Next, $\partial P/\partial c$ is zero or negative, and the higher the infection rate the more injurious is the happening.

$\partial P/\partial V$ is zero when $t = 0$. It generally increases at first with t and afterwards decreases; it is negative when $x = L$ provided $1 - c(1-L)/K$ is negative. In other words, in these cases an increase of V is beneficial at the beginning of the outbreak, but injurious later on. Now $V = N-M+I-E$ and diminishes if the case-mortality M increases; in fact $\partial P/\partial M = -\partial P/\partial V$. Thus an increase in the case-mortality is injurious at first, but may be ultimately *beneficial,* because an affected individual who dies ceases to be infective.

An Application of the Theory of Probabilities to the Study of a priori *Pathometry.*—Part III.

By Lieut.-Colonel Sir Ronald Ross, K.C.B., F.R.S., R.A.M.C.T.F., and Hilda P. Hudson, M.A., Sc.D.

(Received October 19, 1916.)

CONTENTS.

XI.

(i) *Variable Happening.*—In Section IX, we commenced by touching upon the necessity of studying the effect of changes in the happening-element, but there dealt only with the case of hypometric happening. We now proceed to examine other changes in this element.

It is unlikely that in any infectious disease the infectivity-element c will always remain constant. It may remain practically such for some length of time; and the hypothetical results of this condition have therefore been analysed above in detail; but nature abhors a straight line, and we may infer from general experience that changes are sure to occur from time to time. As suggested in Section I, changes in infectivity may be due—

(*a*) to the action, so to speak, of the infecting organisms themselves—which is difficult to believe in because it involves the conception that organisms living in large numbers of different hosts should, as it were, make a kind of concerted and simultaneous effort; and

(*b*) to changes in season, climate, or other environment which may (1) increase or diminish the infective strength of the infecting organisms; or (2) facilitate or hinder their transference from one host to another.

At present we can proceed only on the supposition that the change has occurred, without attempting to enquire how or why it has occurred; and only some brief notes on the subject can be given at present.

(ii) *Discontinuous Change.*—We have hitherto supposed that when $t = 0$ the number of affected individuals is Z_0, which may be small or large. If the happening is hypermetric ($KL > 0$) and Z_0 is less than LP, the number of affected individuals, Z, will increase towards the latter figure; but if the happening is isometric ($KL = 0$), or is hypometric ($KL < 0$), Z will diminish to the limits discussed in Section IX. What will happen if the infectivity is suddenly changed during one unit of time from c to c', and thereafter remains at the latter figure?

If $c' > c$, the x-curve suddenly becomes steeper; x itself is continuous, but its gradient is discontinuous. The f-curve is discontinuous, and has a sudden rise at the moment when c changes; thereafter both curves proceed according to the equations with c' substituted for c. If at the moment of discontinuity more than half the population is affected, so that f is descending, there is an epidemic manifestation, a sudden rise followed by a decline. Both L and l are increased.

These are probably just the conditions which apply to many zymotic diseases, in which most of the population appears to remain affected, that is immune, for years or permanently. In these cases, at the moment of discontinuity x is nearly L, and L is nearly unity, and the augmented infectivity c' acts merely upon the small residue of non-affected individuals, let us say, newly-born infants, or children, or others who have hitherto escaped infection by accident or by residence in remote villages. It is well known that at the beginning of the present war recruits from many highland villages were attacked by very virulent measles on joining camps in the lowlands. On the other hand, the same sort of epidemic may occur in numbers of villages and small towns in which the considered infection has died out from want of material, and so allowed of a gradual accumulation of fresh material, which is then suddenly fired (so to speak) by the entry of one or two infective cases from outside.

It is extremely improbable, of course, that the infectivity actually makes a single *per saltum* change; but we may still apply the equations already obtained if we give to c a series of progressive discontinuous augmentations for successive units of time, followed by similar discontinuous diminutions until c returns to its original value. In this case the f-curve will again show epidemic manifestations (if $x > \frac{1}{2}$), and will then decline again towards a lower value. This is one way of approximating to a continuous change in c.

(iii) *Continuous Variation.*—We may, however, obtain new equations on the hypothesis that the infectivity c is a continuous function of the time. Let us suppose that $c = \phi'(t)$, where $\phi'(t) = d\phi(t)/dt$. Then in the fundamental equation 47, namely $dx/dt = \mathrm{K}x(\mathrm{L}-x)$, we have

$$\mathrm{K} = \phi'(t) - \mathrm{D} \qquad \text{and} \qquad \mathrm{KL} = \phi'(t) - \mathrm{D} - \mathrm{R}.$$

