CSC376: Assignment 1: Due by Oct 5, 2019 by 11:59pm

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Part I: Rotation

1. A rotation matrix is a square matrix which is part of the special orthogonal group, in the case of R1 and R2 the SO(3) group which is made up of matricies that are both orthogonal and has a determinant of 1. It must have the following two constraints: $R^T R = I$ and $det(R) = 1$.

$$R_1 = \begin{bmatrix} 0.6314 & 0.6301 & 0.4520 \\ 0.3267 & -0.7448 & 0.5818 \\ -0.7033 & 0.2197 & 0.6761 \end{bmatrix}$$

$$R_1{}^T = \begin{bmatrix} 0.6314 & 0.3267 & -0.7033 \\ 0.6301 & -0.7448 & 0.2197 \\ 0.4520 & 0.5818 & 0.6761 \end{bmatrix}$$

$$R_1{}^T R_1 = \begin{bmatrix} 0.6314(0.6314) + 0.6301(0.6301) + 0.4520(0.4520) & ... \\ ... & \\ & \end{bmatrix}$$

$$= \begin{bmatrix} 0.99999597 & 0.6314(0.3267) + 0.6301(-0.7448) + 0.4520(0.5818) & ... \\ ... & & \\ & & \end{bmatrix}$$

$$= \begin{bmatrix} 0.99999597 & -0.0000465 & 0.6314(-0.70337) + 0.6301(0.2197) + 0.4520(0.6761) \\ ... & & \\ & & \end{bmatrix}$$

$$= \begin{bmatrix} 0.99999597 & -0.0000465 & 0.22812576 \\ ... & & \\ & & \end{bmatrix} \neq I$$

Therefore, even allowing some rounding errors, R_1 is not a rotation matrix because it cannot satisfy the property $R_1{}^T R_1 = I$ since the first row third column value of $R_1{}^T R_1$ is already non-zero and $R_1{}^T R_1 \neq I$.

$$R_2 = \begin{bmatrix} -7 & 5 & 3 \\ 6 & -9 & 2 \\ 1 & 3 & -4 \end{bmatrix}$$

$det(R_2) = -7 \cdot \det\begin{pmatrix} -9 & 2 \\ 3 & -4 \end{pmatrix} - 5 \cdot \det\begin{pmatrix} 6 & 2 \\ 1 & -4 \end{pmatrix} + 3 \cdot \det\begin{pmatrix} 6 & -9 \\ 1 & 3 \end{pmatrix}$
$= -7 \cdot (36 - 6) - 5(-24 - 2) + 3(18 + 9)$
$= -7(30) - 5(-26) + 3(27)$
$= -210 + 130 + 81$
$= 1$

So far R_2 could possibly be a rotational matrix.

$$R_2 = \begin{bmatrix} -7 & 5 & 3 \\ 6 & -9 & 2 \\ 1 & 3 & -4 \end{bmatrix}$$

$${R_2}^T = \begin{bmatrix} -7 & 6 & 1 \\ 5 & -9 & 3 \\ 3 & 2 & -4 \end{bmatrix}$$

$${R_2}^T R_2 = \begin{bmatrix} -7(-7) + 5(5) + 3(3) & ... \\ ... & \\ & \end{bmatrix}$$

$$= \begin{bmatrix} 49 + 25 + 9 & ... \\ ... & \\ & \end{bmatrix}$$

$$= \begin{bmatrix} 83 & ... \\ ... & \\ & \end{bmatrix} \neq I$$

Therefore, R_2 is not a rotation matrix because it cannot satisfy the property ${R_2}^T R_2 = I$ since the first value on the diagonal of ${R_2}^T R_2$ is already not equal to 1 so ${R_2}^T R_2 \neq I$.

