Integer Multiplication

Multiplying two numbers:

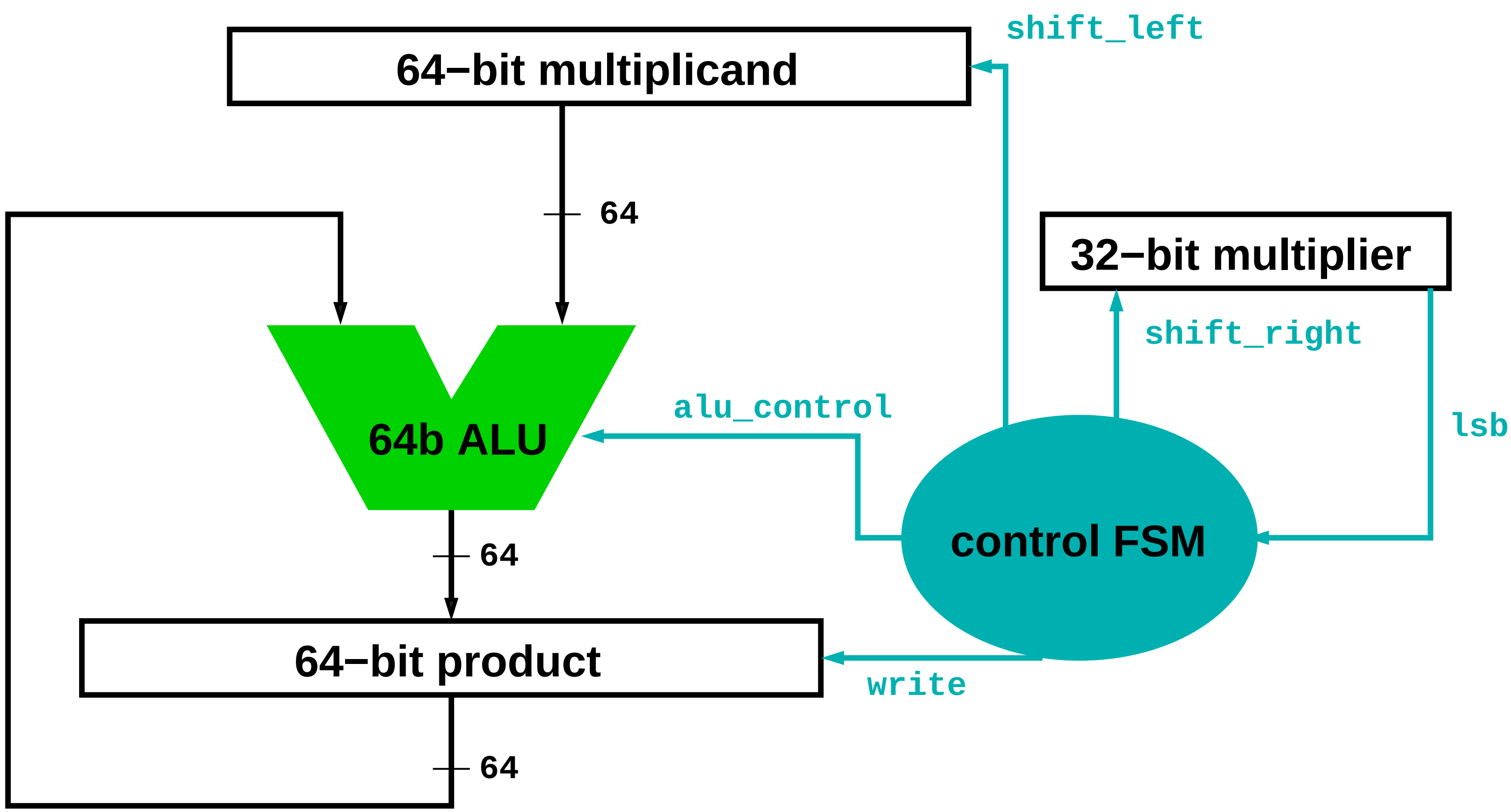
$$
\begin{array}{lrrrrrrrr}
\text{multiplicand} & & & & & 1 & 0 & 1 & 0 \\
\text{multiplier} & & & & \times & 1 & 0 & 0 & 1 \\
\hline
 & & & & & 1 & 0 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 0 & \\
 & & & 0 & 0 & 0 & 0 & & \\
 & + & 1 & 0 & 1 & 0 & & & \\
\hline
 & & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
\hline
\end{array}
$$

m-bits $\times$ n-bits = $(m+n)$-bit result

m-bits: $2^m - 1$ is the largest number

$$\Rightarrow (2^m - 1)(2^n - 1) = 2^{m+n} - 2^m - 2^n + 1$$

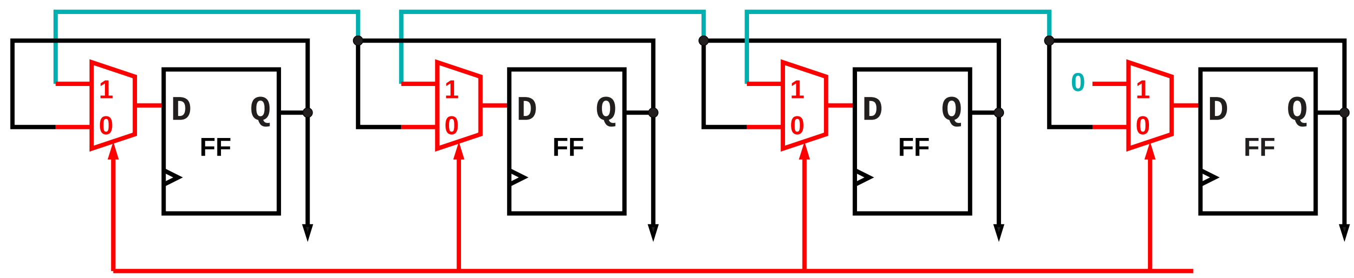
Integer Multiplication: First Try



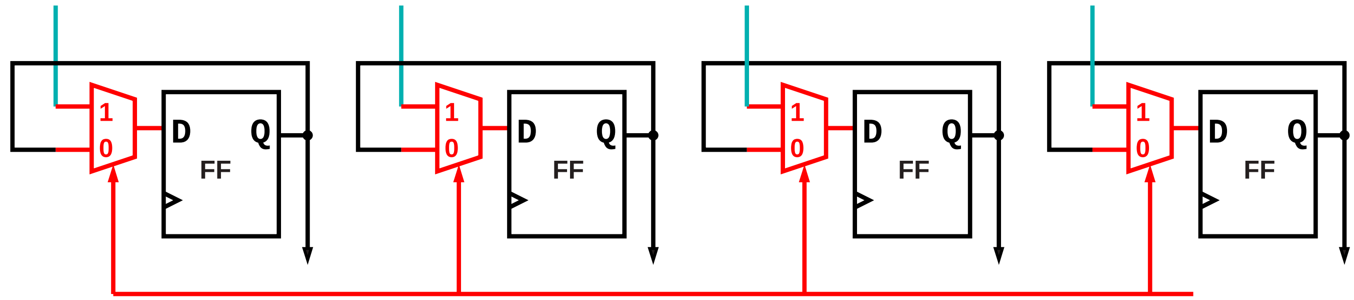
How do we build this?

