Formation of structure in the Universe

Growth of perturbation

⇒ Newtonian eq. of motion for non-relativistic matter.

we will follow space and time dependent density $\rho(\vec{r}, t)$, it's velocity $\vec{v}(\vec{r}, t)$ and the gravitational potential.

The eq. of self gravitating gas dynamics are as follows;

1. Continuity eq.

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0}$$

2. The velocity field $\vec{v}(r, t)$ obey's the euler eq.

$$\boxed{\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{\nabla P}{\rho} - \nabla \phi}$$

3. Gravitational potential obeys. Poisson eq.

$$\boxed{\nabla^2 \phi = 4\pi G \rho}$$

In Case of relativistic matter

1. $$\frac{\partial \rho}{\partial t} + \nabla \cdot \left[\left(\rho + \frac{P}{c^2} \right) \vec{v} \right] = 0$$

2. $$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \nabla \right) \vec{v} = -\frac{\nabla P}{\rho + P/c^2} - \nabla \phi$$

3. $$\nabla^2 \phi = 4\pi G \rho \left(1 + \frac{3P}{\rho c^2} \right).$$

⊛ We can only apply these eq. if the particles form a gas i.e, they have regular collisions so that their energy and momenta are thermalised.
A freely streaming sea of particles such as the CMB or radiation from stars does not act as a gas and we can not apply the above eq. to them.

At ~ 3200 the energy density of the Universe is dominated Cold matter, baryonic matter is ~ 17% of the "Cold" matter in the universe, so the baryonic matter start dominating over the radiation energy density only around $z \sim 550$ i.e, after CMB decoupling.

⊛ linearized eq. for non-relativistic matter

⇒ let us assume that there are small perturbation

1. $\rho(\vec{r},t) = \rho_0(t) + \delta\rho(\vec{r},t)$
2. $\vec{v}(\vec{r},t) = \vec{v}_0(\vec{r},t) + \delta\vec{v}(\vec{r},t)$
3. $\phi(\vec{r},t) = \phi_0(\vec{r},t) + \delta\phi(\vec{r},t)$

Now upto first order:

Continuity eq.:

$$0 = \frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla\cdot\delta\vec{v} + 3H\delta\rho + \vec{v}_0\cdot\nabla\delta\rho$$

Euler eq.:

$$0 = \frac{\partial \delta\vec{v}}{\partial t} + \vec{v}_0\cdot\nabla\delta\vec{v} + \delta\vec{v}\cdot\nabla\vec{v}_0 + \frac{\nabla\delta P}{\rho_0} + \nabla\delta\phi$$

Poisson eq.:

$$0 = \nabla^2\delta\phi - 4\pi G\delta\rho$$

Introduce, $\delta := \frac{\delta\rho}{\rho_0}$

Let us work out the following:

① $\dot{\delta} \equiv \frac{\partial \delta}{\partial t} = \frac{\partial(\delta\rho/\rho_0)}{\partial t} = \frac{1}{\rho_0}\frac{\partial \delta\rho}{\partial t} - \frac{\delta\rho}{\rho_0^2}\frac{\partial \rho_0}{\partial t}.$

Now, $\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial t} = \frac{\partial(\ln\rho_0)}{\partial t}$ $\left[\rho_0 \sim a^{-3} \text{ matter Case}\right]$

$\Rightarrow \quad \ln\rho_0 \sim -3\ln a$

$\Rightarrow \quad \partial(\ln\rho_0) \sim -3\,\partial(\ln a)$

$\Rightarrow \quad \frac{1}{\rho_0}\frac{\partial \rho_0}{\partial t} = \frac{\partial \ln\rho_0}{\partial t} = -3\frac{\partial \ln a}{\partial t} = -3H$

$$\Rightarrow \quad \boxed{\dot{\delta} = \frac{1}{\rho_0}\frac{\partial \delta\rho}{\partial t} + 3H\frac{\delta\rho}{\rho_0}}$$

② $$\boxed{\nabla\delta = \frac{\nabla\delta\rho}{\rho_0}}$$

③ $(\delta\vec{v}\cdot\nabla)\vec{v}_0 = (\delta\vec{v}\cdot\nabla)H\vec{r} = H\delta\vec{v}$

The matter power spectrum.

The power spectrum is simply the amplitude squared of the Fourier components.

$$P(\vec{k}) = \hat{\delta}^*(\vec{k})\,\hat{\delta}(\vec{k}) = |\hat{\delta}(\vec{k})|^2$$

where,

$$\hat{\delta}(\vec{k}, t) = \int \delta(\vec{x}, t)\, e^{i\vec{k}\cdot\vec{x}}\, d^3x .$$

Since, in our case we expect that there is isotropy, we are only interested in $P(|\vec{k}|)$ i.e, in the direction-averaged power spectrum. Define $k := |\vec{k}|$ we then get,

$$P(k) = \frac{1}{4\pi} \oint |\hat{\delta}(\vec{k})|^2 d\Omega$$

The power spectrum function tells us how much "power" there is in perturbations of spatial scales of $\lambda = 2\pi/k$

Matter power spectrum scales roughly as k^1 at low wavenumbers and as k^{-3} for large wavenumbers.

* The relation b/w the power spectrum and the autocorrelation function is given by:

$$\langle \delta^*(\vec{x})\, \delta(\vec{x}+\vec{y}) \rangle = \int \frac{d^3 k}{(2\pi)^3} P(\vec{k})\, e^{-i\vec{k}\cdot\vec{y}}$$

or equivalently,

$$P(\vec{k}) = \int \langle \delta^*(\vec{x})\, \delta(\vec{x}+\vec{y}) \rangle\, e^{i\vec{k}\cdot\vec{x}}\, d\vec{x}$$

The power spectrum is the fourier transform of the autocorrelation function.

* The correlation function has the units of volume. So we can express it in dimensionless form by multiplying by the wavenumber cubed. We assume that the early universe was scale invariant, so that the dimensionless correlation function is constant.

* $\int d\eta = \int_0^t \frac{d\tilde{t}}{a(\tilde{t})}$ but $d_H = a(t) \int_0^t \frac{d\tilde{t}}{a(\tilde{t})}$

and let's say for matter dominated

$$\boxed{d_H = a(t) \underbrace{\int_0^t \frac{d\tilde{t}}{a(\tilde{t})}}_{\eta\,?} = 3t \sim H^{-1}}$$

and $H = \frac{\dot{a}}{a} = \frac{2}{3}\frac{1}{t}$ $\quad \underbrace{H^{-1} = a\eta\,.\,?}_{\text{IDK}}$

* CMB as observed today consist of $\sim 10^5$ regions, which were never in causal contact i.e, if we can trace back how many causally independent patches fit into what we observe today as CMB surface. $\sim 10^5$

* The fact that these two scales: one that is associated with the particle horizon and the inverse Hubble are approximately the same if we take de-accelerated universe. But they are completely different once we have an early phase of inflation.

$\underset{\text{Particle Horizon}}{d_H} \sim H^{-1}$

⊛ Fourier transform of 2-point function

⟶ The power Spectrum $P(k)$

$$\xi(r_{12}) = \langle \delta(\vec{r}_1), \delta(\vec{r}_2) \rangle$$

and
$$\boxed{P(k) = \int \xi(r) e^{-i\vec{k}\vec{r}} d^3\vec{r}}$$

Similarly, Fourier transf. of 3-pt function is called the ⟶ bispectrum.

3-point function is a function of triangle configuration

⊛ when the light reaching us was emitted after the big bang, the sphere radius was: 42 million light years in radius.
But since, the space is expanding it took 13.5 billion light years for that light to reach us.
And also the sphere,
from which the light was emitted also expanded in size and that sphere would now be 46 billion light years away (Radius).

