Counting Combinatorally

Products & Sums

Choose between 2 sets : A, B

choices given by

① Choose from A $\boxed{\text{AND}}$ from B

② Choose from A $\boxed{\text{OR}}$ from B

choices given by

Cartesian Product

$A \times B = \{(a,b) : a \in A, b \in B\}$

$|A \times B| = |A| \cdot |B|$

Union

$A \cup B = \{ c : c \in A \lor c \in B \}$

$|A \cup B| = |A| + |B| - |A \cap B|$

inclusion exclusion principle

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given a set S

Types of Objects to Count

Given S has n-elements

List

all elements of S listed : once each in order

Permuatation list of the set $\{1, \dots, n\}$

 # lists

$n! = (n)(n-1) \cdots (1)$

note: $0! = 1$

Partial List

k elements of S listed : once each in order

 # length-k partial lists

$\frac{n!}{(n-k)!} = (n)(n-1) \cdots (n-k-1)$

Subset

collection of elements in S : once each in no order

 # Subsets

2^n in/out (2 choices) for n-elements

 # k-element Subsets

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ note: $\binom{0}{0} = 1$

 # length-k partial lists = **# k-element Subsets** $\times$ **# lists** of a k element subset

$$\frac{n!}{(n-k)!} = \frac{n!}{k!(n-k)!} \cdot k!$$

partial list = list of a subset

Multiset

collection of elements : more than once (can occur) in no order

A sequence $\underbrace{(m_1, m_2, \dots, m_t)}_{m_1 + \dots + m_t = n}$, where m_i = # of type i

 # Multisets of t-types

$$\binom{n+t-1}{t-1}$$

Combinatorial Proof

Interpret #'s as counting things (+ = or, × = and)

① **Double-Counting:** Show that they count the same thing. (only works when counting exactly the same thing)

② **Bijection:** We can pair all elements between them

Ex

Prove that: $2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$

Given an n-element subset, S:

subsets of S = 2^n

Each subset has a size. Call it K

$\{\text{Subsets}\} = \bigcup_{K \geq 0}^{n} \{\text{Size-K subsets}\}$

Subsets $= \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$

∴ These are both ways to count the # subsets of S.

Ex

Prove that: $\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n-1}{k} = \binom{n}{k+1} \quad \forall\ 0 \leq k \leq n$

RHS Let $S = \{ (k+1)\text{-element subsets of } \{1, \dots, n\} \}$ ∴ $|S| = \binom{n}{k+1}$

LHS $\forall m : k \leq m \leq n-1$

Let $T_m = \{ k\text{-element subsets of } \{1, \dots, m\} \}$ ∴ $|T_m| = \binom{m}{k}$

Let $T'_m = \{ (k+1)\text{-element subset of } \{1, \dots, m+1\} \}$ ∴ $|T'_m| = \binom{m+1}{k}$

(m+1) + k-elements from {1, …, m}

$S = T'_k \cup T'_{k+1} \cup \cdots \cup T'_{n-1}$

So $|S| = \binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n-1}{k}$

Ex

prove: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

k-element subsets of n-element set

(k-1)-element subsets of (n-1) element set

k-element subsets of (n-1) element set

S : {1, 2, …, n}

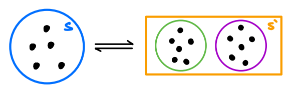
S' : {1, 2, …, n-1}

counts subsets of S

counts subsets of S'

LHS & RHS aren't counting exactly the same thing

However we can show that the objects counted on either side can be paired up w/ each other.



Choosing a: k-element subset of S ⇌ Choosing a: (k-1)-element subset of S' or k-element subset of S'

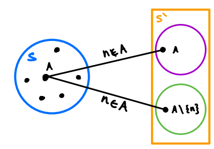
2 cases when choosing a k-element subset of S, A:

① $n \notin A$

$A \setminus \{n\}$: k-element subset of {1, 2, …, n-1}

② $n \in A$

$A \setminus \{n\}$: (k-1)-element subset of {1, 2, …, n-1}



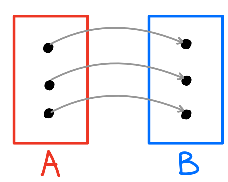
Bijection: rule that ② uniquely maps ① every element of A w/ every element of B

Function: $f : A \to B$

Where

onto — Every $b \in B$ maps to some $a \in A$

one-to-one — Every $a \in A$ maps to only 1 $b \in B$



How to	Prove f is a bijection

f is a bijection ⟺ has an inverse g, where

① $g(f(a)) = a \quad \forall a \in A$

② $f(g(b)) = b \quad \forall b \in B$

Undo each other

Theorem

$A \rightleftharpoons B \Rightarrow |A| = |B|$

"there's a bijection between A & B"

How to | Combinatorial proof w/ bijection

To prove LHS = RHS

1. Find A where $|A| =$ RHS
   Find B where $|B| =$ RHS
2. Find bijection : $f : A \to B$

$\therefore |A| = |B| \quad \therefore$ LHS = RHS

Ex

prove : $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

<u>Proof (using bijections):</u>

$S = \{1, \dots, n\}$, $S' = \{1, \dots, n-1\}$

Let $A = \{$ k-element subsets of S $\}$ $\longrightarrow$ $|A| = \binom{n}{k}$

Let $B = \{$ $(k-1)$ or k element subsets of S' $\}$ $\longrightarrow$ $|B| = \binom{n-1}{k-1} + \binom{n-1}{k}$

