3. **(1p)** (a) Qual é a transformada de Laplace R(s) da função de rampa padrão r(t) = t? Observe que para t < 0, todas as funções são zero. (b) A derivada de r(t) é o passo unitário H(t). Assim, a transformada de Laplace de H(t) pode ser obtida multiplicando R(s) por s. Qual é o resultado? (A função de passo unitário é H(t) em homenagem a Oliver Heaviside.)

A) $R(s) = \mathcal{L}\{r(t)=t\} = \lim_{\alpha\to\infty} \int_0^{\alpha} t.e^{-st}\,dt \rightarrow$ $u=t \mid dv = e^{-st}dt$; $du = dt \mid v = -\frac{e^{-st}}{s}$

$$R(s) = \lim_{\alpha\to\infty}\left[-\frac{t.e^{-st}}{s} + \int \frac{e^{-st}}{s}dt\right] \rightarrow \lim_{\alpha\to\infty}\left[-\frac{t.e^{-st}}{s} + \left[\frac{1}{s}\cdot\left(-\frac{e^{-st}}{s}\right)\right]\right]_0^{\alpha}$$

$$\lim_{\alpha\to\infty}\left[-\frac{\alpha.e^{-s\alpha}}{s} - \frac{e^{-s\alpha}}{s^2} - \left(\cancel{\frac{0.e^{0}}{s}}^{\,0} - \frac{e^0}{s^2}\right)\right] \rightarrow \lim_{\alpha\to\infty}\left[-\cancel{\frac{\alpha.e^{-s\alpha}}{s}}^{\,0} - \cancel{\frac{e^{-s\alpha}}{s^2}}^{\,0} + \frac{1}{s^2}\right] \rightarrow \boxed{R(s) = 1/s^2}$$

B) $\frac{\partial H}{\partial s} = -\int_0^{\infty} r(t).t.e^{-st}dt \;\therefore\; \frac{\partial H}{\partial s} = s \rightarrow \mathcal{L}\{H(t)\} = R(s).s \rightarrow \frac{1}{s^{\cancel{2}}}\cdot\cancel{s} \rightarrow \boxed{\mathcal{L}\{H(s)\} = 1/s}$ p/ s>0 !

4. **(2p)** Resolva esses problemas de valor inicial pela transformação de Laplace:
(a) y′ + y = exp(iwt), y(0) = 8; (b) y′′ - y = exp(t), y(0) = 0, y′(0) = 0;
(c) y′′ + y = 6t, y(0) = 0, y′(0) = 0; (d) y′ - iwy = δ(t), y(0)=0.

OBS $\mathcal{L}\{F'(t)\} = s.\mathcal{L}\{F(t)\} - F(0)$

A) $y' + y = \exp(iwt) \;\therefore\; y(0) = 8 \rightarrow \mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{\exp(iwt)\} \rightarrow s.\mathcal{L}\{y\} - \underbrace{y_0}_{8} + \mathcal{L}\{y\} = \underbrace{\mathcal{L}\{\exp(iwt)\}}_{\frac{1}{s-iw}}$

$$s.\mathcal{L}\{y\} - 8 + \mathcal{L}\{y\} = \frac{1}{s-iw} \rightarrow \mathcal{L}\{y\}.(s+1) = \frac{1}{s-iw} + 8$$

$$\mathcal{L}\{y\} = \frac{1}{(s-iw).(s+1)} + \frac{8}{(s+1)} \rightarrow \mathcal{L}\{y\} = \frac{1}{1+iw}\cdot\left(\underbrace{\frac{1}{s-iw}}_{\exp(iwt)} - \underbrace{\frac{1}{s+1}}_{\exp(-t)}\right) + \underbrace{\frac{8}{(s+1)}}_{8.\exp(-t)}$$

OBS
$$\frac{1}{(s-iw)(s+1)} = \frac{A}{(s+1)} + \frac{B}{(s-iw)}$$
$$\rightarrow \frac{As - Aiw + Bs + B}{(s-iw)(s+1)} \quad \begin{cases} s(A+B) = 0 \\ B - Aiw = 1 \end{cases}$$
$B = -A \;\therefore\; -A(1+iw) = 1$

$-A = \frac{1}{(1+iw)} \;\therefore\; B = \frac{1}{(1+iw)}$

$\mathcal{L}^{-1}\{y\}$

$$\boxed{y(t) = \frac{1}{1+iw}\cdot\left[\exp(iwt) - \exp(-t)\right] + 8.\exp(-t)}$$

B) $y'' - y = \exp(t) \;\therefore\; y(0) = 0 \;\therefore\; y'(0) = 0 \rightarrow s^2.\mathcal{L}\{y\} - s.\cancel{y_0}^{\,0} - \cancel{y'_0}^{\,0} \;\therefore\; \mathcal{L}\{y''\} - \mathcal{L}\{y\} = \mathcal{L}\{\exp(t)\}$

$$s^2.\mathcal{L}\{y\} - \mathcal{L}\{y\} = \exp(t) \rightarrow \mathcal{L}\{y\}.(s^2-1) = \frac{1}{s-1} \rightarrow \mathcal{L}\{y\} = \frac{1}{(s-1).(s^2-1)} \rightarrow \mathcal{L}\{y\} = \frac{1}{(s-1).(s+1).(s-1)}$$

$$\mathcal{L}\{y\} = \frac{1}{4.(s+1)} - \frac{1}{4.(s-1)} + \frac{1}{2.(s-1)^2}$$