Hence 98 billion light years of diameter.

(*) If the spacetime is flat, then the entire universe possess zero net total energy. If the shape of the universe was anything else it would not have zero energy. Zero net energy means that you can create the entire universe out of nothing. Had the universe been filled with net positive or negative energy we would be forced to confront the question of what was the original source of the energy that began the universe. And if it had some amount of energy then why it has that amount and not some other amount.

(*) $\lambda_{obs} = \lambda_{emitted}(1+z)$

$$\Rightarrow \quad 1+z = \frac{1}{a(t)}$$

(*) Note that $d_H \equiv$ Hubble Horizon $\Rightarrow$ Distance over which a photon can travel in one Hubble time. (Claim: d_H/a is increasing)

And, $$\frac{d_H}{a} = \frac{1}{Ha} = \frac{1}{\frac{\dot{a}}{a} \times a} = \frac{1}{\dot{a}}$$

and $\dot{a}$ is decreasing $\therefore \frac{d_H}{a}$ is increasing.

$\Rightarrow$ Horizon expanding faster than scale factor.

(*) If we introduce a classical scalar field

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{2}\partial_\nu\phi\,\partial^\mu\phi - V(\phi)\right]$$

and observe the energy-momentum tensor

i.e., $$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\nu\phi)}\partial_\nu\phi - g_{\mu\nu}\mathcal{L}$$

we will notice that we have off diagonal species in the energy-momentum tensor.

i.e., T_{0i} and T_{ij} are non-zero.

$(i \neq j)$

Therefore scalar field actually doesn't behave as a perfect fluid. But these off diagonal species depend on the gradient and therefore if we are in a region where gradient varies very slowly then we can neglect these terms. Essentially this means, field is homogeneous.

(*) When free electron fraction is 1, everything is fully ionized.

(*) Cross section (= probability of interaction).

* optical depth to Conformal time η is:

$$\tau(\eta) = \int_{\eta}^{\eta_0} a n_e \sigma_T \, d\eta'$$

* $e^{-\tau}$ is the probability that photon has not scattered from time η to today.

* probability of last scattering: visibility

$$g = -\dot{\tau} e^{-\tau}$$

* Inflation predicts a quantity: Comoving Curvature perturbation. (constant and Conserved outside horizon). You can think of this as ~ the potential perturbation initial Condition at early time.

$$\mathcal{R} \sim \Phi \qquad \mathcal{R} = -\frac{(5+3w)}{3+3w} \Phi$$

$\circledast$ Define $\underbrace{\chi_{src}(z) = \int_{t_{src}(z)}^{t_0} \frac{c\,dt}{a(t)}}_{\text{looks like a comoving distance.}} = \int_0^z \frac{c\,dz'}{H(z)}$

we know, $\boxed{a(t) = \frac{1}{1+z}} \Rightarrow \frac{da}{dt} = \frac{-1}{(1+z)^2}\frac{dz}{dt}$.

$\Rightarrow$

$$H = \frac{1}{a}\frac{da}{dt} = \frac{-1}{(1+z)^2}\frac{dz}{dt} \times \frac{1}{a} = \frac{-1}{1+z}\frac{dz}{dt}$$

$\Rightarrow H(z) = \frac{-1}{1+Z}\frac{dz}{dt}$ or $1+Z = -\frac{dz}{dt} \times \frac{1}{H(z)} = \frac{1}{a(t)}$

$$\therefore \int_{t_{src}(z)}^{t_0} \frac{c\,dt}{a(t)} = \int_z^0 \frac{c\,dt}{H(z')} \frac{-dz'}{dt}$$

$$= \int_0^z \frac{c\,dz'}{H(z')}$$

$t \longrightarrow a(t) \longrightarrow z$

⊛ optical depth:

$\tau =$ Scattering rate $\times$ time (Δt)

So that means: optical depth is the no. of collisions a photon has undergone in time Δt.

or if: $\displaystyle \int_{t}^{t_0} dt' \underbrace{n_e(t')\,\sigma_T}_{\Gamma \equiv \text{Scattering rate}} \equiv$ no. of collision photon has undergone from time t to t_0

new: $\boxed{\dot{\tau} = -a n_e \sigma_T}$

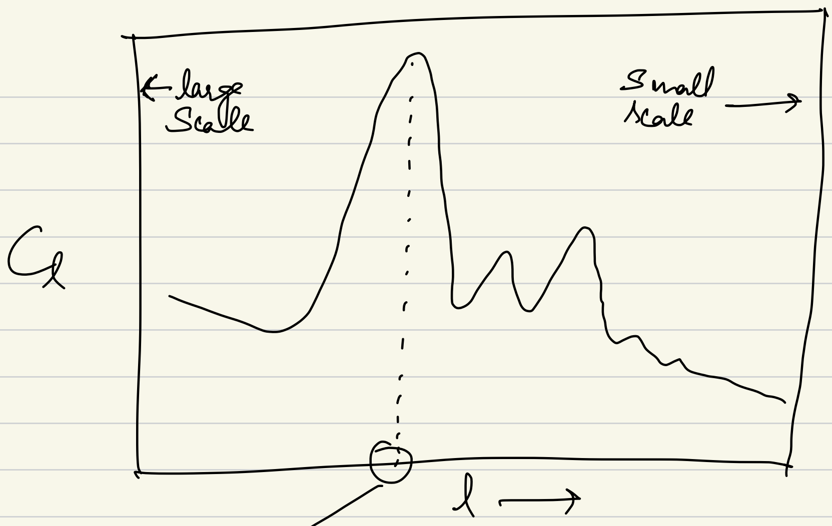
⊛ For photons, $\delta_r = 4\Theta_0$ and $V_r = -3i\Theta_1$

deviation in energy density from the homogeneous energy density in the CMB is related to the monopole.

⊛ · CMB power spectrum has a peak, reflects the fact that there's a characteristic size of the spots $\sim 1^\circ$

Smaller structures ~ peak at higher l

large " ~ " " low l.



Angular size $\sim \frac{360^\circ}{\ell}$

or $\ell \sim \frac{2\pi}{\theta}$

Characteristic size of the spots

$$\Theta(\hat{n}) = \underbrace{\frac{\delta_\gamma}{4} + \Phi_e}_{S} - \underbrace{\hat{n}\cdot v}_{\text{doppler term}} + \int_e^0 d\eta\, 2\Phi'$$

CMB temp anisotropy as a function of direction.

1. How photons propagate to us.
2. Initial Conditions.
3. How these initial conditions evolve in the primordial plasma to give the

* Inflation predicts a quantity called Comoving Curvature perturbation (constant and conserved outside the horizon). You can think of this as ~ the potential perturbation initial conditions at early times.

$$R \sim \Phi \qquad R = -\frac{5+3w}{3+3w}\Phi$$

* $$\langle R(k)\, R^*(k') \rangle = (2\pi)^3 P_R(k)\, \delta(k-k')$$

and $\left[\frac{k^3}{2\pi^2} P_R(k)\right] = \text{Constant}$ (Scale invariant).

* Remember this identity:

$$e^{ik\cdot x} = 4\pi \sum_{lm} i^l \underbrace{j_l(kx)}_{\text{Bessel function}}\, Y^*_{lm}(\hat{k})\, Y_{lm}(\hat{x})$$

* $$C_l = 4\pi \int d\ln k \underbrace{\left[\frac{k^3}{2\pi^2} P_R(k)\right]}\, \underbrace{\left[T_s(k,\eta_*)\right]^2}\, \underbrace{j_l^2}$$

$j_l(kx_*)$ peaks when $kx_* \sim l$.

