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ERNST CASSIRER'S *SUBSTANZBEGRIFF UND FUNKTIONSBEGRIFF*

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Ernst Cassirer's book *Substanzbegriff und Funktionsbegriff* is a difficult book for contemporary readers to understand. Its topic, the theory of concept formation, engages with debates and authors that are largely unknown today. And its "historical" style violates the philosophical standards of clarity first propounded by early analytic philosophers. Cassirer, for instance, never says explicitly what he means by "substance-concept" and "function-concept." In this article, I answer three questions: Why did Cassirer choose to focus on the topic of concept formation? What did Cassirer mean in contrasting "substance-concepts" and "function-concepts"? How does Cassirer's polemic against traditional theories of concept formation lead to the distinctive philosophy of mathematics that he defends in the book? I argue that Cassirer's contrast between substance-concepts and function-concepts includes a series of interrelated contrasts—contrasts that touch on issues in logic, metaphysics, epistemology, and the theory of objectivity. Cassirer's defense of mathematical structuralism flows out of a progressively unfolding and intricate argument that begins with epistemological problems in the theory of concept formation.

1. Introduction

Ernst Cassirer's *Substanzbegriff und Funktionsbegriff*, written in 1910/1923 and translated misleadingly as *Substance and Function*, gives off an odd impression

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to a contemporary reader. Though the book is rich with claims that are highly suggestive and even prescient, many of its central arguments and concepts appear elusive and unclear. A chief explanation for this impression is the historical style in which *Substanzbegriff und Funktionsbegriff* is written. Intentionally blurring the line between history of philosophy and systematic philosophy, Cassirer often argues for a philosophical position by recounting the history of philosophical and scientific work on the topic, shaping the narrative so that his preferred view emerges as the culmination of that history. Cassirer quotes freely from historical figures—even figures with whom he disagrees—and will shift from using one thinker's vocabulary to another thinker's quite different vocabulary as he works his way through the stages of his narrative. Wishing to highlight the historical ancestry of his views, Cassirer will often reuse a philosopher or scientist's technical language in his own way.

This style of argumentation is at odds with the quite different style articulated by the early analytic philosophers contemporary with Cassirer. Despite Reichenbach's intellectual affinity with Cassirer (who was, after all, his teacher in Berlin), Reichenbach famously rejected the historical style employed in Cassirer's work and advocated instead that philosophy give a mathematical "logical analysis of science"—an analysis that uses mathematical techniques to discern the epistemological structure of a scientific theory (Reichenbach 1924/1969, xii). Reichenbach's employment of mathematical argumentation and his adoption of the style of the mathematical sciences as a philosophical paradigm marked a clear break with Cassirer and was surely one of the seminal events in the emergence of analytic philosophy from the more mainstream German academic philosophy represented by Cassirer. Although Cassirer talks about mathematics and mathematicians at great length in his book, he does not use mathematical methods in prosecuting his argument, as Reichenbach did in *Axiomatization of the Theory of Relativity* or as Carnap did in the *Aufbau*. And he does not, as Frege did earlier and the logical empiricists did later, try to import mathematical standards of clarity and rigorous argumentation into his writings. One looks in vain for an explicit explanation of the meaning of many of his key terms—including the words "substance-concept" (*Substanzbegriff*) and "function-concept" (*Funktionsbegriff*) themselves. In his famous review of Cassirer's book *Einstein's Theory of Relativity*, Schlick, with some understandable frustration, feared that there was no clear and nontrivial thesis that Cassirer was propounding and that his arguments—inasmuch as they were meant to support substantive and contentious conclusions—were simply straw man arguments (Schlick 1921/1979). Indeed, one might wonder whether the same charges could be leveled against *Substanzbegriff und Funktionsbegriff* as well.

The difficulties in understanding the book begin with the most fundamental question: What is the book about? In the opening of the "Preface," Cassirer explains:

> The investigations contained in this volume were first prompted by studies in the philosophy of mathematics. In the course of an attempt to comprehend the fundamental concepts of mathematics from the point of view of logic, it became necessary to analyse more closely the function of the concept itself and to trace it back to its presuppositions. Here, however, a peculiar difficulty arose: the traditional logic of the concept, in its well-known features, proved inadequate even to *characterize* completely the problems to which the theory of the principles of mathematics led. It became increasingly evident that exact science had here reached questions for which there existed no precise correlate in the formal language of traditional logic. The material content of mathematical knowledge pointed back to a fundamental form of the concept not clearly characterized and recognized within logic. [These investigations] . . . led to a renewed analysis of the principles of concept formation [*Begriffsbildung*] itself. (Cassirer, 1910/1923, iii)[1]

To readers familiar with the development of analytic philosophy, this sounds like an allusion to an argument propounded by Russell in *Principles of Mathematics* seven years earlier in 1903. Russell argues that the development of mathematics has made it clear that its subject matter includes not only number and magnitude but any system of relations. Traditional logic is wholly inadequate for even characterizing the subject matter of modern mathematics, though: "Some of the most important basic judgments of mathematics do not fit into the traditional predication model, since judgments involving *asymmetric* relations (such as arithmetic's *x is the successor of y*) are not reducible to judgments of the form *S* is *P*" (Russell 1903, secs. 214–15).

Moreover, the early chapters of *Substanzbegriff und Funktionsbegriff* are clearly a further development of the ideas Cassirer gives in his 1907 paper, "Kant und die moderne Mathematik," which is a review of Russell's *Principles*. There he writes:

1. Throughout this article, I will adopt the following practices. If there is a translation listed in the references, I will quote from the listed translation (except for occasional corrections that I make silently). Page citations are from the listed translations unless otherwise noted. Translations from works not in English for which there is no English translation in the list of references are all my own.

> It is, it appears to me, in fact a new and fruitful point of view, which is introduced by Russell in his treatment of formal logic. The entire "classical" logic has concerned itself with nothing but the subsumption of contents, with the super- and sub-ordination of the spheres of two concepts. . . . Syllogistic appears overall as a particularly reactionary and inhibiting moment. Logic remains bound to the point of view of substance and thereby to the fundamental form of the judgment of predication, while living scientific thought more clearly aims at the *concept of function* [*Funktionsbegriff*] as its own systematic middle point. One recognizes in this connection the value and necessity of the new foundation on which Russell is seeking to place logic. (Cassirer 1907, 7)

Cassirer is here using the contrast between "substance" and "function" to cite characteristic logical and metaphysical theses. Russell showed that there is a branch of formal logic, the logic of relations, that is not reducible to syllogistic reasoning about sentences of the form "*S* is *P*" ("substance has attribute").[2] Russell also showed that there are relations among objects not reducible to nonrelational properties of their relata.[3] That Aristotle's logic was interdependent with a metaphysics of Aristotelian substances was again a point Russell himself had made in 1900 in his famous book on Leibniz.

However, a reading of *Substanzbegriff und Funktionsbegriff* in its entirety quickly makes it clear that Cassirer intends to cover much more with his contrast between "substance-concepts" and "function-concepts" than just these Russellian logical and metaphysical theses. In fact, this surface agreement with Russell covers over a deep disagreement. Cassirer regularly criticizes Russell in the book: for his particular brand of logicism, for his platonism, and for his views on acquaintance (1910/1923, 316; Heis 2010). In particular, Cassirer defends what is often called "structuralism" in the philosophy of mathematics.[4] The basic idea, rooted in the mathematician Richard Dedekind's work, is that mathematical objects are just positions in structures: that is, all of the essential properties of, say, a particular natural number are irreducible relational properties between it and the other natural numbers. Although Cassirer, again

2. For similar logical uses of "substance" and "function," see also Cassirer (1910/1923, 37, 71; 1921/1923, 389).

3. See also Cassirer (1907, 6; 1910/1923, 8, 56) for similarly metaphysical uses of "substance" and "function."

4. A clear expression is: "What is here [in Dedekind's work] expressed is just this: that there is a system of ideal objects whose content is exhausted in their mutual relations. The 'essence' of the numbers is completely expressed in their positions" (1910/1923, 39). On the structuralist features of Cassirer's thought, see Gower (2000) and Heis (2011, sec. 3).

following Dedekind, sees this "structuralism" as a kind of logicism, it is nevertheless—as Cassirer very strongly emphasizes—opposed to Russell's and Frege's view (1910/1923, 44–54). In fact, Russell himself had singled out Dedekind's view of the numbers for explicit criticism.[5] What, then, does Cassirer mean when he says that modern mathematics calls into question key features of the traditional logic? How are these failures of the traditional logic (cataloged in *Substance and Function*, chap. 1) supposed to lead to the philosophy of mathematics inspired by Dedekind (sketched at the very beginning of chap. 2)? And perhaps most pressing: What precisely then does Cassirer intend in contrasting *Substanzbegriff* with *Funktionsbegriff* if not a contrast between the old logic and the new logic of relations?

A starting point for an answer to these questions is given in the first paragraph from Cassirer's "Preface," quoted earlier. There, Cassirer says that the precise target of his attack on the traditional logic is its theory of concept formation (*Begriffsbildung*). This was not a topic that Russell discussed explicitly in his logical writings;[6] now we do not usually consider the topic a part of logic. This is partly a reflection of the fact that Cassirer's conception of logic is broader than ours, including topics that fall outside of the boundaries of "formal" logic. The subtitle of the book, after all, is *Investigations into the Fundamental Questions of the Critique of Knowledge*.

