CS280 Homework 1

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1 Perspective Projection

1. Take two arbitrary lines lying in the same plane, $\vec{a_1} + \lambda\vec{d}$ and $\vec{a_2} + \lambda\vec{e}$. We know the vanishing points of these two lines are $f\frac{\vec{d}}{d_z}$ and $f\frac{\vec{e}}{e_z}$. Furthermore, because they lie in the same plane, we know $\vec{d} \times \vec{e} = \vec{n}$, where $\vec{n}$ is the normal vector of the plane. Finally, the vanishing line of a plane is $xn_x + yn_y + fn_z = 0$. Using these facts, we will show the vanishing point of the lines are in fact on the vanishing line of the plane.

$$\vec{n} = \begin{pmatrix} d_y e_z - d_z e_y & d_z e_x - d_x e_z & d_x e_y - e_x d_y \end{pmatrix}$$

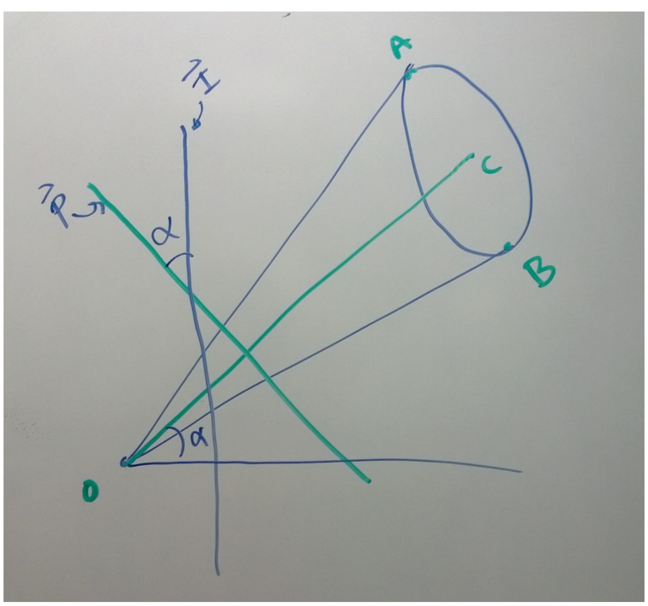
Plugging the vanishing points of the lines into the vanishing line...

$$f\frac{d_x}{d_z}(d_y e_z - d_z e_y) + f\frac{d_y}{d_z}(d_z e_x - d_x e_z) + f(d_x e_y - e_x d_y) = 0$$
$$\frac{d_x d_y e_z}{d_z} - \frac{d_x d_z e_y}{d_z} + \frac{d_y d_z e_x}{d_z} - \frac{d_x d_y e_z}{d_z} + d_x e_y - e_x d_y = 0$$
$$0 = 0$$

$$f\frac{e_x}{e_z}(d_y e_z - d_z e_y) + f\frac{e_y}{e_z}(d_z e_x - d_x e_z) + f(d_x e_y - e_x d_y) = 0$$
$$\frac{e_x d_y e_z}{e_z} - \frac{e_x d_z e_y}{e_z} + \frac{e_y d_z e_x}{e_z} - \frac{e_y d_x e_z}{e_z} + d_x e_y - e_x d_y = 0$$
$$0 = 0$$

Thus, we have confirmed that the vanishing point of any line on a plane must lie on the plane's vanishing line.

2. The eccentricity of an ellipse in a cone can be calculated as $\frac{\sin(\alpha)}{\sin(\beta)}$, where α is as depicted below (we will look at β after).

