Discrete Fibration Semantics of Dependent Types

Zesen Qian(zesenq)

May 15, 2018

1 Category Preliminaries

Definition 1 (Category of Poset)**.** *$\mathfrak{Pos}$ is the category in which objects are partially ordered sets and arrows are monotone mappings.*

Definition 2 (Discrete fibration)**.** *An arrow in $\mathfrak{Pos}$, $f : A \to B$ a* discrete fibration *iff: For any $a \in A$ and $b \in B$, if $f(a) \leq b$, then there exists a unique a' such that $a \leq a'$ and $f(a') = b$.*

We denote such an arrow by $\twoheadrightarrow$ (instead of $\to$ for arbitrary monotone mappings) if it matters in the context.

Definition 3 (Category of fibration)**.** *$Fib(A)$ is the category in which objects are fibrations into a poset A, and an arrow between two fibrations $f : B \twoheadrightarrow A, g : C \twoheadrightarrow A$ is any monotone function $h : B \to C$ such that $f = g \circ h$.*

Lemma 4. *$\mathfrak{Pos}$ has pullbacks.*

Proof. Given $f : D \to C$, $g : E \to C$, define the *canonical pullback* $D \times_C E$ to be

$$\{(d, e) \in D \times E | f(d) = g(e)\}$$

with order

$$(d, e) \leq (d', e') \Leftrightarrow d \leq d' \wedge e \leq e'.$$

, and projection

$$\pi_0 = (d, e) \mapsto d : D \times_C E \to D$$
$$\pi_1 = (d, e) \mapsto e : D \times_C E \to E$$

It's easy to show that it's indeed a pullback. □

We will denote the pulling back of g along f by $f^*(g)$.

Theorem 5. *In $\mathfrak{Pos}$, for $f : B \to A$, $g : C \to A$, if g is fibration, then f^*g is fibration as well.*

Proof. As shown in the figure, the canonical pullback is defined to be $\{(b,c) \in B \times C | f(b) = g(c)\}$, whose order is the conjunction of the component-wise order. $f^*g$is defined to be the first projection π_0. To show it's a fibration, we are given $(b,c) \in B \times C$, $b^* \in B$, and $f(b) = g(c)$. If $\pi_0(b,c) \leq b^*$, we need to give a unique $(b',c') \in B \times C$ which satisfies $f(b') = g(c')$, such that $(b,c) \leq (b',c')$ and $\pi_0(b',c') = b^*$. This condition is equivalent to a unique $c' \in C$ such that $f(b^*) = g(c')$, $c \leq c'$. But we know that $g(c) = f(b) \leq f(b^*)$, and g is fibration, so there is a unique c' such that $c \leq c'$ and $g(c) = f(b^*)$, which is exactly what's needed. $\square$

From now on, the functor f^* might goes $Fib(A) \to Fib(B)$ (instead of $\mathfrak{Pos}/A \to \mathfrak{Pos}/B$) depending on the context.

Theorem 6 (Composition of fibration)**.** *The composition of two fibrations is still fibration.*

Proof. Say there is $f : A \twoheadrightarrow B$ and $g : B \twoheadrightarrow C$. Now for any $a \in A$ and $c \in C$, if $g(f(a)) \leq c$, then there is a unique b' such that $f(a) \leq b'$ and $g(b') = c$, since g is a fibration. By a similar argument, there is a unique a' such that $a \leq a'$ and $f(a') = b'$. In summary, we found a unique a' such that $a \leq a'$ and $g(f(a')) = c$. $\square$

From now on, the functor $f_!$ goes $Fib(A) \to Fib(B)$(instead of $\mathfrak{Pos}/A \to \mathfrak{Pos}/B$) if we already know f is a fibration $A \twoheadrightarrow B$.