Put $x = 1/y$, so that $dy/dx = -\mathrm{KL}y + \mathrm{K}$; then

$$y \exp \int \mathrm{KL}dt = \int (\mathrm{K} \exp \int \mathrm{KL}dt)\, dt + \text{constant}$$

by the well-known solution. As however, $\mathrm{K} = \mathrm{KL} + \mathrm{R}$,

$$\int (\mathrm{K} \exp \int \mathrm{KL}dt)\, dt = \exp \int \mathrm{KL}dt + \mathrm{R} \int (\exp \int \mathrm{KL}dt)\, dt;$$

hence

$$y = 1 + \exp\left(-\int_0^t \mathrm{KL}dt\right)\left\{\mathrm{R}\int_0^t \left(\exp \int_0^t \mathrm{KL}dt\right) dt + y_0 - 1\right\}$$

If R is negligible, as may often happen in short and sharp epidemics with long immunity, we may write, for many diseases at least,

$$\int \mathrm{KL}dt = \phi(t) - \mathrm{D}t,$$

$$x = \frac{1}{1 + \text{const.} \times \epsilon^{-\phi(t) - \mathrm{D}t}}.$$

We have at present no knowledge of the nature of the function $c = \phi'(t)$ or even if there is such a function. But if there is, we may suppose that, for positive values of t, $\phi'(t)$ is small when t is small, increases at first as t increases, reaches a maximum for a certain value of t, and then declines to a small value again when t is larger—as is observed in many phenomena. In other words we should expect a symmetrical or non-symmetrical bell-shaped curve such as those with which we are familiar in statistical work.

(iv) *Graphical Treatment of x.*—We may also suppose that c is given in terms of the time t, not by an analytic function, but by a curve whose equation is unknown. We can then determine the nature of x and f by graphical methods.

Since $dx/dt = \mathrm{K}x(\mathrm{L}-x)$, there could only be a turning value of x when $\mathrm{K} = 0$ or $x = 0$ or $x = \mathrm{L}$. But when $\mathrm{K} = 0$, then $\mathrm{KL} = -\mathrm{R}$ and dx/dt does not vanish; when $x = 0$ the happening ceases altogether; the only turning value of x is when $x = \mathrm{L}$.

Now since by definition x lies between 0 and 1, we can only have $x = \mathrm{L}$ if L also lies between 0 and 1; the diagram of IX (i) shows that this only occurs when $c > \mathrm{D}+\mathrm{R}$. Hence unless the given c-curve rises above the level $\mathrm{D}+\mathrm{R}$, the corresponding x-curve has no fluctuations, and either rises steadily to 1 as its limit, or sinks to 0. But wherever the c-curve has a portion rising above $\mathrm{D}+\mathrm{R}$, we can plot the values of $\mathrm{L} = 1-\mathrm{R}/(c-\mathrm{D})$ corresponding to this portion, which form a cap, part of the L-curve, lying entirely between the levels 0 and 1. If the c-curve fluctuates, there are a series of these L-caps, which are the only parts of the L-curve that the x-curve can possibly cross.

Now, whenever the x-curve meets an L-cap, $x = \mathrm{L}$, $dx/dt = 0$, and x has a turning value. If the x-curve descends to meet the L-cap from outside, it begins to ascend as soon as it is inside; and goes on ascending till it meets

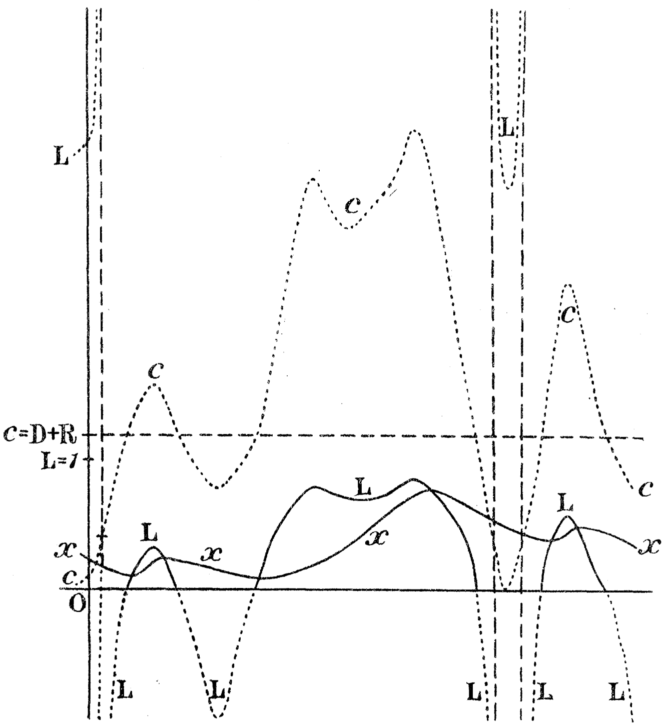

Fig. 1.

the opposite side of the same cap, when it passes outside and immediately begins to descend, until it meets another L-cap. The x-curve descends steadily except where it is inside a cap; and as soon as the L-caps are plotted, the general character of the curve is fairly apparent. If c has a definite limit c_∞ when $t \to \infty$ and this limit is less than $D+R$, then the last L-cap is closed, x is ultimately descending and tends to 0. But if $c_\infty > D+R$, then L has a limit L_∞ and x tends to the same limit as L, and may approach it either from above or from below.

