The midterm will cover up to:

- Soundness theorem
- consistent sets of props
- maximally consistent

but not including:

- completeness theorem
- every consistent set is satisfiable

Week 1

Syntax = Format

Semantics = meaning

Inductive Defⁿ (of a set)

3 parts

1. Domain Set — X
2. Core Set — A
3. Set of Operations — P

larger set

core set = minimal subset

$A \subseteq X$

$\left.\begin{matrix} X \\ A \\ P \end{matrix}\right\} \quad I(X, A, P)$

$P : X \to X$

maps members of X to members of X

means:

- minimal subset of X
- contains A
- closed under the operations in P

The minimalist requirement:

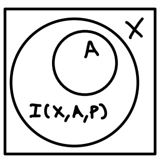
$\forall$ set W that satisfies:

1) $A \subseteq W$

2) W closed under P's operations

Then:

$I(X, A, P) \subseteq W$



$A \subseteq I(X, A, P) \subseteq X$

Note: $I(X, A, P)$ is unique (given any X, A, P)

Ex: Set of multiples of 3

$\left.\begin{array}{l} X = \mathbb{N} \\ A = \{0, 3\} \\ P = \{f\}, \text{ where } f(x,y) = x + y \end{array}\right\} \quad I(X, A, P) = \{\text{all (+) multiples of } 3\}$

ie: $18 = f(f(\overset{6}{f(3,3)}, 3), f(\overset{3}{f(3,3)}, 3))$

Set Theory

Let C be a <u>collection</u> of sets:

$\boxed{\cap C} = \{x : \forall W \in C, x \in W\}$

$\boxed{\cup C} = \{x : \text{for some } W \in C, x \in W\}$

What is the minimal set satisfying:

1. $A \subseteq W$
2. W is closed under P

We can define that minimum:

Let $C = \{W : W \text{ satisfies } ① \,\&\, ②\}$

Let $D = \cap C_{(X,A,P)}$

Then D satisfies ① & ② & is minimal among all sets

Well Formed Formulas

$X = \{\text{all propositions}\}$ = all sequences of the symbols:

- $($, $)$, $\neg$, $\wedge$, $\vee$ $\rightarrow$
- {lower case letters with / without indexes} A

$$P = \left\{ \frac{\alpha, \beta}{(\alpha \wedge \beta)}, \frac{\alpha, \beta}{(\alpha \wedge \beta)}, \frac{\alpha, \beta}{(\alpha \rightarrow \beta)}, \frac{\alpha}{(\neg\, \alpha)} \right\}$$

brackets always applied

$I(X, A, P) = \{\text{W.F.F's}\}$

Week 2

Generation Sequence

Sequence of propositions $\alpha_1, \alpha_2, \dots, \alpha_n \in X$

each α_i either:

1. a member of A
2. the result of applying $f \in P$ to α_i's of smaller indices

Ex:

$((p_1 \to p_2) \to (\neg p_3))$

must be earlier

$p_1, p_2, (p_1 \to p_2), p_3, (\neg p_3), ((p_1 \to p_2) \to (\neg p_3))$

$x \in X$ has a generation sequence w.r.t A, P $\iff$ $x \in I(X, A, P)$

Ex:

x_1 $(p_1 \to (p_2 \to p_3))$ $\quad p_1, p_2, p_3, (p_2 \to p_3), (p_1 \to (p_2 \to p_3))$ legal

x_2 $(\neg(p_2 \wedge p_7))$ $\quad p_2, p_7, (p_2 \wedge p_7), \neg(p_2 \wedge p_7)$ legal

$(p_1 \to (p_2 \to p_3)) \to (\neg(p_2 \wedge p_7))$

$x_1 \to x_2 \longrightarrow$ legal

legal $\quad f(x_1, x_2) \quad$ legal

show that some element is not in $I(X, A, P)$

Prove a set has a property

General Principle:

Given: members in X have a: property

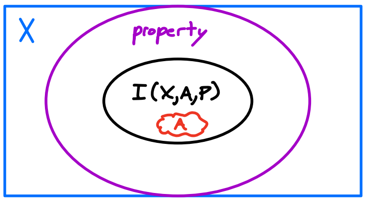
If:

① all members of A satisfy the property

② property preserved with any $f \in P$

Then:

every member of $I(X, A, P)$ satisfy the property



We use...