Lemma 7. *For $f : C \twoheadrightarrow D$, $f_! \dashv f^*$ between $Fib(C)$ and $Fib(D)$.*

Proof. This is easily witnessed by the UMP of pullbacks. $\square$

Definition 8 (Category of elements)**.** *For $P \in \mathfrak{Set}^{\mathbb{C}}$, we define $\int_{\mathbb{C}} P$ to be the category in which elements are*

$$\{(c,x) | c \in \mathbb{C}, x \in P(c)\}$$

with arrows

$$(c,x) \to (c',x') \Leftrightarrow c \to c' \wedge p(c \to c')(x) = x'$$

As a partially ordered set can be seen as a category, the above definition applies to them as well.

Definition 9 (Index projection)**.** *For $P \in \mathfrak{Set}^{\mathbb{C}}$, define the functor*

$$\pi_P = (c,x) \mapsto c : \int_{\mathbb{C}} P \to \mathbb{C}$$

with obvious valuation on arrows. It's trivial to see that it's a functor indeed.

Theorem 10. *For $P : \mathfrak{Set}^C$ where $C : \mathfrak{Pos}$, π_P is a fibration.*

Proof. It's monotone because it's functorial. To see it's a fibration, given (a,x) and a' where $\pi_p(a,x) \leq a'$, there is $(a', p(a \leq a')(x))$ satisfying the condition. The uniqueness is easy to check. $\square$

Theorem 11. *$Fib(C)$ is equivalent to $\mathfrak{Set}^C$.*

Proof. We will construct $F : Fib(C) \to \mathfrak{Set}^C$, $I : \mathfrak{Set}^C \to Fib(C)$.

F's behavious on objects are as follows. For $f : D \twoheadrightarrow C$, $F(f)$ is a functor in $\mathfrak{Set}^C$. For object $c \in C$, define $F(f)(c) = f^{-1}(c)$. For arrows $c_0 \le c_1$, define $F(f)(c_0 \le c_1)(d)$ to be the d' given by the lifting property.

Now define F's behaviour on arrows. For $h : f \to g$ where $f : D \twoheadrightarrow C$ and $g : E \twoheadrightarrow C$ and $f = g \circ h$, $F(h)$ has to be $F(f) \Rightarrow F(g)$. Define $F(h)_c(d) = h(d)$. The naturality condition is easy to check. In particular the commutivity is given by the lifting property.

I's behaviours on objects: for $P \in \mathfrak{Set}^A$, define $I(P) = \pi_P$. For a Nat.Trans $\phi : P \Rightarrow Q$, $I(\phi)$ has to be a monotone function $\int_C P \to \int_C Q$ such that $I(P) = I(Q) \circ I(\phi)$. We define $I(\phi)(c, x) = (c, \phi_c(x))$.

Now we will give $\epsilon : F \circ I \cong 1_{\mathfrak{Set}^C}$ and $\eta : 1_{Fib(C)} \cong I \circ F$. For $P : \mathfrak{Set}^C$, define $(\epsilon_P)_c : (\Pi_P)^{-1}(c) \to P(c)$ to be $(c, x) \mapsto x$. $(\epsilon_P)_c$ is obviously an isomorphism. The naturality of ϵ_P is easy to check. Therefore ϵ_P is an isomorphism. The naturality of ϵ is easy to check.

For $f : D \twoheadrightarrow C$, define $\eta_f : D \to \int_C f^{-1}$ to be $d \mapsto (f(d), d)$. Obviously η_f is an isomorphism, and η is natural. $\square$

The functor F are of course a family of functors relative to poset C, but the poset is usually clear from the context so we will just write F for this specific one. Similar for I.

Lemma 12. *For an equivalence $F, G : \mathbb{C} \cong \mathbb{D}$, we have both $F \dashv G$ and $G \dashv F$, with unit and counit derived from the natural transformation witnessting the equivalence.*

Proof. We kindly direct the reader to [1] for the proof. $\square$

Theorem 13. *For any small category $\mathbb{C}$, the Yoneda embedding*

$$y_{\mathbb{C}} = C \mapsto [C, -] : \mathbb{C} \to \mathfrak{Set}^{\mathbb{C}}$$

or just y when the category is clear from the context, has the following UMP: given any cocomplete category $\mathbb{E}$ and functor $F : \mathbb{C} \to \mathbb{E}$, there is a colimit preserving functor $F_{\hat{!}} : \mathfrak{Set}^{\mathbb{C}} \to \mathbb{E}$ such that

$$F_{\hat{!}} \circ y \cong F$$

and such a functor is unique up to natural isomorphism.