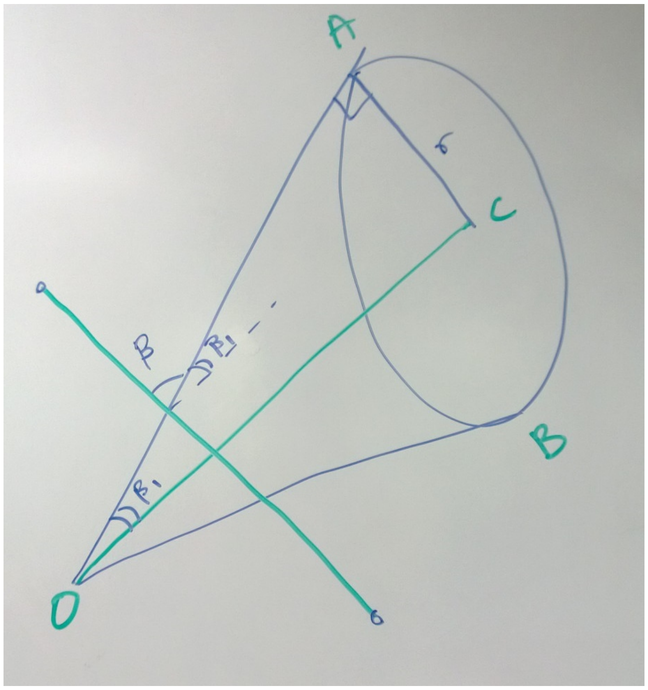


Specifically, α is the angle of the image plane from the $\vec{P}$, which is perpendicular to the ray from the origin to the center of the sphere. β is the angle of the cone's slant from that same line.

Let's calculate α first. Since the image plane is perpendicular to the **Z** axis, and $\vec{p}$ is perpendicular to $\vec{OC}$, it's easy to see that the angle between $\vec{OC}$ and the **Z** axis is also α.

$$\sin(\alpha) = \frac{X}{\sqrt{X^2 + Z^2}}$$

Next, let's look at β.



From the image, we see that β and β_1 are complementary. We can calculate β_1 first.

$$\begin{aligned}\cos(\beta_1) &= \frac{\|\vec{OA}\|}{\|\vec{OC}\|} \\ &= \frac{\sqrt{X^2 + Z^2 - r^2}}{\sqrt{X^2 + Z^2}}\end{aligned}$$

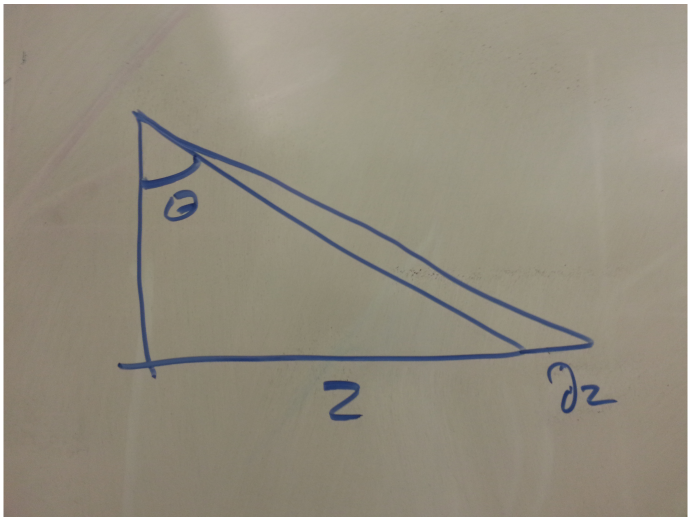
Furthermore,

$$\begin{aligned}\beta &= 90 - \beta_1 \\ \sin(\beta) &= \cos(\beta_1) \\ &= \frac{\sqrt{X^2 + Z^2 - r^2}}{\sqrt{X^2 + Z^2}}\end{aligned}$$

Hence, the eccentricity of the ellipse is

$$\begin{aligned} e &= \frac{\sin(\alpha)}{\sin(\beta)} \\ &= \frac{X}{\sqrt{X^2 + Z^2 - r^2}} \end{aligned}$$