Since $L = 1 - R/(c-D)$, $\partial L/\partial c = R/(c-D)^2$, and is positive, L rises and falls with c and the top of an L-cap corresponds exactly to a maximum of c. But a maximum of x occurs where it crosses a descending branch of L: hence the maxima of x occur somewhat later than the corresponding maxima of c.

Suppose, for example, that c is periodic, rising and falling regularly every year, and crossing the level $D+R$ twice a year. Then L has a series of caps with equal equidistant maxima of height λ, say. Start from a minimum of c, between two L-caps, and suppose the value x_0 of x to be given.

Then x begins to descend. If x_0 is large, x may pass over the top of one or more L-caps, descending all the while, but, sooner or later, it meets an L-cap, passes inside it, and begins to rise. It thereafter meets every cap, for it leaves each cap at a height less than λ, and, since it descends between the caps, it cannot pass over the top of the next. Every year x rises and falls, the maximum occurring later than that of c or L, and we have a yearly epidemic. But there is no reason to make these epidemics equal; though, if any two are equal, they all must be. For if x has equal values for corresponding points of the periods, the whole course following these points is the

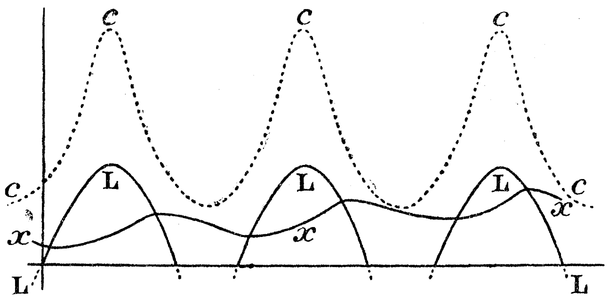

FIG. 2.

same; in other words, if two periods are superposed, and if the two parts of the x-curve have one point in common, they coincide, but, if x meets any L-cap at a point lower than the preceding cap, its whole course inside that cap is lower, and lower still in the next cap, and so on, tending to a limiting form which is periodic, and which may be the axis itself, the happening dying out. But, if x meets any L-cap higher than the last, it meets the next higher still, and so on, and tends to a periodic limiting form of finite fluctuations.

In order to plot x more closely, we have to replace the fundamental differential equation by the approximate difference equation

$$x_{n+1}-x_n = \mathrm{K}x_n(\mathrm{L}-x_n),$$

where K, L are calculated from the value of c measured off the given curve, and x_0 is given; or by some modification of the method.

(v) *Graphical Treatment of f.*—If c is given and x has been obtained, we have next to deal with f, which can be directly calculated from the equation

$$f = cx(1-x) = c\{\tfrac{1}{4}-(x-\tfrac{1}{2})^2\} = c\{\tfrac{1}{4}-(\tfrac{1}{2}-x)^2\}.$$

Hence f always increases with c: as x increases, f increases if $x < \frac{1}{2}$, and decreases if $x > \frac{1}{2}$. If we wish to know the general behaviour of f, the following Table is useful; it allows for a single bell of each of the c- and x-curves, the maximum of x occurring later than that of c.

c.	x.	f.		
		if $x < \frac{1}{2}$.	if $x = \frac{1}{2}$.	if $x > \frac{1}{2}$.
minimum	decreasing	decreasing	minimum	increasing
increasing	decreasing	?	increasing	increasing
increasing	minimum	increasing	increasing	increasing
increasing	increasing	increasing	increasing	?
maximum	increasing	increasing	maximum	decreasing
decreasing	increasing	?	decreasing	decreasing
decreasing	maximum	decreasing	decreasing	decreasing
decreasing	decreasing	decreasing	decreasing	?
minimum	decreasing	decreasing	minimum	increasing

In using this, we follow f down a column until x passes through the value $\frac{1}{2}$, when we cross to the next f-column.

Since in each column f is increasing in the third row and decreasing in the seventh, there is bound to be a maximum of f between them, which occurs before the maximum of c if $x > \frac{1}{2}$ and after if $x < \frac{1}{2}$.

Recrudescence.—It is seen that if c and x are both decreasing and $x > \frac{1}{2}$, we cannot say what f is doing, as the two causes of its change act in opposite ways. It is possible for f to have a minimum followed by a short rise and a maximum, and, if x then becomes equal to $\frac{1}{2}$, f falls again to another minimum. This would happen, for example, if c suddenly became constant after decreasing, before x drops to $\frac{1}{2}$. Then we have a slight recrudescence of the epidemic near its close.

Since
$$\frac{df}{dt}=\frac{\partial f}{\partial c}\frac{dc}{dt}+\frac{\partial f}{\partial x}\frac{dx}{dt}.$$

and in the case considered dc/dt and dx/dt are both negative,

$$\frac{df}{dt} = -x(1-x)\left|\frac{dc}{dt}\right| + c(2x-1)\,\mathrm{K}x(\mathrm{L}-x)$$

and is positive, so that f rises again, provided dc/dt is small enough, but, as x decreases, the factor $2x-1$ decreases and changes sign; the positive second term in df/dt becomes negative, and f sinks.