This subtitle indicates the Kantian elements in Cassirer's work, in particular the Kantian idea of "transcendental logic," at least as this was interpreted by Cassirer's Marburg teachers Hermann Cohen and Paul Natorp.[7] According to this fundamental idea of Marburg neo-Kantianism, philosophy's starting point is our current best mathematical natural science—not psychology or a metaphysics arrived at independently of reflecting on science. Philosophy takes this science as given: it does not try to justify science or revise it but, rather, seeks to isolate the conditions of its possibility.[8] This broader notion of "logic" characteristic of Marburg neo-Kantianism explains why Cassirer's book covers topics that we would not find in Russell's logical writings. However, although Cassirer's neo-Kantian background explains the wide scope of Cassirer's work, it does not explain the particular philosophical positions he defends. In Hermann

5. Russell (1903, sec. 242). I discuss this passage in sec. 3 below.

6. Russell does discuss the theory of concept formation later in Russell (1912, chap. 10). There he argues that certain basic universals (sensible qualities) are derived by abstraction from the particulars with which we are immediately acquainted. Significantly, this is the very theory that Cassirer is so keen to attack in *Substanzbegriff und Funktionsbegriff*. My point here, though, is that the topic does not appear in Russell's logical writings, but only in the writings in the theory of knowledge that Russell turned to after completing *Principia Mathematica*.

7. See Heis (2010) on Cassirer's conception of logic.

8. On the transcendental method, see Cassirer (1912), Natorp (1912), Richardson (2006).

Cohen's own book on logic, *Logik der reinen Erkenntnis* (1902), Cohen insists that the philosophically fundamental mathematical notion—and thus the fundamental notion of transcendental logic that explains the possibility of mathematical natural science—is the notion of an infinitesimal (Cohen 1902, 28–34, 102ff; cf. Cohen 1883/1984). For this reason, Cohen worried that Cassirer's book—privileging as it did the entirely abstract notion of a relation from Russell's logic and defending a structuralist account of mathematical objects—was a departure from Marburg philosophy of mathematics.[9] In fact, as I shall argue, this surface level disagreement with Cohen covers over a deeper level of agreement.

The broad goal of the article is to resolve, at least partially, these mysterious features of Cassirer's book. I begin by addressing in the first two sections the two most basic (and I think most pressing) interpretive questions facing a contemporary reader of *Substance and Function*: Why does Cassirer choose as his central topic the theory of concept formation? What are "substance-concepts" and "function-concepts"? My conclusions will be the following. Cassirer's choice of *Begriffsbildung* was an ingenious and dialectically subtle way to pull together many seemingly unrelated concerns of Cassirer's philosophy. His central distinction between substance-concepts and function-concepts is multifaceted and breaks up into a series of interrelated contrasts between logical, metaphysical, and epistemological theses. Comprehending these contrasts under one umbrella as the contrast between "*Substanzbegriff*" and "*Funktionsbegriff*" allows Cassirer to present his philosophy as the coming together (in a rather complicated way) of distinct lines of reasoning from Russell, Kant, and nineteenth-century German philosopher-logicians.

Substanzbegriff und Funktionsbegriff is a complicated and wide-ranging book. It would be a fool's errand to try to cover all of its themes in one article. But precisely because of this wide range (and the historical style of exposition that Cassirer employs), it is not always easy to see how the various particular positions that Cassirer defends flow out of the main argument of the book—the contrast between "*Substanzbegriff*" and "*Funktionsbegriff*." As a representative illustration—an example of studying a tree without losing the forest—

9. For the history of Cohen's reaction, see Gawronksy (1949, 20–21). The August 24, 1910, letter from Cohen that spells out his reservations to Cassirer is reproduced in the recently released CD-ROM that accompanies volume 18 of Cassirer's *Nachgelassene Manuskripte und Texte* (Cassirer 2009). (I would like to thank Massimo Ferrari for pointing me to these sources.) Since Cassirer does not actually discuss infinitesimals or the foundations of analysis in *Substanzbegriff und Funktionsbegriff*, I will not address in this article the relation between Cassirer's philosophy of mathematics and Cohen's treatment of infinitesimals. Instead, I will focus on the relation between Marburg neo-Kantianism and Cassirer's anti-abstractionism. A helpful recent paper that, like this article, reads *Substanzbegriff und Funktionsbegriff* as continuous with classic Marburg neo-Kantianism is Ferrari (2010).

I explain in section 3 how Cassirer uses the philosophical tools I lay out in the first two sections of the article to argue for Dedekind-style structuralism. My goal throughout is to isolate the main ideas in *Substance and Function* and to show how they interconnect in an interesting and philosophically rich way. For this reason, the reader will have to forgive the many occasions where, inevitably, I pass quickly over ideas and arguments that would otherwise deserve more extended treatment.

2. Why Concept Formation?

Although it might seem odd to modern readers to begin a work in the philosophy of mathematics and science with a discussion of the nature and origin of concepts, as Cassirer does in chapter 1 of *Substanzbegriff und Funktionsbegriff*, this would not have been so strange to Cassirer's audience. The theory of concept formation was traditionally covered in the doctrine of concepts, the first topic discussed in logic texts. Cassirer understood the question "How are concepts formed?" broadly, as a question in scientific methodology—What do scientists do? What are the methods that scientists use to form successful concepts (1910/1923, 26)? For him, it is also a question about the justification of scientific practices: What justifies the formation of new concepts? What constrains it? Indeed, the central question of modern mathematics can be put this way: what justifies mathematics' newfound freedom—as Cantor put it—to form concepts (Cassirer 1907, 47)?

Cassirer was certainly not alone in interpreting the traditional logical question of concept formation in this broad way; by the late nineteenth century such wide-ranging discussions were common in logic texts. For instance, in Wilhelm Wundt's *Logik*, in addition to a discussion of the traditional doctrine of concept formation by abstraction, we find detailed analyses of the ways in which particular sciences form concepts—including 170 pages on mathematical methodology, with detailed discussions of the geometrical principle of duality, Steiner's definition of a conic, the so-called principle of the permanence of form, and histories of the concepts of a mathematical function, and of the differential.[10] An original theory of concept formation in the exact sciences was also given by Benno Erdmann.[11] According to Erdmann, concepts are

10. On abstraction, see Wundt (1880–83/1906–8, vol. 1, chap. 2.1). On concept formation in mathematics, see vol. 2, pt. 2 (74–219 in the 1st ed. [1883]; 101–268 in the 3rd ed. [1907]).

11. Erdmann (1892/1907). The psychology of concept formation by abstraction is presented on 65–92 of the 2nd ed.; the second-level concepts are discussed on 158–75. See also Cassirer's discussion of Erdmann's theory (1910/1923, 23). On the relation of *Substanzbegriff und Funktionsbegriff* to the late nineteenth-century German logic textbook tradition, see Ihmig (1997, 273–80).

formed either in the traditional way from abstraction or by unrestricted set-theoretic comprehension on prior concepts. He argues that this theory of concept formation explains the formation of the concept of transfinite numbers, continuity, Dedekind's *Zahlkörper*, and spaces of *n* dimensions.

It is in this spirit that Cassirer approaches the question of concept formation in mathematics and the mathematical natural sciences. His discussion begins with an extended polemic against the traditional theory of concepts, which derived from Aristotle. Cassirer sees it as containing two interrelated parts. On the traditional theory of *conceptual structure*, concepts are either simple or are composed of simple concepts by conjunction, addition, or exclusion. A corollary of this view of conceptual structure is that there is no need for an independent logic of relations, since conceptual relations in judgments can be reduced to predicating a predicate concept to a subject concept and all valid forms of inference can be reduced to syllogistic (1910/1923, 8; Cassirer 1907, 7). On the traditional theory of *concept formation*, concepts are formed by noticing similarities or differences among particulars and abstracting the concept, as the common element, from these similarities or differences (1910/1923, 5).

From a contemporary perspective, this doctrine seems to paste together a properly logical thesis about formal structure with a quite different epistemological thesis about the origin of conceptual content. But these two parts are not independent, and in the tradition they were thought to be two sides of one coin. On this traditional model, all concepts can be placed onto a Porphyrian genus/species tree. These conceptual structures can be characterized using only conjunction, disjunction, and negation. Specification of a genus can be expressed as the conjunction of a genus and differentia, and the extension of a genus is disjoined into the extensions of its various species. Since the species of a genus are disjoint, each is then the negation of the rest.[12] As a species is formed by adding a differentia to a genus, so too is a concept formed by abstracting away the differentia from two species concepts to leave behind the genus as the common element. Within this model of conceptual structure, then, concept formation by abstraction is just the inverse operation to specification (1910/1923, 5).

Since theories of conceptual structure were intertwined with theories of concept formation in the tradition, Cassirer's discussion of concept formation allows him to consider, as Frege and Russell had done, the expressive and in-

12. Because of the intimate relation between Boolean operations on class concepts and the traditional theory of conceptual structure, Cassirer considered Boolean logic (without a theory of relations) to be just as open to his anti-abstractionist polemics as the traditional Aristotelian logic (Cassirer 1907, 5–6; 2010, 15–16).

ferential limitations of traditional syllogistic as compared to the new logic of relations. But it also allows him to discuss methodological questions within scientific and mathematical practice that were offstage in Russell's writings. What's more—as I will show by the end of my article—a chief conclusion of Cassirer's polemic against abstractionism is that methodological issues within mathematics (and science) play a much larger role in questions of mathematical (and scientific) truth and existence than one might have thought.

