

Formation of structure in the Universe

Growth of Perturbation

⇒ Newtonian eq. of motion for non-relativistic matter.

We will follow space and time dependent density $\rho(\vec{r}, t)$, its velocity $\vec{v}(\vec{r}, t)$ and the gravitational potential.

The eq. of self gravitating gas dynamics are as follows:

1. Continuity eq.

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0}$$

2. The velocity field $\vec{v}(\vec{r}, t)$ obeys the Euler eq.

$$\boxed{\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{\nabla p}{\rho} - \nabla \phi}$$

3. Gravitational potential obeys Poisson eq.

$$\boxed{\nabla^2 \phi = 4\pi G \rho}$$

In Case of relativistic matter

$$1. \frac{\partial f}{\partial t} + \nabla \cdot \left[\left(\rho + \frac{P}{c^2} \right) \vec{v} \right] = 0$$

$$2. \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = - \frac{\nabla P}{\rho + P/c^2} - \nabla \phi$$

$$3. \nabla^2 \phi = 4\pi G \rho \left(1 + \frac{3P}{\rho c^2} \right)$$

We can only apply these equations if the particles form a gas, i.e. if they have regular collisions so that their energy and momenta are thermalized. A freely streaming sea of particles such as the CMB or radiation from stars does not act as a gas and we can not apply the above equations to them. However, before the release of the CMB at $z < 1100$ the electromagnetic radiation is firmly coupled (both energetically and kinetically) to the baryons. For $z \gg 1100$ the baryon-photon sea is therefore very well approximated by a gas, obeying above Eqs. Most of the inertial mass of this gas is dominated by the photons, with a small contribution by the baryons.

Though for $z < 3200$ the energy density of the Universe is dominated cold matter, baryonic matter is only 17% of the "cold" matter in the Universe, so the baryonic matter starts dominating over the radiation energy density only around $z \sim 550$, i.e. after CMB decoupling. During most of the time before $z \sim 1100$ the baryons therefore play a role as the medium that makes the "photon sea" a "photon gas"; their contribution to the inertial or gravitational mass is small. During this time the photon gas can produce sound waves (which would not be possible for a free-streaming photon sea).

- ④ linearized eq. for non-relativistic matter
 \Rightarrow let us assume that there are small perturbations

$$1. \rho(\vec{r}, t) = \rho_0(t) + \delta\rho(\vec{r}, t)$$

$$2. \vec{v}(\vec{r}, t) = \vec{v}_0(\vec{r}, t) + \delta\vec{v}(\vec{r}, t)$$

$$3. \phi(\vec{r}, t) = \phi_0(\vec{r}, t) + \delta\phi(\vec{r}, t)$$

Now upto first order:

Continuity eq.:

$$0 = \frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \delta\vec{v} + 3H\delta\rho + \vec{v}_0 \cdot \nabla \delta\rho$$

Euler eq.:

$$0 = \frac{\partial \delta\vec{v}}{\partial t} + \vec{v}_0 \cdot \nabla \delta\vec{v} + \delta\vec{v} \cdot \nabla \vec{v}_0 + \frac{\nabla \delta P}{\rho_0} + \nabla \delta\phi$$

Poisson eq.:

$$0 = \nabla^2 \delta\phi - 4\pi G \delta\rho$$

Introduce, $\delta := \frac{\delta p}{p_0}$

let us work out the following:

$$\textcircled{1} \quad \dot{\delta} = \frac{\partial \delta}{\partial t} = \frac{\partial(\delta p/p_0)}{\partial t} = \frac{1}{p_0} \frac{\partial \delta p}{\partial t} - \frac{\delta p}{p_0^2} \frac{\partial p_0}{\partial t}.$$

Now, $\frac{1}{p_0} \frac{\partial p_0}{\partial t} = \frac{\partial(\ln p_0)}{\partial t} \quad \left[p_0 \sim a^{-3} \text{ matter Case} \right]$

$$\Rightarrow \ln p_0 \sim -3 \ln a$$

$$\Rightarrow \partial(\ln p_0) \sim -3 \partial(\ln a)$$

$$\therefore \frac{1}{p_0} \frac{\partial p_0}{\partial t} = \frac{\partial \ln p_0}{\partial t} = -3 \frac{\partial \ln a}{\partial t} = -3H$$

$$\Rightarrow \boxed{\dot{\delta} = \frac{1}{p_0} \frac{\partial \delta p}{\partial t} + 3H \frac{\delta p}{p_0}}$$

$$\textcircled{2} \quad \boxed{\nabla \delta = \frac{\nabla \delta p}{p_0}}$$

$$\textcircled{3} \quad (\delta \vec{v} \cdot \nabla) \vec{v}_0 = (\delta \vec{v} \cdot \nabla) \vec{H} \vec{\sigma} = H \delta \vec{v}$$

The matter power Spectrum.

The power spectrum is simply the amplitude squared of the Fourier components.

$$P(\vec{k}) = \hat{\delta}^*(\vec{k}) \hat{\delta}(\vec{k}) = |\hat{\delta}(\vec{k})|^2$$

where, $\hat{\delta}(\vec{k}, t) = \int \delta(\vec{x}, t) e^{i\vec{k}\cdot\vec{x}} d\vec{x}$.

Since, in our case we expect that there is isotropy, we are only interested in $P(\vec{k})$ i.e., in the direction-averaged power spectrum.

Define $k := |\vec{k}|$ we then get,

$$P(k) = \frac{1}{4\pi} \oint |\hat{\delta}(\vec{k})|^2 d\Omega$$

The power spectrum function tells us how much "power" there is in perturbations of spatial scales of $\delta = 2\pi/k$

Matter power spectrum scales roughly as k^1 at low wavenumbers and as k^{-3} for large wavenumbers.

- The relation b/w the power spectrum and the autocorrelation function is given by:

$$\langle \delta(\vec{x}) \delta(\vec{x} + \vec{y}) \rangle = \int \frac{d^3 k}{(2\pi)^3} P(\vec{k}) e^{i \vec{k} \cdot \vec{y}}$$

or equivalently,

$$P(\vec{k}) = \int \langle \delta^*(\vec{x}) \delta(\vec{x} + \vec{y}) \rangle e^{i \vec{k} \cdot \vec{x}} d\vec{x}$$

The power spectrum is the Fourier transform of the autocorrelation function.

- The correlation function has the units of volume. So we can express it in dimensionless form by multiplying by the wavenumber cubed.

We assume that the early universe was scale invariant, so that the dimensionless correlation function is constant.

④ $\int d_n = \int_0^t \frac{dt}{\tilde{a}(t)} \quad \text{but} \quad d_H = a(t) \int_0^t \frac{dt}{\tilde{a}(t)}$

and let's say for matter dominated

$$d_H = a(t) \int_0^t \frac{dt}{\tilde{a}(t)} = 3t \sim H^{-1}$$

$\underbrace{\qquad}_{n?}$

$$\text{and} \quad H = \frac{\dot{a}}{a} = \frac{2}{3} \frac{1}{t} \quad H^2 = a \eta. ?$$

IDK

④ CMB as observed today consists of $\sim 10^5$ regions, which were never in causal contact; i.e., if we can trace back how many causally independent patches fit into what we observe today as CMB surface. $\sim 10^5$

④ The fact that these two scales: one that is associated with the particle horizon and the inverse Hubble are approximately the same if we take de-accelerated universe. But they are completely different once we have an early phase of inflation.

$$d_{\text{H}} \sim H^{-1}$$

particle horizon

④ Fourier transform of 2-point function

→ The power spectrum $P(k)$

$$\xi(r_{12}) = \langle S(\vec{r}_1), S(\vec{r}_2) \rangle$$

and
$$P(k) = \int \xi(r) e^{-ik\vec{r}} d^3r$$

Similarly, Fourier transf. of 3-pt function
is called the \rightarrow bispectrum.

3-point function is a function of triangle configuration

④ When the light reaching us was emitted after
the big bang, the sphere radius was
42 million light years in radius.

