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ALGEBRAIC NUMBER THEORY

LECTURE NOTES

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1 Small prefix

Recall:

- L numberfield : $\iff L$ is a finite extension of \mathbb{Q}
In particular: L/\mathbb{Q} is separable $\Rightarrow L/\mathbb{Q}$ is primitive, i.e. $L = \mathbb{Q}(\alpha), \mathbb{Q}[X] \ni f_\alpha =$ minimal polynomial of α over \mathbb{Q} and $[L : \mathbb{Q}] = \deg(f_\alpha)$.
- $\mathcal{O} := \{\alpha \in L \mid f_\alpha \in \mathbb{Z}[X]\}$ is called *ring of integers* (generalization of $\mathbb{Z} \subseteq \mathbb{Q}$).
 \mathcal{O} is an integral domain.
- Goal: study the ring \mathcal{O}
- Questions:
 1. What is \mathcal{O}^\times ? What is its structure?
 2. What are the prime ideals of \mathcal{O} ?
 3. Do we have a unique prime factorization, i.e. is \mathcal{O} a UFD?

1.1 Motivation

Problem 1.1.1 (Fermat's conjecture, ~ 1640). Show that the equation $x^n + y^n = z^n$ has no nontrivial integer solutions, i.e. solutions (x, y, z) with $x, y, z \in \mathbb{Z} \setminus \{0\}$ for $n \geq 3$.

History:

- 1770: Euler found solution for $n = 3$
- 1825: Dirichlet and Legendre using Germain
- Kummer showed it for many primes, he showed as well that his idea doesn't work for all $n \in \mathbb{N}_{>2}$
- Conjecture was proved by Wiles in 1997

Remark 1.1.2. i) If Fermat's is true for n , then also for nk for all $k \in \mathbb{N}$.

ii) It is sufficient to prove Fermat's conjecture for $n = 4$ and all odd primes.

Proof. i) Suppose (x, y, z) is a nontrivial solution of $x^{nk} + y^{nk} = z^{nk} \Rightarrow (x^k, y^k, z^k)$ is a nontrivial solution to $x^n + y^n = z^n$.

ii) Follows from i).

□

Proposition 1.1.3 ($n = 2$). Suppose $x, y, z \in \mathbb{Z}$, $\gcd(x, y, z) = 1$

- i) x, y, z are pairwise coprime if $x^2 + y^2 = z^2$
- ii) $x^2 + y^2 = z^2 \Rightarrow$ either x or y is even
- iii) $x^2 + y^2 = z^2 \iff \exists r, s \in \mathbb{N}_0, \gcd(r, s) = 1$ s.t. $x = \pm 2rs, y = \pm(r^2 - s^2), z = \pm(r^2 + s^2)$.

Proof. i) clear \checkmark

ii) One of x, y, z has to be even, since $odd + odd \neq odd$. Suppose z is even. Then look at equation mod 4, this gives a contradiction. By i) only one of x and y is even.

iii) „ \Leftarrow “: calculation

„ \Rightarrow “: Wlog. assume $x, y, z \in \mathbb{N}_0$, x even, y, z odd:

$$\begin{aligned} \Rightarrow x = 2u, z + y = 2v, z - y = 2w, \gcd(w, v) = 1 (y, z \text{ are coprime}), x^2 + y^2 = z^2 \\ \Rightarrow 4u^2 = x^2 = z^2 - y^2 = (z - y)(z + y) = 4wv \Rightarrow u^2 = vw \\ \xRightarrow{\gcd(v, w)=1} v = r^2, w = s^2 \Rightarrow z = v + w = r^2 + s^2, y = v - w = r^2 - s^2 \\ \text{and } x = 2u = 2\sqrt{vw} = 2rs \end{aligned}$$

□

Remark. $(x, y, z) \in \mathbb{Z}^3$ with $x^2 + y^2 = z^2$ are called *pythagorean triples*.

Proposition 1.1.4 ($n = 4$). The equation $x^4 + y^4 = z^2$ (and $x^4 + y^4 = z^4$) have no nontrivial integer solutions.

Proof. Suppose $x, y, z \in \mathbb{Z}$ with $x^4 + y^4 = z^2, xyz \neq 0$. Wlog $x, y, z > 0, x, y, z$ coprime, $x = 2\tilde{x}$ for some $\tilde{x} \in \mathbb{N}$. Choose z minimal with this conditions.

$$\begin{aligned} \text{Prop. 1.2} \Rightarrow \exists r, s \in \mathbb{N} \text{ s.t. } x^2 = 2rs, y^2 = r^2 - s^2, z = r^2 + s^2 \text{ and } \gcd(r, s) = 1 \\ \Rightarrow y^2 + s^2 = r^2 \text{ with } y, s, r \text{ coprime.} \end{aligned}$$

$$\text{Prop. 1.2} \Rightarrow \exists a, b \in \mathbb{N} \text{ s.t. } s = 2ab, y = a^2 - b^2, r = a^2 + b^2 \text{ and } \gcd(a, b) = 1.$$

$$\text{plug in} \Rightarrow x^2 = 4ab(a^2 + b^2)$$

$$\Rightarrow \tilde{x}^2 = ab(a^2 + b^2) \text{ and } a, b, a^2 + b^2 \text{ pairwise coprime}$$

As in proof of Prop. 1.2 (they are coprime but a square number)

$$\begin{aligned} \Rightarrow \exists c, d, e \in \mathbb{N} \text{ s.t. } a = c^2, b = d^2, a^2 + b^2 = e^2 \\ \Rightarrow c^4 + d^4 = a^2 + b^2 = e^2 \text{ and } e \leq a^2 + b^2 = r < z \end{aligned}$$

!since z was chosen to be minimal.

□

From now on: $n = p$ odd prime.

Idea 1.1.5 (by Germain). Distinguish 2 cases in Fermat's problem:

1. „First case“: x, y, z with p does not divide xyz .
2. „Second case“: exactly one of x, y, z is divided by p .

Some approach:

- Use primitive p -th root of unity $\zeta = \zeta_p$.
- Reminder: $X^p - 1 = (X - 1)(X - \zeta) \dots (X - \zeta^{p-1})$
- Setting $\tilde{y} = -y$ we get:

$$\begin{aligned}
 x^p + y^p &= x^p - \tilde{y}^p = \tilde{y}^p \left(\left(\frac{x}{\tilde{y}} \right)^p - 1 \right) \\
 &= \tilde{y}^p \left(\frac{x}{\tilde{y}} - 1 \right) \left(\frac{x}{\tilde{y}} - \zeta \right) \dots \left(\frac{x}{\tilde{y}} - \zeta^{p-1} \right) \\
 &= (x - \tilde{y})(x - \tilde{y}\zeta) \dots (x - \tilde{y}\zeta^{p-1}) \\
 &= (x + y)(x + y\zeta) \dots (x + y\zeta^{p-1})
 \end{aligned}$$

Lemma 1.1.6. For $x, y, z \in \mathbb{Z}$ we have $x^p + y^p = z^p \iff (x+y)(x+y\zeta) \dots (x+y\zeta^{p-1}) = z^p$

Idea: Look at prime divisors in $\mathbb{Z}[\zeta]$.

Problem: Would be good to have unique prime factorization. This will not be true in general.

1.2 The ring $\mathbb{Z}[\zeta]$

Suppose ζ is a primitive n -th root of unity

Reminder 1.2.1. i) $\mathbb{Q}(\zeta)/\mathbb{Q}$ is algebraic extension of degree $[\mathbb{Q}(\zeta) : \mathbb{Q}] = \varphi(n)$

ii) $\mathbb{Q}(\zeta)/\mathbb{Q}$ is a Galois extension. In particular:

$$\text{Hom}(\mathbb{Q}(\zeta)/\mathbb{Q}) \cong \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) = \{\sigma_i \text{ with } \sigma_i(\zeta) = \zeta^i \mid i \in (\mathbb{Z}/n\mathbb{Z})^\times\} \cong (\mathbb{Z}/n\mathbb{Z})^\times$$

iii) Consider the norm map $\mathcal{N} : \mathbb{Q}(\zeta) \rightarrow \mathbb{Q}$, $\alpha \mapsto \det(\gamma \mapsto \alpha\gamma)$. We have for $\alpha = r(\zeta)$ ($r \in \mathbb{Q}[X]$ polynomial) with min. polynomial $f_\alpha = X^k + c_{k-1}X^{k-1} + \dots + c_0$:

- If we have $\mathbb{Q}(\alpha) = \mathbb{Q}(\zeta)$, then $\mathcal{N}(\alpha) = (-1)^{\varphi(n)} c_0$
- $\mathcal{N}(\alpha) = \prod_{\sigma \in \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})} \sigma(\alpha) = \prod_{i \in (\mathbb{Z}/n\mathbb{Z})^\times} r(\zeta^i)$
- $\alpha \in \mathbb{Q} \Rightarrow \mathcal{N}(\alpha) = \alpha^{\varphi(n)}$

iv) $X^{n-1} + X^{n-2} + \dots + 1 = \frac{X^n - 1}{X - 1} = (X - \zeta)(X - \zeta^2) \dots (X - \zeta^{n-1})$
 $\xrightarrow{X=1} n = (1 - \zeta)(1 - \zeta^2) \dots (1 - \zeta^{n-1})$

Reminder 1.2.2 (and preview). i) $\mathcal{O} := \mathbb{Z}[\zeta] := \{r(\zeta) \mid r \in \mathbb{Z}[X]\}$

ii) $\mathbb{Z}[\zeta] = \{\alpha \in \mathbb{Q}(\zeta) \mid f_\alpha \in \mathbb{Z}[X]\}$ (proof later)

iii) $\mathbb{Z}[\zeta]$ is a free \mathbb{Z} -module with basis $\{1, \zeta, \dots, \zeta^{d-1}\}$ with $d = \varphi(n)$ (proof later)

iv) $\alpha \in \mathbb{Z}[\zeta] \Rightarrow \mathcal{N}(\alpha) \in \mathbb{Z}$ (proof later)

v) $\{\alpha \in \mathcal{O} \mid |\alpha| = 1\}$ is finite (proof later)

Reminder 1.2.3. Suppose R is an integral domain:

i) $\alpha \in R$ is *irreducible* : \iff If $\alpha = \alpha_1 \alpha_2$ with $\alpha_i \in R$, then $\alpha_1 \in R^\times$ or $\alpha_2 \in R^\times$

ii) $\alpha, \alpha' \in R$ are *associated to each other* : $\iff \exists \varepsilon \in R^\times : \alpha = \varepsilon \alpha'$

iii) R is called *factorial* : \iff each $\alpha \in R, \alpha \neq 0$ can be written in a unique way as $\alpha = \varepsilon \pi_1 \cdot \dots \cdot \pi_r$ with π_i irreducible up to multiplication with $\varepsilon \in R^\times$

iv) $\alpha_1, \alpha_2 \in R$ are called *coprime* : \iff If $\alpha' \in R$ with $\exists \beta_1, \beta_2 \in R : \alpha_1 = \alpha' \beta_1, \alpha_2 = \alpha' \beta_2$ then $\alpha' \in R^\times$.

Remark (and correction). 1. Recall: L/\mathbb{Q} field extensions:

$$\mathcal{O} := \{\alpha \in L \mid f_\alpha \in \mathbb{Z}[X]\}$$

!! Here: f_α is by definition monic, i.e. leading coefficient is 1.

Remark: $\mathcal{O} = \{\alpha \in L \mid \exists f \in \mathbb{Z}[X] \text{ with } f \text{ monic and } f(\alpha) = 0\}$

„ \subseteq “: clear

„ \supseteq “: Lemma of Gauss

2. Recall: Definition of field norm for L/K finite field extension How is norm defined?

$\mathcal{N} : L \rightarrow K$ defined as follows:

Suppose $\alpha \in L \Rightarrow \varphi_\alpha : \beta \mapsto \alpha\beta$ is linear map over K . Then:

$$\mathcal{N}_{L/K}(\alpha) := \det(\phi_\alpha)$$

Properties:

a) If $L = K(\alpha)$ and $X^n + c_{n-1}X^{n-1} + \dots + c_0$ is a minimal polynomial of α over K , then $\mathcal{N}_{L/K}(\alpha) = (-1)^n c_0$.

b) $\mathcal{N}_{L/K}(\alpha) = (\prod_{i=1}^r \sigma_i(\alpha))^q$ with $\text{Hom}_K(L, \bar{K}) = \{\sigma_1, \dots, \sigma_r\}$ and $q = \text{inseparable degree, i.e. } [L : K] = [L : K]_s \cdot q$.

c) $\alpha \in K \Rightarrow \mathcal{N}_{L/K}(\alpha) = \alpha^d$ with $d = [L : K]$ (see Bosch „Algebra“ 4.7).

General reference: NEUKIRCH

This chapter: BOREVICH + SHAFEREVICH Chapter 3.1.

Recall: Goal: prove for p prime and odd

$$x^p + y^p = z^p$$

has no non-trivial solutions. Last time:

$$x^p + y^p = z^p = (x + y)(x + y\zeta)(x + y\zeta^2) \dots (x + y\zeta^{p-1}) \in \mathbb{Z}[\zeta]$$

From now on: p odd prime, $\zeta = e^{\frac{2\pi i}{p}}$ primitive p -th root of unity $\mathcal{O} = \mathbb{Z}[\zeta]$.

Proposition 1.2.4. *For the group of units \mathcal{O}^\times of $\mathcal{O} = \mathbb{Z}[\zeta]$ we have:*

$$\mathcal{O}^\times = \{\alpha \in \mathcal{O} \mid \mathcal{N}(\alpha) = \pm 1\}$$

Notation: $\mathcal{N} = \mathcal{N}_{\mathbb{Q}(\zeta)/\mathbb{Q}}$ in this chapter.

Proof. „ \subseteq “: “ $\alpha \in \mathcal{O}^\times \Rightarrow \exists \beta \in \mathcal{O}$ with $\alpha\beta = 1 \Rightarrow 1 = \mathcal{N}(\alpha\beta) \stackrel{!}{=} \underbrace{\mathcal{N}(\alpha)}_{\in \mathbb{Z}} \underbrace{\mathcal{N}(\beta)}_{\in \mathbb{Z} \text{ by 2.2 v}} \Rightarrow \text{claim}$ “

„ \supseteq “: Suppose $\alpha \in \mathcal{O}$ with $\mathcal{N}(\alpha) = \pm 1$.

$$\Rightarrow \pm 1 = \mathcal{N}(\alpha) = \prod_{\sigma \in \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})} \sigma(\alpha)$$

Note: $\alpha = a_0 + a_1\zeta + \dots + a_{p-2}\zeta^{p-2} \in \mathbb{Z}[\zeta]$

$$\Rightarrow \sigma(\alpha) = a_0 + a_1\zeta^i + \dots + a_{p-2}\zeta^{i(p-2)} \text{ for some } i \in \{1, \dots, p-1\} \Rightarrow \sigma(\alpha) \in \mathbb{Z}[\zeta]$$

$$\Rightarrow \alpha \text{ is a divisor of 1 in } \mathbb{Z}[\zeta] \Rightarrow \alpha \in \mathcal{O}^\times. \quad \square$$

Lemma 1.2.5. *i) $\mathcal{N}(1 - \zeta^s) = p$ for $s \in \mathbb{Z}$ with $s \not\equiv 0 \pmod{p}$*

ii) $1 - \zeta$ is irreducible in $\mathcal{O} = \mathbb{Z}[\zeta]$.

iii) $p = \varepsilon \cdot (1 - \zeta)^{p-1}$ with some $\varepsilon \in \mathcal{O}^\times$.

Proof. i) 2.1. iv) $\Rightarrow p = (1 - \zeta)(1 - \zeta^2) \dots (1 - \zeta^{p-1})$

$$2.1. \text{ iii) } \Rightarrow \mathcal{N}(1 - \zeta^s) = \prod_{\sigma \in \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})} \sigma(1 - \zeta^s) = \prod_{i=1}^{p-1} (1 - \zeta^{si}) = \prod_{j=1}^{p-1} (1 - \zeta^j) = p$$

ii) We obtain from i) that $1 - \zeta \notin \mathcal{O}^\times$. Suppose $1 - \zeta = \alpha\beta$ with $\alpha, \beta \in \mathcal{O}$

$$\Rightarrow p = \mathcal{N}(1 - \zeta) = \mathcal{N}(\alpha)\mathcal{N}(\beta) \Rightarrow \mathcal{N}(\alpha) = \pm 1 \text{ or } \mathcal{N}(\beta) = \pm 1 \xrightarrow{\text{Prop 2.4}} \alpha \in \mathcal{O}^\times \text{ or } \beta \in \mathcal{O}^\times.$$

iii) Use: $1 - \zeta^s = (1 - \zeta) \underbrace{(1 + \zeta + \zeta^2 + \dots + \zeta^{s-1})}_{\varepsilon_s} = (1 - \zeta)\varepsilon_s$

$$\Rightarrow p = \mathcal{N}(1 - \zeta^s) = \underbrace{\mathcal{N}(1 - \zeta)}_{=p} \cdot \mathcal{N}(\varepsilon_s) \Rightarrow \mathcal{N}(\varepsilon_s) = 1 \Rightarrow \varepsilon_s \in \mathcal{O}^\times$$

$$\text{Hence } p = \prod_{s=1}^{p-1} (1 - \zeta^s) = \prod_{s=1}^{p-1} \underbrace{\varepsilon_s}_{\in \mathcal{O}^\times} (1 - \zeta) = (1 - \zeta)^{p-1} \underbrace{\prod_{s=1}^{p-1} \varepsilon_s}_{\in \mathcal{O}^\times}$$

□

Notation: $\varepsilon_s = 1 + \zeta + \dots + \zeta^s$.

Lemma 1.2.6. *i) $a \in \mathbb{Z}$ with $1 - \zeta$ divides a in $\mathcal{O} \Rightarrow p$ divides a .*

ii) An n -th root of unity lies in $\mathbb{Q}(\zeta) \iff n$ divides $2p$.

Proof. i) $a = (1 - \zeta)\beta$ with $\beta \in \mathcal{O} \Rightarrow a^{p-1} = \mathcal{N}(a) = p\mathcal{N}(\beta) \xrightarrow{(\mathcal{N}(\beta) \in \mathbb{Z})} p$ divides a .

ii) „ \Leftarrow “: $-1 \in \mathbb{Q}(\zeta)$ and thus $e^{\frac{2\pi i}{2p}} \in \mathbb{Q}(\zeta)$

„ \Rightarrow “: Consider $H := \{\omega \in \mathbb{Q}(\zeta) \mid \omega \text{ is a root of unity}\}$

- a) $H \subseteq \mathbb{Z}[\zeta]$: Suppose $\omega \in H \Rightarrow \omega^n - 1 = 0$ for some $n \in \mathbb{N} \Rightarrow f_\omega$ is a divisor of $X^n - 1 \Rightarrow f_\omega \in \mathbb{Z}[X] \xrightarrow{2.2ii)} \omega \in \mathbb{Z}[\zeta]$.
- b) $\tilde{\omega}$ some conjugate of $\omega \Rightarrow \tilde{\omega}$ is a root of $X^n - 1 \Rightarrow |\tilde{\omega}| = 1 \xrightarrow{2.2v)} H$ is finite $\Rightarrow H$ is a cyclic subgroup of $\mathbb{Q}(\zeta)^\times$.
 Choose some generator ω_0 of H and denote $m := \text{ord}(\omega_0)$. Since $\zeta \in H$ and $\text{ord}(\zeta) = p \Rightarrow p$ divides m . Decompose $m = p^s \cdot m'$ with $s \geq 1$ and $\gcd(m', p) = 1$. Consider the field extensions chain:

$$\mathbb{Q} \subseteq \mathbb{Q}(\omega_0) \subseteq \mathbb{Q}(\zeta)$$

with degrees $[\mathbb{Q}(\zeta) : \mathbb{Q}] = p - 1 = \varphi(p)$ and $[\mathbb{Q}(\omega_0) : \mathbb{Q}] = \varphi(m) = p^{s-1}(p-1)\varphi(m') \leq p-1 \Rightarrow s = 1$ and $\varphi(m') = 1$ and thus $m' = 1, 2 \Rightarrow \text{ord}(\omega_0) \leq 2p$. □

Notation 1.2.7.

1. L/K field extension, $\alpha \in L, \bar{K}$ given algebraic closure. The elements $\sigma(\alpha)$ with $\sigma \in \text{Hom}_K(L, \bar{K})$ are called *conjugates of α* . In particular: L/K normal \Rightarrow conjugates live in L .
2. R ring, I ideal in R , $p : R \rightarrow R/I$ canonical projection. For $\alpha, \beta \in R$ we denote $\alpha \equiv \beta \pmod{I} : \iff p(\alpha) = p(\beta)$.
 If $I = \langle q \rangle$ is a principal ideal, we denote $\alpha \equiv \beta \pmod{q} : \iff \alpha \equiv \beta \pmod{\langle q \rangle}$

Example 1.2.8. Consider $\mathbb{Q}(\zeta)/\mathbb{Q}$ with $\zeta^p = 1, R = \mathcal{O} = \mathbb{Z}[\zeta], \alpha = a_0 + a_1\zeta + a_2\zeta^2 + \cdots + a_{p-2}\zeta^{p-2}$

- i) The conjugates of α are: $\alpha_h = a_0 + a_1\zeta^h + a_2\zeta^{2h} + \cdots + a_{p-2}\zeta^{h(p-2)}$ with $h \in \{1, \dots, p-1\}$.
- ii) Consider $\lambda = 1 - \zeta$ and $I = \langle \lambda \rangle$.
 $1 \equiv \zeta \pmod{\lambda}$ and $\alpha \equiv a_0 + a_1 + \cdots + a_{p-2} \pmod{\lambda} (\in \mathbb{Z})$.
- iii) $\alpha^p \equiv a_0^p + (a_1\zeta)^p + \cdots + (a_{p-2}\zeta^{p-2})^p = \underbrace{a_0^p + a_1^p + \cdots + a_{p-1}^p}_{\in \mathbb{Z}} \pmod{p}$

Theorem 1 (Kummer's Lemma). *If $\varepsilon \in \mathbb{Z}[\zeta]$ is a unit, i.e. $\varepsilon \in \mathbb{Z}[\zeta]^\times$,*

$$\frac{\varepsilon}{\bar{\varepsilon}} = \zeta^a \quad \text{for some } a \in \mathbb{Z}$$

Here $\bar{\varepsilon} = \tau(\varepsilon)$, where τ is the complex conjugation.

Recall: $\tau \in \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$.

Proof. Denote $\varepsilon = a_0 + a_1\zeta + \cdots + a_{p-2}\zeta^{p-2} = r(\zeta)$ with $r(X) = \sum_{i=0}^{p-2} a_i X^i \in \mathbb{Z}[X]$.

Observe:

1. $\varepsilon \in \mathcal{O}^\times \Rightarrow \exists \varepsilon' \in \mathcal{O}$ s.t. $\varepsilon \varepsilon' = 1 \Rightarrow \bar{\varepsilon} \bar{\varepsilon}' = 1 \Rightarrow \bar{\varepsilon} \in \mathcal{O}^\times$
2. $\mu := \frac{\varepsilon}{\bar{\varepsilon}} = \frac{r(\zeta)}{r(\bar{\zeta})}$ and the conjugate μ_k of μ is $\frac{r(\zeta^k)}{r(\bar{\zeta}^k)} = \frac{r(\zeta^k)}{r(\zeta^k)}$. In particular $|\mu_k| = 1$.
 It follows that $\mu_k \in \{\alpha \in \mathcal{O}^\times \mid |\alpha| = 1\}$ which is by 2.2. v) a finite subgroup of $\mathbb{Q}(\zeta)^\times \Rightarrow \mu$ is a root of unity
 Lemma 2.6 $\Rightarrow \mu = \pm \zeta^a$ for some $a \in \mathbb{Z}$.
Claim: $\mu = \zeta^a$
Proof of claim: suppose $\mu = -\zeta^a$, i.e. $\varepsilon = -\bar{\varepsilon} \zeta^a$ (\star)
Idea: calculation mod $\lambda = 1 - \zeta$ $\varepsilon = a_0 + a_1 \zeta + \dots + a_{p-2} \zeta^{p-2}$
 Ex. 2.8.ii) $\Rightarrow \varepsilon \equiv \underbrace{a_0 + a_1 + \dots + a_{p-2}}_{=: M \in \mathbb{Z}} \equiv \bar{\varepsilon} \pmod{\lambda}$
 $(\star) \Rightarrow \varepsilon \equiv -\bar{\varepsilon} \pmod{\lambda} \Rightarrow M \equiv -M \pmod{\lambda} \Rightarrow 2M \equiv 0 \pmod{\lambda} \xrightarrow{\text{Lemma 2.6 i)}} p \text{ divides } 2M \text{ in } \mathbb{Z} \xrightarrow{p \text{ odd}} p \text{ divides } M$
 $\Rightarrow \lambda = 1 - \zeta$ divides M in \mathcal{O} by Lemma 2.5.
 $\Rightarrow \varepsilon \equiv \bar{\varepsilon} \equiv M \equiv 0 \pmod{\lambda = 1 - \zeta} \Rightarrow$ Contradiction to ε is unit and $1 - \zeta$ is irreducible

□

Corollary 1.2.9. ε unit in $\mathbb{Z}[\zeta] \Rightarrow \varepsilon = r \zeta^s$ with some $r \in \mathbb{R}, s \in \mathbb{Z}$.

Proof. Prop 2.9 $\Rightarrow \exists a \in \mathbb{Z}, \varepsilon = \zeta^a \cdot \bar{\varepsilon}$.

Choose $s \in \mathbb{Z}$ with $2s \equiv a \pmod{p}$

$\Rightarrow \frac{\varepsilon}{\zeta^s} = \zeta^s \cdot \bar{\varepsilon} = \frac{\bar{\varepsilon}}{\zeta^{-s}} = \frac{\bar{\varepsilon}}{\bar{\zeta}^s} = r \in \mathbb{R}$ and $\varepsilon = r \cdot \zeta^s$.

□

Lemma 1.2.10. Suppose $x, y, m, n \in \mathbb{Z}$ with $m \not\equiv n \pmod{p}$. $x + y \zeta^n$ and $x + y \zeta^m$ are relatively prime $\iff (x$ and y are relatively prime) and $(x + y$ not divisible by $p)$

Proof. „ \Rightarrow “:

- $d \mid x$ and $d \mid y \Rightarrow d \mid x + \zeta^n y$ and $d \mid x + \zeta^m y$ \nmid
- „ $p \mid x + y$ “ Recall: $p = \varepsilon(1 - \zeta)^{p-1}$ with $\varepsilon \in \mathcal{O}^\times$
 $\Rightarrow x + \zeta^m y = \underbrace{x + y}_{\text{divisible by } p} + y \cdot \underbrace{(\zeta^m - 1)}_{(\zeta - 1)(1 + \zeta + \zeta^2 + \dots + \zeta^{m-1})} \equiv 0 \pmod{1 - \zeta}$
 same way $x + \zeta^n y \equiv 0 \pmod{1 - \zeta} \nmid$

„ \Leftarrow “: Idea: show: $\exists \alpha_0, \beta_0 \in \mathcal{O}$ with:

$$1 = \alpha_0(x + \zeta^m y) + \beta_0(x + \zeta^n y)$$

Consider: $A := \{\alpha(x + \zeta^m y) + \beta(x + \zeta^n y) \mid \alpha, \beta \in \mathcal{O}\}$

A is an ideal in \mathcal{O} . We have:

1. $(x + \zeta^m y) - (x + \zeta^n y) = \zeta^m(1 - \zeta^{n-m})y = \underbrace{\zeta^n \varepsilon_{n-m}}_{\in \mathcal{O}^\times} (1 - \zeta)y \Rightarrow (1 - \zeta)y \in A$

2. $\zeta^n(x + \zeta^m y) - \zeta^m(x + \zeta^n y) = (\zeta^n - \zeta^m)x = \zeta^n \cdot (1 - \zeta^{n-m})x = \underbrace{\zeta^n \varepsilon_{m-n}}_{\in \mathcal{O}^\times} \cdot (1 - \zeta)x \Rightarrow (1 - \zeta)x \in A.$
3. $\gcd(x, y) = 1 \Rightarrow \exists a, b \in \mathbb{Z} \text{ with } 1 = ax + by \Rightarrow (1 - \zeta)xa + (1 - \zeta)yb = 1 - \zeta \xrightarrow{1. \& 2.} 1 - \zeta \in A$
4. $x + y = \underbrace{x + \zeta^n y}_{\in A} + \underbrace{(1 - \zeta^n)y}_{\in A} \in A$
5. $\gcd(p, x + y) = 1 \Rightarrow \exists \bar{a}, \bar{b} \in \mathbb{Z} : 1 = \underbrace{\bar{a}p}_{\in A} + \underbrace{\bar{b}(x + y)}_{\in A} \in A.$
 \Rightarrow Hence $x + \zeta^n y$ and $x + \zeta^m y$ are coprime.

□

Remark 1.2.11. Suppose $\alpha = a_0 + a_1\zeta + \dots + a_{p-1}\zeta^{p-1} \in \mathcal{O}$ with $a_i \in \mathbb{Z}$ and at least one $a_j \neq 0$.

If $n \in \mathbb{Z}$ with n divides α in \mathcal{O} , then n divides all a_i

Proof. Recall from 2.2 (preview): $1, \zeta, \zeta^2, \dots, \zeta^{p-2}$ is a basis of \mathcal{O} .

Furthermore: $1 + \zeta + \dots + \zeta^{p-1} = 0$

$\Rightarrow \{1, \zeta, \dots, \zeta^{p-1}\} \setminus \{\zeta^j\}$ is a basis \Rightarrow claim.

□

1.3 First case of Fermat in case of $\mathbb{Z}[\zeta]$ is a UFD (unique factorization domain)

Reference: BOREVICH + SHAFEREVIC + WASHINGTON Chapter 1

As before: p odd prime, $\zeta = e^{\frac{2\pi i}{p}}$ p -th root of unity.

Theorem 2. Suppose that $\mathbb{Z}[\zeta]$ is a UFD, then $x^p + y^p = z^p$ has no non-trivial solutions (x, y, z) , such that neither x, y nor z is divisible by p .

Theorem 3 ($p = 3$). Suppose $x, y, z \in \mathbb{Z}$ with $x^3 + y^3 = z^3 \pmod{9} \Rightarrow 3$ divides x, y or z .