Part I

2.) $\theta = \pi$ The exponential coordinates of R_3 are simply

$$\hat{\omega} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \qquad R_3 = \hat{\omega}\theta = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \cdot \pi = \begin{matrix} \frac{\pi}{\sqrt{3}} \\ \frac{\pi}{\sqrt{3}} \\ \frac{\pi}{\sqrt{3}} \end{matrix} = \hat{\omega}\theta$$

$$Rot(\hat{\omega}, \theta) = \begin{bmatrix} \cos\theta + \hat{\omega}_1^2(1-\cos\theta) & \hat{\omega}_1\hat{\omega}_2(1-\cos\theta) - \hat{\omega}_3\sin\theta & \hat{\omega}_1\hat{\omega}_3(1-\cos\theta) + \hat{\omega}_2\sin\theta \\ \hat{\omega}_1\hat{\omega}_2(1-\cos\theta) + \hat{\omega}_3\sin\theta & \cos\theta + \hat{\omega}_2^2(1-\cos\theta) & \hat{\omega}_2\hat{\omega}_3(1-\cos\theta) - \hat{\omega}_1\sin\theta \\ \hat{\omega}_1\hat{\omega}_3(1-\cos\theta) - \hat{\omega}_2\sin\theta & \hat{\omega}_2\hat{\omega}_3(1-\cos\theta) + \hat{\omega}_1\sin\theta & \cos\theta + \hat{\omega}_3^2(1-\cos\theta) \end{bmatrix}$$

$$= \begin{bmatrix} -1 + \frac{1}{3}(2) & \frac{1}{3}(2) - 0 & \frac{1}{3}(2) + 0 \\ \frac{1}{3}(2) + 0 & -1 + \frac{1}{3}(2) & \frac{1}{3}(2) - 0 \\ \frac{1}{3}(2) - 0 & \frac{1}{3}(2) + 0 & -1 + \frac{1}{3}(2) \end{bmatrix}$$

$$= \begin{bmatrix} -1 + \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -1 + \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -1 + \frac{2}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} = R_3$$

$$R_3 R_3 = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 & -\frac{1}{3}\left(\frac{2}{3}\right) + \frac{2}{3}\left(-\frac{1}{3}\right) + \left(\frac{2}{3}\right)^2 & -\frac{1}{3}\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \frac{2}{3}\left(-\frac{1}{3}\right) \\ \frac{2}{3}\left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right)\frac{2}{3} + \left(\frac{2}{3}\right)^2 & \left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 & \left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)\frac{2}{3} + \frac{2}{3}\left(-\frac{1}{3}\right) \\ \frac{2}{3}\left(-\frac{1}{3}\right) + \left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)\frac{2}{3} & \left(\frac{2}{3}\right)^2 + \frac{2}{3}\left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right)\frac{2}{3} & \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$t_1 = \frac{1}{2}\sqrt{1 + \left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right)}$$
$$= 0 \quad \leftarrow \text{first variant is a bad choice}$$

$$t_2 = \frac{1}{2}\sqrt{1 + \left(-\frac{1}{3}\right) - \left(-\frac{1}{3}\right) - \left(-\frac{1}{3}\right)}$$
$$= \frac{1}{2}\sqrt{1 + \frac{1}{3}} = \frac{1}{2}\sqrt{\frac{4}{3}}$$
$$= \frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} = t_2 = t_3 = t_4$$

$$\eta = \frac{1}{4t_2}\left(\frac{2}{3} - \frac{2}{3}\right)$$
$$\eta = 0$$

$$\epsilon_1 = t_2 = \frac{1}{\sqrt{3}}$$

$$\epsilon_3 = \frac{1}{4t_2}\left(\frac{2}{3} + \frac{2}{3}\right)$$
$$= \epsilon_2 \longrightarrow \epsilon_2 = \frac{1}{4t_2}\left(\frac{2}{3} + \frac{2}{3}\right)$$
$$= \left(\frac{4}{\sqrt{3}}\right)^{-1} \cdot \left(\frac{4}{3}\right)$$
$$= \frac{\sqrt{3}}{4} \cdot \frac{4}{3} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{\sqrt{9}}{3\sqrt{3}} = \frac{3}{3\sqrt{3}}$$
$$\epsilon_2 = \frac{1}{\sqrt{3}}$$

$$\epsilon_3 = \frac{1}{\sqrt{3}}$$

$$\xi = \eta + \epsilon_1 i + \epsilon_2 j + \epsilon_3 k$$
$$= 0 + \frac{1}{\sqrt{3}} i + \frac{1}{\sqrt{3}} j + \frac{1}{\sqrt{3}} k$$
$$\xi = \frac{1}{\sqrt{3}} i + \frac{1}{\sqrt{3}} j + \frac{1}{\sqrt{3}} k$$

