Assignments for weak stabilities: genus 2 graph cases

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1 Extending stability conditions on graphs

Does the assignments of I_4 extend to G_{455} and G_{456}? A: Yes Task pull assignments of each to onenote.

There are 6 stability conditions for I_4 and 6 for G_{455} and G_{456}. We see by the diagrams each stability condition of I_4 extends uniquely to a stability condition of G_{455}, and similarly for G_{456}.

My questions

- The graph G_{455} is made of 4 subgraphs of the form I_3 with a leaf). How do the leafs on these subgraphs effect how it extends (holds constant on vertex?)?

Notes

- From examples see assignments $d'_i = d'_j$ if $T_i = T_j$.

1.1 CMSA: $g = 1$ classification

We see that # MSA functions with $f(\{i\}) = 0$ equals the # stability conditions (from a ϕ) up to translation.

We have see that Nicolas previous bijection does not respect the translations.

Define another bijection different from one in paper. There are $4! = 24$ bijections in total (could just focus on I_3 first) want bijection with stability condition with $\underline{0}$ on Γ_1 and function with $f(\{i\}) = 0$.

1.2 Check for extending: Back to coding

Tasks

- Get $n = 3$ trivalent graphs from pdf (or just an example selection)
- check the number of stability conditions.

Notes

- Trivalent graphs (from pdf) have no tails and are connected.
- Note, all graphs can be obtained from the contraction of a trivalent case and for trivalent graphs $\#V = 2g - 2 + n$ and $\#E = 3g - 3 + n$.

2 Graphs with 4 vertices

2.1 I_4

Example 2.1. We now obtain all weak stabilities, $\sigma(\underline{d}')$, for I_4. Where $d_1' = \underline{0} := (0, 0, 0, 0) \in S^0(G)$. We aim to find ϕ satisfying the ϕ-inequalities for $\sigma(\underline{d}')$.

