4.5 The intermediate value theorem

4.5.1 Intermediate Value Theorem (IVT): Let M be a metric space and $K \subset M$ be connected. Let $f: M \longrightarrow \mathbb{R}$ be continuous. Let $x, y \in K$, $c \in \mathbb{R}$, with $f(x) < c < f(y)$. Then there exists $z \in K$ with $f(z) = c$.

Proof Suppose there is no such z. Let $U := (-\infty, c)$, $V := (c, \infty)$. By continuity of f, $f^{-1}(U) \cap K$ and $f^{-1}(V) \cap K$ are open in K and nonempty, because $x \in f^{-1}(U)$, $y \in f^{-1}(V)$ and $x, y \in K$. Moreover

$$\left(f^{-1}(U) \cap K\right) \cap \left(f^{-1}(V) \cap K\right) = \emptyset \quad \text{and} \quad \left(f^{-1}(U) \cap K\right) \cup \left(f^{-1}(V) \cap K\right) = K$$

Hence K is disconnected ↯. □

Alternative proof K connected $\overset{4.2.1}{\Longrightarrow}$ $f(K)$ connected $\overset{\substack{\text{Lemma} \\ \text{page 3.5.1}}}{\Longrightarrow}$ $f(K)$ is an interval. Hence $\forall f(x), f(y) \in f(K)$, $c \in \mathbb{R}$ with $f(x) < c < f(y)$ it follows $c \in f(K)$ and so there exists $z \in K$ with $f(z) = c$. □

Example (i) Let $f: [0,1] \to [0,1]$ be continuous. Then f has a fixed point, i.e. $\exists x \in [0,1]$ with $f(x) = x$.

Proof Set $g(x) := f(x) - x$, $x \in [0,1]$. Then $g: [0,1] \longrightarrow [-1,1]$ is again continuous (use e.g 4.3.3(i)). Suppose $g(0) \neq 0$ and $g(1) \neq 0$ (otherwise we are done). Then $g(1) < 0 < g(0)$, hence by the IVT: $\exists z \in [0,1]$ with $g(z) = 0$. □

(ii) Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be a polynomial of degree n, i.e

$$f(x) = a_n x^n + \ldots + a_1 x + a_0 \quad , \quad a_n \neq 0, \; a_i \in \mathbb{R}, \; n \in \mathbb{N}$$

If n is odd, then f has a real root.

Proof Clearly f is continuous (use 4.1.4 (ii)). For $x \neq 0$ we have

$$f(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n \cdot x} + \frac{a_{n-2}}{a_n \cdot x^2} + \ldots + \frac{a_0}{a_n \cdot x^n}\right)$$

Thus for large $\ell \in \mathbb{N}$

$$f(\ell) \approx a_n \cdot \ell^n$$
$$f(-\ell) \approx -a_n \cdot \ell^n$$

$f(\ell)$ and $f(-\ell)$ have opposite sign $\Rightarrow \exists\, x_0 \in [-\ell, \ell]$ with $f(x_0) = 0$. $\square$