notin$ cycle

$e \neq$ bridge $\iff$ $e \in$ cycle

contrapositive

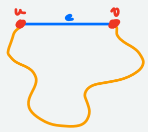


Proof (contrapositive)

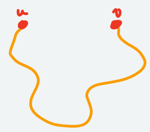
Assume: $e \in$ cycle

$e = uv$

$C = (u, \overset{e}{\frown} v, v_1, v_2, \dots, v_k, u)$



Consider $C \setminus e$



$C \setminus e$ contains a uv path

↳ $G \setminus e$ contains a uv path

uv path $\Rightarrow$ u & v are in the same component in $G \setminus e$

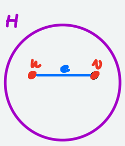
(Lemma) ↳ e is not a bridge in G

Assume: $e \neq$ bridge ($e = uv$)

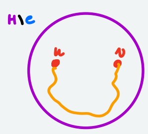
② $\Leftarrow$

Goal: construct a cycle w/ e

Suppose $e \in H$ ← component

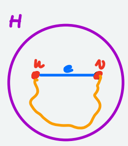


consider $H \setminus e$



e is not a bridge $\Rightarrow$ $H \setminus e$ is connected

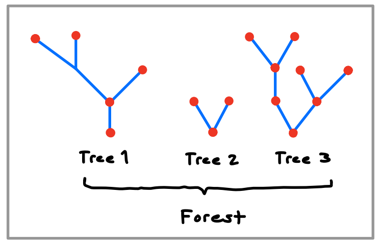
$\therefore$ $\exists$ uv-path : $P = (u, v_1, \dots, v_k, v)$



In H,

$C = (u, v_1, \dots, v_k, v, e, u)$

Trees



Forest no cycles

components are trees

Tree no cycles + connected

- minamally connected graph
  ↳ By removing any edge, T is no longer connected

Lemma

Tree $\iff$ every edge is a bridge

why: For every e : $e \notin$ cycle
For every e : $e =$ bridge

$e \notin$ cycle $\Rightarrow e =$ bridge

Theorem # edges in a tree

$$q = p - 1$$

Proof induction on q

base: $q = 0$ Then $p = 1$ (only tree possible) $\therefore \overset{0}{q} = \overset{1-1}{p-1}$

Inductive: $q \geq 1$

$\boxed{\text{I.H}}$ $(q = p - 1)$ when #edges $< q$

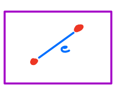
Consider:

T with q edges

Let e be an edge of T

Lemma ↳ e is a bridge

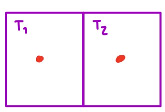
T



$T \setminus e$

- has exactly 2 components (T_1 & T_2)

$T \setminus e$



Show that T_1, T_2 are trees:

① Connected (components are connected) ✓

② No cycles (they are subgraphs of T, w/ no cycles) ✓

$\therefore T_1, T_2$ are trees w/ #edges $< q$

I.H ↳ $q_1 = p_1 - 1$, $q_2 = p_2 - 1$

$q_T = q_1 + q_2 + 1$

$p_T = p_1 + p_2$

$q_T = (p_1 - 1) + (p_2 - 1) + 1$

$= (p_1 + p_2) - 1$

$= p_T - 1$

$\therefore q = p - 1$

Theorem # edges in a Forest

$$q = p - k$$

#components

Proof

each component is a tree

$$\therefore \quad q = \sum_{T \in F} (p_T - 1) = \sum_{i=1}^{k} (p_i - 1) = (p_1 + \dots + p_k) - (1)k = p - k$$

$T \in F$: tree in forest

$i = 1 \dots k$: k components

Leaf

v with degree 1

Theorem

$\forall T$ with $p_T \geq 2$: # leaves ≥ 2

Proof longest path

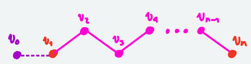
__Assume:__ T has at least 2 vertices

Let $P = v_1, \dots, v_n$ be the longest path in T

V_1 has neighbour U_2

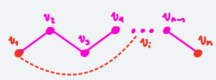
Can V_1 have other neighbours ?

- __Case 1__: neighbour off path



⊗ otherwise P could be longer

- __Case 2__: neighbour on path



⊗ otherwise T would have a cycle

$\therefore$ V_1 has degree 1 => is a leaf

Same argument for v_n (which $\neq v_1$ since T has at least 2 vertices)

$\therefore$ V_n has degree 1 => is a leaf

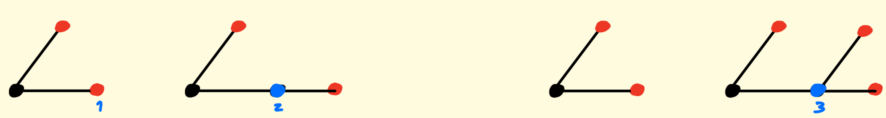
$\therefore$ T has at least 2 leaves

Note

Vertices of degree 2 have no effect on the leaf count

Vertices of higher degree do.