Proof. (Adapted from [2] proposition 9.16). We will give an adjunction $F_{\hat{!}} \dashv F^{\hat{*}}$, then $F_{\hat{!}}$ will of course preserve colimits. Now to define $F_{\hat{!}}$, given a presheaf $P \in \mathfrak{Set}^{\mathbb{C}}$, we first rewrite it as a colimit of representables:

$$P \cong \varinjlim_{j \in J} y(\pi_P(j))$$

where $J = \int_{\mathbb{C}} P$, then define

$$F_{\hat{!}} = P \mapsto \varinjlim_{j \in J} F(\pi_P(j)) : \mathfrak{Set}^{\mathbb{C}} \to \mathbb{E}$$

and

$$F^{\hat{*}} = E \mapsto C \mapsto [F(C), E]_{\mathbb{E}} : \mathbb{E} \to \mathfrak{Set}^{\mathbb{C}}$$

Now check that indeed $F_{\hat{!}} \dashv F^{\hat{*}}$.

$$\begin{aligned}
&[P, F^{\hat{*}}(E)] \\
\cong&[\varinjlim_{j\in J} y(\pi_P(j)), F^{\hat{*}}(E)] \\
\cong&\varprojlim_{j\in J} [y(\pi_P(j)), F^{\hat{*}}(E)] && (y \text{ turns } \lim_{\rightarrow} \text{ to } \lim_{\leftarrow}) \\
\cong&\varprojlim_{j\in J} F^{\hat{*}}(E)(\pi_P(j)) && \text{(Yoneda lemma)} \\
\cong&\varprojlim_{j\in J} [F(\pi_P(j), E] && (\text{def. of } F^{\hat{*}}) \\
\cong&[\varinjlim_{j\in J} F(\pi_P(j)), E] && (y \text{ turns } \lim_{\rightarrow} \text{ to } \lim_{\leftarrow}) \\
\cong&[F_{\hat{!}}(P), E] && (\text{def. of } F_{\hat{!}})
\end{aligned}$$

Now check that indeed $F_{\hat{!}} \circ y \cong F$. Given any $C \in \mathbb{C}$,

$$F_{\hat{!}}(y(C)) = \varinjlim_{j\in J} F(\pi_P(j))$$

where $P = [C, -]$ and $J = \int_{\mathbb{C}} P$. Note there is a terminal object $(C, 1_C) \in J$. By an observation of colimit's definition, the value of the colimit is therefore simply the diagram $F \circ \pi_P$ at the point of terminal object, which gives us FC.

Finally, since $F_{\hat{!}}$ satisifies this equation, its valuation on representable presheaves must be unique up to isomorphism. Moreover, it preserves colimits, and all other presheaves can be represented as colimits of representable ones, so its valuation overall must be unique up to isomorphism. □

Definition 14. *For small categories $\mathbb{C}, \mathbb{D}$, and $f : \mathbb{C} \to \mathbb{D}$, define the functor*

$$f^{\hat{*}} = Q \mapsto c \mapsto Q(f(c)) : \mathfrak{Set}^{\mathbb{D}} \to \mathfrak{Set}^{\mathbb{C}}$$

Theorem 15. *There exist functors $f_{\hat{!}}, f_{\hat{*}} : \mathfrak{Set}^{\mathbb{C}} \to \mathfrak{Set}^{\mathbb{D}}$ such that $f_{\hat{!}} \dashv f^{\hat{*}} \dashv f_{\hat{*}}$.*

Proof. (Adapted from [2] corrolarry 9.17). First define

$$F = y_{\mathbb{D}} \circ f : \mathbb{C} \to \mathfrak{Set}^{\mathbb{D}} : \mathbb{C} \to \mathfrak{Set}^{\mathbb{D}}$$