⊛ $$\underbrace{\left(\frac{\delta_r}{4}\right)'' + \frac{1}{3}k^2\left(\frac{\delta_r}{4}\right) = \phi'' - \frac{1}{3}k^2\phi}_{\text{Acoustic oscillations: approximate eq.}}$$

Radition pressure

Driving force from potential.

During large scales - modes evolving during matter domination.

∴ ϕ be constant.

and eq: becomes,

$$\underbrace{\left(\frac{\delta_r}{4} + \phi\right)'' + \frac{1}{3}k^2\left(\frac{\delta_r}{4} + \phi\right) = 0}_{\text{simple harmonic oscillation}}$$

$v \longrightarrow \frac{k}{\sqrt{3}}$ higher $k \longrightarrow$ high frequency

⊛ Another imp quantity is :

$$R \equiv \frac{\bar{\rho}_b}{(\bar{\rho}_r + \bar{\rho}_r)}$$

Adding baryon Contribution offsets the Sachs wolfe term by Constant R.

$$\frac{\delta_r}{4} + \phi_e = \frac{1}{5}\mathcal{R}\left[(1+3R)\cos(kr_s) - R\right]$$

$\mathcal{R}$ → Comoving Curvature perturb.

Upshot is that baryons enhance Compression.

Therefore,

$$\frac{\delta_r}{4} + \phi \sim \begin{cases} -\frac{1}{5}\mathcal{R}\left[(1+3R)\cos - 3R\right] & \text{Large Scale.} \\ -R\left[\cos\right] & \text{Small Scale.} \end{cases}$$

⊛ Baryons increase the odd peaks which are the Compression peaks → created due to Compress in BAO

⊛ So far we are assuming that we are dealing with the perfect fluid. That is a good approximation on a large scale, on scales that are large compared to the mean free path of the photon that is scattering.

Considering small scales ~ higher k, higher l, that no longer is a good approximation.

Because of the diffusion through the scattering of the photons. So if you set up a tiny variation in the, that's not gonna persist for long cause, the photons are scattering and they are gonna diffuse out these tiny variations.

Now we need to figure out the length scale over which this diffusion process will happen.

Diffusion length $\longrightarrow$ How far the photon has random walked after N scatterings.

$\Rightarrow$ Structures of the size of Diffusion length are erased. This is also known as <u>silk damping</u>

upshot $\longrightarrow$ Small scales are erased $\longrightarrow$ high l, high k are damped away.

⊛ $d_p \equiv$ particle Horizon $= \begin{cases} 3t = 2H^{-1} & \text{matter} \\ 2t = H^{-1} & \text{rad.} \end{cases}$

⊛ Event horizon $\equiv d_e = a(t) \int_t^{\infty} \frac{dt'}{a(t')}$.

For the universe dominated by matter or radiation for all times, d_e is infinite.

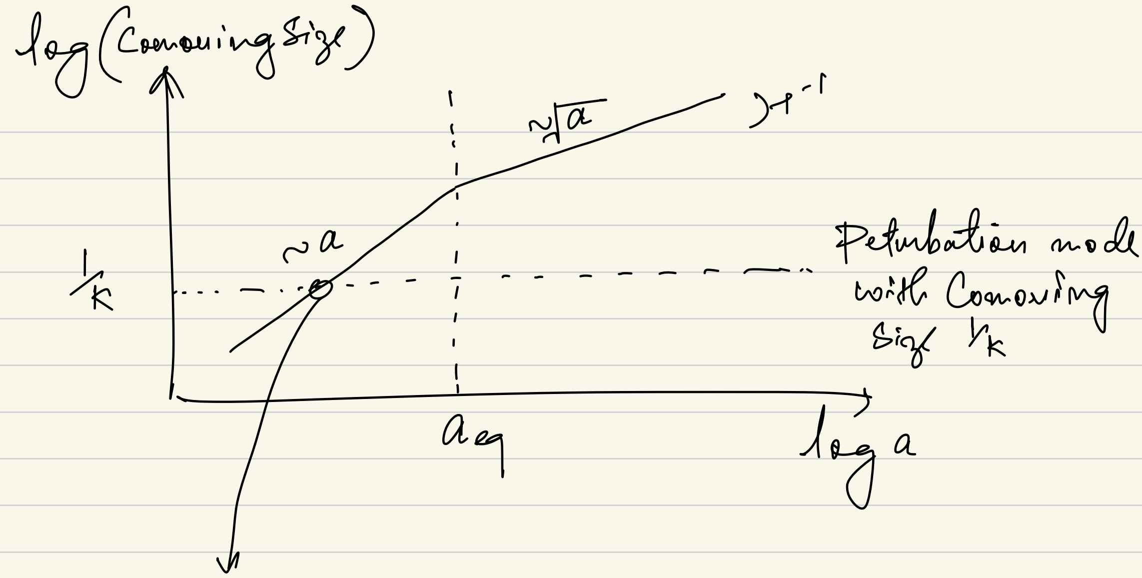
For a de Sitter universe, $d_e = H^{-1}$ is finite and equal to hubble horizon.

Structure formation

* perturbations are characterized by their
  1. size $\equiv a/k$
  2. amplitude $\equiv \delta(\eta, k)$
* The physical size grows $\propto a$ due to expansion
* In the comoving coordinates the size $\sim 1/k$ stays fixed.
* $\mathcal{H}^{-1} \propto a$ radiation domination

  $\mathcal{H}^{-1} \propto \sqrt{a}$ matter domination

  $\mathcal{H}^{-1} \equiv$ Comoving Hubble parameter.
* $$\eta = \begin{cases} \mathcal{H}^{-1} & \text{rad. domination} \\ 2\mathcal{H}^{-1} & \text{matter domination} \end{cases}$$



Horizon entry of the mode with wavenumber k

⊛ Next we need to find how the perturbations evolve?

$\Theta, \mathcal{N}, \underbrace{\dot{\delta}, v}_{\text{Dark matter}}, \delta_b, v_b, \phi, \psi$?

⊛ Evolution of perturbations (outside Horizon, rad. domination)

1. $$v = v_b = v_\gamma = -ik\eta \times \frac{\psi}{p+2}$$

For superhorizon mode $k\eta \ll 1$ is suppressed and we can neglect the velocity in practice.

2. We set also the higher moments of the neutrino temp. perturbation to zero initially.

3. Neglecting the velocity in the Boltzmann eq. for the densities (or monopoles for the photons).

$$\dot{\delta} = \dot{\delta}_b = -3\dot{\phi} \; , \quad \dot{\Theta}_0 \approx \dot{N}_0 = -\dot{\phi}$$

4. From the einstein eq. we get,

$$\phi \approx -\psi \approx \text{Constant} \qquad \text{for } k\eta \ll 1, \; a \ll a_{eq}$$

5. The photon and neutrino density is also constant far outside the horizon.

6. To further reduce the initial conditions we need to make assumptions. The standard assumption are known as adabatic intial condition which posit that perturbation in the entropy density are universal for all species.

7- In summary, adiabatic initial conditions correspond to the choice:

$$\delta = \delta_b = \frac{3}{2}\phi$$

$$\Theta_0 = N_0 = \frac{1}{2}\phi$$

$$\Psi = -\phi$$

$$v = v_b = 0$$

$$\Theta_\ell = N_\ell = 0 \quad (\ell \geq 1)$$

The physical motivation for adiabatic initial conditions is that, in thermal equilibrium, they correspond to a universal temp. perturbation $\delta T/T$ for all species.

