Stability Conditions in Genus 2

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1 Introduction

Given all the mildly superadditive functions f^2 on n indices, with given values on the singletons, we want to obtain all the compatible f^3s, and the polytopes they define.

The idea is that, given f^2, a compatible function f^3 is obtained in the same way as extending f^2 to $\mathcal{P}^+_{n+1}$ and imposing an extra symmetry condition.

2 The functions f^2 and f^3

We would like to extend the bijection that exists in genus 1 between universal weak stability conditions for $\overline{\mathcal{J}}^d_{1,n}$ and mildly superadditive functions $f : \mathcal{P}^+_{n-1} \to \mathbb{Z}$ with fixed $f(i), i \in \{1, \ldots, n-1\}$. In the genus 2 case, we use pairs of functions (f^2, f^3), that encode the admissible multidegrees of line bundles of the Jacobians of vine curves with 2 and 3 edges respectively.

Remark. There still needs to be proven that, given a pair (f^2, f^3), its restriction to a single curve C defines a full set of representatives for line bundle multidegrees.

We know that a universal ϕ-stability always gives a pair (f^2, f^3).

Let us give the relevant definitions:

Definition 2.1. A *mildly superadditive function* $f^2 : \mathcal{P}^+_n \to \mathbb{Z}$ is given by the condition:

Let $A, B \subset [n]$ be disjoint and non empty, then

$$f^2(A) + f^2(B) \leq f^2(A \cup B) \leq f^2(A) + f^2(B) + 1. \tag{1}$$

Definition 2.2. Let $A, B \subset [n]$ be disjoint and such that $B \neq \emptyset$. Given f^2 as above, $f^3 : \mathcal{P}_n \to \mathbb{Z}$ is said to be *compatible with* f^2 if it satisfies the following conditions:

$$f^3(A) + f^2(B) \leq f^3(A \cup B) \leq f^3(A) + f^2(B) + 1; \tag{2}$$

$$f^3(A^c) = -2 - f^3(A). \tag{3}$$

3 Obtaining the stability conditions

In order to adapt Rhys' algorithm for computing the MSA functions in genus 1, we need to express condition (2) in a more suitable form.

Definition 3.1. Let $I \subset \mathcal{P}_n$ and $g : \mathcal{P}_n \to \mathbb{Z}$, we define an *indicator function* to be

$$\varepsilon_I(A) = \begin{cases} g(\emptyset) & \text{if } A \notin I, \\ g(\emptyset) + 1 & \text{if } A \in I. \end{cases}$$

Let f^2 be a MSA. An indicator function is said to be *compatible with* f^2 if it satisfies the condtions of lemma 1.1 in Rhys' notes, that is: for each $A \subsetneq B \in \mathcal{P}_n^+$

- If $A \subsetneq B$ is f^2-minimal and $\varepsilon_I(A) = g(\emptyset) + 1$,then $\varepsilon_I(B) = g(\emptyset) + 1$;
- If $A \subsetneq B$ is f^2-maximal and $\varepsilon_I(B) = g(\emptyset) + 1$,then $\varepsilon_I(A) = g(\emptyset) + 1$.

Proposition 3.1. *Let f^2 be a mildly superadditive function. A function $g : \mathcal{P}_n \to \mathbb{Z}$ satisfies condition* (2) *with respect to f^2 if and only if there exists $I \subset \mathcal{P}_n$ such that ε_I is compatible with f^2 and*

$$g = f^2 + \varepsilon_I. \tag{2'}$$

Proof. Let g satisfy (2). Then, in particular, $f^2(B) + g(\emptyset) \leq g(B) \leq f^2(B) + g(\emptyset) + 1$ for each $B \in \mathcal{P}_n^+$. Define $I \subset \mathcal{P}_n$ such that $B \in I \iff g(B) = f^2(B) + g(\emptyset) + 1$. Then

$$g = f^2 + \varepsilon_I.$$

Let $A \subsetneq B$, then, by condition (2),

$$0 \leq (f^2(B) - f^2(A) - f^2(B \setminus A)) + (\varepsilon_I(B) - \varepsilon_I(A)) \leq 1,$$

which implies that ε_I is compatible with f^2.

Viceversa, let $g = f^2 + \varepsilon_I$ satisfy (2') and let $A \subsetneq B$.

$$g(B) - g(A) - f^2(B \setminus A) = \begin{cases} \varepsilon_I(B) - \varepsilon_I(A), & \text{if } A \subsetneq B \text{ is minimal}, \\ \varepsilon_I(B) - \varepsilon_I(A) + 1, & \text{if } A \subsetneq B \text{ is maximal}. \end{cases}$$

By compatibility of ε_I this is always contained in $\{0, 1\}$, so g satisfies condition (2).

Now, Rhys' algorithm finds all the functions g that satisfy condition (2'). Therefore, it is sufficient to select the ones that additionally satisfy condition (3), to obtain all the pairs (f^2, f^3) we are looking for.

Moreover, we have the following:

Definition 3.2. Let (f^2, f^3) be a pair of functions as above. The *stability polytope* determined by (f^2, f^3) is given my the system of inequalities

$$\begin{cases} f^2(A) < x_A < f^2(A) + 1, & \forall A \in \mathcal{P}_n^+, \\ 1 + 2f^3(A) < x_A - x_{A^c} < 3 + 2f^3(A), & \forall A \in \mathcal{P}_n, \end{cases}$$

where $x_A = \sum_{i \in A} x_i$, with $\{x_1, \ldots, x_n\}$ coordinates in $\mathbb{R}^n$.

If we manage to find a pair (f^2, f^3) such that the stability polytope is empty, we have a counterexample that shows that not all the universal stability conditions are ϕ-stabilities.

3.1 The algorithm

The idea of the algorithm is the following:

- Generate all the f^2s with domain $\mathcal{P}_n^+$ and starting conditions $\{f^2(\{i\}) = 0\}_{1\leq i\leq n}$ and $\{f^2(\{1\}) = 1\} \cup \{f^2(\{i\}) = 0\}_{2\leq i\leq n}$. (This is trivial for $n = 1$).
- Extend them to all the f^2s with domain $\mathcal{P}_{n+1}^+$ via the old algorithm. These are the same as the gs with domain $\mathcal{P}_n$, translated by $g(\emptyset)$, which satisfy condition (2).
- Impose the extra condition (3), to obtain all the pairs (f^2, f^3) on n indices.
- Check if the polytopes are empty.