In fig. 3, c rises and falls in a symmetrical bell and then remains constant;

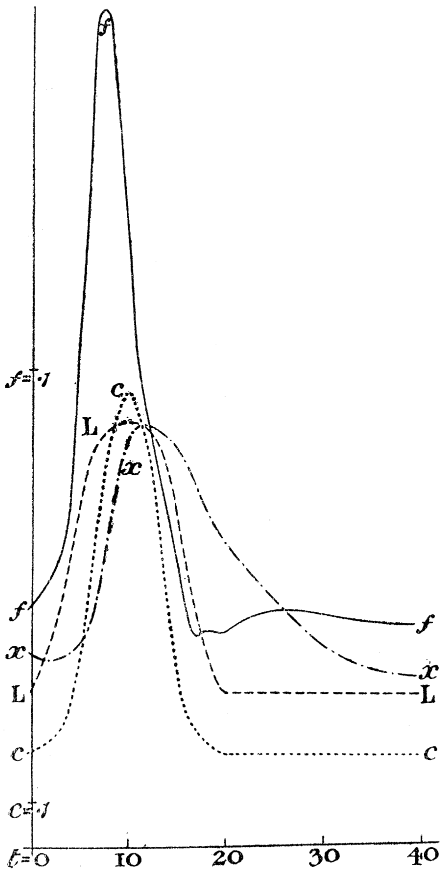

FIG. 3.

L does the same; x has an unsymmetrical bell, rising more steeply than it falls, and sinking to the same limit as L; and f shows an epidemic with slight double recrudescence before sinking to its limit.

The graphs of c, x, and L are on the same scale, and that of f on 10 times the scale.

(vi) *f Assigned.*—The last sub-section started by assuming that c is given in some experimental way. But c is not given except by conjecture, and the method is useful to test whether any particular conjecture leads to numerical results which agree with the records of actual epidemics. Now the most extensive and accurate records are the notifications of fresh cases and of deaths made by the medical officers of health; in particular, the records of measles in Aberdeen for the 20 years 1883–1902 are very suitable material, though even there the numbers are not large enough to be quite satisfactory.

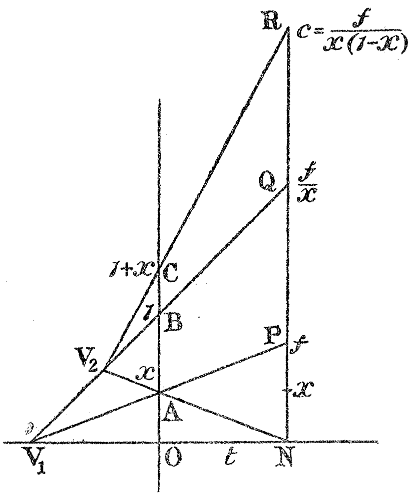

Fig. 4.

The death-rates do not correspond to any of the symbols we have introduced so far; for MZ is the number of deaths of affected persons per unit time, a different thing from the number of deaths due to the affection. We return to this in a later section.

We start therefore with the fresh assumption that the notifications, that is, the values of F, are known. Suppose also that P is given, either by interpolation from census returns, or because in a short and mild epidemic P is nearly constant. Then f is known as a function of t. The fundamental equations are best taken in the forms

$$dx/dt = f - x(\mathrm{D}+\mathrm{R}-\mathrm{D}x), \qquad c = f/x(1-x),$$

from which x and c can respectively be determined, if reasonable estimates can be made of the constants D, R.

When x, f, are found and plotted, we can construct c graphically. P is the point (t, f). Along the vertical axis mark off

$$\mathrm{OA} = x, \quad \mathrm{OB} = 1, \quad \mathrm{OC} = 1+x; \text{ then } \quad \mathrm{AB} = 1-x, \quad \mathrm{AC} = 1.$$

Let PA meet the axis NO in V_1; let $\mathrm{V}_1\mathrm{B}$ meet PN in Q and NA in V_2, and let $\mathrm{V}_2\mathrm{C}$ meet PN in R.

Then $\quad QN/OB = PN/OA, \quad QN = f/x,$

and $\quad RN/AC = QN/AB, \quad RN = f/x(1-x) = c,$

so that R is the point on the c-curve corresponding to P on the f-curve.

If D and R can both be neglected, then

$$x = \int_{t=0}^{t} F dt + x_0.$$

In this case, x steadily increases whatever is the law of f, and the area under the f-curve always < 1.

In a great many cases, the f-curve is a slightly unsymmetrical bell, rising more steeply than it falls. Now

$$x^2(1-x^2)\frac{dc}{dt} = \frac{df}{dt}x(1-x) - f^2(1-2x),$$

and df/dt is positive for the first, smaller part of the range, then vanishes and is negative for the rest of the time. The second term is negative if $x < \frac{1}{2}$, and positive if $x > \frac{1}{2}$.