. Apply 13, we have

$$F_{\hat{!}} \dashv F^{\hat{*}}$$

with $F_{\hat{!}} \circ y \cong F = y_{\mathbb{D}} \circ f$. I claim that $f^{\hat{*}} \cong F^{\hat{*}}$, because:

$$\begin{aligned}
&F^{\hat{*}}(Q)(C) \\
=&[FC, Q] && \text{(def. in 13)} \\
=&[y(f(C)), Q] && (\text{def. of } F) \\
\cong&Q(f(C)) && \text{(Yoneda lemma)} \\
=&f^{\hat{*}}(Q)(C) && (\text{def. of } f^{\hat{*}})
\end{aligned}$$

Define $f_{\hat{!}} = F_{\hat{!}}$ so we have $f_{\hat{!}} \dashv f^{\hat{*}}$. For the other adjunction, define

$$G = f^{\hat{*}} \circ y_{\mathbb{D}} : \mathbb{D} \to \mathfrak{Set}^{\mathbb{C}}$$

and apply 13 again, which gives us

$$G_{\hat{!}} \dashv G^{\hat{*}}$$

with $G_{\hat{!}} \circ y_{\mathbb{D}} \cong G = f^{\hat{*}} \circ y_{\mathbb{D}}$. We need $G_{\hat{!}} \cong f^{\hat{*}}$ so we can just define $f_{\hat{*}} = G^{\hat{*}}$. By the uniqueness given in 13, we only need to show that $f^{\hat{*}}$ satisfy the equation:

$$f^{\hat{*}} \circ y_{\mathbb{D}} \cong G = f^{\hat{*}} \circ y_{\mathbb{D}}$$

which is immediate, and that it preserves colimit, which is also true because for any $\varinjlim_{j \in J} Q_j \in \mathfrak{Set}^{\mathbb{D}}$:

$$\begin{aligned}
&(f^{\hat{*}}(\varinjlim_{j\in J} Q_j)(C) \\
\cong&(\varinjlim_{j\in J} Q_j)(f(C)) && \text{(def. of } f^{\hat{*}}) \\
\cong&\varinjlim_{j\in J} (Q_j(f(C))) && (\varinjlim \text{ in } \mathfrak{Set}^{\mathbb{C}} \text{ is point-wise}) \\
\cong&\varinjlim_{j\in J} ((f^{\hat{*}}(Q_j))(C)) && \text{(def. of } f^{\hat{*}}) \\
\cong&(\varinjlim_{j\in J} (f^{\hat{*}}(Q_j)))(C) && (\varinjlim \text{ in } \mathfrak{Set}^{\mathbb{C}} \text{ is point-wise})
\end{aligned}$$

□

Lemma 16. *For $g : E \twoheadrightarrow D$, $f : C \twoheadrightarrow D$, we have $f^*(g) = I(f^{\hat{*}}(F(g))$, where F, I is the equivalence $Fib(C) \cong \mathfrak{Set}^C$ given in 11.*

Proof. By the definition given in 11, we have $I(f^{\hat{*}}(F(g))) : \int_C(g^{-1} \circ f) \twoheadrightarrow C$ defined to be $(c, e) \mapsto c$. By the canonical definition of f^*, we have $f^*(g) : C \times_D E \twoheadrightarrow C$ to be $(c, e) \mapsto c$. Now, $\int_C(g^{-1} \circ f)$ is defined to be

$$\{(c, e) | e \in g^{-1}(f(c))\}$$

with order

$$(c, e) \leq (c', e') \Leftrightarrow c \leq c' \wedge e' = g^{-1}(f(c \leq c'))$$

, which can paraphrased to

$$\{(c, e) | g(e) = f(c)\}$$

with order

$$(c, e) \leq (c', e') \Leftrightarrow c \leq c' \wedge e \leq e'$$

, which is exactly the definition of $C \times_D E$. So the two fibrations are actaully equal. □

Theorem 17. *For $f : C \twoheadrightarrow D$, we have a chain of adjunction $I \circ f_{\hat{!}} \circ F \dashv f^* \dashv I \circ f_{\hat{*}} \circ F$ between $Fib(C)$ and $Fib(D)$, where $f_{\hat{!}}$ and $f_{\hat{*}}$ are defined in 15.*

Proof.

$$
\begin{aligned}
&[I(f_{\hat{!}}(F(A))), B] \\
\cong&[f_{\hat{!}}(F(A)), F(B)] && \text{(by 11 and 12)} \\
\cong&[F(A), f^{\hat{*}}(F(B))] && \text{(by 15)} \\
\cong&[A, I(f^{\hat{*}}(F(B))] && \text{(by 11 and 12)} \\
=&[A, f^*(B)] && \text{(by 16)}
\end{aligned}
$$

The proof for the other adjunction is similar.

