Supplementary Material (AMA3020): Stiff Differential Equations

rwalsh21

February 2026

1 Purpose of this document

This supplementary document contains supporting material for the main report, including: (i) a self-contained statement of the classical fourth-order Runge Kutta (RK4) scheme, (ii) the derivation of the RK4 amplification factor for the linear test equation $\dot{x} = \lambda x$,

2 Classical RK4 scheme

Runge Kutta methods provide accurate numerical solutions of ordinary differential equations without requiring higher derivatives of the solution. The main idea is to approximate the change in the solution over one step by sampling the slope $f(t, x)$ at several points inside the step and combining these slopes in a weighted average. This yields higher accuracy than using a single slope (as in Euler's method), while still only requiring evaluations of the right-hand side $f(t, x)$ [1, 2].

Consider the initial value problem

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0, \tag{1}$$

and let $x_n \approx x(t_n)$ with $t_{n+1} = t_n + h$. The classical fourth-order Runge–Kutta method (RK4) computes four slope estimates

$$\begin{aligned} k_1 &= f(t_n, x_n), \\ k_2 &= f\left(t_n + \frac{h}{2},\, x_n + \frac{h}{2}k_1\right), \\ k_3 &= f\left(t_n + \frac{h}{2},\, x_n + \frac{h}{2}k_2\right), \\ k_4 &= f(t_n + h,\, x_n + hk_3), \\ x_{n+1} &= x_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right). \end{aligned} \tag{2}$$

The interpretation is straightforward [1, 2],
k_1 is the slope at the start of the step, k_2 and k_3 sample slopes at midpoint predictions, and k_4 is the slope at the end of the step. The weighted average cancels lower-order errors; RK4 has local truncation error $O(h^5)$ and global error $O(h^4)$.

3 Amplification factor for $\dot{x} = \lambda x$

A standard way to analyse stability is to apply a numerical method to the linear test equation [1, 5]

$$\dot{x} = \lambda x, \qquad x(0) = x_0, \tag{3}$$

where $\lambda \in \mathbb{C}$ is a constant. The exact solution is [3]

$$x(t) = e^{\lambda t} x_0,$$

so across a step of length h the exact update multiplies the current value by $e^{\lambda h}$ [1]. It is therefore natural to introduce the nondimensional step parameter [4, 5]

$$z = \lambda h, \tag{4}$$

which combines the problem parameter λ and the chosen step size h into a single quantity.

Applying RK4 to Eq. (3) produces a linear update of the form [1, 2]

$$x(h) = R(z)\, x_0, \tag{5}$$

where $R(z)$ is called the *amplification factor* because it tells how the numerical solution is multiplied across the step [5]. For the test equation, the RK4 stages are [1, 2]

$$k_1 = \lambda x_0, \qquad k_2 = \lambda \left(x_0 + \frac{h}{2} k_1 \right), \qquad k_3 = \lambda \left(x_0 + \frac{h}{2} k_2 \right), \qquad k_4 = \lambda \left(x_0 + h k_3 \right),$$

and substituting these into the RK4 update rule (2) gives the explicit polynomial [1, 2]

$$R(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}. \tag{6}$$

This is the degree 4 Taylor polynomial of e^z, so RK4 can be interpreted as replacing the exact step multiplier e^z with the polynomial multiplier $R(z)$ [1, 4].

3.1 Stability condition

If $\mathrm{Re}(\lambda) < 0$ then the exact solution decays in magnitude, and we want the numerical update to avoid amplifying that decaying mode. A common stability requirement is therefore [5, 4]

$$|R(z)| \leq 1. \tag{7}$$

Along the negative real axis ($z \in \mathbb{R}$, $z \leq 0$), RK4 satisfies (7) only for approximately $z \in [-2.7853, 0]$ [1, 3]. This gives the practical bound used in the main report:

$$h \lesssim \frac{2.7853}{|\lambda|}.$$

References

[1] R. L. Burden and J. D. Faires, *Numerical Analysis*, Cengage Learning.

[2] D. Dundas, AMA3020 lecture notes / handouts on numerical solution of ODEs.

[3] W. E. Boyce, R. C. DiPrima, and D. B. Meade, *Elementary Differential Equations and Boundary Value Problems*, Wiley.

[4] E. Hairer and G. Wanner, *Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems*, Springer.

[5] J. D. Lambert, *Numerical Methods for Ordinary Differential Systems*, Wiley.