Proof. Recall: Little Fermat's theorem $x^p \equiv x, y^p \equiv y, z^p \equiv z \pmod{p}$.

$$\begin{aligned}
 x^3 + y^3 &\equiv z^3 \pmod{3} \Rightarrow x + y \equiv z \pmod{3} \\
 &\Rightarrow z = x + y + 3u \text{ with } u \in \mathbb{Z} \\
 \Rightarrow \underline{x^3 + y^3} &\equiv (x + y + 3u)^3 \equiv \underline{x^3 + y^3} + 3xy^2 + 3x^2y \pmod{9} \\
 &\Rightarrow 0 \equiv xy^3 + x^2y \equiv xy(x + y) \equiv xyz \pmod{3} \\
 &\Rightarrow x, y \text{ or } z \text{ is divisible by } 3
 \end{aligned}$$

□

Lemma 1.3.1. *Let $p \geq 5$. Suppose $x, y, z \in \mathbb{Z}$ with $x^p + y^p = z^p$. If $x \equiv y \equiv -z \pmod{p}$, then $p|z$.*

Proof. $z \equiv z^p = x^p + y^p \equiv -2z^p \equiv -2z \pmod{p} \Rightarrow 3z \equiv 0 \pmod{p} \xrightarrow{p \neq 3} p|z$. \square

Remark 1.3.2. It follows from Lemma 3.2 that in the first case of Fermat we may assume for $p \geq 5$ that $x \not\equiv y \pmod{p}$ because we can replace $x^p + y^p = z^p$ by $x^p + (-z)^p = (-y)^p$ and $x \not\equiv -z \pmod{p}$.

of Thm. 1. $p = 3 \Rightarrow$ claim follows from Prop 3.1.

Now: $p \geq 5$. Suppose $x, y, z \in \mathbb{Z}$ with p divides neither x, y nor z , x, y, z are pairwise coprime and $x \not\equiv y \pmod{p}$. Suppose $z^p = x^p + y^p = (x + y)(x + \zeta y) \dots (x + \zeta^{p-1}y)$.

Apply Lemma 2.11:

- $\gcd(x, y) = 1$ ✓
- Little Fermat $\Rightarrow x + y \equiv x^p + y^p \equiv z^p \not\equiv 0 \pmod{p}$

$\xrightarrow{2.11} x + y, x + \zeta y, \dots, x + \zeta^{p-1}y$ are pairwise coprime.

$\xrightarrow{\mathbb{Z}[\zeta] \text{ UFD}} \text{„}x + \zeta^i y \text{ have to be } p\text{-power“}$ More precisely: $x + \zeta y = \varepsilon \alpha^p$ with $\varepsilon \in \mathcal{O}^\times, \alpha \in \mathcal{O}$, since they are coprime factors of a p -th power.

1. Cor. 2.10 $\Rightarrow \varepsilon = r\zeta^s$ with $r \in \mathbb{R}, s \in \mathbb{Z}$
2. Example 2.8. iii) $\Rightarrow \exists a \in \mathbb{Z}$ with $\alpha^p \equiv a \pmod{p}$.

$$\begin{aligned} x + \zeta y &= r\zeta^s \alpha^p \equiv r\zeta^s a \pmod{p} \\ x + \zeta^{-1}y &= \overline{x + \zeta y} \equiv r\zeta^{-s} a \pmod{p} \\ \Rightarrow \zeta^{-s}(x + \zeta y) &\equiv ra \equiv \zeta^s(x + \zeta^{-1}y) \pmod{p} \\ \Rightarrow \underbrace{x + \zeta y - \zeta^{2s}x - \zeta^{2s-1}y}_{=x \cdot 1 + y\zeta - x\zeta^{2s} - y\zeta^{2s-1}} &\equiv 0 \pmod{p} \end{aligned}$$

Idea: Use Rem. 2.12

Case 1: $1, \zeta, \zeta^{2s-1}, \zeta^{2s}$ are distinct $\xrightarrow{p \geq 5, \text{ Rem 2.12}} p|x$ and $p|y$. Contradiction to first case.

\square

Recall: $L = \mathbb{Q}(\zeta)$, $\mathcal{O} = \mathbb{Z}[\zeta]$, where ζ is a p -th root of unity

Last time:

- (1) $a_1 1 + a_2 \zeta + \dots + a_p \zeta^{p-1} = \alpha$ and at least one $a_j = 0$
If α is divided by $n \in \mathbb{Z}$ then all the a_i are divided by n .
- (2) $x + y\zeta - x\zeta^{2s} - y\zeta^{2s-1} \equiv 0 \pmod{p}$

Continuation of proof of Theorem 1. “Case 2” $1, \zeta, \dots, \zeta^{2s}$ are not distinct.

Observe: $1 \neq \zeta$ and $\zeta^{2s-1} \neq \zeta^{2s}$

“Case 2A” $1 = \zeta^{2s} (\Leftrightarrow p|s)$.

(2) implies $y\zeta - y\zeta^{2s-1} \equiv 0 \pmod{p}$ such that Remark 2.12 yields the contradiction $p|y$.

“Case 2B” $1 = \zeta^{2s-1} (\Leftrightarrow \zeta = \zeta^{2s})$.

(2) implies $(x - y)1 + (y - x)\zeta \equiv 0 \pmod{p}$ such that Remark 2.12 yields $p|y - x$, which contradicts the assumption $x \not\equiv y \pmod{p}$.

“Case 2C” $\zeta = \zeta^{2s-1}$.

(2) implies $x - x\zeta^2 \equiv 0 \pmod{p}$ such that Remark 2.12 yields the contradiction $p|x$. \square

Questions:

(1) Under which assumption is \mathcal{O} a UFD?

(2) What can we do if \mathcal{O} is not a UFD?

→ Idea of Kummer: “calculate with ideals”

Prospect: Theorem (Montgomery, Uchida, 1971)

$\mathbb{Z}[\zeta]$ is a UFD if and only if $p \leq 19$, p prime.

Preview: From Kummer’s idea we obtain a better criterion for p called **regular**, which ensures that Fermat’s conjecture holds for p .

Conjecture. *There are infinitely many regular primes.*

2 Ring of integers

In this chapter, all rings are assumed to be commutative with 1.

2.1 Integral ring extensions

Definition 2.1.1 (“ganze Ringerweiterungen”). Let $A \subset B$ be a ring extension.

- (i) $b \in B$ is **integral** over A if there exists a monic polynomial $f(X) = X^n + a_{n-1}X^{n-1} + \dots + a_0 \in A[X]$ with $f(b) = 0$.
- (ii) B is **integral** over A if all $b \in B$ are integral over A .

Proposition 2.1.2. Let $A \subset B$ be a ring extension and $b_1, \dots, b_n \in B$. Then b_1, \dots, b_n are integral over A if and only if

$$A[b_1, \dots, b_n] = \{f(b_1, \dots, b_n) \mid f \in A[X_1, \dots, X_n]\}$$

is a finitely generated A -module.

Reminder 2.1.3 (“Adjunkte”). Let R be a ring and $A \in R^{n \times n}$

- (i) $A^\# = (a_{i,j}^\#)$ with $a_{i,j}^\# = (-1)^{i+j} \det(A_{j,i})$, where $A_{j,i}$ is obtained from A by deleting the j -th row and i -th column of A .
- (ii) We have $AA^\# = A^\#A = \det(A)I$. In particular, $Ax = 0$ implies $A^\#Ax = 0$ such that $\det(A)x = 0$.

Proof of Proposition 1.2. “ \Rightarrow ” If $n = 1$ and b is integral over A , then there is an $f \in A[X]$ with f monic such that $f(b) = 0$. Let $g \in A[X]$ be arbitrary. Then

$$g(X) = q(X)f(X) + r(X)$$

with $q, r \in A[X]$ and $\deg r < \deg f = d$. Hence $g(b) = r(b)$ with $\deg r < d$. Thus $\{1, b, \dots, b^{d-1}\}$ generate $A[b]$ as an A -module. The case $n \geq 2$ follows by induction.

“ \Leftarrow ” $A[b_1, \dots, b_n]$ is finitely generated as an A -module by w_1, \dots, w_r . If $b \in A[b_1, \dots, b_n]$ then

$$bw_i = \sum_{j=1}^r a_{j,i} w_j$$

such that

$$(bI - (a_{i,j}))w = 0.$$

Thus, $\det(bI - (a_{i,j}))w = 0$ and hence

$$\det(bI - (a_{i,j}))w_i = 0$$

for all $i = 1, \dots, r$. If we now use that

$$1 = c_1w_1 + \dots + c_rw_r$$

we can infer $\det(bI - (a_{i,j}))1 = 0$. Consider

$$M = bI - (a_{i,j}) = \begin{pmatrix} b - a_{1,1} & -a_{1,2} & \cdots & -a_{1,r} \\ -a_{2,1} & b - a_{2,2} & \cdots & -a_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{r,1} & -a_{r,2} & \cdots & b - a_{r,r} \end{pmatrix}.$$

By the Leibniz formula we have

$$\det(M) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n m_{\sigma(i),i}$$

which is a polynomial over b with leading coefficient 1. Hence b is integral over A . \square

Corollary 2.1.4 (And Definition). *(i) If $A \subset B$ is an extension of rings then*

$$\overline{A} = \{b \in B \mid b \text{ is integral over } A\}$$

*is a ring. It is called the **integral closure** of A in B . If $\overline{A} = A$ then A is called **integrally closed** in B .*

(ii) We have transitivity, that is to say, if A, B, C are rings with $A \subset B \subset C$ such that C is integral over B and B is integral over A then C is integral over A .

(iii) The integral closure of A in B is integrally closed, i.e., $\overline{\overline{A}} = \overline{A}$.

Proof. “(i)” If $b_1, b_2 \in \overline{A}$ then $A[b_1], A[b_2]$ are finitely generated A -modules. Hence $A[b_1, b_2]$ is a finitely generated A -module. Thus, by Proposition 1.3, $b_1 + b_2$ and b_1b_2 are integral, i.e., elements of \overline{A} .

“(ii)” If $c \in C$ then c is integral over B and hence there is a monic polynomial $f = X^n + b_{n-1}X^{n-1} + \dots + b_0 \in B[X]$ with $f(c) = 0$. This shows that c is integral over $R = A[b_1, \dots, b_{n-1}]$ such that Proposition 1.3 shows that $R[c]$ is a finitely generated R -module. Furthermore, b_0, \dots, b_{n-1} are integral over A such that another application of Proposition 1.3 shows that R is a finitely generated A -module. Hence, $R[c]$ is a finitely generated A module such that c is integral over A by Proposition 1.3.

“(iii)” Follows from (ii). \square

Definition 2.1.5 (“ganzer Abschluss und normaler Ring”). If A is an integral domain we call its integral closure \overline{A} in $K = \text{Quot}(A)$ the **normalization** or the **integral closure** of A . We say A is **integrally closed** if A is integrally closed in K .

Remark 2.1.6. If A is a UFD then A is integrally closed.

Proof. Suppose $b = \frac{a}{a'} \in \text{Quot}(A)$ with $\gcd(a, a') = 1$ is integral over A . Then there exist $a_0, \dots, a_{n-1} \in A$ with

$$\left(\frac{a}{a'}\right)^n + a_{n-1} \left(\frac{a}{a'}\right)^{n-1} + a_{n-2} \left(\frac{a}{a'}\right)^{n-2} + \dots + a_0 = 0$$

such that

$$a^n + a_{n-1}a'a^{n-1} + a_{n-2}a'^2a^{n-2} + \dots + a_0a'^n = 0.$$

Let $a' = \varepsilon\pi_1 \cdots \pi_r$ be the prime factorization of a' with $\varepsilon \in A^\times$ and π_1, \dots, π_r primes. Since $\pi_i | a'$ the above equation shows that actually $\pi_i | a^n$. But this implies $\pi_i | a$ which is a contradiction to $\gcd(a, a') = 1$. Hence we have $a' = \varepsilon \in A^\times$ such that $b \in A$. \square

2.2 Integral closures in field extensions

Setting:

- A is an integral domain.
- A is integrally closed.
- $K = \text{Quot}(A)$.
- L/K is a finite field extension with $\overline{A}_K = A \subset K = \text{Quot}(A) \hookrightarrow L \supset B = \overline{A}_L$.
- B is the integral closure of A in L . Observe: $B \cap K = A$

Remark 2.2.1. (i) B is integrally closed in L .

(ii) If $\beta \in L$ then there are $b \in B$ and $a \in A \setminus \{0\}$ such that $\beta = \frac{b}{a}$.

In particular, $L = \text{Quot}(B)$.

(iii) For $\beta \in L$ we have $\beta \in B$ if and only if $f_\beta \in A[X]$, where f_β is the minimal polynomial of β over K .

Proof. “(i)” Follows from the transitivity in Corollary 1.4.

“(ii)” Choose $a \in A$ with $a^n f_\beta(X) = a^n X^n + a^{n-1} c_{n-1} X^{n-1} + \dots + c_0 \in A[X]$. Then we have

$$a^n \beta^n + c_{n-1} a^{n-1} \beta^{n-1} + \dots + c_0 = 0$$

and hence

$$(a\beta)^n + c_{n-1} (a\beta)^{n-1} + \dots + c_0 = 0$$

such that $a\beta$ is integral over A . Consequently, $b = a\beta \in B$ and $\beta = \frac{b}{a}$.

“(iii)” “ \Leftarrow ” Obvious. “ \Rightarrow ” Let β be a zero of $g(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in A[X]$. Then f_β divides g . If β_1, \dots, β_n are the zeros of f_β in \overline{K} then they are also zeros of g and thus integral over A . Hence the coefficients of f_β are integral over A and are elements of K such that $f_\beta \in A[X]$ as claimed. \square

Reminder 2.2.2 (Trace, Norm). Let $K \subseteq L$ be a finite field extension. For α in L consider the map $T_\alpha : \beta \mapsto \alpha\beta$. The following holds

i) $\text{Tr}_{L/K}(\alpha) = \text{Tr}(T_\alpha)$ and $\mathcal{N}_{L/K}(\alpha) = \det(T_\alpha)$,

ii) If $L = K(\alpha)$ and $f_\alpha(X) = X^n + \sum_{i=0}^{n-1} a_i X^i$ then

$$\text{Tr}_{L/K}(\alpha) = -a_{n-1} \text{ and } \mathcal{N}_{L/K}(\alpha) = (-1)^n \cdot a_0,$$

iii) Since $T_{\alpha+\beta} = T_\alpha + T_\beta$ and $T_{\alpha\beta} = T_\alpha \circ T_\beta$, we conclude that

$$\text{Tr}_{L/K} : (L, +) \rightarrow (K, +) \text{ and } \mathcal{N}_{L/K} : (L^*, \cdot) \rightarrow (K^*, \cdot)$$

are group homomorphisms,

iv) Suppose $K \subseteq L$ is a separable field extension with $L = K(\alpha)$. Further assume $\text{Hom}_K(L, \overline{K}) = \{\sigma_1, \dots, \sigma_n\}$. Then the following holds

- $f_\alpha = \prod_{i=1}^n (X - \sigma_i(\alpha))$,
- $\text{Tr}_{L/K}(\alpha) = \sum_{i=1}^n \sigma_i(\alpha)$,
- $\mathcal{N}_{L/K}(\alpha) = \prod_{i=1}^n \sigma_i(\alpha)$,

v) Trace and norm are transitive, i.e., for field extensions $K \subseteq L \subseteq M$ it holds

- $\mathcal{N}_{L/K} \circ \mathcal{N}_{M/L} = \mathcal{N}_{M/K}$,
- $\text{Tr}_{L/K} \circ \text{Tr}_{M/L} = \text{Tr}_{M/K}$.

Definition 2.2.3 (Discriminant). Let $K \subseteq L$ be a separable field extension and let $\alpha_1, \dots, \alpha_n$ be a K -basis of L . Further let $\text{Hom}_K(L, \overline{K}) = \{\sigma_1, \dots, \sigma_n\}$. Consider the matrix

$$A := \begin{pmatrix} \sigma_1(\alpha_1) & \sigma_1(\alpha_2) & \cdots & \sigma_1(\alpha_n) \\ \sigma_2(\alpha_1) & \sigma_2(\alpha_2) & \cdots & \sigma_2(\alpha_n) \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_n(\alpha_1) & \sigma_n(\alpha_2) & \cdots & \sigma_n(\alpha_n) \end{pmatrix} = (\sigma_i(\alpha_j))_{i,j} \in L^{n \times n}.$$

We call $d(\alpha_1, \dots, \alpha_n) := \det(A^2)$ the **discriminant** of L over K with respect to the basis $\alpha_1, \dots, \alpha_n$.

Remark 2.2.4. In the situation of Definition (2.2.3) the following holds.

- i) Consider the matrix $B = (\text{Tr}_{L/K}(\alpha_i \alpha_j))_{i,j}$ in $K^{n \times n}$. Then the discriminant is given by $d(\alpha_1, \dots, \alpha_n) = \det(B)$. In particular, the discriminant $d(\alpha_1, \dots, \alpha_n)$ lies in K .
- ii) Suppose we have Θ in L such that $1, \Theta, \dots, \Theta^{n-1}$ forms a basis of L . Then the following equality holds

$$d(1, \Theta, \dots, \Theta^{n-1}) = \prod_{1 \leq i < j \leq n} (\Theta_i - \Theta_j)^2.$$

Here Θ_i denotes $\sigma_i(\Theta)$. If $L = K(\Theta)$ then $d(1, \Theta, \dots, \Theta^{n-1})$ coincides with the discriminant of the minimal polynomial f_Θ . Note that we use the notion of discriminants for polynomials here.

Proof. We begin by proving statement i). One computes

$$\det(A)^2 = \det(A^t) \cdot \det(A) = \det(A^t \cdot A).$$

The following calculation proves the claim

$$\begin{aligned} A^t \cdot A &= (\sigma_j(\alpha_i))_{i,j} \cdot (\sigma_k(\alpha_\ell))_{k,\ell} \\ &= \left(\sum_{j=1}^n \sigma_j(\alpha_i) \cdot \sigma_j(\alpha_\ell) \right)_{i,\ell} \\ &= \left(\sum_{j=1}^n \sigma_j(\alpha_i \cdot \alpha_\ell) \right)_{i,\ell} \\ &= (\text{Tr}_{L/K}(\alpha_i \cdot \alpha_\ell))_{i,\ell} \\ &= B. \end{aligned}$$

For statement ii), we will compute the determinant of the following Vandermonde matrix

$$\det(A) = \det \begin{pmatrix} 1 & \Theta_1 & \dots & \Theta_1^{n-1} \\ 1 & \Theta_2 & \dots & \Theta_2^{n-1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & \Theta_n & \dots & \Theta_n^{n-1} \end{pmatrix} =: V_n(\Theta_1, \dots, \Theta_n).$$

By induction, we prove that $V_n(\Theta_1, \dots, \Theta_n)$ is nonzero and that the following equality holds

$$V_n(\Theta_1, \dots, \Theta_n) = \prod_{1 \leq i < j \leq n} (\Theta_j - \Theta_i).$$

For $n = 2$, we have

$$\det(A) = \det \begin{pmatrix} 1 & \Theta_1 \\ 1 & \Theta_2 \end{pmatrix} = \Theta_2 - \Theta_1 \neq 0.$$

Hence the claim holds for $n = 2$. Now we assume that the claim holds for a $n \in \mathbb{N}_{\geq 2}$. We want to prove that viewed as polynomials in Z the following equality holds

$$V_{n+1}(\Theta_1, \dots, \Theta_n, Z) = V_n(\Theta_1, \dots, \Theta_n) \cdot \prod_{i=1}^n (Z - \Theta_i). \quad (2.1)$$

This implies that

$$V_n(\Theta_1, \dots, \Theta_{n+1}) = V_n(\Theta_1, \dots, \Theta_n) \cdot \prod_{i=1}^n (\Theta_{n+1} - \Theta_i) = \prod_{1 \leq i < j \leq n} (\Theta_j - \Theta_i).$$

To show equality (2.1), recall that

$$V_{n+1}(\Theta_1, \dots, \Theta_n, Z) = \det \begin{pmatrix} 1 & \Theta_1 & \dots & \Theta_1^n \\ 1 & \Theta_2 & \dots & \Theta_2^n \\ \vdots & \vdots & \dots & \vdots \\ 1 & \Theta_n(\alpha_2) & \dots & \Theta_n^n \\ 1 & Z & \dots & Z^n \end{pmatrix}.$$

One sees that the polynomials on both sides of equality (2.1) have degree n . Moreover, $\{\Theta_1, \dots, \Theta_n\}$ is the set of zeros for both polynomials. Since the leading coefficient in both cases is $V_n(\Theta_1, \dots, \Theta_n)$, the polynomials are equal. This proves the claim. \square

Example 2.2.5. Consider $L = \mathbb{Q}(\sqrt{D})$ for a square free integer D different from 0 and 1. Then the following holds

- $\mathfrak{B}_1 = \{1, \sqrt{D}\}$ is a \mathbb{Q} -basis of L .
- Define $\sigma_2 : L \rightarrow \overline{\mathbb{Q}}, a + b\sqrt{D} \mapsto a - b\sqrt{D}$. Then we have

$$\text{Hom}_{\mathbb{Q}}(L, \overline{\mathbb{Q}}) = \{\sigma_1 = \text{id}, \sigma_2\}.$$

- $\text{Tr}_{L/\mathbb{Q}}(a + b\sqrt{D}) = a + b\sqrt{D} + a - b\sqrt{D} = 2a$.
- $\mathcal{N}_{L/\mathbb{Q}}(a + b\sqrt{D}) = (a + b\sqrt{D}) \cdot (a - b\sqrt{D}) = a^2 - b^2 \cdot D$.
- $d(\mathfrak{B}_1) = \det \begin{pmatrix} 1 & \sqrt{D} \\ 1 & -\sqrt{D} \end{pmatrix}^2 = (-2\sqrt{D})^2 = 4D$.
- We have

$$(\alpha_i \alpha_j)_{i,j} = \begin{pmatrix} 1 & \sqrt{D} \\ \sqrt{D} & D \end{pmatrix}.$$

Hence we compute

$$\det((\text{Tr}(\alpha_i \alpha_j))_{i,j}) = \det \begin{pmatrix} 2 & 0 \\ 0 & 2D \end{pmatrix} = 4D.$$

- Consider the \mathbb{Q} -basis of L given by $\mathfrak{B}_2 = \{1 + \sqrt{D}, 1 - \sqrt{D}\}$. Computing the discriminant for this basis yields

$$d(1 + \sqrt{D}, 1 - \sqrt{D}) = \det \begin{pmatrix} 1 + \sqrt{D} & 1 - \sqrt{D} \\ 1 - \sqrt{D} & 1 + \sqrt{D} \end{pmatrix}^2 = 16D.$$

Hence we see that the discriminant depends on the basis we choose.

Proposition 2.2.6. *Let $K \subseteq L$ be a separable field extension.*

i) *The bilinear map*

$$h : L^2 \rightarrow K, (x, y) \mapsto \text{Tr}_{L/K}(xy)$$

is non degenerate, i.e., $h(x, y) = 0$ for all $y \in L$ implies that $x = 0$.

ii) *If $\alpha_1, \dots, \alpha_n$ forms a basis of L/K then $d(\alpha_1, \dots, \alpha_n) \neq 0$.*

Proof. For statement i), we choose a primitive element Θ . Then $1, \Theta, \dots, \Theta^{n-1}$ is a K -basis of L . Let B be the matrix representation of h with respect to this basis. We find

$$\begin{aligned} \det(B) &\stackrel{(2.4) \text{ i)}}{=} d(1, \Theta, \dots, \Theta^{n-1}) \\ &\stackrel{(2.4) \text{ ii)}}{=} \prod_{1 \leq i < j \leq n} (\Theta_i - \Theta_j)^2 \neq 0. \end{aligned}$$

Here Θ_i denotes $\sigma_i(\Theta)$. This shows that h is non degenerate. We now prove statement ii). Observe that the matrix $M = (\text{Tr}_{L/K}(\alpha_i \alpha_j))_{i,j}$ is the matrix representation of h with respect to $\alpha_1, \dots, \alpha_n$. By Remark (2.4), we conclude

$$d(\alpha_1, \dots, \alpha_n) = \det(M).$$

Now, i) implies that $\det(M)$ is nonzero. □

Remark 2.2.7. Let $A \subseteq B$ be an integral ring extension with $B \subseteq L$ and $A = B \cap K \subseteq K$. Assuming that $\text{Hom}_K(L, \overline{K}) = \{\text{id} = \sigma_1, \dots, \sigma_n\}$ the following holds

- i) If $x \in B$ then $\sigma_i(x) \in B$ for all $1 \leq i \leq n$.
- ii) For all $x \in B$ the trace $\text{Tr}_{L/K}(x)$ and the norm $\mathcal{N}_{L/K}(x)$ lie in A .
- iii) Let $x \in B$. Then x lies in B^* if and only if the norm $\mathcal{N}_{L/K}(x)$ lie in A^* .

Proof. We start by proving i). Let x in B . By Remark (2.1), we have that the minimal polynomial f_x lies in $A[X]$. Since $\sigma(x)$ is also a zero of f_x , it is contained in B . This shows i). Now, statement ii) follows from i), Remark (2.2) iv) and the fact that $A = B \cap K$. For iii), assume that x is a unit in B , i.e., we find y in B with $xy = 1$. Hence

$$\mathcal{N}_{L/K}(x) \cdot \mathcal{N}_{L/K}(y) = \mathcal{N}_{L/K}(xy) = 1.$$

Using ii), we deduce that $\mathcal{N}_{L|K}(x)$ lies in A^* . This proves one direction. For the other direction, assume that $\mathcal{N}_{L|K}(x)$ lies in A^* , i.e., we find $a \in A$ with

$$\begin{aligned} 1 &= a \cdot \mathcal{N}_{L|K}(x) \\ &= a \cdot \prod_{i=1}^n \sigma_i(x) \\ &= a \cdot x \cdot \underbrace{\prod_{i=2}^n \sigma_i(x)}_{\in B, \text{ by i)}}. \end{aligned}$$

Hence x lies in B^* . This proves iii). \square

Proposition 2.2.8. Suppose $\alpha_1, \dots, \alpha_n \in B$ forms a K -basis of L . Let d denote the discriminant $d(\alpha_1, \dots, \alpha_n) \in A$. Then $d \cdot B$ is contained in $A\alpha_1 + \dots + A\alpha_n$.

Proof. Suppose $\alpha = \sum_{j=1}^n c_j \alpha_j \in B$ for $c_i \in K$. We want to solve for (c_1, \dots, c_n) . Applying the trace to the equalities

$$\alpha_i \alpha = \sum_{j=1}^n c_j \alpha_i \alpha_j, \quad 1 \leq i \leq n,$$

we obtain

$$\text{Tr}_{L/K}(\alpha_i \alpha) = \sum_{j=1}^n c_j \text{Tr}_{L/K}(\alpha_i \alpha_j), \quad 1 \leq i \leq n.$$

Hence $x = (c_1, \dots, c_n)$ is the solution of the linear system $Mx = y$, where

$$M = ((\text{Tr}_{L/K}(\alpha_i \alpha_j)))_{i,j} \in A^{n \times n}, \quad y = (\text{Tr}_{L/K}(\alpha_i \alpha))_i \in A^n.$$

By Remark (1.3), we have

$$\det(M) \cdot x = M^\# Mx = M^\# y \in A^n.$$

Using Remark (2.4), we know $\det(M) = d(\alpha_1, \dots, \alpha_n) =: d$. We conclude that dc_i lies in A for $1 \leq i \leq n$, which proves the claim. \square

Definition 2.2.9 (Ganzheitsbasis). Suppose $\omega_1, \dots, \omega_n \in B$ forms a basis of B over A , i.e., every $\alpha \in B$ can be written in a unique way as an A -linear combination $\sum_{i=1}^n c_i \omega_i$. Then $\omega_1, \dots, \omega_n$ is called an **integral basis** of B over A .

Example 2.2.10. Same situation as in Ex. 2.5. $\mathcal{B}_1 = \{1, \sqrt{D}\} \subseteq B$. Consider:

$$\begin{aligned} \alpha &= \frac{1}{2}(1 + \sqrt{D}) \Rightarrow 2\alpha = 1 + \sqrt{D} \\ \Rightarrow (2\alpha - 1)^2 &= D \Rightarrow 4\alpha^2 - 4\alpha + 1 = D \\ \Rightarrow f_\alpha(X) &= X^2 - X + \frac{1-D}{4} \end{aligned}$$

Hence if $D \equiv 1 \pmod{4} \Rightarrow \alpha \in B$ and \mathcal{B}_1 is not an integral basis.

Proposition 2.2.11. *Let $D \in \mathbb{Z}$, D square-free, $D \neq 0, 1$, $B :=$ integral closure of \mathbb{Z} in $\mathbb{Q}(\sqrt{D}) = L$.*

- i) $D \equiv 2, 3 \pmod{4} \Rightarrow \{1, \sqrt{D}\}$ is an integral basis of B/\mathbb{Z} in particular $B = \mathbb{Z}[\sqrt{D}]$.
- ii) $D \equiv 1 \pmod{4} \Rightarrow \{1, \frac{1}{2}(\sqrt{D}+1)\}$ is an integral basis of B/\mathbb{Z} . and $B = \mathbb{Z}[\frac{1}{2}(1+\sqrt{D})]$.

Proof. Consider $\alpha = a + b\sqrt{D} \in \mathbb{Q}(\sqrt{D})$ with $a, b \in \mathbb{Q}$.

$$\Rightarrow f_\alpha = X^2 - 2aX + a^2 - b^2D.$$

Rem 2.1: $\alpha \in B \iff f_\alpha \in \mathbb{Z}[X] \iff 2a \in \mathbb{Z} \text{ and } a^2 - b^2D \in \mathbb{Z}$.

- (1) Show: $\alpha \in B \Rightarrow 2b \in \mathbb{Z}$.

$$\alpha \in B \Rightarrow 4a^2 - 4b^2D = 4z \text{ with } z \in \mathbb{Z}. \text{ Write } b = \frac{p}{q} \text{ with } p, q \in \mathbb{Z}, \gcd(p, q) = 1$$

$$\Rightarrow 4p^2D = ((2a)^2 - 4z)q^2 \quad (\star)$$

$$\Rightarrow q = 1 \text{ or } 2.$$

- (2) Show: $q = 2 \Rightarrow D \equiv 1 \pmod{4}$

$$(\star) \Rightarrow p^2D = (2a)^2 - 4z \equiv (2a)^2 \pmod{4}$$

$$p \text{ is odd, hence } p^2 \equiv 1 \pmod{4} \Rightarrow (2a)^2 \text{ is odd (i.e. } a = \frac{2n-1}{2} \in \mathbb{Q})$$

$$\Rightarrow (2a)^2 \equiv 1 \pmod{4} \Rightarrow D \equiv 1 \pmod{4}.$$

- (3) It follows from (2) if $D \equiv 1 \pmod{4}$:

$$\alpha \in B \iff \alpha = a + b\sqrt{D} \text{ or } \alpha = \frac{1}{2}(a + b\sqrt{D}) \text{ with } a, b \in \mathbb{Z}. \text{ Hence we obtain:}$$

$$B = \begin{cases} \mathbb{Z}[\sqrt{D}] & , \text{ if } D \equiv 2, 3 \pmod{4} \\ \mathbb{Z}[\frac{1}{2}(1 + \sqrt{D})] & , \text{ if } D \equiv 1 \pmod{4} \end{cases}$$

For the second case observe that $\frac{a}{2} + \frac{b}{2}\sqrt{D} = \frac{a-b}{2} + \frac{b}{2}(1 + \sqrt{D}) \in \mathbb{Z}[\frac{1}{2}(1 + \sqrt{D})]$.