If we prove that $A \rightleftharpoons B$, then $|A| = |B|$ meaning LHS = RHS

Define $f : A \to B$ by :

$$f(x) = \begin{cases} x & : \text{ if } n \notin x \\ x \setminus \{n\} & : \text{ if } n \in x \end{cases}$$

Define $g : B \to A$ by

$$g(y) = \begin{cases} y & : \text{ if } |y| = k \\ y \cup \{n\} & : \text{ if } |y| = k-1 \end{cases}$$

$\left.\begin{array}{ll} g(f(a)) = a & \forall a \in A \\ f(g(b)) = b & \forall b \in B \end{array}\right\}$ inverses

$\therefore f$ is a bijection

$\therefore A \rightleftharpoons B$

$\therefore |A| = |B|$

* it suffices to state f and it's inverse.
  you don't need to prove that they're inverse

We can also use a bijection to make counting easier :

Ex :

<u>Find :</u> # Multisets of t-types

Let $S = \{$ size-n multisets of t-types $\}$

To find $|S|$: put S in bijection w/ a different set whose size is easy to find

Let $T = \{$ size-n strings containing $(n-t-1)$ dots & $(t-1)$ dividers $\}$

define $f : S \to T$ as :

$$f(n_1, n_2, \dots, n_t) = \underbrace{\bullet \bullet \cdots \bullet}_{n_1} \mid \underbrace{\bullet \bullet \cdots \bullet}_{n_2} \mid \cdots \mid \underbrace{\bullet \bullet \cdots \bullet}_{n_t}$$

And $g : T \to S$ as :

$g(t) = (n_1, n_2, \dots, n_t)$

- $n_1 = \# \bullet$ before 1^{st} $\mid$ in t
- $n_2 = \# \bullet$ between 1^{st} & 2^{nd} $\mid$ in t
- $n_t = \# \bullet$ after $(t-1)^{th}$ $\mid$ in t

$\therefore S \rightleftharpoons T$

$\therefore |S| = |T|$

$\therefore$ (# possible multisets) $=$ (# length-$(n+t-1)$ (n $\bullet$'s & $(t-1)$ $\mid$'s))

$= \binom{n+t-1}{t-1}$ choose locations of $\mid$'s

Power Series

infinite polynomial

never ending

$$P(x) = \underbrace{p_0}_{\text{constant}} + \underbrace{p_1 x + p_2 x^2 + \cdots}_{x \text{ terms}} = \sum_{n \ge 0} p_n x^n$$

Needs to be Well-Defined : each coefficient is finite

not ∞

Geometric $\quad \dfrac{1}{1-x} = \sum_{i \ge 0} x^i$

NBT $\quad \dfrac{1}{(1-x)^t} = \sum_{n \ge 0} \binom{n+t-1}{t-1} x^n$

Binomial Theorem $\quad (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

$n \in \mathbb{N}$

Extract Coefficients $\quad F(x) = \sum_{n \ge 0} f_n x^n \qquad [x^k]F(x) = f_k$

1	$[x^k](aF(x) + bG(x)) = a[x^k]F(x) + b[x^k]G(x)$
2	$[x^k](x^l F(x)) = [x^{k-l}]F(x)$
3	$[x^k](F(x)G(x)) = \sum_{l=0}^{k}([x^l]F(x))([x^{k-l}]G(x))$

$(x^l)(x^{k-l}) = x^k$

ex: $[x^n](1+2x)(1-3x)^{-3}$

$= [x^4]\left[(1+2x) \cdot \sum_{n \ge 0} \binom{n+3-1}{3-1}(3x)^n\right]$ NBT

$= [x^4]\left[\sum_{n \ge 0} \binom{n+3-1}{3-1}(3x)^n + 2x \sum_{n \ge 0} \binom{n+3-1}{3-1}(3x)^n\right]$ sum rule

$= [x^4]\sum_{n \ge 0}\binom{n+2}{2}3^n x^n + [x^4]\left[2x\sum_{n \ge 0}\binom{n+2}{2}3^n x^n\right]$ 2^{nd} rule

$= 3^n\binom{n+2}{2} + 2[x^3]\sum_{n \ge 0}\binom{n+2}{2}3^n x^n$

$= 3^n\binom{n+2}{2} + 2\binom{n+2}{2}3^n$

1 FPS

$$a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

$a_0, a_1, a_2, \ldots$ Sequence of $\mathbb{C}$ #'s (we'll only care about $\mathbb{N}$)

Examples:

Geometric Series:

$1 + x + x^2 + \cdots$ FPS associated to sequence of 1's

Polynomials: FPS where eventually, coefficients are all 0 (finite series)

$2 + x + 3x^2 + 4x^3$

Generating Series: is a FPS

Equality of FPS: each coefficient is equal

Coefficient of FPS: $a_k = [x^k] A(x)$ — $A(x)$: FPS

2 ADD/MULTIPLY FPS

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots$$

Add $\quad A(x) + B(x) = (a_0 + b_0) + (a_0 + b_0)x + (a_0 + b_0)x^2 + \cdots$

Multiply $\quad A(x) \cdot B(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + \cdots$

Ex: Find B(x)

① $(1-x)B(x) = 1 + x$

- $1 - 2x + 0x^2 + \ldots$
- $b_0 + b_1 x + b_2 x^2 + \cdots$
- $1 + x + 0x^2 + \cdots$