Registers And Shift Registers

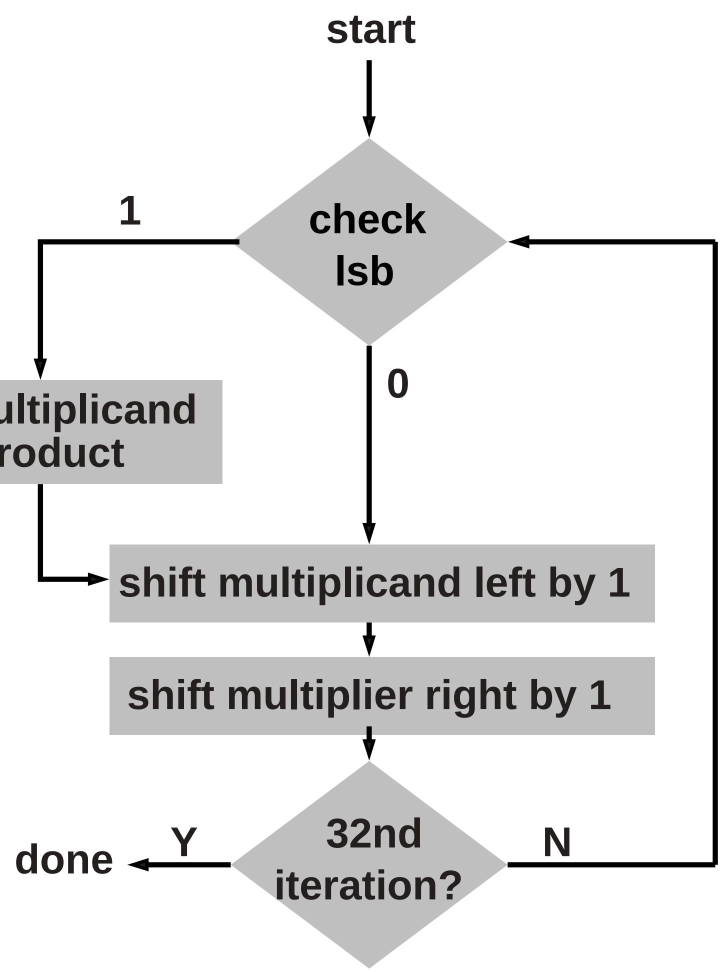
Register with shift left:



Register with write:



Control


start
check lsb
1
0
add multiplicand to product
shift multiplicand left by 1
shift multiplier right by 1
32nd iteration?
Y
N
done

Integer Multiplication

Observations:

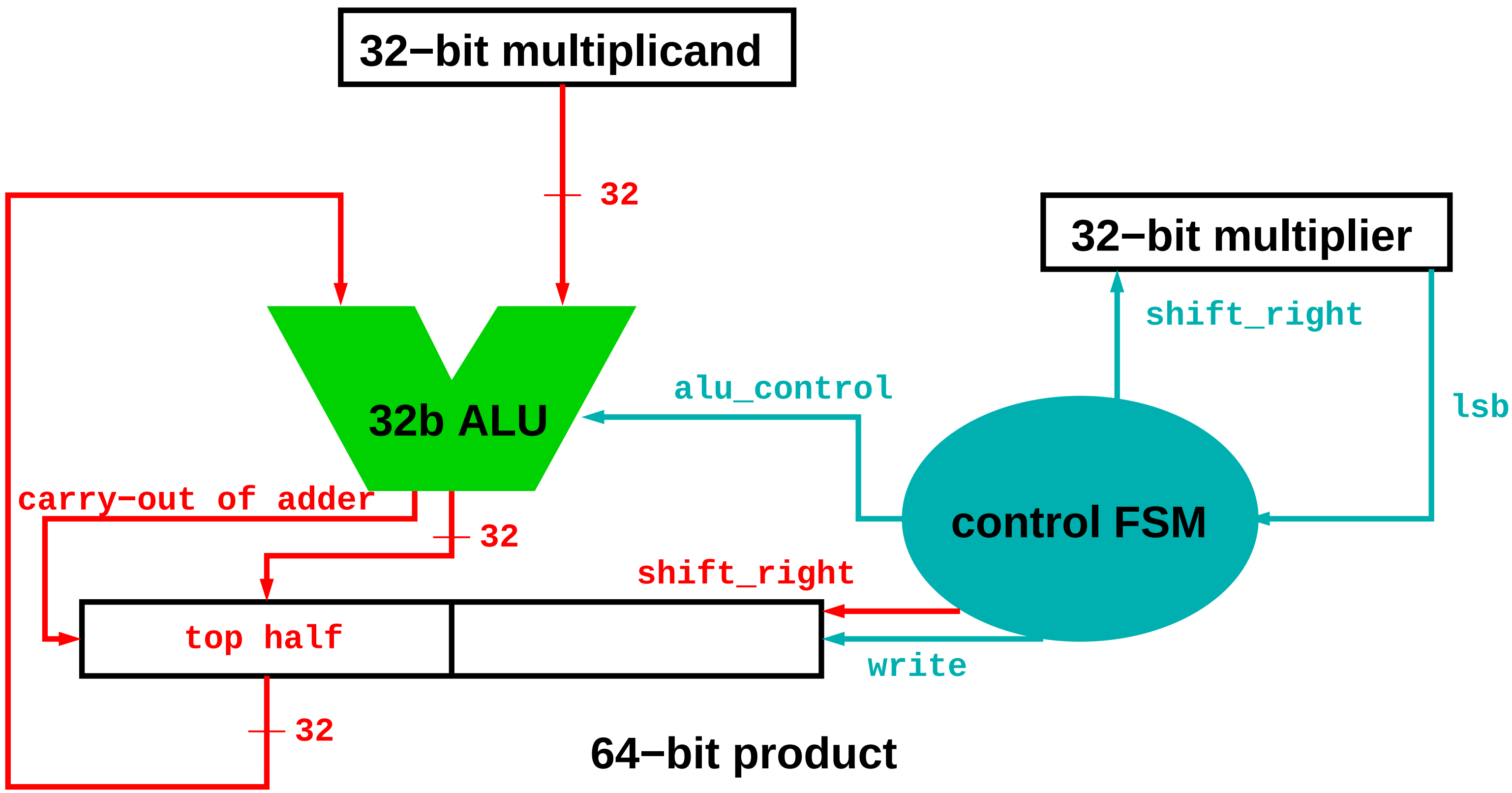
- 32 iterations for multiplication $\Rightarrow$ 32 cycles
- How long does 1 iteration take?
- Suppose 5% of ALU operations are multiply ops, and other ALU operations take 1 cycle.

  $$\Rightarrow CPI_{alu} = 0.05 \times 32 + 0.95 \times 1 = 2.55!$$
- Half of the bits of the multiplicand are zero

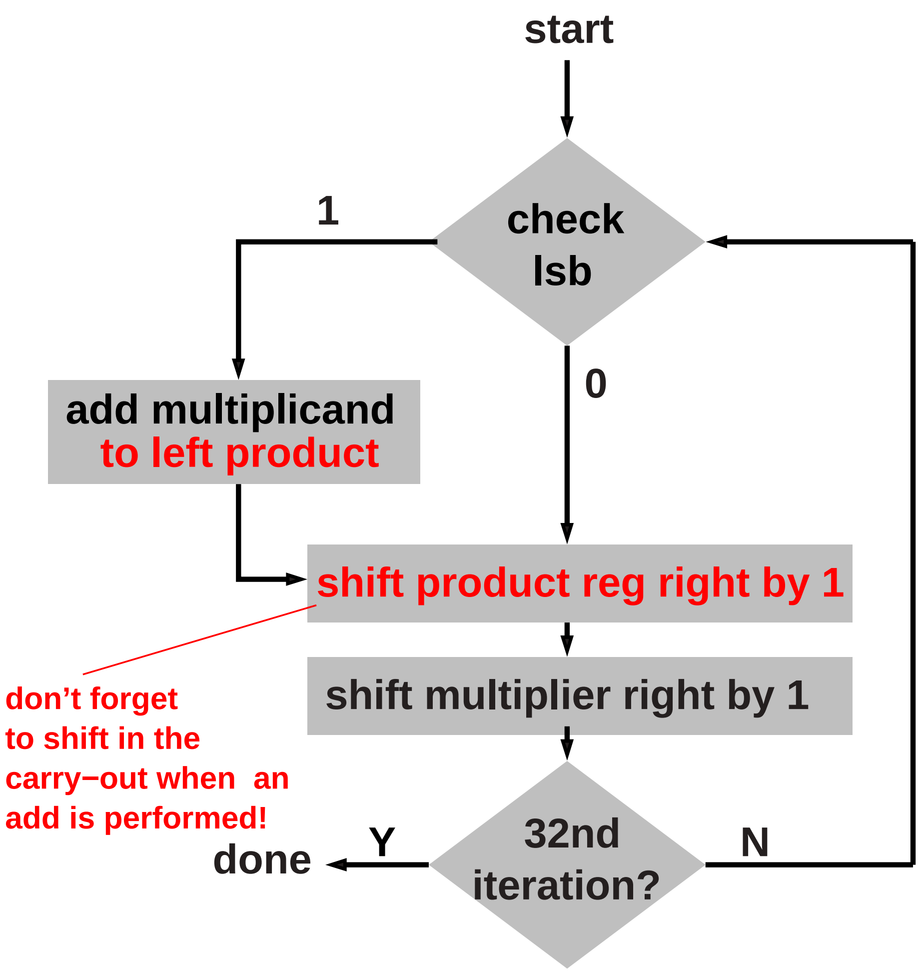
  $\Rightarrow$ 64-bit adder is wasted
- 0's inserted when multiplicand shifted left