But since, the space is expanding it took
13.5 billion light years for that light to reach us.
And also the sphere,

from which the light was emitted also
expanded in size and that sphere would
now be 46 billion light years away (Radius).

Hence 98 billion light years of diameter.

④ If the spacetime is flat, then the entire universe possess zero net total energy. If the shape of the universe was anything else it would not have zero energy. Zero net energy means that you can create the entire universe out of nothing. Had the universe been filled with net positive or negative energy we would be forced to confront the question of what was the original source of the energy that began the universe. And if it had some amount of energy then why it has that amount and not some other amount.

$$\textcircled{5} \quad \lambda_{\text{obs}} = \lambda_{\text{emitted}} (1+z)$$

$$\Rightarrow 1+z = \frac{1}{a(t)}$$

④ Note that $d_H \equiv$ Hubble Horizon \Rightarrow Distance over which a photon can travel in one Hubble time. (Claim: d_H/a is increasing)

$$\text{And, } \frac{d_H}{a} = \frac{1}{Ha} = \frac{1}{\dot{a} \times a} = \frac{1}{\dot{a}}$$

and \dot{a} is decreasing $\therefore \frac{d_H}{a}$ is increasing.

\Rightarrow Horizon expanding faster than scale factor.

④ If we introduce a classical scalar field

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} \partial_\nu \phi \partial^\nu \phi - V(\phi) \right]$$

and observe the energy-momentum tensor

$$\text{i.e., } T_{\mu\nu} = \frac{\partial L}{\partial (\partial_\mu \phi)} \partial_\nu \phi - g_{\mu\nu} \text{ grad}$$

we will notice that we have off-diagonal species in the energy-momentum tensor.

i.e., T_{0i} and T_{ij} are non-zero.

$(i \neq j)$

Therefore scalar field actually doesn't behave as a perfect fluid. But these off-diagonal species depend on the gradient and therefore if we are in a region where gradient varies very slowly then we can neglect these terms. Essentially this means, field is homogeneous.

④ When free electron fraction is 1, everything is fully ionized.

④ Cross Section (= probability of interaction).

④ optical depth to Conformal time η is:

$$\tau(\eta) = \int_{\eta_0}^{\eta} \alpha n_e \sigma_T d\eta'$$

④ $e^{-\tau}$ is the probability that photon has not scattered from time η to today.

④ probability of last scattering: visibility

$$\eta = -\tau e^{-\tau}$$

④ Inflation predicts a quantity: Comoving Curvature perturbation. (constant and conserved outside horizon). You can think of this as \sim the potential perturbation initial condition at early time.

$$R \sim \dot{\Phi} \quad R = -\frac{(5+3w)}{3+3w} \dot{\Phi}$$

$$\textcircled{*} \quad \text{Define } \chi_{\text{src}}(z) = \int_{t_{\text{src}}(z)}^{t_0} \frac{c dt}{a(t)} = \int_0^z \frac{c dz'}{H(z')}$$



looks like a
comoving distance.

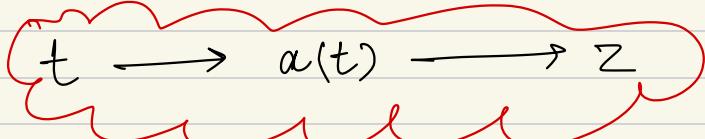
we know, $\left[a(t) = \frac{1}{1+z} \right] \Rightarrow \frac{da}{dt} = \frac{-1}{(1+z)^2} \frac{dz}{dt}$

$$\Rightarrow H = \frac{1}{a} \frac{da}{dt} = \frac{-1}{(1+z)^2} \frac{dz}{dt} \times \frac{1}{a} = \frac{-1}{1+z} \frac{dz}{dt}$$

$$\Rightarrow H(z) = \frac{-1}{1+z} \frac{dz}{dt} \quad \text{or} \quad 1+z = -\frac{dz}{dt} \times \frac{1}{H(z)} = \frac{1}{a(t)}$$

$$\therefore \int_{t_{\text{src}}(z)}^{t_0} \frac{c dt}{a(t)} = \int_z^0 \frac{c dt}{H(z')} - \frac{dz'}{dt}$$

$$= \int_0^z \frac{c dz'}{H(z')}$$


 $t \rightarrow a(t) \rightarrow z$

④ optical depth:

$$\tau = \text{Scattering rate} \times \text{time} (\Delta t)$$

So that means: optical depth is the no. of collisions a photon has undergone in time Δt .

or if: $\int_t^{t_0} dt' n_e(t') \sigma_T =$ no. of collision
photon has undergone
from time t to t_0

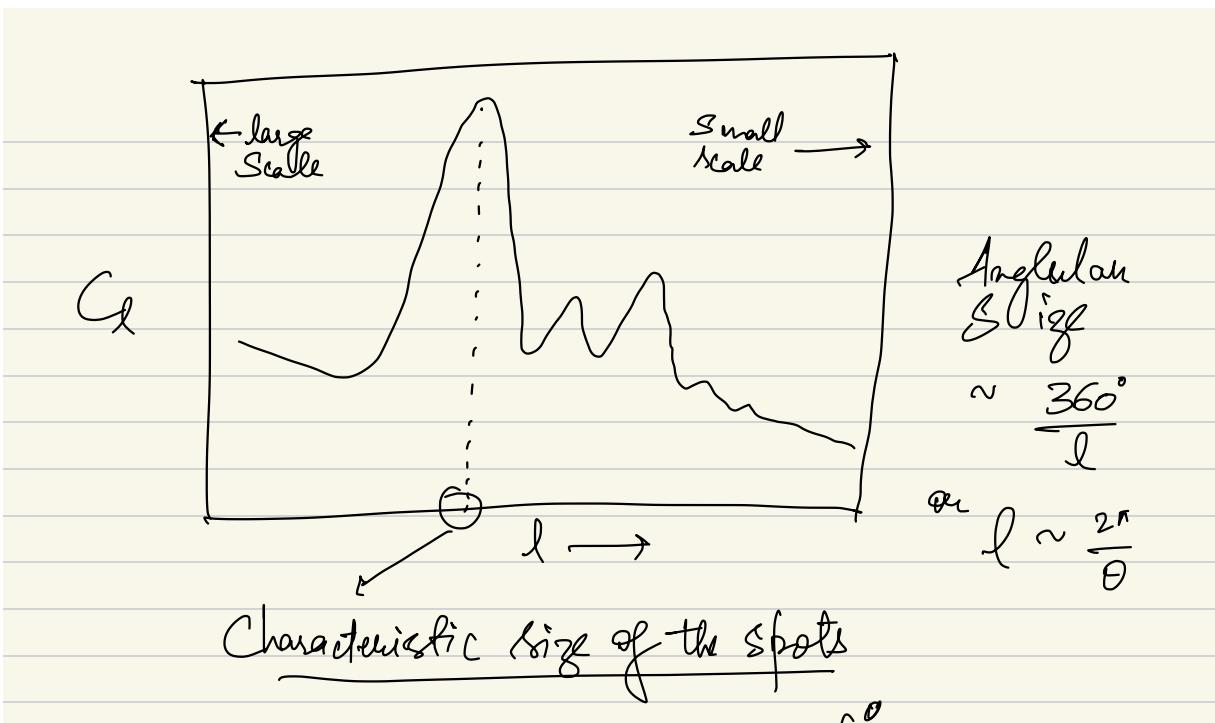
$$\Gamma = \text{Scattering rate}$$

now:- $\dot{\tau} = -\alpha n_e \sigma_T$

④ For photons, $(S_r = 4\Theta_0)$ and $V_r = -3i\Theta_1$

deviation in energy density from the homogeneous energy density in the CMB is related to the monopole.

④ CMB power spectrum has a peak, reflects the fact that there's a characteristic size of the spots $\sim 1^\circ$.
Smaller structures \sim peak at higher l
large " " " " " low l .



$$\Theta(\hat{n}) = \underbrace{\sum_n}_{S} + \phi_e - \hat{n} \cdot \mathbf{v} + \int_e^{\Theta} d\eta \, z\phi'$$

doppler term

CMB temp anisotropy as a function of direction.