Part I

3.) $\xi p \xi^{-1}$, $\boxed{\begin{aligned}\xi &= \eta + \epsilon_1 i + \epsilon_2 j + \epsilon_3 k \\ \xi^{-1} &= \eta - \epsilon_1 i - \epsilon_2 j - \epsilon_3 k \\ p &= 0 + xi + yj + zk\end{aligned}}$ ← This is using ξ

$$\xi\xi' = (\eta\eta' - \epsilon_1\epsilon_1' - \epsilon_2\epsilon_2' - \epsilon_3\epsilon_3') + (\eta\epsilon_1' + \eta'\epsilon_1 + \epsilon_2\epsilon_3' - \epsilon_2'\epsilon_3)i + (\eta\epsilon_2' + \eta'\epsilon_2 + \epsilon_3\epsilon_1' - \epsilon_3'\epsilon_1)j + (\eta\epsilon_3' + \eta'\epsilon_3 + \epsilon_1\epsilon_2' - \epsilon_1'\epsilon_2)k$$

$$\xi p = (\eta(0) - \epsilon_1 x - \epsilon_2 y - \epsilon_3 z) + (\eta x + 0(\epsilon_1) + \epsilon_2 z - \epsilon_3 y)i + (\eta y + (0)\epsilon_2 + \epsilon_3 x - \epsilon_1 z)j + (\eta z + (0)\epsilon_3 + \epsilon_1 y - \epsilon_2 x)k$$

Let $a = (-\epsilon_1 x - \epsilon_2 y - \epsilon_3 z)$
$b = (\eta x - \epsilon_3 y + \epsilon_2 z)$
$c = (\epsilon_3 x + \eta y - \epsilon_1 z)$
$d = (-\epsilon_2 x + \epsilon_1 y + \eta z)$
$\Rightarrow \xi p = a + bi + cj + dk$

$$\xi p \xi^{-1} = (a\eta - b(-\epsilon_1) - c(-\epsilon_2) - d(-\epsilon_3)) + (a(-\epsilon_1) + \eta b + c(-\epsilon_3) - (-\epsilon_2)d)i + (a(-\epsilon_2) + \eta c + d(-\epsilon_1) - (-\epsilon_3)b)j + (a(-\epsilon_3) + \eta d + b(-\epsilon_2) - (-\epsilon_1)c)k$$

$$= (a\eta + b\epsilon_1 + c\epsilon_2 + d\epsilon_3) + (-a\epsilon_1 + b\eta - c\epsilon_3 + d\epsilon_2)i + (-a\epsilon_2 + c\eta - d\epsilon_1 + b\epsilon_3)j + (-a\epsilon_3 + d\eta - b\epsilon_2 + c\epsilon_1)k$$

These are equal.

$\boxed{\begin{aligned}-\xi &= -\eta - \epsilon_1 i - \epsilon_2 j - \epsilon_3 k \\ (-\xi)^{-1} &= -\eta + \epsilon_1 i + \epsilon_2 j + \epsilon_3 k\end{aligned}}$ ← This is using $-\xi$

$$-\xi p = (0 + \epsilon_1 x + \epsilon_2 y + \epsilon_3 z) + (-\eta x + 0 - \epsilon_2 z + y\epsilon_3)i + (-\eta y + 0 - \epsilon_3 x + z\epsilon_1)j + (-\eta z + 0 - \epsilon_1 y + x\epsilon_2)k$$