  $\Rightarrow$ product LSBs don't change

Using A 32-Bit ALU



New Control

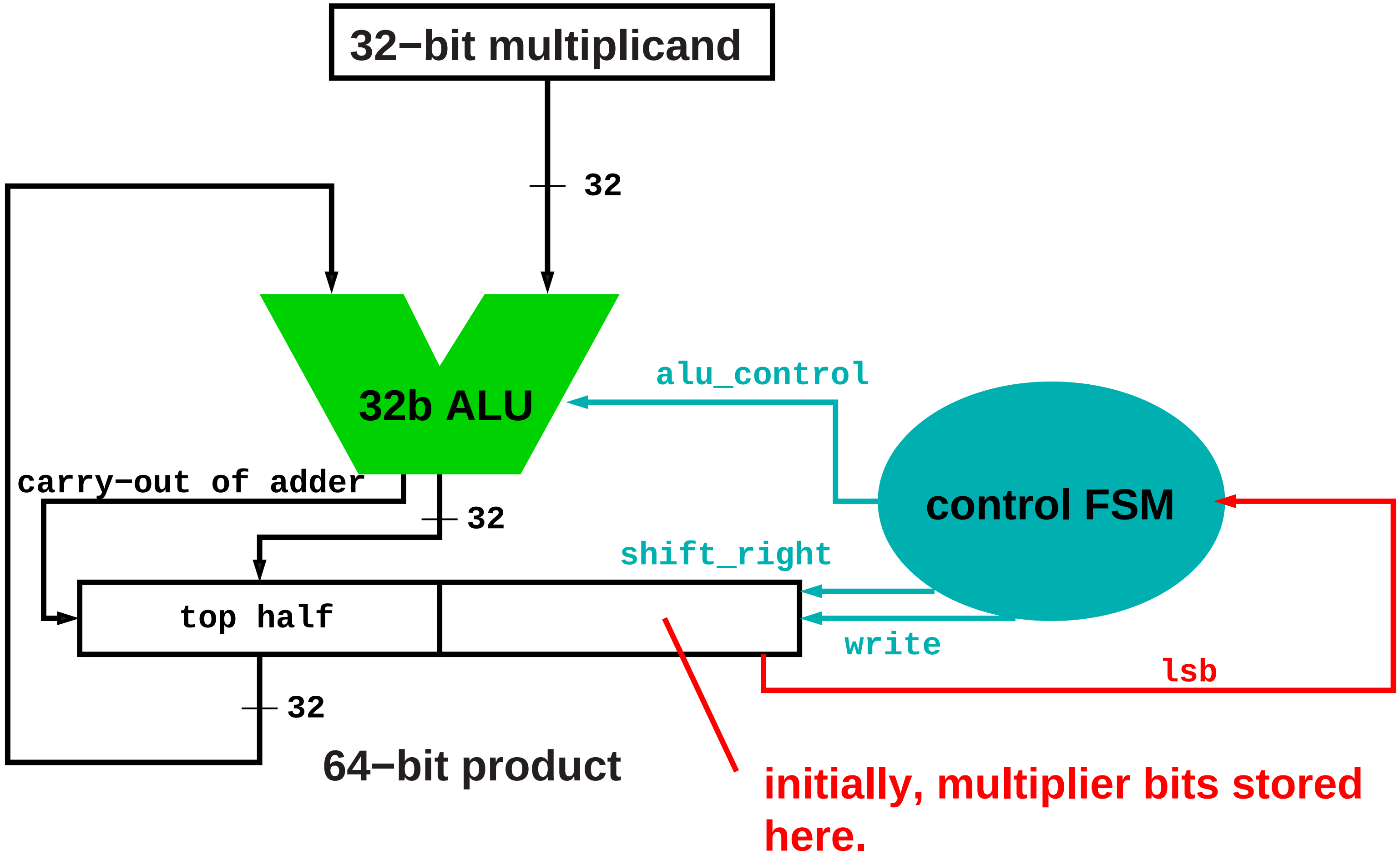


Bottom half of product register is zero initially.

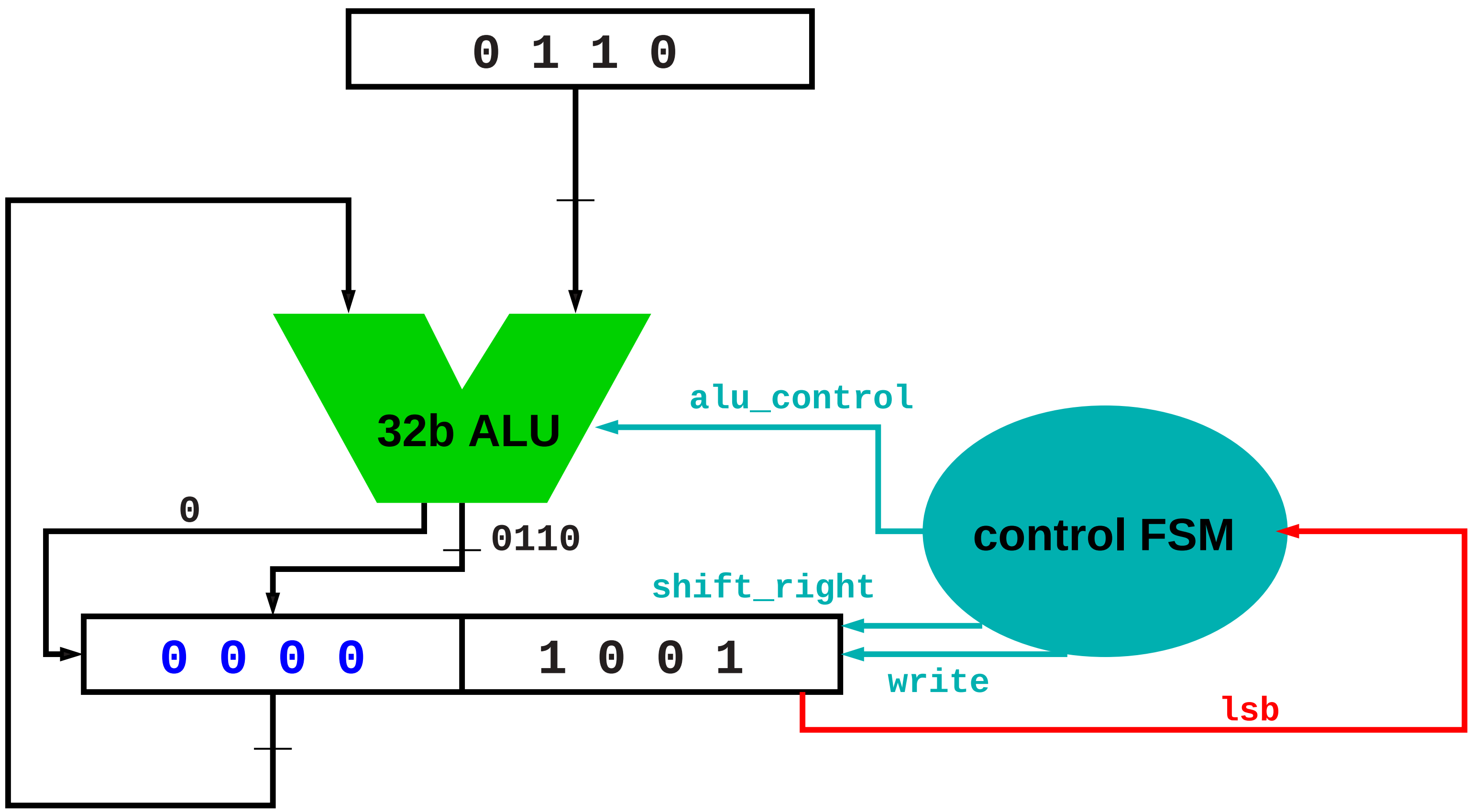
Each iteration:
- **adds 1 product bit**
- **loses one multiplier bit**

Share storage for product register and multiplier!