This implies the claim. □

Proposition 2.2.12. *Suppose L/K separable and A is a principal ideal domain. Let $M \neq 0$ be a finitely generated B -submodule of $L \Rightarrow M$ is a free A -module. In particular: B is a free A -module of rank $n := [L : K]$.*

Reminder 2.2.13. Suppose A is a principal ideal domain and M_0 is a finitely generated free A -module.

- i) Any submodule M of M_0 is free.

- ii) $\text{rank}(M_0) \geq \text{rank}(M)$

of Prop 2.12. Let $\mu_1, \dots, \mu_r \in M \subseteq L$ be generators of M as B -module and let $\alpha_1, \dots, \alpha_n$ be a basis of L/K in B and $d := d(\alpha_1, \dots, \alpha_n) \in A$.

Recall: $L = \{\frac{b}{a} \mid b \in B, a \in A \setminus \{0\}\}$.

- (1) Prop 2.7 $\Rightarrow dB \subseteq A\alpha_1 + \dots + A\alpha_n$

$$(2) \exists a \in A : a\mu_1, \dots, a\mu_r \in B$$

Hence: $daM \subseteq dB \subseteq A\alpha_1 + \dots + A\alpha_n =: M_0$

(M_0 is a free A -module, since $\alpha_1, \dots, \alpha_n$ are basis of L/K).

Reminder 2.13 $\Rightarrow adM$ is a free A -module $\Rightarrow M$ is a free A -module.

Furthermore: $\text{rank}(M) = \text{rank}(adM) \stackrel{\text{Rem. 2.13}}{\leq} \text{rank}(M_0) = n$.

Suppose that $M = B$. So far we got that B is a free A -module and $\text{rank}(B) \leq n$.

Show: $\text{rank}(B) \geq n$.

Let μ_1, \dots, μ_r be a basis of B as A -module. By $L = \{\frac{b}{a} \mid b \in B, a \in A \setminus \{0\}\}$ we have that μ_1, \dots, μ_r generate L over K . \square

Hence: if A is a principal ideal domain, then B has always an integral basis.

Proposition 2.2.14. *Suppose we are in the following situation:*

- L/K and L'/K are Galois extensions of degree n and m in some field E
- A a subring of K such that $K = \text{Quot}(A)$ and B and B' are the integral closures of A in L and L' .
- $\{\omega_1, \dots, \omega_n\}$ and $\{\omega'_1, \dots, \omega'_m\}$ are integral basis for B/A and B'/A .
- $d := d(\omega_1, \dots, \omega_n)$ and $d' := d(\omega'_1, \dots, \omega'_m) \in A$ with d and d' are coprime in A , i.e. $\exists x, x' \in A$ with $1 = dx + d'x'$.
- $K = L \cap L'$

Then we have: $\{\omega_i \omega'_j \mid i \in \{1, \dots, n\}, j \in \{1, \dots, m\}\}$ is an integral basis and its discriminant is $d^m (d')^n$.

Proof. Recall: $L \cap L' = K \Rightarrow [LL' : K] = nm$ and $\{\omega_i \omega'_j\}$ is a basis of the field extension LL'/K .

$\text{Gal}(L/K) = \{\sigma_1, \dots, \sigma_n\}$ and $\text{Gal}(L'/K) = \{\sigma'_1, \dots, \sigma'_m\}$

\Rightarrow obtain unique lifts $\hat{\sigma}_i \in \text{Gal}(LL'/L')$ and $\hat{\sigma}'_j \in \text{Gal}(LL'/L)$ and $\text{Gal}(LL'/K) = \{\hat{\sigma}_i \hat{\sigma}'_j \mid i \in \{1, \dots, n\}, j \in \{1, \dots, m\}\}$.

Consider: $\alpha \in \tilde{B} :=$ integral closure of A in LL' .

Write $\alpha = \sum_{i,j} \alpha_{i,j} \omega_i \omega'_j = \sum_j \beta_j \omega'_j$ with $\alpha_{i,j} \in K$ and $\beta_j = \sum_i \alpha_{i,j} \omega_i \in L$.

$\Rightarrow \hat{\sigma}'_i(\alpha) = \sum_j \beta_j \hat{\sigma}'_i(\omega'_j)$, since $\hat{\sigma}'_i \in \text{Gal}(LL'/L)$.

\Rightarrow We have a linear system:

$$a = Tb \text{ with } a = \begin{pmatrix} \hat{\sigma}'_1(\alpha) \\ \vdots \\ \hat{\sigma}'_m(\alpha) \end{pmatrix} \in \tilde{B}^m, \quad b = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} \in L^m, \quad T = (\hat{\sigma}'_i(\omega'_j))_{(i,j)} \in \tilde{B}^{m \times m}$$

Observe: $\det(T)^2 = d'$

$$\begin{aligned}
 &\Rightarrow \det(T)b = T^\# T b = T^\# a \in \tilde{B}^m && \Rightarrow d'b \in \tilde{B}^m \\
 &\Rightarrow \forall j : d'\beta_j = \sum_i d'\alpha_{i,j}\omega_i \in \tilde{B} \cap L = B \\
 &\Rightarrow d'\alpha_{i,j} \in A, \text{ since } \{\omega_1, \dots, \omega_n\} \text{ is an integral basis.} \\
 &\Rightarrow d\alpha_{i,j} \in A \text{ in the same way} \\
 &\Rightarrow \alpha_{i,j} = (x'd' + xd)\alpha_{i,j} = x'd'\alpha_{i,j} + xd\alpha_{i,j} \in A.
 \end{aligned}$$

Hence: $\{\omega_i\omega'_j \mid (i,j) \in \{(1,1), \dots, (n,m)\}\}$ is an integral basis of \tilde{B}/A .

For calculating the discriminant consider the matrix $M = (\hat{\sigma}_k \circ \hat{\sigma}'_l(\omega_i\omega'_j))_{(k,l),(i,j)} = (\hat{\sigma}_k(\omega_i)\hat{\sigma}'_l(\omega'_j))$.

Consider $Q = (\hat{\sigma}_k(\omega_i))$

$$\Rightarrow M = \begin{pmatrix} Q & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & Q \end{pmatrix} \cdot \begin{pmatrix} I \cdot \hat{\sigma}'_1(\omega'_1) & \dots & I \cdot \hat{\sigma}'_1(\omega'_1) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ I \cdot \hat{\sigma}'_m(\omega'_m) & \dots & I \cdot \hat{\sigma}'_m(\omega'_m) \end{pmatrix}$$

Observe:

$$(1) \det(Q)^2 = d(\omega_1, \omega_n) = d$$

$$(2) \text{ The second matrix can be transformed by switching rows and columns to } \begin{pmatrix} Q' & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & Q' \end{pmatrix}$$

with $Q' = (\sigma'_l(\omega'_j))$ and $\det(Q') = d'$

$$\Rightarrow \det(M)^2 = \det(Q)^{2m} \cdot \det(Q')^{2n} = d^m d'^n. \quad \square$$

Remark 2.2.15 (and Definition). Suppose $K = \mathbb{Q}$, $A = \mathbb{Z}$, L a number field and $B = \mathcal{O}_k$.

(i) There is always an integral basis w_1, \dots, w_n .

(ii) The **discriminant** $d_k = d_k(\mathcal{O}_k) = d(w_1, \dots, w_n)$ does not depend on the choice of integral basis.

Proof. “(i)” Proposition 2.12 “(ii)” Let w'_1, \dots, w'_n be another integral basis. Then there exists a base change matrix $T \in \text{GL}_n(\mathbb{Z})$ with

$$\begin{pmatrix} w'_1 \\ \vdots \\ w'_n \end{pmatrix} = T \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}.$$

Hence

$$\begin{pmatrix} \sigma(w'_1) \\ \vdots \\ \sigma(w'_n) \end{pmatrix} = T \begin{pmatrix} \sigma(w_1) \\ \vdots \\ \sigma(w_n) \end{pmatrix}.$$

such that

$$d(w'_1, \dots, w'_n) = \underbrace{\det T}_{\in \{1, -1\}}^2 d(w_1, \dots, w_n) = d_k.$$

□

Example 2.2.16. Let $L = \mathbb{Q}(\sqrt{D})$ with $D \in \mathbb{Z}$ square-free. By Proposition 2.14 we have:

- (i) $\mathcal{O}_k = \mathbb{Z}[\sqrt{D}]$ and $\{1, \sqrt{D}\}$ is an integral basis for $D \equiv 2, 3 \pmod{4}$ and $d_k = 4D$.
- (ii) $\mathcal{O}_k = \mathbb{Z}\left[\frac{1+\sqrt{D}}{2}\right]$ and $\left\{1, \frac{1+\sqrt{D}}{2}\right\}$ is an integral basis for $D \equiv 1 \pmod{4}$ and $d_k = D$.

In particular, this holds for $D = -1$, i.e., the Gaussian integers $\mathbb{Z}[i]$.

2.3 Ideals

Let R be a commutative ring with 1.

Problem: \mathcal{O}_k is not a UFD in many cases, e.g. in $\mathbb{Z}[\sqrt{-5}]$ we have

$$(1 + \sqrt{-5})(1 - \sqrt{-5}) = 1 + 5 = 6 = 2 \cdot 3,$$

that is, two different ways to factor 6 in irreducible elements.

Idea:

- (1) Maybe we have too few elements, i.e.,

$$1 + \sqrt{-5} = p_1 p_2, 1 - \sqrt{-5} = p_3 p_4 \text{ and } 2 = p_2 p_3, 3 = p_1 p_4$$

for some primes p_i .

- (2) An element is determined (up to units) by the set of elements it divides, e.g.

$$p_i \longleftrightarrow \{x \in \mathcal{O}_k; p_i | x\} = p_i \mathcal{O}_k \text{ (this is an ideal)}.$$

Notation 2.3.1. Let $I, J \subset R$ be ideals. We define

- $I + J = \{a + b; a \in I, b \in J\}$,
- $IJ = \{\sum_i a_i b_i; a_i \in I, b_i \in J\}$.

Definition 2.3.2 (and Reminder). Let $I \subsetneq R$ be an ideal.

- (a) I is called **prime** if for all $a, b \in R$ with $ab \in I$ we already have $a \in I$ or $b \in I$.
 \Leftrightarrow For all ideals $A, B \subset R$ with $AB \subset I$ we have $A \subset I$ or $B \subset I$.
- (b) I is called **maximal** if for any ideal $I \subset J \subset R$ we have $J = I$ or $J = R$.
 $\Leftrightarrow R/I$ is a field.
- (c) R is called **Noetherian** if every ascending chain of ideals

$$I_1 \subset I_2 \subset \dots$$

becomes stationary, i.e., if there is an $N \in \mathbb{N}$ such that $I_n = I_N$ for all $n \geq N$.

\Leftrightarrow Every ideal in R is finitely generated.

- (d) R is called a **Dedekind domain** if
- R is an integral domain,
 - R is integrally closed,
 - R is Noetherian, and
 - every prime ideal in R is maximal.

Proposition 2.3.3. *If \mathcal{O} is the integral closure of \mathbb{Z} in a number field then \mathcal{O} is a Dedekind domain.*

Proof. It is clear that \mathcal{O} is an integral domain and integrally closed. Furthermore, by Proposition 2.12 each \mathbb{Z} -submodule is finitely generated as a \mathbb{Z} -module, thus also as an \mathcal{O} -module. Hence \mathcal{O} is Noetherian.

Now, let $I \subset \mathcal{O}$ be a prime ideal. Then $I \cap \mathbb{Z} \subset \mathbb{Z}$ is a prime ideal such that $\mathbb{Z}/(I \cap \mathbb{Z}) = \mathbb{F}_p$. Using $\mathcal{O} = \mathbb{Z}[w_1, \dots, w_n]$ we conclude

$$\mathcal{O}/I = \mathbb{Z}/(I \cap \mathbb{Z})[w'_1, \dots, w'_n] = \mathbb{F}_p[w'_1, \dots, w'_n] = \mathbb{F}_p(w'_1, \dots, w'_n),$$

where $w'_i \equiv w_i \pmod{I}$. Thus \mathcal{O}/I is a field and hence I maximal. \square

From now on: Let \mathcal{O} denote a Dedekind domain.

Theorem 4. *Every ideal $0 \neq I \subset \mathcal{O}$ has a unique factorization*

$$I = P_1 \cdots P_n$$

into prime ideals $P_i \subset \mathcal{O}$.

Lemma 2.3.4. *For every ideal $0 \neq I \subset \mathcal{O}$ there exist nonzero prime ideals $P_i \subset \mathcal{O}$ such that*

$$P_1 \cdots P_n \subset I.$$

Proof. Set $M = \{0 \neq I \subset \mathcal{O} \text{ ideal; } I \text{ does not have such } P_i\}$ and suppose $M \neq \emptyset$. Then M is partially ordered by inclusion and since \mathcal{O} is Noetherian, every chain in M has an upper bound. Thus, the Lemma of Zorn yields a maximal element $I_0 \in M$. Since I_0 cannot be prime there are $a, b \in \mathcal{O}$ such that $ab \in I_0$ but $a, b \notin I_0$. Consider the ideals $I_1 = (a) + I_0$ and $I_2 = (b) + I_0$ which satisfy $I_0 \subsetneq I_1$, $I_0 \subsetneq I_2$ and $I_1 I_2 \subset I_0$. Since I_0 is a maximal ideal in M , we have $I_{1,2} \notin M$ hence we find prime ideals $P_1, \dots, P_n, P'_1, \dots, P'_m \subset \mathcal{O}$ with

$$P_1 \dots P_n \subset I_1 \text{ and } P'_1 \dots P'_m \subset I_2.$$

Finally, we conclude $P_1 \dots P_n P'_1 \dots P'_m = I_1 I_2 \subset I_0 \Rightarrow I_0 \notin M \nRightarrow M = \emptyset$. \square

Lemma 2.3.5. Let $0 \neq P \subset \mathcal{O}$ be a prime ideal, $I \subset \mathcal{O}$ an ideal and $K = \text{Quot}(\mathcal{O})$. Then:

$$(i) \ P^{-1} := \{x \in K; xP \subset \mathcal{O}\} \supsetneq \mathcal{O}$$

$$(ii) \ I \subsetneq P^{-1}I := \{\sum_i a_i x_i; a_i \in I, x_i \in P^{-1}\}$$

Proof. “(i)” Let $0 \neq a \in P$, $P_1 \dots P_n \subset (a) \subset P$ as in Lemma 3.5 with n minimal.

Claim: Without loss of generality we can assume that $P_1 = P$.

Proof of the claim: Since $P_1 \dots P_n \subset P$ and P is prime, there is an index i such that $P_i \subset P$, by reindexing we may assume that $i = 1$. However, we assumed \mathcal{O} to be Dedekind, hence P_1 is a maximal ideal in \mathcal{O} . Thus, $P_1 \subset P \subsetneq \mathcal{O}$ implies that $P_1 = P$ as claimed.

Now, since n was chosen minimal we have $P_2 \dots P_n \not\subset (a)$, i.e., there exists an element $b \in (a) \setminus P_2 \dots P_n$. On the one hand we thus have

$$a^{-1}b \notin \mathcal{O}$$

and on the other hand $bP \subset (a)$ such that $a^{-1}bP \subset \mathcal{O}$ and hence

$$a^{-1}b \in P^{-1}.$$

Both of this together shows that $P^{-1} \supsetneq \mathcal{O}$.

“(ii)” Assume there is an ideal $I \subset \mathcal{O}$ such that $P^{-1}I \subset I$. Let $\{\alpha_1, \dots, \alpha_n\} \subset I$ be a generating set and choose $x \in P^{-1} \setminus \mathcal{O}$. Then,

$$x\alpha_i = \sum_j a_{ij}\alpha_j$$

for some $a_{ij} \in \mathcal{O}$. Consider the matrix $A = xE_n - (a_{ij})_{i,j}$, which satisfies

$$A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = 0.$$

Since $A^\# A = \det A$ we conclude $\det A = 0$ such that x is a zero of the monic polynomial $\det(XE_n - (a_{ij})_{i,j})$ over \mathcal{O} . But since \mathcal{O} is integrally closed this implies $x \in \mathcal{O}$, a contradiction. \square

Proof of Theorem 3.4. Existence of a factorization: Let

$$M = \{0 \neq I \subset \mathcal{O} \text{ ideal}; I \text{ has no factorization}\}$$

and assume that $M \neq \emptyset$. As in Lemma 3.5, let $I_0 \in M$ be a maximal element and let $P \supset I_0$ be a maximal ideal containing I_0 . Since I_0 is not prime we have $I_0 \neq P$ such that by Lemma 3.6,

$$I_0 \subsetneq P^{-1}I_0 \subset P^{-1}P = \mathcal{O}.$$

Note that $I_0 = I_0\mathcal{O} = I_0P^{-1}P$ and $I_0 \neq P$ imply $P^{-1}I_0 \subsetneq \mathcal{O}$. Since I_0 was maximal in M we thus have $P^{-1}I_0 \notin M$, i.e., there are prime ideals $P_1, \dots, P_n \subset \mathcal{O}$ with $P^{-1}I = P_1 \cdots P_n$. This leads to the contradiction $I = PP_1 \cdots P_n$.

Uniqueness of the factorization: Suppose that

$$I = P_1 \cdots P_n = Q_1 \cdots Q_m$$

are two prime factorizations. Then $P_1 \supset I = Q_1 \cdots Q_m$, hence without loss of generality we can assume that $Q_1 \subset P_1$. Since \mathcal{O} is Dedekind we conclude $Q_1 = P_1$ such that

$$P_2 \cdots P_n = P_1^{-1}I = Q_2 \cdots Q_m.$$

The claim follows by induction. □

Definition 2.3.6. We call two ideals $0 \neq I, J \subset \mathcal{O}$ **coprime** $:\Leftrightarrow I + J = \mathcal{O}$. For example, one could take two distinct prime ideals in a Dedekind ring.

Remark 2.3.7. Let $P_1, \dots, P_n \subset \mathcal{O}$ be pairwise coprime. Then P_1 and $P_2 \cdots P_n$ are coprime and we have $\prod_{i=1}^n P_i = \bigcap_{i=1}^n P_i$.

Proof. Induction on n : The case $n = 2$ is clear. Let $n > 2$. Since P_1 and P_2 are coprime, $\exists p_1 \in P_1, p_2 \in P_2$, such that we can write $1 = p_1 + p_2$. By induction hypothesis, $\exists p'_1 \in P_1, p \in P_3 \cdots P_n$, such that $1 = p'_1 + p$. It follows

$$1 = p_1 + p_2 \cdot (p'_1 + p) = \underbrace{p_1 + p_2 p'_1}_{\in P_1} + \underbrace{p_2 p}_{\in P_2 \cdots P_n},$$

which yields the first claim.

For the second claim, first note that $\prod P_i \subset \bigcap P_i$ is clear.

For the converse, let $a \in \bigcap P_i$, which of course implies that $a \in P_i$ for all i . As above, we write $1 = p_1 + p$, $p_1 \in P_1, p \in P_2 \cdots P_n$. We get $a = ap_1 + ap$, which implies that $a \in aP_1 + P_1 \cdot \prod_{i=2}^n P_i$ for all i and by induction hypothesis, we get $a \in \prod P_i$. □

Theorem 5 (Chinese Remainder Theorem). *Let $P_1, \dots, P_n \subset \mathcal{O}$ be pairwise coprime ideals, $I = \bigcap_{i=1}^n P_i$. Then we have*

$$\mathcal{O}/I \cong \bigoplus_{i=1}^n \mathcal{O}/P_i$$

Proof. Consider the map

$$\phi : \mathcal{O} \longrightarrow \bigoplus_i \mathcal{O}/P_i, \quad a \mapsto \bigoplus_i a \pmod{P_i}.$$

Obviously, $\ker(\phi) = I$. It remains to show, that ϕ is surjective. Let first $n = 2$: For $p_1 \in P_1, p_2 \in P_2$ let $1 = p_1 + p_2$ and for any $a_1, a_2 \in \mathcal{O}$ write $a = a_2 p_1 + a_1 p_2$. Then $\phi(a) = a_1 \oplus a_2 \in \mathcal{O}/P_1 \oplus \mathcal{O}/P_2$.

In general, by 3.8, we know that $\exists y_i \in \mathcal{O}$ with $y_i \equiv 1 \pmod{P_i}$ and $y_i \equiv 0 \pmod{\bigcap_{j \neq i} P_j}$. Hence the element $a = \sum_{i=1}^n a_i y_i$ is mapped to $\bigoplus_{i=1}^n a_i \pmod{P_i}$ \square

Definition 2.3.8. A **fractional ideal** of K is a finitely generated \mathcal{O} -module $0 \neq I$ of K . Since \mathcal{O} is noetherian, this is equivalent to: $\exists c \in \mathcal{O}$, such that $c \cdot I \subset \mathcal{O}$ is an ideal (since every submodule of \mathcal{O} is finitely generated). The product of two fractional ideals is denoted in the same way as introduced in 3.3. Ideals in \mathcal{O} are called **integral ideals**.

Theorem 6. *The fractional ideals of K , together with the product, form an abelian group, which we denote by \mathcal{J}_K .*

Proof. Commutativity and associativity are clear. The unit in \mathcal{J}_K is given by \mathcal{O} . We define $I^{-1} := \{x \in K \mid x \cdot I \subset \mathcal{O}\}$ and show, that this defines an inverse for all $I \in \mathcal{J}_K$.

For a prime ideal $P \subset \mathcal{O}$, we have already seen in 3.4 that $P^{-1}P = \mathcal{O}$ and for an integral ideal $I = P_1 \cdots P_n$, we have $J = P_1^{-1} \cdots P_n^{-1}$ as an inverse:

$J \subset I^{-1}$ is clear. For the converse, let $x \in I^{-1}$, we then have $x \cdot IJ \subset \mathcal{O}$, with $x \cdot I \subset \mathcal{O}$ and $IJ = \mathcal{O}$, therefore $x \cdot 1 \in J$ and $I^{-1} \subset J$ follows.

Let now I be fractional. Then $\exists c \in \mathcal{O}$, such that cI is integral. But then $(cI)^{-1} = c^{-1}I^{-1}$ and hence $II^{-1} = (cI)(c^{-1}I^{-1}) = \mathcal{O}$ \square

Corollary 2.3.9. *Every fractional ideal I has a unique factorization $I = \prod P_i^{n_i}$, with $n_i \in \mathbb{Z}$, $P_i \subset \mathcal{O}$ distinct prime ideals and only finitely many $n_i \neq 0$. In particular, \mathcal{J}_K is a free abelian group on the prime ideals of \mathcal{O} .*

Proof. By 3.11, every element $I \in \mathcal{J}_K$ can be written as $I = AB^{-1}$ for some integral ideals $A, B \subset \mathcal{O}$. Therefore, by 3.4, we get $I = \prod P_i^{n_i}$ and by multiplying denominators, we see that this presentation is unique. \square

Definition 2.3.10. The principle ideals generate a subgroup \mathcal{P}_K of \mathcal{J}_K . We call the quotient group $\text{Cl}_K := \mathcal{J}_K/\mathcal{P}_K$ the **ideal class group**. We have an exact sequence of groups

$$1 \longrightarrow \mathcal{O}^\times \longrightarrow K^\times \xrightarrow{a \mapsto a\mathcal{O}} \mathcal{J}_K \longrightarrow \text{Cl}_K \longrightarrow 1.$$

2.4 Lattices and Minkowski

Definition 2.4.1. Let V be an n -dimensional \mathbb{R} -vector space. A **lattice** $\Lambda \subset V$ is a subgroup of the form $\mathbb{Z}v_1 + \dots + \mathbb{Z}v_m$, where v_1, \dots, v_m are linearly independent over V . We call (v_1, \dots, v_m) a **basis** of Λ and $\phi := \{x_1v_1 + \dots + x_mv_m \mid x_i \in [0, 1)\}$ a **fundamental domain** of Λ . We call Λ **complete**, if $n = m$.

CAUTION: For many people, lattices are always complete!

Example 2.4.2. (a) $\mathbb{Z} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbb{Z} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \subset \mathbb{R}^2$ is a complete lattice

(b) $\mathbb{Z} + \mathbb{Z}\sqrt{2} \subset \mathbb{R}$ is not a lattice, since 1 and $\sqrt{2}$ are not linearly independent.

(c) $\mathbb{Z} \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \subset \mathbb{R}^2$ is a non-complete lattice.

Proposition 2.4.3. A subgroup $\Lambda \subset V$ is a lattice $\Leftrightarrow \Lambda$ is a discrete subgroup of V .

Proof. " \Rightarrow ": Take $\{\lambda + x_1v_1 + \dots + x_nv_n + \text{rest of basis} \mid |x_n| < 1\}$ as a neighbourhood for $\lambda \in \Lambda$.

" \Leftarrow ": Let $V_0 = \langle \Lambda \rangle_{\mathbb{R}}$. Then we can choose a basis v_1, \dots, v_m of V_0 in Λ , such that $\Lambda_0 := \mathbb{Z}v_1 + \dots + \mathbb{Z}v_m$ is a lattice in V_0 .

Claim: The index $[\Lambda : \Lambda_0]$ is finite.

Proof of the claim: Since Λ_0 is complete, $V = \bigsqcup_{\lambda \in \Lambda_0} \phi_0 + \lambda$. Since Λ is discrete and ϕ_0 bounded, $\Lambda \cap \phi_0$ is finite. Hence we have only finitely many residue classes $\lambda + \Lambda_0$ of Λ and therefore $[\Lambda : \Lambda_0] =: d < \infty$.

From this follows, that $\Lambda \subset \frac{1}{d}\Lambda_0 = \mathbb{Z}(\frac{1}{d}v_1) + \dots + \mathbb{Z}(\frac{1}{d}v_m)$. Therefore, Λ has a \mathbb{Z} -basis $w_1 = v_1n_1, \dots, w_r = v_rn_r$ for some $n_i \in \frac{1}{d}\mathbb{N}$ and since Λ spans V_0 , we get $r = m$ and they are linearly independent. \square

Let $\Gamma = v_1\mathbb{Z} + \dots + v_n\mathbb{Z} \subset \mathbb{R}^n$ be a complete lattice. We define

$$\text{vol } \Gamma = \text{vol } \phi = |\det(v_1, \dots, v_n)|.$$

Note that this definition is independent of the chosen basis since for a transformation

$$A(v_1, \dots, v_n) = (v'_1, \dots, v'_n)$$

between two bases we have $\det A = \pm 1$.

Theorem 7 (Minkowski). Let $X \subset \mathbb{R}^n$ be a convex, symmetric central (i.e., $x \in X$ implies $-x \in X$) subset and let $\Gamma \subset \mathbb{R}^n$ be a complete lattice. If

$$\text{vol } X > 2^n \text{vol } \Gamma$$

then there exists some $\gamma \in \Gamma \setminus \{0\}$ such that $\gamma \in X$.

Proof. Claim: It suffices to show that there are $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$, such that

$$\left(\frac{1}{2}X + \gamma_1\right) \cap \left(\frac{1}{2}X + \gamma_2\right) \neq \emptyset.$$

Proof of claim: Let $x = \frac{1}{2}x_1 + \gamma_1 = \frac{1}{2}x_2 + \gamma_2$ with some $x_1, x_2 \in X$. Then

$$y = \frac{1}{2}(x_1 - x_2) = \gamma_2 - \gamma_1 \in \Gamma \setminus \{0\}$$

with $y \in X$ since X is symmetrical central.

Now let us assume that the family $\left(\frac{1}{2}X + \gamma\right)_{\gamma \in \Gamma}$ is pairwise disjoint. Then

$$\left(\left[\frac{1}{2}X + \gamma\right] \cap \phi\right)_{\gamma \in \Gamma}$$

also consists of pairwise disjoint sets such that we obtain the contradiction

$$\begin{aligned} \text{vol } \Gamma = \text{vol } \phi &\geq \sum_{\gamma \in \Gamma} \text{vol} \left(\left[\frac{1}{2}X + \gamma\right] \cap \phi \right) = \sum_{\gamma \in \Gamma} \text{vol} \left(\frac{1}{2}X \cap [\phi - \gamma] \right) \\ &= \text{vol} \left(\frac{1}{2}X \right) = \frac{1}{2^n} \text{vol } X. \end{aligned}$$

□

2.5 Minkowski theory

Let $[K : \mathbb{Q}] = n$ be a field extension, $\tau_i: K \hookrightarrow \mathbb{C}$ different embeddings and consider the embedding

$$j: K \hookrightarrow K_{\mathbb{C}} = \prod_{\tau_i} \mathbb{C}, \quad a \mapsto (\tau_1(a), \dots, \tau_n(a)).$$

Define a hermitian scalar product on $K_{\mathbb{C}}$ by

$$\langle (x_{\tau_i}), (y_{\tau_i}) \rangle = \sum_{\tau_i} x_{\tau_i} \overline{y_{\tau_i}}$$

and consider the complex conjugation $F \in \text{Gal}(\mathbb{C}/\mathbb{R})$ given by $\mathbb{C} \rightarrow \mathbb{C}, z \mapsto \bar{z}$. Let

$$F(\tau) = \bar{\tau}: a \mapsto \overline{\tau(a)}$$

and extend it to $K_{\mathbb{C}}$ by

$$F: K_{\mathbb{C}} \rightarrow K_{\mathbb{C}}, (x_{\tau}) \mapsto (\bar{x}_{\bar{\tau}}).$$

Example. Let $D > 0$ be square-free. Consider

$$\mathbb{Q}(\sqrt{D}) \hookrightarrow \mathbb{Q}(\sqrt{D})_{\mathbb{C}} = \prod_{\tau_i} \mathbb{C}$$

with

$$\tau_1(a + b\sqrt{D}) = a + b\sqrt{D} \quad \text{and} \quad \tau_2(a + b\sqrt{D}) = a - b\sqrt{D}.$$

Then

$$j(a + b\sqrt{D}) = (a + b\sqrt{D}, a - b\sqrt{D})$$

and $F(\tau_1) = \tau_1, F(\tau_2) = \tau_2$ such that

$$F(x_{\tau_1}, x_{\tau_1}) = (\bar{x}_{\tau_1}, \bar{x}_{\tau_2}).$$

Remark. • $F(\langle x, y \rangle) = \langle F(x), F(y) \rangle$

• $\text{Tr}: K_{\mathbb{C}} \rightarrow \mathbb{C}, (x_{\tau}) \mapsto \sum_{\tau} x_{\tau}$ such that $(\text{Tr} \circ j)(a) = \text{Tr}_{K/\mathbb{Q}}(a)$

Now define the F -invariant \mathbb{R} -vector space

$$K_{\mathbb{R}} = K_{\mathbb{C}}^+ = \{x \in K_{\mathbb{C}} \mid F(x) = x\} = \{x \in K_{\mathbb{C}} \mid x_{\bar{\tau}} = \overline{x_{\tau}} \text{ for all } \tau\}.$$

Since $\bar{\tau}(a) = \overline{\tau(a)}$ for all $a \in K$ and all τ , we have $j(K) \subset K_{\mathbb{R}}$. We call $K_{\mathbb{R}}$ the **Minkowski space** and $\langle \cdot, \cdot \rangle|_{K_{\mathbb{R}}}$ the **canonical metric**.