More deeply, the motivation is that then all perturbations were seeded by a single fluctuating degree of freedom.

Boltzmann eq. for photons.

$$\mathcal{L}[f] = C[f]$$

Note
Temp perturbation Θ is a Scalar Quantity

$\Rightarrow$

$$\frac{\partial \Theta}{\partial t} + \underbrace{\frac{\partial \Theta}{\partial x^i} \frac{\hat{p}^i}{a}}_{\text{free streaming}} + \underbrace{\frac{\hat{p}^i}{a} \frac{\partial \psi}{\partial x^i} + \frac{\partial \phi}{\partial t}}_{\text{effect of gravity.}}$$

$$= n_e \sigma_T \left(\underbrace{\Theta_0(t, \vec{x}) - \Theta(t, \vec{x}, \hat{p})}_{\text{Isotropization}} + \underbrace{\hat{p} \cdot \vec{v}_e}_{\text{Drag term}} \right)$$

Free streaming :- photons probe different regions after sometime.

Effect of gravity : Gravitational RedShift + Change with time of the gravitational potential

Isotropization : Compton Scattering changes the direction of the photons but not energy. Therefore if the Compton Scattering is very efficient then the angular direction would be washed out. And this Converges to monopole Contribution which sort of average over all angles.

Conformal time

⇒ Conformal time is defined in such a way that the spacetime metric becomes conformally flat.
(particularly in the context of FRWL metric.)

$$d\eta = \frac{dt}{a(t)} \quad \text{or} \quad \eta = \int \frac{dt}{a(t)}$$

In terms of conformal time, the FLRW metric takes the simple form:

$$ds^2 = a(\eta)^2 \left(-d\eta^2 + d\mathbf{x}^2\right)$$

The advantage of using conformal time is that it simplifies the eq. of motion for the scalar, vector, and tensor perturbations in the early universe.

One key feature is that conformal time extends from $-\infty$ to 0 for a finite interval of proper time, making it useful coordinate for describing both the early and late-time behaviour of the universe.

Fourier transform.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t}\, d\omega$$

In our Case

$$\Theta(\eta, \vec{x}, \hat{p}) = \int \frac{d^3 k}{(2\pi)^3}\, e^{ik\cdot x}\, \Theta(\eta, \vec{k}, \hat{p})$$

Note that this effects only the spatial dependence.

Also, the gradient in fourier space is just ik

i.e, $\quad v_e'' = \nabla \chi = ik\chi$

Boltzmann eq. for Cold Dark matter.

$\Rightarrow$ we assume that at the time of recombination any interactions of Cold dark matter are negligible and therefore no

$$\mathcal{L}[f_{dm}] = p^0 \frac{df_{dm}}{dt} = 0$$

Boltzmann eq. after simplification: we have Continuity eq. and Euler eq.

$$\dot{\delta}_{dm} + ikv_{dm} + 3\dot{\phi} = 0 \quad \text{linearized Continuity eq.}$$

$$\dot{v}_{dm} + \mathcal{H} v_{dm} + ik\phi = 0 \quad \text{linearlized Euler eq.}$$

where, $\mathcal{H} = \frac{1}{a}\frac{da}{d\eta} = aH$ (Conformal Hubble rate)

and density Contrast $\delta_{dm} \equiv \frac{n_{dm} - n^{(0)}_{dm}}{n^{(0)}_{dm}}$

Lecture 1.

Cosmological principle : Universe is homogeneous and isotropic.

isotropic :- Same in every direction

homogeneous: Same in every place.

Now if it's isotropic then the only way it could not be homogeneous if it forms a ring or shell like structure. Then obv. the place at the centre is different from any other place and therefore not homogeneous.

Therefore as far as we can see the space is uniformly filled with particles.

particles — galaxies, stars etc.

I mean if you average over billions of ligh years then it's look pretty much the same.

Hubble parameter : $\dfrac{\text{Velocity}}{\text{Distance}} = \dfrac{\dot{a}(t)}{a(t)}$.

There is not reason for hubble parameter / constant to be constant in time. It is constant in the sense that it has same value throughout the universe at a given moment

$\therefore$ velocity b/w galaxies = $H \times$ Distance b/w them.

⇒ Mass in the each grid remains same but since the universe expand therefore density decreases.

$$\rho(t) = \frac{\nu}{a^3(t)}$$

$\nu \longrightarrow$ mass per grid box

⇒ Just by Newton's gravitational law and the 2nd law of motion, we can derive the following equation.

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi}{3} G\rho}$$

That's why this is an acceleration eq.

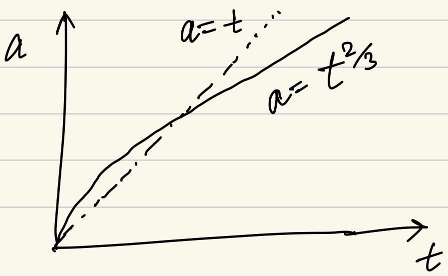
⇒ From energy conservation we get:

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho}$$

at escape velocity (energy conservation)

Now, the solving the energy conservation law

$a \sim c t^{2/3}$ c is some constant.



Lecture 2

⇒ We would run into trouble if the galaxies or particles or whatever is present in the universe are moving pass each other at a significant fraction of the speed of light.

⇒ By including the cases where $E \neq 0$ we get the Friedmann equation.

i.e,
$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi}{3}\rho G = \frac{C}{a^2}}$$

Note: $\rho(t) = \dfrac{\nu}{a^3(t)}$

RHS can be both +ve or −ve dependending on the energy.

3 Cases

1. Positive energy: $a \sim t^{2/3}$ (small a) $a \sim t$ (large a)

2. Negative energy: collapse.

3. Zero energy: Asomtotically slow down but never stop moving.

Note: Positiveness/negativeness/zero energy is associated with geometry of the universe.

For radiation dominated universe the friedmann eq. would be:

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G}{3}\frac{v}{a^4} = \frac{c}{a^2}$$

$\longrightarrow$ only this changes.

Solving this for zero energy case we get:

$$a \sim t^{1/2}$$

For zero energy case with both matter and radiation we have:

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{c_1}{a^3} + \frac{c_2}{a^4}}$$

Note

One can even see that friedmann eq. actually looks like einstein eq.

$$\underbrace{\left(\frac{\dot{a}}{a}\right)^2 - \frac{c}{a^2}}_{\text{geometry}} = \underbrace{\frac{8\pi}{3}G\rho}_{\text{energy momentum}}$$

Lecture 3.

Metric in polar Coordinates

$$ds^2 = dr^2 + r^2 d\theta^2$$

$d\theta^2$ is the metric of one circle of unit radius.
So let's call is $d\Omega_1^2$

$$\therefore \quad ds^2 = dr^2 + r^2 d\Omega_1^2$$

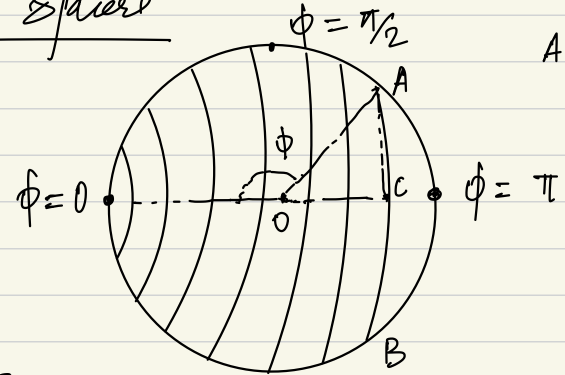
⇒ Also ds^2 is essentially the metric of a flat space.