LHS: $(1-2x)(b_0 + b_1 x + b_2 x^2 + \cdots)$

$b_0 + (b_1 - 2b_0)x + (b_2 - 2b_1)x^2 + \cdots$

RHS: $1 + x$

LHS = RHS when sequence of coefficients are equal

$[x^k]\,(1-2x)B(x) = [x^k]\,(1+x)$ for $k \geq 0$

- $k = 0$: $b_0 = 1$
- $k = 1$: $b_1 - 2b_0 = 1 \rightarrow b_1 = 3$
- $k \geq 2$: $b_k - 2b_{k-1} = 0 \rightarrow b_k = 2b_{k-1}$

$\therefore\ B(x) = 1 + 3x + 6x^2 + \cdots + 3\,2^{k-1}x^k + \cdots$

② $(x - 2x^2)B(x) = 1 + x$

For any $B(x)$: $\left.\begin{array}{l}[x^0]\,(x-2x^2)B(x) = 0 \\ \text{but } [x^0]\ 1 + x = 1\end{array}\right\}$ $\therefore$ no solution

Theorem 2.1 $\forall A(x), C(x)$:

$A(0) \neq 0 \implies$ There's a unique sol^n $B(x)$ to:

$$A(x)B(x) = C(x)$$

3 INVERSE FPS

Inverse of FPS $A(x)$: $B(x) = A(x)^{-1} \rightleftharpoons A(x)B(x) = 1$

Ex:

$(1-x)(1+x+x^2+\cdots) = 1$

$\hookrightarrow \therefore \frac{1}{1-x} = 1 + x + x^2 + \cdots$ (geo Series)

Theorem 3.1

Inverse of FPS $A(x)$ exists $\Longleftrightarrow A(0) \neq 0$

Inverse exists $\Rightarrow$ unique

Composition

For FPS $A(x), B(x)$

$A(B(x)) = a_0 + a_1 B(x) + a_2 B(x)^2 + \cdots$

Note

$A(x) = \underset{a_0}{1} + \underset{a_1}{x} + \underset{a_2}{x^2} + \cdots$

$B(x) = \underset{b_0}{1} + \underset{b_1}{x}$

$\Big\} A(B(x)) = 1 + (1+x) + (1+x)^2$

The constant term of $A(B(x)) = A(B(0)) = \underbrace{1 + B(0) + B(0)^2 + \cdots}_{\text{infinite}}$

not well defined !

unless $B(0) = 0$ (const-term of $B(x) = 0$)

Theorem 3.2

$B(0) = 0 \Rightarrow A(B(x))$ is well defined FPS

How to | Find Inverse

① Theorem 2.1 w/ $C(x) = 1$

② Geometric Series + Composition

Ex: Inverse of $(1-x+2x^2)$

$$\frac{1}{(1-x+2x^2)} = \frac{1}{1-(x-2x^2)} \overset{GS}{=} 1 + (x-2x^2) + (x-2x^2)^2 + \cdots$$

$A(B(x))$

geo Series $\quad B(0) = 0$

3 Rational / Irrational Series

Rational $\overset{FPS}{A(x)} = \frac{P(x)}{Q(x)}$ $\begin{matrix}\rightarrow \text{polynomial} \\ \rightarrow \text{polynomial}\end{matrix}$

Theorem 4.1 $\underset{\text{Squared}}{A(x)^2} = \underset{\substack{\text{odd} \\ \text{degree}}}{1+x} \Rightarrow A(x)$ <u>not rational</u>

$A(x)^2$ is odd degree $\Rightarrow A(x)$ <u>not rational</u>

Theorem 4.2 $A(x)$ <u>rational</u> FPS $\Rightarrow A(x)$ is a polynomial

$A(x)^2$ polynomial $\quad \downarrow$ rational

Theorem 4.2 $\Rightarrow$ **Theorem 4.1**

any polynomial squared has even degree.

But $(1+x)$ has odd degree

$A(x)^2 = 1+x$

$A(x)^2$ polynomial $\Rightarrow A(x)$ <u>not rational</u>

Generating Series

an FPS (power series, but only care about coefficients)

We can encode answers to counting problems in a generating series.

2 Things Define a Generating Series:

① Set (S)
② weight function (ω)

} FPS where coefficients mean something

Weight Function

$\omega : A \longrightarrow \mathbb{N} \quad : \quad \omega(a) \Rightarrow$ weight

$\omega^{-1} : \mathbb{N} \longrightarrow A \quad : \quad \omega^{-1}(n) = A_n$

How to	Prove that ω Is a weight function

① $\omega(a) \geq 0 \quad \forall a$

② $\forall n \geq 0 : \ \omega^{-1}(n)$ is finite

if A has $\omega_A : A \to \mathbb{N}$:

$$A = \bigcup_{n=0}^{\infty} A_n$$

Finite or countably infinite

Finite

elements in A of weight n

$$[x^n]\, \Phi_A(x) = |A_n|$$

$$\Phi_S^{\omega}(x) = \sum_{\sigma \in S} x^{\omega(\sigma)}$$

big set

element in S

the type of element

Let $S_n = \{\sigma \in S : \omega(\sigma) = n\}$

$$\Phi_S^{\omega}(x) = \sum_{n \geq 0} |S_n|\, x^n$$

Purpose:

Combinatorial Setup $\xrightarrow{\text{geo-series}}$ algebraic object

$S = \{$ multisets on t-types $\}$, $\omega(M) = \lvert M \rvert$	$S = \{$ all binary Strings $\}$, $\omega(\sigma) =$ length of σ	$S = \{$ Subsets of $\{1,2,\ldots,t\}\}$, $\omega(\sigma) = \lvert\sigma\rvert$
$\Phi_S(x) = \sum_{\sigma \in S} x^{\lvert M \rvert} = \sum_{\sigma \in S} \lvert\sigma_n\rvert x^n$	$\Phi_S(x) = \sum_{\sigma \in S} x^{(\text{length of } \sigma)} = \sum_{\sigma \in S} \lvert\sigma_n\rvert x^n$	$\Phi_S(x) = \sum_{\sigma \in S} x^{\omega(\sigma)}$
$= \sum_{n \geq 0} (\#\, n\text{-element multisets on t-types})\, x^n$	$= \sum_{n \geq 0} (\#\text{binary Strings of length } n)\, x^n$	$= \sum_{n \geq 0} (\#\, n\text{-element Subsets of } \{1,\ldots,t\})\, x^n$
$= \sum_{n \geq 0} \binom{n+t-1}{t-1} x^n$ NBT	$= \sum_{n \geq 0} 2^n x^n$ geo-series	$= \sum_{n \geq 0} \binom{t}{n} x^n$
$= (1-x)^{-t}$	$= \dfrac{1}{1-2x}$	$= (1+x)^t$

Lemmas

combinatorial meaning of adding/multiplying generating series

Sum Lemma

1) $A \cap B = \emptyset$

2) $\omega_1(\sigma_1) = \omega_2(\sigma_2) = \omega(\sigma_1, \sigma_2)$

$$\left.\begin{matrix} 1) \\ 2) \end{matrix}\right\} \quad \Phi_{S_1}(x) + \Phi_{S_2}(x) = \Phi_{S_1 \cup S_2}(x)$$

GS of the union

Product Lemma

1) $\omega_1(\sigma_1) + \omega_2(\sigma_2) = \omega(\sigma_1, \sigma_2) \longrightarrow \Phi^{\omega_1}_{S_1}(x) \times \Phi^{\omega_2}_{S_2}(x) = \Phi_{S_1 \times S_2}(x)$

gen-series of cartesian product

Stars

$$S^* = S^0 \cup S^1 \cup S^2 \cdots = \bigcup_{n \geq 0} S^n$$

Sum lemma

ω of S^*

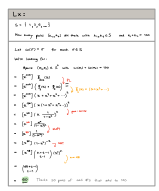
$$\omega_{S^*} = \omega_{S^0} = \omega_{S^1} = \omega_{S^1}$$

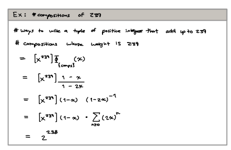
no elements S can have weight -0

Φ_S can't have constant term

String Lemma

$\omega_S \neq 0$: $\quad \Phi_{S^*}(x) = \dfrac{1}{1 - \Phi_S(x)}$





2 Applications:

Composition

$$\Phi_S(x) = \frac{x}{1-x}$$

$S = \{1, 2, 3, \ldots\}$

ω_S = "size" = sum of parts

$S^* = \{\text{all compositions}\}$

$$\Phi_{S^*}(x) = \sum_{n \geq 0} |S_n| x^n \quad \begin{cases} 1 & n = 0 \\ 2^{n-1} & n \geq 1 \end{cases}$$

ε $\{\}$, $\omega(\{\}) = 0$

Binary String

$$\Phi_S(x) = 2x$$

$S = \{0, 1\}$

ω_S = length

$S^* = \{\text{all strings}\}$

$$\Phi_{S^*}(x) = \sum_{n \geq 0} |S_n| x^n \longrightarrow 2^n$$

ε $\{\}$, $\omega(\{\}) = 0$

Note: $\displaystyle \sum_{(s,t) \in S \times T} f(s,t) = \sum_{s \in S} \sum_{t \in T} f(s,t)$

Stars

$S^* = S^0 \cup S^1 \cup S^2 \cdots = \bigcup_{n \geq 0} S^n$

Sum lemma

ω of S^*

$\omega_{S^*} = \omega_{S^0} = \omega_S = \omega_{S^1}$

$\omega = 0$

no elements S can have weight - 0

Φ_S can't have constant term

String Lemma

$\omega_S \neq 0$: $\quad \Phi_{S^*}(x) = \dfrac{1}{1 - \Phi_S(x)}$

Composition

$\Phi_S(x) = \dfrac{x}{1-x}$

$S = \{1, 2, 3, \ldots\}$

ω_S = "size" = sum of parts

any # parts

$S^* = \{\text{all compositions}\}$

Binary String

$\Phi_S(x) = 2x$

$S = \{0, 1\}$

ω_S = length

any # parts

$S^* = \{\text{all strings}\}$

Restrictions on Compositions: 2 main ways

S^*

1) length of compositions restricted
2) S restricted
3) $S = A \cup B$, $A \times B$

Ex 1: Find # compositions with odd # parts

$S = \mathbb{Z}_{\geq 1}$

$S' = S^1 \cup S^3 \cup \cdots = \bigcup_{k \geq 0} S^{2k+1}$

Sum Lemma

$\Phi_{S'} = \sum_{k \geq 0} \Phi_S(x)^{2k+1}$

$= \Phi_S(x) \sum_{k \geq 0} \left[\Phi_S(x)^2\right]^k$

GS

$= \dfrac{\Phi_S(x)}{1 - \left[\Phi_S(x)^2\right]}$

Ex 2: Find # compositions where all parts are odd

$S = \{1, 3, 5, \dots\}$ $\quad\Phi_S(x) = \sum_{n\geq 0} x^{2n+1} = \frac{x}{1-x^2}$