□

Lemma 18. *For $f : C \twoheadrightarrow D$, we have $f_! \cong I \circ f_{\hat{!}} \circ F : Fib(C) \to Fib(D)$.*

Proof. By 7 and 17, adjoint is unique up to isomorphism. □

Definition 19. *For $f : C \twoheadrightarrow D$, define*

$$f_* = I \circ f_{\hat{*}} \circ F : Fib(C) \to Fib(D)$$

Corollary 20. *For $f : C \twoheadrightarrow D$, we have $f_! \dashv f^* \dashv f_*$ between $Fib(C)$ and $Fib(D)$.*

2 Semantics

The reader shall recall the syntax of type theory. As this paper intends to introduce a semantics adapted from the classical semantics in locally cartesian closed categories, we will only focus on the different parts and kindly ask the reader to refer to other sources[3][4] for common parts.

The results in this section, including definitions and proofs, might refer to each other. This might seem circular, but I should remind the reader that such results are all defined or proved by induction, and whenever a cross-reference appears, the result referred to is always prior to the result referring in the order of induction. So overall the arguments are all well-formed. Also we may implicitly use some results about the syntax, but they shoud be obvious and not hard to prove.

Here is the general scheme of our semantics:

intuition	syntax	semantics
Γ is a valid context	$\Gamma \vdash$	$[\![\Gamma]\!] \in \mathfrak{Pos}$
A is a valid type in Γ	$\Gamma \vdash A$	$[\![\Gamma, A]\!] \overset{[\![A]\!]}{\twoheadrightarrow} [\![\Gamma]\!]$
t is a valid term of type A in Γ	$\Gamma \vdash t : A$	$[\![\Gamma, A]\!] \overset{[\![t]\!]}{\leftarrow} [\![\Gamma]\!]$

A context Γ will be an object in $\mathfrak{Pos}$. A type A under context Γ will be a fibration going into $[\![\Gamma]\!]$, and we will take the interpretation for Γ, A to be the domain of the fibration. A term of type A under context Γ will be a section to $[\![A]\!]$ which doesn't have to a fibration.

Theorem 21 (Beck-Chevalley condition)**.** *Given the derivation of $\Gamma \vdash s : A$, $\Gamma, x : A \vdash B$, and $\Gamma \vdash B[s/x]$, we claim that $[\![B[s/x]]\!] = [\![s]\!]^*([\![B]\!])$.*

Proof. Recall that $B[s/x]$ is defined inductively on B. For example, if $B = \Sigma_{y:C} D$, then

$$\begin{aligned} &(\Sigma_{y:C} D)[s/x] \\ =&\Sigma_{y:C[s/x]} D[s/x] && \text{(def. of subst.)} \end{aligned}$$

It suffices to show that(after applying some permutation and weakening rules)

$$[\![s]\!]^*([\![C]\!]_!([\![D]\!])) = ([\![s]\!]^*([\![C]\!]))_!(([\![C]\!]^*([\![s]\!]))^*([\![D]\!]))$$

which can be shown by gluing pullbacks. The case for Π is similar:

$$[\![s]\!]^*([\![C]\!]_*([\![D]\!])) = ([\![s]\!]^*([\![C]\!]))_*(([\![C]\!]^*([\![s]\!]))^*([\![D]\!]))$$

$\square$

Definition 22. *The semantics(adapted from [4]) is defined inductively on derivation of judgments.*

- $$\frac{\Gamma, x : A \vdash B}{\Gamma \vdash \Sigma_{x:A} B}$$

  Given $[\![A]\!] : [\![\Gamma, A]\!] \twoheadrightarrow [\![\Gamma]\!]$ *and* $[\![B]\!] : [\![\Gamma, A, B]\!] \twoheadrightarrow [\![\Gamma, A]\!]$, *we take* $[\![\Sigma_{x:A} B]\!]$ *to be* $[\![A]\!]_!([\![B]\!])$.