Remark. Note that $j: K \rightarrow K_{\mathbb{R}} \cong K \otimes_{\mathbb{Q}} \mathbb{R}$, where the isomorphism is given by $a \otimes x \mapsto j(a)x$ for $x \in \mathbb{R}$.

Explicit description of $K_{\mathbb{R}}$: Let $n = r + 2s$, where r and s are the number of embeddings

$$\varphi_1, \dots, \varphi_r: K \hookrightarrow \mathbb{R}$$

and

$$\sigma_1, \overline{\sigma_1}, \dots, \sigma_s, \overline{\sigma_s}: K \hookrightarrow \mathbb{C},$$

respectively. Notice that $F(\varphi_i) = \varphi_i$ and $F(\sigma_j) = \overline{\sigma_j}$. Then elements of $K_{\mathbb{C}}$ are of the form

$$x = (x_{\varphi(1)}, \dots, x_{\varphi(r)}, x_{\sigma_1}, x_{\overline{\sigma_1}}, \dots, x_{\sigma_s}, x_{\overline{\sigma_s}})$$

with

$$F(x) = (\overline{x_{\varphi_1}}, \dots, \overline{x_{\varphi_r}}, \overline{x_{\sigma_1}}, \overline{x_{\sigma_1}}, \dots, \overline{x_{\sigma_s}}, \overline{x_{\sigma_s}}).$$

Hence we have

$$K_{\mathbb{R}} = \{x \in K_{\mathbb{C}} \mid x_{\varphi_i} \in \mathbb{R}, x_{\overline{\sigma_j}} = \overline{x_{\sigma_j}}\}.$$

Proposition 2.5.1. *The map*

$$f: K_{\mathbb{R}} \xrightarrow{\cong} \mathbb{R}^{r+2s} = \prod_{\tau} \mathbb{R},$$

$$x \mapsto (x_{\varphi_1}, \dots, x_{\varphi_r}, \operatorname{Re} x_{\sigma_1}, \operatorname{Im} x_{\sigma_1}, \dots, \operatorname{Re} x_{\sigma_s}, \operatorname{Im} x_{\sigma_s}).$$

is an isomorphism. It transforms the canonical metric into the scalar product

$$\langle x, y \rangle = \sum_{\tau} \alpha_{\tau} x_{\tau} y_{\tau},$$

where

$$\alpha_{\tau} = \begin{cases} 1, & \tau = \varphi_i \text{ for some } i, \\ 2, & \tau = \sigma_j \text{ for some } j. \end{cases}$$

Proof. Obviously, f is an isomorphism. For $x = (x_{\tau}), y = (y_{\tau}) \in K_{\mathbb{R}}$ we have

$$\begin{aligned} \langle x, y \rangle|_{K_{\mathbb{R}}} &= \sum_{\tau} x_{\tau} \overline{y_{\tau}} \\ &= \sum_{\varphi_i} x_{\varphi_i} y_{\varphi_i} + \sum_{\sigma_j} x_{\sigma_j} \overline{y_{\sigma_j}} + \sum_{\overline{\sigma_j}} \overline{(x_{\sigma_j} y_{\sigma_j})} \\ &= \dots = (f(x), f(y)). \end{aligned}$$

□

Remark. • The canonical metric induces a volume $\operatorname{vol}_{\text{can}}$ on $K_{\mathbb{R}}$ and thus on \mathbb{R}^{r+2s} .

- If we denote the Lebesgue measure on \mathbb{R}^{r+2s} by $\operatorname{vol}_{\text{Leb}}$ then, for $X \subset K_{\mathbb{R}}$,

$$2^s \operatorname{vol}_{\text{Leb}} f(X) = \operatorname{vol}_{\text{can}} X.$$

- We will thus consider $K \supset U \xrightarrow{j} j(U) \xrightarrow{f} \mathbb{R}^{r+2s}$.

Example. Let $e_j = (0, \dots, 1, \dots, 0)$. Note that we have $\langle e_{\varphi_i}, e_{\varphi_i} \rangle = 1$ and $\langle e_{\sigma_j}, e_{\varphi_j} \rangle = 2$, such that $\langle \frac{e_{\sigma_j}}{\sqrt{2}}, \frac{e_{\sigma_j}}{\sqrt{2}} \rangle = 1$. Hence

$$\left\{ e_{\varphi_1}, \dots, e_{\varphi_r}, \frac{e_{\sigma_1}}{\sqrt{2}}, \frac{e_{\overline{\sigma_1}}}{\sqrt{2}}, \dots \right\}$$

is an orthonormal basis. Using the correspondence

$$X \subset K_{\mathbb{R}} \leftrightarrow f(X) \subset \mathbb{R}^{r+2s}$$

we thus define

$$\operatorname{vol}_{\text{can}} X = \operatorname{vol}_{\text{can}} f(X) = 2^s \operatorname{vol}_{\text{Leb}} f(X).$$

Proposition 2.5.2. *If $I \neq 0$ is an \mathcal{O}_k -ideal then $\Gamma = j(I)$ is a complete lattice in $K_{\mathbb{R}}$. Its fundamental domain has volume*

$$\text{vol } \Gamma = \text{vol } \phi = \sqrt{|d_k|} \cdot [\mathcal{O}_k : I].$$

Proof. Choose α_i such that $I = \alpha_1\mathbb{Z} + \cdots + \alpha_n\mathbb{Z}$. Then $\Gamma = j(I) = j(\alpha_1)\mathbb{Z} + \cdots + j(\alpha_n)\mathbb{Z}$. Define

$$A = (j(\alpha_1), \dots, j(\alpha_n))^T = \begin{pmatrix} \tau_1(\alpha_1) & \cdots & \tau_n(\alpha_1) \\ \vdots & \ddots & \vdots \\ \tau_1(\alpha_n) & \cdots & \tau_n(\alpha_n) \end{pmatrix}$$

such that

$$\text{vol } \phi = |\det A| = \sqrt{|d_k|} \cdot [\mathcal{O}_k : I].$$

Furthermore,

$$d(I) = d(\alpha_1, \dots, \alpha_n) = |\det A|^2 = d(\mathcal{O}_k) \cdot [\mathcal{O}_k : I]^2,$$

with $[\mathcal{O}_k : I] = |\det M|$ for the change of basis M from \mathcal{O}_k to I . \square

Theorem 8. *Let $I \neq 0$ be an ideal in \mathcal{O}_k . Let $(c_\tau)_\tau$ be a collection of real number such that $c_\tau > 0$, $c_\tau = c_{\bar{\tau}}$ and*

$$\prod_{\tau} c_\tau > \left(\frac{2}{\pi}\right)^s \sqrt{|d_k|} \cdot [\mathcal{O}_k : I].$$

Then there exists $a \in I \setminus \{0\}$ such that

$$|\tau(a)| < c_\tau$$

for all $\tau \in \text{Hom}(K, \mathbb{C})$.

Proof. Consider the convex, central symmetric set

$$X = \{(x_\tau) \in K_{\mathbb{R}} \mid |x_\tau| < c_\tau \text{ for all } \tau\}$$

and let $f: K_{\mathbb{R}} \rightarrow \mathbb{R}^n$, $n = r + 2s$, as in Proposition 5.1. Notice that for $x \in X$ we have $f(x) = (x_{\varphi_1}, \dots, x_{\varphi_r}, a_1, b_1, \dots, a_s, b_s)$ with $|x_{\varphi_i}| < c_{\varphi_i}$ and $a_j^2 + b_j^2 < c_{\sigma_j}^2$. Hence

$$\text{vol}_{\text{can}} X = 2^s \text{vol}_{\text{Leb}} f(X) = 2^s \left(\prod_{i=1}^r 2c_{\varphi_i} \right) \left(\prod_{j=1}^s \pi c_{\sigma_j}^2 \right) = 2^{r+s} \pi^s \prod_{\tau} c_\tau,$$

and thus, by Proposition 5.2,

$$\begin{aligned} 2^n \text{vol } \Gamma &= 2^{r+2s} \sqrt{|d_k|} \cdot [\mathcal{O}_k : I] \\ &= 2^{r+s} \pi^s \left[\left(\frac{2}{\pi}\right)^s \sqrt{|d_k|} \cdot [\mathcal{O}_k : I] \right] \\ &< 2^{r+s} \pi^s \prod_{\tau} c_\tau \\ &= \text{vol}_{\text{can}} X. \end{aligned}$$

Consequently, by Minkowski's theorem, there exists $j(a) \in \Gamma \setminus \{0\}$ with $j(a) \in X$ and $|\tau(a)| < c_\tau$ for all τ . \square

Multiplicative Minkowsky theory

Define

$$j: K^* \hookrightarrow K_{\mathbb{C}}^* = \prod_{\tau} \mathbb{C}^*, a \mapsto (\tau(a))_{\tau}$$

and

$$\mathcal{N}: K_{\mathbb{C}}^* \rightarrow \mathbb{C}^*, (x_{\tau}) \mapsto \prod_{\tau} x_{\tau}.$$

Denote the composition of these maps by $\mathcal{N}_{K/\mathbb{Q}} = \mathcal{N} \circ j$. Furthermore, consider

$$l: \mathbb{C}^* \rightarrow \mathbb{R}, z \mapsto \log |z|$$

and its extension

$$l: K_{\mathbb{C}}^* \rightarrow \prod_{\tau} \mathbb{R}, (x_{\tau}) \mapsto (\log |x_{\tau_1}|, \dots, \log |x_{\tau_n}|).$$

All in all, we have

$$\begin{array}{ccccc} K^* & \xrightarrow{j} & K_{\mathbb{C}}^* & \xrightarrow{l} & \prod_{\tau} \mathbb{R} \\ \mathcal{N}_{K/\mathbb{Q}} \downarrow & & \downarrow \mathcal{N} & & \downarrow \text{Tr} \\ \mathbb{Q}^* & \xrightarrow{i} & \mathbb{C}^* & \xrightarrow{l} & \mathbb{R} \end{array}$$

with

$$\left[\prod_{\tau} \mathbb{R} \right]^+ = \prod_{\varphi_i} \mathbb{R} \times \prod_{\sigma_j} [\mathbb{R} \times \mathbb{R}]^+ \xrightarrow{\cong} \mathbb{R}^{r+s},$$

where the isomorphism is given by

$$(x_{\varphi_1}, \dots, x_{\varphi_r}, x_{\sigma_1}, x_{\overline{\sigma_1}}, \dots, x_{\sigma_s}, x_{\overline{\sigma_s}}) \mapsto (x_{\varphi_1}, \dots, x_{\varphi_r}, 2x_{\sigma_1}, \dots, 2x_{\sigma_s}),$$

and we have

$$K_{\mathbb{R}} \rightarrow \mathbb{R}^{r+s}, (x_{\tau}) \mapsto (\log |x_{\varphi_1}|, \dots, \log |x_{\varphi_r}|, \log |x_{\sigma_1}|^2, \dots, \log |x_{\sigma_s}|^2).$$

2.6 The class number

Let $n = [K : \mathbb{Q}]$, denote by J_K the group of fractional ideals of K , by P_k its subgroup of principal ideals and by $\text{Cl}_k = J_k/P_k$ the ideal class group. Define the **absolute norm** of an ideal $I \subset \mathcal{O}_k$ by

$$n(I) = [\mathcal{O}_k : I].$$

For $I = (\alpha)$, we have $n(I) = \mathcal{N}_{K/\mathbb{Q}}(\alpha)$. If $\mathcal{O}_k = w_1\mathbb{Z} + \dots + w_n\mathbb{Z}$ and $I = \alpha w_1\mathbb{Z} + \dots + \alpha w_n\mathbb{Z}$ we have

$$\alpha w_i = \sum_j a_{ij} w_j$$

for some matrix $A = (a_{ij})$ such that $\mathcal{N}_{K/\mathbb{Q}}(\alpha) = |\det A| = [\mathcal{O}_k : I]$.

Proposition 2.6.1. If $I = P_1^{\nu_1} \cdots P_r^{\nu_r}$ then $n(I) = n(P_1)^{\nu_1} \cdots n(P_r)^{\nu_r}$.

Proof. By the Chinese remainder theorem,

$$\mathcal{O}_k/I \cong (\mathcal{O}_k/P_1^{\nu_1}) \oplus \cdots \oplus (\mathcal{O}_k/P_r^{\nu_r}),$$

such that

$$n(I) = [\mathcal{O}_k : I] = \prod_j [\mathcal{O}_k : P_j^{\nu_j}] = \prod_j n(P_j)^{\nu_j}.$$

Claim: $P \supsetneq P^2 \supsetneq \cdots \supsetneq P^\nu$ and P^i/P^{i+1} is a (\mathcal{O}_k/P) -vector space of dimension 1

Proof of Claim: Let $a \in P^i/P^{i+1}$. Then we have

$$P^i \supset J = (a) + P^{i+1} \supsetneq P^{i+1}$$

and

$$\mathcal{O}_k \supset J' = JP^{-i} \supsetneq P = P^{i+1}P^{-i}.$$

Since $J'|P$ we have $J = P^i$ and thus $[a] \in P^i/P^{i+1}$ is a basis.

Now, the Claim yields

$$n(P^\nu) = [\mathcal{O}_k : P^\nu] = [\mathcal{O}_k : P] [P : P^2] \cdots [P^{\nu-1} : P^\nu] n(P)^\nu.$$

□

In particular, for integral ideals I, J we have $n(IJ) = n(I)n(J)$ such that we can extend n to J_k by

$$n: J_k \rightarrow \mathbb{R}_+^*, I = P_1^{\nu_1} \cdots P_r^{\nu_r} \mapsto n(I) = n(P_1)^{\nu_1} \cdots n(P_r)^{\nu_r}.$$

Reminder 2.6.2. \mathcal{J}_K = group of fractional ideals = abelian group generated by all prime ideals

\mathcal{P}_K = group of all principal fractional ideals.

$\text{Cl}_K := \mathcal{J}_K/\mathcal{P}_K$

\Rightarrow obtain following exact sequence:

$$1 \rightarrow \underbrace{\mathcal{O}_K^\times}_{\text{How big?}} \rightarrow K^\times \rightarrow \mathcal{J}_K \rightarrow \underbrace{\text{Cl}_K}_{\text{How big?}} \rightarrow 1$$

$$a \mapsto (a) = a\mathcal{O}_K$$

Last Time: \mathfrak{a} ideal in $\mathcal{O}_K, \mathfrak{a} \neq 0$.

- $\mathcal{N}(\mathfrak{a}) = (\mathcal{O}_K : \mathfrak{a})$ absolute norm.

In particular: $\mathcal{N}((a)) := |\mathcal{N}_{K/\mathbb{Q}}(a)|$.

- $\mathfrak{a} = \mathcal{P}_1^{\nu_1} \cdots \mathcal{P}_r^{\nu_r}$ decomposition into primes
 $\Rightarrow \mathcal{N}(\mathfrak{a}) = \mathcal{N}(\mathcal{P}_1)^{\nu_1} \cdots \mathcal{N}(\mathcal{P}_r)^{\nu_r}$

In particular: $\mathcal{N}(\mathfrak{a}_1\mathfrak{a}_2) = \mathcal{N}(\mathfrak{a}_1)\mathcal{N}(\mathfrak{a}_2)$.

- Hence \mathcal{N} can be extended to fractional ideals: $\mathcal{N} : \mathcal{J}_K \rightarrow \mathbb{R}_+^\times$.

Goal: Show that Cl_K is finite.

Idea:

- Find in each integral ideal \mathfrak{a} an element $a \neq 0$ of norm bounded by $\mathcal{N}(\mathfrak{a})$.
- Show: For $M > 0$ there are only finitely many integral ideals \mathfrak{a} with $\mathcal{N}(\mathfrak{a}) \leq M$.
- Show: Each class $[\mathfrak{a}] \in \text{Cl}_K$ contains an integral ideal \mathfrak{a}_1 s.t. $\mathcal{N}(\mathfrak{a}_1) \leq M_0 = \left(\frac{2}{\pi}\right)^s \sqrt{|d_K|}$.
Recall: s = number of not-real embeddings of K into \mathbb{C} .

Lemma 2.6.3. *Suppose: $\mathfrak{a} \neq 0$ is an integral ideal $\Rightarrow \exists a \in \mathfrak{a}, a \neq 0$ s.t. $|\mathcal{N}_{K/\mathbb{Q}}(a)| \leq \left(\frac{2}{\pi}\right)^s \sqrt{|d_K|} \mathcal{N}(\mathfrak{a})$.*

Proof. $M_0 := \left(\frac{2}{\pi}\right)^s \sqrt{|d_K|}$

Idea: Use „Thm. 5.3“

given: $c_\tau \in \mathbb{R}_{>0} (\tau \in \text{Hom}(K, \mathbb{C}))$ with $c_\tau = c_{\bar{\tau}}$ and $\prod_\tau c_\tau > M_0 \mathcal{N}(\mathfrak{a})$

$\Rightarrow \exists a \in \mathfrak{a}, a \neq 0$ with $|\tau(a)| < c_\tau$ for all τ .

For each $\varepsilon > 0$ choose a sequence $c_\tau \in \mathbb{R}_{>0}$ with $c_\tau = c_{\bar{\tau}}$ and $\prod_\tau c_\tau = M_0 \mathcal{N}(\mathfrak{a}) + \varepsilon$

$\xRightarrow{\text{Thm 5.3}}$ Find $a_\varepsilon \neq 0$ in \mathfrak{a} with

$$|\mathcal{N}_{K/\mathbb{Q}}(a)| = \prod_\tau |\tau(a)| < M_0 \mathcal{N}(\mathfrak{a}) + \varepsilon$$

Since left side is integer, we obtain: $\exists a \neq 0$ in \mathfrak{a} with $|\mathcal{N}_{K/\mathbb{Q}}(a)| \leq M_0 \mathcal{N}(\mathfrak{a})$. \square

Lemma 2.6.4. *Let $M \in \mathbb{R}_{>0}$. There are only finitely many integral ideals \mathfrak{a} with $\mathcal{N}(\mathfrak{a}) \leq M$.*

Proof. (1) Consider first only prime ideals $\mathcal{P} \neq 0$: Suppose $\mathcal{N}(\mathcal{P}) \leq M$

Recall: $\mathcal{P} \cap \mathbb{Z} = p\mathbb{Z}$ with p prime number (Prop. 3.3)

\Rightarrow obtain embedding $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathcal{O}_K/\mathcal{P} \Rightarrow \mathcal{N}(\mathcal{P}) = (\mathcal{O}_K : \mathcal{P}) = \#\mathcal{O}_K/\mathcal{P} = p^f$

Hence: $p^f \leq M$. In particular P is bounded.

Furthermore: There are only finitely many prime ideals \mathcal{P} with $\mathcal{P} \cap \mathbb{Z} = p\mathbb{Z}$.

Since $\mathcal{P} \cap \mathbb{Z} = p\mathbb{Z} \Rightarrow p \in \mathcal{P} \Rightarrow (p) \subseteq \mathcal{P}$ But there are only finitely many prime ideals in \mathcal{O}_K which divide (p) .

(2) Suppose now \mathfrak{a} is an arbitrary integral ideal, $\mathfrak{a} \neq 0$:

$\Rightarrow \mathfrak{a} = \mathcal{P}_1^{\nu_1} \cdots \mathcal{P}_r^{\nu_r}$ with \mathcal{P}_i prime ideal and $\nu_i \in \mathbb{N}$ and $\mathcal{N}(\mathfrak{a}) = \mathcal{N}(\mathcal{P}_1)^{\nu_1} \cdots \mathcal{N}(\mathcal{P}_r)^{\nu_r}$.

Now the claim follows from (1). \square

Theorem 9 (Finiteness of Cl_K). *The ideal class group of $\text{Cl}_K = \mathcal{J}_K/\mathcal{P}_K$ is finite.*

Proof. Let $M_0 := \left(\frac{2}{\pi}\right)^s \sqrt{|d_K|}$

Show that each class $[\mathfrak{a}] \in \text{Cl}_K$ contains an integral ideal \mathfrak{a}_1 with $\mathcal{N}(\mathfrak{a}_1) \leq M_0$. Then the

claim follows from Lemma 6.3.

Let $[a] \in \text{Cl}_K$. Choose $\gamma \in \mathcal{O}_K, \gamma \neq 0$ with γa^{-1} is integral.

$$\begin{aligned} \text{Lemma 6.2} &\Rightarrow \exists b \in \mathfrak{b} := \gamma a^{-1} \text{ with } b \neq 0 \text{ and } |\mathcal{N}_{K/\mathbb{Q}}(b)| \leq M_0 \mathcal{N}(\mathfrak{b}) \\ &\Rightarrow \mathcal{N}((b)\mathfrak{b}^{-1}) = \mathcal{N}((b))\mathcal{N}(\mathfrak{b}^{-1}) \leq M_0 \end{aligned}$$

Observe: The factorial ideal $(b)\mathfrak{b}^{-1} = (b)\gamma^{-1}a \in [a]$, hence $a_1 := b\gamma^{-1}a$ does the job. a_1 is an integral ideal, since $(b) \subseteq \gamma a^{-1}$ \square

Definition 2.6.5 („Klassenzahl“). $h_K := \# \text{Cl}_K := (\mathcal{J}_K : \mathcal{P}_K)$ is called the class number of K .

Proposition 2.6.6. Suppose R is a Dedekind domain.

R is a UFD $\iff R$ is a PID (principal ideal domain).

Proof. „ \Leftarrow “: true for general domains.

„ \Rightarrow “: Suppose R is a UFD.

Step 1: Every prime ideal is principal.

Let \mathcal{P} be a prime ideal, $\mathcal{P} \neq 0$. Choose $a \in \mathcal{P}, a \neq 0$. Let $a = p_1 \cdots p_n$ be its prime factor decomposition. \mathcal{P} prime $\Rightarrow p_i \in \mathcal{P}$ for one of the i 's $\Rightarrow \mathcal{P} \supseteq (p_i) \Rightarrow \mathcal{P} = (p_i)$, since (p_i) is a prime ideal and R is a Dedekind domain.

Step 2: \mathfrak{a} arbitrary ideal.

$\Rightarrow \mathfrak{a} = \mathcal{P}_1 \cdots \mathcal{P}_n$ is a product of prime ideals

$\Rightarrow \mathfrak{a}$ is principal, since each \mathcal{P}_i is. \square

Corollary 2.6.7. We have for a number field K :

$h_K = 1 \iff \mathcal{O}_K$ is a principal domain $\iff \mathcal{O}_K$ is a UFD.

2.7 The theorem of Dirichlet

Goal: Describe \mathcal{O}_K^\times

Recall:

- $\mathcal{O}^\times = \{\varepsilon \in \mathcal{O}_K \mid \mathcal{N}_{K/\mathbb{Q}}(\varepsilon) = \pm 1\}$
- $\mu(K) := \{x \in \mathcal{O}_K \mid \exists n \in \mathbb{N} \text{ with } x^n = 1\} \subseteq \mathcal{O}_K^\times$ is a finite subgroup.

Idea: Use multiplicative Minkowsky theory:

- $\text{Hom}(K, \mathbb{C}) = \{\tau_1, \dots, \tau_r, \tau_{r+1}, \overline{\tau_{r+1}}, \tau_{r+s}, \overline{\tau_{r+s}}\}$
- $j : K^\times \hookrightarrow K_\mathbb{R}^\times = \{x \in \prod_\tau \mathbb{C}^\times \mid x_{\bar{\tau}} = \overline{x_\tau}\}, a \mapsto (\tau(a))_\tau$
- $l : K_\mathbb{R}^\times \rightarrow [\prod_\tau \mathbb{R}]^+ := \{z \in \prod_\tau \mathbb{R} \mid z_{\bar{\tau}} = z_\tau\}, x = (x_\tau) \mapsto (\log |x_\tau|)_\tau$

\Rightarrow commutative diagramm:

$$\begin{array}{ccccc}
 \mathcal{O}_K^\times & & S & & H \\
 \text{in} & & \text{in} & & \text{in} \\
 K^\times & \xrightarrow{j} & K_{\mathbb{R}}^\times & \xrightarrow{l} & [\prod_{\tau} \mathbb{R}]^+ \\
 \downarrow \mathcal{N}_{K/\mathbb{Q}} & & \downarrow \mathcal{N} & & \downarrow \text{Tr} \\
 \mathbb{Q}^\times & \longrightarrow & \mathbb{R} & \xrightarrow{\log|\cdot|} & \mathbb{R}
 \end{array}$$

with $\mathcal{N}(x) = \prod_{\tau} x_{\tau}$, $\text{Tr}(z) = \sum_{\tau} z_{\tau}$.

Consider the three groups:

- (1) $\mathcal{O}_K^\times = \{\varepsilon \in \mathcal{O}_K \mid \mathcal{N}_{K/\mathbb{Q}}(\varepsilon) = \pm 1\}$
- (2) $S := \{x \in K_{\mathbb{R}}^\times \mid \mathcal{N}(x) = \pm 1\}$ „Norm 1 hyper surface“
- (3) $H := \{z \in [\prod_{\tau} \mathbb{R}]^+ \mid \text{Tr}(z) = 0\}$ „Trace 0 hypersurface“

\Rightarrow Morphisms restrict to

$$\mathcal{O}_K^\times \xrightarrow{j} S \xrightarrow{l} H.$$

Define $\Gamma := l \circ j(\mathcal{O}_K^\times) = \text{image of } l \circ j$.

Recall from additive Minkowski-Theory: $j(\mathcal{O}_K)$ is a complete lattice in $K_{\mathbb{R}}$

Proposition 2.7.1. *The sequence*

$$1 \rightarrow \mu(K) \rightarrow \mathcal{O}_K^\times \xrightarrow{l \circ j} \Gamma \rightarrow 1$$

is an exact sequence.

Proof. $\lambda := l \circ j$

We have to show: $\ker(\lambda) = \mu(K)$.

Observe: $a \in \ker(\lambda) \iff \forall \tau \in \text{Hom}(K, \mathbb{C}) : \log |\tau(a)| = 0 \iff |\tau(a)| = 1$

Hence: $\ker(\lambda) = \{a \in \mathcal{O}^\times \mid |\tau(a)| = 1\}$.

„ \supseteq “: \checkmark

„ \subseteq “: $j(\ker(\lambda))$ is bounded as subset of $K_{\mathbb{R}}^\times$. Furthermore: $j(\ker(\lambda)) \subseteq j(\mathcal{O})$ which is a lattice in $K_{\mathbb{R}} \Rightarrow j(\ker(\lambda))$ is finite and thus also $\ker(\lambda)$.

Altogether: $\ker(\lambda)$ is a finite subgroup of $K^\times \Rightarrow$ every element in $\ker(\lambda)$ has finite order \Rightarrow every element is a root of unity. \square

Goal: Describe Γ

Recall: $\alpha, \alpha' \in \mathcal{O}_K$ are associated : $\iff \exists \varepsilon \in \mathcal{O}_K^\times$ s.t. $\alpha' = \alpha \cdot \varepsilon$.

Proposition 2.7.2. *Let $a \in \mathbb{Z}$. There are at most $(\mathcal{O}_K : a\mathcal{O}_K) = \mathcal{N}((a))$ elements $\alpha \in \mathcal{O}_K$ up to associates with $\mathcal{N}_{K/\mathbb{Q}}(\alpha) = \pm a$.*

Proof. Suppose w.l.o.g.: $a > 1$.

Consider the cosets of \mathcal{O}_K modulo the subgroup $a\mathcal{O}_K$. Show that each coset contains at most one such α up to associates.

Suppose: $\alpha \in \mathcal{O}$ with $\mathcal{N}_{K/\mathbb{Q}}(\alpha) = \pm a$ and suppose $\beta = \alpha + a\gamma$ with $\gamma \in \mathcal{O}_K$ also satisfies $\mathcal{N}_{K/\mathbb{Q}}(\beta) = \pm a \Rightarrow \frac{\beta}{\alpha} = 1 \pm \frac{\mathcal{N}_{K/\mathbb{Q}}(\alpha)}{\alpha} \gamma$.

Recall: $\frac{\mathcal{N}(\alpha)}{\alpha} \in \mathcal{O}_K \Rightarrow \frac{\beta}{\alpha} \in \mathcal{O}_K$.

Obtain in the same way $\frac{\alpha}{\beta} \in \mathcal{O}_K$. Hence $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ are in $\mathcal{O}_K^\times \Rightarrow \alpha$ and β are associated. \square

Lemma 2.7.3. *Let V be an \mathbb{R} -vector space of dimension n , Γ a lattice in V .*

Γ is complete $\iff \exists M \subseteq V$ with M bounded s.t. $\bigcup_{\gamma \in \Gamma} M + \gamma = V$.

Proof. „ \Rightarrow “: $\Gamma = \mathbb{Z}v_1 + \dots + \mathbb{Z}v_n \Rightarrow M := \phi := \{r_1v_1 + \dots + r_nv_n \mid 0 \leq r_i < 1\}$ does it.

„ \Leftarrow “: Consider: $V_0 := \mathbb{R}$ -vector space generated by Γ . Have to show: $V_0 = V$.

Let $v \in V$. Consider the sequence kv ($k \in \mathbb{N}$).

Precondition $\Rightarrow \forall k \exists a_k \in M$ and $\gamma_k \in \Gamma$ with $kv = a_k + \gamma_k$

M bounded $\Rightarrow \frac{1}{k}a_k \rightarrow 0 \Rightarrow v = \lim_{k \rightarrow \infty} \frac{1}{k}a_k + \frac{1}{k}\gamma_k = \lim_{k \rightarrow \infty} \frac{1}{k}\gamma_k \Rightarrow v \in V_0$, since V_0 is closed. \square

Theorem 10. *The group Γ is a complete lattice in $H = \{x \in [\prod_\tau \mathbb{R}]^+ \mid \text{Tr}(x) = 0\} \cong \mathbb{R}^{r+s-1}$. Hence Γ is isomorphic to \mathbb{Z}^{r+s-1} .*

Proof. Step 1: Show that Γ is a lattice, i.e. show that Γ is a discrete subgroup of H .

More precisely: show that $\forall c > 0$:

$$\Gamma \cap \{(z_\tau)_\tau \in \prod_\tau \mathbb{R} \mid |z_\tau| \leq c\} =: Q_c$$

is finite.

Observe: $l^{-1}(Q_c) = \{(x_\tau)_\tau \in \prod_\tau \mathbb{C}^\times \mid e^{-c} \leq |x_\tau| \leq e^c\}$ since $l((x_\tau)_\tau) = (\log|x_\tau|)_\tau$.

$\Rightarrow l^{-1}(Q_c) \cap j(\mathcal{O}_K^\times)$ is finite, since $j(\mathcal{O}_K)$ is a lattice in $K_\mathbb{R}$. This shows the claim.

Step 2: Show that Γ is complete.

Idea: Use Lemma 7.3.

Hence: find $M \subseteq H$ as required in the lemma.

Equivalently: find $T \subseteq S$, s.t. $S = \bigcup_{\varepsilon \in \mathcal{O}_K^\times} T \cdot j(\varepsilon)$ and T is bounded.