Cassirer's main argument in the first chapters of the book is that the traditional model does not accord with the function and structure of concepts as they are found in the exact sciences of his day: he asks, "Is the theory of the concept, as here developed, an adequate and faithful picture of the procedure of the concrete sciences?" (1910/1923, 11).

Cassirer sees serious difficulties with the traditional model from the outset. When we form a concept by abstraction from our knowledge of a particular, we remove the elements that are not common with other particulars and thereby reduce the content of our knowledge of the particular. How could reducing the content of our knowledge be a cognitive achievement (1910/1923, 18)? This was a familiar argument among nineteenth-century logicians. There were two solutions extant in the literature at the end of the nineteenth century. The first, due to Friedrich Trendelenburg, argued that the theory of concept formation needed to be supplemented by an Aristotelian metaphysics of substantial forms. Abstraction then removes the accidental features in our representations of various particulars in order to isolate their essences, and surely the identification of essences is a cognitive achievement.[13] The second approach, due principally to Hermann Lotze, argued that—if the structure of the concept is rich enough—the formation of a successful concept can allow us to retain all our knowledge of particulars. Lotze's example is a curve in analytic geometry. If we first know all of the points that the curve occupies in space and then ascend to a general formula—that is, a mathematical function describing the curve—we retain all of our particular knowledge, since all of the positions of the curve in space can be derived from the functional formula itself. So given the right kind of concepts, concept formation does not im-

13. See Cassirer (2010, 3; 1910/1923, 6–7). Of course there needs also to be a story about why the procedure *compare, reflect, abstract* hits on essences and not unimportant common features. Again, the Aristotelian metaphysics will fill in the gap: perception is a causal relation between the object and the mind; in this causal interaction, the substantial forms of perceived objects are transferred into the medium of the mind. See Cassirer (1906/1922, 382; 2010, 3; 1910/1923, 7–8). In each of these three works, Cassirer cites Trendelenburg. See Trendelenburg (1870, 18–21, chap. 10). On Trendelenburg's philosophy of concept formation, see Heis (2012, sec. 4).

poverish our representation of particulars: it hits on mathematical laws from which we can rederive all of the properties of the various particulars that fall under the newly formed concept.[14]

This simple example is a paradigm for Cassirer. Instead of supplementing a logical doctrine with metaphysics, observe the actual concepts that are employed in our most successful exact science. And the example of mathematical functions contradicts the traditional theory of conceptual structure, since—as Russell showed definitively—mathematical functions can, in general, be expressed only using the new logic of relations.

When Cassirer tests the traditional theory of concepts against the actual procedure of our best current science, he is applying the method of transcendental logic—first introduced by his teacher Cohen. According to this method, philosophy takes our best sciences as a "fact" and investigates the conditions of their possibility. However, the relationship between Cassirer's polemic against abstractionism and Marburg neo-Kantianism runs much deeper than the fact that Cassirer's investigation is an instantiation of the transcendental method.[15] The reason is that the abstractionist theory impinges on a characteristically Marburg neo-Kantian preoccupation: the distinction between the Kantian faculties of sensibility and understanding. Recall that for Kant sensibility is passive (or "receptive") and the understanding—the faculty of concepts—is active (or "spontaneous"). That all our concepts can be formed by abstraction from given representations requires that there be a strata of representations that are entirely independent of concepts that can be the material for abstraction. That is, there must be representations of sensibility whose epistemic efficacy is entirely independent of the understanding. Natorp and Cohen were famously hostile toward the idea that sensibility—the capacity for objects to be given to us through their affecting us—makes a contribution to our knowledge that is independent of the understanding (Friedman 2000, chap. 3; Kim 2003). Moreover, on the abstractive view, the concept <red> is derived from a prior awareness of red things *as red*. But this abstraction is a way of forming concepts for the first time only if the primitive awarenesses are themselves possible prior to the subject's possessing any concepts. Abstraction then requires a kind of epistemic atomism: that a subject could have, for example, an experience of a red thing *as red* without the subject knowing any other facts, possessing any other concepts,

14. Lotze (1880/1888). See also Cassirer's discussion of Lotze's solution (1910/1923, 7, 19–23). Lotze recognized that mathematical functions characterize concepts whose structures cannot be captured using the traditional conceptual forms of conjunction, disjunction, and exclusion of marks. I discuss Lotze's argument in Heis (2013).

15. As I explain below in sec. 4, there is in fact a deep interconnection between Cassirer's advocacy of the transcendental method and his rejection of abstractionism.

or having any other similar noninferential acts of awareness. But, as Cohen argued, there is not and cannot be something "given" (*gegeben*) in an experience (Cohen 1902, 24–5, 48–51).

Before writing *Substanzbegriff und Funktionsbegriff*, Cassirer argued at length in *Erkenntnisproblem*, volume 2, that Kant's philosophy definitively refuted abstractionism. Commenting on the transcendental deduction, Cassirer writes:

> If, according to the traditional logical doctrine, the concept is merely the result of "abstraction" from a plurality of sensory data, so has it now been shown that "similar" impressions must be placed under a determinate rule of judging, before they—as is necessary for the process of "abstraction"—can be cognized as similar and be comprehended in a common genus. The unity of a genus presupposes the unity of an ideal norm, and the abstractive comparison presupposes a constructive connection. In its proper fundamental meaning, a concept is nothing other than the consciousness of this unity of synthesis. (Cassirer 1907/1922, 676; cf. 667)

Cassirer here presents the positive argument of Kant's transcendental analytic as beginning by locating a gap in abstractionism: between the subject's having similar impressions and forming a concept from them, the subject must also cognize the impressions as similar. To arrive at an empirical concept, it is not enough that the objects sensed are similar, and it is not even enough that the sensations produced by those objects have similar sensory qualities. The subject must herself recognize the similarity. The failure to acknowledge this middle step in the process of concept formation, Cassirer argues, prevented pre-Kantian philosophers from seeing the necessity of two fundamental Kantian ideas. The first is that every intuition (e.g., the intuition of a red thing as *red*, as opposed to a mere impression of red) requires the application of a concept as the rule for ordering the manifold in that intuition into one representation. And so intuitions, which are the only plausible material basis for abstraction, already presuppose concepts.[16] This shows that not every concept can be formed by abstraction, and it suggests the possibility that there are some privileged concepts that are never formed by abstraction. This first idea thus opens the way for a second, that there are preconditions of experience—categories that

16. Cassirer quotes Kant, *Critique of Pure Reason*, A103—a concept is the "*one* consciousness that unifies the manifold that has been successively intuited, and then also reproduced, into one representation"—to make this point (1907/1922, 676). See also Cassirer (1910/1923, 16–17; 1929/1957, 315). Cassirer goes on to draw from these attacks on abstractionism the familiar Marburg conclusion that the "division between intuition and concept resolves itself ever more clearly into a purely logical correlation" (1907/1922, 698).

a subject must possess antecedently to any experience and principles that the subject must know prior to any experience. And so there are some concepts, the categories, that are presupposed in any cognitive representation—even in the awareness of a red thing *as red*—and so must already be in place prior to abstracting (Cassirer 1907/1922, 677, 698; 1918/1981, 167, 176).

In chapter 1 of *Substanzbegriff und Funktionsbegriff*, although Cassirer does not mention Kant by name, he repeats the argument, in his own voice, that the theory of abstraction presupposes (without explaining) the capacity to represent similar impressions as similar (15). To this argument, Cassirer adds a suite of supplementary arguments. In abstraction, we are told, we take some representations and abstract out the common *marks*; but this just presupposes that our representations already contain marks (17; Cassirer 1928/1993, 161). In abstraction, we take some given *grouping* of representations and abstract the common element; but this just presupposes that—before the process of comparing, reflecting, and abstracting even begins—we have a principle for grouping together just these representations (15; Cassirer 1923/1955, 278–79).

3. Substance-Concept and Function-Concept

The second interpretive question we posed concerned the meaning of Cassirer's key terms *Substanzbegriff* and *Funktionsbegriff*. Given Cassirer's historical method, we should not expect him to use his terms with the same meaning in every context but to vary the meanings depending on which historical tradition he is discussing or appropriating. (And we should remember that Cassirer was not being sloppy: his historical method of philosophizing put a premium on presenting his view as the culmination of distinct lines of reasoning within the history of philosophy.) Using the words "function" and "function-concept" in many ways advances his purpose, even if it would frustrate the purposes of a Frege or a Carnap. This is certainly true of the word "function" (*Funktion*).[17] In many instances, of course, he uses it in its ordinary mathematical meaning.[18] But the word takes on three other meanings depending on which tradition in the theory of concept formation and structure he is engaging with. First, when

17. Guido Kreis (2010, chap. 1.2) also acknowledges that Cassirer uses the word in different ways and draws on different historical sources in his use.