- $$\frac{\Gamma \vdash s : A \quad \Gamma \vdash t : B[s/x]}{\Gamma \vdash (s, t) : \Sigma_{x:A} B}$$

  Given $[\![A]\!] : [\![\Gamma, A]\!] \twoheadrightarrow [\![\Gamma]\!]$, $[\![s]\!] : [\![\Gamma]\!] \to [\![\Gamma, A]\!]$, $[\![B]\!] : [\![\Gamma, A, B]\!] \twoheadrightarrow [\![\Gamma, A]\!]$, $[\![B[s/x]]\!]$ *will be* $[\![s]\!]^*([\![B]\!])$, *and we are further given section* $[\![t]\!] : [\![\Gamma]\!] \to [\![\Gamma, B[s/x]]\!]$. *Take* $[\![(s, t)]\!]$ *to be* $[\![B]\!]^*([\![s]\!]) \circ [\![t]\!]$. *The condition of section is immediate.*

- $$\frac{\Gamma \vdash t : \Sigma_{x:A} B}{\Gamma \vdash \pi_0 t : A}$$

  Given $[\![A]\!] : [\![\Gamma, A]\!] \twoheadrightarrow [\![\Gamma]\!]$, $[\![B]\!] : [\![\Gamma, A, B]\!] \twoheadrightarrow [\![\Gamma, A]\!]$, *and section* $[\![t]\!] : [\![\Gamma]\!] \to [\![\Gamma, A, B]\!]$, *take* $[\![\pi_0 t]\!]$ *to be* $[\![B]\!] \circ [\![t]\!]$. *The section condition is immediate.*

- $$\frac{\Gamma \vdash t : \Sigma_{x:A} B}{\Gamma \vdash \pi_1 t : B[\pi_0 t/x]}$$

  Given $[\![A]\!] : [\![\Gamma, A]\!] \twoheadrightarrow [\![\Gamma]\!]$, $[\![B]\!] : [\![\Gamma, A, B]\!] \twoheadrightarrow [\![\Gamma, A]\!]$, *and section* $[\![t]\!] : [\![\Gamma]\!] \to [\![\Gamma, A, B]\!]$, $[\![B[\pi_0 t/x]]\!]$ *will be* $([\![B]\!] \circ [\![t]\!])^*([\![B]\!])$. *Observe* $1_{[\![\Gamma]\!]} : [\![\Gamma]\!] \to [\![\Gamma]\!]$ *and* $[\![t]\!] : [\![\Gamma]\!] \to [\![\Gamma, A, B]\!]$, *and* $[\![B]\!] \circ [\![t]\!] = ([\![B]\!] \circ [\![t]\!]) \circ 1_{[\![\Gamma]\!]}$, *by the UMP of pullback we can get some arrow* $[\![\Gamma]\!] \to [\![\Gamma, B[\pi_0 t/x]]\!]$, *we take this to be* $[\![\pi_1 t]\!]$. *The condition of section is immediate.*

- $$\frac{\Gamma, x : A \vdash B}{\Gamma \vdash \Pi_{x:A} B}$$

  Given $[\![A]\!] : [\![\Gamma, A]\!] \twoheadrightarrow [\![\Gamma]\!]$ *and* $[\![B]\!] : [\![\Gamma, A, B]\!] \twoheadrightarrow [\![\Gamma, A]\!]$*, we take* $[\![\Pi_{x:A} B]\!]$ *to be* $[\![A]\!]_*([\![B]\!])$*.*