Let $a' = (\epsilon_1 x + \epsilon_2 y + \epsilon_3 z) = -a$ η
$b' = (-\eta x - \epsilon_2 z + y\epsilon_3) = -b$ ϵ_1
$c' = (-\eta y - \epsilon_3 x + z\epsilon_1) = -c$ ϵ_2
$d' = (-\eta z - \epsilon_1 y + x\epsilon_2) = -d$ ϵ_3
$\Rightarrow -\xi p = -a - bi - cj - dk$

$$-\xi p(-\xi)^{-1} = (a\eta + b\epsilon_1 + c\epsilon_2 + d\epsilon_3) + (-a\epsilon_1 + (-\eta)b - c\epsilon_3 + \epsilon_2 d)i + (-a\epsilon_2 + \eta c - d\epsilon_1 + \epsilon_3 b)j + (-a\epsilon_3 + \eta d - b\epsilon_2 + \epsilon_1 c)k$$

$$= \xi p \xi^{-1}$$

$\therefore$ Both quaternions ξ and $-\xi$ produce the same rotated point, shown by using the sandwich product $\xi p \xi^{-1}$

Part I

4.) $p = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\theta = 90° = \frac{\pi}{4}$, rotate about $\hat{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Let $\xi = \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)(1)i + 0j + 0k$ (orientation representation of a unit quaternion)

$p = 0 + 0i + 1j + 0k$

(rule for non-commutative multiplication)

$$\xi p = (0-0-0-0) + (0+0+0-0)i + \left(\cos\left(\tfrac{\pi}{4}\right)(1) + 0 + 0 - 0\right)j + \left(0 + 0 + \sin\left(\tfrac{\pi}{4}\right)(1) - 0\right)k$$
$$= 0 + 0i + \cos\left(\tfrac{\pi}{4}\right)j + \sin\left(\tfrac{\pi}{4}\right)k \longrightarrow \text{(first half of sandwich product)}$$

$$\xi^{-1} = \cos\left(\tfrac{\pi}{4}\right) - \sin\left(\tfrac{\pi}{4}\right)i - 0j - 0k$$

$$\xi p \xi^{-1} = (0-0-0-0) + (0+0+0-0)i + \left(0 + \cos^2\left(\tfrac{\pi}{4}\right) - \sin^2\left(\tfrac{\pi}{4}\right) - 0\right)j$$ (rule for non-commutative multiplication)
$$+ \left(0 + \sin\left(\tfrac{\pi}{4}\right)\cos\left(\tfrac{\pi}{4}\right) + 0 + \sin\left(\tfrac{\pi}{4}\right)\cos\left(\tfrac{\pi}{4}\right)\right)k$$
$$= 0 + 0i + \left(\tfrac{1}{\sqrt{2}}\left(\tfrac{1}{\sqrt{2}}\right) - \tfrac{1}{\sqrt{2}}\left(\tfrac{1}{\sqrt{2}}\right)\right)j + \left(\tfrac{1}{\sqrt{2}}\left(\tfrac{1}{\sqrt{2}}\right) + \tfrac{1}{\sqrt{2}}\left(\tfrac{1}{\sqrt{2}}\right)\right)k$$
$$= 0 + 0i + 0j + 1k \longrightarrow \text{(full sandwich product)}$$

$\hookrightarrow$ After the rotation, the new vector $p' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Let R be the elemental rotation matrix to rotate $90°$ about the x-axis

$$R = Rot(\hat{x}, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ 0 & \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Let p'' be the vector p after rotation R

$$p'' = Rp = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+0+0 \\ 0+0+0 \\ 0+1+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = p' = p''$$

$\therefore$ The rotation computed w/ the sandwich product and the elemental rotation matrix on vector p are equal.

Part I

5.) $R_4 = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Convert R_4 into quaternion ξ $\quad t_1 > t_4 > t_2 = t_3$ => choose the largest trace $t_1 = \frac{\sqrt{2+\sqrt{2}}}{2}$

$t_1 = \frac{1}{2}\sqrt{1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 1} = \frac{\sqrt{2+\sqrt{2}}}{2} = t_1 = \eta$

$\epsilon_1 = \frac{1}{4t_1}(0-0) = 0$

$\epsilon_2 = \frac{1}{4t_1}(0-0) = 0$

$\epsilon_3 = \frac{1}{4t_1}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4t_1} = \frac{\sqrt{2}}{2\sqrt{2+\sqrt{2}}} = \frac{\sqrt{2-\sqrt{2}}}{2}$