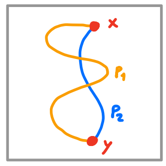
ie :



2 leaves   2 leaves   2 leaves   3 leaves

Theorem

2 distinct path between 2 vertices => cycle

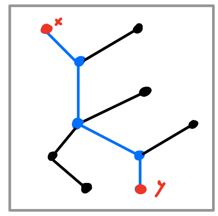


→ no cycles $\not\Rightarrow$ Unique paths

→ no cycles + connected => Unique paths

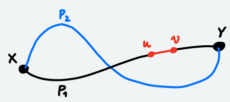
Lemma

tree : unique path between any 2 vertices



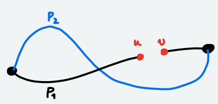
Proof

Assume: 2 unique xy paths exist in T



∴ there's an edge uv in P_1 & not in P_2

Consider T - uv

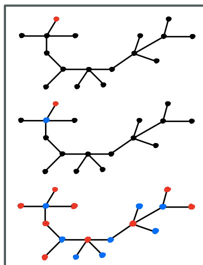


Since there still exists a uv path, then u & v are in the same component in T-uv. Hence uv is not a bridge

Lemma

A tree is bipartite

no cycles => no odd cycles => bipartite



pick v to be in A

make neighbours in B

Repeat

Span Tree

Tree + Spanning Subgraph (minimally connected subgraph)

Theorem 2^{nd} way to characterize connectedness

Connected $\iff$ has Span Tree

Proof

Assume: G has a span-tree (T) ① $\Leftarrow$

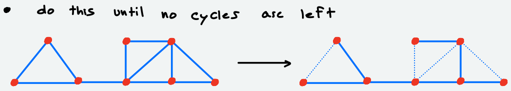
↳ there's a path between any 2 vertices in T

↳ there's a path between any 2 vertices in G $\quad V(T) = V(G)$

↳ G is connected

Idea: remove edges until you get a tree ② $\Rightarrow$

- don't remove bridges (remove edges in cycles)
- do this until no cycles are left



Since we repeat something: use induction

Since we are removing cycles: induct on the # cycles

Assume: G is connected ② $\Rightarrow$

Induction: on # cycles $= K$

$\boxed{\text{Base}}$ $K = 0$

connected + no cycles $\Rightarrow$ G is a span tree

$\boxed{\text{Inductive}}$ $K \geq 1$

$\boxed{\text{I.H}}$ #cycles $< K$ $\Rightarrow$ G has a S.T

G: K-cycles
cycle C
$e =$ edge in C

$G \setminus e$:

- connected (since e is not a bridge)
- # cycles $< K$ (since C is no longer a cycle)

I.H: $G \setminus e$ has Span-Tree T

G has a span tree, T

$V(T) = V(G \setminus e) = V(G)$

Corollary

① Connected
② $q = p - 1$ $\Bigg\} \Rightarrow$ Tree

If we don't know that there are no cycles, we can instead look at the vertex & edge count

Proof using our theorem

Assume: G is: ① Connected
② $q = p-1$

① $\Rightarrow$ has a span tree, T

$V(T) = V(G)$: T has p vertices

Theorem (#edges of a tree): T has $p-1$ edges

② $\Rightarrow$ $G = T$

G: p vertices, $p-1$ edges
T: p vertices, $p-1$ edges
$\Big\}$ $G = T$

Corollary

① connected
② no cycles
③ p-1 edges

} 2/3 hold => G is a tree

①+② : by def^{n} of a tree

①+③ : by previous corollary

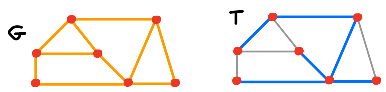
②+③ : ???

② no cycles => forest (k components)
$\therefore q = p - k$

③ $q = p-1$ => $p-1 = p-k$
$k = 1 \longrightarrow$ tree

Going from spantree-to-spantree

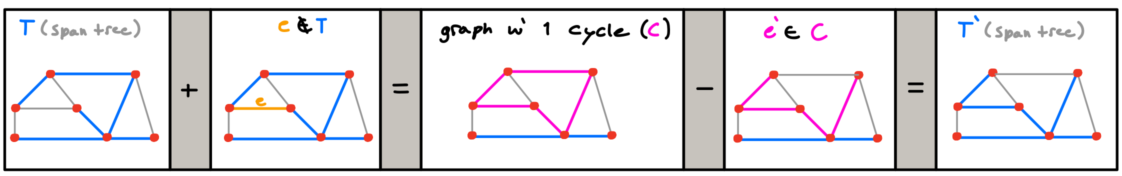
Given: T : span tree of G



Corollary : Method 1

① Add : $e \notin E(T)$ => $T + e$ has 1 cycle (C)

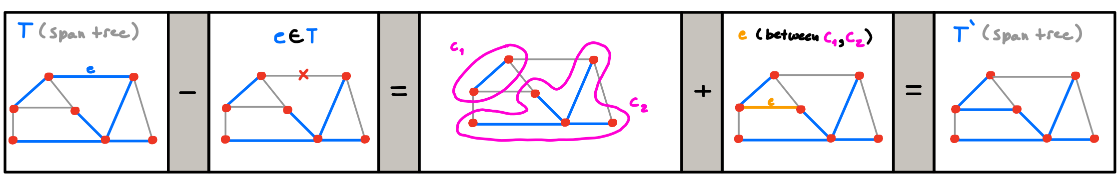
② Remove : $e' \in C$ => $T + e - e'$: new span tree of G



Corollary : Method 2

① Remove : $e \in E(T)$ => $T - e$ has 2 components (C_1 , C_2)

② Add : $e' \in C.I(V(C_1))$ or $C.I(V(C_2))$ => $T - e + e'$: new span tree of G



Bipartite Characterization

Theorem Bipartite Characterization

bipartite $\iff$ no odd cycles

Proof

Assume: G has bipartition (A, B) $\Rightarrow$

Let C : cycle, length k
represented by the closed walk : $(v_1, \dots, v_k, v_1)$

Assume (WLOG) : $v_1 \in A$
$\implies v_i \in A \iff i$ is odd
$\implies v_i \in B \iff i$ is even