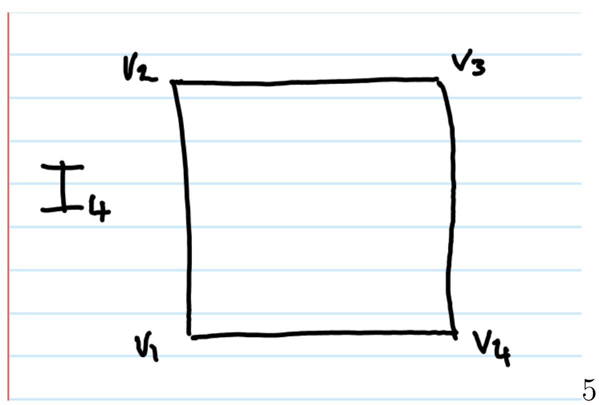


Figure 1: Labelled I_4

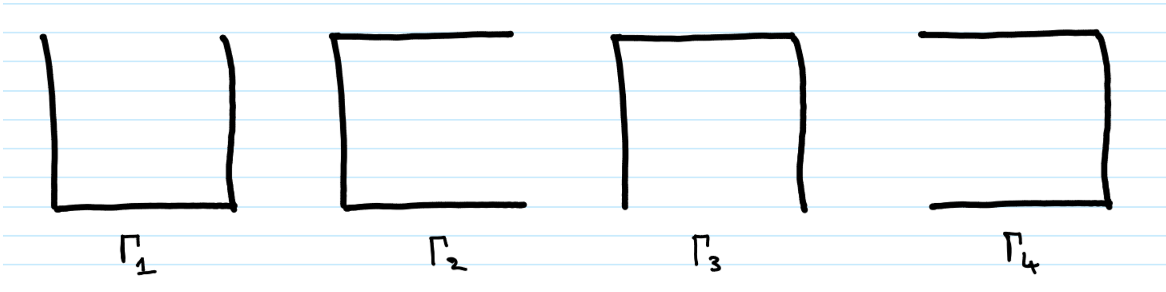
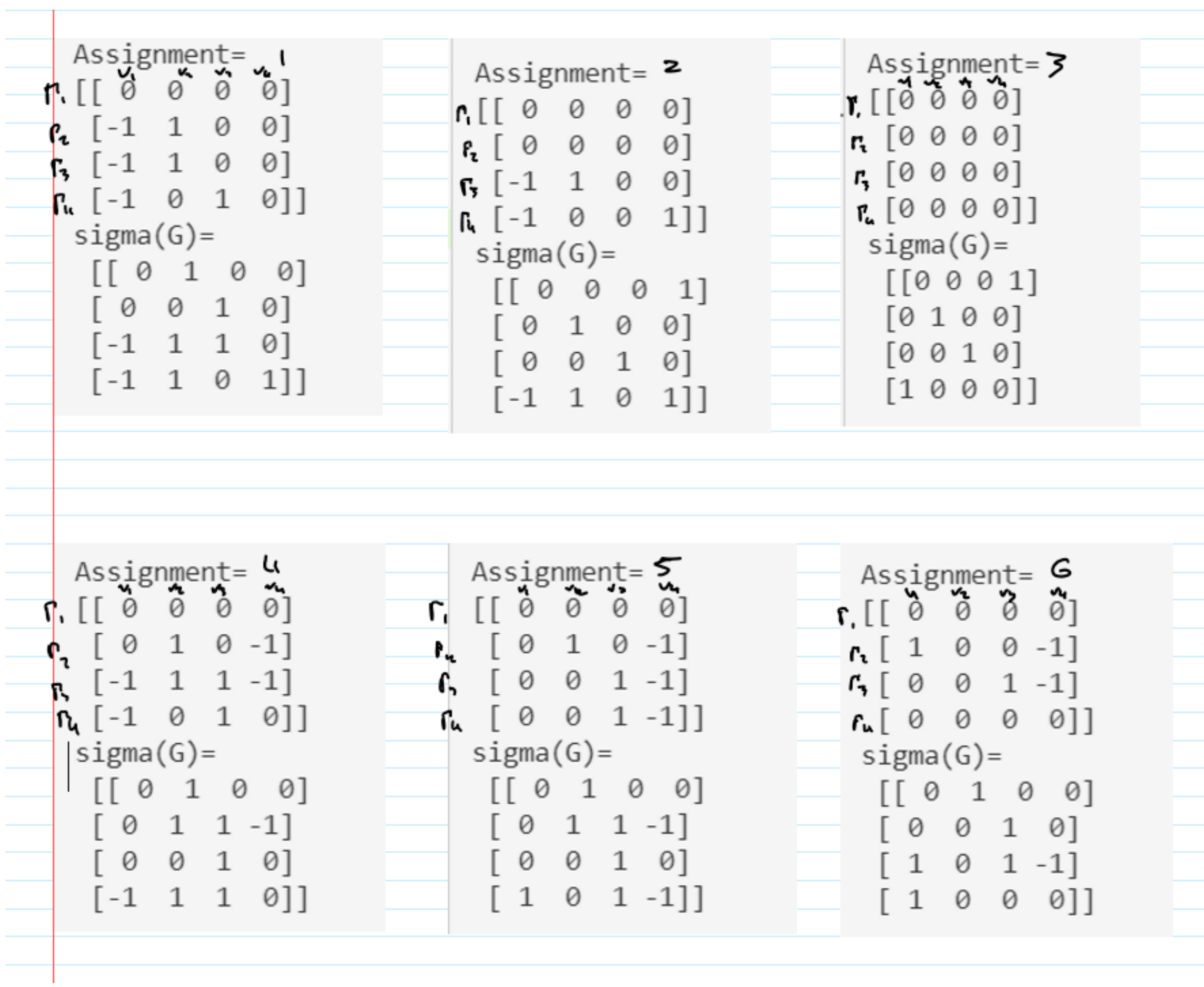


Figure 2: Ordered spanning trees of I_4

$$|d_1 - \phi_1| < 1 \quad |d_2 - \phi_2| < 1 \quad |d_3 - \phi_3| < 1 \quad |d_4 - \phi_4| < 1$$

$$|d_1 + d_2 - \phi_1 - \phi_2| < 1 \quad |d_1 + d_4 - \phi_1 - \phi_4| < 1.$$

For each can pull ϕ.



```
Assignment= 1
[[ 0  0  0  0]
 [-1  1  0  0]
 [-1  1  0  0]
 [-1  0  1  0]]
sigma(G)=
[[ 0  1  0  0]
 [ 0  0  1  0]
 [-1  1  1  0]
 [-1  1  0  1]]

Assignment= 2
[[ 0  0  0  0]
 [ 0  0  0  0]
 [-1  1  0  0]
 [-1  0  0  1]]
sigma(G)=
[[ 0  0  0  1]
 [ 0  1  0  0]
 [ 0  0  1  0]
 [-1  1  0  1]]

Assignment= 3
[[0 0 0 0]
 [0 0 0 0]
 [0 0 0 0]
 [0 0 0 0]]
sigma(G)=
[[0 0 0 1]
 [0 1 0 0]
 [0 0 1 0]
 [1 0 0 0]]

Assignment= 4
[[ 0  0  0  0]
 [ 0  1  0 -1]
 [-1  1  1 -1]
 [-1  0  1  0]]
sigma(G)=
[[ 0  1  0  0]
 [ 0  1  1 -1]
 [ 0  0  1  0]
 [-1  1  1  0]]

Assignment= 5
[[ 0  0  0  0]
 [ 0  1  0 -1]
 [ 0  0  1 -1]
 [ 0  0  1 -1]]
sigma(G)=
[[ 0  1  0  0]
 [ 0  1  1 -1]
 [ 0  0  1  0]
 [ 1  0  1 -1]]

Assignment= 6
[[ 0  0  0  0]
 [ 1  0  0 -1]
 [ 0  0  1 -1]
 [ 0  0  0  0]]
sigma(G)=
[[ 0  1  0  0]
 [ 0  0  1  0]
 [ 1  0  1 -1]
 [ 1  0  0  0]]
```

Figure 3:

Remark. We also obtain these non-trivial assignments by our method (see method sheet).