Then we have for $M := l(T) : H = \bigcup_{\varepsilon \in \mathcal{O}_K^\times} M + l(j(\varepsilon)) = \bigcup_{\gamma \in \Gamma} M + \gamma$.

Furthermore: T bounded $\Rightarrow \exists C > 0 : \forall x \in T : \forall \tau : |x_\tau| < C$.

Since $\prod_\tau |x_\tau| = 1 \Rightarrow \exists c > 0 : \forall x \in T : \forall \tau : |x_\tau| > c \Rightarrow M = l(T)$ is bounded in H .

Step 3: Definition of T

- Choose sequence (c_τ) with $c_\tau > 0$, $c_{\bar{\tau}} = c_\tau$ and $C := \prod c_\tau > M_0 = (\frac{2}{\pi})^s \sqrt{d_K}$ and define $X := \{(x_\tau)_\tau \mid |x_\tau| < c_\tau\}$.
- Choose $\alpha_1, \dots, \alpha_N \in \mathcal{O}_K$ s.t. each $\alpha \in \mathcal{O}_K, \alpha \neq 0$ with $|\mathcal{N}_{K/\mathbb{Q}}(\alpha)| \leq C$ is associated to one α_i (by Prop 7.2. possible).

Define $T := S \cap \bigcup_{i=1}^n X \cdot j(\alpha_i)^{-1}$.

Step 4: T does the job:

- (1) X is bounded $\Rightarrow Xj(\alpha_i)^{-1}$ is bounded $\Rightarrow T$ is bounded.
- (2) Observe: $y = (y_\tau) \in S \Rightarrow Xy = \{(x_\tau) \in K_{\mathbb{R}} \mid |x_\tau| < c'_\tau\}$ with $c'_\tau = c_\tau \cdot |y_\tau|$
 $\Rightarrow c'_\tau = c'_\tau$ and $\prod_\tau c'_\tau = \prod_\tau c_\tau \underbrace{\prod_\tau |y_\tau|}_{=1(y \in S)} = C$.
 $\Rightarrow \exists \alpha \in \mathcal{O}_K$ with $|\tau(\alpha)| < c'_\tau \forall \tau \Rightarrow j(\alpha) \in Xy$
- (3) Show that: $S = \bigcup_{\varepsilon \in \mathcal{O}_K^\times} Tj(\varepsilon)$
 Suppose $y \in S \stackrel{(2)}{\Rightarrow} \exists \alpha \in \mathcal{O}_K \setminus \{0\}$ with $j(\alpha) \in Xy^{-1} \Rightarrow j(\alpha) = xy^{-1}$ for some $x \in X$.
 Furthermore: $|\mathcal{N}_{K/\mathbb{Q}}(\alpha)| = |\mathcal{N}(xy^{-1})| = |\mathcal{N}(x)| < \prod_\tau c_\tau = C$.
 $\Rightarrow \alpha$ is associated to some α_i , hence $\alpha_i = \varepsilon \alpha$ with $\varepsilon \in \mathcal{O}_K^\times$.
 $\Rightarrow y = xj(\alpha)^{-1} = xj(\alpha_i^{-1}\varepsilon)$.
 Finally: y and $j(\varepsilon) \in S \Rightarrow xj(\alpha_i)^{-1} \in S \cap Xj(\alpha_i)^{-1} \subseteq T \Rightarrow y \in Tj(\varepsilon)$.

□

Corollary 2.7.4. $\mathcal{O}_K^\times \cong \mathbb{Z}^{r+s-1} \times \mu(K)$.

Proof. We have the exact sequence

$$1 \rightarrow \mu(K) \rightarrow \mathcal{O}_K^\times \xrightarrow{l} \Gamma \cong \mathbb{Z}^{r+s-1} \rightarrow 1$$

Fix a basis v_1, \dots, v_t ($t := r + s - 1$) of Γ and preimages $\varepsilon_1, \dots, \varepsilon_t$ in \mathcal{O}_K^\times .

Let $A := \langle \varepsilon_1, \dots, \varepsilon_t \rangle \subseteq \mathcal{O}_K^\times$.

Then $\lambda|_A$ is an isomorphism and thus $A \cap \mu(K) = \{1\}$. In particular every $\alpha \in \mathcal{O}_K^\times$ decomposes in a unique way as $\alpha = \nu \cdot \mu$ with $\nu \in A$ and $\mu \in \mu(K)$. □

2.8 Prime ideals in \mathcal{O}_K

Question: Describe the prime ideals in \mathcal{O}_K that "live above a prime ideal $\mathfrak{p} \subset \mathbb{Z}$ ".

Consider the following, more general situation: Let

- \mathcal{O} be a Dedekind domain,
- $K = \text{Quot}(\mathcal{O})$,
- $L \mid K$ a finite and separable field extension,
- $\hat{\mathcal{O}}$ the integral closure of \mathcal{O} in L .

Definition 2.8.1. In the setting above, we say that a prime ideal $\hat{\mathfrak{p}} \in \hat{\mathcal{O}}$ lies above a prime ideal $\mathfrak{p} \in \mathcal{O} : \Leftrightarrow \hat{\mathfrak{p}} \cap \mathcal{O} = \mathfrak{p}$.

Proposition 2.8.2. $\hat{\mathcal{O}}$ is a Dedekind domain.

Proof. (1) $\hat{\mathcal{O}}$ is an integral domain and is integrally closed (see **Remark 2.1**).

- (2) We show, that every prime ideal $0 \neq \hat{\mathfrak{p}} \in \hat{\mathcal{O}}$ is maximal: We know that $\mathfrak{p} := \hat{\mathfrak{p}} \cap \mathcal{O}$ is a prime ideal in \mathcal{O} .

(Claim:) $\mathfrak{p} \neq 0$. Choose $0 \neq x \in \hat{\mathfrak{p}}$. Since $\hat{\mathcal{O}}$ is integrally closed, $\exists a_0, \dots, a_{n-1}$, such that

$$x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0.$$

We may assume that the equation is minimal, i.e. $a_0 \neq 0$. Then we have

$$0 \neq a_0 = -a_1x - \dots - a_{n-1}x^{n-1} - x^n \in \hat{\mathfrak{p}} \cap \mathcal{O} = \mathfrak{p}.$$

Since \mathfrak{p} is a prime ideal of \mathcal{O} , it is also maximal, i.e. \mathcal{O}/\mathfrak{p} is a field. Hence $\hat{\mathcal{O}}/\hat{\mathfrak{p}}$ is a finite extension of \mathcal{O}/\mathfrak{p} as an \mathcal{O}/\mathfrak{p} -algebra. Therefore \mathcal{O}/\mathfrak{p} a field $\Rightarrow \hat{\mathcal{O}}/\hat{\mathfrak{p}}$ is a field $\Rightarrow \hat{\mathfrak{p}}$ is a maximal ideal.

- (3) $\hat{\mathcal{O}}$ is Noetherian: Choose a basis $\alpha_1, \dots, \alpha_n$ of $L \mid K$ with $\alpha_1, \dots, \alpha_n \in \hat{\mathcal{O}}$. Let $d := d(\alpha_1, \dots, \alpha_n) \neq 0$ (**Proposition 2.6**). Recall that $d \cdot \hat{\mathcal{O}} \subset \mathcal{O}\alpha_1 + \dots + \mathcal{O}\alpha_n$ (**Proposition 2.8**) and that therefore $\hat{\mathcal{O}} \subset \mathcal{O}\frac{\alpha_1}{d} + \dots + \mathcal{O}\frac{\alpha_n}{d}$. Hence every ideal $I \subset \hat{\mathcal{O}}$ can be regarded as a submodule of the \mathcal{O} -module $\mathcal{O}\frac{\alpha_1}{d} + \dots + \mathcal{O}\frac{\alpha_n}{d}$. But since this module is finitely generated and \mathcal{O} is Noetherian, I must be finitely generated as well.

□

Proposition 2.8.3. *Let $\mathfrak{p} \subset \mathcal{O}$ be a prime ideal. Then $\mathfrak{p} \cdot \hat{\mathcal{O}} \subsetneq \hat{\mathcal{O}}$.*

Proof. We may assume $\mathfrak{p} \neq 0$.

- (1) Choose $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$. Then we can write $\pi \cdot \mathcal{O} = \mathfrak{p} \cdot \mathfrak{u}$ with $\mathfrak{p}, \mathfrak{u}$ coprime, i.e. $\mathcal{O} = \mathfrak{p} + \mathfrak{u} \Rightarrow \exists s \in \mathfrak{u}, t \in \mathfrak{p} : 1 = s + t$. In particular, $s \notin \mathfrak{p}$ since $1 \notin \mathfrak{p}$ and $s \cdot \mathfrak{p} \subset \mathfrak{u} \cdot \mathfrak{p} = \pi \cdot \mathcal{O}$.
- (2) Suppose $\mathfrak{p}\hat{\mathcal{O}} = \hat{\mathcal{O}}$. Then $s \cdot \hat{\mathcal{O}} = s\mathfrak{p}\hat{\mathcal{O}} \subset \pi\hat{\mathcal{O}} \Rightarrow s = \pi x$ with some $x \in \hat{\mathcal{O}} \cap K = \mathcal{O} \Rightarrow s \in \pi\mathcal{O} \subset \mathfrak{p}$, a contradiction.

□

Remark 2.8.4. Let $\mathfrak{p} \neq 0$ be a prime ideal in \mathcal{O} . Then:

- (i) $\mathfrak{p} \cdot \hat{\mathcal{O}} = \mathfrak{p}_1^{e_1} \cdot \dots \cdot \mathfrak{p}_r^{e_r}$ with $e_1, \dots, e_r \in \mathbb{N}$ and $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ prime ideals in $\hat{\mathcal{O}}$.
- (ii) A prime ideal $\hat{\mathfrak{p}}$ in $\hat{\mathcal{O}}$ satisfies: $\hat{\mathfrak{p}} \cap \mathcal{O} = \mathfrak{p} \Leftrightarrow \hat{\mathfrak{p}} = \mathfrak{p}_i$ for some i .

Proof. (i) follows from **Proposition 8.2+8.3**.

- (ii) " \Leftarrow ": $\mathfrak{p}\hat{\mathcal{O}} = \mathfrak{p}_1^{e_1} \cdot \dots \cdot \mathfrak{p}_r^{e_r} \Rightarrow \mathfrak{p}\mathcal{O} \subset \mathfrak{p}_i \Rightarrow \mathfrak{p} \subset \mathfrak{p}_i \cap \mathcal{O}$. We have $\mathfrak{p}_i \cap \mathcal{O} \neq 0$, $1 \notin \mathfrak{p}_i \cap \mathcal{O}$ and \mathfrak{p} is maximal, hence $\mathfrak{p} = \mathfrak{p}_i \cap \mathcal{O}$.
- " \Rightarrow ": $\hat{\mathfrak{p}} \cap \mathcal{O} = \mathfrak{p} \Rightarrow \mathfrak{p}\hat{\mathcal{O}} \subset \hat{\mathfrak{p}} \Rightarrow \hat{\mathfrak{p}}$ divides $\mathfrak{p}\hat{\mathcal{O}}$.

□

Definition 2.8.5. Let $0 \neq \mathfrak{p}$ be a prime ideal in \mathcal{O} and $\mathfrak{p}\hat{\mathcal{O}} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$ the decomposition into prime ideals.

- (i) e_i is called **ramification index** of \mathfrak{p}_i .
 \mathfrak{p}_i is called **unramified** $:\Leftrightarrow e_i = 1$.
 \mathfrak{p} is called **unramified**, if all \mathfrak{p}_i are unramified.
 \mathfrak{p} is called **totally ramified** $:\Leftrightarrow r = 1$.
- (ii) $f_i := \dim_K \hat{\mathcal{O}}/\mathfrak{p}_i$ with $K := \mathcal{O}/\mathfrak{p}$ is called **local degree** or **relative degree** of \mathfrak{p}_i .

Theorem 11. In the situation of **Definition 8.5**, we have the fundamental equation:

$$\sum_{i=1}^r e_i \cdot f_i = n \quad \text{with } n = [L : K]$$

Proof. We can write

$$\hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}} = \bigoplus_{i=1}^r \hat{\mathcal{O}}/\mathfrak{p}_i^{e_i}$$

by the Chinese Remainder Theorem. Let $k = \mathcal{O}/\mathfrak{p}$

Step 1: We show, that $\dim_k \hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}} = n$. Choose a basis $\bar{\omega}_1, \dots, \bar{\omega}_m$ of $\hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}}$ over k and choose preimages $\omega_1, \dots, \omega_m$ in $\hat{\mathcal{O}}$. We will show, that $\omega_1, \dots, \omega_m$ is a basis of $L \mid K$, i.e $m = n$, from which the claim follows.

- (1) Suppose $\omega_1, \dots, \omega_m$ are linearly dependant, i.e $\exists \alpha_1, \dots, \alpha_m \in K$, not all zero and such that

$$\alpha_1 \omega_1 + \cdots + \alpha_m \omega_m = 0. \quad (*)$$

Since $K = \text{Quot}(\mathcal{O})$, we may choose $\alpha_1, \dots, \alpha_m \in \mathcal{O}$, since we can just clear denominators. Consider the ideal $\mathfrak{u} := \langle \alpha_1, \dots, \alpha_m \rangle \subset \mathcal{O}$. $\mathfrak{p} \neq 0 \Rightarrow \mathfrak{u}^{-1}\mathfrak{p} \subsetneq \mathfrak{u}^{-1}$. Choose some $\alpha \in \mathfrak{u}^{-1} \setminus \mathfrak{u}^{-1}\mathfrak{p} \Rightarrow \alpha \cdot \mathfrak{u} \not\subseteq \mathfrak{p} \Rightarrow \alpha\alpha_1, \dots, \alpha\alpha_m \in \mathcal{O}$, but not all lie in \mathfrak{p} .

$\xRightarrow{(*)} \alpha\alpha_1\omega_1 + \cdots + \alpha\alpha_m\omega_m = 0 \pmod{\mathfrak{p}}$ with at least one of the $\alpha\alpha_i \notin \mathfrak{p}$. Hence $\alpha\alpha_1\bar{\omega}_1 + \cdots + \alpha\alpha_m\bar{\omega}_m = 0$ with at least one $\alpha\alpha_i \neq 0$, which contradicts the assumption that $\bar{\omega}_1, \dots, \bar{\omega}_m$ is a basis.

- (2) Consider $M := \mathcal{O}\omega_1 + \cdots + \mathcal{O}\omega_m$ and $N := \hat{\mathcal{O}}/M$. Since $\hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}} = K\bar{\omega}_1 + \cdots + K\bar{\omega}_m$, we have $\hat{\mathcal{O}} = M + \mathfrak{p}\hat{\mathcal{O}} \xrightarrow{\text{mod } M} N = \mathfrak{p}N$. The proof of **Proposition 8.2** implies, that $\hat{\mathcal{O}}$ and N are finitely generated as \mathcal{O} -modules. Choose generators $\bar{\alpha}_1, \dots, \bar{\alpha}_s$ of N . $N = \mathfrak{p}N \Rightarrow \exists \alpha_{i,j} \in \mathfrak{p}$ with $\bar{\alpha}_i = \sum_{j=1}^s \alpha_{i,j} \bar{\alpha}_j$. Consider $A = (\alpha_{i,j})_{i,j=1}^s - I$. Then

$$A \cdot \begin{pmatrix} \bar{\alpha}_1 \\ \vdots \\ \bar{\alpha}_s \end{pmatrix} = 0.$$

Furthermore, $d := \det(A) = (-1)^s \pmod{\mathfrak{p}} \Rightarrow d \neq 0$. We now see

$$0 = A^\# A \begin{pmatrix} \bar{\alpha}_1 \\ \vdots \\ \bar{\alpha}_s \end{pmatrix} = d \cdot \begin{pmatrix} \bar{\alpha}_1 \\ \vdots \\ \bar{\alpha}_s \end{pmatrix} \Rightarrow d \cdot N = 0,$$

hence $d \cdot \hat{\mathcal{O}} \subset M = \mathcal{O}\omega_1 + \dots \mathcal{O}\omega_m$. Now, for some $\beta \in L$, we have $\beta = d \underbrace{\beta'}_{\in L} = d \cdot \frac{b}{a} = \frac{1}{a}db$, with $b \in \hat{\mathcal{O}}$ and $a \in \mathcal{O}$. Hence $\beta \in K\omega_1 + \dots + K\omega_m \Rightarrow m = n$ and $\omega_1, \dots, \omega_m$ generate $L \mid K$.

Step 2: We show, that $\dim_K \hat{\mathcal{O}}/\mathfrak{p}_i^{e_i} = e_i f_i$. Consider the chain

$$\hat{\mathcal{O}}/\mathfrak{p}_i^{e_i} \supsetneq \mathfrak{p}_i/\mathfrak{p}_i^{e_i} \supsetneq \dots \supsetneq \mathfrak{p}_i^{e_i-1}/\mathfrak{p}_i^{e_i} \supsetneq 0$$

as a chain of K -vectorspaces. Choose an $\alpha \in \mathfrak{p}_i^j \setminus \mathfrak{p}_i^{j+1}$ and consider the homomorphism

$$\begin{aligned} \hat{\mathcal{O}} &\longrightarrow \mathfrak{p}_i^j/\mathfrak{p}_i^{j+1} \\ a &\longmapsto \alpha \cdot a, \end{aligned}$$

which is surjective with kernel \mathfrak{p}_i (since \mathfrak{p}_i^{j+1} is coprime to $\alpha\hat{\mathcal{O}}$). Therefore $\mathfrak{p}_i^j/\mathfrak{p}_i^{j+1} \cong \hat{\mathcal{O}}/\mathfrak{p}_i$ and we have

$$\dim_K \hat{\mathcal{O}}/\mathfrak{p}_i^{e_i} = \sum_{j=0}^{e_i-1} \dim_K \mathfrak{p}_i^j/\mathfrak{p}_i^{j+1} = e_i \cdot f_i$$

□

Next, we will examine the example of the Gaussian integers $\mathbb{Z}[i]$. By **Proposition 2.10**, $\mathbb{Z}[i]$ is the ring of integers $\hat{\mathcal{O}}$ of the field extension $\mathbb{Q}[i] \mid \mathbb{Q}$.

Reminder 2.8.6. (i) $\mathbb{Z}[i]$ is an euclidean ring $\Rightarrow \mathbb{Z}[i]$ is a PID $\Rightarrow \mathbb{Z}[i]$ is an UFD

(ii) In particular, all prime ideals $\mathfrak{p} = \langle \pi \rangle$ with π prime.

Remark 2.8.7. Let R be a domain, $a, b \in R$. Then $\langle a \rangle = \langle b \rangle \Leftrightarrow a$ and b are associated.

Proof. " \Rightarrow ": $\langle a \rangle = \langle b \rangle \Rightarrow \exists r, r' \in R : b = ra$ and $a = r'b \Rightarrow b = rr'b \Rightarrow (1 - rr')b = 0 \xrightarrow{R \text{ domain}} r, r' \in R^\times$.

" \Leftarrow ": $a = \epsilon b$ with $\epsilon \in R^\times \Rightarrow b = \epsilon^{-1}a \Rightarrow \langle a \rangle = \langle b \rangle$. \square

Remark 2.8.8. For $L = \mathbb{Q}[i]$ and $K = \mathbb{Q}$, we have

- (i) $\text{Gal } L | K = \{\text{id}, (a + bi \mapsto a - bi)\}$
- (ii) $\mathcal{N}_{L|K}(a + bi) = (a + bi) \cdot (a - bi) = a^2 + b^2$.
- (iii) Since $\mathbb{Z}[i]$ is a UFD, an element is prime \Leftrightarrow it is irreducible.
- (iv) $\mathbb{Z}[i]^\times = \{\alpha \in \mathbb{Z}[i] \mid \mathcal{N}_{L|K}(\alpha) = 1\} = \{1, -1, i, -i\}$.
- (v) For $\alpha = a + bi$, its associated elements are $-a - bi, ai - b, -ai + b$.

Proposition 2.8.9 (Theorem of Wilson). *Let $p \in \mathbb{Z}$ be a prime number. Then:*

- (i) $(p - 1)! \equiv -1 \pmod{p}$.
- (ii) If $p = 4n + 1$ with $n \in \mathbb{N}$, then $(2n)!^2 \equiv -1 \pmod{p}$.

Proof. (i) Since the statement is obvious for $p = 2$, let $p > 2$. Consider $X^{p-1} - 1 \in \mathbb{Z}/p\mathbb{Z}[x]$. Then $1, \dots, p-1$ are all zeroes and

$$X^{p-1} - 1 = (x - 1) \cdot (x - 2) \cdot \dots \cdot (x - (p - 1)) \in \mathbb{Z}/p\mathbb{Z}[X].$$

When we look at the constant term, we see that $-1 = (-1)^{p-1} \cdot (p - 1)! = (p - 1)!$

- (ii) $(-1) \equiv (p-1)! \equiv (4n)! = 1 \cdot 2 \cdot \dots \cdot 2n \cdot (p-1) \cdot \dots \cdot (p-2n) \equiv (2n)! \cdot (-1)^{2n} \cdot (2n)! \equiv (2n)!^2 \pmod{p}$.

\square

Proposition 2.8.10. *If p is a prime in \mathbb{Z} with $p \equiv 1 \pmod{4}$, then p is not a prime in $\mathbb{Z}[i]$.*

Proof. Write $p = 4n + 1$. By the Theorem of Wilson, we have $X^2 \equiv -1 \pmod{p}$ for $x = (2n)!$. Then $p \mid X^2 + 1 = (x + i)(x - i) \in \mathbb{Z}[i]$, but $\frac{x \pm i}{p} \notin \mathbb{Z}[i]$. \square

Proposition 2.8.11. *Each prime element $\pi \in \mathbb{Z}[i]$ is associated to one of the following prime elements of $\mathbb{Z}[i]$:*

- (1) $\pi = 1 + i$.
- (2) $\pi = a + bi$, with $a^2 + b^2 = p$ prime in \mathbb{Z} and $p \equiv 1 \pmod{4}$.
- (3) $\pi = p$ prime in \mathbb{Z} and $p \equiv 3 \pmod{4}$.

Proof. We proof the proposition in 3 steps.

Step 1: If π is as in (1) or (2), then π is prime. Suppose $\pi = \alpha\beta$. Then $p = \mathcal{N}(\pi) = \mathcal{N}(\alpha) \cdot \mathcal{N}(\beta) \in \mathbb{Z}$, so either $\mathcal{N}(\alpha) = 1$ or $\mathcal{N}(\beta) = 1$, i.e. α or β is a unit.

Step 2: If π is as in (3), then π is a prime in \mathbb{Z} . Suppose $\pi = \alpha\beta \in \mathbb{Z}[i]$. Then $p^2 = \mathcal{N}(\pi) = \mathcal{N}(\alpha) \cdot \mathcal{N}(\beta)$. If $\alpha, \beta \notin \mathbb{Z}[i]^\times$, then $\mathcal{N}(\alpha) = \mathcal{N}(\beta) = p$. Write $\alpha = a + bi$. Then $p = \mathcal{N}(\alpha) = a^2 + b^2 \not\equiv 3 \pmod{4}$, since it is always $a^2 + b^2 \equiv 0, 1 \pmod{4}$, a contradiction.

Step 3: We have now shown, that the elements (1) – (3) are prime. Let now $\pi_0 \in \mathbb{Z}[i]$ be a prime element. We will show, that π_0 is associated to one of the three elements above. Look at $\mathcal{N}(\pi_0) = p_1 \cdots p_r$ with p_1, \dots, p_r primes in \mathbb{Z} . Since π_0 is prime, it divides $p := p_i$, $1 \leq i \leq r \Rightarrow \mathcal{N}(\pi_0)$ divides $\mathcal{N}(p) = p^2$, i.e. $\mathcal{N}(\pi_0) = p$ or p^2 .

Case 1: $\mathcal{N}(\pi_0) = p$. If $p = 2$, then $\pi_0 \in \{1 + i, 1 - i, -1 + i, -1 - i\}$, i.e. π_0 is associated to $1 + i$. If $p > 2$, then $p = \mathcal{N}(\pi_0) = a^2 + b^2 \equiv 1 \pmod{4} \Rightarrow \pi_0$ is associated to an element as in (2).

Case 2: $\mathcal{N}(\pi_0) = p^2 \Rightarrow \pi_0 | p^2 \Rightarrow \pi_0 | p \Rightarrow \frac{p}{\pi_0} \in \mathbb{Z}[i]$ and $\mathcal{N}(\frac{p}{\pi_0}) = \frac{\mathcal{N}(p)}{\mathcal{N}(\pi_0)} = \frac{p^2}{p^2} = 1$, i.e. $\frac{p}{\pi_0}$ is a unit, hence π_0 is associated to p . By **Proposition 8.10**, $p \not\equiv 1 \pmod{4}$. Also $p \neq 2$, since $2 = (1 + i)(1 - i)$ is not prime in $\mathbb{Z}[i]$. Hence $p \equiv 3 \pmod{4}$ and π_0 is associated to an element as in (3).

□

Corollary 2.8.12 (Fermat). (i) If p is prime then $p = a^2 + b^2 \Leftrightarrow p \not\equiv 3 \pmod{4}$

(ii) $\forall n \in \mathbb{N} : n = a^2 + b^2 \Leftrightarrow \nu_p(n)$ is even for all primes $p \equiv 3 \pmod{4}$ ($\nu_p(n)$ = exponent of p in prime factorization of n over \mathbb{Z}).

Proof. (i) " \Rightarrow ": Same as in Step 2 of 8.11

" \Leftarrow ": If $p = 2$, then $2 = 1 + 1$. If $p \equiv 1 \pmod{4}$, then by **Proposition 8.10**, $p = \alpha\beta \in \mathbb{Z}[i]$ with $\mathcal{N}(\alpha) = \mathcal{N}(\beta) = p$. Write $\alpha = a + bi$ and get $p = \mathcal{N}(\alpha) = a^2 + b^2$.

(ii) " \Rightarrow ": $n = a^2 + b^2 \Rightarrow n = \mathcal{N}(\alpha)$ with $\alpha = a + bi \in \mathbb{Z}[i]$. Write $\alpha = \epsilon \cdot \pi_1 \cdots \pi_r \cdot \pi_{r+1} \cdots \pi_{r+s}$ with π_1, \dots, π_r as in (3) and $\pi_{r+1}, \dots, \pi_{r+s}$ as in (1) or (2). Then $\mathcal{N}(\alpha) = \prod_{i=1}^r \mathcal{N}(\pi_i) = p_1^2 \cdots p_r^2 \cdot p_{r+1} \cdots p_{r+s}$ with $p_1, \dots, p_r \equiv 3 \pmod{4}$ and $p_{r+1}, \dots, p_{r+s} \not\equiv 3 \pmod{4}$.

" \Leftarrow ": $n = p_1^2 \cdots p_r^2 \cdot p_{r+1} \cdots p_{r+s}$ as above. By (i), $p_j \not\equiv 3 \pmod{4}$ and hence $p_j = a_j^2 + b_j^2$ for $r+1 \leq j \leq r+s$. Define $\alpha := p_1 \cdots p_r \cdot (a_{r+1} + ib_{r+1}) \cdots (a_{r+s} + ib_{r+s})$. Then $\mathcal{N}(\alpha) = n$.

□

Corollary 2.8.13. The prime ideals \mathfrak{p}_i in $\mathbb{Z}[i]$ that lie over a prime ideal $\mathfrak{p} = \langle p \rangle$ in \mathbb{Z} are obtained as follows:

(i) $p = 2 \Rightarrow \langle 2 \rangle \mathbb{Z}[i] = \langle 1 + i \rangle \langle 1 - i \rangle = \langle 1 + i \rangle^2$. Hence $r = 1$, $e_1 = 2$, $f_1 = 1$.

(ii) $p \equiv 1 \pmod{4} \xrightarrow{p=a^2+b^2} \langle p \rangle \mathbb{Z}[i] = \langle a+bi \rangle \langle a-bi \rangle$. Hence $r = 2$, $e_1 = e_2 = 1$, $f_1 = f_2 = 1$.

(iii) $p \equiv 3 \pmod{4} \Rightarrow \langle p \rangle \mathbb{Z}[i]$ is a prime ideal. Hence $r = 1$, $e_1 = 1$, $f_1 = 2$.

□

GOAL: Describe prime ideals explicitly for all simple extensions $L = K[\Theta]$ with $\Theta \in \hat{\mathcal{O}}$.

Caution: Before, we had $\mathbb{Z}[i] = \hat{\mathcal{O}}$. In general, we might have $\hat{\mathcal{O}}' := \mathcal{O}[\Theta] \subsetneq \hat{\mathcal{O}}$.

Idea: Take the largest ideal of $\hat{\mathcal{O}}$ which also lies in $\hat{\mathcal{O}}'$.

Definition 2.8.14. The set $\mathcal{F} := \{ \alpha \in \hat{\mathcal{O}} \mid \alpha \hat{\mathcal{O}} \subset \hat{\mathcal{O}}' \}$ is called **conductor**.

Example 2.8.15. If $\hat{\mathcal{O}} = \mathbb{Z}[i]$ and $\Theta = i$, then $\hat{\mathcal{O}}' = \mathcal{O}[\Theta] \Rightarrow \mathcal{F} = \hat{\mathcal{O}}$.

Proposition 2.8.16. In the situation above, let $f(X) := f_{\Theta}(X)$ be the minimal polynomial of Θ . Let \mathfrak{p} be a prime ideal in \mathcal{O} and $K := \mathcal{O}/\mathfrak{p}$. Consider the image \bar{f} of f in $K[X]$ and let $\bar{f} = \bar{f}_1^{e_1} \cdots \bar{f}_r^{e_r}$ be the prime factorization in $K[X]$. Choose preimages $f_1, \dots, f_r \in \mathcal{O}[X]$. Then:

If \mathfrak{p} is coprime to \mathcal{F} , i.e. $\mathfrak{p} + \mathcal{F} \cap \mathcal{O} = \mathcal{O}$, then the ideals in $\hat{\mathcal{O}}$ which lie over \mathfrak{p} are given as follows: $\mathfrak{p}_i := \mathfrak{p}\hat{\mathcal{O}} + f_i(\Theta)\hat{\mathcal{O}}$, $1 \leq i \leq r$ and the local degree of \mathfrak{p}_i is equal to $\deg(\bar{f}_i)$.

Proposition 2.8.17. Let R and S be rings and $\varphi: R \rightarrow S$ a ring homomorphism.

(i) If \mathfrak{q} is a prime ideal in S then $\varphi^{-1}(\mathfrak{q})$ is a prime ideal in R .

(ii) If φ is surjective and \mathfrak{p} is a prime ideal in R with $\ker \varphi \subset \mathfrak{p}$ then $\varphi(\mathfrak{p})$ is a prime ideal in S .

Proof. “(i)” Preimages of ideals are ideals. Suppose $ab \in \varphi^{-1}(\mathfrak{q})$. Then $\varphi(a)\varphi(b) \in \mathfrak{q}$ such that, without loss of generality, $\varphi(a) \in \mathfrak{q}$ and hence $a \in \varphi^{-1}(\mathfrak{q})$.