$\therefore$ $v_k v_1$ is an edge $\wedge$ $v_1 \in A \implies v_k \in B \implies k$ is even

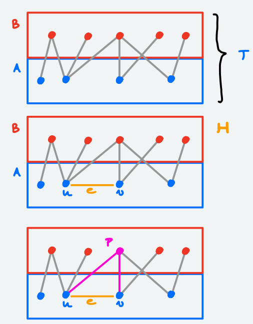
$\therefore$ Any cycle has even length

Assume: G is not bipartite (contrapositive) $\Leftarrow$

not bipartite $\implies$ non-bipartite component, H

H : component $\implies$ connected $\implies$ has a span-tree, T

T : span-tree $\implies$ bipartite $\implies$ bipartition (A, B)



H : non-bipartite $\implies$ (A, B) isn't a bipartition of H
$\implies \exists : e = uv$ in H w/ $u, v \in A$

T : span-tree $\implies$ connected
$\implies \exists$ uv-path in T (call it P)

$P = (\overset{u}{v_0}, v_1, \dots, \overset{v}{v_k})$

$v_i \in A \iff i$ is even
$\therefore$ $v_k \in A \implies k$ is even

Let $C = P + e$ be a cycle in H

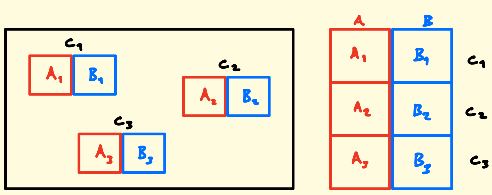
C is odd length

Note

bipartite $\iff$ every component is bipartite

Why?

If all components are bipartite, each has a bipartition
we can combine them to make a bipartition for G



bipartite $\iff$ every Subgraph is bipartite

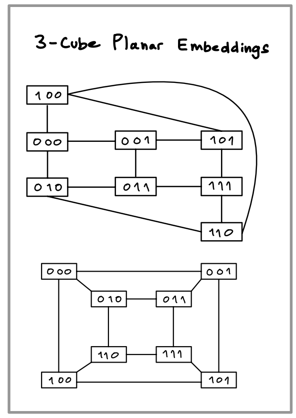
Planarity

Planar Embedding

structure　　　　specific representation

drawing of a graph :　　　　recall: graph $\neq$ drawing

- edges only intersect at vertices



Planar

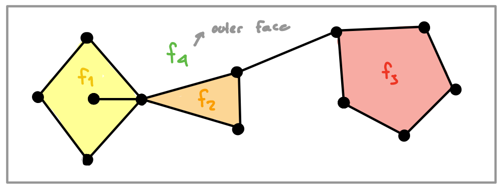
planar $\iff$ can be represented w/ a planar embedding

planar $\iff$ each component is planar

Face

region on the plane not seperated by edges

Outer face not bounded



Face Notation :

1 Boundary

Subgraph (V', E')

touch the face

2 Adjacency

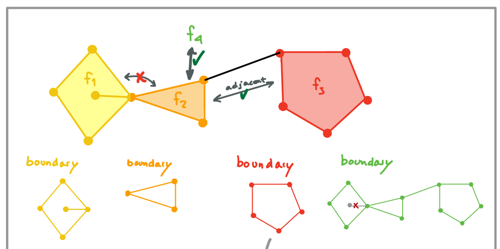
2 faces are **adjacent** : boundaries share an edge

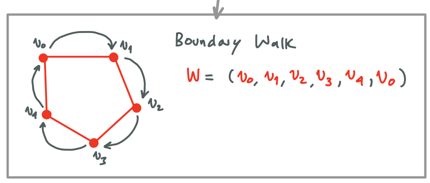
3 Boundary Walk

Closed walk , representing the boundary

4 Degree

Length of boundary walk



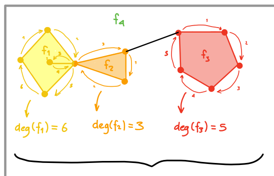


Lemma Handshake Lemma for faces

$$\sum_{f \in F} \deg(f) = 2|E|$$

disconnected or connected

Sum of face degrees : each edge has a face on each side
↳ each edge is counted twice



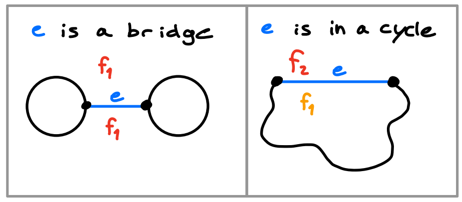
$$\sum_{i=1}^{4} \deg(f_i) = 6 + 3 + 5 + 14 = 28$$

$$2|E| = 2(14) = 28$$

Lemma

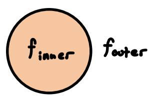
bridge ⟺ Same face on both sides

non-bridge ⟺ different faces on either side



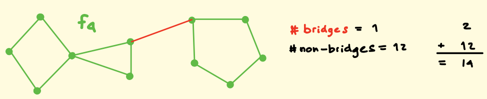
Theorem

cycle seperates plane into : ① inner face
② outer face



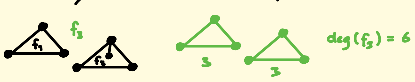
Note

deg(f) = # non bridges in boundary
\+
2(#bridges)



Note

if boundary is disconnected , sum the lengths of each walk



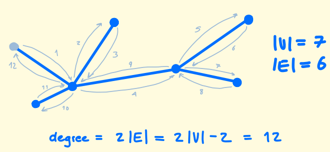
Note

Planar Embedding of a Tree :