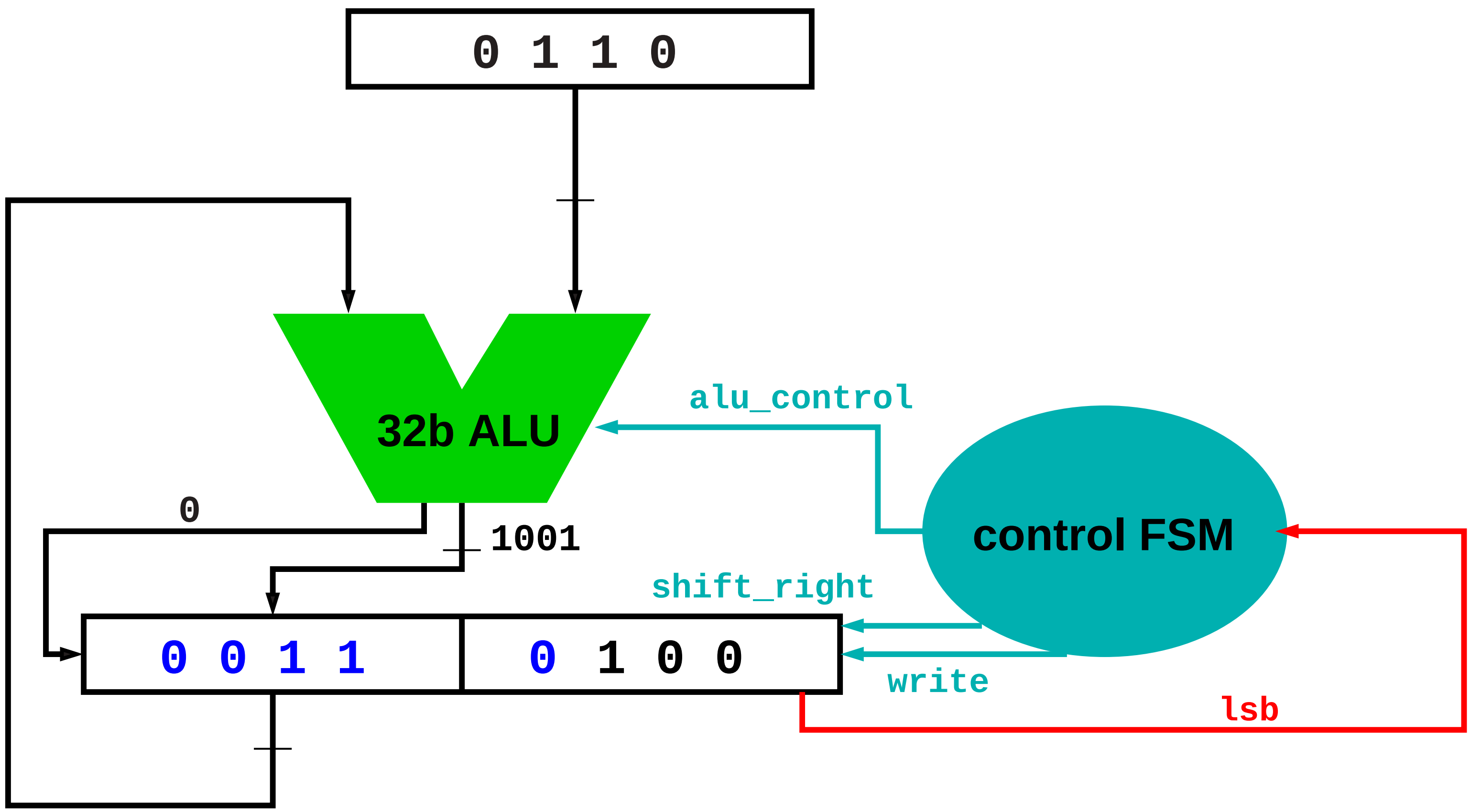
Integer Multiplication Hardware



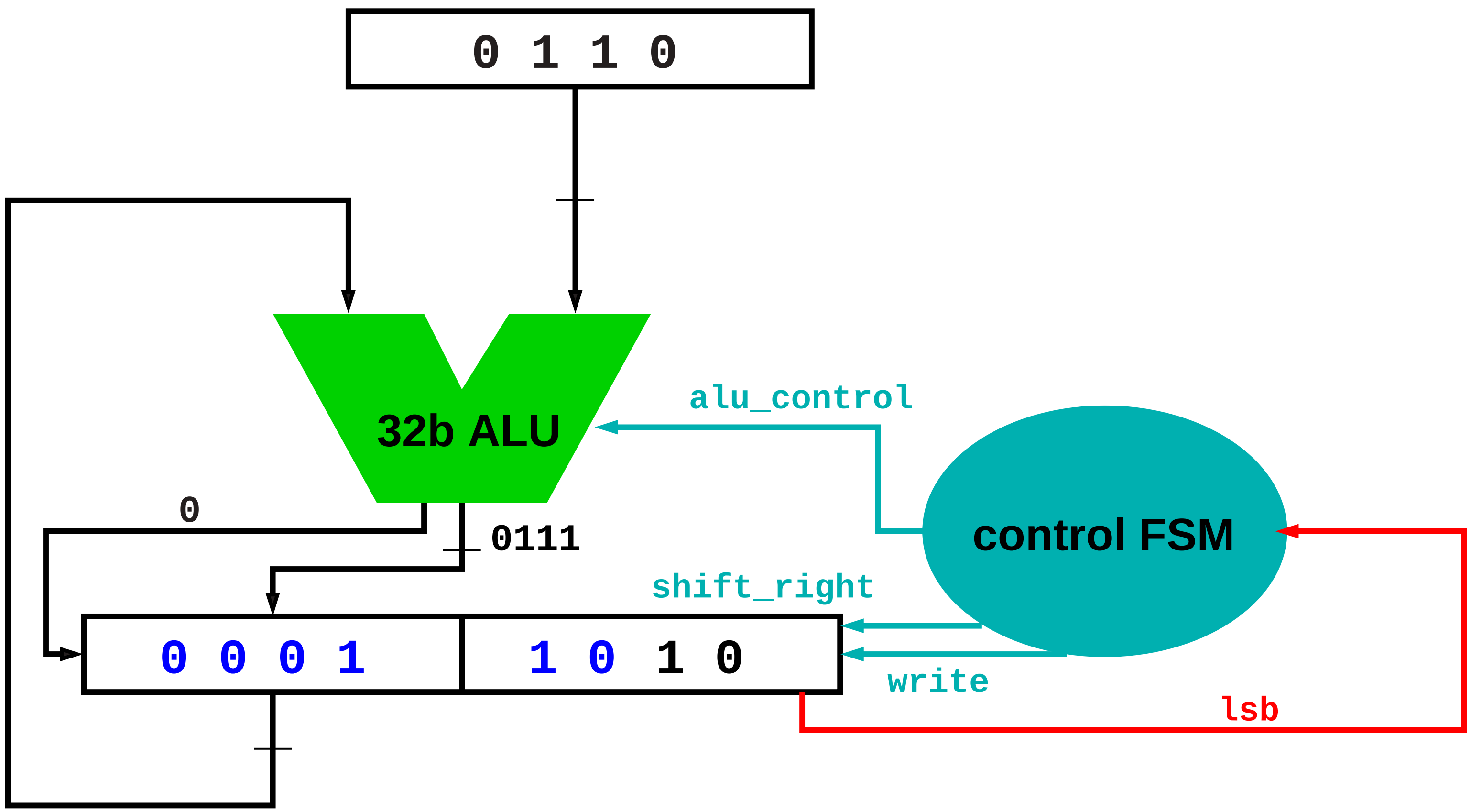
Integer Multiplication Hardware



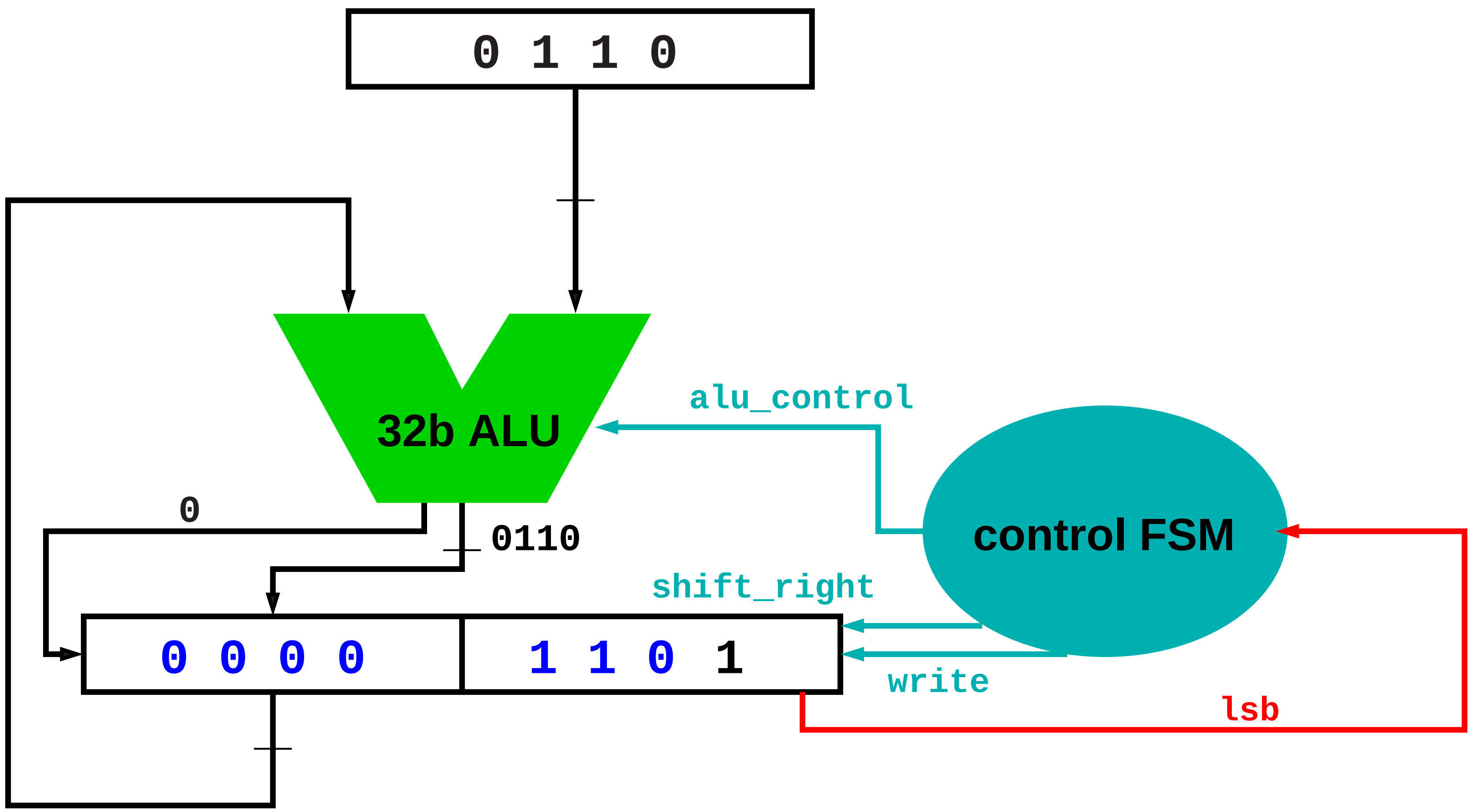
Integer Multiplication Hardware



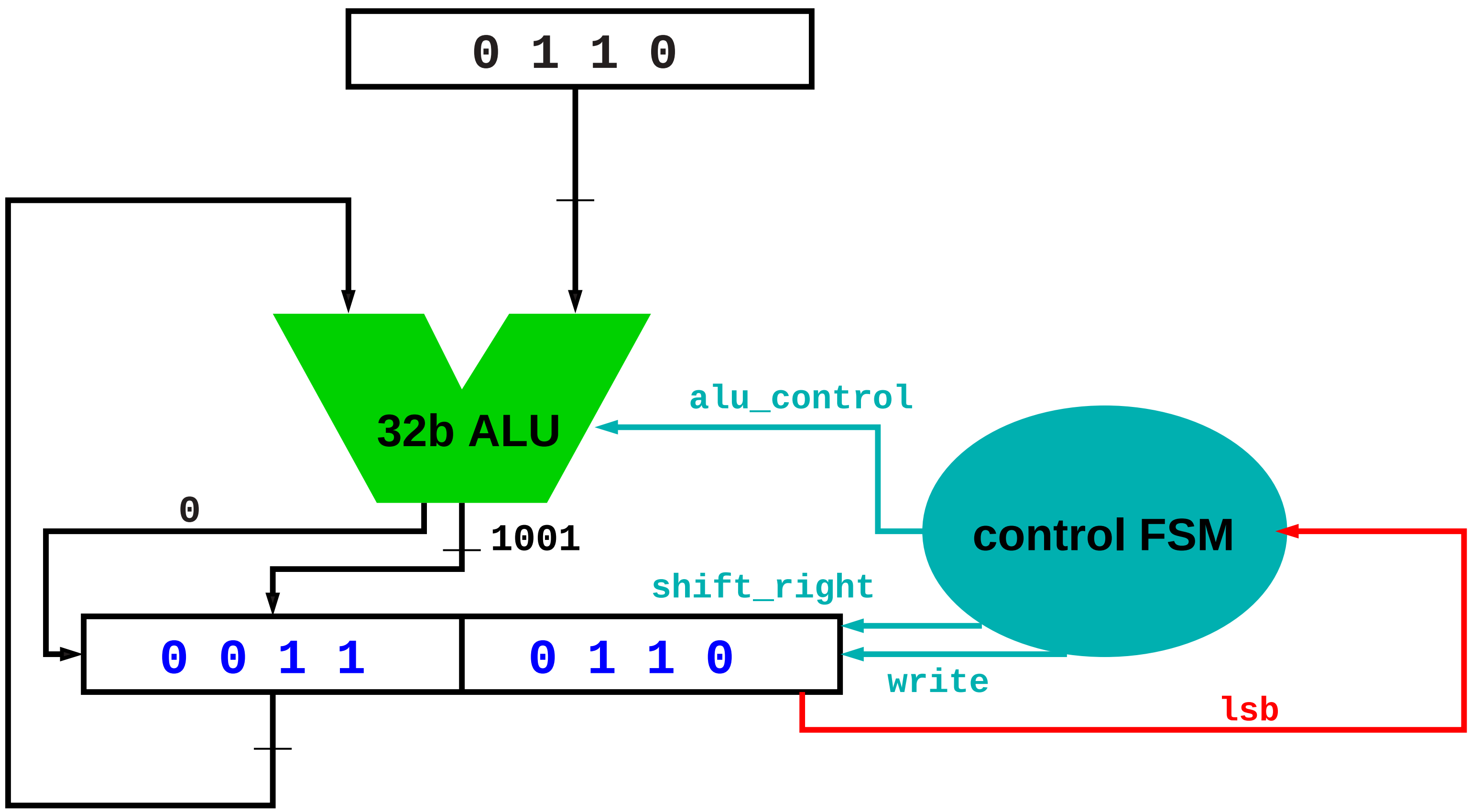
Integer Multiplication Hardware



Integer Multiplication Hardware



Integer Multiplication Hardware



Integer Multiplication

- Each step requires an add and shift
- MIPS: `hi` and `lo` registers correspond to the two parts of the product register
- Hardware implements `multu`
- Signed multiplication:
  - Determine sign of the inputs, make inputs positive
  - Use `multu` hardware, fix up sign
  - Better: Booth's algorithm

Booth Multiplication

Example:

$$
\begin{array}{lrrrrrrr}
\text{multiplicand} & & & & 1 & 0 & 1 & 0 \\
\text{multiplier} & & & \times & 0 & 1 & 1 & 0 \\
\hline
 & & & & 0 & 0 & 0 & 0 \\
 & & & 1 & 0 & 1 & 0 & \\
 & & 1 & 0 & 1 & 0 & & \\
+ & 0 & 0 & 0 & 0 & & & \\
\hline
 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
\hline
\end{array}
$$

Instead we could subtract early and add later...

$$6x = 2x + 4x = -2x + 8x$$

$$11110000 = 10000XXXX - 0001XXXX$$

Booth Multiplication

run of 1s

0 1 1 1 1 0

first "0" after run of 1s add back!

first "1" after 0 subtract!

Current	Right	Explanation
1	0	beginning of run of 1s
0	1	end of run of 1s
1	1	middle of run of 1s
0	0	middle of run of 0s

Originally for speed: shifts faster than adds

CSL

Booth Multiplication

Depending on current and previous bits, do one of the following:

- 00: middle of a run of 0s $\Rightarrow$ no operation
- 01: end of a run of 1s $\Rightarrow$ add multiplicand to left half of product
- 10: start of a run of 1s $\Rightarrow$ subtract multiplicand from left half of product
- 11: middle of a run of 1s $\Rightarrow$ no operation

As before, shift product register right by 1 bit per step.