$w_S = \text{size} \neq 0$

$S^* = \bigcup_{k\geq 0} S^k$

String lemma

$$\Phi_{S^*} = \frac{1}{1-\Phi_S(x)}$$

$$= \frac{1}{1-\frac{x}{1-x^2}}$$

Ex 3: Find # compositions where each part is at least 2 & #parts is even

$S = \{2, 3, \dots\}$ $\quad\Phi_S(x) = \sum_{n\geq 2} x^n = x^2 \sum_{n\geq 2} x^{n-2} = x^2 \sum_{n\geq 0} x^n = \frac{x^2}{1-x}$

$w_S = \text{size} \neq 0$

$S' = S^0 \cup S^2 \cup \dots = \bigcup_{k\geq 0} S^{2k}$

Sum Lemma

$$= \sum_{k\geq 0} \Phi_S(x)^{2k}$$

$$= \sum_{k\geq 0} \left[\Phi_S^2(x)\right]^k$$

$$= \frac{1}{1-\Phi_S^2(x)}$$

$$= \frac{1}{1-\frac{x^2}{1-x}}$$

Ex 3: How many compositions with $2k$ parts where: 1^{st} k parts at least 5; 2^{nd} k parts are multiples of 3

Let

$A = \{5, 6, 7, 8, \dots\}$
$B = \{3, 6, 9, 12, \dots\}$

weight preserving bijection

Our set is

$S = A^k \times B^k$ ☆ All compositions satisfying the properties.

$$\Phi_A(x) = \frac{x^5}{1-x}$$

$$\Phi_B(x) = \frac{x^3}{1-x^3}$$

By product lemma,

$$\Phi_S(x) = (\Phi_A(x))^k (\Phi_B(x))^k$$

$$= \left(\frac{x^5}{(1-x)}\right)^k \left(\frac{x^3}{1-x^3}\right)^k$$

$$= \frac{x^{8k}}{(1-x)^k (1-x^3)^k}$$

Answer is $[x^n]\Phi_S(x)$

Ex 4: # compositions where: 1st & every other part is greater than 4
every other part is in $\{1,2,3\}$

$A = \{4,5,\dots\}$ $\omega_A(a) = a$

$B = \{1,2,3\}$ $\omega_B(b) = b$

$S = A \times B \times A \times B \times \cdots$

$D = (A \times B)^*$

bijection splits the set D by parity of parts

weight preserving

even # parts

odd # parts

$$S \rightleftharpoons D^* \cup (D^* \times A)$$

bijection alone

$$\therefore \; |S| = \left| D \cup (D \times A) \right|$$

weight preserving

$$\Phi_D(x) = \underbrace{\Phi_S(x) + \Phi_S(x)\, \Phi_A(x)}_{\text{now we can use simple sum, product and string lemmas}}$$

A bijection between A & B is enough to say that $|A| = |B|$

However, you need a weight-preserving bijection to say that $\Phi_A(x) = \Phi_B(x)$

$\boxed{\text{Concatenation}}$ $\sigma_1\sigma_2 = a_1 \dots a_n b_1 \dots b_m$

$\boxed{\text{Set}}$ $S_1 S_2 = \{ \sigma_1\sigma_2 : \sigma_1 \in S_1, \sigma_2 \in S_2 \}$ $\xrightarrow{\text{ambiguity}}$ unambigous if $AB \rightleftharpoons A \times B$

no duplicates

for every ab, unique pair (a,b)

$\boxed{\text{Union}}$ $S_1 \cap S_2$ $\xrightarrow{\text{ambiguity}}$ un ambigous if $A \cap B = \emptyset$

$\boxed{\text{Star}}$ S^* $\xrightarrow{\text{ambiguity}}$ unambigous if $\varepsilon \cap A \cap A^2 \cap A^3 = \emptyset$

And $A^2, A^3, \dots$ are ambigous

$\boxed{\text{Expressions}}$ Describes a set of strings

- $\{\varepsilon\}$
- $\{0\}$ or $\{1\}$
- $S_1 \cup S_2$
- $S_1 S_2$
- S^*

- $\{\varepsilon\} \Rightarrow 1$
- $\{0\}$ or $\{1\}$ $\Rightarrow x$
- $S_1 \cup S_2 \Rightarrow \Phi_{S_1}(x) + \Phi_{S_2}(x)$ (sum lemma)
- $S_1 S_2 \Rightarrow \Phi_{S_1}(x)\,\Phi_{S_2}(x)$ (product lemma)
- $S^* \Rightarrow \dfrac{1}{1 - \Phi_S(x)}$ (string lemma)

unambigous $\{0,1\}^*$ $\rightleftharpoons$ $\{$binary strings$\}$ by α

000 1 001 | (000, 1, 001)

ambigous $\{0,1\}^*$ $\not\rightleftharpoons$ $\{$binary strings$\}$ by α

Block Decomposition

Block of a string is a maximal substring of 1's or 0's

$\{0\}^*\{1\{1\}^*0\{0\}^*\}^*\{1\}^*$ break <u>after</u> each block of 0's

$\{1\}^*\{0\{0\}^*1\{1\}^*\}^*\{0\}^*$ break <u>after</u> each block of 1's

Start end

Prefix Decomposition

end

prefix A^*B

$(\{0\}^*\{1\})^*\{0\}^*$ break <u>after</u> each occurrence of 1

$(\{1\}^*\{0\})^*\{1\}^*$ break <u>after</u> each occurrence of 0

postfix BA^*

$\{0\}^*(\{1\}\{0\}^*)^*$ break <u>before</u> each occurrence of 1

$\{1\}^*(\{0\}\{1\}^*)^*$ break <u>before</u> each occurrence of 0

Start

Break after the 1^{st} occurrence

We can use these decompositions to prove unambiguity for other sets.