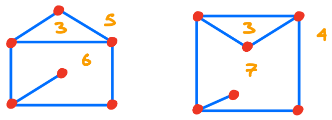
Tree :

- every edge a bridge
- 1 face
- q = p - 1

$\therefore \quad \deg(f) = 2|E| = 2(p-1)$



1 graph => many planar embeddings



<u>Why are they different embeddings</u> :

face-adjacency <u>property</u> doesn't hold

↓

Face degrees : 3, 5, 6 vs 3, 4, 7

<u>How are they the Same</u> ?

faces (3)

Theorem Euler's Formula

① connected

② Planar : s faces

} => $n - m + s = 2$

$n - m$ ↓ Structural ↓ Same, independent on drawing

s ↓ drawing ↓ Same, independent on drawing

Corollary

All planar embeddings have the same # faces

Proof

<u>Idea</u> :

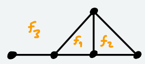
Use <u>induction</u> (weak induction)

- Fix n, induct on m, leave S as is
- inductive Step :

remove non-bridge to keep G connected

$\boxed{\text{Connected, planar embedding}} \xrightarrow{\text{remove } e} \boxed{\text{Connected, planar embedding}}$

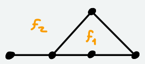
Connected, planar embedding:
- n vertices
- m edges
- s faces



Connected, planar embedding:
- n vertices
- $(m-1)$ edges → $\boxed{\text{I.H}}$
- $(s-1)$ faces ← we removed a cycle

remains constant

$\overbrace{s + n - m}^{\text{remains constant}} = 2$



Note

Connected => Span-tree => $(p-1)$ edges

Non-Planar Graphs

To prove : G is planar => give a planar embedding

To prove : G is not planar => ???

Lemma connect faces to required edge degrees

degree (every-face) $\geq d$ => $q \leq \frac{d(p-2)}{d-2}$

given a planar graph

Lemma

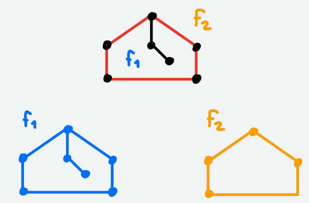
G contains a Cycle => every face boundary contains a cycle

P.E has a cycle => at least 2 faces

=> faces are different

=> completely seperated by a boundary

=> boundary must contain a cycle



Theorem to prove non-planar

$p \geq 3$: planar => $q \leq 3p-6$ — dependent of embedding

Contrapositive

$p \geq 3$: $q \geq 3p-6$ => not planar — independent of embedding

Proof

Case 1 : G has no cycles

G is a forest
↳ # edges = $n-1$

For $n \geq 3$ $(n-1) \leq 3n-6$ ✓

Case 2 : G has a cycle

(Lemma =>) every face boundary has a cycle
every cycle has length ≥ 3
} every face has degree ≥ 3

(Lemma =>) degree (every-face) ≥ 3 => $m \leq \frac{3(n-2)}{3-2}$ $m \leq 3n-6$

Corollary

K_5 is not planar

Proof

$p = 5$
$q = 10$ $\quad p \geq 3: \quad q \leq (3p-6)$
$\leq 9 \longrightarrow$ non planar

modify the theorem for bipartite graphs ?

- all cycles have even length
- cycles have length ≥ 3

$\Big\}$ every cycle has length ≥ 4
$\downarrow$
degree(every face) ≥ 4

Theorem prove non-planar (bipartite version)

$p \geq 3$ bipartite : planar $\Rightarrow q \leq 2p - 4$ — less edges allowed

Contrapositive

$p \geq 3$ bipartite : $q \geq 2p - 4 \Rightarrow$ non-planar

Proof

<u>Case 1</u> : G has no cycles

G is a forest $\Rightarrow m = n-1 \Rightarrow$ For $n \geq 3$ $(n-1) \leq 3n-6$ ✓

<u>Case 2</u> : G has a cycle

(Lemma $\Rightarrow$) every face boundary has a cycle
cycle length ≥ 3
(bipartite $\Rightarrow$) only even cycles

$\Big\}$ every face has degree ≥ 4

(Lemma $\Rightarrow$) $\exists$: Planar Embedding : degree(every-face) $\geq 4 \Rightarrow m \leq \frac{4(n-2)}{4-2}$

$m \leq 2n-4$

Corollary

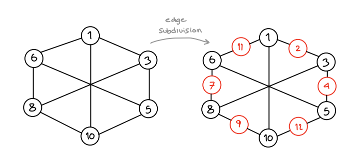
$K_{3,3}$ is not planar

Proof

$n = 6$
$m = 9$ $\quad 2(6) - 4 = 8 \quad 9 > 8 \quad$ not planar
bipartite

Any graph containing K_5 or $K_{3,3}$ is non-planar

any graph that "looks" like K_5 or $K_{3,3}$ is non-planar



Edge Subdivision

Replacing an edge with a path (vertices of degree 2 only!)

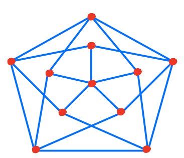
planar $\iff$ all edge subdivisions are planar

Theorem Kuratowski's Theorem

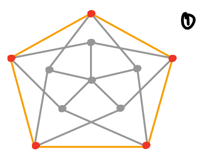
planar $\iff$ not an edge subdivision of K_5 or $K_{3,3}$

characterizes planar graphs :

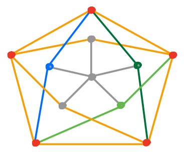
edge subdivision of K_5



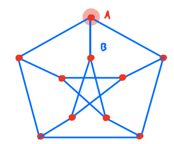
① identify a cycle



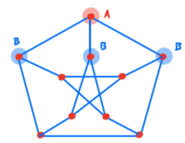
② Find 3 crossing paths



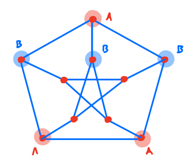
edge subdivision of $K_{3,3}$



① Pick an A



② pick 3 adjacent to A to be in B



pick 2 vertices to be in A such that the 3 vertices in B have a path to it.