Now if x_0 is small, at the summit where $df/dt = 0$, $x < \frac{1}{2}$ and dc/dt is negative. In this case, the c-curve reaches its summit sooner than that of f, and is more unsymmetrical. But if x_0 is larger, $1-2x$ changes its sign earlier, and the summit of the c-curve is later in consequence. These results are an extension of those in Part I (p. 224), where we found that in the same case where D, R, are neglected, f is perfectly symmetrical when x is constant.

But if we do not wish entirely to exclude reversions, we assume

$$D = 0, \qquad R > 0.$$

Then
$$dx/dt + Rx = f,$$

$$x = \epsilon^{-Rt}\int_{t=0}^{t} f\epsilon^{Rt}dt + x_0\epsilon^{-Rt}.$$

For example, let $f = at(T-t)$, which is the simplest function that rises and falls, vanishing at $t = 0$ and $t = T$, with a maximum at $t = \frac{1}{2}T$. Then

$$x = \frac{at}{R^2}\left\{R(T-t)+2\right\} - \frac{a}{R^3}(RT+2)(1-\epsilon^{-Rt}) + x_0\epsilon^{-Rt}.$$

Or we may allow for asymmetry by an exponential factor; $f = at(T-t)\epsilon^{-\lambda t}$, which still vanishes at $t = 0$ and $t = T$, but has a maximum before $t = \frac{1}{2}T$;

$$x = \frac{at}{(R-\lambda)^2}\left\{(R-\lambda)(T-t)+2\right\}\epsilon^{-\lambda t}$$
$$- \frac{a}{(R-\lambda)^3}\left\{(R+\lambda)T+2\right\}(\epsilon^{-\lambda t} - \epsilon^{-Rt}) + x_0\epsilon^{-Rt}.$$

If f is periodic, let

$$f = a+b\sin pt, \qquad \text{where } a \geqq b\,;$$

$$x = \frac{a}{\mathrm{R}}(1-\epsilon^{-\mathrm{R}t})+\frac{b}{p^2+\mathrm{R}^2}(\mathrm{R}\sin pt-p\cos pt+p\epsilon^{-\mathrm{R}t})+x_0\,\epsilon^{-\mathrm{R}t}.$$

In all these, the negative exponentials decrease rapidly; in the last, x oscillates about a/R as mean, the phase lagging behind that of f by $\frac{1}{p}\tan^{-1}\frac{p}{\mathrm{R}}$. Then c has the same period.

(vii) *More General Equations.*—In many diseases, for example those in which there is a period of incubation, the infectivity is variable, and depends not only on the absolute time but also on the *duration of the case,* which is the time elapsed since the case was first infected. In order to take this into account, c must be considered as a function of two independent variables, t the time, and s the duration; fluctuations of the infectivity of the disease as a whole cause c to vary as a function of t; episodes in the course of a single case cause c to vary as a function of s, very possibly a discontinuous function. For example, an incubation period of q days is represented by taking $c = 0$ when $s < q$.

For greater generality, suppose that V, I, E, M, N, c, r, D, R, all depend on t and s; but v, i, e, m, n on t only.

Let $\mathrm{F}_{t,0}dt$ be the number of fresh cases occurring between times t and $t+dt$.

Let $\mathrm{F}_{t+s,s}dt$ be the number of the batch of $\mathrm{F}_{t,0}dt$ cases which survive, having neither died, emigrated, nor reverted, at time $t+s$, being then of duration s. Thus $\mathrm{F}_{t,s}$ is the number of cases existing at time t whose duration is s. The total number of cases existing at time t is

$$\mathrm{Z} = \int_{s=0}^{\infty}\mathrm{F}_{t,s}ds. \tag{86}$$

As before, $\mathrm{P} = \mathrm{Z}+\mathrm{A}$, and the number of fresh cases is now

$$\mathrm{F}_{t,0} = \frac{\mathrm{A}}{\mathrm{P}}\int_{s=0}^{\infty}c\,\mathrm{F}_{t,s}ds.$$

The rate of change of the particular batch of cases which started at time t_0 is given by

$$\frac{d}{dt}\mathrm{F}_{t,\,t-t_0} = (\mathrm{V}-\mathrm{N}-r)\,\mathrm{F}_{t,\,t-t_0},$$

which may be written

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)\mathrm{F}_{t,s} = (\mathrm{V}-\mathrm{R})\,\mathrm{F}_{t,s} \tag{87}$$

where $$s = t-t_0.$$

The fundamental equations now take the forms

$$\left.\begin{aligned}\frac{dP}{dt} &= vA + \int_{s=0}^{\infty} VF_{t,s}ds \\ \frac{dZ}{dt} &= \int_{s=0}^{\infty} (V-R)\,F_{t,s}ds + F_{t,0} \\ \frac{dA}{dt} &= vA + \int_{s=0}^{\infty} RF_{t,s}ds - F_{t,0}\end{aligned}\right\} \qquad (88)$$

These equations are not independent. If 87 is integrated with respect to s from 0 to ∞, since $F_{t,\infty} = 0$ we have

$$\frac{d}{dt}\int_{s=0}^{\infty} F_{t,s}ds - F_{t,0} = \int_{s=0}^{\infty} (V-R)\,F_{t,s}ds,$$

which is equivalent to 88 by virtue of 86.