Integer Division

```
                      0101        quotient
divisor       0010 | 1011         dividend
                 0010
               - 0010
                   ----
                   0011
                   0010
                 - 0010
                   ----
                   0001           remainder
```

Red: steps where subtracting would result in a negative number, i.e. quotient bit is zero.

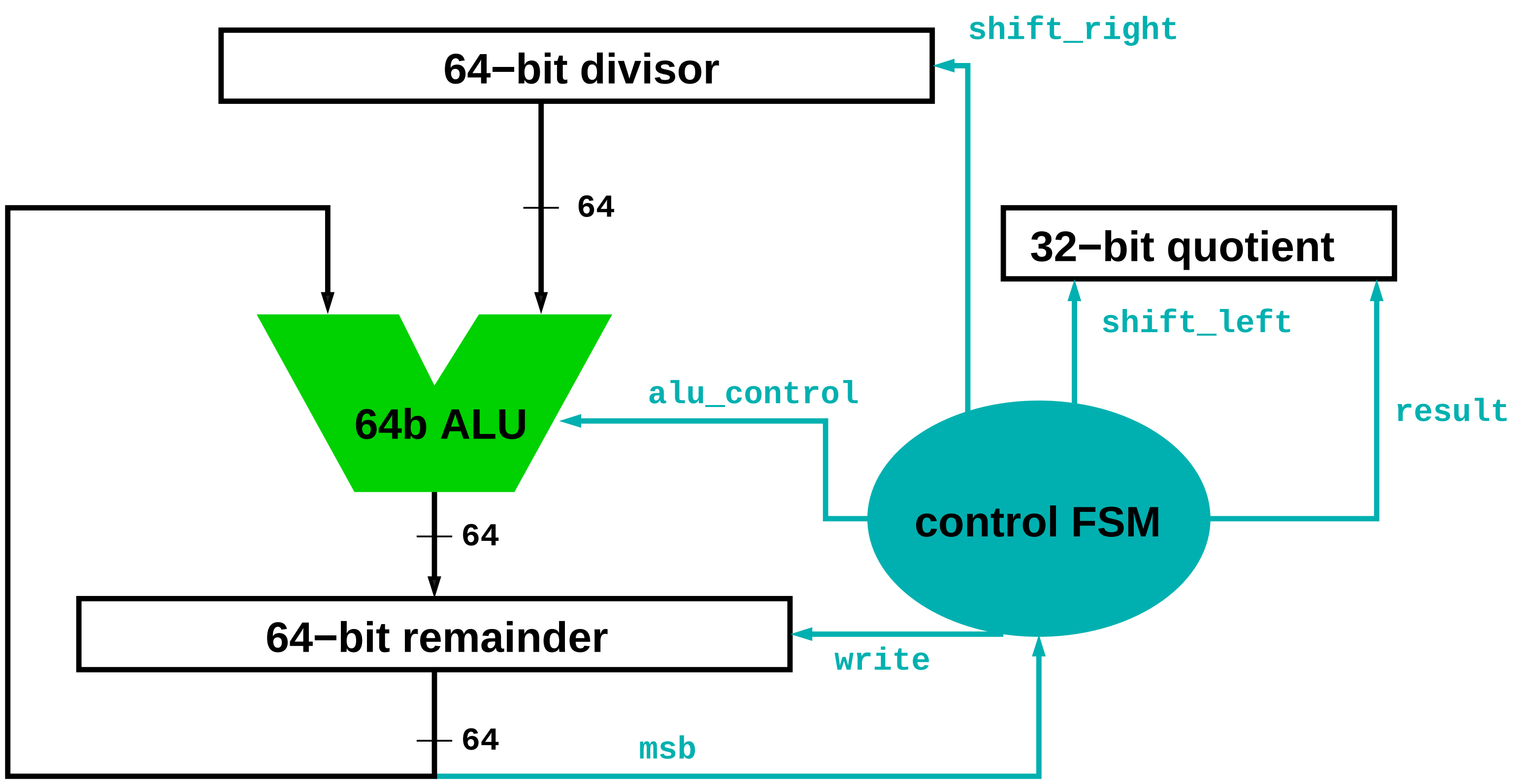
Integer Division

		0101	*quotient*
divisor	0010	1011	*dividend*

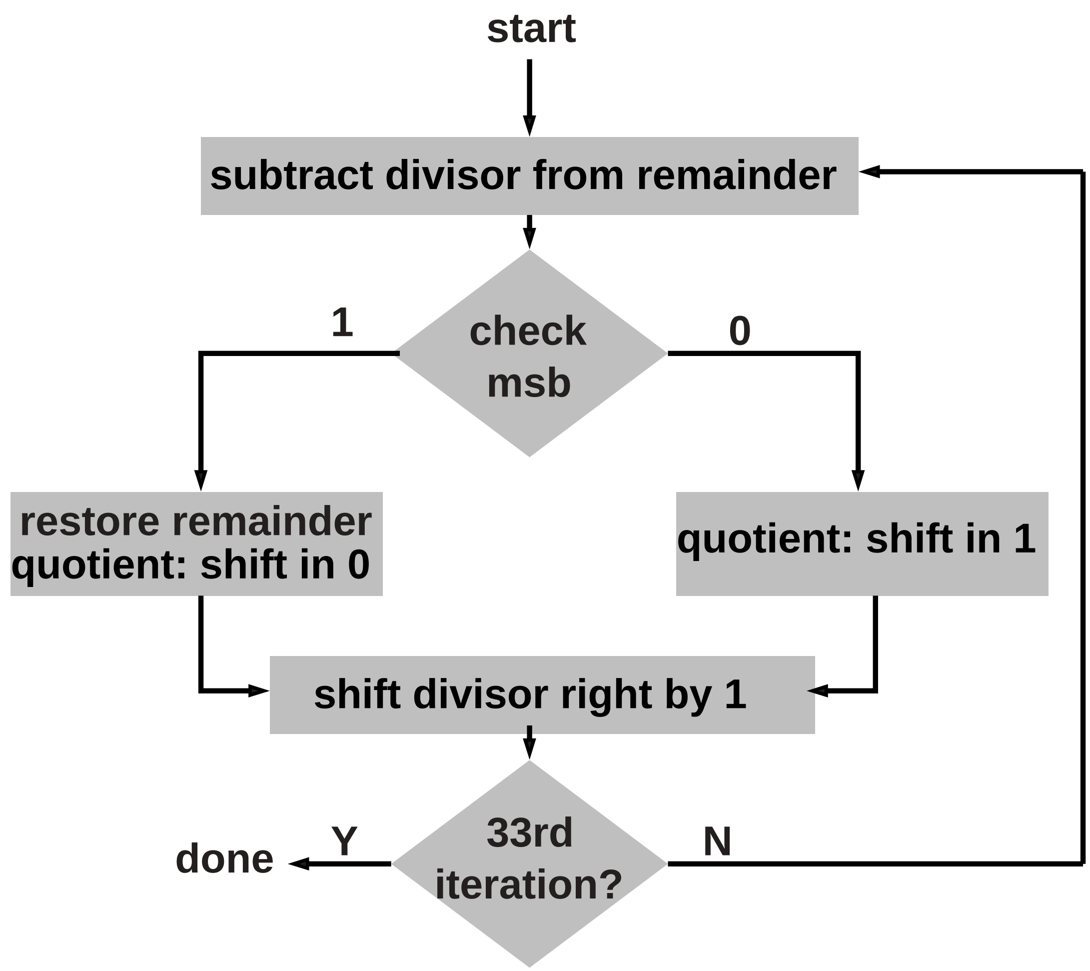
```
  00010000
- 00001000
----------
  00000011
  00000100
- 00000010
----------
  00000001   remainder
```

Pad out the dividend and divisor to 8 bits.