18. This is especially true in Cassirer (2010), a lecture delivered in Berlin in 1907 that first publicly put forward the main idea of the book that was to appear 3 years later. But in the lecture, Cassirer draws primarily on the pre-Kantian historical figures discussed in Cassirer (1906/1922, 1907/1922), such as Kepler and Galileo, and has not yet fully thematized the abstract notion of relation that becomes central in the philosophy of mathematics in *Substanzbegriff und Funktionsbegriff*. In *Substanzbegriff und Funktionsbegriff*, the distinctly mathematical notion of a function becomes less prominent.

speaking Russellian, Cassirer will use the word synonymously with "relation," since in the new logic a function can be defined as a one-one or many-one relation. So, for instance, he claims that Dedekind's *Was sind und was sollen die Zahlen?* (Dedekind 1888/1963) shows that the concept <natural number> is a "pure functional concept" (*Funktionalbegriff*), since the "presuppositions of the derivation of the concept of number are given in the general *logic of relations*" (1910/1923, 36; see also 1907, 7; 1910/1923, 265). Second, when Cassirer is investigating the methodology of the various exact sciences, he will make claims about the function, that is the "role" or "purpose," of particular mathematical or scientific concepts within this or that particular scientific field.[19]

The third use of "function" derives from Kant. For Kant, a "function" is a rule-governed activity of the mind. He attributes them exclusively to the understanding, the faculty of concepts. As active or "spontaneous," they are contrasted with the "affections" of sensibility, which are passive and "receptive" (Kant 1998, A68/B93). Thus, Cassirer will talk of "functions" when he is speaking of various mental or intellectual "activities" (*Tätigkeiten*) or "procedures." For instance, when Cassirer gives the objection to abstractionism that he attributes to Kant —that similar representations can give rise to a common concept only if the subject already judges them to be similar—he concludes that there are other "pure conscious functions" or "intellectual functions" that are at play in concept formation prior to the activities of comparison, reflection, and abstraction (1910/1923, 337, 16; cf. 14).

However, in his more careful moments, Cassirer claims that these "intellectual functions" are not really mental acts, because they are not psychological states at all. According to the particular, strongly antipsychologistic reading of Kant favored by Cassirer and the other Marburg neo-Kantians, philosophy is not directly concerned with "subjective 'representation [*Vorstellung*]'" or with the causal and temporal relations among the states of "particular thinking subjects." Rather, the goal is to identify "certain axioms and norms of scientific knowledge" on which the rest of our scientific knowledge is "logically dependent."[20] These "principles and universal laws" constitute the "general 'form' of

19. For example, Cassirer says that one of the fundamental thoughts of his book is that "what the concept is and means in its general function can only be shown by tracing this function through the most important fields of scientific investigation" (1910/1923, iv). See also Cassirer on the "functions" (or "roles") of irrational numbers in number theory and imaginary points in geometry (1910/1923 61, 83).

20. Cassirer (1910/1923, 298; cf. 315). Continuing this point on the same page, Cassirer writes: "The proposition, that being is a 'product' of thought, thus contains no reference to any physical or metaphysical causal relation, but signifies merely a purely functional relation, a relation of superordination and subordination in the validity of certain judgments." Here Cassirer is exploiting the semantic flexibility of the word "function" to make his point. An idealism like Cassirer's does not assert that objects

experience" (1910/1923, 268). Kant, on this reading, successfully identified the categories and principles as preconditions of Newtonian science. Marburg neo-Kantianism departs from Kant, however, in recognizing that in periods of scientific revolution, even these "intellectual functions" can be modified.

> Such principles as, for example, those on which Newton founds his mechanics, do not need to be taken as absolutely unchanging dogmas; they can rather be regarded as the temporarily simplest intellectual "hypotheses," by which we establish the unity of experience. We do not relinquish the content of these hypotheses, as long as any less sweeping variation, concerning a *derived* moment, can reestablish the harmony between theory and experiment. But if this way has been closed, criticism is directed back to the presuppositions [*Voraussetzungen*] themselves and to the demand for their reshaping. Here it is the "functional form" [*Funktionsform*] itself, that changes into another. (1910/1923, 268)

Here Cassirer speaks equivalently of "principles and universal laws," the "'form' of experience," "presuppositions," and "functional form." In this sense, then, the investigation into the "functions" of knowledge is really an investigation into the epistemic preconditions of our current best science.

Given these various uses of "function," it is not surprising, then, that there is no one, unambiguous contrast that Cassirer draws between "function-concept" (*Funktionsbegriff*) and "substance-concept" (*Substanzbegriff*).[21] In some cases, Cassirer speaks of function-concepts as a particular kind of a concept—a concept whose proper explication requires mathematical functions or the Russellian logic of relations. In section 3 of chapter 1, for instance, Cassirer repeats the argument from Lotze that the most successful concepts—like concepts expressible as mathematical functions—allow us to derive all of the particulars that fall under the concept, say, by substituting various constants for the variables in the function's analytic expression (see above, sec. 2). Cassirer calls such concepts "function-concepts" and contrasts them with "substance-concepts,"

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are dependent on "intellectual functions" as a product is dependent on an act of production. Rather, statements about objects are logically dependent on (and so "functions of") higher laws. This logical dependence is more like the dependence of a conclusion on its premises. (Though, as we will see, the dependence is actually semantic rather than inferential: these intellectual functions are conditions for the sentences of science having meaning at all.)

21. *Substanzbegriff* in German can just mean "the concept substance" ("the concept <substance>"). But as will become clear, Cassirer often uses it to mean "concept as understood by a philosophy that makes the notion of a substance central," as opposed to "concept as understood by a philosophy that makes the notion of a function central." Since that is a mouthful, I shall stick to the admitted Germanism, "substance-concept." For a similar point, see Ihmig (1997, 256).

whose proper explication only requires giving a genus and a differentia (1910/1923, 19–21).

However, the most significant contrasts that Cassirer draws are not between kinds of concepts, but between various opposing philosophical theories about concepts. Again, there is no single determinate contrast that Cassirer draws here, but he instead uses the opposing terms to pick out a series of interrelated contrasting philosophical views—some of which are logical, some of which are metaphysical, and the bulk of which are epistemological. As we saw earlier, Cassirer will use the contrast to advocate for the new logic over the old. According to the logic of substance-concepts, there is no independent logic of relations and logic is exhausted by syllogistic. The contrasting view, as it were the logic of function concepts, is the opposite view, given by Russell. Again, Cassirer, like Russell, associates this logical contrast with the contrast between the two answers to the metaphysical question whether every object is an Aristotelian substance—an object with no essential, irreducibly relational properties (Cassirer 1906/1922, 381–83).

The fundamental use that Cassirer makes of "*Substanzbegriff*" and "*Funktionsbegriff*" is, though, not Russellian, but Kantian: it contrasts philosophical views that overlook the epistemic preconditions of various kinds of knowledge with those that recognize the "functions" that make certain kinds of knowledge possible. Cassirer's argument for the point of view of *Funktionsbegriff* is then a progressive argument articulating the epistemic preconditions of various epistemic states, thereby opposing the epistemological atomisms of the point of view of *Substanzbegriff*. This argument begins in chapter 1 with the attacks on the abstractionist theory of concept formation mentioned in section 1 of this article. Abstractionism for Cassirer is a paradigm of *Substanzbegriff* thinking, and his other claims flow step by step out of this original polemic, moving from the theory of concept formation, to the role of sensations in perceptual knowledge, to confirmation in empirical science and the justification of scientific existence claims. In the rest of this section, I will present the main steps in this unfolding argument for a philosophy of *Funktionsbegriff*.

According to Cassirer's diagnosis, the appeal of concept formation atomism derives from the similarly atomistic view that sensations are the "self-evident, given starting point" of all empirical knowledge—that their epistemic efficacy does not depend on the subject's having any concepts or knowing any facts (1910/1923, 279). This naive conception of the epistemological role of sensations in perceptual experience has a sibling theory within the philosophy of science—an atomistic account of the basic, foundational facts of experimental science: measurements. After analyzing various physical theories in chapter 4, "Concept Formation in Natural Science," Cassirer concludes: "All mea-

surement, however, presupposes certain theoretical principles and in the latter certain universal functions of connection [*Funktionen der Verknüpfung*], of shaping and coordination. We never measure mere sensations, and we never measure with mere sensations, but in general to gain any sort of relations of measurement we must transcend the 'given' of perception and replace it by a conceptual symbol, which possesses no copy in what is immediately sensed."[22] The various cases of measurement together show that the most basic results of scientific experimentation presuppose not only concepts and laws of pure mathematics but also laws of nature and natural scientific concepts.[23] For example, a particular experience of a particular electric current using a galvanometer presupposes a host of other concepts—mathematical concepts like <shape>, <number>, and <direction>, and physical concepts like <motion>, <time>, <distance>, and <force>.[24] Interpreting the motion of a magnetic needle as the change in the electric current itself presupposes various physical laws. (These reflections on measurement then reinforce the attack on abstraction, since the concept <electric current> could not be formed in the atomistic way that traditional abstractionism requires.) Similarly, Poincaréan reflections on the measurement of time show that the concept <duration> could only be formed together with a system of other concepts, including <position>, <inertia>, <force>, and <motion>.