2.2 G_{456}

Example 2.2. Let $G := G_{456}$. We now obtain all weak stabilities, $\sigma(\underline{d}')$, for G_{456}. Where $d_1' = \underline{0} = (0,0,0,0) \in S^0(G)$. We also aim to find ϕ satisfying the ϕ-inequalities for $\sigma(\underline{d}')$.

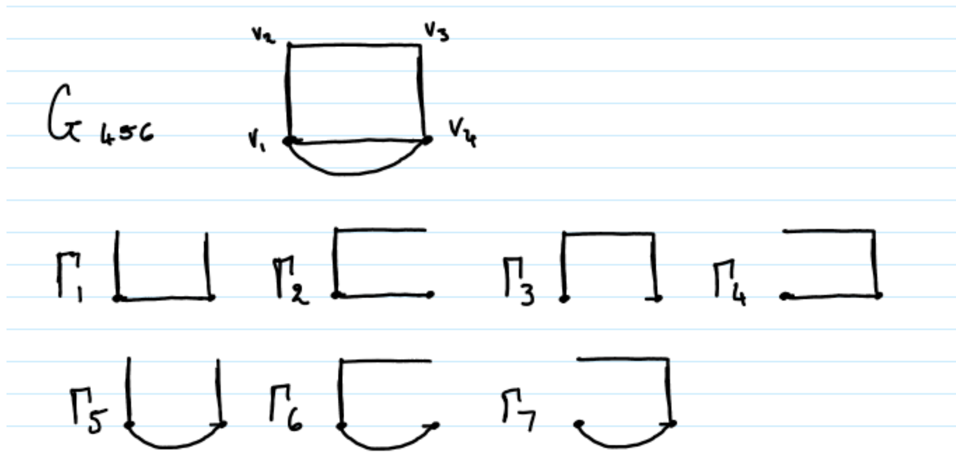


Figure 4: Graph G_{456} with ordered spanning trees

The ϕ-inequalities for G are as follows,

$$|d_1 - \phi_1| < \frac{3}{2}, \quad |d_2 - \phi_2| < 1, \quad |d_3 - \phi_3| < 1, \quad |d_4 - \phi_4| < \frac{3}{2},$$

$$|d_1 + d_2 - \phi_1 - \phi_2| < \frac{3}{2}, \quad |d_1 + d_4 - \phi_1 - \phi_4| < 1$$

```
Assignment= 1
[[0 0 0 0]
 [0 0 0 0]
 [0 0 0 0]
 [0 0 0 0]
 [0 0 0 0]
 [0 0 0 0]
 [0 0 0 0]]
sigma(G)=
 [[0 1 0 1]
 [1 1 0 0]
 [1 0 1 0]
 [2 0 0 0]
 [1 0 0 1]
 [0 0 0 2]
 [0 0 1 1]]
```

```
Assignment= 2
[[ 0  0  0  0]
 [ 0  0  0  0]
 [-1  1  0  0]
 [-1  0  0  1]
 [ 0  0  0  0]
 [ 0  0  0  0]
 [-1  0  0  1]]
sigma(G)=
 [[ 0  1  0  1]
 [ 1  1  0  0]
 [-1  1  0  2]
 [ 1  0  1  0]
 [ 1  0  0  1]
 [ 0  0  0  2]
 [ 0  0  1  1]]
```

```
Assignment= 3
[[ 0  0  0  0]
 [-1  1  0  0]
 [-1  1  0  0]
 [-1  0  1  0]
 [ 0  0  0  0]
 [-1  1  0  0]
 [-1  0  1  0]]
sigma(G)=
 [[ 0  1  0  1]
 [ 1  1  0  0]
 [-1  1  0  2]
 [-1  1  1  1]
 [ 0  1  1  0]
 [ 1  0  1  0]
 [ 0  0  1  1]]
--------------------
```

```
Assignment= 4
[[ 0  0  0  0]
 [ 0  1  0 -1]
 [ 0  0  1 -1]
 [ 0  0  1 -1]
 [ 0  0  0  0]
 [ 0  1  0 -1]
 [ 0  0  1 -1]]
sigma(G)=
 [[ 0  1  0  1]
 [ 1  1  0  0]
 [ 0  1  1  0]
 [ 1  0  1  0]
 [ 2  0  1 -1]
 [ 1  1  1 -1]
 [ 0  0  1  1]]
--------------------
```

```
Assignment= 5
[[ 0  0  0  0]
 [ 0  1  0 -1]
 [-1  1  1 -1]
 [-1  0  1  0]
 [ 0  0  0  0]
 [ 0  1  0 -1]
 [-1  0  1  0]]
sigma(G)=
 [[ 0  1  0  1]
 [ 1  1  0  0]
 [-1  1  1  1]
 [ 0  1  1  0]
 [ 1  0  1  0]
 [ 1  1  1 -1]
 [ 0  0  1  1]]
```