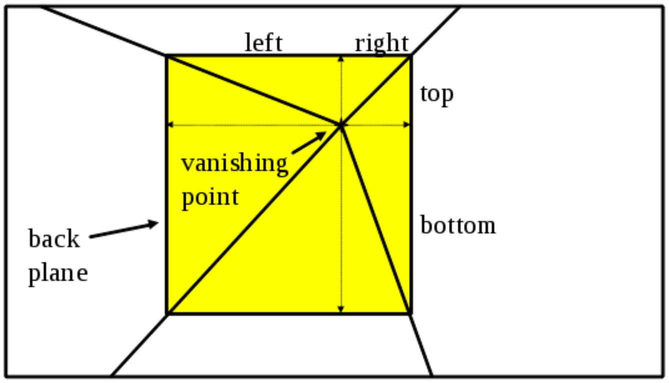
3. Define θ as the angle between the observer's body and the point.



$$\begin{aligned} \theta &= arctan(\frac{z}{h}) \\ tan(\theta + 1') &= \frac{x + \delta z}{h} \\ \delta z &= h * tan(tan^{-1}(\frac{z}{h}) + 1') - z \end{aligned}$$

2 Tour into the Picture

1. To estimate the 3D geometry of the room, we take advantage of the fact that the back plane is parallel to the image plane, and the other walls are perpendicular to it. If we know the back plane along with the vanishing point, we know the relative camera position in the room. Furthermore, if we assume a focal length, we can calculate the depth of the room, using similar triangles.



2. To compute each of the homographies, we solve for H given s and d, source and destination points. A homography must map s to d.

$$\vec{d} = H\vec{s}$$

Writing this out explicitly:

$$\begin{pmatrix} wx' \\ wy' \\ w \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$wx' = ax + by + c$$

$$wy' = dx + ey + f$$

$$w = gx + hy + 1$$

If we plug in the equation for w into the first two, we get:

$$ax + by + c - gxx' - hyx' = x'$$

$$dx + ey + f - gxy' - hyy' = y'$$

Given a pair of source/destination points, we get two equations. Thus, we need four pairs of points to generate the eight equations we need to solve for the homography. Writing those equations in matrix form, with (x'_i, y'_i) as the ith destination point:

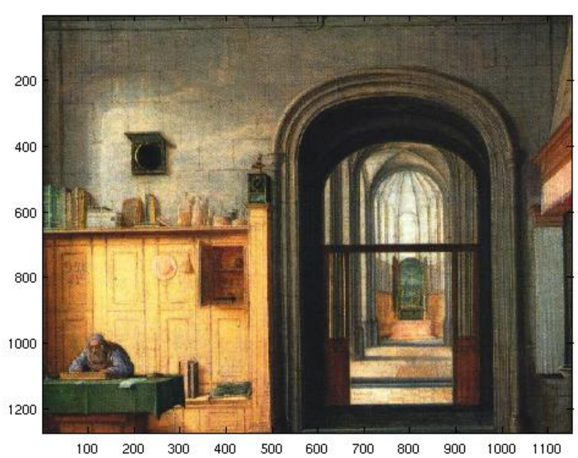
$$\begin{pmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1x'_1 & y_1x'_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -x_1y'_1 & y_1y'_1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 & -x_2x'_2 & y_2x'_2 \\ 0 & 0 & 0 & x_2 & y_2 & 1 & -x_2y'_2 & y_2y'_2 \\ \dots \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{pmatrix} = \begin{pmatrix} x'_1 \\ y'_1 \\ x'_2 \\ y'_2 \\ \dots \end{pmatrix}$$