$\xi = \frac{\sqrt{2+\sqrt{2}}}{2} + 0i + 0j + \frac{\sqrt{2}}{2\sqrt{2+\sqrt{2}}} \cdot k$

$R_4 = Rot(\hat{z}, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ => $\frac{\sqrt{2}}{2} = \cos\theta$ => $\arccos\frac{\sqrt{2}}{2} = \theta$ => $\theta = \frac{\pi}{4}$

=> $\left(\hat{w} = \hat{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \theta = \frac{\pi}{4}\right)$ is the axis-angle representation

$= \begin{bmatrix} 0 \\ 0 \\ \frac{\pi}{4} \end{bmatrix}$

$\xi' = \cos\left(\frac{\pi}{8}\right) + 0i + 0j + (1)\sin\left(\frac{\pi}{8}\right)k$ (orientation representation of unit quaternion)

$\xi' = \frac{\sqrt{2+\sqrt{2}}}{2} + 0i + 0j + \frac{\sqrt{2-\sqrt{2}}}{2} k$

=> $\xi = \xi'$ so the check for my calculations on R_4 is passed

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Convert R_5 into quaternion ξ $\quad t_2 > t_1 = t_3 = t_4$ => choose the largest trace $t_2 = 1$

$t_2 = \frac{1}{2}\sqrt{1 + 1 - (-1) - (-1)} = \frac{1}{2}\sqrt{4} = 1 = t_2$

$\epsilon_1 = t_2 = 1$

$\epsilon_2 = \frac{1}{4t_2}(0+0) = 0$ => $\xi = 0 + 1i + 0j + 0k$

$\epsilon_3 = \frac{1}{4t_2}(0+0) = 0$

$\eta = \frac{1}{4t_2}(0-0) = 0$

$R_5 = Rot(\hat{x}, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$ => $\arccos(-1) = \theta$

$\theta = \pi$

=> $\left(\hat{w} = \hat{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \theta = \pi\right) = \begin{bmatrix} \pi \\ 0 \\ 0 \end{bmatrix}$ this is the axis-angle representation

$\xi' = \cos\left(\frac{\pi}{2}\right) + (1)\sin\left(\frac{\pi}{2}\right)i + 0j + 0k$

$\xi' = 0 + 1i + 0j + 0k$

(orientation representation of unit quaternion)

$\xi = \xi'$ so the check for my calculations on R_5 is passed

Resource used for simplified $R_z(\alpha)R_y(\beta)R_x(\gamma)$
"Computing Euler angles from a rotation matrix"
by Gregory G. Slabaugh

Part I

6.) One way to compute $R_z(\alpha)R_y(\beta)R_x(\gamma)$ is to left multiply the result of $R_y(\beta)R_x(\gamma)$ by $R_z(\alpha)$. To save some trivial calculations, I will use the formula from the resource:

$$\alpha=\beta=\gamma=\frac{\pi}{2}$$

$$R = R_z(\alpha)R_y(\beta)R_x(\gamma)$$

$$= \begin{bmatrix} \cos\beta\cos\alpha & \sin\gamma\sin\beta\cos\alpha-\cos\gamma\sin\alpha & \cos\gamma\sin\beta\cos\alpha+\sin\gamma\sin\alpha \\ \cos\beta\sin\alpha & \sin\gamma\sin\beta\sin\alpha+\cos\gamma\cos\alpha & \cos\gamma\sin\beta\sin\alpha-\sin\gamma\cos\alpha \\ -\sin\beta & \sin\gamma\cos\beta & \cos\gamma\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} 0(0) & 1(1)(0)-0(1) & 0(1)(0)+1(1) \\ 0(1) & 1(1)(1)+0(0) & 0(1)(1)-1(0) \\ -1 & 1(0) & 0(0) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = R$$

$$R\hat{x} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = -\hat{z} \qquad (Rn=n)$$

$$R\hat{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \hat{y} \quad \text{—> this must be the axis of rotation}$$

