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The simple stochastic epidemic: a complete solution in terms of known functions†

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1. THE MATHEMATICAL MODEL

The so-called 'simple stochastic epidemic' is a simplified epidemic model which involves infection but not recovery (see Bailey, 1950, 1957). It might well be applicable, for example, to the spread of a mild upper respiratory infection, at least in the preliminary stages of rapid spread before recovery or removal from circulation could occur.

We suppose that initially, when the time $t = 0$, there is a group of n susceptibles and one infective. These are assumed to mix together homogeneously and randomly with contact-rate β. This means that the chance of a contact, sufficient to transmit infection, in an interval Δt between any two *specified* individuals is $\beta\Delta t$. Let the probability that there are still r susceptibles uninfected at time t be $p_r(t)$. Thus the chance of one new infection in the whole group in an interval Δt is $\beta r(n-r+1)\Delta t$, since there are r susceptibles and $n-r+1$ infectives. If we change the time scale to $\tau = \beta t$, the chance becomes $r(n-r+1)\Delta\tau$. The basic differential-difference equations for the process are easily found to be

$$\left.\begin{aligned} \frac{dp_r}{d\tau} &= (r+1)(n-r)p_{r+1} - r(n-r+1)p_r \quad (0 \leqslant r \leqslant n-1), \\ \frac{dp_n}{d\tau} &= -np_n, \end{aligned}\right\} \tag{1}$$

with initial condition

$$p_n(0) = 1. \tag{2}$$

We are concerned to find a compact, easily manageable solution of equation (1) in terms of known functions.

2. SOLUTION BY LAPLACE TRANSFORMS

In principle, the system of equations in (1) can be solved successively, starting with the equation for p_n. But in practice such a method is not satisfactory. Suppose, however, we use the Laplace transform defined by

$$\left.\begin{aligned} \phi^*(s) &= \int_0^\infty e^{-s\tau}\phi(\tau)\,d\tau, \qquad R(s) > 0, \\ \phi(\tau) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{s\tau}\phi^*(s)\,ds, \end{aligned}\right\} \tag{3}$$

† The work on which this paper is based was partially completed while the author was a Visiting Professor in the Department of Statistics, Stanford University, during 1962.

where c is positive and greater than the real parts of all the singularities of $\phi^*(s)$. Applying the transformation (3) to the equations in (1), and using (2), gives the recurrence relations

$$\left.\begin{aligned} p_r^* &= \frac{(r+1)(n-r)}{s+r(n-r+1)} p_{r+1}^* \quad (0 \leqslant r \leqslant n-1), \\ p_n^* &= \frac{1}{s+n}, \end{aligned}\right\} \tag{4}$$

for the transformed probabilities $p_r^* \equiv p_r^*(s)$.

An explicit solution to (4) is easily obtained in the form

$$p_r^* = \frac{n!(n-r)!}{r!} \prod_{j=r}^{n} \{s+j(n-j+1)\}^{-1} \quad (0 \leqslant r \leqslant n). \tag{5}$$

In principle we have only to invert (5) to derive $p_r(\tau)$. The right side of (5) can be expanded as a sum of partial fractions containing terms like $\{s+j(n-j+1)\}^{-1}$ and $\{s+j(n-j+1)\}^{-2}$, the latter arising when there are repeated factors, which happens if $r < \frac{1}{2}(n+1)$. Thus the terms just noted lead to $e^{-j(n-j+1)\tau}$ and $\tau e^{-j(n-j+1)\tau}$. Unfortunately, the work involved in evaluating the partial fraction expansions, and determining the various coefficients is highly laborious, and the results are very untidy when obtained (see Bailey, 1957).

Obviously, a more convenient method of investigating the whole process is very desirable. We can, for instance, attempt to find a compact expression for the probability-generating function in terms of known functions, instead of having a lot of special expressions for individual probabilities. This can be done as follows.

3. THE PROBABILITY-GENERATING FUNCTION

The main obstacle to reasonably elegant algebraic derivations from the expression in (5) is the presence of repeated factors on the right if $r < \frac{1}{2}(n+1)$. This complication can be avoided by the following device.