$$Ah = b$$

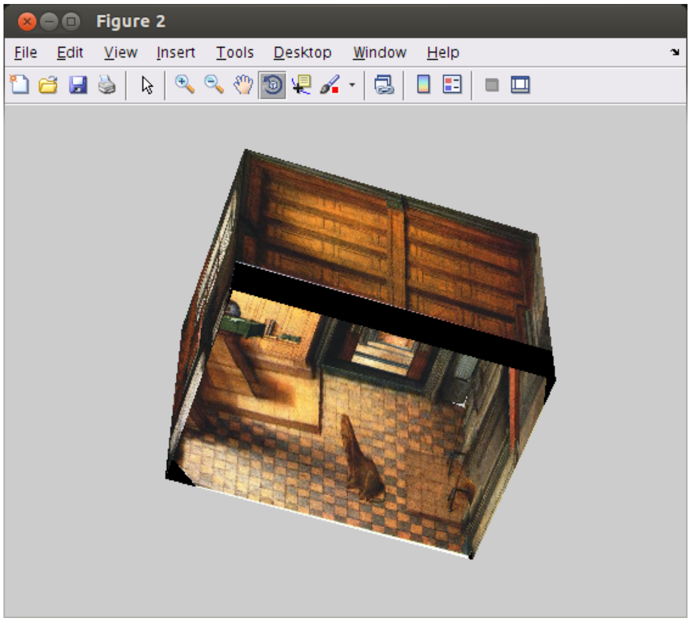
$$h = A^{-1}b$$

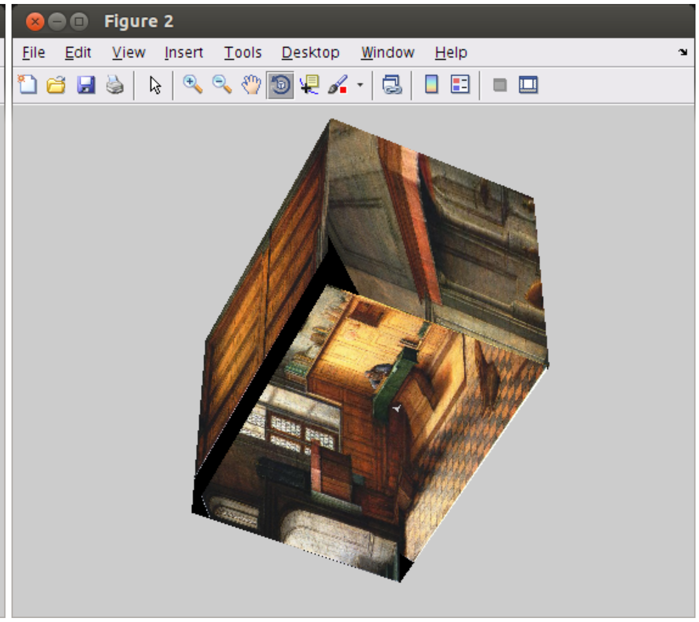
We perform this operation for each of the planes. The vertices obtained from TIP_get5rects are the source, and we create destination vertices such that the resulting rectangle is proportioned correctly according to the 3d geometry calculated.