- ④
 1. How photons propagates to us.
 2. Initial Conditions.
 3. How these initial conditions evolve in the primordial plasma to give the

④ Inflation predicts a quant'le called Comoving Curvature perturbation (constant and conserved outside the horizon). You can think of this as \sim the potential perturbation initial conditions at early times.

$$R \sim \dot{\phi} \quad R = -\frac{5+3w}{3+3w} \dot{\phi}$$

④ $\langle R(k) R^*(k') \rangle = (2\pi)^3 P_R(k) \delta(k - k')$

and $\left[\frac{k^3}{2\pi^2} P_R(k) \right] = \text{constant (Scale invariant)}$

④ Remember this identity:

$$e^{ik \cdot x} = 4\pi \sum_{lm} i^l j_l(kx) Y_{lm}^*(\hat{k}) Y_{lm}(\hat{x})$$

Bessel function

④ $C_l = 4\pi \int dk k \left[\frac{k^3}{2\pi^2} P_R(k) \right] * \left[T_s(k, \eta_*) \right]^2 j_l^2(k\eta_*)$

$j_l(k\eta_*)$ peaks when $k\eta_* \sim l$.

$$\textcircled{4} \quad \left(\frac{\delta_r}{\eta}\right)'' + \frac{1}{3} K^2 \left(\frac{\delta_r}{\eta}\right) = \phi'' - \frac{1}{3} K^2 \phi$$

Radiation pressure

Driving force from potential.



Acoustic oscillations: approximate eq.

During large scales - modes evolving during matter domination.
 $\therefore \phi$ be constant.

and eq: becomes,

$$\left(\frac{\delta_r}{\eta} + \phi\right)'' + \frac{1}{3} K^2 \left(\frac{\delta_r}{\eta} + \phi\right) = 0$$

Simple harmonic oscillator.

$v \rightarrow \frac{k}{\sqrt{3}}$ higher $k \rightarrow$ high frequency

④ Another imp quantity is :

$$R = \frac{\bar{P}_b}{(\bar{P}_r + \bar{P}_r)}$$

Adding baryon contribution offsets the Sachs Wolfe term by constant R .

$$\frac{\partial \sigma}{\eta} + \phi_e = \frac{1}{5} R [(1+3R) \cos(Kr_s) - R]$$

Comparing
Curvature perturb.

Upshot is that baryons enhance compression.
Therefore,

$$-\frac{1}{5} R [(1+3R) (\alpha_s - 3R)] \quad \begin{matrix} \text{large} \\ \text{scale} \end{matrix}$$

$$\frac{\partial \sigma}{\eta} + \phi \sim -R [\alpha_s] \quad \begin{matrix} \text{small} \\ \text{scale} \end{matrix}$$

④ Baryons increase the odd peaks which are the compression peaks \rightarrow created due to compression

④ So far we are assuming that we are dealing with the perfect fluid. That is a good approximation on a large scale, on scales that are large compared to the mean free path of the photon that is scattering.

Considering small scales \sim higher T , higher l , that no longer is a good approximation.

Because of the diffusion through the scattering of the photons. So if you set up a tiny variation in the, that's not gonna persist for long cause, the photons are scattering and they are gonna diffuse out these tiny variations.

Now we need to figure out the length scale over which this diffusion process will happen.

Diffusion length \rightarrow How far the photon has wandered walked after N scatterings.

\Rightarrow Structures of the size of diffusion length are erased. This is also known as silk damping.

upshot \rightarrow Small scales are erased \rightarrow high l , high T are damped away.

$$\textcircled{4} \quad d_p \equiv \text{particle horizon} = \left\{ \begin{array}{l} 3t = 2H^{-1} \text{ matter} \\ 2t = H^{-1} \text{ rad.} \end{array} \right.$$

$$\textcircled{5} \quad \text{Event horizon} = d_e = a(t) \int_t^\infty \frac{dt'}{a(t')}$$

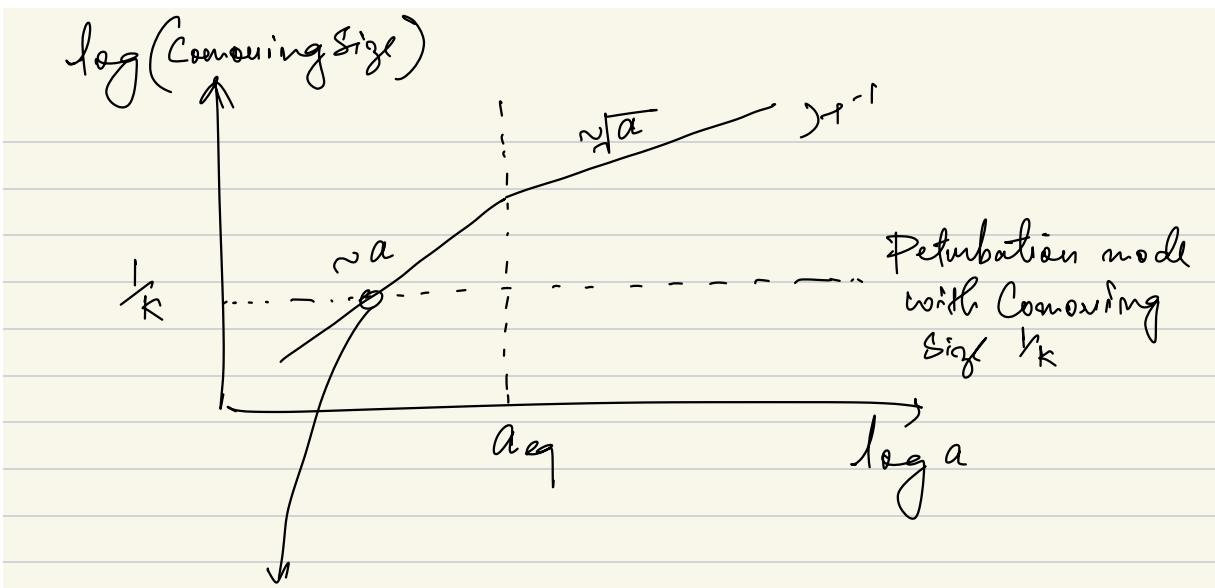
For the universe dominated by matter or radiation for all times, d_e is infinite.

For a de Sitter universe, $d_e = r^{-1}$ is finite and equal to hubble horizon.

Structure formation

- ④ Perturbations are characterized by their
 - ① Size $\equiv a/k$
 - ② amplitude $\equiv \delta(\eta, k)$
- ④ The physical size grows $\propto a$ due to expansion
- ④ In the Comoving Coordinates the size $\sim 1/k$ stays fixed,
- ④ $H^{-1} \propto a$ radiation domination
 $H^{-1} \propto \sqrt{a}$ matter domination
- ④ $H^{-1} \equiv \text{Comoving Hubble parameter}$

$$\eta = \begin{cases} H^{-1} & \text{rad. domination} \\ 2H^{-1} & \text{matter domination} \end{cases}$$



Horizon entry of the mode with wavenumber k

④ Next we need to find how the perturbations evolve?

$\Theta, N, \dot{\delta}, v, \delta_b, v_b, \phi, \psi ?$
 Dark matter

④ Evolution of perturbations → (outside horizon)
 (rad. domination)

$$1. \quad \dot{\psi} = \dot{\psi}_b = \dot{\psi}_\gamma = -ik_\gamma \times \frac{\psi}{p+2}$$

For superhorizon mode $k_\gamma \ll 1$ is supposed and we can neglect the velocity in practice.

2. we set also the higher moments of the neutrino temp. perturbations to zero initially.
3. Neglecting the velocity in the Boltzmann eq. for the densities (or monopoles for the photons).

$$\dot{\delta} = \dot{\delta}_b = -3\dot{\phi}, \quad \dot{\theta}_o \approx \dot{N}_o = -\dot{\phi}$$

4. From the einstein eq. we get,

$$\dot{\phi} \approx -\dot{\psi} \approx \text{constant} \quad \text{for } k_\gamma \ll 1, \quad a \ll a_{eq}$$

5. The photon and neutrino density is also constant just outside the horizon.

6. To further reduce the initial conditions we need to make assumptions. The standard assumptions are known as adiabatic initial condition which posit that perturbation in the entropy density are universal for all species.