Let us consider a slightly modified model in which the chance of one new infection in $\Delta\tau$ is *not* $r(n-r+1)\Delta\tau$ but $r(N-r+1)\Delta\tau$, where N is not an integer, but may for convenience be thought of as only slightly different from n, i.e. $N = n+\epsilon$, for example. The previous equations in (1) are now replaced by

$$\left.\begin{aligned} \frac{dp_r}{d\tau} &= (r+1)(N-r)p_{r+1} - r(N-r+1)p_r \quad (0 \leqslant r \leqslant n-1), \\ \frac{dp_n}{d\tau} &= -n(N-n+1)p_n, \end{aligned}\right\} \tag{6}$$

with initial condition $p_n(0) = 1$, as before. The transformed equations in (4) become

$$\left.\begin{aligned} p_r^* &= \frac{(r+1)(N-r)}{s+r(N-r+1)} p_{r+1}^* \quad (0 \leqslant r \leqslant n-1), \\ p_n^* &= \frac{1}{s+n(N-n+1)}, \end{aligned}\right\} \tag{7}$$

with solution

$$p_r^* = \frac{n!(N-r)\dots(N-n+1)}{r!} \prod_{j=r}^{n} \{s+j(N-j+1)\}^{-1} \quad (0 \leqslant r \leqslant n). \tag{8}$$

When $N = n$, (8) reduces of course to (5).

Since N is now not an integer, there are no repeated factors on the right of (8), and we may write p_r^* in the form

$$p_r^* = \sum_{j=r}^{n} \frac{c_{rj}}{s+j(N-j+1)} \quad (0 \leqslant r \leqslant j \leqslant n), \tag{9}$$

where
$$\begin{aligned} c_{rj} &= \lim_{s\to -j(N-j+1)} \{s+j(N-j+1)\}p_r^* \\ &= \frac{(-1)^{j-r}(N-2j+1)\,n!(N-r)!}{r!(j-r)!\,(n-j)!\,(N-n)!\,(N-j-r+1)\dots(N-j-n+1)}. \end{aligned} \tag{10}$$

The individual probabilities are now obtained directly from (9) by inversion, namely

$$p_r(\tau) = \sum_{j=r}^{n} c_{rj}\, e^{-j(N-j+1)\tau} \quad (0 \leqslant r \leqslant j \leqslant n). \tag{11}$$

A more compact result for the whole probability-generating function $P(x,\tau) = \sum_{r=0}^{n} p_r(\tau)\,x^r$ can be derived as follows. The Laplace transform of $P(x,\tau)$ can be written down in terms of the p_r^* in (9). Thus

$$\begin{aligned} P^*(x,s) &\equiv \sum_{r=0}^{n} p_r^* x^r \\ &= \sum_{r=0}^{n}\sum_{j=r}^{n} \frac{c_{rj}\,x^r}{s+j(N-j+1)} \\ &= \sum_{j=0}^{n}\sum_{r=0}^{j} \frac{c_{rj}\,x^r}{s+j(N-j+1)} \\ &= \sum_{j=0}^{n} \frac{1}{s+j(N-j+1)} \sum_{r=0}^{j} c_{rj}\,x^r. \end{aligned}$$

Therefore it follows that

$$P(x,\tau) = \sum_{j=0}^{n} e^{-j(N-j+1)\tau} G_j(x), \tag{12}$$

where
$$G_j(x) = \sum_{r=0}^{j} c_{rj}\,x^r. \tag{13}$$

We now examine the polynomial $G_j(x)$. Consider the ratio of the $(r+1)$th and rth terms, namely $c_{r+1,j}/c_{rj}$. From (10) it follows that

$$\frac{c_{r+1,j}}{c_{rj}} = \frac{(-j+r)\,(j-N-1+r)}{(-N+r)\,(r+1)}. \tag{14}$$

Hence
$$G_j(x) \equiv c_{0j}F(-j, j-N-1;\ -N, x), \tag{15}$$

where $F(-j, j-N+1;\ -N, x)$ is a terminating hypergeometric series. Writing $c_{0j} \equiv c_j$, we have from (10), with $r = 0$,

$$c_j = \frac{(-1)^j\,(N-2j+1)\,n!\,N!}{j!(n-j)!\,(N-n)!\,(N-j+1)\dots(N-j-n+1)} \tag{16}$$

and so (12) may be expressed as

$$P(x,\tau) = \sum_{j=0}^{n} c_j\, e^{-j(N-j+1)\tau} F(-j, j-N-1;\ -N, x). \tag{17}$$

It may be noticed that $c_0 = 1$, as expected since $P(x,\tau) \to 1$ as $\tau \to \infty$.

The special case in which $N = n$ is obtained simply by letting $\epsilon \to 0$. Algebraic complications now arise because one of the factors in the product $(N-j-r+1)\dots(N-j-n+1)$ in the denominator of (8) may be ϵ, and so $c_{rj} \to \infty$ as $\epsilon \to 0$. But, by pairing off terms involving

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c_{rj} and $c_{r,\, n-j+1}$, a finite limit is achieved. We finally obtain the awkward explicit expressions for the $p_r(\tau)$ previously given by Bailey (1957, pp. 40–41). Similarly, when dealing with the probability-generating function in (17), we combine terms involving c_j and c_{n-j+1}.

It therefore looks as though greater elegance and tractability are achieved by keeping N in as a non-integral quantity until the latest possible stage when we let $N \to n$. An example of this procedure applied to the calculation of the stochastic mean is given in § 5.