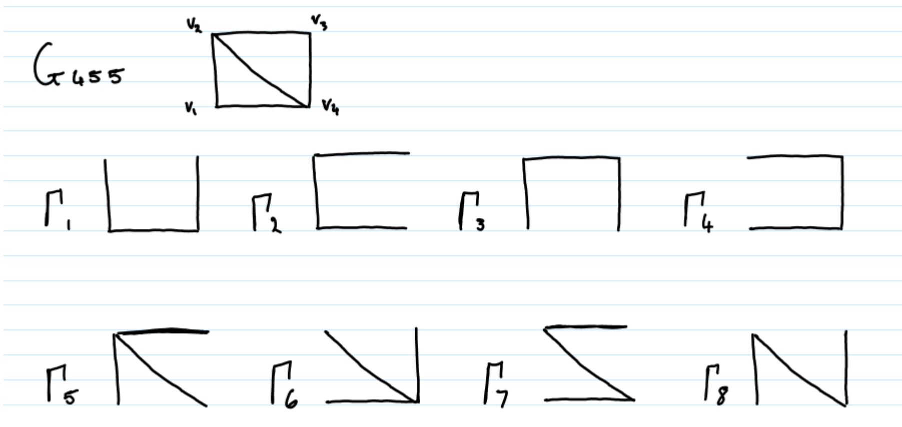
```
Assignment= 6
[[ 0  0  0  0]
 [ 1  0  0 -1]
 [ 0  0  1 -1]
 [ 0  0  0  0]
 [ 0  0  0  0]
 [ 1  0  0 -1]
 [ 0  0  0  0]]
sigma(G)=
 [[ 0  1  0  1]
 [ 1  1  0  0]
 [ 1  0  1  0]
 [ 2  0  0  0]
 [ 1  0  0  1]
 [ 2  0  1 -1]
 [ 0  0  1  1]]
--------------------
```

Figure 5:

We see there are 3 stability conditions up to translation. Each $\sigma(G)$ satisfy the ϕ-inequalities for some ϕ (can add ϕ terms if necessary). The elements of $\sigma(G)$ are pairwise chip-firing independent (checked in Sage).

2.3 G_{455}

Example 2.3. Let $G := G_{455}$. We now obtain all weak stabilities, $\sigma(\underline{d}')$, for G_{455}. Where $d_1' = \underline{0} = (0,0,0,0) \in S^0(G)$. We aim to find ϕ satisfying the ϕ-inequalities for $\sigma(\underline{d}')$.



The ϕ-inequalities are:

$$|d_1 - \phi_1| < 1, |d_2 - \phi_2| < \frac{3}{2}, |d_3 - \phi_3| < 1, |d_4 - \phi_4| < \frac{3}{2}$$

$$|d_1 + d_2 - \phi_1 - \phi_2| < \frac{3}{2}, |d_1 + d_4 - \phi_1 - \phi_4| < \frac{3}{2}.$$

Assignment= 1

$$\begin{array}{c} \\ \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \\ \Gamma_5 \\ \Gamma_6 \\ \Gamma_7 \\ \Gamma_8 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \end{array}$$

sigma(G)= 1

$$\begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array}$$
$$\left[\begin{array}{cccc} 0 & 1 & 0 & 1 \\ -1 & 2 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ -1 & 1 & 0 & 2 \\ -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Assignment= 2

$$\begin{array}{c} \\ \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \\ \Gamma_5 \\ \Gamma_6 \\ \Gamma_7 \\ \Gamma_8 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

sigma(G)= 2

$$\begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array}$$
$$\left[\begin{array}{cccc} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Assignment= 3

$$\begin{array}{c} \\ \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \\ \Gamma_5 \\ \Gamma_6 \\ \Gamma_7 \\ \Gamma_8 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \end{array}$$

sigma(G)= 3

$$\begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array}$$
$$\left[\begin{array}{cccc} 0 & 1 & 0 & 1 \\ -1 & 2 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Assignment= 1