Structural Induction

To prove that some property holds for all members of $I(X, A, P)$

① Base Case:

Show that the property holds for any $a \in A$

② Inductive Step:

Inductive-Hypothesis: Assume the Prop holds for some $x, y \in X$:

Show that property preserved with any $f \in P$

Ex: For proposition α : $\#_{)}(\alpha) = \#_{(}(\alpha)$

<u>Base Cases</u>: $\forall a \in A$, $\#_{)}(a) = \#_{(}(a) = 0$

<u>Inductive Step</u>: assume α, β both satisfy the property

- $(\neg \alpha)$: $\#_{(}(\neg\alpha) = \#_{)}(\neg\alpha)$
- $(\alpha \to \beta)$: $\#_{(}(\alpha \to \beta) = \#_{)}(\alpha \to \beta)$
- $(\alpha \vee \beta)$: $\#_{(}(\alpha \vee \beta) = \#_{)}(\alpha \vee \beta)$
- $(\alpha \wedge \beta)$: $\#_{(}(\alpha \wedge \beta) = \#_{)}(\alpha \wedge \beta)$

by inductive hypothesis $\#_{(}(\alpha) = \#_{)}(\alpha)$ for any α

Proper Initial Segment

Segment of proposition from the start up to anywhere but the end

$$\underbrace{\underbrace{(\ p \to}\ q \vee p_1)\ \vee}_{\text{proper initial segment}})$$

Ex: every proper initial segment has more left brackets then right brackets

Base Case: prop 4 holds for all prop variables

Inductive Step: Assume α, β Satisfying the property

① $\dfrac{\alpha}{(\neg\alpha)}$

Let γ be a proper initial segment of α

Case		# Left	# right
1	(	1	0
2	($\neg$	1	0
3	($\neg$ γ	$1 + \#_(\gamma$	$\#_) \gamma$
4	($\neg$ α	$1 + \#_(\alpha$	$\#_) \alpha$

by I.H, γ has more L.B than R.B

if α is a proposition

Then $\#_((\alpha) = \#_)(\alpha)$

Since we added a Left-B

$\#_(> \#_)$

② $\dfrac{\alpha, \beta}{\alpha \vee \beta}$

Case	P.I.S
1	(
2	(γ
3	(α
4	(α $\vee$
5	(α $\vee$ γ
6	(α $\vee$ β

γ = initial seg of α

γ = initial seg of β

doesn't matter

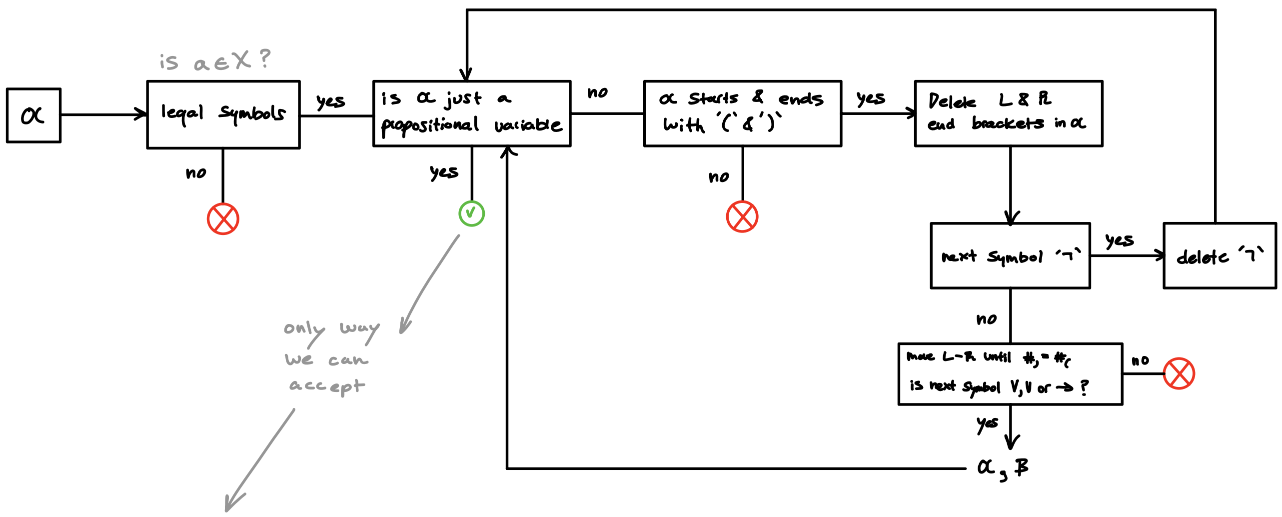
we still need to prove for other operations, but the proofs will be similair.