7- In Summary, adiabatic initial conditions correspond to the choice:

$$\delta = \delta_b = \frac{3}{2} \phi$$

$$D_b = N_b = \frac{1}{2} \phi$$

$$\Psi = -\phi$$

$$v = v_b = 0$$

$$\Theta_l = N_l = 0 \quad (l \geq 1)$$

The physical motivation for adiabatic initial conditions is that, in thermal equilibrium, they correspond to a universal temp. perturbation $\delta T/T$ for all species.

More deeply, the motivation is that these all perturbations were seeded by a single fluctuating degree of freedom.

Boltzmann eq. for photons

$$d[f] = Cf$$

2

$$\frac{\partial \Theta}{\partial t} + \frac{\partial \Theta}{\partial x^i} \frac{p^i}{a} + \frac{p^i}{a} \frac{\partial \psi}{\partial x^i} + \frac{\partial \phi}{\partial t}$$

free streaming

effect of gravity.

Note
Temp perturbation
 Θ is a Scalar Quantity

$$= n_e \sigma_T \left(\Theta_0(t, \vec{x}) - \Theta(t, \vec{x}, \vec{p}) + \vec{p} \cdot \vec{v}_e \right)$$

Isotropization
Drag terms

Free Streaming :- Photons probe different regions after sometime.

Effect of gravity: Gravitational Redshift
+ Change with time
of the gravitational potential

Isotropization: Compton Scattering changes the direction of the photons but not energy. Therefore if the Compton Scattering is very efficient then the angular distribution would be washed out. And this converges to monopole contribution which is sort of average over all angles.

Conformal time

⇒ Conformal time is defined in such a way that the spacetime metric becomes conformally flat.
(Particularly in the context of FRWL metric.)

$$d\eta = \frac{dt}{a(t)} \quad \text{or} \quad \eta = \int \frac{dt}{a(t)}$$

In terms of Conformal time, the FLRW metric takes the simple form:

$$ds^2 = a(\eta)^2 (-d\eta^2 + d\mathbf{x}^2)$$

The advantage of using Conformal time is that it simplifies the eq. of motion for the scalar, vector, and tensor perturbations in the early universe.

One key feature is that Conformal time extends from $-\infty$ to 0 for a finite interval of proper time, making it useful coordinate for describing both the early and late-time behaviour of the universe.

Fourier transform.

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

In our Case

$$\Theta(\eta, \vec{x}, \hat{p}) = \int \frac{d^3 k}{(2\pi)^3} e^{ik \cdot x} \Theta(\eta, \vec{k}, \hat{p})$$

Note that this affects only the spatial dependence.

Also, the gradient in Fourier space is just $i\vec{k}$

i.e., $\nabla''_c = \nabla x = i\vec{k}x$

Boltzmann eq. for Cold Dark matter.

→ we assume that at the time of recombination any interactions of Cold dark matter are negligible and therefore no

$$d[f_{dm}] = P^0 \frac{df_{dm}}{dt} = 0$$

Boltzmann eq. after simplification: we have
Continuity eq. and Euler eq.

$$\dot{\delta}_{dm} + iKv_{dm} + 3\dot{\phi} = 0 \quad \text{linearized Continuity eq.}$$

$$\dot{v}_{dm} + H v_{dm} + iK\phi = 0 \quad \text{linearized Euler eq.}$$

where, $H = \frac{1}{a} \frac{da}{d\eta} = aH \quad (\text{Conformal Hubble rate})$

and density contrast $\delta_{dm} \equiv \frac{n_{dm} - n_{dm}^{(0)}}{n_{dm}^{(0)}}$

Lecture 1.

Cosmological principle: Universe is homogeneous and isotropic.

isotropic :- Same in every direction

homogeneous: Same in every place.

Now if it's isotropic then the only way it could not be homogeneous is if it forms a ring or shell like structure. Then observe the place at the centre is different from any other place and therefore not homogeneous.

Therefore as far as we can see the space is uniformly filled with particles.
Particles — galaxies, stars etc.

I mean if you average over billions of light years then it's look pretty much the same.

Hubble parameter: $\frac{\text{Velocity}}{\text{Distance}} = \frac{\dot{a}(t)}{a(t)}$.

There is no reason for hubble parameter / constant to be constant in time. It is constant in the sense that it has same value throughout the universe at a given moment
 \therefore velocity b/w galaxies = $H \times$ Distance b/w them.

⇒ Mass in the each grid remains same but since the universe is expand therefore density decreases.

$$\rho(t) = \frac{v}{a^3(t)}$$

$v \rightarrow$ mass per grid box

⇒ Just by Newton's gravitational law and the 2nd law of motion, we can derive the following equation.

$$\boxed{\ddot{a} = -\frac{4\pi}{3} GP}$$

That's why this is an acceleration eq.

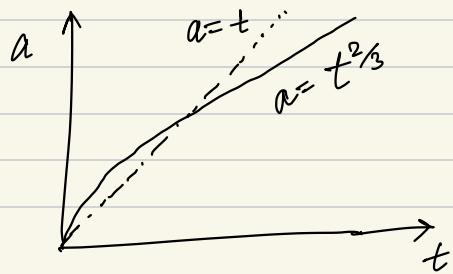
⇒ From energy Conservation we get:

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} GP}$$

at escape velocity
(energy conservation)

Now, solving the energy Conservation law

$$a \sim ct^{2/3} \quad c \text{ is some constant.}$$



Lecture 2

- ⇒ We would run into trouble if the galaxies or particles or whatever is present in the universe are moving past each other at a significant fraction of the speed of light.
- ⇒ By including the cases where $E \neq 0$ we get the Friedmann equation.

i.e,

$$\left[\left(\frac{\dot{a}}{a} \right)^2 - \frac{8\pi}{3} PG = \frac{C}{a^2} \right] \quad \text{Note: } p(t) = \frac{v}{a^3(t)}$$

RHS can be both +ve or -ve depending on the energy.

3 Cases

1. positive energy: $a \sim t^{2/3}$ (small a) $a \sim t$ (large a)
2. Negative energy: collapse.
3. Zero energy: Asymptotically slow down but never stop moving.

Note: Positiveness/negativeness/zero energy is associated with geometry of the universe.

For radiation dominated universe the friedmann eq. would be:

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G}{3} \frac{v}{a^4} = \frac{C}{a^2}$$

Only this changes

Solving this for zero energy case we get:

$$a \sim t^{1/2}$$

For zero energy case with both matter and radiation we have:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{C_1}{a^3} + \frac{C_2}{a^6}$$

Note

~~We can even see that friedmann eq. actually looks like einstein eq.~~

$$\underbrace{\left(\frac{\dot{a}}{a}\right)^2}_{\text{geometry}} - \frac{C}{a^2} = \underbrace{\frac{8\pi G}{3} v}_{\text{energy momentum}}$$

Lecture 3.

Metric in polar coordinates

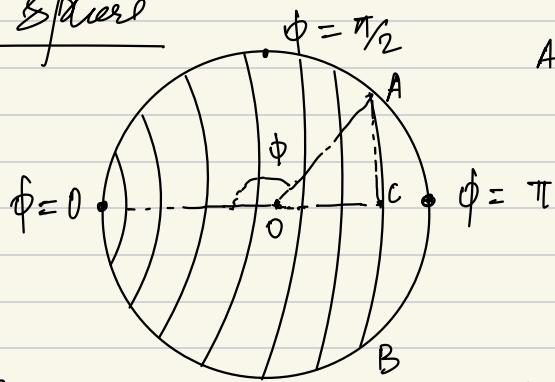
$$ds^2 = dr^2 + r^2 d\theta^2$$

$d\theta^2$ is the metric of one circle of unit radius.
so let's call it $d\Omega^2$,

$$\therefore ds^2 = dr^2 + r^2 d\Omega^2$$

- ⇒ Also ds^2 is essentially the metric of a flat space.
- ⇒ Think of flat space as a metric of nested Circles.
- ⇒ Circle is also called one sphere.
- ⇒ 2-Sphere is also an homogeneous surface -

Metric of a Sphere



$$AC = \text{radius} = \sin \phi$$

$$OA = \text{radius} = r$$

$$d\Omega_2^2 = r^2 (d\phi^2 + \sin^2 \phi d\Omega_1^2)$$

(Similarly for 3-sphere)

$$d\Omega_3^2 = r^2 (d\phi^2 + \sin^2 \phi d\Omega_2^2)$$

(Flat 3-D here r is different.)