Since even these most elementary experimental results presuppose that an investigator already possesses a system of concepts and presupposes certain theoretical laws, it is easy to see that the point can be generalized from measurements to all scientific claims. Indeed, as Cassirer learned from Duhem, no

22. This quotation appears in Cassirer (1921/1923, 427), which picks up the argument from *Substanzbegriff und Funktionsbegriff* (especially chap. 4 of *Substanzbegriff und Funktionsbegriff*; see 267) and extends it to a discussion of general relativity.

23. Here are some of the cases Cassirer discusses: assigning a real number to a temperature by measuring a volume of mercury presupposes laws of geometry and the law relating temperature to the expansion of volume of mercury (1910/1923, 142–43); when Regnault measured the volume of a gas in his tests of Boyle's law, he used an instrument the design of which just presupposes the "abstract principles of general mechanics and celestial mechanics" (143; cf. Duhem 1906/1977, 145–47); the measurement of time requires the identification of a unit, the choice of which presupposes the law of conservation of energy, or the principle of inertia (145; cf. Poincaré 1905/2001, 210–22); the measurement of the curvature of space presupposes the choice of a "rigid body" (107; cf. Poincaré 1902/2001); the determination of the position of a body requires constructing a Langean "inertial system" (182); Ampère's measurement of the intensity of an electric current required a galvanometer, whose operations presuppose various physical laws (1910/1923, 280). As even the example of temperature—an apparent sensory quality—shows, the most basic scientific measurements go well beyond noting the intensities of this or that sensory quality.

24. I refer to concepts using angle brackets.

single empirical statement of natural science can be confirmed atomistically.[25] This confirmation anatomism, then, reinforces the attacks on abstractionism, since if an empirical concept <F> could be derived from given experiences independently of other concepts or beliefs, then an empirical judgment like "This is an F" could be confirmed directly by comparing the concept with the particular representation from which it was abstracted.[26]

A particular variety of confirmation atomism has appealed to many philosophers: that we can at least know of some things that they exist simply through encountering them, without needing to possess any concepts or hold any beliefs. Against this atomistic view of existence claims, Cassirer claims that we could not know that an object exists outside of a system of concepts and judgments. For example, we are justified in claiming that certain kinds of numbers or points exist only when we can show that they perform "an indispensable function" in our system of mathematical propositions (1910/1923, 83). Similarly, when discussing Macquord Rankine's writings on energetics, Cassirer argues that a "substantial interpretation of energy" would require that energy is a determination of things that "belongs to them directly in their sensuous appearance," that it is an "isolated sensuous property to be perceived for itself." On the opposed "functional theory of the concept" of energy, asserting the "being" of energy "would obviously lose its meaning, if we wished to separate it from the whole system of judgments, in which it has arisen" (197). There is no direct, presuppositionless knowledge of objects, not even the knowledge that something exists. An epistemology of acquaintance is just as hopeless as the Aristotelian metaphysics that it requires.[27]

25. See, e.g., Cassirer (1910/1923, 147). Cassirer alludes to Duhem (1906/1977) frequently (1910/1923,143–47, 280). See also Cassirer (1929/1957, 421, 416, 461). On Cassirer's appropriation of Duhem, see Ferrari (1995).

26. Some commentators worry that Cassirer's view leaves little room for experience in scientific theories at all (see, e.g., Mormann 1999, 302). I think this worry is less pressing than it might first appear. That sensations are not sufficient to ground any knowledge does not show that they do not play an essential role in grounding scientific knowledge. That there cannot be a measurement without a system of concepts and laws does not show that measurements cannot confirm or disconfirm a theory; it just shows that they cannot do so by themselves.

27. Cassirer thinks that an Aristotelian metaphysics of substantial forms tries to avoid recognizing the epistemic preconditions of our knowledge by claiming that it is the nature of the mind to be able to take on the form of the substances that affect it in perception (see 1910/1923, 7). On Cassirer's view, this does not explain the possibility of experience: it just restates the problem in metaphysical terms. (Using Kant's apt phrase, the perceived object cannot just "migrate" into our mind [Kant 1902, 4:282; Cassirer 1907/1922, 690].) Here, of course, we see a connection between the epistemological use of "function" (as epistemic precondition) and the metaphysical use of "substance."

Recognizing these epistemic preconditions necessitates, Cassirer argues, inverting the conceptual dependencies among the key concepts of epistemology—the concepts <object>, <truth>, <knowledge>, and <objectivity>. According to the "substance" or "copy" theory of knowledge, the primitive concept is the metaphysical concept of an <object>.[28] <Truth> is explained in terms of the concept <object>, where a representation is true if it faithfully mirrors the properties of objects. Knowledge is then a certain species of *true* representation—a "copy" of objects, as it were. Last, objective knowledge is a kind of knowledge whose special status is explained in terms of the peculiarities of its object: objective knowledge is about "external" objects, not the inner states of a subject.[29] However, this theory of knowledge is undermined by the polemics against abstractionism. For instance, the "substance" theory has it that objectivity is a two-place relation between a representation and a certain independent object. But we have just seen that it is impossible to know any fact about an object—even the simple fact that the object exists—outside of a system of concepts and judgments. Thus, we cannot determine whether a particular representation is objective without bringing in many other concepts and judgments. In particular, the Duhemian arguments against confirmation atomism seem to leave open the possibility that one subject could assert some fact, and another deny it, by making different compensatory adjustments in the rest of their theories. This seems, in fact, to threaten to make knowledge claims within exact science "unrestrained" and open to subjective "caprice" (1910/1923, 187). (This danger would not arise if the theory of abstraction were true, because the given representations that play the role as the base for abstraction could be the direct, external check on our knowledge.) Cassirer therefore advocates the "functional" theory of objectivity, where objectivity is fundamentally a feature of a scientific theory as a whole, and individual concepts and judgments are objective inasmuch as they are part of objective total theories. He states, "The content of experience becomes 'objective' for us when we understand how each element is woven into the whole" (284). A total theory, further, is objective if its concepts and judgments have a systematic form, what Cassirer, following Kant, calls "unity" (322). This unity is expressed in judgments—since judgments give the interrelations among particular concepts—and ultimately in general laws, since laws provide a logical structure to the theory as a whole (Cassirer 1921/1923, 388). Objectivity has traditionally been associated with permanence: the

28. On the "copy theory of knowledge" and the "functional" theory in later works, see Cassirer (1921/1923, 391ff.; 1923/1955, 77; 1950, 61ff.).

29. Cassirer argues at length in *Substanzbegriff und Funktionsbegriff*, chaps. 6–7, that objectivity cannot be explained in terms of a certain kind of object, and he particularly rejects explaining the objective/subjective distinction in terms of outer/inner.

spatial layout of a room, for instance, is objective while the visual experience of that room is not, since the same layout can give rise to changing experiences as a subject moves about a room. Cassirer explains this permanence with a law: a judgment in mathematical form that will predict the look of the room as a function of the subject's position and orientation—the perspectives change, but the law remains the same (1910/1923, 273, 261). Moreover, even the highest laws of a science change as our theories are refined or replaced, and no theory ever achieves the ideal of complete systematicity and coherence. Our current theories are then objective only inasmuch as there are "laws"—that is, rules or methods—that determine how theories are to be refined and replaced in the face of new facts.[30] Understanding the objectivity of a theory, then, will ultimately require isolating the methods of the various exact sciences, tasks that Cassirer carries out in chapters 2–4 of *Substance and Function*.

The functional theory of objectivity, then, explains objectivity in terms of systematic features of a total theory, instead of appealing—as the substance theory does—to an antecedently intelligible notion of an object. With this new notion, Cassirer then introduces the functional theory of knowledge: that a representation is knowledge if it plays the sort of role within a system of representations that would make objectivity possible.[31] The concept <object>, finally, is understood in terms of <objectivity> and <knowledge>, as that which is represented by fully objective knowledge (1910/1923, 314). We cannot know what an object is without knowing what it is for our representations to be subject to an external constraint, and this external constraint is only intelligible in terms of the rule-governed systematic "functions" of objectivity.[32] An object is

30. Cassirer (1910/1923, 287): "[Thought's] spontaneity is not unlimited and unrestrained; it is connected, although not with the individual perception, with the *system* of perceptions in their order and connection. It is true this order is never to be established in a single system of concepts, which excludes any choice, but it always leaves room for different possibilities of exposition; insofar as our intellectual construction is extended and takes up new elements into itself, it appears that it does not proceed according to caprice, but follows a certain law of progress. This law is the ultimate criterion of 'objectivity.'"

31. See Cassirer (1910/1923, 315): "The two elements of structure and function in their interpenetration determine the complete concept of knowledge." Cassirer (1929/1957, 4–5): "We say that we know the object when we have achieved a systematic unity in the manifold of intuition."

32. Cassirer (1921/1923, 391): The functional theory of knowledge "does not measure the truth of fundamental cognitions by transcendent objects, but it grounds conversely the meaning of the concept of the object on the meaning of the concept of truth. Only the idealistic concept of truth overcomes finally the conception which makes knowledge a copying, whether of absolute things or of immediately given 'impressions.' The 'truth' of knowledge changes from a mere pictorial to a pure functional expression."

For an illuminating discussion of Cassirer on objectivity, objecthood, and systematicity, see Ihmig (1997, 158–81).

not simply given in intuition; it is that in whose concept the manifold of intuition is united (Kant 1998, B137; cf. Cassirer 1921/1923, 393; 1929/1957, 4).