$\mathcal{L}^{-1}\{y\}$

$$\boxed{y(t) = \frac{\exp(-t)}{4} - \frac{\exp(t)}{4} + \frac{t.\exp(t)}{2}}$$

C) $y'' + y = 6t \;\therefore\; y(0) = 0 \;\therefore\; y'(0) = 0 \rightarrow \mathcal{L}\{y''\} + \mathcal{L}\{y\} = 6t \rightarrow s^2.\mathcal{L}\{y\} + \mathcal{L}\{y\} = \mathcal{L}\{6t\} \rightarrow \mathcal{L}\{y\}.(s^2+1) = \frac{6}{s^2}$

$\mathcal{L}\{y\} = \frac{1}{s^2+1}\cdot\frac{6}{s^2} \rightarrow$ Aplicando $\mathcal{L}^{-1}\{y\} \rightarrow \boxed{y(t) = 6t - 6.\text{sen}(t)}$

D) $y' - iw.y = \delta(t) \;\therefore\; y(0) = 0 \rightarrow \mathcal{L}\{y'\} - \mathcal{L}\{iw.y\} = \delta(t) \rightarrow s.\mathcal{L}\{y\} - iw.\mathcal{L}\{y\} = \mathcal{L}\{\delta(t)\}$

$\mathcal{L}\{y\}.(s-iw) = \mathcal{L}\{\delta(t)\} \rightarrow \mathcal{L}\{y\} = \frac{\cancel{\mathcal{L}\{\delta(t)\}}^{\,1}}{s-iw} \rightarrow$ Aplicando $\mathcal{L}^{-1}\{y\} \rightarrow \boxed{y(t) = \exp(iwt)}$

5. **(2p)** Transforme a equação variável de tempo de Bessel **ty′′+ y′ + ty = 0** usando $\mathscr{L}$[ty] = -dY/ds para encontrar uma equação de primeira ordem para Y. Separando variáveis ou substituindo **Y(s) = C/√1+s²** , encontre a transformação de Laplace da função Bessel **y=J₀**.

$t.y'' + y' + ty = 0 \;\therefore\; \mathcal{L}\{ty\} = -\frac{dy}{ds} \;\therefore\; Y(s) = \frac{C}{\sqrt{1+s^2}}$

$\hookrightarrow \mathcal{L}\{t.y''\} + \mathcal{L}\{y'\} + \mathcal{L}\{t.y\} = \mathcal{L}\{0\} \rightarrow \frac{-d}{ds}(s^2.\mathcal{L}\{y\} - s.y_0 - y'_0) + s.\mathcal{L}\{y\} - y_0 - \frac{d\mathcal{L}\{y\}}{ds} = 0$

$\rightarrow -2s.\mathcal{L}\{y\} - s^2.\frac{d\mathcal{L}\{y\}}{ds} + \cancel{y_0} + s.\mathcal{L}\{y\} - \cancel{y_0} - \frac{d\mathcal{L}\{y\}}{ds} = 0 \rightarrow -s.\mathcal{L}\{y\} - \frac{d\mathcal{L}\{y\}}{ds}.(s^2+1) = 0$

$\rightarrow \mathcal{L}\{y\} = -\frac{d\mathcal{L}\{y\}}{ds}\cdot\frac{s^2+1}{s}$ ... Logo: $y(s) = -\frac{dy(s)}{ds}\cdot\frac{s^2+1}{s} \;\therefore\; -\int\frac{s}{s^2+1}ds = \int\frac{1}{y(s)}dy(s)$ $\rightarrow \ln(y(s))$

$\hookrightarrow u = s^2+1$, $\frac{du}{2} = s\,ds$ $\Big\}$ $\int\frac{1}{2u}du \rightarrow \frac{1}{2}.\ln(u) \rightarrow \frac{1}{2}.\ln(s^2+1) + K$

$\rightarrow -\frac{1}{2}.\ln(s^2+1) + K = \ln(y(s)) \rightarrow \exp(-\frac{1}{2}.\ln(s^2+1) + K) = y(s)$

$\rightarrow y(s) = \overbrace{\exp(K)}^{C}.\exp(-\frac{1}{2}.\ln(s^2+1)) \rightarrow y(s) = C.\cancel{\exp}(\cancel{\ln}(s^2+1)^{-\frac{1}{2}}) \rightarrow y(s) = C.(s^2+1)^{-\frac{1}{2}} \rightarrow \boxed{y(s) = \frac{C}{\sqrt{1+s^2}}}$ OK!