proof Complete

Parsing Algorithm

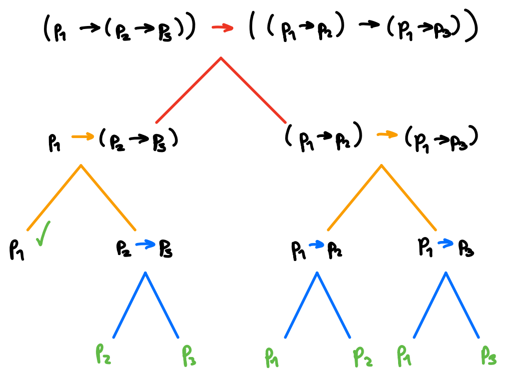
Input : proposition , α

Output : Legal or Not



Parse Tree

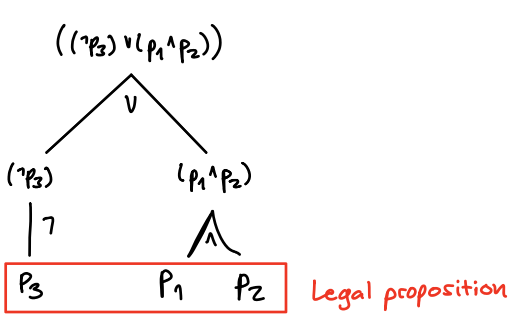
Only accept proposition when the leaves of the tree are prop-variables.



ex: $((\neg p_3) \vee (p_1 \wedge p_2)) \xrightarrow{\text{generating Seq}} p_3,\ (\neg p_3),\ p_2,\ p_1,\ (p_1 \wedge p_2),\ ((\neg p_3) \vee (p_1 \wedge p_2))$

(not unique)

parse tree (unique)



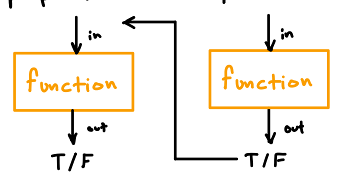
Legal proposition

Week 3

Semantics

<u>Semantics</u>: (of prop-logic)

propositions prop-variables $\longrightarrow$ Statements (either true or false)



Truth Assignment

A function $S: p \rightarrow \{T, F\}$, $S(p) = \begin{cases} T \\ F \end{cases}$

proposition variable

truth value

Note: $\bar{S}(p) = S(p)$

prop variable

Truth Evaluation

Extends S to a statement proposition α

$$\bar{S}: \alpha \longrightarrow \{T, F\} \quad , \quad \bar{S}(\alpha) = \begin{cases} T \\ F \end{cases}$$

statement

Ex:

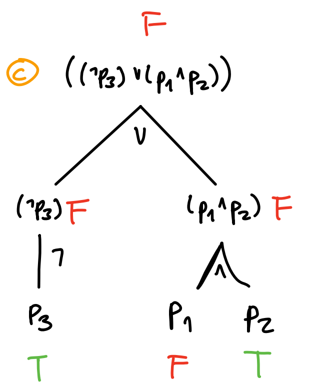
$$\left.\begin{array}{l} S: p = T,\ q = F \\ \alpha = ((\neg p) \rightarrow q) \end{array}\right\} \quad \bar{S}(\alpha) = T$$

Ways to Find $\bar{s}(\alpha)$

s: $p_1 = T$
$p_2 = F$
$p_3 = T$

$\alpha = ((\neg p_3) \vee (p_1 \wedge p_2))$

① Parse Tree :



Start from the leaves and go up

② Truth Table

p_1	p_2	p_3	$(\neg p_3)$	$(p_1 \wedge p_2)$	$(\neg p_3) \vee (p_1 \wedge p_2)$
T	T	T	F	T	T
T	T	F	T	T	T
T	F	T	F	F	F
F	T	T	F	F	F
T	F	F	T	F	T
F	T	F	T	F	T
F	F	T	F	F	F
F	F	F	T	F	F

$\underbrace{p_1 \quad p_2 \quad p_3}_{s(p_i)}$ $\qquad$ $\downarrow \bar{s}(\alpha)$

For any truth assignment S & proposition α, we only want one possible truth evaluation $\bar{S}(\alpha)$

Solution: given α, there is no ambiguity where it came from.
guarantees that extending every S to $\bar{S}$ is well defined.