4. The Theory of Abstraction and Cassirer's Argument for Dedekind's Structuralism

The conclusion of our discussion of Cassirer's various uses of the terms *Substanzbegriff* and *Funktionsbegriff*, then, is that the fundamental use of the terms is Kantian, as a way of identifying epistemic preconditions and of criticizing the "Aristotelian" view that there is knowledge that can be acquired atomistically. His fundamental argument against the abstractive theory of concept formation derives from neo-Kantian polemics against the possibility of sensations whose epistemic efficacy does not depend on a perceiver having any concepts or knowing any facts. This polemic is an opening salvo in a progressive argument culminating in nonatomistic theories of concept formation, confirmation, and ontological commitment. These nonatomistic theories, then—pushing the argument yet further—motivate Cassirer to invert the traditional order of explanatory dependence among the concepts of objectivity, judgment, knowledge, truth, and object. This inverted order of explanation Cassirer calls the "functional" theory of knowledge.

In the introduction to this article, I noted that Cassirer's book begins with an attack on the traditional logic in chapter 1—and thus with an argument that would seem to be familiar to Russell and all of us post-Russellians who have embraced the new logic of relations—and then quickly moves to a defense of a Dedekindian—and so anti-Russellian—view that the "'essence' of the numbers is completely expressed in their positions" (1910/1923, 39). Although Cassirer of course agrees that Russell has identified expressive limitations in the traditional logic and has shown that there are some irreducible relations in mathematics, it should now be abundantly clear that Cassirer's attack on the theory of concept formation implicit in the traditional logic is intended to go far beyond Russell's contentions. Is there then a connection between this attack on abstractionism and the structuralist philosophy of arithmetic?

There is one connection that becomes apparent rather quickly. According to the abstractionist view of concept formation, representation of particulars is prior to the representation of the concept that comprehends them, since one must grasp the particulars that fall under the concept (or at least some of them) before abstracting the new concept from them.[33] On Dedekind's view, however,

33. See Cassirer (1910/1923, 8). See also Cassirer (2010, 3), on Aristotle's view that individual substances are first in knowledge and first in time.

the particular numbers "gain their whole being [*Bestand*], so far as it comes within the scope of the arithmetician, first in and with the relations that are predicated of them. Such 'things' are *terms of relations*, which can never be 'given' in isolation but only in ideal community with each other" (1910/1923, 36). Since the numbers are just positions in the natural number structure, it is impossible to grasp a particular number without first grasping the whole number structure.[34] The concept <natural number> is not then grasped after grasping particular numbers—as the abstractionist theory would have it—but before.

This argument shows that if numbers are just positions in the natural number structure, then the theory of abstraction is false. But this still leaves open our question: Does Cassirer think that his attack on abstractionism leads to his defense of Dedekind's structuralism (and not the other way)? And, if so, how? Now, Cassirer clearly does think that reflecting on what is wrong with the abstraction theory (and what is right with the "functional theory of knowledge") will make apparent the virtues of Dedekind's approach over Russell's.

> What was at stake [in the debate between Dedekind and Frege and Russell] was . . . the universal question of how knowledge is actually related to "objects" and what conditions it must fulfill in order to acquire "objective meaning." The question is "What is meant by mathematical 'existence,' and how can there be any meaningful question about the proof of such existence?" For with regard to the view represented by Cantor, Frege, and Russell, the fact is that every concept must correspond with something real if it is to have any objective reality whatsoever, and that the task of the logical deduction and justification of the concept can be none other than that of the producing and characterization of this reality.[35]

There is a rather straightforward—but ultimately, I think, mistaken—reading of Cassirer's reasoning here that would move from the epistemic thesis that no knowledge is "'given' in isolation" directly to ontological structuralism. Such a direct argument would proceed like this. The possibility of "given" sensations can be expressed as the claim that before the subject synthesizes or relates con-

34. For this reason, Cassirer argues that Dedekind—though he uses the language of "abstraction" and "copying" (*Abbildung*)—is using that language in a way that undermines the traditional abstractionist theory of concept formation and thereby also the "copy" theory of knowledge (1910/1923, 36).

35. Cassirer (1950, 61, 63). Although this passage was written almost 30 years after *Substanzbegriff und Funktionsbegriff*, Cassirer makes it clear there that he is simply recapitulating the argument from *Substanzbegriff und Funktionsbegriff*.

tents to each other, there are antecedently available sensations for the mind to relate. In opposition a philosopher might argue—as Natorp puts it—that the mind can relate elements to one another only if the *termini* of the relations are "generated" by the relation itself. Applying this thought to arithmetic, this might seem to mean that the successor relation "generates [*erzeugt*] all the members of the complex" of natural numbers: that numbers are just positions in the natural number structure (1910/1923, 36).[36] Moreover, one of the conclusions that Cassirer drew from the failure of abstractionism is that there cannot be any knowledge independent of a system of concepts and judgments. Might we move directly then to the conclusion that there are no numbers independent of the whole system of numbers?

Cassirer sometimes writes as if he could move immediately from this epistemic holism to an ontological holism. For instance, he writes,

> The attempt to present the entirety of cognition in a systematic unity ends in final *Form-concepts* that bring to expression the possible kinds of relation between contents in general. In these fundamental relations are given the final invariants to which cognition is able to advance; therefore also the "objective" standing [*Bestand*] of being is grounded in them. For objectivity is—according to the critical analysis and meaning of this concept—itself only another designation for the validity of determinate combinatory connections that are to be separately discovered and are to be investigated in their structure. The task of *Erkenntniskritik* consists in this, to go back from the unity of the general concept of the object to the manifold of necessary and sufficient conditions that constitute it. In this sense the thing that cognition calls its 'object' is resolved into a web of relations that are themselves held together through the highest rules and principles. (Cassirer 1927/1993, 13)

36. Kant of course described the characteristic activity of the understanding as "synthesis." Both Cohen and Natorp thought this a poor choice of words because it suggests the existence of independently given contents that the mind takes up passively and combines (Cohen 1902, 23ff.; Natorp 1910, 46ff.). Natorp preferred instead to say that thinking is "the positing of a relation" (*Setzen von Beziehung*) and that "the relation first posits the terms" of the relation, since "nothing can be given to thinking that is more original than thinking" (Natorp 1910, 103). Cohen prefers to describe thinking as "generating" (*Erzeugen*), insisting that the thing generated *is* the generating ("Die Erzeugung selbst ist das Erzeugniss"), since this expression makes clear the "creative sovereignty of thinking" (Cohen 1910, 25–26). It is certainly striking, then, that Cassirer self-consciously uses both the language of "positing relations" and "generating" to describe the structuralist view—derived from Dedekind—that mathematical objects are just positions in structures.

Cassirer here gives the "functional" theory of objectivity we saw earlier, and he asserts that all of our knowledge of objects requires a system of concepts and principles that make that knowledge possible. However, if Cassirer's conclusion—that the object is "resolved into a web of relations"—amounts to the ontological view that all objects are positions in a relational structure, then the conclusion simply does not follow. It does not follow from the fact that *we cannot know one thing* without knowing many things that *there cannot be* one thing without there being many things. That we cannot know any property of an object without that knowledge being related back to a system of other cognitions does not imply that the property known is itself just a relation of that object to other objects.

Interpretive charity alone should prevent us from attributing this direct argument to Cassirer. Luckily, though, there is ample textual evidence that Cassirer's argument is dialectically more subtle and draws on a richer set of mathematical facts to motivate Dedekind's position. Cassirer states that his evidence for the view that numbers are simply "terms of relations" is derived from "the development of scientific arithmetic in the last decades," as he puts it (1910/1923, 35). Indeed, after the attacks on abstractionism in chapter 1, Cassirer's argument continues in the next two chapters with 100 pages recounting the history of mathematics in the nineteenth century. Cassirer's goal in these chapters is to show that Dedekind's view (now generalized even to geometry) develops naturally and inevitably from these historical developments. In chapter 2, he argues that, for example, Gauss's explanation of negative and imaginary numbers and Dedekind's definition of irrationals as cuts are best understood as the view that these various domains of numbers are relational structures. It is hopeless, for instance, to define the irrationals in geometrical terms as lengths of lines and areas in space, or to try to find a privileged physical interpretation of complex numbers. Dedekind's view that the natural numbers are positions in a progression, on Cassirer's view, simply makes explicit what was implicit in the earlier work by number theorists on these "extended" number domains, and it allows a unitary coherent conceptual framework in which to talk about all of the various kinds of "numbers" (56). Further confirmation for this view is given in chapter 3, where Cassirer recounts the development of geometry up through Hilbert's *Foundations of Geometry*. There he argues that the fruitful use of ideal points in geometry, the gradual acceptance of non-Euclidean geometry, and Klein's Erlanger Program all point toward the view that the objects of geometry are not points or lines in empirical space, but objects whose existence is secured and whose properties are determined entirely by specifying a system of concepts (paradigmatically in an axiom sys-

tem) that describes them (110, 88ff.). This history culminates in Hilbert's foundations of geometry, which Cassirer interprets not as a kind of formalism, but as the view that axiom systems pick out abstract structures. These structures then supplant empirical space as the real object of geometry.[37]

Each of these developments within number theory and geometry—despite the issues that are unique to each—nevertheless exhibit a few fundamental features that point toward the structuralist view. Cassirer argues that in each field mathematical research in even elementary subjects (like elementary geometry or arithmetic) requires employing kinds of mathematical objects that have no obvious spatiotemporal interpretation; that the vocabulary of the logic of relations provides a unified conceptual framework in which to discuss all of these areas of modern mathematics; that mathematicians are no longer constrained by whether or not their theories will find applications in physics (although surprisingly, many theories do find unexpected applications); that mathematicians often find it necessary to specify their subject matter in an axiom system—though without giving any meaning to the terms of that system beyond the intertheoretic relations specified by the axioms. All of these genuinely new and revolutionary aspects of modern mathematics are well understood if we say that mathematics is now just the study of positions in relational structures.