These equations as they stand are quite unwieldy, but they are capable of being applied to a large number of simpler cases.

If V, R, are independent of t and constant for a particular range of s, say $s = q_1$ to q_2, then for any intermediate value of s, from 87

$$F_{t,s} = F_{t-q_1,\,q_1}\,\epsilon^{(V-R)(t-q_1)}$$

and if V, R have other constant values V′, R′ from $s = q_2$ to q_3, then for any value of s in the latter range,

$$\begin{aligned}F_{t,s} &= F_{t-q_2,\,q_2}\,\epsilon^{(V'-R')(t-q_2)} \\ &= F_{t-q_1,\,q_1}\,\epsilon^{(V'-R')t+(V-V'-R+R')q_2-(V-R)q_1},\end{aligned}$$

and so on.

The simplest case is when c is constant from $s = q_1$ to q_2, zero for all other values of s, and independent of t, while all the other elements are neglected. Since there are then no deaths nor recoveries, $F_{t,s} = F_{t,0}$ and P is constant. Then

$$\begin{aligned}\frac{dZ_t}{dt} &= F_{t,0} = c\,\frac{A}{P}\int_{s=q_1}^{q_2} F_{t-s,0}ds \\ &= c\,(1-Z_t/P)\,(Z_{t-q_1} - Z_{t-q_2}),\end{aligned}$$

the suffixes showing the value to be substituted for t in Z.

Let $c\,(1-Z_t/P) = Y_t$, then this difference equation takes the simpler form

$$\frac{dY_t}{dt} = Y_t\,(Y_{t-q_1} - Y_{t-q_2}).$$

The best way of dealing with this equation is to assume a Dirichlet series for Y, substitute and equate coefficients.

Let $$Y_t = B_0 + B_1\epsilon^{-pt} + B_2(B_1\epsilon^{-pt})^2 + B_3(B_1\epsilon^{-pt})^3 + \ldots$$

and let $$\Delta_n = \epsilon^{nq_2} - \epsilon^{nq_1},$$

then p, B_1 are arbitrary, $B_0 = p/\Delta_1$, and the general reduction formula for the coefficients is

$$-B_n p(\Delta_n - n\Delta_1)/\Delta_1 = B_{n-1}\Delta_1 + B_{n-2}B_2\Delta_2 + B_{n-3}B_3\Delta_3 + \ldots + B_{n-1}\Delta_{n-1}.$$

Since $\Delta_n - n\Delta_1$ is positive, the signs of the coefficients alternate; $B_0 = Y_\infty$ and is essentially positive, and therefore p is positive;

$$B_1 = -\frac{1}{p}\frac{dY_\infty}{dt} = \frac{c}{Pp}\frac{dZ_\infty}{dt}$$

and is positive, for Z always increases. Therefore B_{2n} is negative and B_{2n+1} positive.

The series for F is

$$F_{t,0} = \frac{pP}{c}\left\{B_1\epsilon^{-pt} + 2B_2(B_1\epsilon^{-pt})^2 + 3B_3(B_1\epsilon^{-pt})^3 + \ldots\right\}.$$

XII.

(i) *Mortality: First Modification of the Equations of Part I.*—It has been assumed that the average death-rate of the affected population is a constant M; part of this, m say, is due to other causes, and $M-m$ is due to the disease. If m is constant, so is $M-m$, and the deaths from disease are simply proportional to Z and their ratio to the total population P is proportional to x. It therefore increases steadily; this does not agree with the records.

It is more reasonable to assume that the deaths from the disease all occur soon after infection and that their number is proportional to F; let it be μF. Then its ratio to the total population is μf, and rises and falls in proportion to f. This is much more nearly true. But now the fundamental equations are altered. We assume that the μF deaths are in addition to MZ deaths due to other causes, and that they all happen before the cases have had time to infect an appreciable number of other people. Then

$$dP/dt = VZ + vA - \mu F,$$
$$dZ/dt = (V-N-r)Z + (1-\mu)F,$$
$$F = cZA/P.$$

As before write

$$Z = xP, \qquad A = (1-x)P, \qquad F = cx(1-x)P,$$
$$v - V = D, \qquad N + r = R.$$

The result of elimination is

$$dx/dt = -\{D+R-(1-\mu)c\}x + \{D-(1-2\mu)c\}x^2 - \mu cx^3$$
$$= -\mu cx(x-\alpha)(x-\beta), \text{ say, where } \alpha > \beta,$$

of which the integral is

$$\epsilon^{-\mu c\alpha\beta(\alpha-\beta)t} = x^{\alpha-\beta}(x-\alpha)^{\beta}(x-\beta)^{-\alpha}/\{x_0^{\alpha-\beta}(x_0-\alpha)^{\beta}(x_0-\beta)^{-\alpha}\}.$$

The form of this curve depends on the signs of α, β.