$$R\hat{z} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \hat{x}$$

R looks like it is a rotation of $\frac{\pi}{2}$ about the y axis. We can verify by calculating $Rot(\hat{y},\theta)$

$$Rot(\hat{y},\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = R$$

Hence the angle of rotation $\theta=\frac{\pi}{2}$ and the rotation axis $\hat{u}=\hat{y}=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Equivalent to this rotation of $\frac{\pi}{2}$ about $\hat{y}$ is a rotation of $\frac{3\pi}{2}$ in the opposite direction about $\hat{y}$, so $\theta' = -\frac{3\pi}{2}$

Hence, an alternate set can be $\alpha=\beta=\gamma=-\frac{3\pi}{2}$ (based on my findings)

$\alpha = \text{atan2}(r_{21}, r_{11}) = \text{atan2}(0,0) = \text{undefined}$ (based on the computed matrix)

$\beta = \text{atan2}(-r_{31}, \sqrt{r_{11}^2+r_{21}^2}) = \text{atan2}(1, \sqrt{0+0}) = \frac{\pi}{2}$

$\gamma = \text{atan2}(r_{32}, r_{33}) = \text{atan2}(0,0) = \text{undefined}$

The name of the phenomenon that arises through an unfortunate rotation sequence and angles is called gimbal lock.

Part II

1.) $T_{bg} = \begin{bmatrix} 1 & 0 & 0 & F \\ 0 & -1 & 0 & -A \\ 0 & 0 & -1 & H \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad T_{bp} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & G \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & -C \\ 0 & 0 & 1 & h_p \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$T_{ba} = \begin{bmatrix} -1 & 0 & 0 & D \\ 0 & -1 & 0 & E+4B \\ 0 & 0 & 1 & K \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad T_{bi} = \begin{bmatrix} 1 & 0 & 0 & D-L \\ 0 & 1 & 0 & E+2B \\ 0 & 0 & 1 & K+M \\ 0 & 0 & 0 & 1 \end{bmatrix}$ work

$T_{bh} = \begin{bmatrix} 1 & 0 & 0 & D-L \\ 0 & -1 & 0 & E+B \\ 0 & 0 & -1 & K \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$T_{bp} = (R, p)$ Since the frame $\{p\}$ is rotated about the co-ordinate frame axis $\hat{z}$: $R = Rot(\hat{z}, \theta)$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos 135° & -\sin 135° & 0 \\ \sin 135° & \cos 135° & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad p = \begin{bmatrix} G \\ -C \\ h_p \end{bmatrix}$$

Part II

2.) Using subscript cancellation rule: $T_{gp} = T_{gb} T_{bp}$

we have T_{bp} from II-1, so we need to find T_{gb}

$$T_{gb} = \begin{bmatrix} 1 & 0 & 0 & -F \\ 0 & -1 & 0 & -A \\ 0 & 0 & -1 & H \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_{gp} = T_{gb}T_{bp} = \begin{bmatrix} 1 & 0 & 0 & -F \\ 0 & -1 & 0 & -A \\ 0 & 0 & -1 & H \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & G \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & -C \\ 0 & 0 & 1 & h_p \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\left(\frac{1}{\sqrt{2}}\right)+0+0+0 & 1\left(\frac{1}{\sqrt{2}}\right)+0+0+0 & 0+0+0+0 & G+0+0-F \\ 0-1\left(\frac{1}{\sqrt{2}}\right)+0+0 & 0-1\left(\frac{1}{\sqrt{2}}\right)+0+0 & 0+0+0+0 & 0-1(C)+0-A \\ 0+0+0+0 & 0+0+0+0 & 0+0-1+0 & 0+0-1(h_p)+H \\ 0+0+0+0 & 0+0+0+0 & 0+0+0+0 & 0+0+0+1 \end{bmatrix}$$

$$T_{gp} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & G-F \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & C-A \\ 0 & 0 & -1 & H-h_p \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$d = \sqrt{x^2+y^2+z^2}$

$d = \sqrt{(G-F)^2 + (C-A)^2 + (H-h_p)^2}$

$\therefore$ the transformation matrix T_{gp} is equal to and the Euclidean distance between the tool-center-point and the handle of the peg is $\sqrt{(G-F)^2 + (C-A)^2 + (H-h_p)^2}$

3.) $$T_{ai} = \begin{bmatrix} -1 & 0 & 0 & L \\ 0 & -1 & 0 & 2B \\ 0 & 0 & 1 & M \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_{ih} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -B \\ 0 & 0 & -1 & -M \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ The computed transformation matricies in terms of the dimensions given in Fig. 1 are above.