Colouring

K – Colouring

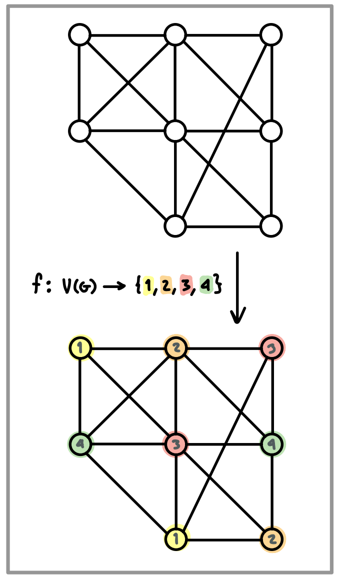
A function :

① Gives vertices a colour

$f : V \to \{ K \text{ colours} \}$

② Adjacent vertices get different colours

$f(u) \neq f(v)$ if $uv \in E$



G has a <u>K-colouring</u> : G is <u>K-colourable</u>

Note

K-colourable ~~⇒~~ (K−1) – colourable
K-colourable ⇒ (K+1) – colourable
p vertices ⇒ p – colourable

Q	What is the minimum # of colours needed ?

<u>Extreme cases</u> :

min

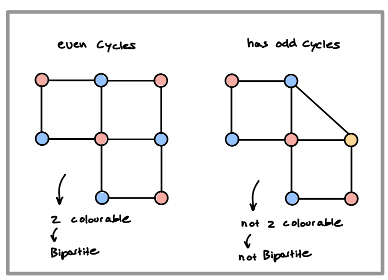
2

Theorem

Bipartite ⟺ 2 – colourable

no cycles of odd length

only even cycles ⟺ 2 – colourable

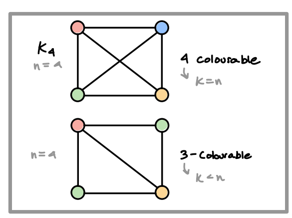


p

max

Theorem

Complete ⟺ p – colourable & not K – coulourable for $K < p$



Between these 2 extremes, determining minimum # of colours needed is hard.

We will focus on colouring planar graphs

Corollary

planar => has a vertex of degree ≤ 5

Proof

Assume: G : planar
p-vertices

If All vertices are degree ≥ 6
↳ $6p = 2q$ (Handshake)
$q = 3p$

But :
planar => $q \leq 3p-6$

⊗ $q \geq 3p-6$

Theorem * no matter the graph

planar => is 6-colourable	* proof easy
planar => is 5-colourable	* proof requires contraction
planar => is 4-colourable	* proof is too long

We can't say that <u>every</u> planar graph is 3-colourable

<u>6-Colourable Proof:</u>

Proof Induction on # vertices $p \geq 1$

<u>Base Case</u>: $p=1$

Graphs with 1 vertex are 6-colourable ✓

<u>Induction</u>: $p \geq 1$

I.H : assume H with $p \leq k$ ($k \geq 1$) is 6 colourable & planar

Show that : planar G w/ $p = k+1$ is 6-colourable

Since G is planar : it has a vertex v with $\deg(v) \leq 5$

G': remove v & all edges incident w/ v

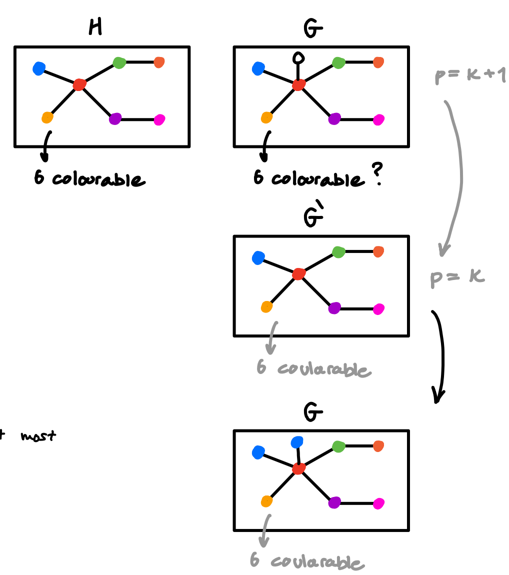
- k vertices
- planar (subgraph of a planar graph)

} G' is 6 colourable

Since v in G is adjacent to 5 vertices, they can be at most 5 colours. So give v a colour not in N(v).