Let
$$Q = \mu c(x-\alpha)(x-\beta)$$
$$= \mu c x^2 - \{D-(1-2\mu)c\}x + D + R - (1-\mu)c.$$

Case (i).—If the last term is positive, $(1-\mu)c < D+R$, then α, β, have the same sign; and if x_0 is a small positive quantity, less than $|\alpha|$ or $|\beta|$, dx_0/dt is negative, x begins to decrease, Q cannot change sign and x decreases steadily to zero as its limit.

Case (ii).—If the last term in Q is negative, $(1-\mu)c > D+R$, then α is positive and β is negative. When $x = 1$, $Q = R$ and is positive; therefore $0 < \alpha < 1$. Now dx_0/dt is positive, x begins to increase, but it does not reach α till $t = \infty$; hence Q cannot change sign and x increases steadily to α as its limit.

Now $f = cx(1-x)$, and as x varies from 0 to 1, f varies from 0 to 0 in a bell-shaped curve with a maximum at $x = \frac{1}{2}$.

In case (i) f decreases steadily from f_0 to 0, in a curve of the same general character as the x-curve, for the factor $(1-x)$ does not vary much.

In case (ii) if $\alpha < \frac{1}{2}$, f increases steadily to $c\alpha(1-\alpha)$ as its limit, but if $\alpha > \frac{1}{2}$, f has a maximum $\frac{1}{4}c$ when $x = \frac{1}{2}$ and then sinks to its limit $c\alpha(1-\alpha)$.

If D, R are both negligible, $Q = c(x-1)(\mu x + 1 - \mu)$, and $\alpha = 1$; then f has a maximum and sinks to 0 as its limit.

(ii) *Second Modification.*—A popular assumption about infectious diseases is that the case ceases to be infective after a certain time, when also there ceases to be danger of death from the disease, but remains immune permanently. To represent this, suppose the population P to consist of three classes: $P = A + X + Y$, where A are unaffected, X infective and immune, Y immune. For simplicity, neglect all "natural" variations, but suppose there are MX deaths from the disease per unit time, rX recoveries, and $F = cXA/P$ fresh cases. The fundamental equations are

$$dP/dt = -MX, \qquad dX/dt = cXA/P - (M+r)X,$$
$$dY/dt = rX, \qquad dA/dt = -cXA/P.$$

Hence P decreases and Y increases steadily. Suppose $Y_0 = 0$. Then

$$dY/dP = -r/M, \qquad Y = (r/M)(P_0 - P);$$
$$dA/dP = c/M \,.\, A/P, \qquad A = A_0(P/P_0)^{c/M};$$
$$-(1/M)\,dP/dt = X = (1+r/M)P - (r/M)P_0 - A_0(P/P_0)^{c/M};$$
$$t = -\frac{1}{M}\int_{P=P_0}^{P} dP/\{(1+r/M)P - (r/M)P_0 - A_0(P/P_0)^{c/M}\};$$

which is integrable in finite form if c/M is rational, and leads to algebraic functions and logarithms.

In the final steady state, $X = 0$ and P is given by

$$(1+r/M)P-(r/M)P_0-A_0(P/P_0)^{c/M} = 0,$$

which has a real root P_∞ between P_0 and 0. A definite number of persons $A_\infty = A_0(P_\infty/P_0)^{c/M}$ escape the disease altogether.

Now, dX/dt is of the same sign as $cA-(M+r)P$, and is positive at first, provided $c > (M+r)P_0/A_0$; in this case X, the total number of infective persons, rises to a maximum before sinking to 0.

dx/dt is found to be of the same sign as $(c-M)A-rP_0$, which is positive at first if $c > M+rP_0/A_0$, and steadily decreases.

dF/dt is of the same sign as $cA-(M+r)P-(c-M)X$, and, if $c > M$, this is less than dX/dt; the maximum of F (if it has one) is therefore earlier than that of X or of MX, that is to say, the greatest number of deaths occurs later than the height of the epidemic, which is the time of greatest number of notifications.