Integer Division



Integer Division



Integer Division

Observations:

- Half the bits in the divisor are zero

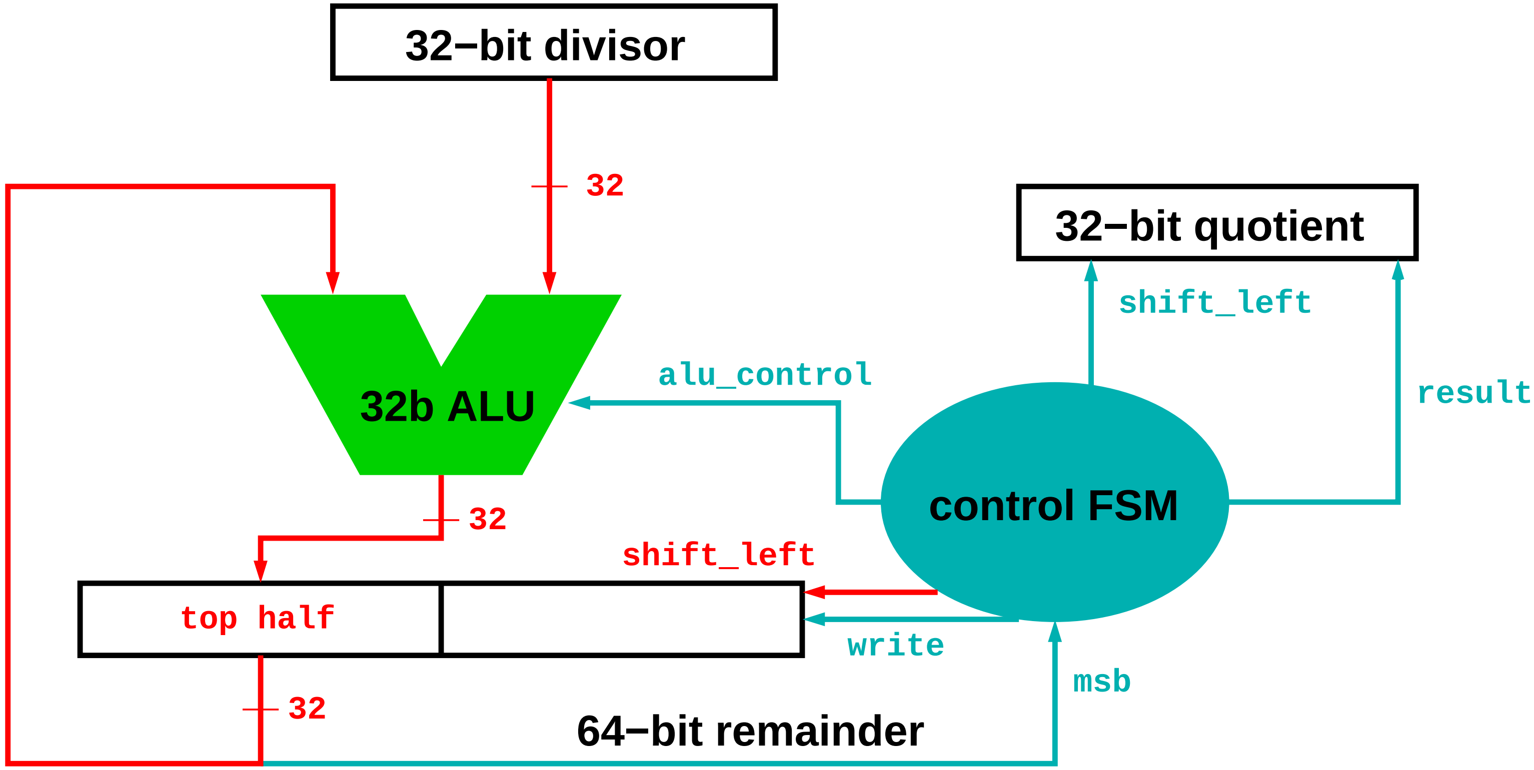
  $\Rightarrow$ 64-bit ALU wasted

- Instead of shifting divisor right, we can shift remainder left
- When does the first iteration shift in a 1 into the quotient?

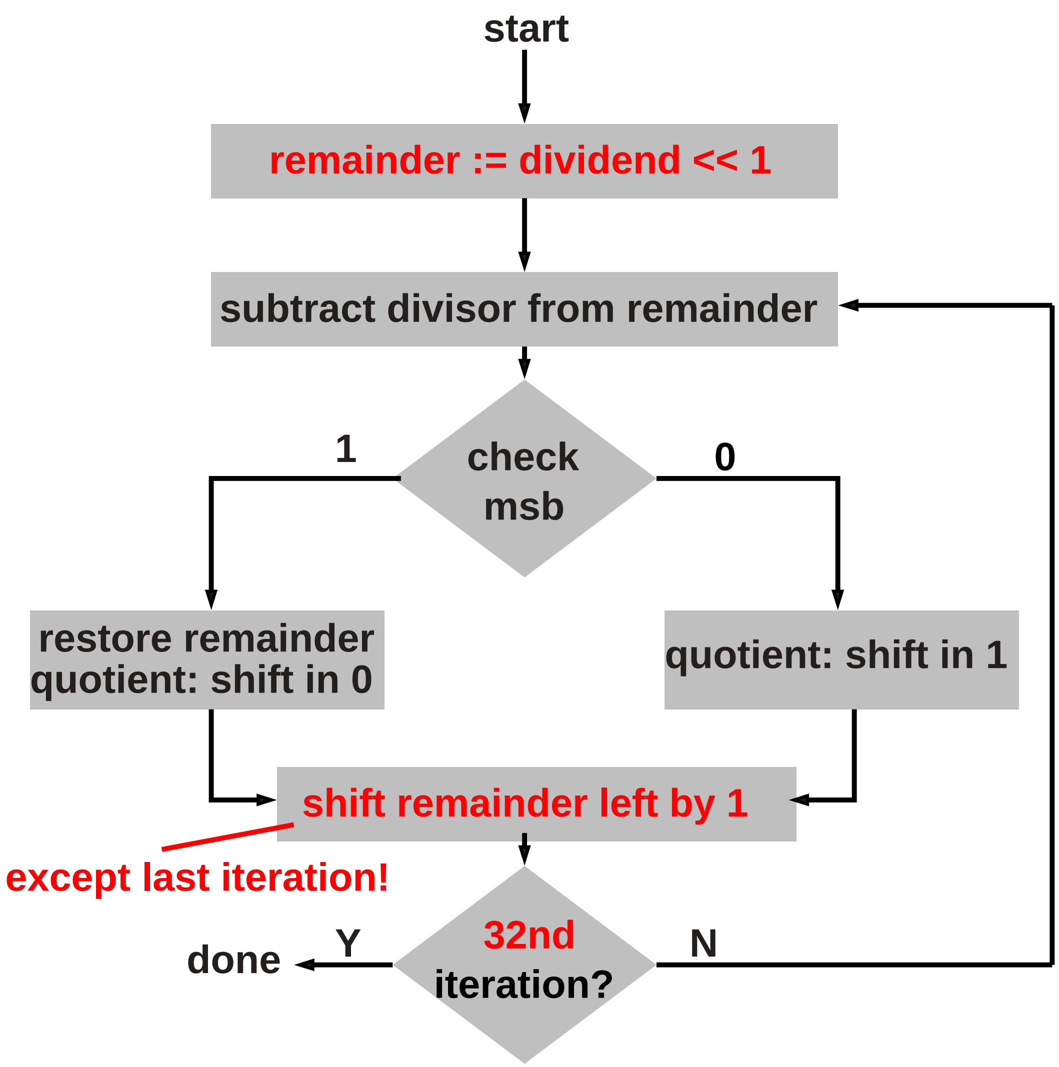
  $\Rightarrow$ save 1 iteration

What is the initial value of the divisor?