Although the details of the development of mathematics itself provide Cassirer with his fundamental argument for mathematical structuralism, this hardly means that the epistemological reflections, beginning with the attack on abstractionism and culminating in the "functional" theory of knowledge, have no role to play. This is because a philosophy that takes Dedekind's view of mathematical objects seriously has to overcome long-standing metaphysical objections. In fact, Russell himself objected to Dedekind on precisely these grounds: "It is impossible that the ordinals should be, as Dedekind suggests, nothing but the terms of such relations as constitute a progression. If they are to be anything at all, they must be intrinsically something; they must differ from other entities as points from instants, or colors from sounds" (Russell 1903, sec. 242; cf. Cassirer 1910/1923, 39). But how does Russell know that it is impossible that numbers could be nothing but terms of a relation? On what grounds is he asserting that the numbers must have some intrinsic properties? Certainly, Cassirer thinks, not from the "the development of scientific

37. Cassirer (1910/1923, 94): In Hilbert's geometry, "intuition seems to grasp the content [*Inhalt*] as an isolated self-contained existence [*Bestand*]; but as soon as we go on to characterize the existence in judgment, it resolves into a web of related structures which reciprocally support each other. Concept and judgment know the individual only as a member, as a point in a systematic manifold, which, here, as in arithmetic, as opposed to the all particular structures, appears as the real logical *prius*."

arithmetic." Whatever intrinsic properties Russell might find for the numbers are surely, he thinks, irrelevant to pure mathematics itself. In fact, Russell's reasoning here violates the lessons drawn from Cassirer's polemic against abstractionism. Scientific arithmetic—and, for that matter, scientific geometry, number theory, and analysis—are paradigm objective disciplines that progress historically by employing identifiable and clear standards. According to the functional theory, this objectivity allows us to say that mathematics provides genuine knowledge—true claims about independently existing objects. There is then no wiggle room for Russell to object to the procedure of mathematicians based on some prior metaphysical convictions. For how else could Russell be so confident that numbers cannot be as Dedekind claims—except by *per impossibile* stepping outside of the epistemic system of scientific arithmetic and apprehending directly what the objects of mathematics must be like. Russell's procedure seems to be the opposite of Cassirer's—he begins with an assertion about what objects must be like, and then—based on that—tries to determine which knowledge claims mirror that reality.

On Cassirer's diagnosis, Russell's felt need to found arithmetic in something besides the numerical structure described by the axioms of arithmetic derives from a prior attachment to the copy theory of knowledge. For both Frege and Russell, he argues, "our every concept must correspond with something real": Russell comes to arithmetic with a purely philosophical conception of what is real and interprets scientific arithmetic to fit that mold (Cassirer 1950, 63). Indeed, we can be more specific: Russell's desire to identify intrinsic properties seems to be motivated by something like the abstractive theory itself. On the epistemological picture that underlies the abstractive theory of concept formation, it is necessary to represent particulars prior to grasping the concepts (and so also the systems) they fall under. To do so, the objects themselves must have intrinsic properties—or else, one could not represent one such particular without representing many, in violation of the atomism inherent in the traditional picture of concept formation. But if we drop the copy theory of knowledge and the abstractive theory of concept formation, then Russell's worries become unmotivated (1910/1923, 316–17).

In the passage from Cassirer (1927/1993) quoted above, it seemed that Cassirer was drawing a substantive and controversial metaphysical conclusion—that an object is "resolved into a web of relations"—directly from the epistemological conclusions that were spun out from the arguments against abstractionism. In fact, though, the role of the functional theory of knowledge is not to support robust metaphysical reflection, but to deflate it. We say that a mathematical object *exists* "insofar as it fulfills a logically indispensable function in the system of [mathematical] propositions" (1910/1923, 83). This means

that questions about what there is are reduced to questions about the structure and method of mathematics itself. Once all of the relevant facts about mathematics are in—what concepts mathematicians employ, what standards they use, what allows mathematics to be objective—there is no further work for metaphysical reflection.

On Cassirer's view, structuralist philosophy of mathematics emerges naturally out of reflections on the mathematical practice of his recent contemporaries. One might think, then, that Cassirer infers directly from the details of the "fact" of modern mathematics to structuralism by a simple application of the transcendental method: that mathematical objects have no intrinsic nonrelational properties is a condition of the possibility of modern mathematics.[38] On this reading, then, the polemics against abstractionism would play no role in Cassirer's argument. I do not believe, however, that this more direct reading gives the whole story. First, Russell (and Frege and Cantor) is just as much a mathematician as Dedekind, and so the injunction "take modern mathematics as a fact and investigate its conditions" is no help until we can decide which mathematics to take as our starting point. Second, Cassirer is very clear (see Cassirer [1950, 61–63], partially quoted above) that the resolution of the dispute between Russell and Dedekind depends on a prior decision in favor of the functional over the substance theory of knowledge, existence, and objectivity. And Cassirer is explicit in *Substanzbegriff und Funktionsbegriff* and again in Cassirer (1950) that the attack on the substance view implicit in Russell's approach has to draw heavily on Cassirer's rejection of epistemological atomism. An acceptable reading has to make sense of Cassirer's claims. In particular, I noted, Russell's commitment to numbers having intrinsic properties is motivated (on Cassirer's view) by the epistemological atomism inherent in traditional abstractionism. Indeed, we can say more. Given the incompatibility that I highlighted above between a structuralist view of mathematical objects and an atomistic epistemology, Russell might well think that taking modern science as a given would in fact require rejecting Dedekind's approach—if Russell believed that the epistemology of science requires an atomistic theory of sensory experience. Without Cassirer's extended attacks, drawn from Duhem and others, on atomistic theories of measurement and confirmation, the transcendental method would be powerless to decide between Dedekind and Russell.

The attacks on abstractionism buttress the application of the transcendental method in a yet further fundamental way. If Russell could be convinced that measurement and confirmation in empirical science do not require or even

38. Thanks to an anonymous referee for challenging me to defend my reading against this alternative.

allow for an atomistic epistemology, he still might cling to his requirement that every object "must be intrinsically something" by rejecting the transcendental method altogether and appealing to some other way of knowing (psychology? metaphysical insight?). However, since the polemics on abstractionism ultimately call into question epistemological atomisms, any investigation of the nature and existence of mathematical objects would have to start "in the middle"—within a system of interconnected concepts and laws. The failure of abstractionism demonstrates that philosophical questions about mathematics must begin within (and not prior to) a worked out systematic body of thought about mathematical objects. And this is just what the transcendental method does. A direct reading of Cassirer's argument that downplays the role of Cassirer's anti-atomism would thus undermine the chief argument for the necessity of a philosophical method like the transcendental method. (Cassirer, like the other Marburg philosophers, appropriated from their reading of Kant both the transcendental method and the opposition to the "given." We can now see that these two aspects of the Kantian heritage are not separate for Cassirer: the opposition to the given shows that we must begin our reflection with the concrete system of science itself; the application of the transcendental method to science shows that abstractionism is untenable. Each stands or falls together.)

In section 2 of this article, we saw that Cassirer chose to organize his book around the question of concept formation because it allowed him to bring together three historically distinct lines of research: Russell and Frege's new logic of relations (or "functions"), investigations into the details of mathematical methodology by late nineteenth-century German logicians, and neo-Kantian epistemological reflections. We can now see that these three lines of research come together for Cassirer in his argument that mathematical objects are positions in relational structures. Cassirer's primary evidence in favor of structuralism is given by the details of mathematical history and practice—the kind of details that were cataloged and discussed by logicians like Wundt and Erdmann. These facts about mathematics can bear the weight of Cassirer's argument because traditional metaphysical scruples about intrinsic and relational properties are defused by the Kantian reflections on concept use and objectivity. The functional theory of knowledge gives the frame in which facts about mathematical methodology can come to the foreground. And clearly this view of mathematics can be coherently formulated only with a unified conceptual vocabulary—given by the new logic of relations—for characterizing relational structures.

A premise in Cassirer's argument is that mathematics itself requires no more than what the structuralist account provides. For instance, he argues that defining numbers in terms of classes serves no mathematical purpose and that the

characterization of the concept of a progression suffices for arithmetic (1910/1923, 49). Russell and Frege's attempt to give the numbers specific intrinsic properties, as certain kinds of classes, is then on Cassirer's view motivated solely by some prior ontology.[39] However, it requires some further argument (not provided by Cassirer) that there really is no compelling mathematical reason to define numbers in class-theoretic terms—and many philosophers may want to take issue with Cassirer at this point. Whether or not such a claim can be sustained, though, we can be clear on what kind of evidence Cassirer would accept as relevant given the functional theory of knowledge: evidence drawn from within the internal concerns of exact science itself.

