Profile Analysis

Another useful application of Test Problem 6 is the repeated measurements problem applied to two independent groups. This problem arises in practice when we observe repeated measurements of characteristics (or measures of the same type under different experimental conditions) on the different groups which have to be compared. It is important that the p measures (the "profile") are comparable and in particular are reported in the same units. For instance, they may be measures of blood pressure at p different points in time, one group being the control group and the other the group receiving a new treatment. The observations may be the scores obtained from p different tests of two different experimental groups. One is then interested in comparing the profiles of each group: the profile being just the vectors of the means of the p responses (the comparison may be visualized in a two dimensional graph using the parallel coordinate plot introduced in Section 1.7).

We are thus in the same statistical situation as for the comparison of two means:

$$X_{i1} \sim N_p(\mu_1, \Sigma) \quad i = 1, \ldots, n_1$$
$$X_{i2} \sim N_p(\mu_2, \Sigma) \quad i = 1, \ldots, n_2$$

where all variables are independent. Suppose the two population profiles look like Figure 7.1.

The following questions are of interest:

1. Are the profiles similar in the sense of being parallel (which means no interaction between the treatments and the groups)?
2. If the profiles are parallel, are they at the same level?
3. If the profiles are parallel, is there any treatment effect, i.e., are the profiles horizontal?

The above questions are easily translated into linear constraints on the means and a test statistic can be obtained accordingly.

Parallel Profiles

Let $\mathcal{C}$ be a $(p-1) \times p$ matrix defined as $\mathcal{C} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix}$.

The hypothesis to be tested is

$$H_0^{(1)} : \mathcal{C}(\mu_1 - \mu_2) = 0.$$

From (7.11), (7.12) and Corollary 5.4 we know that under H_0:

$$\frac{n_1 n_2}{(n_1+n_2)^2}(n_1+n_2-2)\left\{\mathcal{C}(\bar{x}_1 - \bar{x}_2)\right\}^\top (\mathcal{C}\mathcal{S}\mathcal{C}^\top)^{-1}\mathcal{C}(\bar{x}_1 - \bar{x}_2) \sim T^2(p-1, n_1+n_2-2) \quad (7.21)$$

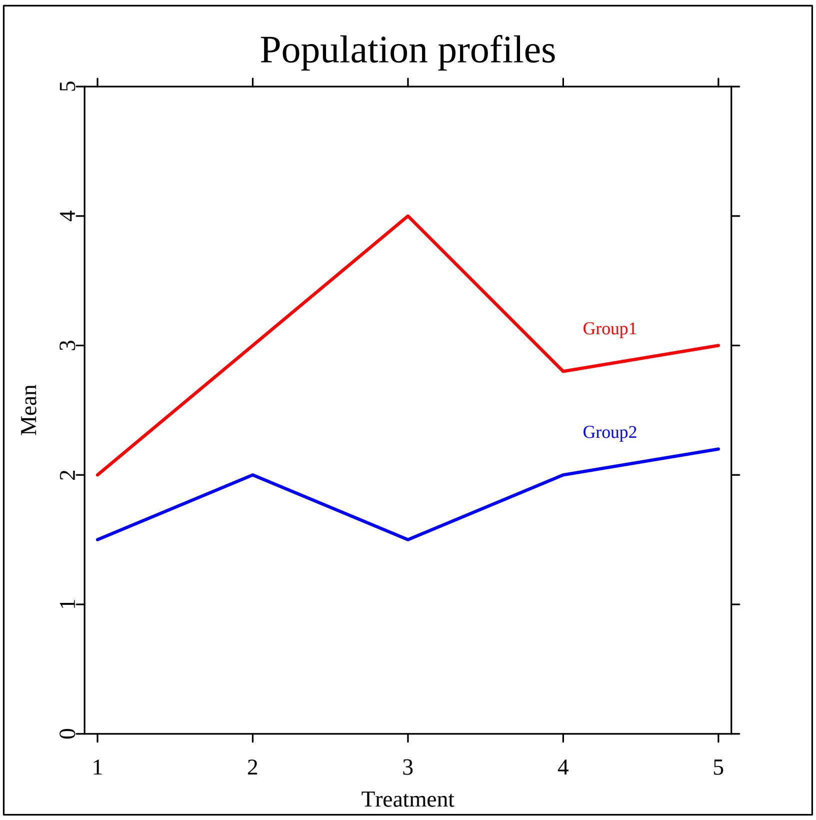

Figure 7.1. Example of population profiles MVAprofil.xpl

where $\mathcal{S}$ is the pooled covariance matrix. The hypothesis is rejected if