Unique Readability Theorem

For every valid prop α, only 1 of the following holds:

1. α = prop variable
2. $\alpha = (\neg \beta)$
3. $\alpha = (\beta \vee \gamma)$
4. $\alpha = (\beta \wedge \gamma)$
5. $\alpha = (\beta \rightarrow \gamma)$

Tautology

$[\bar{S}(\alpha) = T \quad \forall s] \Rightarrow \alpha$ is a tautology

Show that α is a tautology:

$\alpha = p \vee (\neg p)$

① Truth table

p	$(\neg p)$	$p \vee (\neg p)$
T	F	T
F	T	T

every S — α = tautology

② for $\alpha \rightarrow \beta$

only False when $\alpha = T$ & $\beta = F$

Just show that that case can never happen.

Ex:

$((\neg(\alpha \wedge \beta)) \rightarrow ((\neg \alpha) \vee (\neg p)))$

Let $\neg(\alpha \wedge \beta) = F$	$\therefore ((\neg \alpha) \vee (\neg p)) = T$
$\therefore \alpha \wedge \beta = T$	$\therefore$ tautology
$\therefore \alpha = T, \beta = T$	

$\boxed{\text{Contradiction}}$ $\left[\bar{S}(\alpha) = F \quad \forall s\right] \Rightarrow \alpha$ is a contradiction

$\boxed{\text{Satisfiable}}$ $\left[\exists s \text{ s.t } \bar{S}(\alpha) = T\right] \Rightarrow \alpha$ is satisfiable

<u>Semantic Relations Among Propositions:</u>

$\boxed{\text{Logical Implication}}$ $\left[\forall s : \bar{S}(\alpha) = T \Rightarrow \bar{S}(\beta) = T\right] \Rightarrow \alpha \models \beta$

in other words:

$\alpha \models \beta \iff (\alpha \rightarrow \beta)$ is a tautology

$\boxed{\text{Logical Equivalence}}$ $\left[\forall s : \bar{S}(\alpha) = \bar{S}(\beta)\right] \Rightarrow \alpha \equiv \beta$

in other words:

$\alpha \equiv \beta \iff \alpha \models \beta \wedge \beta \models \alpha$

$\boxed{\text{Syntactic Equivalence}}$

$\alpha = p \wedge q$ different propositions <u>Syntactically</u> $\alpha = \beta$: Syntactically Equivalent

$\beta = q \wedge p$ Logically equivalent $\alpha \equiv \beta$: logical equivalence

<u>Connections:</u>

α is a tautology $\iff$ $(\neg\alpha)$ is a contradiction

α is satisfiable $\iff$ α is not a contradiction

2 tautology's are logically equivalent

2 contradiction's are logically equivalent

α = Contradiction $\Rightarrow$ $\alpha \models \beta \quad \forall \beta$

β = tautology $\Rightarrow$ $\alpha \models \beta \quad \forall \alpha$

Semantics over Sets of Formulas

Let $\Sigma = \{\text{propositions}\}$

$\exists s$ s.t $\forall \alpha \in \Sigma$, $\bar{s}(\alpha) = T \implies \Sigma$ is satisfiable

$\forall s, [\forall \beta \in \Sigma, s(\beta) = T \implies \bar{s}(\alpha) = T] \implies \Sigma \models \alpha$

Σ: set of formulas, assumptions

α: single formula, conclusion

Ex:

① $\Sigma = \{p_1, p_2, p_3, \dots, p_n\}, n \in \mathbb{N}$

Σ is satisfiable: let $s(p_i) = T \ \forall i$

② $\Sigma = \{\neg p_1, \neg p_2, \dots, \neg p_n\}$

Σ is satisfiable: let $s(p_i) = F \ \forall i$

③ $\Sigma = \{p_i \to (\neg p_{i+1}) : i \in \mathbb{N}\}$

Σ is satisfiable: let $s(p_i) = F \ \forall i$

④ $\Sigma = \{p, \neg p\}$

Σ is not satisfiable

Ex:

Show $\underbrace{\{p, q\}}_{\Sigma} \models (p \wedge q)$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

→ check since s satisfies Σ

not applicable

Week 4

Σ is satisfiable $\iff$ $\exists\, \alpha$ such that $\Sigma \nvDash \alpha$

$\Sigma \nvDash \alpha$ only if Σ is satisfiable & α is not. So then, Σ

Σ is unsatisfiable $\implies$ $\forall \alpha : \Sigma \vDash \alpha$ → If $\Sigma \vDash \alpha$, then Σ can be satisfiable or not

$\Sigma \nvDash \alpha$ can be shown with a single s

$\Sigma \vDash \alpha$ is a statement about all s's

$\Sigma = \phi \implies \Sigma \nvDash \alpha, \ \forall \alpha$

every s vacously satisfies Σ (since Σ has no members)

in particular, an s for which $s(p) = F$.