“(ii)” Images of ideals under surjective homomorphisms are ideals. Let $\bar{a}\bar{b} \in \varphi(\mathfrak{p})$. Since φ is surjective there are $a, b \in R$ with $\varphi(a) = \bar{a}$, $\varphi(b) = \bar{b}$ and there is $c \in \mathfrak{p}$ with $\varphi(c) = \bar{a}\bar{b}$. Hence

$$ab - c \in \ker \varphi \subset \mathfrak{p}$$

such that $ab \in \mathfrak{p}$. We may assume that $a \in \mathfrak{p}$ and conclude $\bar{a} = \varphi(a) \in \varphi(\mathfrak{p})$. □

Definition 2.8.18. In the situation of Proposition 2.8.17 we define:

(i) $\text{Spec}(R) = \{ \mathfrak{p} \subset R \mid \mathfrak{p} \text{ is a prime ideal} \}$

(ii) $\text{Spec}_S(R) = \{ \mathfrak{p} \subset \text{Spec}(R) \mid \mathfrak{p} \supset \ker \varphi \}$

Corollary 2.8.19. In the situation of Proposition 2.8.17 we have:

(i) If $\varphi: R \rightarrow S$ is a homomorphism of rings then φ induces a map

$$\varphi^*: \text{Spec}(S) \rightarrow \text{Spec}_S(R), \mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p}).$$

(ii) If φ is surjective then φ^* is a bijection with inverse map

$$\varphi_*: \operatorname{Spec}_S(R) \rightarrow \operatorname{Spec}(S), \mathfrak{p} \mapsto \varphi(\mathfrak{p}).$$

Reminder 2.8.20. For $a \in \mathbb{Z}$ and p prime in \mathbb{Z} the **Legendre symbol** is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 0, & p \text{ divides } a, \\ 1, & \text{there is an } x \in \mathbb{Z}/p\mathbb{Z} \text{ such that } x^2 \equiv a \pmod{p}, \\ -1, & \text{else.} \end{cases}$$

Example 2.8.21. Apply Proposition 8.15 for quadratic number fields, D square-free:

$$\begin{array}{ccccc} \hat{\mathcal{O}} & = & \mathbb{Z}[\theta] & \subset & \mathbb{Q}(\sqrt{D}) \\ & & \uparrow & & \uparrow \\ \mathcal{O} & = & \mathbb{Z} & \subset & \mathbb{Q} \end{array}$$

Reminder 2.8.22. If $D \not\equiv 1 \pmod{4}$ then we can choose $\theta = \sqrt{D}$ and obtain $f = f_\theta = X^2 - D$ and $d(f_\theta) = 4D$.

If $D \equiv 1 \pmod{4}$ then we can choose $\theta = \frac{1}{2}(1 + \sqrt{D})$ and obtain $f = f_\theta = X^2 - X - \frac{D-1}{4}$ and $d(f_\theta) = D$.

Consider $p \in \mathbb{Z}$ prime and define $\bar{f} = \bar{f}_\theta$ as the image of f in $\mathbb{Z}/p\mathbb{Z}[X]$.

Observe: \bar{f} has two equal zeroes in $\mathbb{Z}/p\mathbb{Z}$ iff $d(f) = 0$ in $\mathbb{Z}/p\mathbb{Z}$ iff

$$\begin{cases} \left(\frac{4D}{p}\right) = 0, & D \not\equiv 1 \pmod{4}, \\ \left(\frac{D}{p}\right) = 0, & D \equiv 1 \pmod{4}. \end{cases}$$

\bar{f} has two different zeroes in $\mathbb{Z}/p\mathbb{Z}$ iff $d(f)$ is a non-zero square in $\mathbb{Z}/p\mathbb{Z}$ iff

$$\begin{cases} \left(\frac{4D}{p}\right) = 1, & D \not\equiv 1 \pmod{4}, \\ \left(\frac{D}{p}\right) = 1, & D \equiv 1 \pmod{4} \end{cases} \Leftrightarrow \left(\frac{D}{p}\right) = 1.$$

\bar{f} has no zeroes in $\mathbb{Z}/p\mathbb{Z}$ iff $\left(\frac{D}{p}\right) = -1$.

Proposition 8.15 then implies in the first case that $p\hat{\mathcal{O}} = \hat{\mathcal{P}}_1^2$ with

$$\mathcal{P}_1 = \begin{cases} p\hat{\mathcal{O}} + \theta\hat{\mathcal{O}}, & D \not\equiv 1 \pmod{4}, \\ p\hat{\mathcal{O}} + \left(\theta - \frac{1}{2}\right)\hat{\mathcal{O}}, & D \equiv 1 \pmod{4}, \end{cases}$$

In the second case we obtain $p\hat{\mathcal{O}} = \hat{\mathcal{P}}_1\hat{\mathcal{P}}_2$ with $\hat{\mathcal{P}}_{1,2} = p\hat{\mathcal{O}} + (\theta \pm x)\hat{\mathcal{O}}$, where $x^2 \equiv D \pmod{p}$.

In the third case $p\hat{\mathcal{P}}$ is a prime ideal.

Example. Let $D \not\equiv 1 \pmod{p}$, $\left(\frac{4D}{p}\right) = 0$ and $p \neq 2$. Consider the map $\pi: \hat{\mathcal{O}} \rightarrow \hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}}$ with $\hat{\mathcal{O}} = \mathbb{Z}[\sqrt{D}]$ and $\mathfrak{p}\hat{\mathcal{O}} = \{a + b\sqrt{D} \mid p|a \text{ and } p|b\}$ and thus

$$\begin{aligned} \hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}} &\cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}[\sqrt{D}] \cong (\mathbb{Z}/p\mathbb{Z}[X])/(X^2 - D), \\ \theta &\leftrightarrow (0, \sqrt{D}) \leftrightarrow \bar{X}. \end{aligned}$$

We have

$$\hat{\mathcal{P}}_1 = \pi^{-1}((\bar{\theta})) = \{a + b\sqrt{D} \mid p \text{ divides } a\}.$$

Example. Let $D \not\equiv 1 \pmod{p}$ and $\left(\frac{4D}{p}\right) = 1$. Then there exists $x \in \mathbb{Z}$ with $x^2 \equiv D \pmod{p}$ and $p \nmid x$. Here, $\pi: \hat{\mathcal{O}} \rightarrow \hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}}$ is the map

$$\begin{aligned} \mathbb{Z}[\sqrt{D}] &\rightarrow \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}[\sqrt{D}] \\ &\cong (\mathbb{Z}/p\mathbb{Z}[X])/(X - x)(X + x) \\ &\cong (\mathbb{Z}/p\mathbb{Z}[X])/(X - x) \oplus (\mathbb{Z}/p\mathbb{Z}[X])/(X + x) \end{aligned}$$

given by

$$a + b\sqrt{D} \mapsto \bar{a} + \bar{b}\sqrt{D} \cong \bar{a} + \bar{b}X \cong (\bar{a} + \bar{b}x, \bar{a} - \bar{b}x).$$

Recall that $\bar{f}(X) = (X - x)(X + x) = \bar{f}_1\bar{f}_2$ with $\bar{f}_1, \bar{f}_2 \in \mathbb{Z}[X]$ and

$$f_1(\theta) = \theta - x = \sqrt{D} - x = -x + \sqrt{D}$$

with $\pi(f_1(\theta)) \leftrightarrow (0, -2\bar{x})$. Observe that for $\bar{x} \in \mathbb{F}_p^\times$ we have the correspondence

$$(\pi(f_1(\theta))) \leftrightarrow \mathcal{O} \oplus (\mathbb{Z}/p\mathbb{Z}[X])/(X + p) \cong \mathcal{O} \oplus \mathbb{Z}/p\mathbb{Z}$$

and hence $\hat{\mathcal{P}}_1 = \pi^{-1}(\mathcal{O} \oplus \mathbb{Z}/p\mathbb{Z})$.

Proof of Prop. 8.16. Consider the map $\pi: \hat{\mathcal{O}} \rightarrow \hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}}$. By Corollary 8.19 we have a bijection

$$\{\hat{\mathcal{P}} \mid \hat{\mathcal{P}} \text{ prime ideal in } \hat{\mathcal{O}} \text{ with } \hat{\mathcal{P}} \cap \mathcal{O} = \mathfrak{p}\} \leftrightarrow \{\mathfrak{q} \mid \mathfrak{q} \text{ prime ideal in } \hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}}\}.$$

We show:

$$\hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}} \cong \hat{\mathcal{O}}'/\mathfrak{p}\hat{\mathcal{O}}' \cong k[X]/(\bar{f}),$$

where $k = \hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}}$ and $\hat{\mathcal{O}}' = \mathcal{O}[\theta]$.

Step 1: $\hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}} \cong \hat{\mathcal{O}}'/\mathfrak{p}\hat{\mathcal{O}}'$

Consider the homomorphism $\varphi: \hat{\mathcal{O}}' \rightarrow \hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}}$ induced by the inclusion $\hat{\mathcal{O}}' \hookrightarrow \hat{\mathcal{O}}$.

“(1)” φ is surjective: If $\mathfrak{p} + (\mathbb{F} \cap \mathcal{O}) = \mathcal{O}$ then $\mathfrak{p}\hat{\mathcal{O}} + \mathbb{F} = \hat{\mathcal{O}}$ and hence $\mathfrak{p}\hat{\mathcal{O}} + \hat{\mathcal{O}}' = \hat{\mathcal{O}}$ (multiply both sides of first equation with $\hat{\mathcal{O}}$).

“(2)” $\ker \varphi = \mathfrak{p}\hat{\mathcal{O}}'$: “ \supset ” Clear. “ \subset ” We have $\ker \varphi = \hat{\mathcal{O}}' \cap \mathfrak{p}\hat{\mathcal{O}}$. Use $\mathfrak{p} + (\mathbb{F} \cap \mathcal{O}) = \mathcal{O}$ and write $1 = p + a$ with $p \in \mathfrak{p}$ and $a \in \mathbb{F} \cap \mathcal{O}$. For $x \in \hat{\mathcal{O}}' \cap \mathfrak{p}\hat{\mathcal{O}}$ we have:

$$x = 1 \cdot x = (p + a)x = px + ax \in \mathfrak{p}\hat{\mathcal{O}}'.$$

Step 2: $\hat{\mathcal{O}}'/\mathfrak{p}\hat{\mathcal{O}}' \cong k[X]/(\bar{f})$

Recall that $\hat{\mathcal{O}}' = \mathcal{O}[\theta] \cong \mathcal{O}[X]/(f)$. Consider $\Psi: \mathcal{O}[X] \rightarrow k[X]/(\bar{f})$, which is surjective. It holds that $\ker \Psi = (\mathfrak{p}, f)$ and hence Ψ induces an isomorphism $\hat{\mathcal{O}}'/\mathfrak{p}\hat{\mathcal{O}}' \rightarrow k[X]/(\bar{f})$.

Step 3: Consider now $R = k[X]/(\bar{f})$ and determine $\text{Spec}(R)$.

“(1)” Recall the prime decomposition $\bar{f} = \bar{f}_1^{e_1} \cdots \bar{f}_r^{e_r}$ in $k[X]$ and consider the projection $k[X] \twoheadrightarrow k[X]/(\bar{f})$. By Corollary 8.19 we have the correspondence

$$\text{Spec}(R) \leftrightarrow \{\mathfrak{p} \text{ prime ideal in } k[X] \mid \bar{f} \in \mathfrak{p}\}$$

and hence $\text{Spec}(R) = \{(\bar{f}_i) \mid i = 1, \dots, r\}$.

“(2)” Notice that

$$R/(\bar{f}_i) = (k[X]/(\bar{f})) / (\bar{f}_i) \cong k[X]/(\bar{f}_i)$$

is a k -vector space of dimension $\deg(\bar{f}_i)$ such that

$$[R/(\bar{f}_i) : k] = \deg(\bar{f}_i).$$

“(3)” In R we have

$$\bigcap_{i=1}^r (\bar{f}_i)^{e_i} = (\bar{f}) = 0.$$

Step 4: Use the isomorphism

$$k[X]/(\bar{f}) \rightarrow \hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}}, g \mapsto g(\theta)$$

and obtain from Step 3 with $\mathcal{P}_i = (f_i(\theta))$ that:

$$(i) \quad \text{Spec}(\hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}}) = \{\mathcal{P}_i \mid i = 1, \dots, r\}$$

$$(ii) \quad \left[(\hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}}) / \mathcal{P}_i : k \right] = \deg(\bar{f}_i)$$

$$(iii) \quad \bigcap_{i=1}^r \mathcal{P}_i^{e_i} = 0$$

Step 5: Take preimages in $\hat{\mathcal{O}}$ via $\hat{\mathcal{O}} \rightarrow \hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}}$ and observe that (iii) implies $\bigcap_{i=1}^r \hat{\mathcal{P}}_i^{e_i} \subset \mathfrak{p}\hat{\mathcal{O}}$ such that $\mathfrak{p}\hat{\mathcal{O}}$ divides $\bigcap_{i=1}^r \hat{\mathcal{P}}_i^{e_i}$. Furthermore,

$$[L : K] = n = \deg(f) = \sum_{i=1}^r e_i f_i$$

such that by Theorem 11, $\mathfrak{p}\hat{\mathcal{O}} = \prod_{i=1}^r \hat{\mathcal{P}}_i^{e_i}$. □

$$\begin{array}{ccccccc}
& & \hat{\mathcal{P}} & \subseteq & \hat{\mathcal{O}} & \subseteq & L \\
& & \uparrow & & \uparrow & & \uparrow \\
\hat{\mathcal{O}}\mathcal{P} = \hat{\mathcal{P}}_1^{e_1} \cdots \hat{\mathcal{P}}_r^{e_r} & & \mathcal{P} & \subseteq & \mathcal{O} & \subseteq & K
\end{array}$$

Proposition 2.8.23. *There are only finitely many prime ideals $\hat{\mathcal{P}}$ in $\hat{\mathcal{O}}$ which are ramified over $\mathcal{P} = \hat{\mathcal{P}} \cap \mathcal{O}$.*

Proof. Choose primitive element θ of $L|K$ in $\hat{\mathcal{O}}$. Let $f_\theta \in \mathcal{O}[X]$ be the minimal polynomial of θ and $d := \text{discr}(f_\theta) = \text{discr}(1, \theta, \dots, \theta^{n-1}) = \prod_{i < j} (\theta_i - \theta_j)^2 \in \mathcal{O}$.

Here θ_i, θ_j are the zeroes of f_θ in the algebraic closure.

Claim: If \mathcal{P} is a prime ideal in \mathcal{O} s.t.

- \mathcal{P} is coprime to (d) and
- \mathcal{P} is coprime to $\mathbb{F} \cap \mathcal{O}$

then \mathcal{P} is unramified, i.e. all $\hat{\mathcal{P}}$ lying above \mathcal{P} are unramified.

From the claim we obtain that there are only finitely many \mathcal{P} which allow ramification.

Proof of the claim: Write $\hat{\mathcal{O}}\mathcal{P} = \hat{\mathcal{P}}_1^{e_1} \cdots \hat{\mathcal{P}}_r^{e_r}$. Consider $\bar{f}_\theta \in \mathcal{O}/\mathcal{P}[X]$. As in Prop. 8.15

$$\bar{f}_\theta = \bar{f}_1^{e_1} \cdots \bar{f}_r^{e_r} \quad (\star)$$

a prime decomposition. (d) and \mathcal{P} are coprime $\Rightarrow \bar{d} = \text{image of } d \text{ in } \mathcal{O}/\mathcal{P} \neq 0 \Rightarrow \bar{f}_\theta$ has only single zeroes in an algebraic closure of $\mathcal{O}/\mathcal{P} \xrightarrow{(\star)} e_1 = \cdots = e_r = 1$ \square

Definition 2.8.24.

- \mathcal{P} is said to split completely or to be totally split : $\iff e_i = f_i = 1 \ \forall i \in \underline{r}$.
- \mathcal{P} is said to be indecomposed, nonsplit or totally ramified : $\iff r = 1$.

2.9 Hilbert's theorem of ramification

Idea: Consider Galois extensions $L|K \rightarrow$ life becomes much nicer.

Same setting as in 8. Suppose further that $L|K$ normal and consider $G = \text{Gal}(L|K)$.

Remark 2.9.1. i) $\hat{\mathcal{P}}$ prime ideals in $\hat{\mathcal{O}}$ with $\mathcal{P} := \hat{\mathcal{P}} \cap \mathcal{O}$. For $\sigma \in \text{Gal}(L|K)$ we have $\sigma(\hat{\mathcal{P}})$ is a prime ideal in $\hat{\mathcal{O}}$ above \mathcal{P} .

ii) $\text{Gal}(L|K)$ acts transitively on the set of prime ideals $\hat{\mathcal{P}}$ in $\hat{\mathcal{O}}$ over \mathcal{P} .

Proof. i) Recall from Rem 2.1 iii) that $\sigma(\hat{\mathcal{O}}) = \hat{\mathcal{O}}$
 $\Rightarrow \sigma(\hat{\mathcal{P}})$ is again a prime ideal in $\hat{\mathcal{O}}$.
 $\sigma(\hat{\mathcal{P}}) \cap \mathcal{O} = \sigma(\hat{\mathcal{P}} \cap \mathcal{O}) = \sigma(\mathcal{P}) = \mathcal{P}$
 $\Rightarrow \sigma(\hat{\mathcal{P}})$ lies above \mathcal{P} .

- ii) follows from i) that we have such an action. Let $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}'$ be prime ideals above $\mathcal{P} = \hat{\mathcal{P}} \cap \mathcal{O} = \hat{\mathcal{P}}' \cap \mathcal{O}$. Assume that $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}'$ are not in the same G -orbit. Hence $\hat{\mathcal{P}}'$ and $\sigma(\hat{\mathcal{P}})$ are coprime for each $\sigma \in G$.
 $\Rightarrow \hat{\mathcal{P}}'$ is coprime to $\sigma_1(\hat{\mathcal{P}}) \cdot \dots \cdot \sigma_n(\hat{\mathcal{P}})$, where $G = \{\sigma_1, \dots, \sigma_n\}$.
 CRT $\Rightarrow \exists x \in \hat{\mathcal{O}}$ with $x \equiv 0 \pmod{\hat{\mathcal{P}}'}$ and $x \equiv 1 \pmod{\sigma(\hat{\mathcal{P}})}$ for all $\sigma \in G$.
 In particular: $\mathcal{N}_{L|K}(x) = \prod_{\sigma \in G} \sigma(x) \in \hat{\mathcal{P}}' \cap \mathcal{O} = \mathcal{P}$
 Also: $\forall \sigma \in G : x \notin \sigma(\hat{\mathcal{P}}) \Rightarrow \forall \sigma \in G : \sigma(x) \notin \mathcal{P}$
 $\Rightarrow \mathcal{N}_{L|K}(x) = \prod_{\sigma \in G} \sigma(x) \notin \hat{\mathcal{P}} \cap \mathcal{O} = \mathcal{P} \nmid$.

□

Definition 2.9.2. Let $\hat{\mathcal{P}}$ be a prime ideal of $\hat{\mathcal{O}}$ above \mathcal{P} .

- i) $G_{\hat{\mathcal{P}}} := \text{Stab}_G(\hat{\mathcal{P}}) = \{\sigma \in G \mid \sigma(\hat{\mathcal{P}}) = \hat{\mathcal{P}}\}$ is called decomposition group („Zerlegungsgruppe“)
 ii) $Z_{\hat{\mathcal{P}}} := \{x \in L \mid \sigma(x) = x \text{ for all } \sigma \in G_{\hat{\mathcal{P}}}\}$ is called decomposition field („Zerlegungskörper“)

Remark 2.9.3. Let $\hat{\mathcal{P}}_0$ be a prime ideal which lies above \mathcal{P} .

- i) $G/G_{\hat{\mathcal{P}}_0} := \{gG_{\hat{\mathcal{P}}_0} \mid g \in G\} \xleftrightarrow{1:1} \{\hat{\mathcal{P}} \mid \hat{\mathcal{P}} \text{ lies above } \mathcal{P}\}$
 ii) $G_{\hat{\mathcal{P}}_0} = \{1\} \iff [G : G_{\hat{\mathcal{P}}_0}] = [L : K] = n \iff \mathcal{P} \text{ is totally split} \iff Z_{\hat{\mathcal{P}}_0} = L$ ($r = [G : G_{\hat{\mathcal{P}}_0}]$)
 iii) $G_{\hat{\mathcal{P}}_0} = G \iff [G : G_{\hat{\mathcal{P}}_0}] = 1 \iff \mathcal{P} \text{ is nonsplit} \iff Z_{\hat{\mathcal{P}}_0} = K$
 iv) $G_{\sigma(\hat{\mathcal{P}}_0)} = \sigma \circ G_{\hat{\mathcal{P}}_0} \circ \sigma^{-1}$

Proof. Follows from Prop 9.1 + definitions + group actions. □

Remark 2.9.4. Suppose $\mathcal{P}\hat{\mathcal{O}} = \hat{\mathcal{P}}_1^{e_1} \cdot \dots \cdot \hat{\mathcal{P}}_r^{e_r}$ with local degrees $f_i = [\hat{\mathcal{O}}/\hat{\mathcal{P}}_i : \mathcal{O}/\mathcal{P}]$. Then $e_1 = \dots = e_r$ and $f_1 = \dots = f_r$.

Proof. Prop. 9.1 $\Rightarrow \exists \sigma_i \in G$ s.t. $\sigma_i(\hat{\mathcal{P}}_1) = \hat{\mathcal{P}}_i$
 $\Rightarrow \hat{\mathcal{O}}/\hat{\mathcal{P}}_1 \cong \hat{\mathcal{O}}/\hat{\mathcal{P}}_i$, $a \pmod{\hat{\mathcal{P}}_1} \mapsto \sigma_i(a) \pmod{\hat{\mathcal{P}}_i}$ as $k = \mathcal{O}/\mathcal{P}$ -vectorspaces $\Rightarrow f_1 = f_i$ and $\hat{\mathcal{P}}_i^k \supseteq \mathcal{P}\hat{\mathcal{O}} \iff \hat{\mathcal{P}}_i^k = (\sigma_i(\hat{\mathcal{P}}_1))^k \supseteq \mathcal{P}\hat{\mathcal{O}} = \sigma_i(\mathcal{P}\hat{\mathcal{O}}) \Rightarrow e_i = e_1$. □

Consider the field extensions $K \subseteq Z_{\hat{\mathcal{P}}} \subseteq L$. We have:

$$\begin{array}{ccccccc}
 & & \hat{\mathcal{P}} & \subseteq & \hat{\mathcal{O}} & \subseteq & L \\
 & & | & & | & & | \\
 \hat{\mathcal{P}}_Z & := & \hat{\mathcal{P}} \cap Z_{\hat{\mathcal{P}}} & \subseteq & \hat{\mathcal{O}} \cap Z_{\hat{\mathcal{P}}} & \subseteq & Z_{\hat{\mathcal{P}}} \\
 & & | & & | & & | \\
 & & \mathcal{P} & \subseteq & \mathcal{O} & \subseteq & K
 \end{array}$$

Observe $\hat{\mathcal{O}} \cap Z_{\hat{\mathcal{P}}}$ is the integral closure of \mathcal{O} in $Z_{\hat{\mathcal{P}}}$.

Proposition 2.9.5. Suppose $\mathcal{P}\hat{\mathcal{O}} = (\prod_{\sigma} \sigma(\hat{\mathcal{P}}))^e$ with local degree f .

- i) $\hat{\mathcal{P}}_Z$ is non-split in $\hat{\mathcal{O}}$, i.e. $\hat{\mathcal{P}}$ is the only prime ideal above $\hat{\mathcal{P}}_Z$.
- ii) $\hat{\mathcal{P}}/\hat{\mathcal{P}}_Z$ has ramification index e and local degree f .
- iii) $\hat{\mathcal{P}}_Z/\mathcal{P}$ has ramification index 1 and local degree 1, i.e. $\hat{\mathcal{P}}_Z/\mathcal{P}$ is totally split.

Proof. i) $Z_{\hat{\mathcal{P}}} = L^{G_{\hat{\mathcal{P}}}} \Rightarrow \text{Gal}(L/Z_{\hat{\mathcal{P}}}) = G_{\hat{\mathcal{P}}}$. Now statement follows from 9.3 iii)

ii)+iii) Let $e' = \text{ramification index of } \hat{\mathcal{P}}/\hat{\mathcal{P}}_Z$ and $e'' = \text{ramification index of } \hat{\mathcal{P}}_Z/\mathcal{P}$

Let $f' = \text{local degree of } \hat{\mathcal{P}}/\hat{\mathcal{P}}_Z$ and $f'' = \text{local degree of } \hat{\mathcal{P}}_Z/\mathcal{P}$.

Hence: $\hat{\mathcal{P}}_Z\hat{\mathcal{O}} = \hat{\mathcal{P}}^{e'}$ and $\mathcal{P}(\hat{\mathcal{O}} \cap Z_{\hat{\mathcal{P}}}) = \hat{\mathcal{P}}_Z^{e''} \cdot \dots \Rightarrow \mathcal{P}\hat{\mathcal{O}} = (\hat{\mathcal{P}}^{e'})^{e''} \cdot \dots$

$\Rightarrow e = e' \cdot e'' \quad (\star).$

Also we have for the field extensions

$$\hat{\mathcal{O}}/\hat{\mathcal{P}} \supseteq \underbrace{\hat{\mathcal{O}} \cap Z_{\hat{\mathcal{P}}}/\hat{\mathcal{P}}_Z}_{f'} \supseteq \underbrace{\mathcal{O}/\mathcal{P}}_{f''}$$

$\Rightarrow f = f' \cdot f'' \quad (\star\star).$

Thm. 11 \Rightarrow 1) For $L|K : n = [L : K] = e \cdot f \cdot r$ with $r = [G : G_{\hat{\mathcal{P}}}] \quad (n = |G|).$

2) For $L|Z_{\hat{\mathcal{P}}} : |G_{\hat{\mathcal{P}}}| = \frac{n}{r} \stackrel{\text{Thm. 11}}{=} e' \cdot f' \cdot \underbrace{r'}_{=1(\text{by i})} \stackrel{1)}{=} e \cdot f \Rightarrow e' = e, f' = f$ and

$e'' = 1 = f'' \Rightarrow \text{Claim.}$

□

Definition 2.9.6. In our general setting we call $\kappa(\hat{\mathcal{P}}) := \hat{\mathcal{O}}/\hat{\mathcal{P}}$ the residue class field („Restklassenkörper“).

Remark 2.9.7. Prop 9.5 iii) $\Rightarrow [\kappa(\hat{\mathcal{P}}_Z) : \kappa(\mathcal{P})] = 1$ hence, $\kappa(\hat{\mathcal{P}}_Z) = \kappa(\mathcal{P}) = \mathcal{O}/\mathcal{P} =: k$.

Proposition 2.9.8. If $\hat{\mathcal{P}}/\mathcal{P}$ is non-split, i.e. $\hat{\mathcal{P}}$ is the only prime ideal over \mathcal{P} , then we obtain the following surjective group homomorphism: $\varphi : G = \text{Gal}(L/K) \rightarrow \text{Aut}(\kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P})).$

Proof. Step 1: φ is well-defined:

Since $\hat{\mathcal{P}}/\mathcal{P}$ is totally split, we have $\sigma(\hat{\mathcal{P}}) = \hat{\mathcal{P}}$. Therefore $\sigma \in \text{Gal}(L/K)$ induces an automorphism of $\kappa(\hat{\mathcal{P}})$.

Step 2: $\kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P})$ is a normal extension:

Denote $k := \kappa(\mathcal{P})$ and $\kappa := \kappa(\hat{\mathcal{P}})$. Consider $\bar{\theta} \in \kappa$ and let $\bar{g} \in k[X]$ be its minimal polynomial over k . Have to show that \bar{g} decomposes into linear factors over κ . Let θ be a preimage of $\bar{\theta}$ in $\hat{\mathcal{O}}$ and $f \in \mathcal{O}[X]$ its minimal polynomial $\Rightarrow f(\bar{\theta}) = 0$. Let \bar{f} be the image of f in $k[X]$, hence $\bar{f}(\bar{\theta}) = 0$ and thus \bar{g} divides \bar{f} .

Furthermore: L/K is normal $\Rightarrow f$ decomposes into linear factors over $L \Rightarrow$ also over $\hat{\mathcal{O}}$, since Galois-Automorphisms preserve $\hat{\mathcal{O}} \Rightarrow \bar{f}$ decomposes into linear factors over $\kappa = \hat{\mathcal{O}}/\hat{\mathcal{P}} \Rightarrow \bar{g}$ does so.

Step 3: φ is surjective:

Let $\bar{\sigma} \in \text{Aut}(\kappa/k)$. Consider the field extension: $k \subseteq E \subseteq \underbrace{\kappa}_{\text{purely inseparable} \Rightarrow \text{Aut}(\kappa/E)=\{1\}}$ (\star)

with E is the maximal separable field extension.

$\Rightarrow \exists \bar{\theta} \in E$ with $E = k(\bar{\theta})$ and $\theta \in \hat{\mathcal{O}}$ a preimage. Let again $\bar{g} \in k[X]$ be the minimal polynomial of $\bar{\theta}$ and f, \bar{f} as in Step 2.

$\Rightarrow \bar{\sigma}(\bar{\theta})$ is a zero of \bar{g} , hence $(X - \bar{\sigma}(\bar{\theta}))$ divides \bar{g} and hence \bar{f} since \bar{g}, f and \bar{f} decompose into linear factors.

$\Rightarrow \exists \theta' \in \hat{\mathcal{O}}$ with $\theta' \bmod \hat{\mathcal{P}} = \bar{\sigma}(\bar{\theta})$ and θ' is a zero of f (there is a linear factor $(X - \theta')$ of f which is sent to the factor $(X - \bar{\sigma}(\bar{\theta}))$ of \bar{f})

$\Rightarrow \exists \sigma \in \text{Gal}(L/K)$ with $\sigma(\theta) = \theta'$ and thus $\sigma(\theta) \equiv \theta' \equiv \bar{\sigma}(\bar{\theta}) \bmod \hat{\mathcal{P}}$.

