Homework #17 Key

7.11

The original (primal) linear program is the following:

$$\max\ x + y$$

$$2x + y \leq 3$$

$$x + 3y \leq 5$$

$$x, y \geq 0$$

This can be written in vector/matrix form as:

$$\max\ \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which, in turn, can be written compactly as:

$$\max\ \mathbf{c}^T\mathbf{x}$$

$$A\mathbf{x} \leq \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

where $\mathbf{c}^T = \begin{pmatrix} 1 & 1 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\mathbf{0}$ means a vector of 0s of appropriate size.

Given this, the dual program is constructed as

$$\min\ \mathbf{w}^T\mathbf{b}$$

$$\mathbf{w}^T A \geq \mathbf{c}$$

$$\mathbf{w} \geq \mathbf{0}$$

Letting $\mathbf{w}^T = \begin{pmatrix} w & z \end{pmatrix}$, this can be expanded to

$$\max \begin{pmatrix} w & z \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} w & z \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} w \\ z \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which, finally, can be written as:

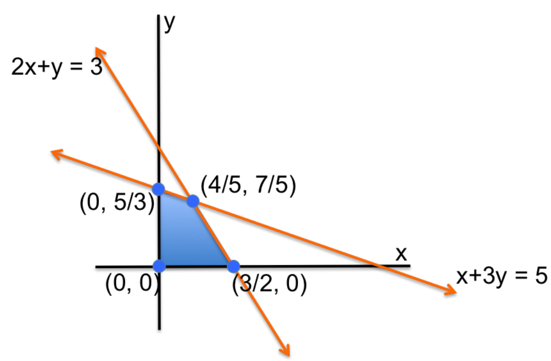
$$\min 3w + 5z$$

$$2w + z \geq 1$$

$$w + 3z \geq 1$$

$$w, z \geq \mathbf{0}$$

Now, to solve the primal problem, we can use a Simplex-based LP solver, of course, but since it only involves 2 variables, we can also do it visually as follows. The constraints and feasible region can be visualized as below:



And, given that we know that the solution must lie at one of the four vertices where constraints intersect, we can simply find the value at each of these points, and take the maximum:

$$(0, 0) = 0 + 0 = 0$$

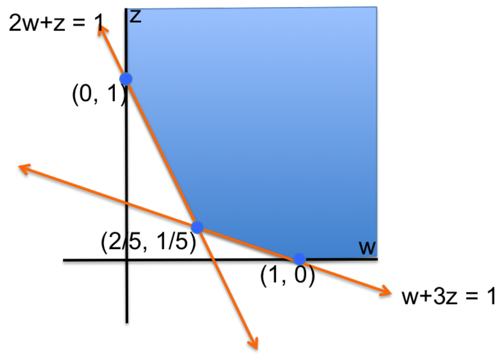
$$(\frac{3}{2}, 0) = \frac{3}{2} + 0 = \frac{3}{2}$$

$$(0, \frac{5}{3}) = 0 + \frac{5}{3} = \frac{5}{3}$$

$$(\frac{4}{5}, \frac{7}{5}) = \frac{4}{5} + \frac{7}{5} = \frac{11}{5}$$

which, of course, is $\frac{11}{5}$.

Finally, to solve the dual problem, we can also use a Simplex-based LP solver, of course, but again, since it only involves 2 variables, we can also do it visually as above. The constraints and feasible region are now visualized as:



And, this time we find the minimum value for any of the vertices:

$$(1,0) = 3*1+5*0 = 3$$
$$(0,1) = 3*0+5*1 = 5$$
$$(\frac{2}{5},\frac{1}{5}) = 3*\frac{2}{5}+5*\frac{1}{5} = \frac{11}{5}$$

which is also $\frac{11}{5}$. Note that for the dual problem, the feasible region is unbounded above, but since we are are minimizing in this case, this does not preclude us from obtaining a solution.