⇒ Think of flat space as a metric of nested Circles.

⇒ Circle is also called one sphere.

⇒ 2-Sphere is also an homogeneous Surface.

Metric of a Sphere



AC = radius = $\sin\phi$
OA = radius = r

$d\Omega_2^2 = r^2 (d\phi^2 + \sin^2\phi \, d\Omega_1^2)$ & Similarly for 3-sphere

$d\Omega_3^2 = r^2 (d\phi^2 + \sin^2\phi \, d\Omega_2^2)$ $\left(ds^2 = dr^2 + r^2 d\Omega_2^2 \right)$
(3-Sphere) Flat 3-D Here r is different.

Dimension:	Flat	Curved (1, 2, 3 - sphere)
1	dr^2	$r^2 d\theta^2 = r^2 d\Omega_1^2$
2.	$dr^2 + r^2 d\Omega_1^2$	$r^2 (d\phi^2 + \sin^2\phi \, d\Omega_1^2)$
3.	$dr^2 + r^2 d\Omega_2^2$	$r^2 (d\phi^2 + \sin^2\phi \, d\Omega_2^2)$

Another Homogeneous geometry is Hyperbolic Space.

$$dH^2 = dr^2 + \sinh^2 r \, d\Omega_1^2$$

$$dH^3 = dr^2 + \sinh^2 r \, d\Omega_2^2$$

Metric for different geometries.

$$ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2) \qquad \text{flat}$$

$$ds^2 = -dt^2 + a^2(t) \, d\Omega_3^2 \qquad \text{Spherical}$$

$$ds^2 = -dt^2 + a^2(t) \, dH_3^2 \qquad \text{Hyperbolic}$$

lecture 4.

related to Curvature

Starting with:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

$$k := \begin{cases} +1 & \text{+ve} \\ 0 & \text{flat} \\ -1 & \text{-ve} \end{cases}$$

This eq. comes from the time-time component of the einstein equation.

1. Matter dominated universe

$$\rho = \frac{\rho_0}{a^3}$$

2. Radiation Dominated

$$\rho = \frac{\rho_0}{a^4}$$

Note uptill now for $k \equiv \begin{cases} +ve \longrightarrow \text{Collapse} \\ 0 \longrightarrow \text{Asymptotically Slow down} \\ -ve \longrightarrow \text{continues to increase linearly at later time.} \end{cases}$

Generally the eq. of states take this form.

$\Rightarrow$ $P = w\rho$ $\quad P \equiv$ pressure

$w = 0$ for matter dominated

$w = \frac{1}{3}$ for radition. $\quad w = -1$ for Dark energy

Now $\quad E = \rho V$

$dE = -p\,dv$

Also, $dE = \rho\,dv + V\,d\rho = -p\,dv$

$\Rightarrow$ $V\,d\rho = -(p+\rho)\,dv$

But $p = w\rho$

$\therefore$ $V\,d\rho = -(1+w)\rho\,dv$

$\Rightarrow$ $\frac{d\rho}{\rho} = -(1+w)\frac{dv}{V}$

$\Rightarrow$ $\log\rho = -(1+w)\log V$

$\Rightarrow$ $\rho = \frac{c}{V^{(1+w)}} = \frac{c}{a^{3(1+w)}}$

let's Say a box of gas.

Volume = V

$$\boxed{\rho = \frac{c}{a^{3(1+w)}}}$$

Note: A negative pressure like tension.

and Dark energy is the property of empty space and empty space doesn't dilute over streching.

Lecture 5

matter: $a(t) \sim t^{2/3}$

Radition: $a(t) \sim t^{1/2}$

Since $p = w\rho \Rightarrow w = 0 \Rightarrow$ pressure $= p = 0$

$w = -1 \Rightarrow$ tension

and for radiation $w = \frac{1}{3}$ (massless particles)

Vacuum energy is simply there in the empty space.

Vacuum energy density of space doesn't change.

$$\rho_0 = \Lambda \frac{3}{8\pi G} \qquad \Lambda \equiv$$

Case with only Vacuum energy density.

$\Lambda := \pm, 0$ and $k = +ve, 0, -ve$

So there are various Cases. obviously some of them doesn't make sense.

Case 1 ($k = 0$ and +ve Λ)

$$\left(\frac{\dot{a}}{a}\right)^2 = \Lambda \Rightarrow \frac{da}{dt} = \sqrt{\Lambda}\, a$$

$$a = c\, e^{\sqrt{\Lambda} t}$$

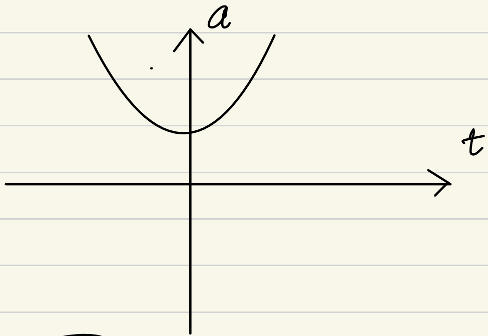
Hubble Constant $= \sqrt{\Lambda}$

This space exponentially expand and this is called de-sitter space.

Case 2 (k +ve and Λ +ve)

$$\left(\frac{\dot{a}}{a}\right)^2 = \Lambda - \frac{1}{a^2} \Rightarrow \dot{a}^2 - \Lambda a^2 = -1$$

$\Rightarrow$



Picture of the universe in this case

And this is also called de-sitter space.

Case 3 k -ve and Λ -ve.

$\Rightarrow \dot{a}^2 = -a^2 + 1$ Harmonic Oscillator?

Big Crunch.

Friedmann eq and Continuity eq.

1. $H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$.

2. $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$

Continuity eq (not independent of above two)

$$D_\mu T^{\mu\nu} = 0 \Rightarrow \boxed{\frac{d\rho}{dt} + 3H(\rho + P) = 0}$$

and Remember,

$P = w\rho$ and $\rho \propto \frac{1}{a^{3(1+w)}}$.

lecture 6. (Red Shift)

$$\frac{-K}{a^2} + \frac{8\pi G}{3}\rho = \frac{C_R}{a^4} + \frac{C_m}{a^3} + \Lambda - \frac{K}{a^2} = H^2$$

$$\Omega_R + \Omega_m + \Omega_\Lambda + \Omega_K = 1$$

So basically in past

$\Omega_R \sim 0$

Ω_Λ was unknown

$\Omega_m \sim \frac{1}{20}$ or $\frac{1}{30}$ something

$\therefore$ Only K factor remains and therefore:

K: is -ve. (open infinite universe)

$\Rightarrow$ So for the most time in the history of the universe

$\frac{C_m}{a^3}$ would dominate $\frac{-K}{a^2}$

$\therefore a \sim t^{2/3}$ $\therefore H = \frac{\dot{a}}{a} = 2/3t \Rightarrow H_{Today} = \frac{2}{3T_{age}}$

Dark matter

$$\Rightarrow \quad \frac{mMG}{r^2} = \frac{mv^2}{r} \Rightarrow \frac{MG}{r} = v^2$$

or $\quad v \propto \frac{1}{\sqrt{r}}$

* Dark matter is probably not electrically charged, otherwise we should see it radiate.

* One fact about the dark matter particles is that they cluster. Things cluster less when they have high velocities.

  # Cold dark matter.

Red Shift.