$$\begin{array}{cc} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \\ \Gamma_5 \\ \Gamma_6 \\ \Gamma_7 \\ \Gamma_8 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

sigma(G)= 1

$$\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 2 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ -1 & 1 & 0 & 2 \\ -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Assignment= 2

$$\begin{array}{cc} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \\ \Gamma_5 \\ \Gamma_6 \\ \Gamma_7 \\ \Gamma_8 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

sigma(G)= 2

$$\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Assignment= 3

$$\begin{array}{cc} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \\ \Gamma_5 \\ \Gamma_6 \\ \Gamma_7 \\ \Gamma_8 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

sigma(G)= 3

$$\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 2 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Each $\sigma(G)$ satisfy the ϕ-inequalities for some ϕ (can add ϕ terms if necessary). The elements of $\sigma(G)$ are pairwise chip-firing independent (checked in Sage).

2.4 Vertex 4, extensions of stability conditions

```
Assignment=
[[0 0 0 0]
 [0 0 0 0]
 [0 0 0 0]
 [0 0 0 0]]
```

```
Assignment=
 [[0 0 0 0]
 [0 0 0 0]
 [0 0 0 0]
 [0 0 0 0]
 [0 0 0 0]
 [0 0 0 0]
 [0 0 0 0]]
```

```
Assignment=
[[ 0  0  0  0]
 [ 1  0  0 -1]
 [ 0  0  1 -1]
 [ 0  0  0  0]]
```

```
Assignment= 6
r1 [[ 0  0  0  0]
r2 [ 1  0  0 -1]
r3 [ 0  0  1 -1]
r4 [ 0  0  0  0]
5 [ 0  0  0  0]
6 [ 1  0  0 -1]
7 [ 0  0  0  0]]
```

```
Assignment=
[[ 0  0  0  0]
 [ 0  1  0 -1]
 [ 0  0  1 -1]
 [ 0  0  1 -1]]
```

```
Assignment= 4
 [[ 0  0  0  0]
 [ 0  1  0 -1]
 [ 0  0  1 -1]
 [ 0  0  1 -1]
 [ 0  0  0  0]
 [ 0  1  0 -1]
 [ 0  0  1 -1]]
```

```
Assignment=
[[ 0  0  0  0]
 [ 0  0  0  0]
 [-1  1  0  0]
 [-1  0  0  1]]
```

```
Assignment=
 [[ 0  0  0  0]
 [ 0  0  0  0]
 [-1  1  0  0]
 [-1  0  0  1]
 [ 0  0  0  0]
 [ 0  0  0  0]
 [-1  0  0  1]]
```

```
Assignment=
[[ 0  0  0  0]
 [-1  1  0  0]
 [-1  1  0  0]
 [-1  0  1  0]]
```

```
Assignment=
 [[ 0  0  0  0]
 [-1  1  0  0]
 [-1  1  0  0]
 [-1  0  1  0]
 [ 0  0  0  0]
 [-1  1  0  0]
 [-1  0  1  0]]
```

```
Assignment=
[[ 0  0  0  0]
 [ 0  1  0 -1]
 [-1  1  1 -1]
 [-1  0  1  0]]
```

```
Assignment=
 [[ 0  0  0  0]
 [ 0  1  0 -1]
 [-1  1  1 -1]
 [-1  0  1  0]
 [ 0  0  0  0]
 [ 0  1  0 -1]
 [-1  0  1  0]]
```