3. Here we show the fronto-parallel views of the ceiling, floor, left wall, right wall, and back wall.

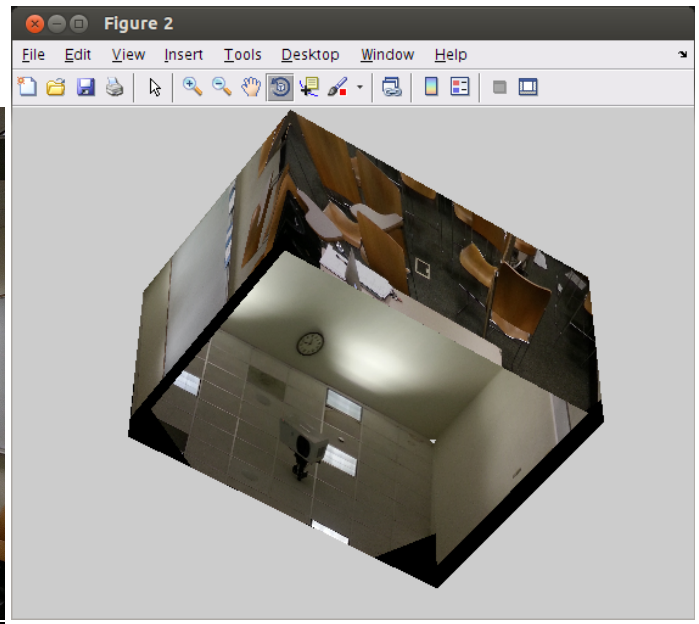


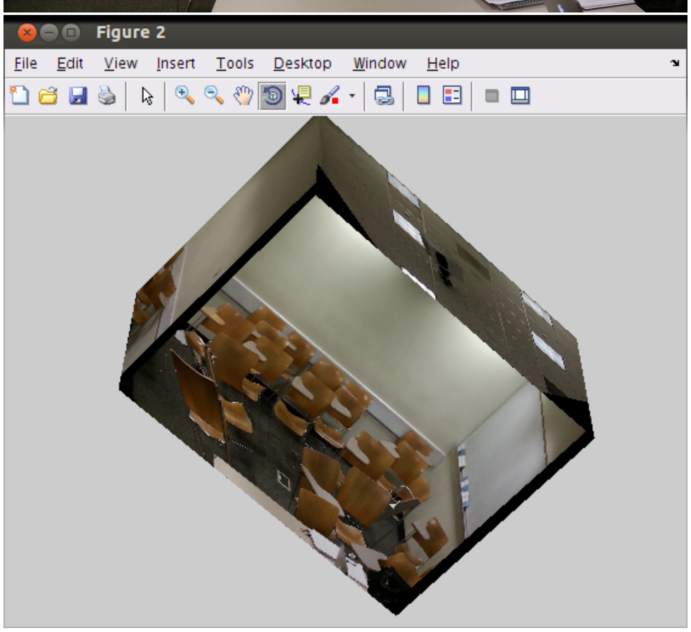
4. Views from sjerome.jpg:



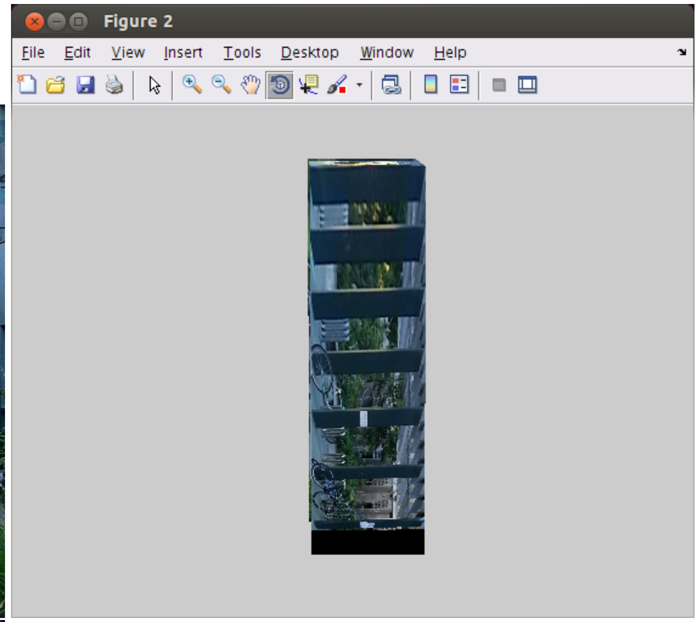


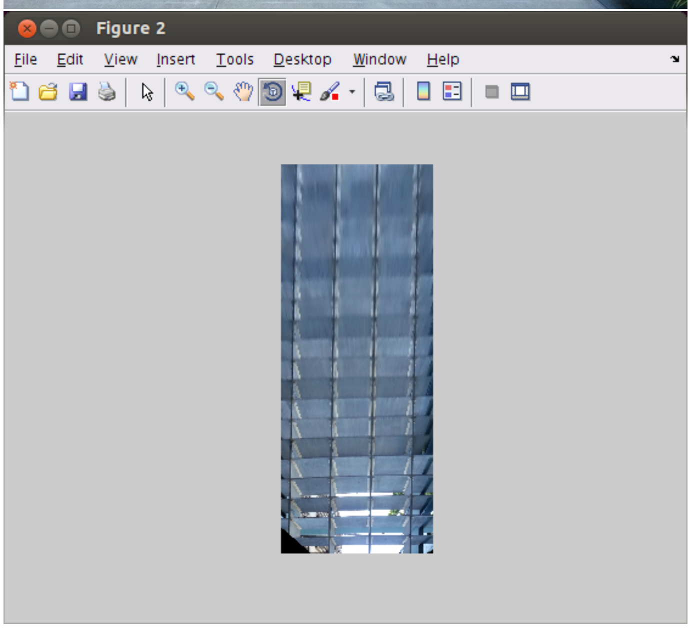
Views from 320 Soda:





Views from the Soda Etcheverry Breezeway:





3 Geometric Transformations

1. We multiply the two reflection matrices and then use trigonometric sum/difference formulas.

$$\begin{pmatrix} cos(2\beta) & sin(2\beta) \\ sin(2\beta) & -cos(2\beta) \end{pmatrix} \begin{pmatrix} cos(2\alpha) & sin(2\alpha) \\ sin(2\alpha) & -cos(2\alpha) \end{pmatrix} =$$

$$\begin{pmatrix} cos(2\beta)cos(2\alpha) + sin(2\beta)sin(2\alpha) & sin(2\alpha)cos(2\beta) - cos(2\alpha)sin(2\beta) \\ sin(2\beta)cos(2\alpha) - cos(2\beta)sin(2\alpha) & cos(2\alpha)cos(2\beta) + sin(2\alpha)sin(2\beta) \end{pmatrix} =$$

$$\begin{pmatrix} cos(2(\beta - \alpha)) & -sin(2(\beta - \alpha)) \\ sin(2(\beta - \alpha)) & cos(2(\beta - \alpha)) \end{pmatrix}$$

2. Consider a skew-symmetric matrix and its powers.

$$\hat{s} = \begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix}$$

$$\hat{s}^2 = \hat{s}^2$$
$$\hat{s}^3 = (a^2 + b^2 + c^2) * -\hat{s}$$
$$\hat{s}^4 = (a^2 + b^2 + c^2) * -\hat{s}^2$$
$$\hat{s}^5 = (a^2 + b^2 + c^2)^2 * \hat{s}$$
$$...$$

We know $||s|| = 1$, so we recognize a pattern.

$$\hat{s} = \hat{s}$$
$$\hat{s}^2 = \hat{s}^2$$
$$\hat{s}^3 = -\hat{s}$$
$$\hat{s}^4 = -\hat{s}^2$$
$$...$$

Now, we consider Roderigues' formula.

$$R = e^{\phi \hat{s}}$$
$$R = I + \phi\hat{s} + \frac{\phi^2 \hat{s}^2}{2!} + \frac{\phi^3 \hat{s}^3}{3!} + \frac{\phi^4 \hat{s}^4}{4!} ...$$

Subbing in the powers of $\hat{s}$,

$$R = I + \phi\hat{s} + \frac{\phi^2 \hat{s}^2}{2!} - \frac{\phi^3 \hat{s}}{3!} - \frac{\phi^4 \hat{s}^2}{4!} ...$$
$$R = I + (\phi - \frac{\phi^3}{3!} + ...)\hat{s} + (\frac{\phi^2}{2!} - \frac{\phi^4}{4!} + ...)\hat{s}^2$$
$$R = I + sin(\phi)\hat{s} + (1 - cos(\phi))\hat{s^2}$$

3. The general form of a Euclidean planar transformation is E:

$$\begin{pmatrix} cos\theta & -sin\theta & t_x \\ sin\theta & cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