$\Sigma = \{\text{all plopositions}\} \implies \Sigma \vDash \alpha, \ \forall \alpha$

Σ is never satisfiable (since if $p \in \Sigma$, $(\neg p) \in \Sigma$)

Complete

Σ is complete $\iff \forall \alpha : [\Sigma \models \alpha] \vee [\Sigma \models \neg\alpha]$

when Σ is satisfied, either α is always satisfied or $\neg\alpha$ is always satisfied

Ex:

$\Sigma = \{(p \vee q)\}$

$\{(p \vee q)\} \not\models p$ — p: F, q: T; p: F

$\{(p \vee q)\} \not\models \neg p$ — p: T, q: F; $\neg p$: F

Σ is not complete

p	q	¬p	¬q	(p ∨ q)
T	T	F	F	T
T	F	F	T	T
F	T	T	T	T
F	F	T	F	F

$\uparrow$ Σ

Proven Claims:

1) Σ is satisfiable & complete $\iff$ there's only 1 S that satisfies Σ

2) Σ is not satisfiable $\implies$ Σ is complete

Connectives

- $\vee, \wedge, \rightarrow, \neg, \ldots$

  Syntax: generates new propositions
  - operations that take statements as input and output a new statement

  Semantics: truth functions : $\alpha \rightarrow \beta \implies f(\alpha, \beta) = T$ (inputs: α, β; output: T)
  - can be described by a truth table

Adequate

A {connectives} is adequate if you can express any truth table with only these connectives.

if set A is adequate then set A' where $A \subseteq A'$ is adequate

Truth Table $f : \{T, F\}^n \longrightarrow \{T, F\}$

Ex:

	a	b	c	f(a,b,c)
1	T	T	T	T
2	T	T	F	T
3	T	F	T	T
4	F	T	T	T
5	T	F	F	F
6	F	T	F	F
7	F	F	T	F
8	F	F	F	F

$f(a,b,c) \longrightarrow$ Spits out majority of input values

$$\left.\begin{aligned} \alpha_1 &= a \wedge b \wedge c \\ \alpha_2 &= a \wedge b \wedge (\neg c) \\ \alpha_3 &= a \wedge (\neg b) \wedge c \\ \alpha_4 &= (\neg a) \wedge b \wedge c \end{aligned}\right\} \alpha = \alpha_1 \vee \alpha_2 \vee \alpha_3 \vee \alpha_4$$

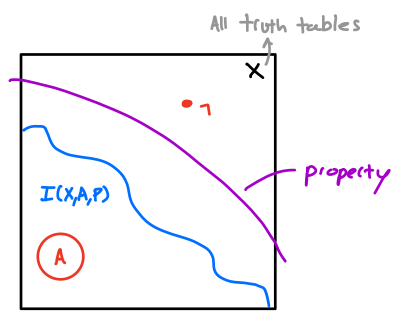
Ex: is $\{\vee, \neg\}$ adequate?

Since $\{\wedge, \vee, \neg\}$ is adequate :

& $A \wedge B = (\neg((\neg A) \vee (\neg B)))$

Then $\{\vee, \neg\}$ is adequate

Ex: is $\{\vee, \wedge\}$ adequate?



__Disproof__:

to show that $\neg$ cannot be expressed, consider the property :

when all inputs are T, the output is also T

all truth tables, prop variables, $\vee, \wedge$

proof by structural induction over $I(X, A, P)$ $\rightarrow$ truth tables made from $\{\vee, \wedge\}$

__Base Case__:

prop variables satisfy the property.