Figure 6: I_4 to G_{456} extensions

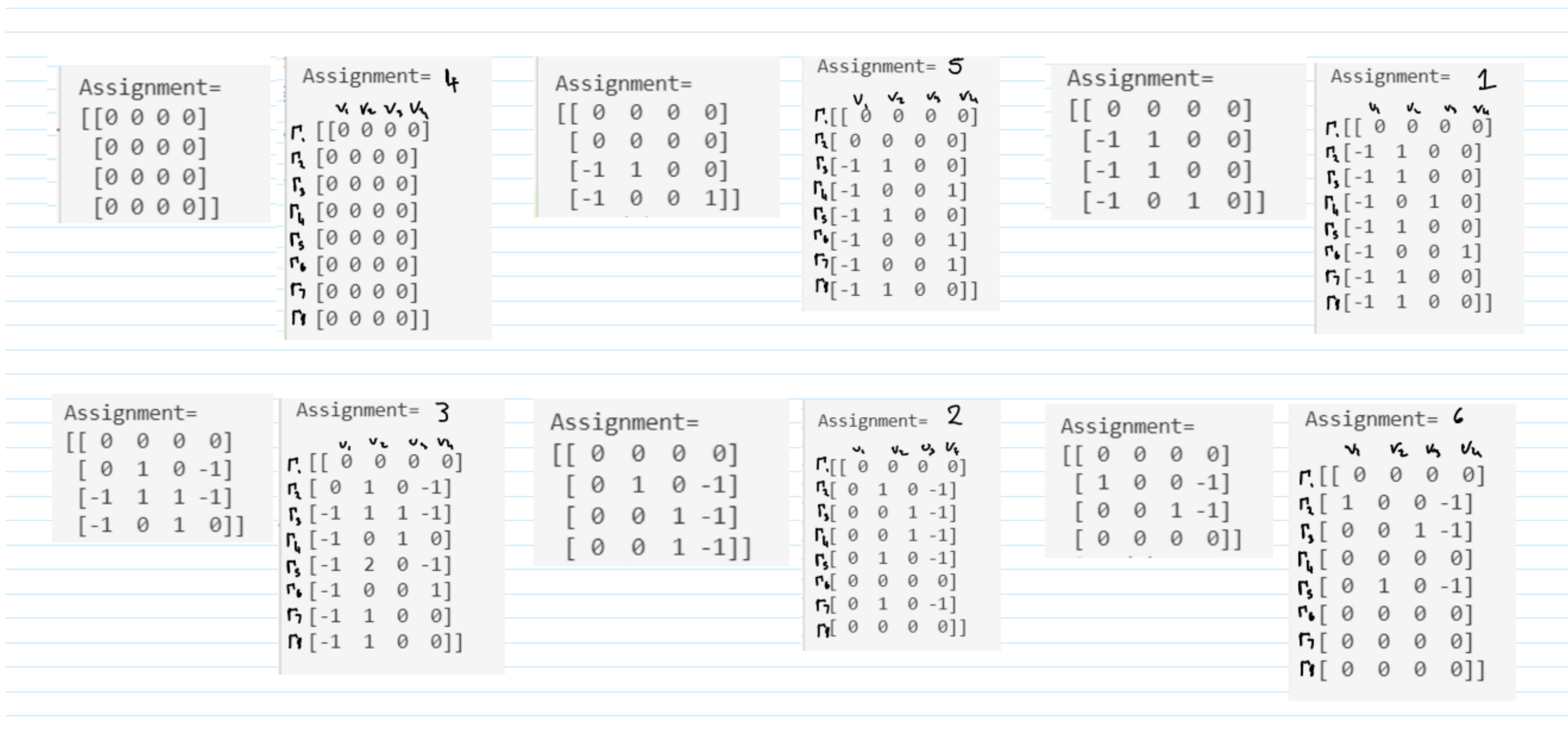


Figure 7: I_4 to G_{455} extensions

3 Graphs with 3 vertices

3.1 I_3

Example 3.1. We now obtain all weak stabilities, $\sigma(\underline{d}^{'})$, for I_3. Where $d_1^{'} = \underline{0} := (0,0,0) \in S^0(G)$. We aim to find ϕ satisfying the ϕ-inequalities for $\sigma(\underline{d}^{'})$.