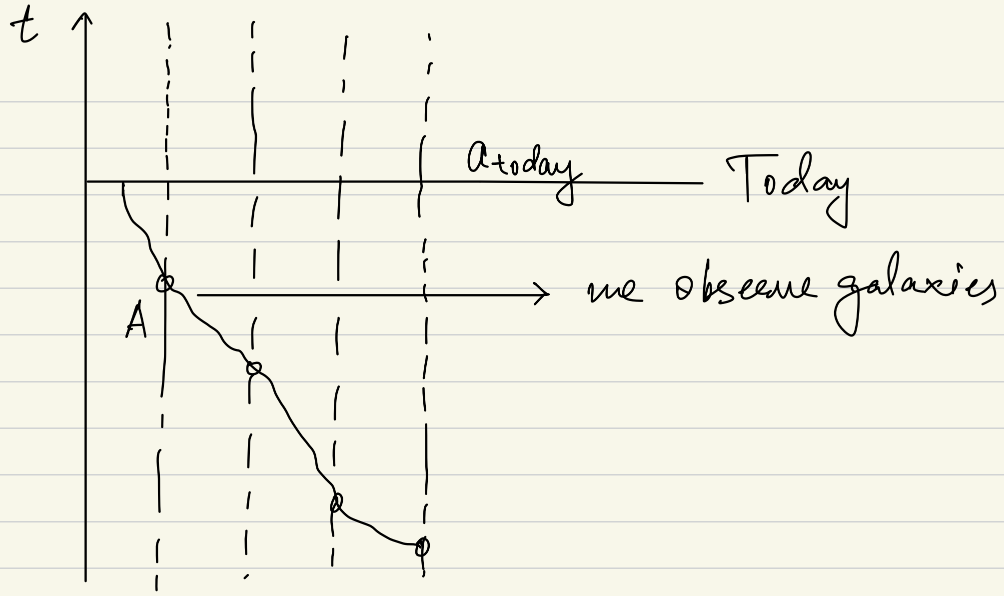
$\lambda_{emitted} \equiv$ Wavelength when emitted from the Source.

When detected it may or may not have a same wavelength due to many reasons:

a) doppler shift

b) gravitational redshift

c) expansion of the universe.



meanwhile light emitted at point A reaches today the universe has streached by a factor:

$$\frac{a_{today}}{a_A}$$

So this is the factor by which E.M wave would streach going from A to today.

$$z = \frac{\lambda_{detected}}{\lambda_{emitted}} - 1$$

redshift factor.

Q. what is the number of galaxies within redshift z and $z + dz$. that is no of galaxies per unit redshift.

$$:= dN/dz$$

Now $\frac{dN}{dZ} \propto \frac{\text{Area}}{dZ} = \frac{dr\, \xi(r)^2}{?}$

$$Z = \frac{a_{today}}{a(t)} - 1 \Rightarrow -\frac{a_{today}}{a^2(t)} da = dZ$$

$$\Rightarrow \frac{dN}{dZ} \approx \frac{\xi^2(r)}{(Z+1) \cdot \frac{da}{dt}}$$

Also using $-dt^2 = a^2(t)\, dr^2$ (eq. for light ray)

or $dr = \frac{-dt}{a(t)}$

for matter dominated universe:

$a \sim t^{2/3}$.

$\therefore \dot{a} = \frac{2}{3} t^{-1/3}$ and

$L(Z)$ and dN/dZ together and enough to determine $a(t)$ and K.

<u>observational fact</u>

$\Omega_R \approx 0$, $\Omega_M \approx 0.3$ (luminous + Dark), $\Omega_\Lambda \approx 0.7$ & $\Omega_K = 0$

Lecture 24 (Fredric Schuller)

⇒ Strictly speaking metric is the gravitational potential and the gravitational field strength is the Reimann tensor.

⇒ In general relativity the energy-momentum tensor feels the mass current (rotating mass).

Peturbation.

$$G_{ab}[g+\delta g] = T_{ab}[g] + \delta T_{ab}[g]$$

$$\Rightarrow \quad G_{ab}[g] + \delta G_{ab}[g, \delta g] = T_{ab}[g] + \delta T_{ab}[g]$$

linear dependence

∴ Solve.

$$\delta G_{ab}[g, \delta g] = \delta T_{ab}[g]$$

⇒ we choose a Coordinate System where:

$$g_{\alpha\beta} = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & -\gamma_{\alpha\beta} & \\ 0 & & & \end{array}\right]$$

Riemanian 3-metric

<u>let's peturb this now</u>

$$\delta g = 2a\, dx^0 \otimes dx^0 - b_\alpha \left(dx^0 \otimes dx^\alpha + dx^\alpha \otimes dx^0\right)$$
$$- \left[2c\,\gamma_{\alpha\beta} + 2e_{\alpha\beta}\right] dx^\alpha \otimes dx^\beta$$

	<u>Degrees of freedom</u>
Scalar field: a, c	1 + 1
Vector field: b_α	3
tensor field: $e_{\alpha\beta}$ $\gamma^{\alpha\beta} e_{\alpha\beta} = 0$	5 (because trace free)

<u>Note</u>: I think the minus sign in δg is because of the choosen style of metric $(1, -1, -1, -1)$.

Helmholtz - Hodge Theorem

It is mainly useful to further decompose

a) Vector field b_α as

$$b_\alpha = \underbrace{D_\alpha B}_{\text{Scalar field}} + \underbrace{B_\alpha}_{\substack{\text{vector field} \\ \text{Divergence free} \\ D_\alpha B^\alpha = 0}}$$

(where D is the Covariant derivative of γ
Something like ∇)

b) $$e_{\alpha\beta} = \left(D_\alpha D_\beta - \frac{1}{3}\gamma_{\alpha\beta}\Delta\right)E \quad \longrightarrow \text{Scalar field,} \quad \boxed{\Delta = \gamma^{\mu\nu} D_\mu D_\nu}$$

$$+ 2D_\alpha E_\beta \longrightarrow \text{vector field Divergence free}$$

$$+ E_{\alpha\beta}$$

Symmetric tensor field.

trace free : $\gamma^{\alpha\beta} E_{\alpha\beta} = 0$

Divergence free : $D_\alpha E^{\alpha\beta} = 0$

Thus a general perturbation δg uniquely decomposes into three independent types of perturbations.

$$\delta g = \underset{\substack{\uparrow \\ A,\, B,\, C,\, E}}{\delta g_{scalar}} + \underset{\substack{\uparrow \\ \underbrace{B_\alpha,\ E_\alpha}_{\substack{\text{Divergence} \\ \text{free}}}}}{\delta g_{vector}} + \underset{\substack{\uparrow \\ \underbrace{E_{\alpha\beta}}_{\substack{\text{Symmetric} \\ \text{trace free} \\ \text{Divergence free}}}}}{\delta g_{tensor}}$$

<u>Therefore</u>.

$$\delta g_{scalar} := 2A\, dx^0 \otimes dx^0 - D_\alpha B \left(dx^0 \otimes dx^\alpha + dx^\alpha \otimes dx^0 \right) - 2\left[C \gamma_{\alpha\beta} + \left(D_\alpha D_\beta - \frac{1}{3} \gamma_{\alpha\beta} \Delta \right) E \right] dx^\alpha \otimes dx^\beta$$

$$\delta g_{vector} := -B_\alpha \left(dx^\alpha \otimes dx^0 + dx^0 \otimes dx^\alpha \right) - 4 D_\alpha E_\beta\, dx^\alpha \otimes dx^\beta$$

$$\delta g_{tensor} := -2 E_{\alpha\beta}\, dx^\alpha \otimes dx^\beta$$

$\Rightarrow$ For Scalar perturbations, only two specific contributions of the fields A, B, C, E will be uneffected (and thus be not removable) by general Coord. transformation. These two Combinations are called gauge invariant. Only these can be taken seriously.
other Combination can always be removed by a Coordinate transformation.