<u>Dimension</u>	<u>Flat</u>	<u>Curved (1, 2, 3 - sphere)</u>
1.	ds^2	$r^2 ds^2 = r^2 d\Omega_1^2$
2.	$ds^2 = dr^2 + r^2 d\Omega_2^2$	$r^2 (d\phi^2 + \sin^2 \phi d\Omega_1^2)$
3.	$ds^2 = dr^2 + r^2 d\Omega_3^2$	$r^2 (d\phi^2 + \sin^2 \phi d\Omega_2^2)$

Another homogeneous geometry is hyperbolic space.

$$ds^2 = dr^2 + \sinh^2 r d\Omega_1^2$$

$$ds^2 = dr^2 + \sinh^2 r d\Omega_2^2$$

Metric for different geometries .

$$ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2) \quad \text{flat}$$

$$ds^2 = -dt^2 + a^2(t) d\Omega_3^2 \quad \text{spherical}$$

$$ds^2 = -dt^2 + a^2(t) d\Omega_2^2 \quad \text{hyperbolic}$$

Lecture 4

Starting with:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} P - \frac{k}{a^2}$$

$\underbrace{\hspace{10em}}$

related to
Curvature

$k := \begin{cases} +1 & \text{+ve} \\ 0 & \text{flat} \\ -1 & \text{-ve} \end{cases}$

This eq. comes from the time-time component of the Einstein equation.

1. Matter Dominated universe

$$\rho = \frac{\rho_0}{a^3}$$

2. Radiation Dominated

$$\rho = \frac{\rho_0}{a^4}$$

Note up till now for $k \equiv \begin{cases} +ve & \rightarrow \text{Collapse} \\ 0 & \rightarrow \text{Asymptotically} \\ & \text{Slow down} \\ -ve & \rightarrow \text{Continuous} \\ & \text{to increase} \\ & \text{linearly at later} \\ & \text{time.} \end{cases}$

Generally the eq. of states take this form

$$\Rightarrow P = w\rho \quad P \equiv \text{pressure}$$

$w=0$ for matter dominated

$w=\frac{1}{3}$ for radiation. $w=-1$ for Dark energy

Now,

$$E = PV$$

$$dE = -PdV$$

$$\text{Also, } dE = PdV + Vdp = -PdV$$

$$\Rightarrow Vdp = -(P+\rho)dV$$

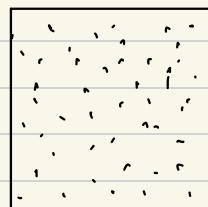
$$\text{But } P = w\rho$$

$$\therefore Vdp = -(1+w)\rho dV$$

$$\Rightarrow \frac{dp}{\rho} = -(1+w) \frac{dV}{V}$$

$$\Rightarrow \log \rho = -(1+w) \log V$$

$$\Rightarrow \rho = \frac{C}{V^{(1+w)}} = \frac{C}{a^{3(1+w)}}$$



Let's say a box
of gas.

Volume = V

$$\rho = \frac{C}{a^{3(1+w)}}$$

Note: A negative pressure like tension.

and Dark energy is the property of empty space
and empty space doesn't dilute over stretching.

Lecture 5

matter: $a(t) \sim t^{2/3}$

Radiation: $a(t) \sim t^{1/2}$

Since $P = w\rho \Rightarrow w=0 \Rightarrow$ pressure $= P = 0$

$w=-1 \Rightarrow$ tension

and for radiation $w=\frac{1}{3}$ (massless particles)

Vacuum energy is simply there in the empty space.

Vacuum energy density of space doesn't change.

$$f_0 = 1 - \frac{3}{8\pi G} \quad \Lambda \equiv$$

Case with only vacuum energy density.

$\Lambda := \pm, 0$ and $k = +ve, 0, -ve$

So there are various cases. Obviously some of them doesn't make sense.

Case 1 ($k=0$ and $\Lambda \neq 0$)

$$\left(\frac{\dot{a}}{a}\right)^2 = \Lambda \Rightarrow \frac{da}{dt} = \sqrt{\Lambda} a$$

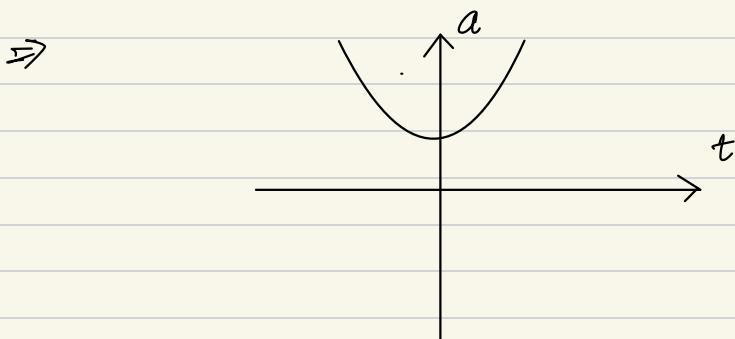
$$a = c e^{\sqrt{\Lambda} t}$$

Hubble $\equiv \sqrt{\Lambda}$
Constant

This space exponentially expand and this is
Called de-sitter Space.

Case 2 ($k > 0$ and $\Lambda > 0$)

$$\left(\frac{\dot{a}}{a}\right)^2 = \Lambda - \frac{1}{a^2} \Rightarrow \dot{a}^2 - \Lambda a^2 = 1$$



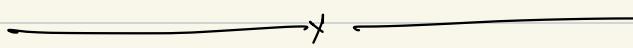
Picture of the universe in this case

And this is also called de-sitter Space.

Case 3 $k \neq 0$ and $\lambda \neq 0$.

$$\Rightarrow \dot{a}^2 = -\dot{a}^2 + 1 \quad \text{Harmonic Oscillator?}$$

Big Crunch .



Friedmann eq and Continuity eq.

$$1. \quad H^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} .$$

$$2. \quad \frac{\dot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3P)$$

Continuity eq (not independent of above two)

$$D_u T^{uv} = 0 \Rightarrow \underbrace{\frac{df}{dt} + 3H(\rho + P) = 0}_{}$$

and Remember,

$$P = w\rho \quad \text{and} \quad f \propto \frac{1}{a^{3(1+w)}}$$

Lecture 6. (Red Shift)

$$-\frac{K}{a^2} + \frac{8\pi G\rho}{3} = \frac{C_L}{a^4} + \frac{C_m}{a^3} + 1 - \frac{K}{a^2} = H^2$$

$$\Omega_R + \Omega_m + \Omega_n + \Omega_k = 1$$

So basically in past

$$\Omega_R \approx 0$$

$$\Omega_n \text{ was unknown}$$

$$\Omega_m \approx \frac{1}{20} \text{ or } \frac{1}{30} \text{ something}$$

\therefore Only K factor remains and therefore:

K : is -ve. (open infinite universe)

\Rightarrow So far the most time in the history of the universe

$$\frac{C_m}{a^3} \text{ would dominate } -\frac{K}{a^2}$$

$$\therefore a \sim t^{2/3} \quad \therefore H = \frac{\dot{a}}{a} = \frac{2}{3t} \Rightarrow H_{\text{Today}} = \frac{2}{3T_{\text{age}}}$$

Dark matter

$$\Rightarrow \frac{m M G}{r^2} = \frac{m v^2}{r} \rightarrow \frac{M G}{r} = v^2$$

On $v \propto \frac{1}{\sqrt{r}}$

① Dark matter is probably not electrically charged. Otherwise we should see it radiate.