__Inductive Step__:

Assume α, β both satisfy the property

① $(\alpha \wedge \beta)$

② $(\alpha \vee \beta)$

A_1	A_2	$\cdots$	A_n	α	β	$\alpha \wedge \beta$	$\alpha \vee \beta$
T	T		T	T	T	T	T

$\underbrace{\alpha \quad \beta}_{\text{inductive hypothesis}}$ $\underbrace{\alpha \wedge \beta \quad \alpha \vee \beta}_{\text{by truth tables of } \vee, \wedge}$

$\therefore$ holds for all $I(X, A, P)$:

Proof of property complete

If α has the property, $\neg\alpha$ doesn't have the property. '$\neg$' does not have this property.
Hence, $\{\vee, \wedge\}$ is not adequate
Disproof Complete

Formal Proofs (SYNTAX)

3 properties we require from a proof:

1) Soundness: if some statement has a proof, it is true
2) Completeness: if a statement is true, it has a proof
3) there is a clear way of checking what is a valid proof

Proof Systems from Propositional Logic:

We will use the term "**Formal Proof**" for a proof in our system as opposed to proof for statements about our system

Our proof system consists of 'axioms' and 'deduction rule'

Axioms

every proposition of 1 of the following forms:

1) AX I $\quad \alpha \to (\beta \to \alpha)$
2) AX II $\quad ((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
3) AX III $\quad (((\neg\alpha) \to (\neg\beta)) \to (\beta \to \alpha))$

Deduction Rule

$\alpha \wedge [\alpha \to \beta] \implies \beta$

Week 5

The set of all Formal Theorems: I(propositions, Axioms, Modus Ponens)
X A P

Formal Proof System uses only $\{\to, \neg\}$ (adequate set)

Formal Proofs

Prove that α is a formal Theorem:
prove that $\alpha \in I(X, A, P)$
provide a generation sequence for the set of formal theorems

A formal proof of proposition α is a sequence of propositions: $\beta_1 \beta_2 \cdots \beta_n$

Such that each β_i is either:

1) an axiom
2) the result of applying M.P to previous β_j's
3) $\beta_n = \alpha$

Ex: For every proposition α, $(\alpha \to \alpha)$ is a formal theorem

proof: (by generation sequence)

① $\alpha \to ((\alpha\to\alpha)\to\alpha)$ — AX I: $\alpha \to (\beta \to \alpha)$ $\beta = (\alpha \to \alpha)$

② $((\alpha \to ((\alpha\to\alpha)\to\alpha))) \to ((\alpha \to (\alpha\to\alpha)) \to (\alpha\to\alpha))$ — AX II: $((\alpha \to (\beta \to \gamma))) \to ((\alpha\to\beta)\to(\alpha\to\gamma))$ $\gamma = \alpha$

③ $((\alpha \to (\alpha\to\alpha)) \to (\alpha\to\alpha))$ — M.P

④ $(\alpha \to (\alpha\to\alpha))$ — AX I: $\alpha \to (\beta\to\alpha)$ $\beta = \alpha$

⑤ $(\alpha\to\alpha)$ — M.P

Ex: $\forall\ \alpha, \beta$ (propositions) $((\neg\alpha) \to (\alpha\to\beta))$ is a formal theorem.

Proof: by formal proof

1) $(\neg\beta \to \neg\alpha) \to (\alpha\to\beta)$ — AX III

2) $[(\neg\beta \to \neg\alpha) \to (\alpha\to\beta)] \to [(\neg\alpha) \to ((\neg\beta\to\neg\alpha) \to (\alpha\to\beta))]$ — AX I

M.P

3) $(\neg\alpha) \to ((\neg\beta\to\neg\alpha)\to(\alpha\to\beta))$

4) $[(\neg\alpha) \to ((\neg\beta\to\neg\alpha)\to(\alpha\to\beta))] \to [(\neg\alpha \to ((\neg\beta\to\neg\alpha)) \to (\neg\alpha \to (\alpha\to\beta))]$ — AX II

M.P

5) $[(\neg\alpha \to ((\neg\beta\to\neg\alpha)) \to (\neg\alpha \to (\alpha\to\beta))]$

6) $(\neg\alpha \to (\neg\beta\to\neg\alpha))$ — AX I

M.P

7) $(\neg\alpha \to (\alpha\to\beta))$

Formal Proofs From Assumptions

Γ = set of propositions (a.k.a assumptions)

Set of Formal Consequences of Γ : $I(\overset{X}{\text{propositions}}, \overset{A}{\text{Axioms} \cup \Gamma}, \overset{P}{M.P})$

A formal proof from assumptions Γ is a gen-sequence for this set : $\beta_1, \beta_2, \dots, \beta_n$