Thus, G is 6-colourable

So the result is true for $p = k+1$

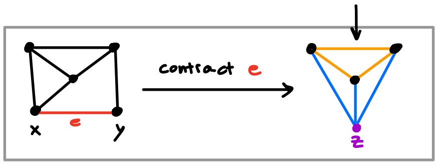


Edge Contraction

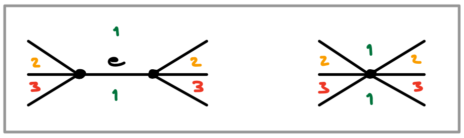
G/e : contract edge 'e' in graph 'G' (Squeeze 2 ends of an edge 2 1 vertex)

1 less vertice

at least 1 less edge



$\boxed{G \text{ is planar} \Rightarrow G/e \text{ is planar}}$



Proof

<u>Proof:</u> induction on #vertices $p \geq 1$

Let G be planar w/ p-vertices

<u>Base</u>: $p = 1$

Holds : graph w/ 1 vertice is 5 colourable

<u>Induction</u>: Let $K \geq 1$

<u>I.H</u>: assume planar graph w/ $p \leq K$ vertices is 5-colourable

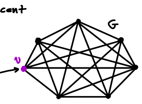
Let : G be planar w/ $p = K+1$ vertices

<u>Case 1</u>: G has v with $\deg(v) \leq 4$

Then G is 5 colourable (same argument as 6-colourable proof)
- remove this edge => I.H holds
- add back edge => still planar
- give new edge 1 of colours not in 4 adjacent

<u>Case 2</u>: no v with $\deg(v) \leq 4$

by theorem 7.55 : G has a v w/ $\deg(v) = 5$

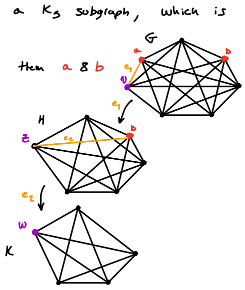


If every edge in $N(v)$ were adjacent, this would be a K_5 subgraph, which is not allowed in a planar graph.

∴ there's at least 2 non-adjacent v's in $N(v)$. Call them a & b

Contract edge $e_1 = \{a, v\} \longrightarrow H = G/e_1$
w/ new vertex 'z'

Contract edge $e_2 = \{b, z\} \longrightarrow K = H/e_2$
w/ new vertex 'w'



G planar => H planar => K planar

$\downarrow$ $K+1$ $\downarrow$ K $\downarrow$ $K-1$

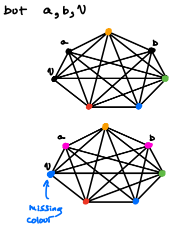
∴ K is 5-colourable (by I.H)

Use this 5-colouring & colour all but a, b, v
- a, b can have same colour
- colour a, b the same

∴ at most 4 colours appear on $N(v)$

Colour v with 1 absent colour

∴ G has a valid 5-colouring



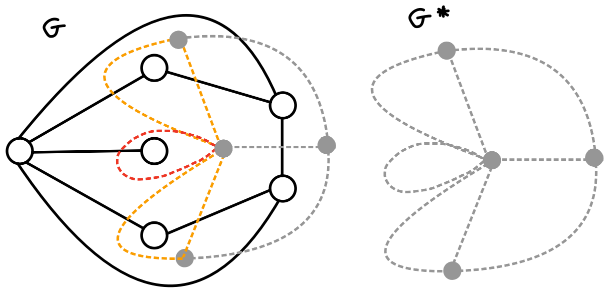
Planar Duals

given:

planar embedding of G

the dual graph G^* :

- vertex for every face
- edge for every adjacent faces



G	G^*
bridge	loop
f_1 adjacent to 2^+ faces	multiple edge
v, $\deg(v) = k$	f, $\deg(f) = k$
f, $\deg(f) = k$	v, $\deg(v) = k$

$\boxed{G \text{ is connected} \Rightarrow (G^*)^* = G}$

$\boxed{\text{Dual graph is planar}}$

↓

4/5/6 colour theorems holds for duals

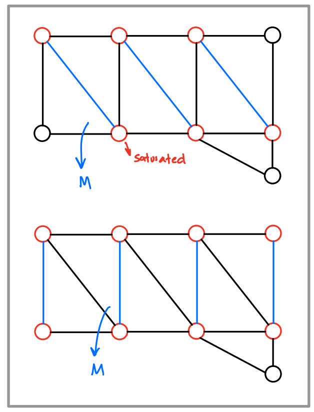
Matchings

Matching

Matching = { edges : no 2 edges have a common vertex }

Saturated

v is <u>saturated</u> if v is incident with an edge in M

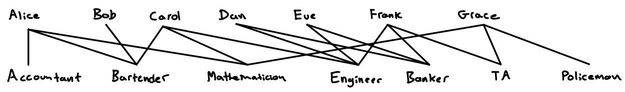


Application job assignment

<u>People</u> : Alice Bob Carol Dan Eve Frank Grace

<u>Jobs</u> : Accountant Bartender Mathematician Engineer Banker TA Policeman

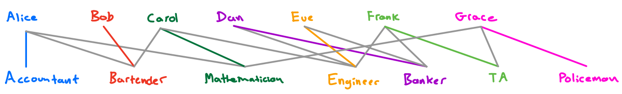
They apply to jobs :



1) each person can work only 1 job

2) each job can accept only one perso

A <u>perfect matching</u> would be a way to satisfy these 2 requirements



Q	What is the maximum matching of a graph ?

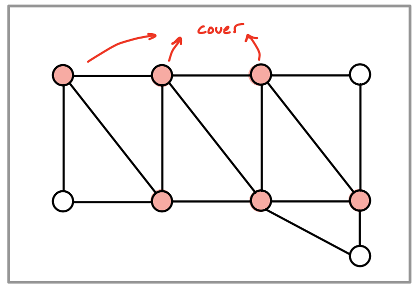
$\boxed{\emptyset}$ smallest matching

Maximum Matching — <u>Saturates</u> max # vertices

└→ Perfect Matching — <u>Saturates</u> every vertex

Cover

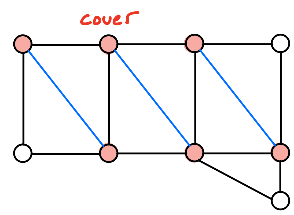
$C = \{$ vertices : every edge has an end in $C \}$



Q	What is the minimum cover of a graph ?