② One fact about the dark matter particles is that they cluster. Things cluster less when they have high velocities.

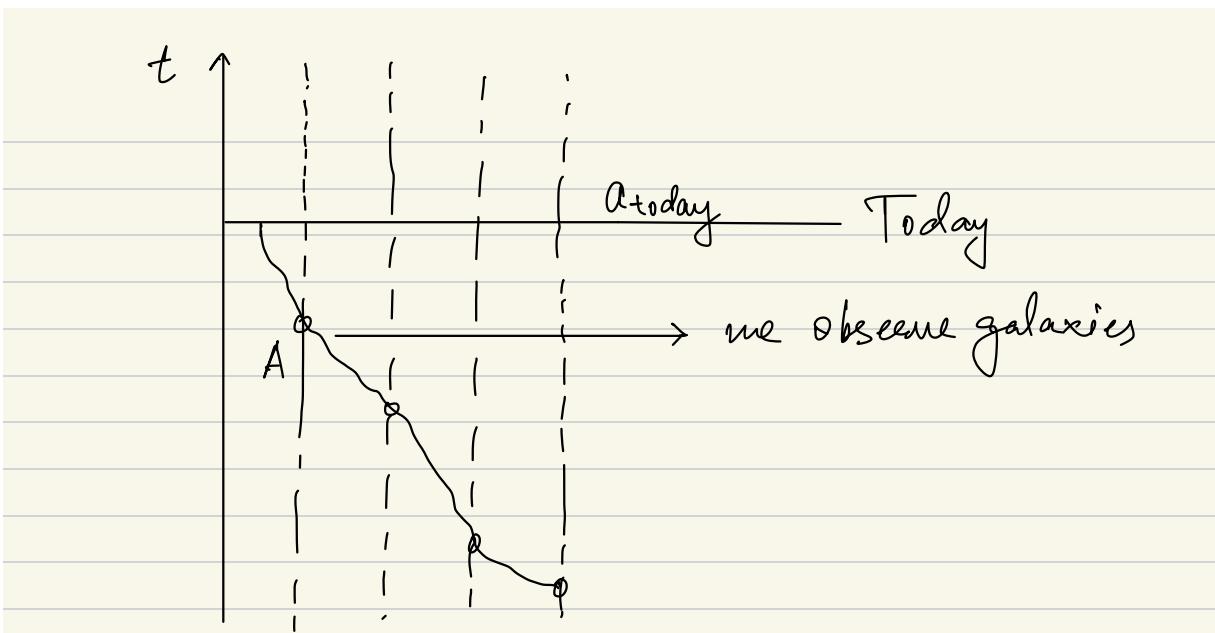
Cold dark matter

Red shift

$\lambda_{\text{emitted}} \equiv$ wavelength when emitted from the source.

When detected it may or may not have a same wavelength due to many reasons:-

- doppler shift
- gravitational redshift
- expansion of the universe



meanwhile light emitted at point A reaches today. The universe has stretched by a factor:

$$\frac{a_{\text{today}}}{a_A} \quad \left. \right\} \text{So this is the factor by which E.M wave would stretch going from A to today.}$$

$$z = \frac{\lambda_{\text{detected}}}{\lambda_{\text{emitted}}} - 1 \quad \# \text{redshift factor.}$$

Q. what is the number of galaxies within redshift Z and $Z + dZ$. that is no of galaxies per unit redshift.

$$:= dN/dz$$

$$\text{Now } \frac{dN}{dz} \propto \frac{\text{Area}}{dz} = \frac{dr \xi(r)^2}{?}$$

$$z = \frac{a_{\text{today}}}{a(t)} - 1 \Rightarrow -\frac{a_{\text{today}}}{a^2(t)} da = dz$$

$$\Rightarrow \frac{dN}{dz} \approx \frac{\xi(r)^2}{(z+1) \cdot \frac{da}{dt}} \quad \text{Also using } -\frac{dt^2}{a^2(t)} = \frac{dr^2}{a^2(t)}$$

(eq. for light
ray)

for matter dominated
universe:

$$or \quad dr = -\frac{dt}{a(t)}$$

$$a \sim t^{2/3}$$

$$\therefore \dot{a} = \frac{2}{3} t^{-1/3} \quad \text{and}$$

$L(z)$ and dN/dz together are enough to determine $a(t)$ and K .

observational fact

$$\Omega_R \approx 0, \quad \Omega_M \approx 0.3, \quad \Omega_\Lambda \approx 0.7 \quad \& \quad \Omega_K = 0$$

(Luminous + Dark)

Lecture 24 (Frederic Schuller)

- ⇒ Strictly speaking metric is the gravitational potential and the gravitational field strength is the Riemann tensor.
- ⇒ In general relativity the energy-momentum tensor feels the mass current (rotating mass).

Perturbation:

$$G_{ab}[g + \delta g] = T_{ab}[g] + \delta T_{ab}[g]$$

$$\Rightarrow G_{ab}[g] + \delta G_{ab}[g, \delta g] = T_{ab}[g] + \delta T_{ab}[g]$$

↑
linear
dependence

∴ Solve:

$$\delta G_{ab}[g, \delta g] = \delta T_{ab}[g]$$

→ we choose a coordinate system where:

$$g_{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\gamma_{\alpha\beta} & & \\ 0 & & -\gamma_{\alpha\beta} & \\ 0 & & & -\gamma_{\alpha\beta} \end{bmatrix} \xrightarrow{\text{Riemannian } 3\text{-metric}}$$

let's perturb this now

$$\delta g = 2a dx^0 \otimes dx^0 - b_\alpha (dx^0 \otimes dx^\alpha + dx^\alpha \otimes dx^0) \\ - [2c \gamma_{\alpha\beta} + 2e_{\alpha\beta}] dx^\alpha \otimes dx^\beta$$

Scalar field: a, c

Vector field: b_α

Tensor field: $e_{\alpha\beta}$

$$\gamma^{\alpha\beta} e_{\alpha\beta} = 0$$

Degrees of freedom
1+1

3

5 (because trace free)

Note: I think the minus sign in δg is because of the chosen style of metric $(1, -1, -1, -1)$.

Helmholtz - Hodge Theorem

It is mainly useful to further decompose

a) Vector field b_a as

$$\vec{b}_\alpha = D_\alpha B + B_\alpha$$

↑ ↑
 Scalar field Vector field
 Divergence-free
 $D_\alpha B^\alpha = 0$

(where D is the
 Covariant derivative of γ
 Something like ∇)

$$\stackrel{b)}{=} C_{\alpha\beta} = \left(D_\alpha D_\beta - \frac{1}{3} \gamma_{\alpha\beta} \Delta \right) E \quad \begin{array}{l} \text{Scalar field,} \\ \boxed{\Delta = \gamma^{\mu\nu} D_\mu D_\nu} \end{array}$$

$$+ 2 D_\alpha E_\beta \rightarrow \text{Vector field}$$

$$+ E_{\alpha\beta} \quad \text{Divergence free}$$

Symmetric tensor field.

$$\text{trace free: } \gamma^{\alpha\beta} f_{\alpha\beta} = 0$$

$$\text{Divergence free: } D_\alpha E^{\alpha\beta} = 0$$

Thus a general perturbation δg uniquely decomposes into three independent types of perturbations.

$$\delta g = \underbrace{\delta g}_{A, B, C, E}^{\text{scalar}} + \underbrace{\delta g}_{B_\alpha, E_\alpha}^{\text{vector}} + \underbrace{\delta g}_{E_{\alpha\beta}}^{\text{tensor}}$$

Divergence free Symmetric Trace free
 Divergence free

Therefore

$$\delta g_{\text{scalar}} := 2A dx^\alpha \otimes dx^\beta - D_\alpha B (dx^\alpha \otimes dx^\beta + dx^\beta \otimes dx^\alpha) - 2 \left[C \gamma_{\alpha\beta} + \left(D_\alpha D_\beta - \frac{1}{3} \gamma_{\alpha\beta} \Delta \right) E \right] dx^\alpha \otimes dx^\beta$$

$$\delta g_{\text{vector}} := -B_\alpha (dx^\alpha \otimes dx^\beta + dx^\beta \otimes dx^\alpha) - 4 D_\alpha E_\beta dx^\alpha \otimes dx^\beta$$

$$\delta g_{\text{tensor}} := -2 E_{\alpha\beta} dx^\alpha \otimes dx^\beta$$

For scalar perturbations, only two specific contributions of the fields A, B, C, E will be unaffected (and thus be not removable) by general coord. transformation. These two combinations are called gauge invariant. Only these can be taken seriously. Other combinations can always be removed by a coordinate transformation.

$$\Psi := A + \dot{B} - \ddot{E}$$

$$\Phi := C - \frac{1}{3} \Delta E$$

For vector perturbations there is only one combination of B_α, E_α that is gauge-invariant.

$$\Theta_\alpha := B_\alpha - \dot{E}_\alpha$$

For the tensor transformation:-
All $E_{\alpha\beta}$ are gauge invariant.