For example, let

$$r = 0{\cdot}01, \qquad M = 0{\cdot}02, \qquad c = 0{\cdot}04, \qquad x_0 = 0{\cdot}0102.$$

Then $\qquad A_0/P_0 = 0{\cdot}9898. \qquad$ Let $\quad P/P_0 = p,$

$$Mt = \int_{p=1}^{p} \frac{dp}{0{\cdot}9898p^2-1{\cdot}5p+0{\cdot}5} = \frac{-1}{0{\cdot}52}\log_e\left\{\frac{2{\cdot}02p-1}{0{\cdot}98p-1}\cdot\frac{-0{\cdot}02}{1{\cdot}02}\right\}$$

$$p = \frac{1+51\epsilon^{-0{\cdot}0104t}}{2{\cdot}02+49{\cdot}98\epsilon^{-0{\cdot}0104t}}$$

$$P_\infty = \tfrac{1}{2}P, \qquad A_\infty = \tfrac{1}{2}P_\infty \text{ approximately.}$$

X is stationary when $A = \frac{3}{4}P$ or $1/p = 1{\cdot}32$, leading to $t = 300$ approximately, the time which elapses before the maximum of the mortality.

XIII.

(i) *Discussion.*—In the foregoing work, very little attention has been paid to the actual values of the constants. We have rather been preparing apparatus which can be used in a great variety of special cases.

It will be seen that the number of dependent variables is large, and that a fluctuation in any one can be accounted for in a great many different ways by supposing that there is a suitably adjusted variation in almost any one of the parameters. And the variation of any one parameter when it is over a range small compared with the mean value can be represented approximately by one of several analytic functions, which would lead to the most diverse results if the variation were to continue over a wider range. It is therefore extremely rash to rule out *a priori* any one possible cause of fluctuation of any one phenomenon.

The cases already considered have led exactly to the series of curves required by the facts:

1. The steadily rising curve of a happening that gradually permeates the whole population [VII (iii)].

2. The symmetrical bell-shaped curve of an epidemic that dies away entirely [VII (v)].

3. The unsymmetrical bell of a new happening that begins with an epidemic, and settles down to a steady endemic level [VII (v)].

4. The periodic curve with regular rise and fall due to seasonal disturbances [XI (iv)].

5. The more irregular curve where there is recrudescence before the end of an epidemic, or where outbreaks differing in violence occur at unequal intervals [XI (v)].

This suggests that, as was stated in the Preface of Part I, the rise and fall of epidemics as far as we see at present can be explained by the general laws of happenings, as studied in this paper.

The methods which have been used call, however, for the following remarks:—

A. In the equations of Section VII, the case-mortality and infectivity were both regarded as constants. It is very unlikely that this is strictly true; and it is much more probable that their values are highest at some definite short period after infection and then gradually die away.

B. In Section XI some attempt is made to meet this objection by introducing the case-duration as a second independent variable, in addition to the time, but there are innumerable other considerations that may also be taken into account, such as the age, sex, and social position of the patients, climate, temperature, and, in the case of insect-borne diseases, the habits of the carrier.

C. Except in XII (ii), the whole population has been divided into two classes only, the unaffected, and the affected who are considered to be both infective and also immune. But it is again unlikely that this is strictly true, and, in order to represent the facts accurately, a much larger number of classes would have to be considered.

D. Although it would probably be easy to write down equations on any given hypothesis as to the number and behaviour of the variables, these equations, like those of XI (vii), would probably be quite unsuited to numerical work, and we should have to fall back on laborious approximats methods such as expansion in series or mechanical quadratures.

E. The quantities x and L have been treated as continuous functions of the time, but, as they represent numbers of persons, they can only take integral values, and must be discontinuous unless they are constant; in the same way, f and l can only vary by integral multiples of $1/P$. This is a serious drawback when the numbers are not large. In a small village, the average number of cases in the period between two epidemics may be only two or three, and, if these are removed by any chance, the whole course of events is entirely changed; there is no one to keep alive the infection, and a new generation may grow up entirely non-immune. Then the entry of a fresh source of infection has the same effect as if c rose suddenly from 0 to its normal value. In fact, in the case of small populations, other than the most probable values may often obtain, so that the whole of this paper is applicable to large numbers rather than to small ones.

The Initial Wave Resistance of a Moving Surface Pressure.

By T. H. Havelock, F.R.S., Professor of Applied Mathematics in the University of Durham.

(Received January 18, 1917.)

1. The study of the water waves produced by the motion of an assigned pressure distribution over the surface has hitherto been limited to the steady state attained when the system has been moving with uniform velocity for a very long time. In his latest series of papers on water waves, Lord Kelvin* made an elaborate graphical and numerical study of cognate problems, and expressed the hope of applying his methods to calculate the initiation and continued growth of canal ship-waves due to the sudden commencement and continued application of a moving, steady surface pressure.

In the following paper, I have not attempted any analysis of the surface elevation itself, but I have proceeded directly to the calculation of the corresponding wave resistance. At present the wave resistance is known only for the steady state for certain localised pressure systems in uniform motion, and it seems desirable to attempt some estimate of the time taken to attain this state when we take into account the beginnings of the motion. One might examine the effect of initial acceleration, but I have limited the problem by considering only the case of a system which is suddenly established, and is at the same instant set in motion with uniform velocity.

* Kelvin, 'Math. and Phys. Papers,' vol. 4, p. 456 (1906).