Integer Division



New Control

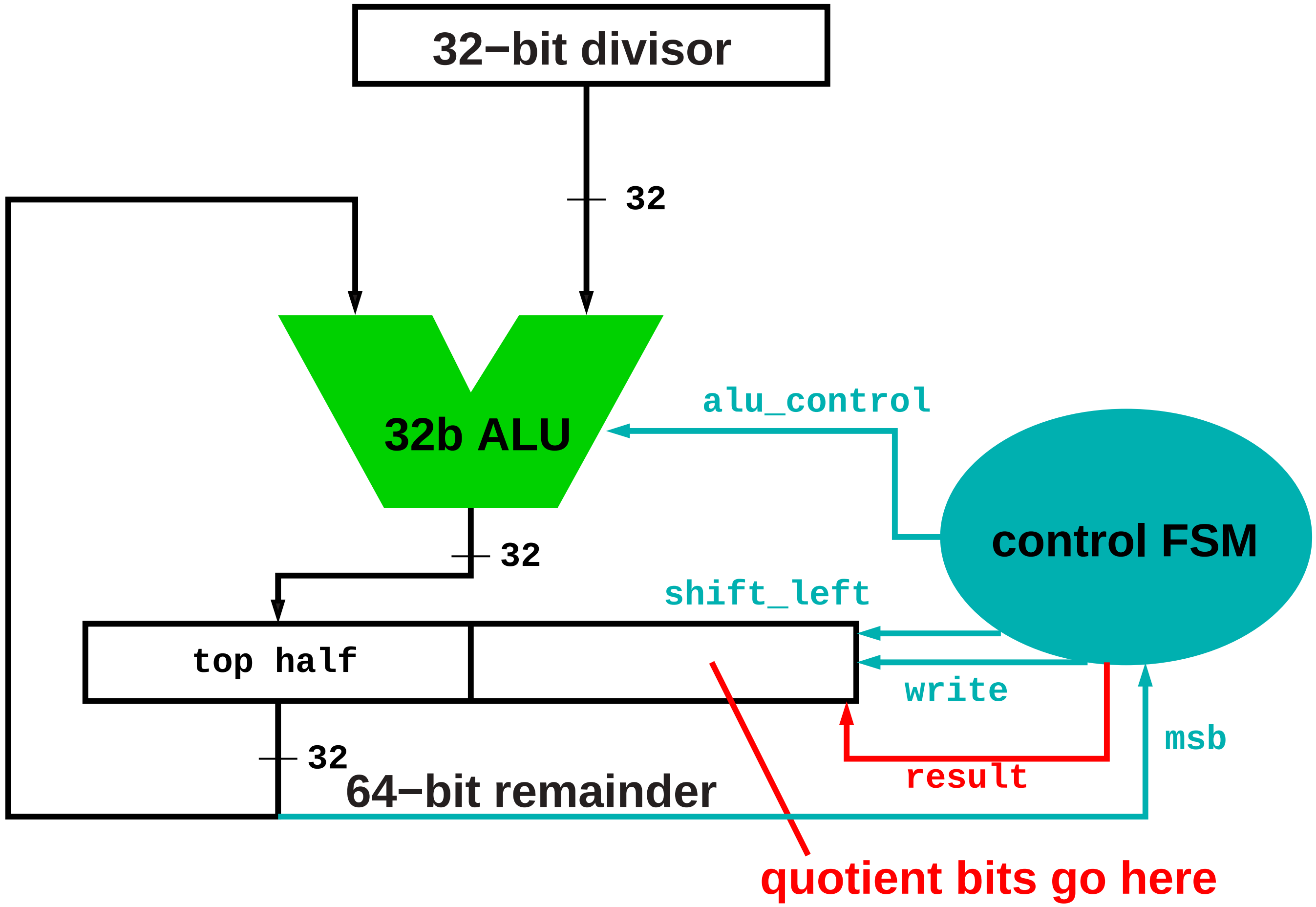


Remainder loses one bit per iteration;

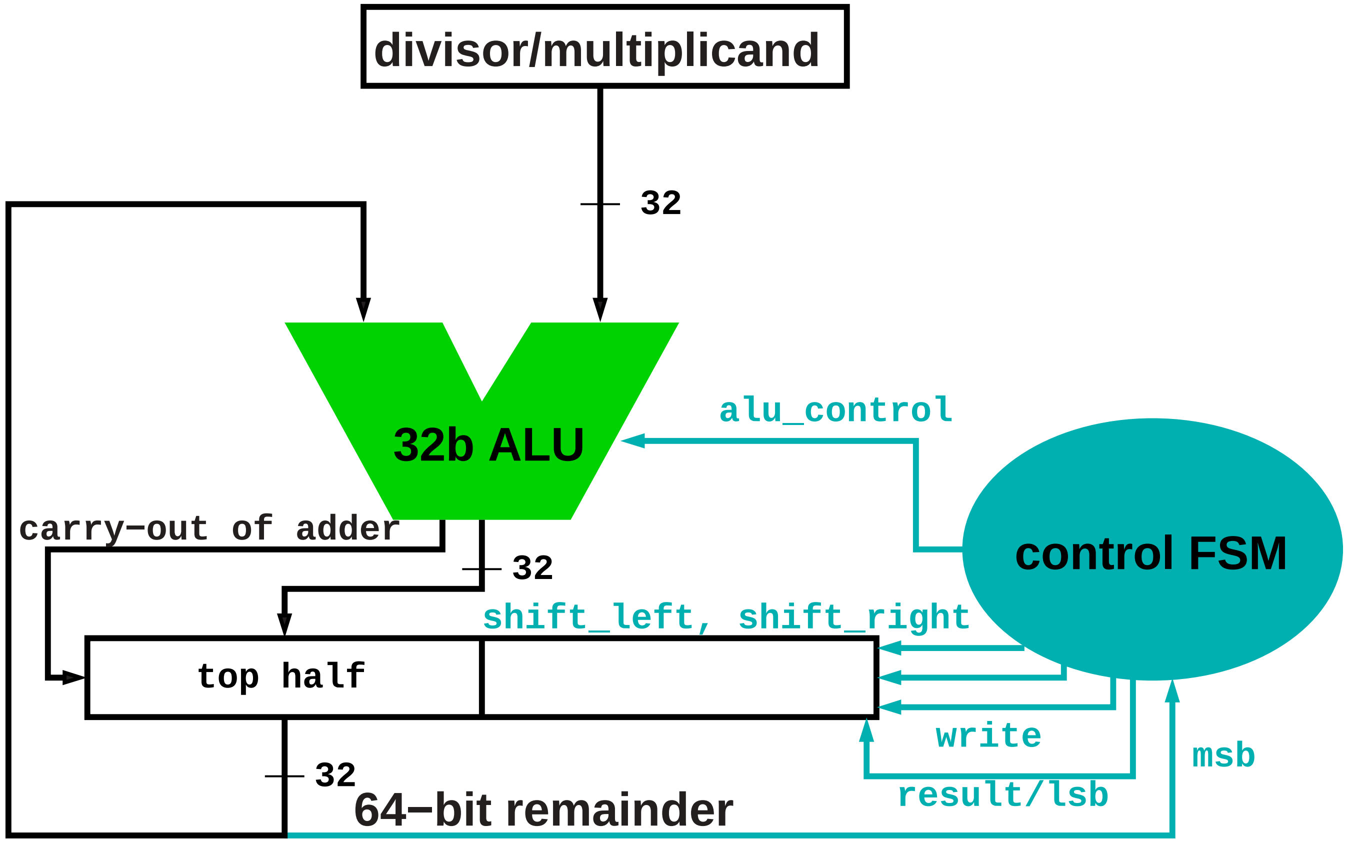
Quotient gains one bit per iteration.

=> share registers!

Final Divider Hardware



Mult/Div



It's the same hardware...

Real Numbers

How do we represent real numbers?

Several issues:

- How many digits can we represent?
- What is the range?
- How accurate are mathematical operations?
- Consistency...

Is $a + b = b + a$?

Is $(a + b) + c = a + (b + c)$?

Is $(a + b) - b = a$?

Fixed Point

Basic idea:

0 1 0 0 1 0 1 0

radix point is here

Choose a *fixed* place in the binary number where the radix point is located.

For the example above, the number is

$$(010.01010)_2 = 2 + 2^{-2} + 2^{-4} = (2.3125)_{10}$$

How would you do mathematical operations?

Floating-Point

Some problematic numbers....

$$6.023 \times 10^{23}$$
$$6.673 \times 10^{-11}$$
$$6.62607 \times 10^{-34}$$

Scientific computations require a number of digits of precision...

But they also need *range*

$\Rightarrow$ permit the radix point to move

$\Rightarrow$ *floating-point numbers*

Floating-Point: Scientific Notation

$$\overbrace{\pm}^{\text{sign}} \underbrace{6.023}_{\text{mantissa}} \times \underbrace{10}_{\text{radix (base)}}{}^{\overbrace{23}^{\text{exponent}}}$$

- Number represented as:
  - mantissa, exponent
- Arithmetic
  - multiplication, division: perform operation on mantissa, add/subtract exponent
  - addition, subtraction: convert operands to have the same exponent value, add/subtract mantissas