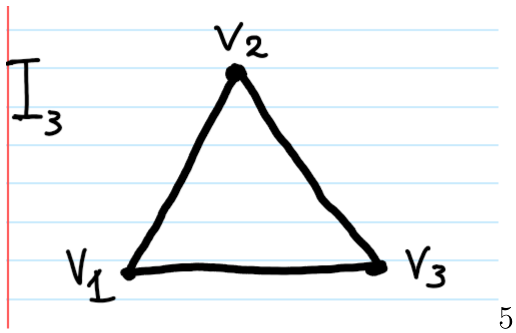


Figure 8: Labelled I_3

Figure 9: Ordered spanning trees of I_3

$$|d_1 - \phi_1| < 1, \quad |d_2 - \phi_2| < 1, \quad |d_3 - \phi_3| < 1.$$

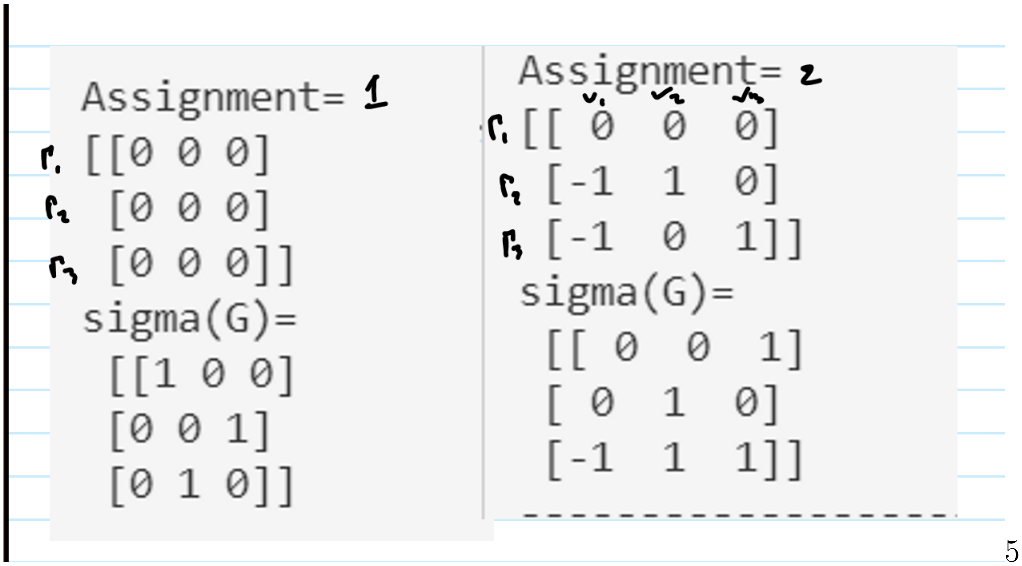
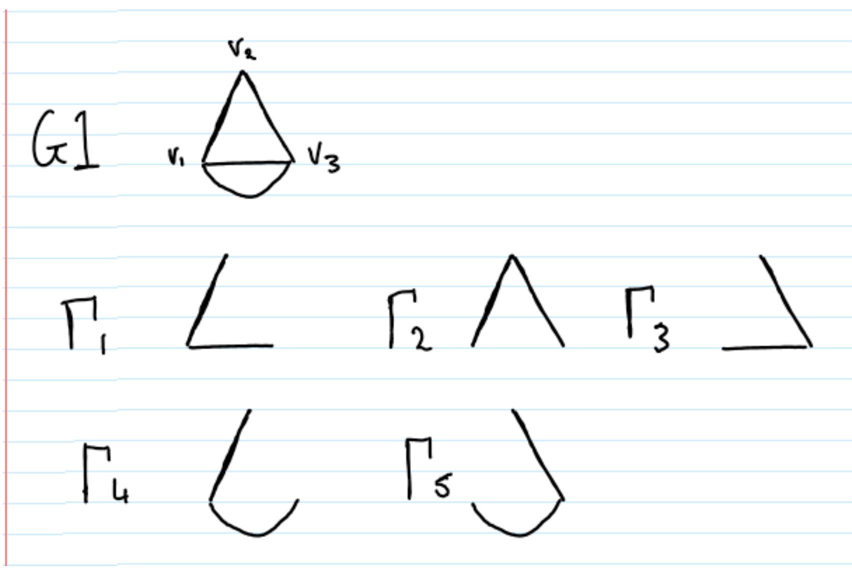


Figure 10: Assignments

3.2 $G1$

Example 3.2. Let $G := G1$ and fix $\underline{0} := (0,0,0) \in S^0(G)$ (allowed modulo translation). We will present the non-constant weak stabilities, $\sigma(\underline{d}')$, for G, where $d_1' = \underline{0}$. And for $\sigma(\underline{d}')$ we aim to find a ϕ satisfying the ϕ-inequalities.



The ϕ-inequalities for G are as follows.

$$|d_1 - \phi_1| < \frac{3}{2}, |d_2 - \phi_2| < 1, |d_3 - \phi_3| < \frac{3}{2}.$$

```
Assignment= 1
     v1 v2 v3
r1 [[0 0 0]
r2  [0 0 0]
r3  [0 0 0]
r4  [0 0 0]
r5  [0 0 0]]
 sigma(G)=
 [[1 0 1]
  [1 1 0]
  [0 0 2]
  [2 0 0]
  [0 1 1]]
 -------------------
```

```
Assignment= 2
     v1  v2  v3
r1 [[ 0   0   0]
r2  [-1   1   0]
r3  [-1   0   1]
r4  [ 0   0   0]
r5  [-1   0   1]]
 sigma(G)=
 [[-1  1  2]
  [ 1  0  1]
  [ 1  1  0]
  [ 0  0  2]
  [ 0  1  1]]
```

The elements of $\sigma(G)$ are pairwise chip-firing independent (checked in Sage).

3.3 Vertex 3 extensions of stability conditions

```
Assignment= 1
r1 [[0 0 0]
r2  [0 0 0]
r3  [0 0 0]]
```

```
Assignment=
[[0 0 0]
 [0 0 0]
 [0 0 0]
 [0 0 0]
 [0 0 0]]
```

```
Assignment= 2
     v1  v2  v3
r1 [[ 0   0   0]
r2  [-1   1   0]
r3  [-1   0   1]]
```

```
Assignment=
[[ 0  0  0]
 [-1  1  0]
 [-1  0  1]
 [ 0  0  0]
 [-1  0  1]]
```

Figure 11: I_3 to $G1$ extensions

4 Contraction on stability conditions

Given σ_{I_n} does contraction of an edge give $\sigma_{I_{n-1}}$? (where for subgraph Γ_0 of I_n, we have $\sigma_{I_n}(\Gamma_0)^{contraction}$ $= \sigma_{I_n}(\Gamma_0^{contract})$).