<u>ie</u>: $\{1\}^*(\{0\}\{1,11,111\})^*$: b.s's where every 0 is followed by 1, 2 or 3 1's

$\{1\}^*(\{0\}\{1\}^*)^*$

Subset

$\{1\}^*(\{0\}\{1,11,111\})^*$

Restriction on the 0-decomposition
Unambigous because it's a subset of an ambigous decomposition

Recursive Decomposition

decomposition of S in terms of itself S

$S = \{\varepsilon\} \cup S\{0,1\} = \{\text{all strings}\}$

unambigous each side of the equation produces each string at most once.

1) Excluded Substring

Ex:

Find g.s for the set S of strings not containing 1100 as a substring

$S = \{0\}^* \{\{111\}^*\{0\}, \{1000\}^*, \{10\}\}^* \{1\}^*$

$$\frac{1}{(1-x)(1-x+x^2-3x^3)} = \Phi_S(x)$$

w` $w(\sigma) = \ell(\sigma)$

a) no overlap

Ex: Find # Strings without 11100 as a substring

$T = \{$ strings with exactly 1 occurence of 11100, in the final 5 bits $\}$

1) $S \cup T = \varepsilon \cup S(0 \cup 1) = \{$all binary strings$\}$

2) $T = S11100$

unambigous

$$1 + 2x\,\Phi_S(x) = \Phi_S(x) + \Phi_T(x)$$

$$\Phi_T(x) = x^5\,\Phi_S(x)$$

$$\Rightarrow \quad \Phi_S(x) = \frac{1}{1-2x+x^5}$$

a) overlap

$S = \{$strings without 0110110$\}$

$T_1 = \{$ strings w` exactly 1 0110110 in the last 7 bits $\}$ ... | 0110110

$T_2 = \{$ strings w` exactly 2 0110110 in the last 13 bits $\}$ 0110110110110

$T_3 = \{$ strings w` exactly 2 0110110 in the last 10 bits $\}$ 0110110110

$T_1\{110110\} = T_2$

$T_1\{110\} = T_3$

$T_1 \cup T_2 \cup T_3 = S\{0110110\}$

$\therefore\ T_1 \cup T_1\{110110\} \cup T_1\{110\} = T_1\{\varepsilon, 110110, 110\} = S\{0110110\}$

Appending 0 or 1 to S: $S\{0,1\}$

<u>case 1:</u> creates a string in S

<u>case 2:</u> Creates a string in T_1

$$S \cup T_1 = \{\varepsilon\} \cup S\{0,1\}$$

<u>Prove Unambiguity:</u> (in order to use lemmas)

- $S \cup T_1$ is unambigous since $T_1 \cap S = \emptyset$
- $\{\varepsilon\} \cup S\{0,1\}$ is unambigous since we decompose before the last bit
- $T_1\{\varepsilon, 110110, 110\}$ ambigous. decomposes before 0110110 is seen
- $S\{0110110\}$ clearly Unambigous

So by the sum & product lemmas:

① $\Phi_S(x) + \Phi_{T_1}(x) = 1 + 2x\,\Phi_S(x)$

② $\Phi_S(x)\,x^7 = \Phi_{T_1}(x)(1 + x^6 + x^3)$

$$\boxed{\Phi_S(x) = \frac{1+x^3+x^6}{1-2x+x^3-2x^4+x^6-x^7}}$$

2) Nesting

$B = \{\varepsilon, 01, 0011, 000111, \dots\}$

$B = \{\varepsilon\} \cup \{0\}B\{1\}$

$B(x) = 1 + x^2 B(x)$

$$B(x) = \frac{1}{1-x^2}$$

Partial Fractions

when it's hard simplify a gen-series

no repeated factors

$$\frac{P(x)}{(1-ax)(1-bx)\cdots(1-nx)} = \frac{A}{1-ax} + \frac{B}{1-bx} + \cdots + \frac{N}{1-nx}$$

To find $A, B, C, \dots, N$: $P(x) = A(1-ax) + B(1-bx) + \cdots + N(1-nx)$

repeated factors

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(1-ax)^{d_a}(1-bx)^{d_b}\cdots(1-nx)^{d_n}} = \left[\frac{A}{(1-ax)^1} + \cdots + \frac{A_n}{(1-ax)^{d_a}}\right] + \left[\frac{N_1}{(1-bx)^1} + \cdots + \frac{N_n}{(1-bx)^{d_b}}\right] + \cdots$$

Recurrence Relations

Homogeneous Linear Reccurrence Relation

$a = (a_0, a_1, a_2, \dots)$

the sequence 'a' satisfies a homogeneous linear reccurence relation :

$$a_n + q_1 a_{n-1} + q_2 a_{n-2} + \cdots + q_k a_{n-k} = 0 \qquad \forall n > k$$

initial conditions

\# initial conditions

\# terms

How to | Find a Formula for a Homogeneous Reccurence Relation

1 | Theorem

Main Theorem

* must be homogeneous

$$a_n + q_1 a_{n-1} + q_2 a_{n-2} + \cdots + q_k a_{n-k} = 0 \quad \forall n \geq k$$

rational

$$Q(x) = 1 + q_1 x + q_2 x^2 + \cdots + q_k x^k$$

$$P(x) = \deg(Q(x)) - 1$$

$$A(x) = \frac{P(x)}{Q(x)}$$

determine w' initial conditions

Formula

$$Q(x) = (1-\lambda_1 x)^{\alpha_1}(1-\lambda_2 x)^{\alpha_2}\cdots(1-\lambda_s x)^{\alpha_s}$$