4. SOLUTION OF PARTIAL DIFFERENTIAL EQUATION FOR PROBABILITY-GENERATING FUNCTION

An alternative approach to many stochastic processes of the general type under discussion is, as is well known, via a partial differential equation for the probability-generating function (or the corresponding functions for moments and cumulants). It is easily shown that for equations (1), the appropriate probability-generating function $P(x, \tau)$ satisfies

$$\frac{\partial P}{\partial \tau} = (1-x)\left(n\frac{\partial P}{\partial x} - x\frac{\partial^2 P}{\partial x^2}\right), \tag{18}$$

with initial condition

$$P(x, 0) = x^n. \tag{19}$$

More generally, the partial differential equation corresponding to equations (6) is

$$\frac{\partial P}{\partial \tau} = (1-x)\left(N\frac{\partial P}{\partial x} - x\frac{\partial^2 P}{\partial x^2}\right), \tag{20}$$

with the same initial condition as in (19). Now a standard method of solving such equations is to attempt a separation of variables followed by a summation of components depending on the appropriate eigenvalues (see, for example, Sagan, 1961). Suppose we try

$$P(x, \tau) = X(x)\,T(\tau), \tag{21}$$

where X is a function of x only, and T a function of τ only. Substituting (21) in (20) gives

$$\frac{T'}{T} = \frac{N(1-x)\,X' - x(1-x)\,X''}{X} = -\lambda, \quad \text{say}, \tag{22}$$

where λ is some suitable constant independent of x and τ.

From the first and third elements of (9) we obtain

$$T(\tau) = e^{-\lambda\tau}, \tag{23}$$

and from the second and third elements we have the second-order differential equation

$$x(1-x)\,X'' - N(1-x)\,X' - \lambda X = 0, \tag{24}$$

a solution of which is the hypergeometric function

$$X(x) = F(\alpha, \beta; \gamma, x), \tag{25}$$

where

$$\gamma = -N; \quad \alpha+\beta = -N-1, \quad \alpha\beta = \lambda. \tag{26}$$

The eigenvalues λ are determined by the fact that the general solution must be a polynomial of at most degree n in x. Hence the hypergeometric functions must terminate, and α, say, must be a negative integer subject to $\alpha+\beta = -N-1$. Therefore we can write the permissible values as

$$\left.\begin{aligned} \alpha = -j, \quad \beta = j-N-1, \\ \lambda_j = j(N-j+1) \quad (j = 0, 1, \ldots, n). \end{aligned}\right\} \tag{27}$$

A general solution of the type sought can now be expressed in the form

$$P(x,\tau) = \sum_{j=0}^{n} d_j\, e^{-j(N-j+1)\tau} F(-j, j-N-1;\ -N, x). \tag{28}$$

This is essentially the same as the result previously obtained in (17). In the present derivation we have not as yet determined the constants d_j, although we know from the previous discussion that $d_j = c_j$.

It should now be clearer why the use of non-integral N makes the investigation more manageable. If N is not an integer, all the eigenvalues λ_j are distinct, corresponding in fact to the $n+1$ possible states of the system. When N is an integer repeated eigenvalues appear, e.g. $\lambda_j \equiv \lambda_{n-j+1}$, and the general solution must contain terms in $\tau e^{-j(N-j+1)}$. The subsequent treatment is then not nearly so simple. On the other hand, with non-integral N and distinct eigenvalues, the d_j can easily be determined from the orthogonality relations obeyed by the hypergeometric functions, using the initial condition.

Thus when $\tau = 0$ the d_j must satisfy

$$x^n = \sum_{j=0}^{n} d_j F(-j, j-N-1;\ -N, x). \tag{29}$$

Now the standard theory of such functions is best pursued in terms of Jacobi polynomials (see Erdélyi, 1953, vol. 2, Chapter 10), represented by $P_j^{\mu,\nu}(y)$ $(j \geqslant 0;\ \mu, \nu > -1)$, where

$$P_j^{\mu,\nu}(y) = \binom{j+\mu}{j} F\left(-j, j+\mu+\nu+1;\ \mu+1,\ \frac{1-y}{2}\right). \tag{30}$$

We require the special values $\mu = -N-1$, $\nu = -1$. Admittedly, the standard results are restricted to $\mu > -1$, but it can be seen that if we calculate the coefficients for the restricted case, they will also hold more generally.

We consider, therefore, the coefficients d_j satisfying

$$\left(\frac{1-y}{2}\right)^n = \sum_{j=0}^{n} d_j P_j^{\mu,\nu}(y) \Big/ \binom{j+\mu}{j}, \tag{31}$$

with $\frac{1}{2}(1-y) = x$ to complete the correspondence between (29) and (31).