- $$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : \Pi_{x:A} B}$$

  By the adjunction $[\![A]\!]^* \dashv [\![A]\!]_*$*, we have* $[[\![A]\!]^*(1_{[\![\Gamma]\!]}), [\![B]\!]] \cong [1_{[\![\Gamma]\!]}, [\![A]\!]_*([\![B]\!])]$*. Note* $[\![t]\!]$ *can be seen as an element on the left side. Push it to the right . This will be a monotone mapping* $e : [\![\Gamma]\!] \to [\![\Gamma, \Pi_{x:A} B]\!]$ *with* $[\![\Pi_{x:A} B]\!] \circ e = 1_{[\![\Gamma]\!]}$*. Take this section to be* $[\![\lambda x.t]\!]$

- $$\frac{\Gamma \vdash s : \Pi_{x:A} B \quad \Gamma \vdash t : A}{\Gamma \vdash st : B[t/x]}$$

  Pull back $[\![A]\!]_*([\![B]\!])$ *along* $[\![A]\!]$*. Observe* $1_{[\![\Gamma, A]\!]} : [\![\Gamma, A]\!] \twoheadrightarrow [\![\Gamma, A]\!]$ *and* $[\![s]\!] \circ [\![A]\!] : [\![\Gamma, A]\!] \to [\![\Gamma, \Pi_{x:A} B]\!]$*. By the UMP of pullback we have an arrow* $g : [\![\Gamma, A]\!] \to [\![A]\!]^*([\![\Gamma, \Pi_{x:A} B]\!])$*.*

  From the adjunction $[\![A]\!]^* \dashv [\![A]\!]_*$ *we have the counit* $\epsilon : [\![A]\!]^* \circ [\![A]\!]_* \Rightarrow 1_{Fib([\![\Gamma, A]\!])}$*. In particular, the component* $\epsilon_{[\![B]\!]} : [\![A]\!]^*([\![A]\!]_*([\![B]\!])) \to [\![B]\!]$ *is actually a monotone mapping* $[\![A]\!]^*([\![\Gamma, \Pi_{x:A} B]\!]) \to [\![\Gamma, A, B]\!]$ *with* $[\![B]\!] \circ \epsilon_{[\![B]\!]} = [\![A]\!]^*([\![A]\!]_*([\![B]\!]))$*.*

  Now obeserve $1_{[\![\Gamma]\!]} : [\![\Gamma]\!] \to [\![\Gamma]\!]$ *and* $\epsilon_{[\![B]\!]} \circ g \circ [\![t]\!] : [\![\Gamma]\!] \to [\![\Gamma, A, B]\!]$*. Also* $[\![t]\!] \circ 1_{[\![\Gamma]\!]} = [\![B]\!] \circ \epsilon_{[\![B]\!]} \circ g \circ [\![t]\!]$*, which is easy to check. By the UMP of pullback we have an arrow* $e : [\![\Gamma]\!] \to [\![\Gamma, B[t/x]]\!]$ *with* $[\![B[t/x]]\!] \circ e = 1_{[\![\Gamma]\!]}$*. Take this section to be* $[\![st]\!]$*.*

Theorem 23 (Soundness)**.** *If* $\bullet \vdash t : A$ *is derivable, then in any model for the theory,* $[\![t]\!] : [\![\bullet]\!] \to [\![A]\!]$ *must be a section of* $[\![A]\!] : [\![A]\!] \to [\![\bullet]\!]$*.*

Proof. By induction on derivation. For axioms such as $\bullet \vdash$, the interpretation is given by the model's parameters and we require it to satisfy the general scheme(at the begining of the section); for rules, the interpretation is given by the inductive definition above and they preserve the scheme. In particular, this theorem is the third row of the scheme. □

References

[1] Adjoint equivalences. `https://ncatlab.org/nlab/show/adjoint+equivalence`.

[2] Steve Awodey. *Category theory*. Oxford University Press, 2010.

[3] Martin Hofmann. Syntax and semantics of dependent types. In *Extensional Constructs in Intensional Type Theory*, pages 13–54. Springer, 1997.

[4] Peter T Johnstone. *Sketches of an elephant: A topos theory compendium*, volume 2, chapter D4.4. Oxford University Press, 2002.