REFERENCES

Cassirer, Ernst. 1906/1922. *Das Erkenntnisproblem in der Philosophie und Wissenschaft der neuren Zeit*. Vol. 1. 3rd ed. Repr. Berlin: Cassirer.

———. 1907. "Kant und die moderne Mathematik." *Kant-Studien* 12:1–40.

———. 1907/1922. *Das Erkenntnisproblem in der Philosophie und Wissenschaft der neuren Zeit*. Vol. 2. 3rd ed. Repr. Berlin: Cassirer.

———. 1910/1923. *Substanzbegriff und Funktionsbegriff: Untersuchungen über die Grundfragen der Erkenntniskritik*. Berlin: Cassirer. Trans. William Curtis Swabey and Marie Collins Swabey in *Substance and Function and Einstein's Theory of Relativity*. Chicago: Open Court.

———. 1912. "Hermann Cohen und die Erneuerung der Kantischen Philosophie." *Kant-Studien* 17:252–73.

———. 1918/1981. *Kants Leben und Lehre*. Berlin: Cassirer. Trans. James Haden as *Kant's Life and Thought*. New Haven, CT: Yale University Press.

———. 1921/1923. *Zur Einstein'schen Relativitätstheorie*. Berlin: Cassirer. Trans. William Curtis Swabey and Marie Collins Swabey in *Substance and Function and Einstein's Theory of Relativity*. Chicago: Open Court.

———. 1923/1955. *Philosophie der symbolischen Formen*. Erster Teil, *Die Sprache*. Berlin: Cassirer. Trans. Charles W. Hendel and Ralph Manheim as *Philosophy of Symbolic Forms*. Vol. 1, *Language*. New Haven, CT: Yale University Press.

———. 1927/1993. "Erkenntnistheorie nebst den Grenzfragen der Logik und Denkpsychologie." In *Erkenntnis, Begriff, Kultur*, ed. R. Bast. Repr. Hamburg: Meiner.

———. 1928/1993. "Zur Theorie des Begriffs." In *Erkenntnis, Begriff, Kultur*, ed. R. Bast. Repr. Hamburg: Meiner.

———. 1929/1957. *Philosophie der symbolischen Formen*. Dritter Teil, *Phänomenologie der Erkenntnis*. Berlin: Cassirer. Trans. Charles W. Hendel and Ralph Manheim as *Philos-*

39. Cassirer assumes that the Dedekind-style view that numbers are positions in structures is opposed to founding arithmetic in a theory of classes. Cassirer thus does not consider the possibility of a structuralist theory of classes. Furthermore, Cassirer does not distinguish between Frege's idea of a concept extension and Russell's idea of a class.

ophy of Symbolic Forms. Vol. 3, *The Phenomenology of Knowledge*. New Haven, CT: Yale University Press.

———. 1950. *The Problem of Knowledge: Philosophy, Science, and History since Hegel*. Trans. Woglom and Hendel. New Haven, CT: Yale University Press.

———. 2009. *Ausgewählter wissenschaftlicher Briefwechsel*. Vol. 18, *Nachgelassene Manuskripte und Texte*. Ed. John Michael Krois et al. Hamburg: Meiner.

———. 2010. "Substanzbegriff und Funktionsbegriff." In *Nachgelassene Manuskripte und Text*, vol. 8, ed. J. Fingerhut, G. Hartung, and R. Kramme, 3–16. Hamburg: Meiner.

Cohen, Hermann. 1883/1984. *Das Prinzip der Infinitesimal-Methode und seine Geschichte in Werke*. Vol. 5. Repr. New York: Olms.

———. 1902. *Logik der reinen Erkenntnis*. Berlin: Cassirer.

Dedekind, Richard. 1888/1963. "The Nature and Meaning of Numbers." In *Essays on the Theory of Numbers*. Trans. Wooster W. Beman, 29–115. New York: Dover.

Duhem, Pierre. 1906/1977. *The Aim and Structure of Physical Theory*. Trans. Philip P. Weiner. New York: Athenaum.

Erdmann, Benno. 1892/1907. *Logische Elementarlehre*. 2nd ed. Repr. Halle: Niemeyer.

Ferrari, Massimo. 1995. "Ernst Cassirer und Pierre Duhem." In *Kulturkritik nach Ernst Cassirer*, ed. Rudolph Küppers, 177–96. Hamburg: Meiner.

———. 2010. "Is Cassirer a Neo-Kantian Methodologically Speaking?" In *Neo-Kantianism in Contemporary Philosophy*, ed. Rudolf A. Makkreel and Sebastian Luft, 293–313. Bloomington: Indiana University Press.

Friedman, Michael. 2000. *A Parting of the Ways: Carnap, Cassirer, and Heidegger*. Chicago: Open Court.

Gawronsky, Dimitry. 1949. "Ernst Cassirer: His Life and Works." In *The Philosophy of Ernst Cassirer*, ed. Paul Arthur Schilpp, 1–38. La Salle: Open Court.

Gower, Barry. 2000. "Cassirer, Schlick and 'Structural' Realism: The Philosophy of the Exact Sciences in the Background to Early Logical Empiricism." *British Journal for the History of Philosophy* 8 (1): 71–106.

Heis, Jeremy. 2010. "'Critical Philosophy Begins at the Very Point Where Logistic Leaves Off': Cassirer's Response to Frege and Russell." *Perspectives on Science* 18:383–408.

———. 2011. "Ernst Cassirer's Neo-Kantian Philosophy of Geometry." *British Journal for the History of Philosophy* 19:759–94.

———. 2012. "Attempts to Rethink Logic." In *The Cambridge History of Philosophy in the 19th Century*, ed. Susan Hahn and Allen Wood, 95–132. Cambridge: Cambridge University Press.

———. 2013. "Frege, Lotze, and Boole." In *The Historical Turn in Analytic Philosophy*, ed. Erich Reck, 113–38. New York: Macmillan.

Ihmig, Karl-Norbert. 1997. *Cassirers Invariantentheorie der Erfarhung und seine Rezeption des "Erlangers Programms."* Hamburg: Meiner.

Kant, Immanuel. 1902. *Gesammelte Schriften*. Ed. Königlich Preußischen Akademie der Wissenschaft. 29 vols. Berlin: DeGruyter.

———. 1998. *Critique of Pure Reason*. Trans. Paul Guyer and Allen Wood. Cambridge: Cambridge University Press.

Kim, Alan. 2003. "Paul Natorp." In *Stanford Encyclopedia of Philosophy*. http://plato.stanford.edu/entries/natorp.

Kreis, Guido. 2010. *Cassirer und die Formen des Geistes*. Berlin: Suhrkamp.

Lotze, Hermann. 1880/1888. *Logik: Drei Bücher vom Denken, vom Untersuchen und vom Erkennen*. 2nd ed. Leipzig: S. Hirzel. Trans. from the second German ed. under the editorship of Bernard Bosanquet as *Logic*. 2 vols. Oxford: Clarendon.

Mormann, Thomas. 1999. "Critical Idealism Revisited: Recent Work on Cassirer's Philosophy of Science." *Yearbook of the Vienna Circle* 6:295–306.

Natorp, Paul. 1910. *Die logischen Grundlagen der exakten Wissenschaft*. Leipzig: Teubner.

———. 1912. "Kant und die Marburger Schule." *Kant-Studien* 17:193–221.

Poincaré, Henri. 1902/2001. *Science and Hypothesis*. Trans. W. J. Greenstreet in *The Value of Science: Essential Writings of Henri Poincaré*. New York: Modern Library.

———. 1905/2001. *The Value of Science*. Trans. W. J. Greenstreet in *The Value of Science: Essential Writings of Henri Poincaré*. New York: Modern Library.

Reichenbach, Hans. 1924/1969. *Axiomatik der relativistischen Raum-Zeit-Lehre*. Braunschweig: Vieweg. Trans. M. Reichenbach as *Axiomatization of the Theory of Relativity*. Berkeley: University of California Press.

Richardson, Alan. 2006. "'The Fact of Science' and Critique of Knowledge: Exact Science as Problem and Resource in Marburg Neo-Kantianism." In *The Kantian Legacy in Nineteenth-Century Science*, ed. Michael Friedman and Alfred Nordmann, 211–26. Cambridge, MA: MIT Press.

Russell, Bertrand. 1903. *The Principles of Mathematics*. New York: Norton.

———. 1912. *The Problems of Philosophy*. London: Oxford University Press.

Schlick, Moritz. 1921/1979. "Kritizistische oder Empiristische Deutung der Neuen Physik?" *Kant-Studien* 26:96–111. Trans. as "Critical or Empiricist Interpretation of Modern Physics?" In *Moritz Schlick: Philosophical Papers*, vol. 2, ed. H. Mulder and B. van de Velde-Schlick. Dordrecht: Reidel.

Trendelenburg, A. 1870. *Logische Untersuchungen*. 3rd ed. Leipzig: S. Hirzel.

Wundt, Wilhelm. 1880–83/1906–8. *Logik: Eine Untersuchung der Principien der Erkenntnis und der Methoden wissenschaftlicher Forschung*. 3rd ed. 3 vols. Repr. Stuttgart: Enke.