$\Rightarrow \varphi(\sigma)|_E = \bar{\sigma}|_E \xrightarrow{(\star)} \varphi(\sigma) = \bar{\sigma}$ □

Remark 2.9.9. Observe that for Step 2 we did not need that $\hat{\mathcal{P}}/\mathcal{P}$ is non-split. Hence we have in the general situation of this section:

$$L/K \text{ normal} \Rightarrow \kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P}) \text{ is normal.}$$

Proposition 2.9.10. *In general, we obtain the following surjective group homomorphism:*

$$G_{\hat{\mathcal{P}}} \rightarrow \text{Aut}(\kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P})) , \sigma \mapsto (a \bmod \hat{\mathcal{P}} \mapsto \sigma(a) \bmod \hat{\mathcal{P}})$$

Proof. Idea: Consider $K \subseteq Z_{\hat{\mathcal{P}}} \subseteq \underbrace{L}_{\text{non-split}}$. Remark 9.7 $\Rightarrow \kappa(\hat{\mathcal{P}}_Z) = k := \kappa(\mathcal{P})$

Lemma 9.8 $\Rightarrow \underbrace{\text{Gal}(L/Z_{\hat{\mathcal{P}}})}_{=G_{\hat{\mathcal{P}}}} \twoheadrightarrow \underbrace{\text{Aut}(\kappa(\hat{\mathcal{P}})/\kappa(\hat{\mathcal{P}}_Z))}_{\text{Aut}(\kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P}))} \Rightarrow \text{Claim.}$ □

Definition 2.9.11 („Trägheitsgruppe“/„Trägheitskörper“). Let $\varphi : G_{\hat{\mathcal{P}}} \twoheadrightarrow \text{Gal}(\kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P}))$ be the surjective group homomorphism from Prop. 9.10.

i) $I_{\hat{\mathcal{P}}} := \ker(\varphi)$ is called inertia group.

ii) $T_{\hat{\mathcal{P}}} := \{x \in L \mid \sigma(x) = x \ \forall \sigma \in I_{\hat{\mathcal{P}}}\}$ is called inertia field.

Remark 2.9.12. i) We obtain the following chain of field extensions:

$$K \subseteq Z_{\hat{\mathcal{P}}} \subseteq T_{\hat{\mathcal{P}}} \subseteq L$$

ii) We have the following short exact sequence:

$$1 \rightarrow I_{\hat{\mathcal{P}}} \rightarrow G_{\hat{\mathcal{P}}} \rightarrow \text{Aut}(\kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P})) \rightarrow 1$$

Proposition 2.9.13. *In the situation of 9.12 we have:*

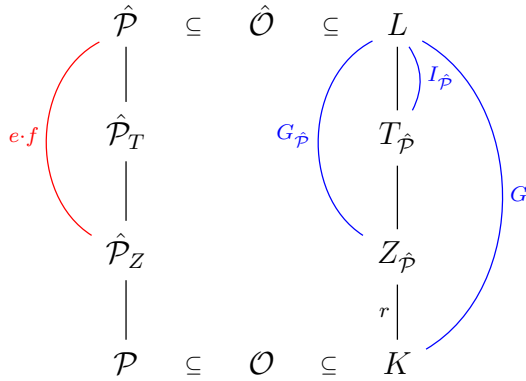
i) $T_{\hat{\mathcal{P}}}/Z_{\hat{\mathcal{P}}}$ is normal and $\text{Gal}(T_{\hat{\mathcal{P}}}/Z_{\hat{\mathcal{P}}}) \cong \text{Aut}(\kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P}))$.

Furthermore: $\text{Gal}(L/T_{\hat{\mathcal{P}}}) \cong I_{\hat{\mathcal{P}}}$.

ii) If $\kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P})$ is separable, then: $\#I_{\hat{\mathcal{P}}} = [L : T_{\hat{\mathcal{P}}}] = e$ and $[G_{\hat{\mathcal{P}}} : I_{\hat{\mathcal{P}}}] = [T_{\hat{\mathcal{P}}} : Z_{\hat{\mathcal{P}}}] = f$

iii) If $\kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P})$ is separable and $\hat{\mathcal{P}}_T := \hat{\mathcal{P}} \cap T_{\hat{\mathcal{P}}}$, then we have

- The ramification index of $\hat{\mathcal{P}}$ over $\hat{\mathcal{P}}_T$ is e and the local degree is 1.
- The ramification index of $\hat{\mathcal{P}}_T$ over $\hat{\mathcal{P}}_Z$ is 1 and the local degree is f .



Proof. i) • $I_{\hat{\mathcal{P}}}$ is normal in $G_{\hat{\mathcal{P}}}$.

- $\text{Gal}(T_{\hat{\mathcal{P}}}/Z_{\hat{\mathcal{P}}}) \cong G_{\hat{\mathcal{P}}}/I_{\hat{\mathcal{P}}} \stackrel{\text{Rem 9.12}}{\cong} \text{Aut}(\kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P}))$
- $T_{\hat{\mathcal{P}}}$ is the fixed field of $I_{\hat{\mathcal{P}}}$

$$\text{ii) } \kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P}) \text{ is separable} \Rightarrow \# \text{Aut}(\kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P})) = \underbrace{[\kappa(\hat{\mathcal{P}}) : \kappa(\mathcal{P})]}_{\hat{\mathcal{O}}/\hat{\mathcal{P}}} \stackrel{9.12}{=} \underbrace{\#G_{\hat{\mathcal{P}}}}_{e.f} / \#I_{\hat{\mathcal{P}}} = f$$

iii) We will show below that $\kappa(\hat{\mathcal{P}}_T) = \kappa(\hat{\mathcal{P}})$. This implies:

- local degree of $\hat{\mathcal{P}}_T/\hat{\mathcal{P}}$ is 1
- ramification index of $\hat{\mathcal{P}}_T/\hat{\mathcal{P}}$ is e since $[L/T_{\hat{\mathcal{P}}}] = \#I_{\hat{\mathcal{P}}} = e$
- multiplicativity of e and $f \Rightarrow \text{rest } \checkmark$

Show that $\kappa(\hat{\mathcal{P}}_T) = \kappa(\hat{\mathcal{P}})$:

Use Lemma 9.8 \Rightarrow Obtain surjective group homomorphism

$$I_{\hat{\mathcal{P}}} = \text{Gal}(L/T_{\hat{\mathcal{P}}}) \xrightarrow{\varphi} \text{Aut}(\kappa(\hat{\mathcal{P}})/\kappa(\hat{\mathcal{P}}_T))$$

By definition of $I_{\hat{\mathcal{P}}}$ the image of this homomorphism is trivial.

$$\Rightarrow \text{Aut}(\kappa(\hat{\mathcal{P}})/\kappa(\hat{\mathcal{P}}_T)) = \{1\} \stackrel{\text{normal}+\text{separable}}{\implies} [\kappa(\hat{\mathcal{P}}) : \kappa(\hat{\mathcal{P}}_T)] = 1.$$

□

2.10 Cyclotomic Fields

In this section, we have

- $\zeta = \zeta_n$ = primitive n -th root of unity

- $L = \mathbb{Q}(\zeta)$
- \mathcal{O} = ring of integers in L
- $d = \varphi(n) = [L : \mathbb{Q}]$.

GOAL:

- (1) Show, that $\mathcal{O} = \mathbb{Z}[\zeta]$
- (2) Describe the prime ideals in \mathcal{O}

Lemma 2.10.1. Suppose $n = l^k$ with l prime and hence $d = \varphi(n) = l^k - l^{k-1} = l^{k-1}(l-1)$.

- The minimal polynomial $\phi(X)$ of ζ is $\phi(X) = X^{(l-1)l^{k-1}} + X^{(l-2)l^{k-1}} + \dots + X^{l^{k-1}} + 1$.
- We have $l = \prod_{g \in (\mathbb{Z}/n\mathbb{Z})^\times} (1 - \zeta^g)$.
- $1 - \zeta^g = \epsilon_g(1 - \zeta)$ with $\epsilon_g \in \mathcal{O}^\times$ for $g \not\equiv 0 \pmod{l}$.
- $l = \epsilon(1 - \zeta)^d$ with $\epsilon \in \mathcal{O}^\times$.
- $\mathcal{N}_{L|\mathbb{Q}}(1 - \zeta) = l$.

Proof. (i)

$$\begin{aligned} \phi(x) &= \prod_{g \in (\mathbb{Z}/n\mathbb{Z})^\times} (X - \zeta^g) = \frac{\prod_{g \in (\mathbb{Z}/n\mathbb{Z})} (X - \zeta^g)}{\prod_{g \in (\mathbb{Z}/l^{k-1}\mathbb{Z})} (X - \zeta^{gl})} = \frac{X^{l^k} - 1}{X^{l^{k-1}} - 1} \\ &= X^{(l-1)l^{k-1}} + X^{(l-2)l^{k-1}} + \dots + X^{l^{k-1}} + 1 \end{aligned}$$

(ii) Follows from (i) with $X = 1$.

(iii) Observe

$$\epsilon_g := \frac{1 - \zeta^g}{1 - \zeta} = 1 + \zeta + \dots + \zeta^{g-1} \in \mathcal{O}$$

and

$$\frac{1}{\epsilon_g} = \frac{1 - \zeta}{1 - \zeta^g}$$

Since $g \not\equiv 0 \pmod{l}$, we can choose some $g' \in \mathbb{Z}$ with $gg' \equiv 1 \pmod{l^k}$. Hence

$$\frac{1}{\epsilon_g} = \frac{1 - \zeta^{gg'}}{1 - \zeta^g} = 1 + \zeta^g + \dots + (\zeta^g)^{g'-1} \in \mathcal{O}.$$

(iv) Follows from (ii) and (iii) with $\epsilon := \prod_{g \in (\mathbb{Z}/n\mathbb{Z})^\times} \epsilon_g$.

(v) Follows from (ii). □

Proposition 2.10.2. *Suppose again that $n = l^k$ with l prime. Set $\lambda := 1 - \zeta$. Then*

(i) $\Pi := (\lambda)$ is a prime ideal of local degree 1.

(ii) $l \cdot \mathcal{O} = \Pi^d$. In particular, $l\mathcal{O}$ is non-split.

Proof. 10.1 (iv) $\Rightarrow l\mathcal{O} = (\lambda)^d$. Let $l\mathcal{O} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$ be the decomposition into prime ideals. By Theorem 11, $d = e_1 f_1 + \cdots + e_r f_r$, where f_i = local degree of \mathfrak{p}_i , hence the above is already the prime decomposition and the local degree is 1. □

Remark 2.10.3. 10.1 and 10.2 generalize Lemma I.25.

Proposition 2.10.4. *Let $n = l^k$, l prime. The basis $1, \zeta, \zeta^2, \dots, \zeta^{d-1}$ of $\mathbb{Q}(\zeta)|\mathbb{Q}$ has the discriminant $d(1, \zeta, \dots, \zeta^{d-1}) = (-1)^a l^s$ with $s = l^{k-1}(kl - k - 1)$ and $a \in \{0, 1\}$.*

Proof. Step 1: Show $d(1, \dots, \zeta^{d-1}) = \pm \mathcal{N}(\phi'(\zeta))$.

Let $\zeta = \zeta_1, \zeta_2, \dots, \zeta_d$ be the conjugates of ζ .

$$\text{Remark 2.4} \Rightarrow d(1, \dots, \zeta^{d-1}) = d(\phi) = \prod_{1 \leq i < j \leq d} (\zeta_i - \zeta_j) = \pm \prod_{\substack{i,j=1 \\ i \neq j}}^d (\zeta_i - \zeta_j).$$

Observe

$$\phi(X) = \prod_{i=1}^d (X - \zeta_i) \Rightarrow \phi'(X) = \sum_{m=1}^d \prod_{\substack{i=1 \\ i \neq m}}^d (X - \zeta_i)$$

and therefore

$$\phi'(\zeta_j) = \prod_{\substack{i=1 \\ i \neq j}}^d (\zeta_j - \zeta_i).$$

Hence we have $d(1, \dots, \zeta^{d-1}) = \pm \prod_{j=1}^d \phi'(\zeta_j) = \pm \mathcal{N}(\phi'(\zeta))$.

Step 2: Calculate $\mathcal{N}(\phi'(\zeta))$ partially.

Observe: $(X^{l^{k-1}} - 1)\phi(X) = X^{l^k} - 1$. Differentiating yields $(X^{l^{k-1}} - 1)\phi'(X) + \phi(X)(\dots) = l^k X^{l^k-1}$. Plugging in $X = \zeta$ gives $(\zeta^{l^{k-1}} - 1)\phi'(\zeta) = l^k \zeta^{l^k-1} = l^k \zeta^{-1}$. Set $\xi := \zeta^{l^{k-1}}$. Then ξ is a root of unity of order l and we have $\mathcal{N}(\phi'(\zeta)) = \frac{(l^k)^d}{\mathcal{N}(\xi-1)}$.

Step 3: Calculate $\mathcal{N}(\xi - 1)$.

Lemma 10.1 $\Rightarrow \mathcal{N}_{\mathbb{Q}(\xi)|\mathbb{Q}}(\xi - 1) = l$. Hence $\mathcal{N}_{\mathbb{L}|\mathbb{Q}}(\xi - 1) = (\mathcal{N}_{\mathbb{Q}(\xi)|\mathbb{Q}}(\xi - 1))^{l^{k-1}} = l^{l^{k-1}}$.

Now combining all 3 steps yields: $d(1, \dots, \zeta^{d-1}) = \pm \frac{l^{kd}}{l^{l^{k-1}}} = \pm l^s$. □

Proposition 2.10.5. *Let n be some natural number. Then $1, \zeta, \dots, \zeta^{d-1}$ is an integral basis of \mathcal{O} .*

Proof. Step 1: Show the claim for $n = l^k$ with l prime.

- (1) Proposition 2.7 $\Rightarrow \pm l^s = d(1, \dots, \zeta^{d-1}) \Rightarrow l^s \cdot \mathcal{O} \subset \mathbb{Z} + \dots + \mathbb{Z}\zeta^{d-1} = \mathbb{Z}[\zeta] \subset \mathcal{O}$.
- (2) Consider $\lambda := (1 - \zeta)$. Proposition 10.2 \Rightarrow local degree of (λ) is 1 $\Rightarrow \mathcal{O}/(\lambda) = \mathbb{Z}/(l)$
 $\Rightarrow \mathcal{O} = \mathbb{Z} + \lambda\mathcal{O}$ (every element of $\mathcal{O} \bmod (\lambda)$ has a representant in \mathbb{Z})
 $\Rightarrow \mathcal{O} = \mathbb{Z}[\zeta] + \lambda\mathcal{O} \quad (*)$.
 Multiplying with λ yields $\lambda\mathcal{O} = \lambda\mathbb{Z}[\zeta] + \lambda^2\mathcal{O} \xrightarrow{(*)} \mathcal{O} = \mathbb{Z}[\zeta] + \lambda^2\mathcal{O} \Rightarrow \dots$
 $\Rightarrow \mathcal{O} = \mathbb{Z}[\zeta] + \lambda^t\mathcal{O} \quad \forall t \geq 1$.
- (3) Plug in $t = s\varphi(l^k)$ and by Proposition 10.2 $l\mathcal{O} = \lambda^{\varphi(l^k)}\mathcal{O}$:
 $\mathcal{O} = \mathbb{Z}[\zeta] + \lambda^{s\varphi(l^k)}\mathcal{O} = \mathbb{Z}[\zeta] + l^s\mathcal{O} = \mathbb{Z}[\zeta]$.

Step 2: Generalize to arbitrary $n = l_1^{k_1} \cdot \dots \cdot l_r^{k_r}$.

Consider $\zeta_i := \zeta^{n_i}$ with $n_i := \frac{n}{l_i^{k_i}}$, a primitive $l_i^{k_i}$ -th root of unity. Then $\text{ord}(\zeta_1), \dots, \text{ord}(\zeta_r)$ are relatively prime. Hence:

- (1) $\mathbb{Q}(\zeta) = \mathbb{Q}(\zeta_1) \cdot \dots \cdot \mathbb{Q}(\zeta_r)$.
- (2) $\mathbb{Q}(\zeta_1) \cdot \dots \cdot \mathbb{Q}(\zeta_{i-1}) \cap \mathbb{Q}(\zeta_i) = \mathbb{Q}$.
- (3) Apply Proposition 2.13 to $\mathbb{Q}(\zeta_1) \cdot \dots \cdot \mathbb{Q}(\zeta_r)$ successively. We obtain, that

$$\{\zeta_1^{j_1}, \dots, \zeta_r^{j_r} \mid 0 \leq j_i \leq d_i - 1\}$$

with $d_i = \varphi(l_i^{k_i})$ is an integral basis of $\mathbb{Q}(\zeta_1, \dots, \zeta_r) = \mathbb{Q}(\zeta)$.

Hence $\mathcal{O} = \mathbb{Z}[\zeta] = \mathbb{Z} + \mathbb{Z}\zeta + \dots + \mathbb{Z}\zeta^{d-1}$, since all ζ_i 's are powers of ζ .

□

Lemma 2.10.6. *Let p be a prime which does not divide n . Then we have in $\mathcal{O} = \mathbb{Z}[\zeta]$:*

$$p\mathcal{O} = \hat{\mathcal{P}}_1 \cdot \dots \cdot \hat{\mathcal{P}}_r$$

with $\hat{\mathcal{P}}_i$ different prime ideals in \mathcal{O} and the local degree of each $\hat{\mathcal{P}}_i$ is $f = \min(\{k \in \mathbb{N} \mid p^k \equiv 1 \pmod{n}\})$.

Proof. Idea: Use Proposition 8.15.

Observe: Since $\mathcal{O} = \mathbb{Z}[\zeta]$, Proposition 8.15 can be applied to all prime ideals of \mathcal{O} .

- Consider $f(X) = \phi_n(X)$.
- Take the image $h(X) := f(\bar{X}) \in \mathbb{F}_p[X]$ and decompose it as $h(X) = h_1^{e_1} \cdot \dots \cdot h_r^{e_r}$ into irreducible factors over \mathbb{F}_p .

Then we have: $p\mathcal{O} = \mathfrak{p}_1^{e_1} \cdot \dots \cdot \mathfrak{p}_r^{e_r}$ with prime ideals \mathfrak{p}_i of local degree $f_i := \deg h_i$.

Step 1: Show $e_1 = \dots = e_r = 1$.

Consider $q(X) := X^n - 1 \in \mathbb{F}_p[X]$. Since $p \nmid n$, $q'(X) = nX^{n-1}$ and q have no common zeroes in $\mathbb{F}_p \Rightarrow q(X)$ has no multiple zeroes in $\mathbb{F}_p \Rightarrow$ The same must be true for $h(x) \Rightarrow e_1 = \dots = e_r = 1$.

Step 2: Show: $f_1 = f_2 = \dots = f_r = k_0 := \min\{k \mid p^k \equiv 1 \pmod{n}\}$

Recall: $f(X) = \phi_n(X)$, $h(X) := \text{image in } \mathbb{F}_p[X] = h_1^{l_1}(X) \cdot \dots \cdot h_r^{l_r}(X)$

Consider the field $L := \mathbb{F}_{p^{k_0}}$ with p^{k_0} elements as field extension of \mathbb{F}_p . Write $p^{k_0} - 1 = nw$ with $w \in \mathbb{N}$.

Observe: $L^\times = \langle a \rangle$ with $\text{ord}(a) = nw \Rightarrow \bar{\zeta} = a^w$ is a primitive n -th root of unity and h decomposes into linear factors over L .

Furthermore: $L = \mathbb{F}_p(\bar{\zeta})$ by minimality of k_0 , since $\#\mathbb{F}_p[\bar{\zeta}] = p^M$ for some M and $\text{ord}(\bar{\zeta}) = n$ divides $p^M - 1 \Rightarrow k_0 = M$.

Let $\bar{f}_1(X)$ be the minimal polynomial of $\bar{\zeta}$ over $\mathbb{F}_p \Rightarrow$

- \bar{f}_1 is an irreducible divisor of $h(X) \Rightarrow \text{w.l.o.g. } \bar{f}_1 = h_1$
- $f_1 = \deg(h_1) = \deg(\bar{f}_1) = [L : \mathbb{F}_p] = k_0 \Rightarrow f_1 = k_0$

□

Proposition 2.10.7 (CHARACTERISATION OF PRIME IDEALS). *Let $n = p_1^{k_1} \cdot \dots \cdot p_r^{k_r}$ be the prime decomposition of n and p some arbitrary prime number.*

Then $p\mathcal{O} = (\hat{\mathcal{P}}_1 \cdot \dots \cdot \hat{\mathcal{P}}_r)^{e_p}$ with $e_p = \varphi(p^{k_p})$ is the factorisation into prime ideals and each prime ideal $\hat{\mathcal{P}}_i$ is of local degree $f_p := \min\{k \in \mathbb{N} \mid p^k \equiv 1 \pmod{\frac{n}{p^{k_p}}}\}$

Proof. Again: Use Prop. 8.15 which applies to all prime ideals in \mathcal{O}

$\Rightarrow \phi_n(X) \in \mathbb{Z}[X]$ min. polynomial of $\zeta \Rightarrow \bar{\phi}_n(X) \in \mathbb{F}_p[X]$ image in $\mathbb{F}_p[X]$.

Denote $n = mp^a$ with $\gcd(p, m) = 1$, i.e. $a = k_p$.

Remember $U_m^\times = \{\text{primitive } m\text{-th roots of unity}\} \cong ((\mathbb{Z}/m\mathbb{Z})^\times, \cdot) \quad (\zeta^k \leftrightarrow k)$.

Use the isomorphism:

$$\begin{aligned} U_m^\times \times U_{p^a}^\times &\rightarrow U_n, \quad (\xi, \eta) \mapsto \xi \cdot \eta \\ \Rightarrow \phi_n(X) &= \prod_{g \in (\mathbb{Z}/n\mathbb{Z})^\times} (X - \zeta^g) = \prod_{\substack{\xi \in U_m^\times, \\ \eta \in U_{p^a}^\times}} (X - \xi\eta) \end{aligned}$$

Step 1: Show that $\phi_n(X) \equiv \phi_m(X)^{\varphi(p^a)} \pmod{p}$

(1) Observe: $X^{p^a} - 1 \equiv (X - 1)^{p^a} \pmod{p}$. For prime ideal $\hat{\mathcal{P}}$ over (p) :

$$X^{p^a} - 1 \equiv (X - 1)^{p^a} \pmod{\hat{\mathcal{P}}}$$

Let $\eta_1, \dots, \eta_{\varphi(p^a)}$ be the primitive p^a -th roots of unity.

$$0 = \eta_j^{p^a} - 1 \equiv (\eta_j - 1)^{p^a} \pmod{\hat{\mathcal{P}}} \Rightarrow \eta_j \equiv 1 \pmod{\hat{\mathcal{P}}}.$$

(2)

$$\begin{aligned}\phi_n(X) &= \prod_{\substack{\xi \in U_m^\times, \\ \eta \in U_{p^a}^\times}} (X - \xi\eta) = \prod_{g \in (\mathbb{Z}/m\mathbb{Z})^\times} (X - \xi)^{\varphi(p^a)} = \phi_m^{\varphi(p^a)} \pmod{\hat{\mathcal{P}}} \\ \Rightarrow \phi_n(X) &\equiv \phi_m(X)^{\varphi(p^a)} \pmod{p}\end{aligned}$$

Step 2: Use Lemma 10.5:

Proof of Lemma 10.5 \Rightarrow exponents of $\phi_m(X) \pmod{p}$ are all 1 \Rightarrow all exponents of $\phi_n(X) \pmod{p}$ are $\varphi(p^a)$. The local degree of the prime factors are by Lemma 10.5 $f = \min\{k \in \mathbb{N} \mid p^k \equiv 1 \pmod{\underbrace{m}_{=n/p^a}}\}$. \square

Corollary 2.10.8. *i) p is ramified in $\mathbb{Q}(\zeta) \iff n \equiv 0 \pmod{p}$ and we have not $p = 2 = \gcd(4, n)$.*

ii) $p \neq 2$. Then p is totally split $\iff p \equiv 1 \pmod{n}$.

Proof. i) Prop. 10.6 $\Rightarrow p$ is unramified $\iff e = 1 \xleftrightarrow{\text{Prop 10.6}} \varphi(p^{k_p}) = 1 \iff k_p = 0$ or $p^{k_p} - p^{k_p-1} = p^{k_p-1}(p - 1) = 1 \iff k_p = 0$ or $(p = 2 \text{ and } 2 = \gcd(4, n))$.

ii) $p \neq 2 : e = 1 \iff k_p = 0 \iff p \nmid n$
 $f = 1 \iff \min\{k \mid p^k \equiv 1 \pmod{\frac{n}{p^k}}\} = 1 \iff p \equiv 1 \pmod{n}$. \square

Remark 2.10.9. We have now in particular proved I.2.2.

3 Fermat's theorem for regular primes

3.1 The proof using a lemma of Kummer

Setting: K -number field, \mathcal{O} = ring of integers

Recall: \mathcal{J}_K := group of fractional ideals, \mathcal{P}_K = subgroup of principal ideals, $\text{Cl}_K = \mathcal{J}_K / \mathcal{P}_K$, $h_K = \# \text{Cl}_K$

Definition 3.1.1. A prime $p \in \mathbb{N}$ is regular : $\iff h_K$ is not divisible by p where $K = \mathbb{Q}(\zeta_p)$.

Remark 3.1.2. Suppose p regular. Then we have for each ideal I in \mathcal{O} = ring of integers in K :

If I^p is a principal ideal, then I is a principal ideal.

Proof. $p \nmid h_K \Rightarrow$ No element of Cl_K has order p . □

Recall: (Lemma I.2.11) $x, y \in \mathbb{Z}$, $\gcd(x, y) = 1$, $x + y \not\equiv 0 \pmod{p}$
 $\Rightarrow x + \zeta^i y$ and $x + \zeta^j y$ are coprime, if $i \not\equiv j \pmod{p}$.

Theorem 12. If p is a regular prime, then Fermat's theorem holds, i.e.

$$x^p + y^p = z^p \text{ in } \mathbb{Z} \Rightarrow xyz = 0.$$

Recall:

$$(1) \quad x^p + y^p = (x + y)(x + \zeta y) \cdots (x + \zeta^{p-1} y) \text{ in } \mathbb{Z}[\zeta].$$

$$(2) \quad \lambda = 1 - \zeta \text{ is prime in } \mathcal{O} = \mathbb{Z}[\zeta]$$

$$(3) \quad 1 - \zeta \sim 1 - \zeta^g \text{ for all } g \not\equiv 0 \pmod{p}$$

Lemma 3.1.3. Suppose that $x, y \in \mathcal{O}$ with x, y are coprime and p does not divide y .

Then we have: either the ideals $(x + \zeta^i y)$ (with $i \in \{0, \dots, p-1\}$) are relatively prime or they all have $(1 - \zeta)$ as a common factor and the ideals $(\frac{x + \zeta^i y}{1 - \zeta})$ (with $i \in \{0, \dots, p-1\}$) are relatively prime.

Proof. Use from the proof of Lemma I.2.11: Let $0 \leq j < i \leq p-1$. $A := (x + \zeta \cdot y, x + \zeta^j \cdot y) \Rightarrow$

$$(1) \quad (1 - \zeta) \cdot y \in A$$

$$(2) (1 - \zeta) \cdot x \in A$$

$$(3) 1 - \zeta \in A \text{ and thus } p \in A$$

$$(4) x + y \in A$$

Suppose q is a prime ideal with $q|(x + \zeta^i \cdot y)$ and $q|(x + \zeta^j \cdot y)$.

Hence $q \supseteq A \stackrel{(3)}{\ni} 1 - \zeta \stackrel{1-\zeta \text{ prime}}{\implies} q = (1 - \zeta)$.

Hence $q = (1 - \zeta)$ is the only prime ideal which possibly divides $(x + \zeta^i \cdot y), (x + \zeta^j \cdot y)$.

Show: If $q = (1 - \zeta)$ divides $(x + \zeta^i \cdot y)$, then it divides $(x + \zeta^{i+1} \cdot y)$.

This follows from the following calculation: $x + \zeta^{i+1} \cdot y = x + \zeta^i \cdot y + \zeta^i(\zeta - 1) \cdot y$

Finally show: If $(1 - \zeta)$ divides $x + \zeta^i \cdot y$, then the $(\frac{x+\zeta^i \cdot y}{1-\zeta})$ and $(\frac{x+\zeta^j \cdot y}{1-\zeta})$ are coprime for $0 \leq j < i \leq p-1$.

Recall: $p \nmid y \Rightarrow 1 - \zeta \nmid y$

Proof: $x + \zeta^i \cdot y - (x + \zeta^j \cdot y) = \zeta^j \cdot y \underbrace{(\zeta^{i-j} - 1)}_{\sim (\zeta-1)} \Rightarrow \frac{x+\zeta^i \cdot y}{1-\zeta} - \frac{x+\zeta^j \cdot y}{1-\zeta} \sim y$.

But $(1 - \zeta) \nmid y \Rightarrow$ Claim. □

Proposition 3.1.4 (“First Case”). *Suppose p is a regular prime with $p \geq 5$ such that $x^p + y^p = z^p$ and $p \nmid xyz$ with $x, y, z \in \mathbb{Z}$. Then $xyz = 0$.*

Proof. Without loss of generality we may assume that x, y, z are coprime. Proceed as in the proof of Theorem 1:

- $z^p = x^p + y^p = (x + y)(x + \zeta y) \cdots (x + \zeta^{p-1}y)$
- Since $p \nmid z$ we have $x + y \equiv x^p + y^p = z^p \equiv z \not\equiv 0 \pmod p$ by little Fermat’s theorem such that $p \nmid x + y$.
- Lemma 2.11 implies that $(x + y), (x + \zeta y), \dots, (x + \zeta^{p-1}y)$ are pairwise coprime such that the first bullet point together with the regularity of p and Remark 1.2 yields $(x + \zeta^i y) = (\alpha_i)^p$ for some $\alpha_i \in \mathcal{O}$. Thus $x + \zeta^i y = \varepsilon_i \alpha_i^p$ with $\varepsilon_i \in \mathcal{O}^\times$.

Now continue as in the proof of Theorem 1. □

Recall (Example 1.2.8). If $\alpha \in \mathcal{O}$ then $\alpha = a_0 + a_1 \zeta + \cdots + a_{p-2} \zeta^{p-2}$ such that

$$\alpha^p \equiv \underbrace{a_0^p + a_1^p + \cdots + a_{p-2}^p}_{=a \in \mathbb{Z}} \pmod p.$$

Lemma 3.1.5 (Kummer’s Lemma II). *Suppose p is a regular prime. If $u \in \mathcal{O}^\times$ such that $u \equiv a \pmod p$ for some $a \in \mathbb{Z}$ then there is an $\alpha \in \mathcal{O}^\times$ such that $u = \alpha^p$.*

The proof is hard and needs more theory.

Remark 3.1.6. $1, 1 - \zeta, \dots, (1 - \zeta)^{p-2}$ is an integral basis of $\mathcal{O} = \mathbb{Z}[\zeta]$.

Proof. $1, \zeta, \dots, \zeta^{p-2}$ is an integral basis by Proposition 2.10.4. Furthermore,

$$\zeta^i = (1 - (1 - \zeta))^i = \sum_{k=0}^i \binom{k}{i} (-1)^{i-k} (1 - \zeta)^{i-k}$$

and $1 - \zeta$ has minimal polynomial of degree lesser equal than $p - 1$. \square

Lemma 3.1.7. *If $\alpha \in \mathcal{O} \setminus (1 - \zeta)$ then there exist $a \in \mathbb{Z}$ and $l \in \mathbb{N}_0$ such that*

$$\zeta^l \alpha \equiv a \pmod{(1 - \zeta)^2}.$$

Proof. We do the proof in multiple steps:

(1) Since $1, 1 - \zeta, \dots, (1 - \zeta)^{p-2}$ is an integral basis of \mathcal{O} we have

$$\alpha \equiv a_0 1 + a_1 (1 - \zeta) \pmod{(1 - \zeta)^2}$$

with $a_0, a_1 \in \mathbb{Z}$.