And a least squares problem is formulated as

$$X\beta = y$$

where β consists of the unknown parameters. Thus, we massage the problem into that form, saying $cos\theta$, $sin\theta$, T_x, and T_y are our parameters, so the problem is linear.

$$\begin{pmatrix} x_1 & -y_1 & 1 & 0 \\ y_1 & x_1 & 0 & 1 \\ \cdots & & & \\ x_n & -y_n & 1 & 0 \\ y_n & x_n & 0 & 1 \end{pmatrix} \begin{pmatrix} cos\theta \\ sin\theta \\ T_x \\ T_y \end{pmatrix} = \begin{pmatrix} x'_1 \\ y'_1 \\ \cdots \\ x'_n \\ y'_n \end{pmatrix}$$

Now we can plug in the given points to find an estimate for E.

$$\beta = \begin{pmatrix} cos\theta \\ sin\theta \\ T_x \\ T_y \end{pmatrix} = (X^T X)^{-1} X^T y$$

For the given set of u and v, we get:

$$\beta = \begin{pmatrix} 0 \\ -0.7857 \\ 0 \\ 0.75 \end{pmatrix}$$

Thus, our Euclidean planar transformation is:

$$E = \begin{pmatrix} 0 & 0.7857 & 0 \\ -0.7857 & 0 & 0.75 \\ 0 & 0 & 1 \end{pmatrix}$$

4. The function is in compute_rotation_matrix.m. From finding the eigenvalues and eigenvectors of $\mathbf{R}$, we see that the eigenvector paired with the postive, purely real eigenvalue is s.

   Eigenvalues and eigenvectors. The eigenvector corresponding to the positive eigenvalue is the normalized axis of rotation.

```
>> R = compute_rotation_matrix(pi/5, [0; 1; 0]);
>> [v,d] = eig(R)

v =

   0.7071            0.7071                 0
        0                 0            1.0000
        0 + 0.7071i       0 - 0.7071i       0


d =

   0.8090 + 0.5878i       0                 0
        0            0.8090 - 0.5878i       0
        0                 0            1.0000

>> R = compute_rotation_matrix(2*pi/3, [1; 0; 0])
>> [v,d] = eig(R)

v =

        0                 0            1.0000
   0.7071            0.7071                 0
        0 - 0.7071i       0 + 0.7071i       0


d =

  -0.5000 + 0.8660i       0                 0
        0           -0.5000 - 0.8660i       0
        0                 0            1.0000
```

$cos\phi$ formula. We tried a couple values out...

```
>> R_x = compute_rotation_matrix(pi, [1;0;0]);
>> R_y = compute_rotation_matrix(pi/2, [0;1;0]);
>> R_z = compute_rotation_matrix(-pi/3, [0;0;1]);
>> 0.5 * (trace(R_x)- 1)

ans =

    -1

>> cos(pi)

ans =

    -1

>> 0.5 * (trace(R_y)- 1)

ans =

     0

>> cos(pi/2)

ans =

   6.1232e-17

>> 0.5 * (trace(R_z)- 1)

ans =

    0.5000

>> cos(-pi/3)

ans =

    0.5000
```

Applying rotation to some sample points...

```
>> R_x * [0; 0; 1]

ans =

         0
   -0.0000
   -1.0000

>> R_x * [1; -1; 1]

ans =

    1.0000
    1.0000
   -1.0000

>> R_y * [1; 0; 1]

ans =

    1.0000
         0
   -1.0000

>> R_z * [3; -1; 0]

ans =

    0.6340
   -3.0981
         0
```

5. The function is in compute_phi_axis.m.

   Recall, from part 4 that the eigenvector of R corresponding to the purely real, postive eigenvalue *is* the axis of rotation of R. Also, we know ϕ from $cos\phi = \frac{1}{2}\{trace(\mathbf{R}) - 1\}$.