$p(y) = \text{prior}$

Assume $p(t|\phi) = \phi^{t}(1-\phi)^{1-t}$ (prior is Bernoulli)

Prove $\phi = \frac{1}{N}\sum_{n=1}^{N} \mathbb{I}[t^{(n)}=1]$

$$l(\phi, \mu_0, \mu_1, \Sigma) = -\ln \prod_{n=1}^{N} p(x^{(n)}|t^{(n)}, \mu_0, \mu_1, \Sigma)\, p(t^{(n)}|\phi) \quad \text{(slide 17)}$$

Choose arbitrary class $k \in \{0,1\}$, preserving generality, Assume $\Sigma_1 = \Sigma_2$
(classes share same covar. matrix)

$$= -\ln \prod_{n=1}^{N} p(x^{(n)}|t^{(n)}, \mu_K, \Sigma)\, p(t^{(n)}|\phi)$$

(Eqn from slide 6)

$$= -\ln \prod_{n=1}^{N} \left( \frac{1}{(2\pi)^{d/2} |\Sigma_K|^{\frac{1}{2}}} \cdot e^{\left(-(x-\mu_K)^T \Sigma_K^{-1} (x-\mu_K)\right)} \right) \cdot \phi^{t}(1-\phi)^{1-t}$$

$$= -\Bigg[\sum_{n=1}^{N} \cancelto{0}{\ln(1)} - \ln\left((2\pi)^{\frac{d}{2}} |\Sigma_K|^{\frac{1}{2}}\right) + \ln\left(e^{-(x-\mu_K)^T \Sigma_K^{-1} (x-\mu_K)}\right)$$

$$+ t\ln\phi + (1-t)\ln(1-\phi)\Bigg]$$

$$\left.\begin{aligned} &= \sum_{n=1}^{N} \frac{d}{2}\ln(2\pi) + \frac{1}{2}\ln|\Sigma_K| + (x-\mu_K)^T \Sigma_K^{-1} (x-\mu_K) \\ &\quad + t\ln\phi + (1-t)\ln(1-\phi) \end{aligned}\right\} \text{Eqn } 1$$

$\boxed{\text{Now equate derivative to } 0}$

$$\frac{d}{d\phi} l(\phi, \mu_K, \Sigma_K) = 0 = \sum_{n=1}^{N} 0 + 0 + 0 + \frac{d}{d\phi} t^{(n)}\ln\phi + \frac{d}{d\phi}(1-t^{(n)})\ln(1-\phi)$$

$$= \sum_{n=1}^{N} \frac{t^{(n)}}{\phi} + \cancelto{0}{\frac{d}{d\phi}(1-t^{(n)})} + \cancelto{0}{0 \cdot \ln(1-\phi)} + (1-t^{(n)}) \cdot \frac{1}{(1-\phi)}(-1) \quad \text{(chain rule)}$$

$$= \sum_{n=1}^{N} \frac{t^{(n)}}{\phi} - \frac{(1-t^{(n)})}{(1-\phi)} = \sum_{n=1}^{N} \frac{t^{(n)}(1-\phi) - \phi(1-t^{(n)})}{\cancel{\phi(1-\phi)}} = 0$$

$$0 = \sum_{n=1}^{N} t^{(n)} - \cancel{t\phi} - \phi + \cancel{t\phi} = \sum_{n=1}^{N} (t^{(n)} - \phi) = \sum_{n=1}^{N} t^{(n)} - N\phi$$

$$\Rightarrow \phi = \frac{1}{N}\sum_{n=1}^{N} t^{(n)}$$

$$\Rightarrow \phi = \frac{1}{N}\sum_{n=1}^{N} \mathbb{I}[t^{(n)}=1] \quad \text{(since prior is Bernoulli, } t^{(n)} \in \{0,1\})$$

$$\therefore \phi = \frac{1}{N}\sum_{n=1}^{N} \mathbb{I}[t^{(n)}=1] \quad \blacksquare$$