The following standard results for Jacobi polynomials are conveniently applied in the present context

$$\left.\begin{aligned} \int_{-1}^{1} w(y)\, P_j^{\mu,\nu}(y)\, P_k^{\mu,\nu}(y)\, dy &= \frac{2^{\mu+\nu+1}\Gamma(j+\mu+1)\,\Gamma(j+\nu+1)}{(2j+\mu+\nu+1)\,j!\,\Gamma(j+\mu+\nu+1)} \quad (k=j); \\ &= 0 \quad (k \neq j); \end{aligned}\right\} \tag{32}$$

and

$$\int_{-1}^{1} w(y) f(y)\, P_j^{\mu,\nu}(y)\, dy = \frac{2^{-j}}{j!} \int_{-1}^{1} f^{(j)}(y)\, w(y)\, (1-y^2)^j\, dy, \tag{33}$$

where $w(y)$ is the appropriate weight-function $(1-y)^{\mu}(1+y)^{\nu}$. If we put $f(y) = \{\frac{1}{2}(1-y)\}^n$ in (33), we obtain the special result

$$\begin{aligned} \int_{-1}^{1} w(y) \left(\frac{1-y}{2}\right)^n P_j^{\mu,\nu}(y)\, dy &= \frac{(-1)^j\, 2^{-n-j}\, n!}{j!\,(n-j)!} \int_{-1}^{1} (1-y)^{n+\mu}\, (1+y)^{j+\nu}\, dy \\ &= \frac{(-1)^j\, 2^{\mu+\nu+1} n!\, \Gamma(n+\mu+1)\, \Gamma(j+\nu+1)}{j!(n-j)!\, \Gamma(n+j+\mu+\nu+2)}. \end{aligned} \tag{34}$$

We now simply multiply (31) through by $w(y)\,P_j^{\mu,\nu}(y)$, and integrate from -1 to $+1$. Using (32) and (34) then quickly gives

$$d_j = \frac{(-1)^j\, n!(2j+\mu+\nu+1)\,(n+\mu)\ldots(\mu+1)}{j!(n-j)!\,(n+j+\mu+\nu+1)\ldots(j+\mu+\nu+1)}, \tag{35}$$

after cancelling common factors, and in particular replacing ratios of gamma functions by appropriate products. If we now put $\mu = -N-1$ and $\nu = -1$, equation (35) reduces to the quantity c_j already given in (16), as it must.

It is suggested that the methods of the present section may be useful in tackling other stochastic processes in which an explicit solution by other methods is not already available.

5. Derivation of the stochastic mean

A closed expression for the stochastic mean in the special case $N = n$ was first given by Haskey (1954), who derived it by means of some very skilful, but extremely heavy, algebra based on partial fraction expansions of the Laplace transforms of probabilities.

In the general case of non-integral N, we use the expression for $P(x,\tau)$ in (17) to yield

$$\begin{aligned} m(\tau) &= P'(1,\tau)\\ &= \sum_{j=0}^{n} c_j\, e^{-j(N-j+1)\tau}\frac{(-j)\,(j-N-1)}{(-N)}\,F(-j+1, j-N;\ -N+1;\ 1)\\ &= -\sum_{j=1}^{n} c_j\, e^{-j(N-j+1)\tau}\frac{\Gamma(N-j+2)\,\Gamma(j+1)}{\Gamma(N+1)}, \end{aligned} \tag{36}$$

since
$$F(-j+1, j-N;\ -N+1, 1) \equiv \frac{\Gamma(N-j+1)\,\Gamma(j)}{\Gamma(N)}.$$

Substituting in (36) the value of c_j given by (16) yields the simple expression

$$m(\tau) = \sum_{j=1}^{n} e^{-j(N-j+1)\tau}\frac{(-1)^{j+1}(N-2j+1)\,n!}{(n-j)!\,(N-n)\ldots(N-j-n+1)}. \tag{37}$$

For the special case $N \to n$ we put $N = n+\epsilon$ in (37) and let $\epsilon \to 0$. The factor $(N-n)^{-1}$ on the right of (37) yields a factor ϵ^{-1}, but if we combine the terms for j and $n-j+1$ a finite limit is obtained. We quickly arrive at Haskey's result

$$m(\tau) = \sum_{j=1} \frac{n!}{(n-j)!\,(j-1)!}\, e^{-j(n-j+1)\tau}\left\{(n-2j+1)^2\tau + 2 - \sum_{v=j}^{n-j} v^{-1}\right\}, \tag{38}$$

where the upper limit is $\tfrac{1}{2}n$ for n even; and $\tfrac{1}{2}(n+1)$ for n odd with the introduction of a factor of $\tfrac{1}{2}$ for the term given by $j = \tfrac{1}{2}(n+1)$.

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