$\deg = \alpha_1 - 1$ $\quad \deg = \alpha_2 - 1$ $\quad \deg = \alpha_s - 1$

$$a_n = P_1(n)\lambda_1^n + P_2(n)\lambda_2^n + \cdots + P_s(n)\lambda_s^n \quad \forall n \geq 0$$

determine w' initial conditions

① Recurrence $\quad a_n - a_{n-1} - 2a_{n-2} = 0, \ n \geq 2$

② Rational Expression $\quad A(x) = \dfrac{C_1 + C_2 x}{1 - x - 2x^2}$ $\quad \deg(P(x)) < \deg(\text{denom})$

③ PF $\quad A(x) = \dfrac{C_1 + C_2 x}{(1-2x)(1+x)} = \dfrac{D_1}{(1-2x)} + \dfrac{D_2}{(1+x)} = D_1 \sum 2^n x^n + D_2 \sum (-1)^n x^n$

④ Formula $\quad a_n = [x^n]A(x) = D_1 \cdot 2^n + D_2 \cdot (-1)^n$

⑤ Initial Conditions $\quad a_n = 3 \cdot 2^n - (-1)^n, \quad n \geq 2$

2 | Long Way

Homogeneous

sequence $f = (f_0, f_1, f_2, f_3, \ldots)$ where:

- $f_0 = 1$
- $f_1 = 1$
- $f_n = f_{n-1} + f_{n-2} \quad \forall n \geq 2$

① let $F(x) = \sum_{n=0}^{\infty} f_n x^n$

$$= f_0 + f_1 x + \sum_{n=2}^{\infty} f_n x^n$$

$$= 1 + x + \sum_{n=2}^{\infty} (f_{n-1} + f_{n-2}) x^n$$

$$= 1 + x + \sum_{n=2}^{\infty} f_{n-1} x^n + \sum_{n=2}^{\infty} f_{n-2} x^n$$

$$= 1 + x + \sum_{n=1}^{\infty} f_n x^{n+1} + \sum_{n=0}^{\infty} f_n x^{n+2}$$

$$= 1 + x + x\sum_{n=1}^{\infty} f_n x^n + x^2 \sum_{n=0}^{\infty} f_n x^n$$

$$F(x) = 1 + x + x(F(x) - f_0) + x^2 F(x)$$

$$F(x) = \frac{1}{1 - x - x^2}$$

② Partial Fractions:

$1 - x - x^2 = (1-\alpha x)(1-\beta x)$ $\quad$ (where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$)

$\dfrac{1}{1-x-x^2} = \dfrac{A}{1-\alpha x} + \dfrac{B}{1-\beta x}$ $\quad$ (where $A = \frac{5+\sqrt{5}}{10}$, $B = \frac{5-\sqrt{5}}{10}$)

② Find Formula

$$F(x) \overset{GS}{=} A\sum_{n=0}^{\infty} \alpha^n x^n + B\sum_{n=0}^{\infty} \beta^n x^n$$

$$= \sum_{n=0}^{\infty} (A\alpha^n + B\beta^n) x^n$$

$$[x^n]F(x) = A\alpha^n + B\beta^n$$

$$= \frac{5+\sqrt{5}}{10}\left[\frac{1+\sqrt{5}}{2}\right]^n + \frac{5-\sqrt{5}}{10}\left[\frac{1-\sqrt{5}}{2}\right]^n$$

* non-homogeneous L.R:

$b_n - 4b_{n-1} + 5b_{n-2} - 2b_{n-3} = 24(-1)^n$ $\quad$ [$b_0 = -1$, $b_1 = -3$, $b_2 = 2$]

Let $B(x) = \sum_{n \geq 0} b_n x^n$

1) $B(x)$ as a rational FPS

$$\sum_{n\geq 3} b_n x^n - 4\sum_{n\geq 3} b_{n-1} x^n + 5\sum_{n\geq 3} b_{n-2} x^n - 2\sum_{n\geq 3} b_{n-3} x^n = 24\sum_{n\geq 3} (-1)^n x^n$$

$\vdots$

$$(1 - 4x + 5x^2 - 2x^3)B(x) - 9x^2 - x + 1 = 24\left(\frac{1}{1+x} - (1 - x + x^2)\right)$$

$$\therefore \quad B(x) = \frac{P(x)}{(1+x)(1-4x+5x^2-2x^3)}$$

$$Q(x) = (1+x)(1-x)^2(1-2x)$$

Use theorem From here

2) coefficients

$$b_n = [x^n]B(x) = A(-1)^n + (B + Cn)1^n + D2^n$$

3) Solve for A, B, C, D w' initial conditions

$A = 2$, $B = 3$, $C = -2$, $D = -1$

$$b_n = 2(-1)^n + 3n - 2 - 2^n \quad \forall n \geq 0$$

How to	Find a Reccurence Relation from a rational

① We have $G(x) = \dfrac{P(x)}{Q(x)}$

$\therefore\ g_n = [x^n]\,G(x), \quad \forall n \geq 0$

② Solve $[x^n]\,G(x)\,Q(x) = [x^n]P(x)$

$\hookrightarrow g_n Q(x) = [x^n]P(x)$

③ Determine minimum n for g_n

④ Initial Conditions

⑤ Write it out

$$G(x) = \frac{1}{1-5x+8x^2-4x^3}$$

$$g_n = [x^n]\,\frac{1}{1-5x+8x^2-4x^3} \quad \forall n \geq 0 .$$

$$G(x)\,(1-5x+8x^2-4x^3) = 1$$

$$[x^n]\left[G(x)\,(1-5x+8x^2-4x^3)\right] = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n \geq 1 \end{cases}$$

$$(*)\quad g_n - 5g_{n-1} + 8g_{n-2} - 4g_{n-3} = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n \geq 1 \end{cases}$$

$$g_n - 5g_{n-1} + 8g_{n-2} - 4g_{n-3} = 0 \quad \forall n \geq 3$$

With initial conditions, g_0, g_1, g_2 given by:

$n=0$: $g_0 = 1$ by $(*)$ with $n=0$

$n=1$: $g_1 - 5g_0 = 0$ by $(*)$ w/ $n=1$

$n=2$: $g_2 - 5g_1 + 8g_0 = 0$ by $(*)$ w/ $n=2$

$g_0 = 1,\ g_1 = 5,\ g_2 = 17$

$\therefore$ (g_n) Satisfies the Linear Reccurence:

- $g_0 = 1$
- $g_1 = 5$
- $g_2 = 17$
- $g_n - 5g_{n-1} + 8g_{n-2} - 4g_{n-3} = 0 \quad \forall n \geq 3$

Beyond Rational Gen-series

Irrational Generating Series

An FPS that can't be expressed as $\frac{P(x)}{Q(x)}$ (2 polynomials)

Weight Preserving Bijection

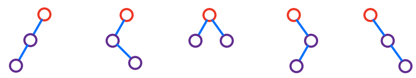
$f: A \to B$ is weight-preserving bijection if:

$$\omega_B(f(a)) = \omega_A(a) \quad \forall a \in A$$

$$\boxed{\text{weight-preserving bijection } A \rightleftharpoons B \implies \Phi_A^{\omega_A}(x) = \Phi_B^{\omega_B}(x)}$$

1 Binary Rooted Trees

Ex: Binary rooted trees with 3-nodes



How many binary trees are there with n nodes ?
Let $T = \{\text{binary rooted trees}\}$, $\omega(t) = \#$ nodes in t $\Big\}\ \Phi_T(x)$

① Construct a weight-preserving bijection f:

$$f : T \longrightarrow \{\varepsilon\} \cup (T \times \{0\} \times T)$$

$\{0\}$ ↳ b.r.t with 1 node

$\longrightarrow (L, 0, R)$

$\omega(t) = \#$nodes in T

$$\omega(L, 0, R) = \omega(t_1) + 1 + \omega(t_2)$$

$$f(t) = \begin{cases} \varepsilon & \text{if } t = \varepsilon \\ (L(t) + 1 + R(t)) & \text{if } t \neq \varepsilon \end{cases}$$

$L(t)$: left tree, $R(t)$: right tree

f is a bijection ✓

f is weight preserving ✓: $\omega(t) = \omega(f(t)) = \omega(L(t)) + 1 + \omega(R(t))$

② Generating Series:

$$\Phi_T(x) = \Phi_{\{\varepsilon\}\cup(T\times\{0\}\times T)}(x)$$

Sum Lemma
Product lemma

$$= 1 + \Phi_T(x) \cdot x \cdot \Phi_T(x)$$

$$= 1 - \Phi_T(x) + x\,\Phi_T^2(x)$$

$$= 4x\left(1 - \Phi_T(x) + x\,\Phi_T^2(x)\right)$$

$$= (2x\Phi_T(x) - 1)^2 + (4x - 1)$$

$$4x - 1 = (2x\Phi_T(x) - 1)^2$$

2	Well-Formed-Parenthesizations

how many W.F.P's are there with n-pairs of parentheses ?

Let $P = \{W.F.P\text{'s}\}$
$\omega(p) = \#$ of pairs
$\Phi_p(x) = ?$

① Weight Preserving bijection :

$\forall \alpha \in P:\ \alpha = \varepsilon$ or $\alpha = (\beta)\gamma$ where $\beta, \gamma \in P$

$f : P \longrightarrow \{\varepsilon\} \cup (P \times \{()\} \times P)$

$(\beta)\gamma \longrightarrow (\beta, (), \gamma)$

$\omega = \#$pairs $\qquad \omega = \begin{cases} 0 \\ \omega(\beta) + 1 + \omega(\gamma) \end{cases}$

$$f(\alpha) = \begin{cases} \varepsilon & , \alpha = \varepsilon \\ (\beta, (), \gamma) & , \text{otherwise} \end{cases}$$

Check that f is weight preserving :

1) f is a bijection $P \rightleftharpoons (\beta, \{()\}, \gamma)$

2) $\forall p \in P$, $\omega_B(f(p)) = \omega_p(p)$

$\omega_p(p) = \#$pairs that p contains

$= \omega_p(\beta) + \omega_p(\gamma) + 1$ or 0

$= \omega_B(f(p))$

$\therefore$ f is weight preserving

② generating Series

$\Phi_p(x) = \Phi_{\{\varepsilon\} \cup (P \times \{()\} \times P)}(x)$

$= \Phi_\varepsilon(x) + \Phi_p^2(x) \cdot \Phi_{()}(x)$

$\Phi_p(x) = 0 + x\,\Phi_p^2(x)$

$$\boxed{\Phi_p(x) = x\,\Phi_p^2(x)}$$