$\boxed{V}$ largest cover

Matchings vs Covers



Lemma

M : matching
C : cover
$\Big\} \; |M| \leq |C|$

$\forall\, e = uv \in M$, at least 1 of u, v is in C
No 2 edges in M share a vertex
$\Big\} \Rightarrow$ C must have at least $|M|$ vertices

$C_1 \quad C_2 \quad C_3$ we want to minimize the size of covers

these sizes never cross

$M_1 \quad M_2 \quad M_3$ we want to maximize the size of matchings

Lemma

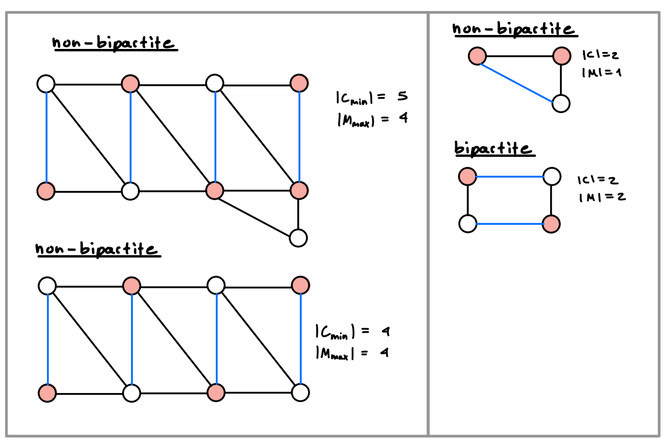
$|M| = |C| \Rightarrow$ M is maximum
C is minimum

For what graphs does converse apply

Theorem Konig's Theorem

bipartite $\Rightarrow |M_{max}| = |C_{min}|$

not bipartite $\Rightarrow |M_{max}| \leq |C_{min}|$

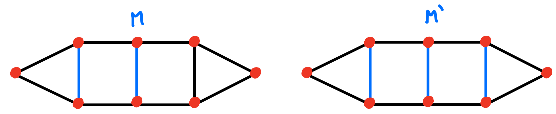


Augment Matchings

Given a matching, how could we get a larger matching

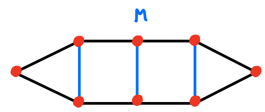
Case 1 : there's another edge we can include

Simply add it



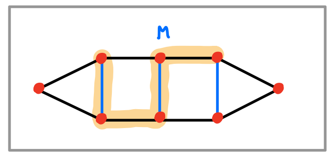
Case 2 : no other edge we could add

???



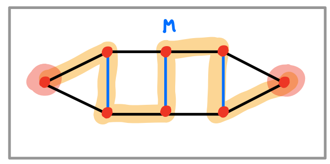
Alternating path

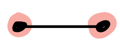
path where consecutive edges alternate from : $\in M$ to $\notin M$

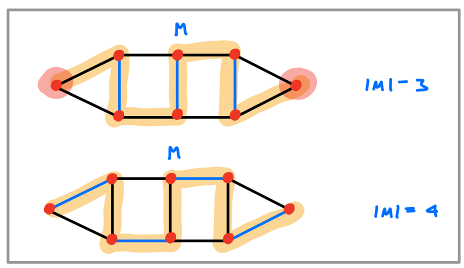


Augmented Path

alternating path that starts & ends in distinct unsaturated vertices



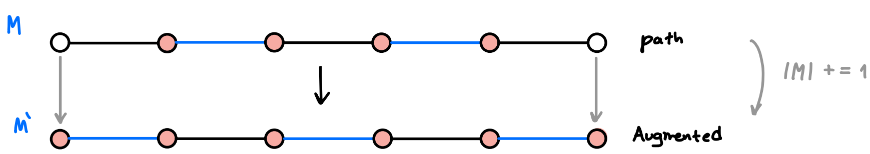
-  is the smallest augmented path
- The 1st & last edges $\notin M$
- always odd length



How to | find a larger M

① Find augmented path

② Swap edges in the path



Lemma

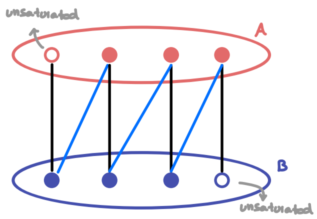
has an augmented path => M isn't max
M is max => no augmented path

converse

To find a maximum matching of a bipartite graph

Bipartite Matching Algorithm XY – construction

Idea:

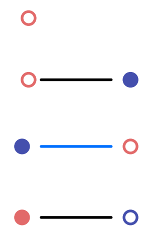


Suppose Augmented path P starts in A

=> Every time we go to B : $\notin M$

=> Every time we go to B : $\in M$

=> we end in B



P starts & ends w/ vertices in a different partition

Algorithm:

c) no such vertices exist

① $X = X_0$, $Y = \emptyset$

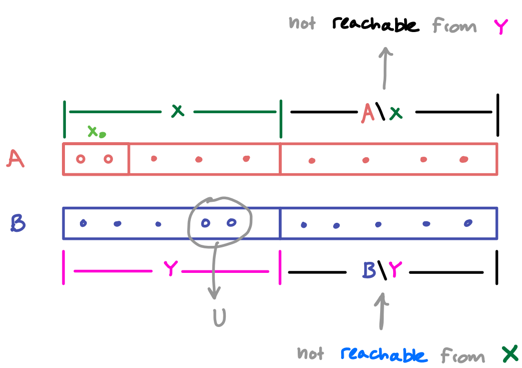
② Find neighbours of X in $B \setminus Y$