7.13

(a) The payoff matrix for the matching pennies game looks like this:

$$\begin{array}{cc} & \begin{array}{cc} h & t \end{array} \\ \begin{array}{c} h \\ t \end{array} & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array}$$

(b) Since the row player is trying to maximize the value of this game, while the row player is trying to minimize it, the row player would like to compute

$$\max \{\min \{x_1 - x_2, -x_1 + x_2\}\}$$

Letting $z = \min \{x_1 - x_2, -x_1 + x_2\}$, we can then construct the following linear program to solve for the optimal strategy and its value:

$$\max z$$

$$x_1 - x_2 \geq z$$

$$-x_1 + x_2 \geq z$$

$$x_1 + x_2 = 1$$

$$x_1, x_2 \geq 0$$

A little algebra turns this into

$$\max z$$

$$-x_1 + x_2 + z \leq 0$$

$$+x_1 - x_2 + z \leq 0$$

$$x_1 + x_2 = 1$$

$$x_1, x_2 \geq 0$$

and, because $x_2 = 1 - x_1$, we can further simplify this as

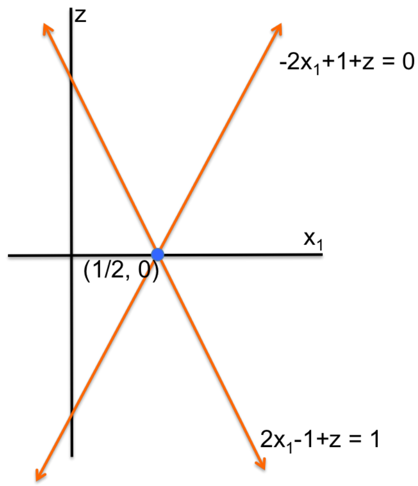
$$\max z$$

$$-2x_1 + 1 + z \leq 0$$

$$+2x_1 - 1 + z \leq 0$$

$$x_1 \geq 0$$

Now, we can graph this.



Note that no feasible region is shown, as there is only a single feasible point, at $(\frac{1}{2}, 0)$, whose value is $z = 0$. This is the value of the game and the optimal strategy for the row player is, therefore, to play the mixed strategy $(0.5, 0.5)$ (that is, to play x_1 50% of the time and x_2 50% of the time.

Complementarily, for the column player, who tries to minimize the value of the game, we want to compute

$$\min \{\max \{y_1 - y_2, -y_1 + y_2\}\}$$

Letting $w = \max \{y_1 - y_2, -y_1 + y_2\}$, we can then construct the following linear program to solve for the optimal strategy and its value:

$$\min w$$

$$y_1 - y_2 \leq w$$
$$-y_1 + y_2 \leq w$$
$$y_1 + y_2 = 1$$
$$y_1, y_2 \geq 0$$

A little algebra turns this into

$$\min w$$

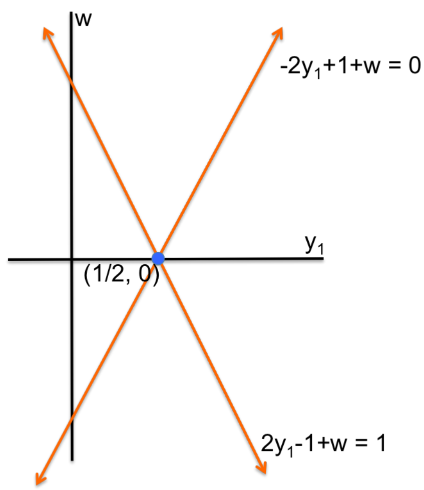
$$-y_1 + y_2 + w \geq 0$$
$$+y_1 - y_2 + w \geq 0$$
$$y_1 + y_2 = 1$$
$$y_1, y_2 \geq 0$$

and, because $y_2 = 1 - y_1$, we can further simplify this as

$$\min w$$

$$-2y_1 + 1 + w \geq 0$$
$$+2y_1 - 1 + z \geq 0$$
$$y_1 \geq 0$$

As expected, this is exactly the dual of the program for the row player. Also, note the symmetry in the programs, which is due to the symmetry in the payoff matrix (in the more general case of non-symmetric payoffs, this symmetry in the linear programs will no longer be evident). Now, we can graph this.



Again, note the symmetry with the primal problem, with, in this case, only the variable names changing. Of course, there is still only a single feasible point, at $(\frac{1}{2}, 0)$, whose value is $w = 0$, which is to be expected—the value of the game is the same for both players (and this is true even for non-symmetric payoffs). Unsurprisingly, the optimal strategy for the column player is, therefore, also to play the mixed strategy $(0.5, 0.5)$ (that is, to play y_1 50% of the time and y_2 50% of the time.