Note: For $\Gamma_0 \subseteq \Gamma$, we have σ_Γ, we also can consider σ_{Γ_0} and for $\Gamma_0' \subseteq \Gamma_0 \subseteq \Gamma$, $\sigma_{\Gamma_0}(\Gamma_0') = \sigma_\Gamma(\Gamma_0')$

5 Contraction: I_3 and I_4

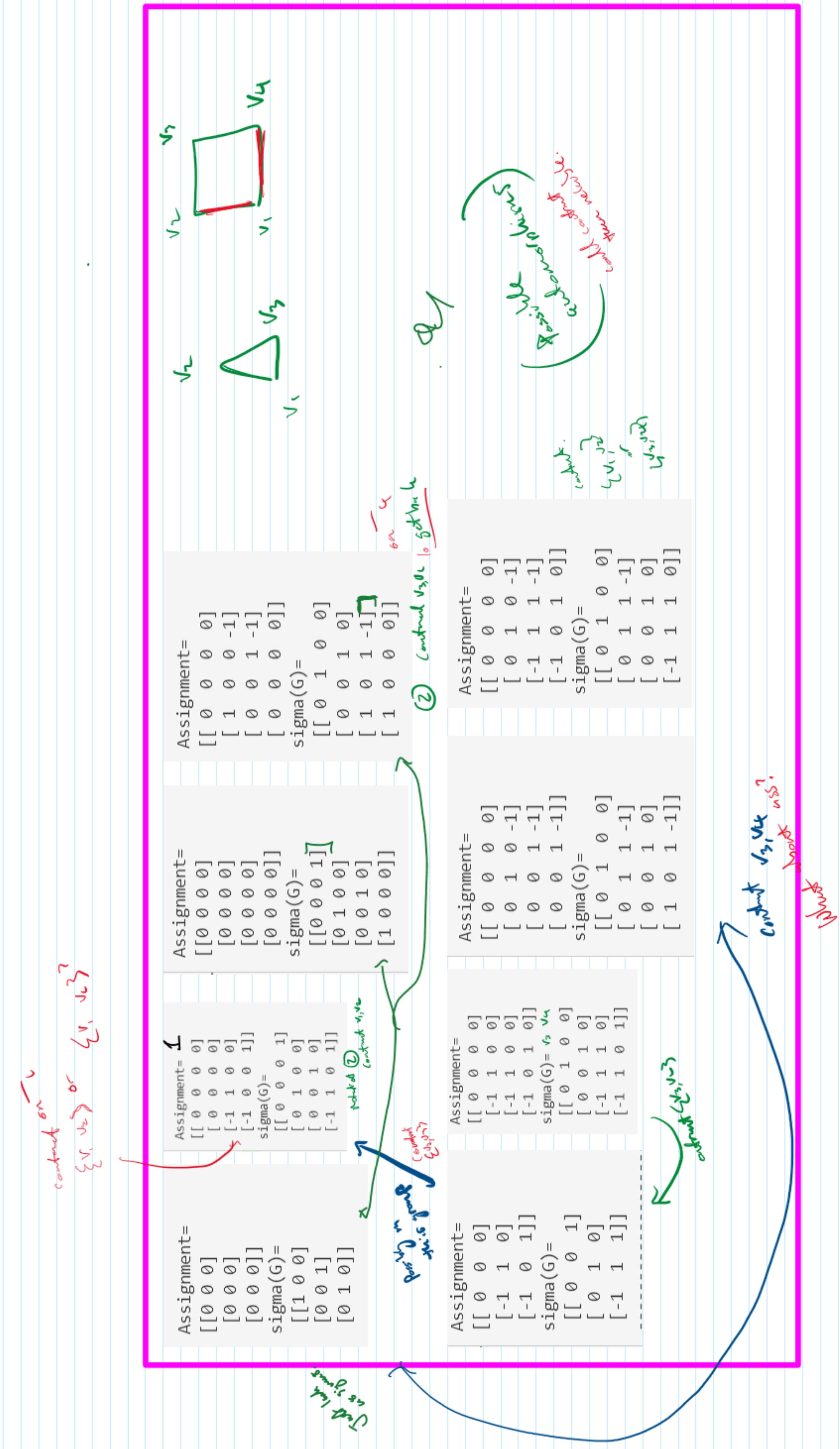


Figure 12: Extending from I_3 to I_4.