$$\frac{n_1 n_2 (n_1 + n_1 - p)}{(n_1 + n_2)^2 (p-1)} (\mathcal{C}\bar{x})^\top \left(\mathcal{C}\mathcal{S}\mathcal{C}^\top\right)^{-1} \mathcal{C}\bar{x} > F_{1-\alpha;p-1,n_1+n_2-p}.$$

Equality of Two Levels

The question of equality of the two levels is meaningful only if the two profiles are parallel. In the case of interactions (rejection of $H_0^{(1)}$), the two populations react differently to the treatments and the question of the level has no meaning.
The equality of the two levels can be formalized as

$$H_0^{(2)} : 1_p^\top (\mu_1 - \mu_2) = 0$$

since

$$1_p^\top (\bar{x}_1 - \bar{x}_2) \sim N_1 \left(1_p^\top (\mu_1 - \mu_2), \frac{n_1 + n_2}{n_1 n_2} 1_p^\top \Sigma 1_p \right)$$

and

$$(n_1+n_2)1_p^\top \mathcal{S} 1_p \sim W_1(1_p^\top \Sigma 1_p, n_1+n_2-2).$$

Using Corollary 5.4 we have that:

$$\begin{aligned} \frac{n_1 n_2}{(n_1+n_2)^2}(n_1+n_2-2)\frac{\left\{1_p^\top(\bar{x}_1-\bar{x}_2)\right\}^2}{1_p^\top \mathcal{S} 1_p} &\sim T^2(1, n_1+n_2-2) \\ &= F_{1,n_1+n_2-2}. \end{aligned} \tag{7.22}$$

The rejection region is

$$\frac{n_1 n_2(n_1+n_2-2)}{(n_1+n_2)^2}\frac{\left\{1_p^\top(\bar{x}_1-\bar{x}_2)\right\}^2}{1_p^\top \mathcal{S} 1_p} > F_{1-\alpha;1,n_1+n_2-2}.$$

Treatment Effect

If it is rejected that the profiles are parallel, then two independent analyses should be done on the two groups using the repeated measurement approach. But if it is accepted that they are parallel, then we can exploit the information contained in both groups (eventually at different levels) to test a treatment effect, i.e., if the two profiles are horizontal. This may be written as:

$$H_0^{(3)}: \mathcal{C}(\mu_1+\mu_2) = 0.$$

Consider the average profile $\bar{x}$:

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1+n_2}.$$

Clearly,

$$\bar{x} \sim N_p\left(\frac{n_1\mu_1+n_2\mu_2}{n_1+n_2}, \frac{1}{n_1+n_2}\Sigma\right).$$

Now it is not hard to prove that $H_0^{(3)}$ with $H_0^{(1)}$ implies that

$$\mathcal{C}\left(\frac{n_1\mu_1+n_2\mu_2}{n_1+n_2}\right) = 0.$$

So under parallel, horizontal profiles we have

$$\sqrt{n_1+n_2}\,\mathcal{C}\bar{x} \sim N_p(0, \mathcal{C}\Sigma\mathcal{C}^\top).$$

From Corollary 5.4 we again obtain

$$(n_1+n_2-2)(\mathcal{C}\bar{x})^\top(\mathcal{C}\mathcal{S}\mathcal{C}^\top)^{-1}\mathcal{C}\bar{x} \sim T^2(p-1, n_1+n_2-2). \tag{7.23}$$

This leads to the rejection region of $H_0^{(3)}$, namely

$$\frac{n_1+n_2-p}{p-1}(\mathcal{C}\bar{x})^\top(\mathcal{C}\mathcal{S}\mathcal{C}^\top)^{-1}\mathcal{C}\bar{x} > F_{1-\alpha;p-1,n_1+n_2-p}.$$