- $\Psi := A + \dot{B} - \ddot{E}$

- $\Phi := C - \frac{1}{3} \Delta E$

$\Rightarrow$ For vector perturbations there is only one Combination of B_α, E_α that is gauge-invariant.

$$\Theta_\alpha := B_\alpha - \dot{E}_\alpha$$

$\Rightarrow$ For the tensor transformation:-
All $E_{\alpha\beta}$ are gauge invariant.

$\Rightarrow$ Eq. for the evolution of photon energy along it's trajectory.

$$\frac{dp}{dt} = -p\left(\frac{\hat{p}^i}{a}\frac{\partial\psi}{\partial x^i} + \frac{\partial\phi}{\partial t} + H\right)$$

Linear Cosmological peturbation theory

$A(t, \vec{x})$: Density, pressure, velocity, metric peturbation

2point Correlation function

$$\underbrace{\langle A(t, \vec{x}), A(t, \vec{x}') \rangle} = \xi_A(t, \vec{x}, \vec{x}') = \xi_A(t, |\vec{x} - \vec{x}'|)$$

→ This Could be thought of as the Clumpiness factor
→ The higher the value for some distance scale, the more clumpy the universe is at that distance scale.

→ Given a random galaxy in a location, the Correlation function describes the probability that another galaxy will be found within the given distance.

In fourier Space

$$\langle A(t, \vec{k}), A^*(t, \vec{k}') \rangle = \delta_D(\vec{k}' - \vec{k}) P_A(k)$$

modolous of K

Power Spectrum.

It represents the contribution to the fluctuation field from waves of wavenumber k. A completely flat power spectrum corresponds to white noise, while a power spectrum that is sharply peaked at some particular value of k has a characteristic length scale.

A single plane wave has a power spectrum consisting of a single spike.

The Collisionless Boltzmann eq. for photons.

1. $g_{00}(\vec{x},t) = -\left(1 + 2\psi(\vec{x},t)\right)$

2. $g_{0i} = 0$

3. $g_{ij} = a^2\delta_{ij}\left(1 + 2\phi(\vec{x},t)\right)$

⇒ In the absence of ψ & ϕ this is FRW metric

⇒ In the absence of expansion $(a=1)$ this metric describes weak gravitational field.

⇒ $\psi \longrightarrow$ newtonian potential
$\phi \longrightarrow$ Spacial Curvature.

⇒ These are the Scalar peturbations only.

⊛ Boltzmann eq.

$$\frac{df}{dt} = C[f]$$

f: is the distribution function
$C[f]$: Collision term

The Collisionless Boltzmann eq. is defined by the Liouville theorem:

$$0 = \frac{d}{d\lambda} f\left(x^\mu(\lambda), p^\mu(\lambda)\right) \qquad \left[\, p^2 = g_{ij} p^i p^j \,\right]$$

But for our purpose: $f(t, \vec{x}, p, \hat{p})$, [No p^0]

$$\Rightarrow \frac{df}{dt} = \underbrace{\frac{\partial f}{\partial t}}_{} + \underbrace{\frac{\partial f}{\partial x^i} \cdot \frac{dx^i}{dt}}_{} + \underbrace{\frac{\partial f}{\partial p}\frac{dp}{dt}}_{} + \underbrace{\frac{\partial f}{\partial \hat{p}^i} \cdot \frac{d\hat{p}^i}{dt}}_{}$$

$\Rightarrow$ Here, we need not include the term proportional to $\partial f / \partial p^0$ when expanding the total time derivative

$\Rightarrow$ The easiest term is the last one since it does not contribute at first order in perturbation theory.

$\Rightarrow$ Also, $\frac{dx^i}{dt} = \frac{dx^i}{d\lambda}\frac{d\lambda}{dt} = \frac{dx^i}{d\lambda} \Big/ \frac{dt}{d\lambda} = \frac{p^i}{p^0}$

Now we know that, $p^2 = g_{\mu\nu} p^\mu p^\nu$

$$\Rightarrow \quad p^2 = 0 = -(1+2\psi)(p^0)^2 + p^2 = 0 \quad \left(\begin{array}{c}\text{zero for}\\ \text{photons}\end{array}\right).$$

where we defined $p^2 = g_{ij} p^i p^j$

$$\Rightarrow \quad p^0 = \frac{p}{(1+2\psi)^{1/2}} = p(1-\psi)$$

and p^i is proportional to $\hat{p}^i$

$\therefore \quad p^i \equiv C\hat{p}^i$

and $\quad p^2 = g_{ij}p^ip^j \Rightarrow p^2 = g_{ij}\hat{p}^i\hat{p}^j C^2$

$$= a^2(1+2\phi)\delta_{ij}\hat{p}^i\hat{p}^j C^2$$

$$= a^2(1+2\phi)C^2$$

Note.

For comoving observ. four velocity is:

$U^\mu = \left(\frac{1}{a}, 0, 0, 0\right)$

$U_\mu = (-a, 0, 0, 0)$

$$\Rightarrow \quad C = p(1-\phi)/a$$

$$\therefore \quad p^i = p\hat{p}^i \frac{1-\phi}{a}.$$

$$\therefore \quad \frac{dx^i}{dt} = \frac{\hat{p}^i}{a}(1+\psi-\phi)$$

we can neglect the potentials in the above eq. and substitute that in the boltzmann eq.

$$\Rightarrow \quad \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial p}\frac{dp}{dt}.$$

now we need to calculate $\frac{dp}{dt}$ only:

To begin with, let us recall that the time Component of the geodesic eq.

$$\frac{dp^0}{d\lambda} = -\Gamma^0_{\alpha\beta}\, p^\alpha p^\beta \qquad \frac{dt}{d\lambda} = p^0$$

But $p^0 = p(1-\psi)$

$$\Rightarrow \quad \frac{d\left[p(1-\psi)\right]}{dt} = -\Gamma^0_{\alpha\beta}\,\frac{p^\alpha p^\beta}{p}(1+\psi)$$

Now we multiply both sides by $(1+\psi)$; drop all terms quadratic in ψ; and reexpress the total time derivative of ψ in terms of partial derivativ

$$\frac{dp}{dt} = p\left\{\frac{\partial\psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\psi}{\partial x^i}\right\} - \Gamma^0_{\alpha\beta}\,\frac{p^\alpha p^\beta}{p}(1+2\psi)$$

Then we calculate all the christoffel symbols and other terms and get:

$$\boxed{\frac{1}{p}\frac{dp}{dt} = -H - \frac{\partial\phi}{\partial t} - \frac{\hat{p}^i}{a}\frac{\partial\psi}{\partial x^i}}$$

Finally the boltzmann eq. becomes:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - p\frac{\partial f}{\partial p}\left[H + \frac{\partial \phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \psi}{\partial x^i}\right]$$

To go further we expand the photon distribution function f about it's zeroth order bose–Einstein value.

$$f(\vec{x}, p, \hat{p}, t) = \left[\exp\left\{\frac{p}{T(t)[1+\Theta(\vec{x},\hat{p},t)]}\right\} - 1\right]^{-1}$$

Here the zeroth order temp T is a function of time only (i.e, scales as a^{-1}) not space.

In the smooth zeroth order universe, photons are distributed homogeneously, so T is independent of $\vec{x}$ and isotropically so T is independent of the direction of propagation $\hat{p}$.

Θ depends on $\vec{x}$ $\hat{p}$ and t. $\Rightarrow$ magnitude of the photon momentum is virtually unchanged during the Compton scattering.