Part II

4.) First find T_{ha}. $T_{ha} = \begin{bmatrix} -1 & 0 & 0 & L \\ 0 & 1 & 0 & -3B \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Now we can calculate the distance

$$d = \|p\| = \left\| \begin{bmatrix} L \\ -3B \\ 0 \end{bmatrix} \right\| = \sqrt{L^2 + (-3B)^2 + 0^2} = \sqrt{L^2 + 9B^2} = d$$

Since the bar is a rigid body, any change on its frames such as $T_{ba}^{(new)}$ will force a corresponding change on $T_{bh}^{(new)}$ and $T_{bi}^{(new)}$ because a rigid body cannot compress or flex.

changes to   let a' represent {a} after the bar was moved.

$$T_{ba} \xrightarrow{\text{changes to}} T_{ba}^{(new)} = T_{ba'}$$

this means T_{ba} underwent the following transformation:

$$T_{ba} T_{aa'} = T_{ba'}$$

$$T_{bh} T_{hh'} = T_{bh'}$$

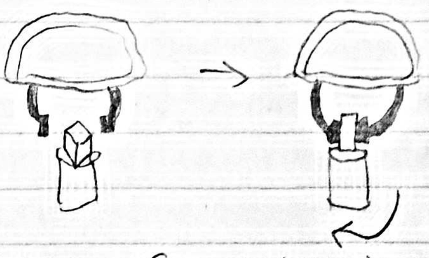
$(T_{hh'})^T T_{ha} T_{aa'} = T_{h'a'}$ Here we can see that the transformation T_{ha} still conserves the Euclidean distance between the origin of {h} and the origin of {a}

Part II

5. My approach is to use the end effector gripper to pick up the peg and place it in the hole of the {h} frame. The peg will be picked up without rotating the gripper over z. At this point, the robot's software interface can provide T_{bg} which will be used in a sequence to calculate $T_{bh}^{(new)}$, $T_{bi}^{(new)}$, and $T_{ba}^{(new)}$. The sequence will be as follows:

1. Pick up peg
2. Insert peg into hole containing origin of $\{h\}^{(moved)}$
3. Get T_{bg} from software interface
4. Compute T_{gp}
5. Compute T_{ph}
6. Compute T_{hi}
7. Compute T_{ia}

T_{gp} can be avoided as having an angular component to the rotation by keeping the gripper's orientation the same so that the peg will match the gripper



This will result in a rotation of the peg about the z axis as the gripper clamps on the peg. This removes the angular component.

8. $T_{bh}^{(new)} = \overset{(3)}{T_{bg}}\,\overset{(4)}{T_{gp}}\,\overset{(5)}{T_{ph}}$

9. $T_{bi}^{(new)} = \overset{(8)}{T_{bh}^{(new)}}\,\overset{(6)}{T_{hi}}$

10. $T_{ba}^{(new)} = \overset{(9)}{T_{bi}^{(new)}}\,\overset{(7)}{T_{ia}}$

$$T_{gp} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ the gripper and peg frame share the same origin once the peg is gripped

$$T_{ph} = \begin{bmatrix} -1 & 0 & 0 & h_p-K \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ the peg is inside the hole so the origin of {p} is above the origin of h

$$T_{hi} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -B \\ 0 & 0 & -1 & -M \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{ia} = \begin{bmatrix} -1 & 0 & 0 & L \\ 0 & -1 & 0 & 2B \\ 0 & 0 & 1 & -M \\ 0 & 0 & 0 & 1 \end{bmatrix}$$