\Rightarrow Eq. for the evolution of photon energy along its trajectory.

$$\frac{d\phi}{dt} = -\beta \left(\frac{\beta^i}{a} \frac{\partial \psi}{\partial x^i} + \frac{\partial \phi}{\partial t} + n \right)$$

Linear Cosmological perturbation theory

$A(t, \vec{x})$: Density, pressure, velocity, metric perturbation

2 point Correlation function

$$\langle A(t, \vec{x}), A(t, \vec{x}') \rangle = \sum_A (t, \vec{x}, \vec{x}') = \sum_A (t, |\vec{x} - \vec{x}'|)$$

- This could be thought of as the clumpiness factor
- The higher the value for some distance scale, the more clumpy the universe is at that distance scale.
- Given a random galaxy in a location, the correlation function describes the probability that another galaxy will be found within the given distance.

In Fourier Space

$$\langle A(t, \vec{k}), A^*(t, \vec{k}') \rangle = S_D(\vec{k}' - \vec{k}) P_A(k) \downarrow \text{modulation of } k$$

Power Spectrum

It represents the contribution to the fluctuation field from waves of wavenumber k . A completely flat power spectrum corresponds to white noise, while a power spectrum that is sharply peaked at some particular value of k has a characteristic length scale.

A single plane wave \mathbf{J} has a power spectrum consisting of a single spike.

The Collisionless Boltzmann eq. for photons.

$$1. \quad g_{00}(\vec{x}, t) = -(1 + 2\psi(\vec{x}, t))$$

$$2. \quad g_{0i} = 0$$

$$3. \quad g_{ij} = a^2 \delta_{ij} (1 + 2\phi(\vec{x}, t))$$

- ⇒ In the absence of ψ & ϕ this is FRW metric
- ⇒ In the absence of expansion ($a=1$) this metric describes weak gravitational field.
- ⇒ $\psi \rightarrow$ newtonian potential
 $\phi \rightarrow$ Spatial Curvature.
- ⇒ These are the scalar perturbations only.

Boltzmann eq.

$$\frac{df}{dt} = C[f] \quad f: \text{in the distribution function}$$

$C[f]$: Collision term.

The Collisionless Boltzmann eq. is defined by the Liouville theorem:

$$0 = \frac{d}{d\lambda} f(x^u(\lambda), p^u(\lambda)) \quad [p^2 = g_{ij} p^i p^j]$$

But for our purpose: $f(t, \vec{x}, \phi, \hat{p})$, [No p^0]

$$\Rightarrow \frac{df}{dt} = \underbrace{\frac{\partial f}{\partial t}}_{\text{time derivative}} + \underbrace{\frac{\partial f}{\partial x^i} \cdot \frac{dx^i}{dt}}_{\text{position derivative}} + \underbrace{\frac{\partial f}{\partial p} \frac{dp}{dt} + \frac{\partial f}{\partial p^i} \cdot \frac{dp^i}{dt}}_{\text{momentum derivative}}$$

\Rightarrow Here, we need not include the term proportional to $\partial f / \partial p^0$ when expanding the total time derivative.

\Rightarrow The easiest term is the last one since it does not contribute at first order in perturbation theory.

$$\Rightarrow \text{Also, } \frac{dx^i}{dt} = \frac{dx^i}{d\lambda} \frac{d\lambda}{dt} = \frac{dx^i}{d\lambda} / dt/d\lambda = \frac{p^i}{p^0}$$

Now we know that. $p^2 = g_{ij} p^i p^j$

$$\Rightarrow p^2 = 0 = -(1+2\psi)(p^0)^2 + p^2 = 0 \quad (\begin{matrix} \text{zero four} \\ \text{photons} \end{matrix})$$

where we defined $p^2 = g_{ij} p^i p^j$

$$\Rightarrow p^0 = \frac{p}{(1+2\psi)^{1/2}} = p(1-\psi)$$

and p^i is proportional to \hat{p}^i

$$\therefore p^i \propto C \hat{p}^i$$

$$\text{and } p^2 = g^{ij} p^i p^j \Rightarrow \hat{p}^2 = g_{ij} \hat{p}^i \hat{p}^j C^2$$

$$= a^2 (1 + 2\phi) \delta_{ij} \hat{p}^i \hat{p}^j C^2$$

$$= a^2 (1 + 2\phi) C^2$$

$$\Rightarrow C = \phi(1 - \phi)/a$$

$$\therefore p^i = \phi \hat{p}^i \frac{1 - \phi}{a}$$

$$\therefore \frac{dx^i}{dt} = \hat{p}^i \frac{1}{a} (1 + \psi - \phi)$$

Note.
For comoving observers
four velocity is:
 $U^{\mu} = \left(\frac{1}{a}, 0, 0, 0 \right)$
 $U_{\mu} = (-a, 0, 0, 0)$

we can neglect the potentials in the above eq.
and substitute that in the boltzmann eq.

$$\Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial p} \frac{dp}{dt}$$

now we need to calculate $\frac{dp}{dt}$ only:

To begin with, let us recall that the time component of the geodesic eq.

$$\frac{dp^0}{d\lambda} = - \Gamma_{\alpha\beta}^0 p^\alpha p^\beta \quad \frac{dt}{d\lambda} = p^0$$

$$\text{But } p^0 = p(1-\psi)$$

$$\Rightarrow \frac{d(p(1-\psi))}{dt} = - \Gamma_{\alpha\beta}^0 \frac{p^\alpha p^\beta}{p} (1+\psi)$$

Now we multiply both sides by $(1+\psi)$; drop all terms quadratic in ψ , and express the total time derivative of ψ in terms of partial derivatives

$$\frac{dp}{dt} = p \left\{ \frac{\partial \psi}{\partial t} + \frac{\dot{p}_i}{a} \frac{\partial \psi}{\partial x^i} \right\} - \Gamma_{\alpha\beta}^0 \frac{p^\alpha p^\beta}{p} (1+2\psi)$$

Then we calculate all the christoffel symbols and other terms and get:

$$\boxed{\frac{1}{p} \frac{dp}{dt} = -H - \frac{\partial \phi}{\partial t} - \frac{\dot{p}_i}{a} \frac{\partial \phi}{\partial x^i}}$$

Finally the boltzmann eq. becomes :

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{\alpha} \frac{\partial f}{\partial x^i} - \frac{p}{\alpha} \frac{\partial f}{\partial p} \left[H + \frac{\partial \phi}{\partial t} + \frac{\hat{p}^i}{\alpha} \frac{\partial \phi}{\partial x^i} \right]$$

To go further we expand the photon distribution function f about its zeroth order base - Einstein value.

$$f(\vec{x}, p, \hat{p}, t) = \left[\exp \left\{ \frac{p}{T(t)[1 + \Theta(\vec{x}, \hat{p}, t)]} \right\} - 1 \right]^{-1}$$

Hence the zeroth order temp T is a function of time only (i.e., scales as α^{-1}) not space-

In the smooth zeroth order universe, photons are distributed homogeneously, so T is independent of \vec{x} and isotropically. So T is independent of - the direction of propagation \hat{p} .