Also, $T\frac{\partial f^{(0)}}{\partial T} = -p\frac{\partial f^{(0)}}{\partial p}$

$$\Rightarrow \quad f \simeq \frac{1}{e^{p/T}-1} + \frac{\partial}{\partial T}\left[\exp\left\{\frac{p}{T}\right\}-1\right]^{-1} T\Theta$$

$$= f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta$$

$$\Rightarrow \quad \boxed{f \simeq f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta}$$

Now everywhere we encounter f in the expression of $\frac{df}{dt}$ on the last page, we substitute the second term of the above eq.

i.e, $\underbrace{-p\frac{\partial f^{(0)}}{\partial p}\Theta}$

Finally we will have,

$$\left.\frac{df}{dt}\right|_{\text{first order}} = -p\frac{\partial f^{(0)}}{\partial p}\left[\underbrace{\frac{\partial\Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i}}_{\text{free streaming}} + \underbrace{\frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}}_{\text{gravity}}\right]$$

<u>Note</u>: The zeroth order distribution function is set precisely by the requirement that the collision term vanishes. The reason being that any collision term includes the rate for the given reaction and for it's inverse. If the distribution function are set to their equilibrium values, the rate for the reaction precisely cancels the rate for it's inverse.

Scattering term

$$e^-(\vec{q}) + \gamma(\vec{p}) \longleftrightarrow e^-(\vec{q}\,') + \gamma(\vec{p}\,')$$

$$E_e(q) - E_e(\vec{q}+\vec{p}-\vec{p}\,') = \frac{q^2}{2m_e} - \frac{(\vec{q}+\vec{p}-\vec{p}\,')^2}{2m_e}$$

$$\simeq \frac{(\vec{p}\,'-\vec{p})q}{m_e}$$

$\vec{p}\,'-p$ is of order p, of order the ambient temperature T,

$\therefore$ RHS of above eq. is of the order;

$\frac{Tq}{m_e} \sim T v_b$ where the baryonic velocity v_b is very small.

$$C[f(\vec{p})] = -p\frac{\partial f^{(0)}}{\partial p} n_e \sigma_T \left[\Theta_0 - \Theta(\hat{p}) - \hat{p}\cdot\vec{v}_b\right]$$

Boltzmann eq. for the photons.

$$\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} + \frac{\partial \phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \psi}{\partial x^i}$$

Thomson cross-section

$$= \underbrace{n_e \sigma_T}_{\Gamma :\equiv \text{scattering rate}} \left[\Theta_0 - \Theta + \hat{p}\cdot\vec{v}_b\right]$$

In the Conformal time setting, the boltzmann eq. becomes:

$$\dot{\Theta} + \hat{p}^i \frac{\partial \Theta}{\partial x^i} + \dot{\phi} + \hat{p}^i \frac{\partial \psi}{\partial x^i} = n_e \sigma_T a\left[\Theta_0 - \Theta + \hat{p}\cdot v_b\right]$$

we also define:

$$\boxed{\Theta_\ell = \frac{1}{(-i)^\ell}\int_{-1}^{1} \frac{d\mu}{2} P_\ell(\mu)\,\Theta(\mu)}$$

$$\Theta_0 = \frac{1}{4\pi}\int d\Omega'\, \Theta(\hat{p}, \vec{x}, t) \qquad \text{[monopole]}$$

The moment Θ_0 and Θ_1 do not completely characterize the photon distribution. More generally it is useful to define the ℓ^{th} multipole moment of the temperature field as above:

$\ell = 2$ (quadrupole)
$\ell = 3$ (octopole).

and P_ℓ is the legendre poly.

Note: The higher legendre polynomial have structures on smaller scales. So the higher moment capture information about the small scale structure of the temperature field.

So the photon distribution can be described either by $\Theta(k, \mu, \eta)$ or by the whole hierarchy of $\Theta_l(k, \eta)$. And ofcourse similar freedom applies to the neutrino distribution.

⇒ optical depth.

$$\tau \equiv \int_t^{t_0} dt' \, n_e(t') \sigma_T = \int_\eta^{\eta_0} d\eta' \, a \, n_e \sigma_T$$

usually one writes:

$$\dot{\tau} = -a n_e \sigma_T$$

$$\boxed{\mu = \frac{\vec{k}}{|\vec{k}|} \cdot \hat{p}}$$

⇒ $\underbrace{n_e \sigma_T}$ is the Scattering rate

⇒ photon decoupling happens when $\tau(z_{dec}) = 1$

⇒ optical depth is zero today.

$\Rightarrow$ Monopole is related to density Contrast

$$\boxed{\delta_\gamma = 4\Theta_0}$$

$\Rightarrow$ The Scalar part of the velocity is related to the first moment,

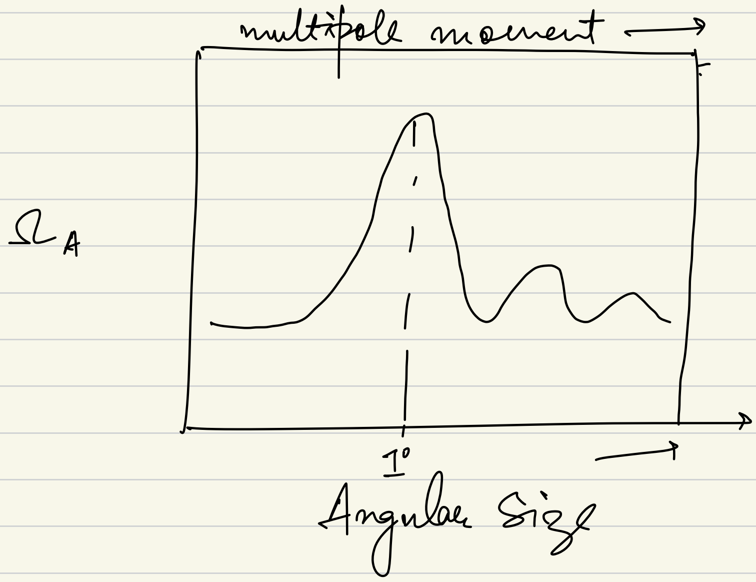
$$\boxed{v_\gamma = -3i\Theta_1}$$

$\Rightarrow$ For dark matter: we assume that dark matter is effectively Collisionless (this is a standard assumption, it is a good approximation eg. for WIMPs

Therefore, Boltzmann eq Reads: $\mathcal{L}[f_{dm}] = 0$

CMB Power Spectrum.

* The entire universe acted as a resonance Cavity.
  ~ with the size of sound Horizon.



* The peaks Correspond to the resonance scale of the Cavity (sound Horizon of the universe).
* About at 1° Correspond to the size of the universe.
* This plot is the fourier transformation of CMB fluctuations.
* Small l (multipole moment) Correspond to large scale and large l Correspond to small scale.

⊛ The first acoustic peak tells us the geometry of the universe. We know the angular size (observation) and we can also calculate the physical size and finally compare both to tell the geometry of the universe. (obviously it's flat).

⊛ By measuring the heights of the peak we know how much Radiation is there in the universe (This is very rough). Ω_R.

Ω_R today is very less because the universe has expanded alot already. $\rho_R \propto a^{-4}$

⊛ $\frac{180}{l}$ = Angular size.

⊛ Peaks at smaller scale are actually earlier oscillations when the sound horizon was smaller.

⊛ In the peaks of the spectrum there was little bit more baryon density therefore in these scales there are little bit more galaxies than other scales.

⊛ more Ω_B means more Silk damping.