2.1 B: any unsaturated => P found =>
- ↳ Augment P
- ↳ Step ①

2.2 B: all saturated
- ↳ put them in Y
- ↳ Follow matching to A
- ↳ Step ②

2.3 none found => DONE

Max Matching. $|M| = |C| = Y \cup (A \setminus X)$



P = path starting at X_0

$X_0 = \{$ unsaturated in $A \}$

$X = \{ v \in A :$ reachable by $P \}$

$Y = \{ v \in B :$ reachable by $P \}$

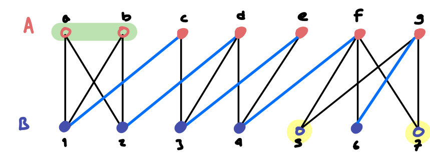
During:

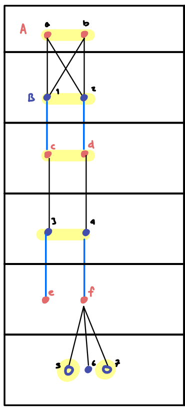
$|C| = Y \cup (A \setminus X)$

$|M| = |C| - U$

$\left.\begin{matrix} |C| \\ |M| \end{matrix}\right\} = Y \cup (A \setminus X)$

Example:

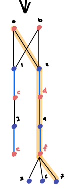


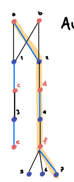


	X	Y
X_0 = { a b } unsaturated vertices in A	a b	$\emptyset$
Unsaturated: { 1 2 }	a b	1 2
Follow matching { c d }	a b c d	1 2
Unsaturated: { 3 4 }	a b c d	1 2 3 4
Follow matching { e f }	a b c d e f	1 2 3 4
{ 5 6 7 } 5, 7 unsaturated	a b c d e f	1 2 3 4 5 6 7

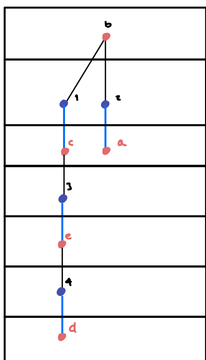
P =

Pick 1 of them (ie: 7)





Augment P



	X	Y
X_0 = { b } unsaturated vertices in A	b	$\emptyset$
	b	1 2
	b c a	1 2
	b c a	1 2 3
	b c a e	1 2 3
	b c a e	1 2 3 4
	b c a e d	1 2 3 4
Stop! neighbours already discovered		

$C_{min} = Y \cup (A \setminus X) = \{1,2,3,4,f,g\}$

$\therefore |C_{min}| = |M_{max}| = 6$

(of Konig's Theorem)

Corollary Bipartite

$\text{deg max} = d \implies |M| \geq \frac{q}{d}$

$|C| \geq \frac{m}{d}$, by Konigs : $|M| = |C| \geq \frac{m}{d}$

every v covers at most d edges

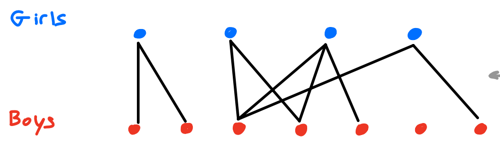
So we need at least $\frac{m}{d}$ vertices to cover all edges

Looking at matching's of bipartite graphs from a different angle

Hall's Theorem bipartition (A,B)

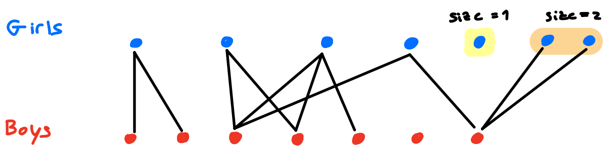
A matching saturates A $\iff$ $\forall D \subseteq A : |N(D)| \geq |D|$

no subset in A whose neighbour set is smaller than itself



← potential mates

What could prevent us from matching every girl with 1 boy



① isolated $|N| = 0, \quad |S| = 1$

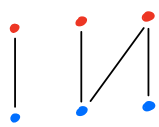
② group of girls only want same boy $|N| = 1, \quad |S| = 2$

Applications:

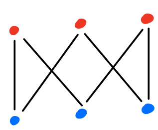
Corollary bipartite :

① $|A| = |B|$
② Halls Condition
$\iff$ perfect matching

ie : k - regular



↙ not regular



↙ regular

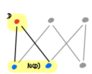
Theorem bipartite :

K regular => perfect matching

If $D \subseteq A$

Each edge w' 1 end in D has the other end in $N(D)$

↳ $\sum_{D} \deg(v) \leq \sum_{N(D)} \deg(v)$



k-reg ↳ $K|D| \leq K|N(D)|$

↳ $|D| \leq |N(D)|$

↓ Hall's

a matching saturates A

↓ regular $|A| = |B|$

perfect matching

Edge Colouring

Edge k-colouring

f : edge $\longrightarrow$ { colours }

- Share a vertex => different colour

partitions the edges of G into k total matchings

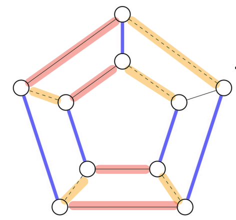
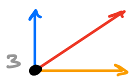


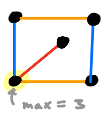
Figure 8.11: A graph with an edge 3-colouring

3 matchings

We need at least deg (v) different colours for edges out of v



to have an edge-k-colouring : $k \geq$ max degree



Theorem bipartite

max degree = k => has edge-k-colouring

Lemma bipartite

$q \geq 1$ => matching saturating each vertex of max degree

regular

every vertex has max degree => perfect matching