Such that each β_i is either :

1) an axiom or assumption $\in \Gamma$
2) the result of applying M.P to previous β_j's
3) $\beta_n = \alpha$

$\boxed{\vdash}$ $\vdash$ = "Formally Proves"

$\vdash \alpha$: "α is a formal theorem" : proof w/ only Axioms

$\Gamma \vdash \alpha$: "α is a formal consequence of Γ" : proof w/ Axioms + Assumptions

Ex: $\forall \alpha, \beta, \gamma$, $\{\alpha, (\alpha \to \beta), (\beta \to \gamma)\} \vdash \gamma$

proof: formal proof from Γ

1)	α	(from Γ)
2)	$(\alpha \to \beta)$	(from Γ)
3)	β	(M.P)
4)	$(\beta \to \gamma)$	(from Γ)
5)	γ	(M.P)

Ex: $\forall \alpha, \beta$: $\{\alpha, (\neg\alpha)\} \vdash \beta$

Formal proof of β from $\Gamma = \{\alpha, (\neg\alpha)\}$:

1)	$(\neg\alpha) \to (\neg\beta \to \neg\alpha)$	Axiom I
2)	$(\neg\alpha)$	assumption
3)	$(\neg\beta \to \neg\alpha)$	M.P (1,2)
4)	$(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$	Axiom III
5)	$\alpha \to \beta$	M.P (3,4)
6)	α	assumption
7)	β	M.P (5,6)

if you assume any contradiction $(\alpha, \neg\alpha)$, you can prove any proposition (β)

Deduction Theorem

(makes Formal Proofs Easier)

$\left.\begin{array}{l}\forall\ \Gamma \text{ (assumptions)} \\ \forall\ \alpha, \beta \text{ (propositions)}\end{array}\right\}\quad [\Gamma \cup \{\alpha\}] \vdash \beta \iff \Gamma \vdash (\alpha \to \beta)$

Ex: $\{(\alpha\to\beta), (\beta\to\gamma)\} \vdash (\alpha\to\gamma)$

by deduction theorem, this is equivalent to : $\underbrace{\{(\alpha\to\beta), (\beta\to\gamma), \alpha\}}_{\Gamma\cup\{\alpha\}} \vdash \gamma$

1) $(\alpha\to\beta)$ (Γ)
2) α (Γ)
3) β (M.P)
4) $\beta\to\gamma$ (Γ)
5) γ (M.P)

Ex: claim : $\vdash (\neg\neg\alpha \to \alpha)$

double negation elimination

by deduction theorem, it suffices to show that : $\{\neg\neg\alpha\} \vdash \alpha$

1) $\neg\neg\alpha \to (\neg\neg\neg\neg\alpha \to \neg\neg\alpha)$ (Ax I)
2) $\neg\neg\alpha$ (assumption)
3) $(\neg\neg\neg\neg\alpha \to \neg\neg\alpha) \to (\neg\alpha \to \neg\neg\neg\alpha)$ (Ax III)
4) $(\neg\alpha \to \neg\neg\neg\alpha)$ (M.P)
5) $(\neg\alpha \to \neg\neg\neg\alpha) \to (\neg\neg\alpha \to \alpha)$ (Ax III)
6) $(\neg\neg\alpha \to \alpha)$ (M.P)
7) α (M.P)

Ex: $\vdash (\alpha \to \neg\neg\alpha)$

double negation introduction

by deduction theorem, it suffices to show that : $\{\alpha\} \vdash \neg\neg\alpha$

1) $(\neg\neg\neg\alpha \to \neg\alpha) \to (\alpha \to \neg\neg\alpha)$ (Ax III)
2) $\neg\neg(\neg\alpha) \to (\neg\alpha)$ (double negation elim)
3) $(\alpha \to \neg\neg\alpha)$ (M.P)
4) α (assumption)
5) $\neg\neg\alpha$ (M.P)

Properties of Proof From Assumptions

1) More Assumptions = more conclusions

for every set of propositions Γ_1, Γ_2 : $\Gamma_1 \subseteq \Gamma_2 \implies \forall \alpha, [\Gamma_1 \vdash \alpha \implies \Gamma_2 \vdash \alpha]$

$\implies \{\alpha : \Gamma_1 \vdash \alpha\} \subseteq \{\alpha : \Gamma_2 \vdash \alpha\}$