Θ depends on \vec{x} , \hat{p} and t . \Rightarrow magnitude of the photon momentum is virtually unchanged during the Compton scattering.

$$\text{Also, } T \frac{\partial f^{(0)}}{\partial t} = - \hat{p} \frac{\partial f^{(0)}}{\partial p}$$

$$\Rightarrow f \approx \frac{1}{e^{p/T_{-1}}} + \frac{2}{\partial T} \left[\exp \left(\frac{p}{T} \right)^{-1} \right]^{-1} T \theta$$

$$= f^{(0)} - p \frac{\partial f^{(0)}}{\partial p} \theta$$

$$\Rightarrow \boxed{f \approx f^{(0)} - p \frac{\partial f^{(0)}}{\partial p} \theta} \quad \checkmark$$

Now everywhere we encounter f in the expression of $\frac{df}{dt}$ on the last page, we substitute the second term of the above eq.

$$\text{i.e., } -p \underbrace{\frac{\partial f^{(0)}}{\partial p}}_{\text{}}$$

Finally we will have,

$$\frac{df}{dt} \Big|_{\text{first order}} = -p \frac{\partial f^{(0)}}{\partial p} \left[\underbrace{\frac{\partial \theta}{\partial t} + \hat{p}_i \frac{\partial \theta}{\partial x^i}}_{\text{free streaming}} + \underbrace{\frac{\partial \phi}{\partial t} + \hat{p}_i \frac{\partial \phi}{\partial x^i}}_{\text{gravity}} \right]$$

Note: The zeroth order distribution function is set precisely by the requirement that the collision term vanishes. The reason being that any collision term includes the rate for the given reaction and for its inverse. If the distribution function are set to their equilibrium values, the rate for the reaction precisely cancels the rate for its inverse.

Scattering term



$$\begin{aligned} E_e(q) - E_e(\vec{q} + \vec{p} - \vec{p}') &= \frac{q^2}{2m_e} - \frac{(\vec{q} + \vec{p} - \vec{p}')^2}{2m_e} \\ &\approx \frac{(\vec{p}' - \vec{p})q}{m_e} \end{aligned}$$

$\vec{p}' - \vec{p}$ is of order p , of order the ambient temperature T ,

\therefore RHS of above eq. is of the order:

$$\frac{Tq}{m_e} \sim TV_b \quad \text{where the baryonic velocity } V_b \text{ is very small.}$$

$$C[f(\vec{p})] = -\vec{p} \frac{\partial f^{(0)}}{\partial \vec{p}} n_e \sigma_T [\Theta_0 - \Theta(\vec{p}) - \hat{\vec{p}}, \vec{V}_b]$$

Boltzmann eq. for the photons.

$$\frac{\partial \Theta}{\partial t} + \hat{p}^i \frac{\partial \Theta}{\partial x^i} + \frac{\partial \phi}{\partial t} + \hat{p}^i \frac{\partial \psi}{\partial x^i} \xrightarrow{\text{Thomson cross-section}} = n_e \sigma_T [\Theta_0 - \Theta + \hat{p} \cdot \vec{v}_b]$$

$\Gamma \equiv \text{scattering rate}$

In the Conformal time setting, the boltzmann eq. becomes:

$$\dot{\Theta} + \hat{p}^i \frac{\partial \Theta}{\partial x^i} + \dot{\phi} + \hat{p}^i \frac{\partial \psi}{\partial x^i} = n_e \sigma_T a [\Theta_0 - \Theta + \hat{p} \cdot \vec{v}_b]$$

we also define:

$$\boxed{\Theta_l = \frac{1}{(-i)^l} \int_{-1}^1 \frac{du}{2} P_l(u) \Theta(u)}$$

$$\Theta_0 = \frac{1}{4\pi} \int d\Omega \Theta(\hat{p}, \vec{x}, t) \quad [\text{monopole}]$$

The moment Θ_0 and Θ_l do not completely characterize the photon distribution. More generally it is useful to define the l^{th} multipole moment of the temperature field as above:
 $l=2$ (quadrupole) and P_l is the legendre poly.
 $l=3$ (octopole).

Note: The higher legendre polynomial have structures on smaller scales. So the higher moment capture information about the small scale structure of the temperature field

So the photon distribution can be described either by $\Theta(\mathbf{k}, \eta)$ or by the whole hierarchy of $\Theta_l(\mathbf{k}, \eta)$. And of course similar freedom applies to the neutrinos distribution.

⇒ optical depth

$$\tau = \int_t^{t_0} dt' n_e(t') \sigma_T = \int_{\eta}^{n_0} dn' n_e \sigma_T$$

usually one writes :

$$\dot{\tau} = -\alpha n_e \sigma_T$$

$$\boxed{m = \frac{\vec{k} \cdot \hat{p}}{|\vec{R}|}}$$

⇒ $n_e \sigma_T$ is the Scattering rate

⇒ photon decoupling happens when $\tau(z_{dec}) = 1$

⇒ optical depth is zero today.

⇒ Monopole is related to density contrast

$$\boxed{\delta_r = 4\Theta_0}$$

⇒ The scalar part of the velocity is related to the first moment.

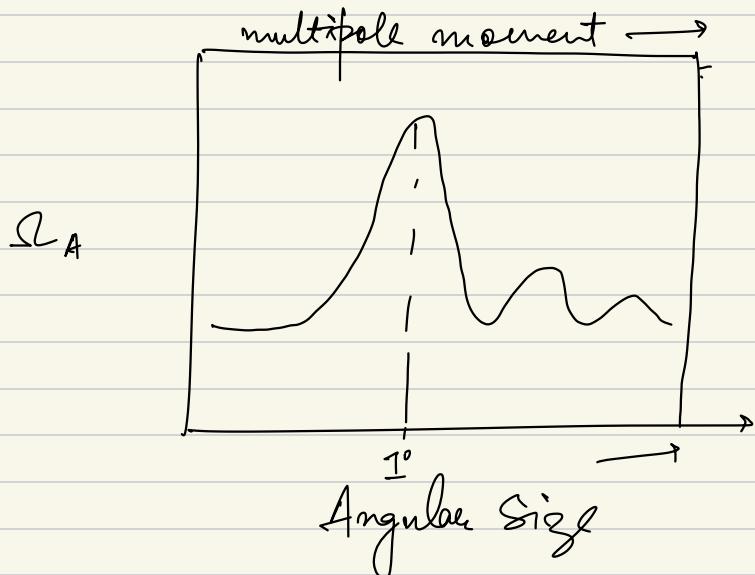
$$\boxed{v_r = -3i\Theta_1}$$

⇒ For dark matter: we assume that dark matter is effectively collisionless (this is a standard assumption, it is a good approximation e.g. for WIMPs)

Therefore, Boltzmann eq reads: $\partial[f_{dm}] = 0$

CMB Power Spectrum

- ④ The entire universe acted as a resonance cavity.
~ with the size of sound horizon.



- ④ The peaks correspond to the resonance scale of the cavity (sound horizon of the universe).
- ④ About at 1° correspond to the size of the universe.
- ④ This plot is the Fourier transformation of CMB fluctuations.
- ④ Small l (multipole moment) correspond to large scale and large l correspond to small scale.

④ The first acoustic peak tells us the geometry of the universe. We know the angular size (observation) and we can also calculate the physical size and finally compare both to tell the geometry of the Universe. (Obviously it's flat).

⑤ By measuring the heights of the peak we know how much radiation is there in the universe (this is very rough). S_R .

S_R today is very less because the universe has expanded a lot already. $S_R \propto a^{-4}$

⑥ $\frac{180}{l} = \text{Angular size}$.

⑦ Peaks at smaller scales are actually earlier oscillations when the sound horizon was smaller.

⑧ In the peaks of the spectrum there was little bit more baryon density therefore in these scales there are little bit more galaxies than other scales.

⑨ more Ω_B means more Silk damping.