2) $\Gamma_2 \trianglerighteq \Gamma_1 \implies \forall \alpha [\Gamma_2 \vdash \alpha \implies \Gamma_1 \vdash \alpha]$

$\downarrow$

$\Gamma_1 \subseteq \{\alpha : \Gamma_2 \vdash \alpha\} \implies \{\alpha : \Gamma_1 \vdash \alpha\} \subseteq \{\alpha : \Gamma_2 \vdash \alpha\}$

Ex:

$\left.\begin{aligned} \Gamma_1 &= \{\alpha, (\alpha \to \beta)\} \\ \Gamma_2 &= \{\beta\} \end{aligned}\right\} \Gamma_1 \trianglerighteq \Gamma_2$

Prove that : $(p \to \beta)$

proof from Γ_2 :		proof from Γ_1 :	
$\beta \to (p \to \beta)$	AxI	$\beta \to (p \to \beta)$	(Ax I)
β	(assumption)	α	(ass)
$(p \to \beta)$	(M.P)	$\alpha \to \beta$	(ass)
		β	M.P
		$p \to \beta$	M.P

Saving Brackets:

We used brackets for unique readability when defining WFF's

But we don't bother writing them in formal proofs if we assume:

Order of Connectives.

1) $\neg$
2) $\wedge$
3) $\vee$
4) $\to$

ie: $p \wedge \neg q \vee t \equiv ((p \wedge (\neg q)) \vee t)$

Week 6

$\vdash$	formally prove	Syntax
$\models$	logically implies	Semantics

Consistency

Γ is consistent if : $\nexists \alpha$ such that : $\underbrace{[\Gamma \vdash \alpha] \wedge [\Gamma \vdash (\neg \alpha)]}_{\text{proves a contradiction}}$

Ex:

Inconsistent Γ:

$\Gamma = \{p, \neg p\}$

$\Gamma = \{$ all propositions $\}$

Consistent Γ:

...

Properties :

1) if Γ is consistent , any subset of Γ is consistent

2) Γ is consistent $\iff$ every finite subset of Γ is consistent

3) If some Γ is consistent $\implies$ $\emptyset$ is consistent

- $\emptyset$ is a subset of any set
- "$\emptyset$ is consistent" means $\nexists \alpha : [\vdash \alpha] \wedge [\vdash (\neg \alpha)]$
- Without assumptions , can't prove a contradiction

4) Γ is consistent $\iff$ there's some prop β s.t $\Gamma \nvdash \beta$ "Some formula it can't prove"

Connecting Syntax & Semantics

Syntax		Semantics
propositions		basic semantic notions
formal proofs		truth tables
$\vdash \alpha$		$\models \alpha$ "α is a tautology"
$\Gamma \vdash \alpha$	$\xrightarrow{\text{soundness}}$ $\xleftarrow{\text{completeness}}$	$\Sigma \models \alpha$
Γ consistency	$\xleftrightarrow{\text{theorem 3}}$	Σ satisfiability
Γ monotonicity		Γ completeness

1) Soundness Theorem

$\forall \Gamma \wedge \forall \alpha$, $\quad \Gamma \vdash \alpha \Rightarrow \Gamma \models \alpha$

in particular: $\underbrace{\vdash \alpha}_{\text{always provable}} \Rightarrow \underbrace{\models \alpha}_{\text{tautology}}$

corollary

ϕ is consistent

Since $\phi \nvdash p$ because p is not a tautology

2) Completeness Theorem

$\forall \Gamma \wedge \forall \alpha$, $\quad \Gamma \models \alpha \Rightarrow \Gamma \vdash \alpha$

in particular: $\underbrace{\models \alpha}_{\text{a tautology}} \Rightarrow \underbrace{\vdash \alpha}_{\text{provable}}$

3) Theorem

Σ is satisfiable $\iff$ Σ is consistent

Maximally Consistent

Σ is maximally consistent if:

1) Σ is consistent

2) $\forall \alpha$: $\quad \Sigma \nvdash \alpha \Rightarrow \Sigma \cup \{\alpha\}$ is inconsistent

Properties:

1) Σ is maximally consistent $\iff$ $\forall \alpha$, either : $\Sigma \vdash \alpha$ $\vee$ $\Sigma \vdash (\neg \alpha)$

2) $\forall$ consistent Σ : $\quad \exists \bar{\Sigma}$ s.t $\bar{\Sigma}$ is maximally consistent

$\Sigma \subseteq \bar{\Sigma}$