(2) Since $1 - \zeta \nmid \alpha$ we have $1 - \zeta \nmid a_0$ such that $p \nmid a_0$ and hence there is $l \in \mathbb{Z}$ with $a_0 l \equiv a_1 \pmod{p}$.

(3) Since $\zeta = 1 - (1 - \zeta)$ we have

$$\zeta^l \equiv 1 - l(1 - \zeta) \pmod{(1 - \zeta)^2}.$$

(4) By (1), (2) and (3) we conclude

$$\begin{aligned} \zeta^l \alpha &\equiv (1 - l(1 - \zeta)) (a_0 + a_1 (1 - \zeta)) \\ &\equiv a_0 + (a_1 - la_0) (1 - \zeta) \\ &\equiv a_0 \pmod{(1 - \zeta)^2}. \end{aligned}$$

\square

Proposition 3.1.8 (“Second case”). *Suppose p is a regular prime with $p \geq 5$ such that $x^p + y^p = z^p$ and $p \mid xyz$ with $x, y, z \in \mathbb{Z}$. Then $xyz = 0$.*

Proof. Without loss of generality x, y, z are pairwise coprime. By changing the role of x, y and z and possibly replacing x by $-x$, y by $-y$ and z by $-z$ we can furthermore assume that $p \nmid z$, $p \nmid x$ and $p \nmid y$. Then, by 2.10.1,

$$z = p^m z_0 = \varepsilon (1 - \zeta)^{(p-1)m} z_0$$

with $z_0 \in \mathbb{Z}$, $m \geq 1$, $\gcd(z_0, p) = 1$ and $\varepsilon \in \mathcal{O}^\times$ such that

$$x^p + y^p = \varepsilon^p (1 - \zeta)^{(p-1)mp} z_0^p.$$

By assumption:

- x, y and z_0 are pairwise coprime since x, y and z are pairwise coprime.
- $1 - \zeta$ and z_0 are coprime since p and z are coprime.
- x and $1 - \zeta$ are coprime since $p \nmid x$. The same holds for y and $1 - \zeta$.

Hence the following Lemma 1.9 yields $xyz_0 = 0$ such that $xyz = 0$ as claimed. \square

Lemma 3.1.9. *Suppose p is a regular prime with $p \geq 5$, $x, y, z_0 \in \mathcal{O}$, $\varepsilon \in \mathcal{O}^\times$ and $x, y, z_0, 1 - \zeta$ are pairwise coprime. If $x^p + y^p = \varepsilon(1 - \zeta)^{kp} z_0^p$ with $k \in \mathbb{N}$, then $xyz_0 = 0$.*

Proof. Assume that there are x, y, z_0 as in the lemma with $xyz_0 \neq 0$. We may assume that k is minimal.

“**Step 1:**” Show that $(1 - \zeta)^2 | x + y$.

(1) By assumption we have

$$\varepsilon(1 - \zeta)^{kp} z_0^p = (x + y)(x + \zeta y) \cdots (x + \zeta^{p-1} y) \quad (*)$$

such that, since $1 - \zeta$ is prime, there is $i \in \{0, \dots, p-1\}$ with $1 - \zeta | x + \zeta^i y$. Hence $1 - \zeta$ divides all $x + \zeta^i y$ by Lemma 1.3, in particular $x + y$.

(2) By Lemma 1.7 there are $a, b \in \mathbb{Z}$ and $l, j \in \mathbb{N}_0$ such that

$$\zeta^l x \equiv a \pmod{(1 - \zeta)^2} \quad \text{and} \quad \zeta^j y \equiv b \pmod{(1 - \zeta)^2}.$$

(3) We may replace x by $x\zeta^l$ and y by $y\zeta^j$ and thus can assume that $x \equiv a, y \equiv b \pmod{(1 - \zeta)^2}$ with $a, b \in \mathbb{Z}$.

(4) $1 - \zeta | x + y$ implies $1 - \zeta | a + b$ such that $(1 - \zeta)^{p-1} | a + b$ (since $a + b \in \mathbb{Z}$ we have also $p | a + b$) and hence $(1 - \zeta)^2 | x + y$. In particular, $k \geq 2$.

“**Step 2:**” Show that $(1 - \zeta)^{(k-1)p+1} | x + y$.

Since the quotients $\frac{x + \zeta^i y}{1 - \zeta}$ are pairwise coprime, all “extra powers” of $1 - \zeta$ have to divide $x + y$. Thus,

$$(1 - \zeta)^{kp-(p-1)} | x + y.$$

Furthermore:

$$1 - \zeta \nmid \frac{x + y}{(1 - \zeta)^{kp-(p-1)}}$$

“**Step 3:**” Show that $\frac{x + \zeta^i y}{1 - \zeta}$ is associated to a p -power.

From (*) we obtain

$$((1 - \zeta)^{k-1} z_0)^p = \prod_{i=0}^{p-1} \left(\frac{x + \zeta^i y}{1 - \zeta} \right).$$

Since the ideals on the right side are pairwise coprime, $\left(\frac{x+\zeta^i y}{1-\zeta}\right)$ is a p -th power. Thus Remark 1.2 yields

$$\frac{x + \zeta^i y}{1 - \zeta} = \varepsilon_i \alpha_i^p$$

with $\alpha_i \in \mathcal{O}$ and $\varepsilon \in \mathcal{O}^\times$. Furthermore, the α_i are pairwise coprime.

“Step 4:” Find $\varepsilon', \eta \in \mathcal{O}^\times$ and $\beta \in \mathcal{O}$ with $\varepsilon'(1 - \zeta)^{(k-1)p} \beta^p = -\alpha_1^p + \eta \alpha_{-1}^p$.

By Step 2, $(1 - \zeta)^{k-1}$ divides α_0 . More precisely, $\alpha_0 = (1 - \zeta)^{k-1} \beta$ with $\beta \in \mathcal{O}$ and $1 - \zeta, \beta$ coprime. Do some ugly calculation:

$$y = \frac{x + y - (x + \zeta y)}{1 - \zeta} = \varepsilon_0 \alpha_0^p - \varepsilon_1 \alpha_1^p = \varepsilon_0 (1 - \zeta)^{(k-1)p} \beta^p - \varepsilon_1 \alpha_1^p \quad (\text{A})$$

$$y = \frac{(x + \zeta^{-1} y) - (x + y)}{\zeta^{-1}(1 - \zeta)} = \zeta \varepsilon_{-1} \alpha_{-1}^p - \zeta \varepsilon_0 \alpha_0^p = \zeta \varepsilon_{-1} \alpha_{-1}^p - \zeta \varepsilon_0 (1 - \zeta)^{(k-1)p} \beta^p \quad (\text{B})$$

Then (B) – (A) yields

$$0 = \zeta \varepsilon_{-1} \alpha_{-1}^p + \varepsilon_1 \alpha_1^p + \varepsilon_0 (1 - \zeta)^{p(k-1)} \beta^p (-\zeta - 1).$$

Now define

$$\varepsilon' = \frac{(1 + \zeta) \varepsilon_0}{-\varepsilon_1} \quad \text{and} \quad \eta = \frac{\zeta \varepsilon_{-1}}{-\varepsilon_1}$$

to obtain

$$\varepsilon' (1 - \zeta)^{p(k-1)} \beta^p = \eta \alpha_{-1}^p - \alpha_1^p. \quad (**)$$

“Step 5:” Show that η is a p -th power.

By (**) we have $0 \equiv \eta \alpha_{-1}^p - \alpha_1^p \pmod{p}$ such that Example 1.2.8 ascertains the existence of $a_{-1}, a_1 \in \mathbb{Z}$ with $\alpha_{-1}^p \equiv a_{-1}, \alpha_1^p \equiv a_1 \pmod{p}$.

“Step 6:” Find a smaller solution to (\star):

$$x' := \alpha_{-1}, y' := v \eta_1, z_0 := \beta.$$

With ($\star\star$) : $\varepsilon' (1 - \zeta)^{p(k-1)} \cdot z_0^p = y'^p + x'^p$ is a smaller solution, a contradiction. \square

4 Geometric aspects

4.1 Localisation

Recall: Here all rings are commutative with 1.

Reminder 4.1.1. (i) Let R be a ring and $S \subseteq R \setminus \{0\}$ be a multiplicative system, i.e.

- (1) $a, b \in S \Rightarrow a \cdot b \in S$ and
- (2) $1 \in S$.

$$R \cdot S^{-1} := \{(a, s) \mid a \in R, s \in S\} / \sim$$

with $(a, s) \sim (a', s')$ if there is $t \in S : t(as' - a's) = 0$.

Denote $\frac{a}{s} := [(a, s)] / \sim$ equivalence class of (a, s) .

RS^{-1} becomes a ring with

$$\begin{aligned} \frac{a_1}{s_1} + \frac{a_2}{s_2} &= \frac{a_1 s_2 + a_2 s_1}{s_1 s_2}, \\ \frac{a_1}{s_1} \cdot \frac{a_2}{s_2} &= \frac{a_1 a_2}{s_1 s_2} \end{aligned}$$

RS^{-1} is called localisation of R by S .

- (ii) The map

$$j_S : R \rightarrow RS^{-1}, \quad r \mapsto \frac{r}{1}$$

is a ring homomorphism with $j_S(S) \subseteq (RS^{-1})^\times$. $\ker(j_S) = \{r \in R \mid \exists a \in S \text{ with } ar = 0\}$. In particular: R is an integral domain $\Rightarrow j_S$ is an embedding and $\frac{a}{b} = \frac{a'}{b'}$ is equivalent to $ab' = a'b$.

Furthermore: R is an integral domain $\Rightarrow RS^{-1} \subseteq \text{Quot}(R)$, $\frac{a}{b} \mapsto \frac{a}{b}$.

- (iii) Localisation has the following universal property: $f : R \rightarrow R'$ a ring homomorphism with $f(S) \subseteq (R')^\times$ then there exists a unique ringhomomorphism $g : RS^{-1} \rightarrow R'$ with $f = g \circ j_S$

$$\begin{array}{ccc} R & \xrightarrow{j_S} & RS^{-1} \\ & \searrow f & \swarrow \exists! g \\ & R' & \end{array}$$

Example 4.1.2. (i) R integral domain, $S = R \setminus 0 \Rightarrow RS^{-1} = \text{Quot}(R)$

- (ii) p prime ideal in R , $S := R \setminus p \Rightarrow R_p := RS^{-1}$.

Proposition 4.1.3 (Description of prime ideals in localisations). *We have the following bijection:*

$$\begin{aligned} \{p \in \operatorname{Spec}(R) \mid p \subseteq R \setminus S\} &\leftrightarrow \{q \in \operatorname{Spec}(RS^{-1})\} \\ \phi : p &\mapsto pS^{-1} = \left\{ \frac{a}{s} \mid a \in p, s \in S \right\} \\ j_S^{-1}(q) &\leftarrow q : \psi \end{aligned}$$

Proof. (1) $\frac{a}{s} = \frac{a'}{s'}$, then $a \in p \iff a' \in P$:

Suppose $a \in p, a' \in R, s, s' \in S$ and $\frac{a}{s} = \frac{a'}{s'} \Rightarrow \exists t \in S : \underbrace{t}_{\notin p} (as' - a's) = 0 \in p$

So $as' - a's \in p$, hence $a's \in p$ and $a' \in p$.

(2) ϕ is well defined, i.e. pS^{-1} is a prime ideal: clear.

(3) ψ is well-defined by Prop. II.8.16.

(4) $\psi \circ \phi(p) = j_S^{-1}(pS^{-1}) = p$:
 $r \in j_S^{-1}(pS^{-1}) \iff j_S(r) \in pS^{-1} \iff \frac{r}{1} \in pS^{-1} \iff r \in p$

(5) $\phi \circ \psi(q) = \psi(j_S^{-1}(q)) = j_S^{-1}(q)S^{-1} = q$:
 $\frac{r}{s} \in j_S^{-1}(q)S^{-1} \iff r \in j_S^{-1}(q) \iff j_S(r) \in q \iff \frac{r}{1} \in q \iff \frac{r}{s} \in q$

□

Definition 4.1.4 (and Prop., lokaler Ring). A ring is a local ring if R has one of the following equivalent properties:

- (i) R has a unique maximal ideal m .
- (ii) $R \setminus R^\times$ is an ideal.
- (iii) $\forall x \in R : x \in R^\times$ or $1 - x \in R^\times$.

In particular we have: If R is a local ring then $m = R \setminus R^\times$ is the unique maximal ideal of R .

Proof. (i) \Rightarrow (ii) : Show that $R = R^\times \cup m$:

(1) $R = R^\times \cup m : a \in R \setminus m$. Hence (a) is not contained in m . So $(a) = R$ and hence $a \in R^\times$.

(2) $R^\times \cap m = \emptyset : a \in R^\times$, so $a \notin m$ since $m \neq R = (a)$. It follows that $m = R \setminus R^\times$ and thus $R \setminus R^\times$ is an ideal.

(ii) \Rightarrow (iii) : Suppose x and $1 - x \in R \setminus R^\times$. Hence $1 = x + (1 - x) \in R \setminus R^\times$.

(iii) \Rightarrow (i) : Suppose that m and m' are two different maximal ideals. Let $a \in m' \setminus m$. Since m is maximal we have $(m, a) = R \Rightarrow \exists b \in m, r \in R$ with $1 = b + ra$. We know $ra \in m'$, hence $ra \notin R^\times$ and by assumption (iii) $\Rightarrow b = 1 - ra \in R^\times$ to $b \in m$. □

Proposition 4.1.5 (localisations by prime ideals are local). *Let R be a ring and $p \in \text{Spec}(R)$. Then R_p is a local ring with maximal ideal pS^{-1} where $S = R \setminus p$.*

Proof. We show that $R_p = R_p^\times \cup pS^{-1}$. Hence $R_p \setminus R_p^\times = pS^{-1}$ is an ideal. Thus R_p is a local ring.

$$(1) R_p = pS^{-1} \cup R_p^\times :$$

Let $a \in R, s \in S = R \setminus p$. Suppose $\frac{a}{s} \notin pS^{-1}$, i.e. $a \notin p$. So $\frac{s}{a} \in R_p$ and $\frac{a}{s} \frac{s}{a} = 1$. Hence $\frac{a}{s} \in R_p^\times$.

$$(2) pS^{-1} \cap R_p^\times = \emptyset :$$

Suppose that $\frac{a}{s} \in R_p^\times$ (with $a \in R, s \in S$) $\Rightarrow \exists a' \in R, s' \in S : \frac{a}{s} \frac{a'}{s'} = 1 \Rightarrow \exists t \in S$ with $t(aa' - ss') = 0 \in p$. Since $t \notin p$ we have $aa' - \underbrace{ss'}_{\notin p} \in p$, so $aa' \notin p$. Since $a \notin p$

it follows $\frac{a}{s} \notin pS^{-1}$.

□

Proposition 4.1.6 (being Dedekind is stable under localisation). *Let \mathcal{O} be a Dedekind domain, $S \subseteq \mathcal{O} \setminus \{0\}$ multiplicative system, then $\mathcal{O}S^{-1}$ is a Dedekind domain.*

Proof. \mathcal{O} is an integral domain, so $\mathcal{O} \subseteq \mathcal{O}S^{-1} \subseteq \text{Quot}(\mathcal{O})$.

$$(1) \mathcal{O}S^{-1} \text{ is an integral domain, since } \mathcal{O}S^{-1} \subseteq \text{Quot}(\mathcal{O}).$$

$$(2) \text{ Show that } \mathcal{O}S^{-1} \text{ is Noetherian, i.e. each ideal is finitely generated:}$$

Let q be an ideal in $\mathcal{O}S^{-1}$ and $p := j_S^{-1}(q)$.

Prop 1.3 says that $q = pS^{-1}$. \mathcal{O} is a Dedekind domain, hence p is finitely generated i.e. $p = (a_1, \dots, a_n) \Rightarrow q = pS^{-1} = (\frac{a_1}{1}, \dots, \frac{a_n}{1})$ is finitely generated.

$$(3) \text{ Show that } \mathcal{O}S^{-1} \text{ is integrally closed:}$$

Suppose $x \in \text{Quot}(\mathcal{O}S^{-1}) = \text{Quot}(\mathcal{O})$ with $x^n + \frac{r_{n-1}}{s_{n-1}}x^{n-1} + \dots + \frac{r_0}{s_0} = 0$ and $r_0, \dots, r_{n-1} \in \mathcal{O}, s_0, \dots, s_{n-1} \in S$.

Let $s := s_0 \cdot \dots \cdot s_{n-1} \in S$, then

$$(sx)^n + \underbrace{s \frac{r_{n-1}}{s_{n-1}}}_{\in \mathcal{O}} (sx)^{n-1} + \dots + \underbrace{s^n \frac{r_0}{s_0}}_{\in \mathcal{O}} = 0$$

$\Rightarrow sx$ is integral over \mathcal{O} and $\hat{x} = sx \in \mathcal{O}$, since \mathcal{O} is integrally closed.

$\Rightarrow x = \frac{\hat{x}}{s} \in \mathcal{O}S^{-1}$. Thus $\mathcal{O}S^{-1}$ is integrally closed.

$$(4) \text{ Prop 1.3 implies that every prime ideal } q \neq 0 \text{ in } \mathcal{O}S^{-1} \text{ is maximal.}$$

□

Definition 4.1.7 („diskreter Bewertungsring“). A ring is called discrete valuation ring (DVR) if

- R is a principal ideal domain and
- R has a (unique) maximal ideal $\mathfrak{m} = (\pi) \neq 0$.

In particular

- R is an integral domain
- R is not a field.

Remark 4.1.8. Let R be a DVR with maximal ideal $\mathfrak{m} = (\pi)$.

- (i) π is prime and any prime π' is associated to π .
- (ii) Any $r \in R \setminus \{0\}$ can be written as $r = \varepsilon \pi^k$ with $\varepsilon \in R^\times$ and $k \in \mathbb{N}$ depending only on r .

Proof. “(i)” Since R is a PID it is also a UFD. Since π is prime, if $\pi = ab$ with $a, b \in R$ then $(\pi) \subset (a)$ such that $(a) = R$ or $(a) = (\pi)$. Hence $a \in R^\times$ or $a = \varepsilon \pi$ with $\varepsilon \in R^\times$. Thus one of a, b must be a unit such that π is prime as a irreducible element of a UFD.

If π' is another prime, then $(\pi') \subset (\pi)$ and we can write $\pi' = a\pi$ with $a \in R$. Since π' is prime, a must be a unit and $\pi' \sim \pi$ follows as claimed.

“(ii)” Since r has a unique prime factorization (up to a unit) the claim follows from (i). \square

Proposition 4.1.9. R is a DVR if and only if R is a local Dedekind domain and not a field.

Proof. “ \Rightarrow ” R is local since it is a DVR, noetherian since it is a PID and integrally closed since it is a UFD. Furthermore, by Remark 1.8 every prime ideal is maximal. Also, since for the maximal ideal $\mathfrak{m} = (\pi)$ we have $\mathfrak{m} \neq 0$, R is not a field.

“ \Leftarrow ” R has a unique maximal ideal \mathfrak{m} since it is a local ring and $\mathfrak{m} \neq 0$ since R is not a field. We need to show that R is a PID:

- (1) Show that \mathfrak{m} is a principal ideal:

Since R is a Dedekind domain it holds that $\mathfrak{m} \neq \mathfrak{m}^2$. Let $\pi \in \mathfrak{m} \setminus \mathfrak{m}^2$ and observe that \mathfrak{m} is the only non-zero prime ideal. Thus, $(\pi) = \mathfrak{m}^k$ and $k = 1$ since $\pi \notin \mathfrak{m}^2$.

- (2) Any ideal \mathfrak{a} is a principal ideal since $\mathfrak{a} = \mathfrak{m}^k = (\pi^k)$.

\square

Definition 4.1.10. Let K be a field. A function $v: K \rightarrow \mathbb{Z} \cup \{\infty\}$ is called **discrete valuation** if for all $x, y \in K$ the following conditions hold:

- (i) $v(xy) = v(x) + v(y)$
- (ii) $v(x + y) \geq \min \{v(x), v(y)\}$

(iii) $v(x) = \infty$ if and only if $x = 0$

(iv) $v \not\equiv 0$ and $v \not\equiv \infty$

Example 4.1.11. Let $p \in \mathbb{Z}$ be prime and $K = \mathbb{Q}$. Define $v_p: \mathbb{Q} \rightarrow \mathbb{Z} \cup \{\infty\}$ by:

(1) If $z \in \mathbb{Z} \setminus \{0\}$ with $z = p^k b$, where $\gcd(p, b) = 1$, then $v_p(z) = k$.

(2) If $x \in \mathbb{Q} \setminus \{0\}$ with $x = \frac{a}{b}$, where $a, b \in \mathbb{Z}$, then $v_p(x) = v_p(a) - v_p(b)$.

Then v_p is a discrete valuation.

Proposition 4.1.12. (i) Let $v: K \rightarrow \mathbb{Z} \cup \{\infty\}$ be a discrete valuation. Then:

- $v(1) = v(-1) = 0$
- $v\left(\frac{a}{b}\right) = v(a) - v(b)$
- $\mathcal{O}_K = \{x \in K; v(x) \geq 0\}$ is a ring with units $\mathcal{O}_K^\times = \{x \in K; v(x) = 0\}$

(ii) The ring \mathcal{O}_K from (i) is a DVR.

(iii) Conversely, if R is a discrete valuation ring, then there exists a discrete valuation $v: K \rightarrow \mathbb{Z} \cup \{\infty\}$ with $K = \text{Quot}(R)$ such that $R = \mathcal{O}_K$ for this valuation.

Proof. “(i)” We have

$$v(1) = v(1 \cdot 1) = v(1) + v(1)$$

such that $v(1) = 0$. Furthermore,

$$0 = v(1) = v((-1)(-1)) = 2v(-1)$$

and hence also $v(-1) = 0$. Next,

$$v(a) = v\left(\frac{a}{b} \cdot b\right) = v\left(\frac{a}{b}\right) + v(b)$$

such that $v\left(\frac{a}{b}\right) = v(a) - v(b)$. Now, if $v(x) = 0$ then $v\left(\frac{1}{x}\right) = v(1) - v(x) = -v(x) = 0$ and hence $\frac{1}{x} \in \mathcal{O}_K$, i.e., $x \in \mathcal{O}_K^\times$. Finally, if $x \in \mathcal{O}_K^\times$ then there is a $y \in \mathcal{O}_K$ with $xy = 1$ such that

$$0 = v(1) = v(xy) = \underbrace{v(x)}_{\geq 0} + \underbrace{v(y)}_{\geq 0}$$

and thus $v(x) = 0$.

“(ii)”

- \mathcal{O}_K is an integral domain since $\mathcal{O}_K \subset K$.
- Show that \mathcal{O}_K is a PID:

Let \mathfrak{a} be an ideal in \mathcal{O}_K . Choose $a \in \mathfrak{a}$ with $v(a)$ minimal. For $b \in \mathfrak{a}$ we have $v(b) \geq v(a)$ and hence $v\left(\frac{b}{a}\right) \geq 0$ such that $\frac{b}{a} \in \mathcal{O}_K$. Thus, $b = \frac{b}{a} \cdot a$ with $\frac{b}{a} \in \mathcal{O}_K$ such that $b \in (a)$ and hence $\mathfrak{a} = (a)$.

- Show that \mathcal{O}_K has a unique maximal ideal:

Define $\mathfrak{m} = \{a \in \mathcal{O}_K; v(a) > 0\}$. Observe that \mathfrak{m} is an ideal in \mathcal{O}_K and $\mathfrak{m} = \mathcal{O}_K \setminus \mathcal{O}_K^\times$ such that \mathfrak{m} is a unique maximal ideal in \mathcal{O}_K .

- \mathcal{O}_K is not a field since the valuation V is not allowed to be 0 everywhere on K^\times .

“(iii)”

- (1) Use Remark 1.8 to define $v: R \setminus \{0\} \rightarrow \mathbb{N}_0$, $r = \varepsilon \pi^k \mapsto k$. Observe that $v(ab) = v(a) + v(b)$, $v(a+b) \geq \min\{v(a), v(b)\}$ and define $v(0) = \infty$.
- (2) Define $v: K = \text{Quot}(R) \rightarrow \mathbb{Z}$ by $v\left(\frac{a}{b}\right) = v(a) - v(b)$ if $a \neq 0$ and $v(0) = \infty$ to obtain a discrete valuation v .
- (3) By definition we have $R \subset \mathcal{O}_K$. Show that $\mathcal{O}_K \subset R$:

Let $\frac{a}{b} \neq 0$ be in \mathcal{O}_K , i.e., $v\left(\frac{a}{b}\right) \geq 0$. Then we have $v(a) \geq v(b)$. Let $\mathfrak{m} = (\pi) \neq 0$ be the unique maximal ideal of R . Then $a = \varepsilon_1 \pi^{k_1}$ and $b = \varepsilon_2 \pi^{k_2}$ with $\varepsilon_1, \varepsilon_2 \in R^\times$, $k_1, k_2 \in \mathbb{N}_0$ and $k_1 \geq k_2$ such that

$$\frac{a}{b} = \frac{\varepsilon_1}{\varepsilon_2} \pi^{k_1 - k_2} \in R.$$

□

Proposition 4.1.13. *Let R be an integral domain. Recall that for $\mathfrak{p} \in \text{Spec } R$ we have $R \subset R_{\mathfrak{p}} \subset \text{Quot}(R)$. In this situation,*

$$R = \bigcap_{\mathfrak{p} \in \text{Spec } R} R_{\mathfrak{p}}.$$

Proof. Let $\frac{a}{b} \in \bigcap_{\mathfrak{p} \in \text{Spec } R} R_{\mathfrak{p}}$. Consider $\mathfrak{a} = \{x \in R; xa \in bR\}$ and observe that \mathfrak{a} is an ideal. We show that $a \notin \mathfrak{p}$ for any $\mathfrak{p} \in \text{Spec } R$:

Since $\frac{a}{b} \in R_{\mathfrak{p}}$ there are $c \in R, s \in R \setminus \mathfrak{p}$ with $\frac{a}{b} = \frac{c}{s}$. Then we have $as = cb$, which implies $s \in \mathfrak{a}$ and $s \notin \mathfrak{p}$.

It follows that $\mathfrak{a} = R$, i.e., $1 \in \mathfrak{a}$ such that $1 \cdot a \in bR$ and thus $\frac{a}{b} \in R$. □

Theorem 13. *Let \mathcal{O} be a noetherian integral domain. Then \mathcal{O} is a Dedekind domain if and only if $\mathcal{O}_{\mathfrak{p}}$ is a DVR for all $\mathfrak{p} \in \text{Spec}(\mathcal{O}) \setminus \{0\}$.*

Proof. “ \Rightarrow ” By Proposition 1.6, $\mathcal{O}_{\mathfrak{p}}$ is a Dedekind domain and by Proposition 1.5, $\mathcal{O}_{\mathfrak{p}}$ is local. Since $\mathfrak{p} \neq 0$ we have $\mathfrak{p}S^{-1} \neq 0$ and hence $\mathcal{O}_{\mathfrak{p}}$ is not a field. Hence, $\mathcal{O}_{\mathfrak{p}}$ is a DVR by Proposition 1.9.

“ \Leftarrow ” By Proposition 1.11 we have $\mathcal{O} = \bigcap_{\mathfrak{p} \in \text{Spec } \mathcal{O}} \mathcal{O}_{\mathfrak{p}}$. Furthermore, Proposition 1.9 implies that $\mathcal{O}_{\mathfrak{p}}$ is integrally closed for any $\mathfrak{p} \in \text{Spec}(\mathcal{O})$ and hence the same holds true for \mathcal{O} .

Show: Every prime ideal $\mathfrak{p} \neq 0$ in \mathcal{O} is maximal.

Consider $\mathfrak{p} \subset \mathfrak{m}$ for some maximal ideal \mathfrak{m} . Consider the localisation $\mathcal{O}_{\mathfrak{m}} = \mathcal{O}S^{-1}$ with $S = \mathcal{O} \setminus \mathfrak{m}$. Then we obtain $\mathfrak{p}S^{-1} \subset \mathfrak{m}S^{-1}$ and $\mathfrak{p}S^{-1}, \mathfrak{m}S^{-1}$ both are prime ideals by Proposition 1.3. Since $\mathcal{O}_{\mathfrak{m}}$ is a DVR, Proposition 1.9 implies that $\mathcal{O}_{\mathfrak{m}}$ is a Dedekind domain, whence $\mathfrak{p}S^{-1} = \mathfrak{m}S^{-1}$, such that by Proposition 1.3 we may finally conclude $\mathfrak{p} = \mathfrak{m}$. \square

Reminder: $v_p : \mathbb{Q} \rightarrow \mathbb{Z} \cup \{\infty\}$, $\mathbb{Z} \ni z = p^k \cdot b \mapsto k$

Proposition 4.1.14. *The discrete valuations in Ex. 1.10 are up to scaling the only discrete valuations on \mathbb{Q} .*

Proof. Let $v : \mathbb{Q} \rightarrow \mathbb{Z} \cup \{\infty\}$ be a discrete valuation. Observe for $n \in \mathbb{N}$ we have $v(n) = v(1 + \dots + 1) \geq v(1) = 0$. For $z \in \mathbb{Z}$ we have $v(z) \geq 0$, since $v(-1) = 0$.

1) Show that $\exists p$ prime with $v(p) < 0$:

Suppose $v(p) = 0$ for all primes p .

$\Rightarrow v(n) = 0$ for all $n \in \mathbb{Z}$.

$\Rightarrow v(x) = 0$ for all $x \in \mathbb{Q} \setminus \mathbb{Z}$

2) Observe $\mathfrak{a} := \{a \in \mathbb{Z} \mid v(a) > 0\}$ is an ideal in \mathbb{Z} . Use that $v(z) \geq 0$ for all $z \in \mathbb{Z}$.

3) Let p be prime with $v(p) > 0$. Such a prime exists by 1). Observe $\mathfrak{a} = (p)$, since $p \in \mathfrak{a}$ and (p) is maximal in \mathbb{Z} . Let $c := v(p) > 0$. Denote $z \in \mathbb{Z}$ as $z = p^k \cdot b$ with $\gcd(b, p) = 1$. $\Rightarrow v(z) = k \cdot v(p) + v(b) = k \cdot c$ since $v(b) = 0, b \notin (p) = \mathfrak{a}$.

4) Obtain the result for $x \in \mathbb{Q}^\times$. \square

Example 4.1.15. Let K/\mathbb{Q} be a number field, \mathcal{O} its ring of integers, $\hat{\mathcal{P}}_0$ a prime ideal in \mathcal{O} above (p) .

Thm. 12 $\Rightarrow \mathcal{O}_{\hat{\mathcal{P}}_0}$ is a discrete valuation ring.

What is the corresponding discrete valuation on $K = \text{Quot}(\mathcal{O}_{\hat{\mathcal{P}}_0}) = \text{Quot}(\mathcal{O})$

$$v_{\hat{\mathcal{P}}_0} : K \rightarrow \mathbb{Z} \cup \{\infty\}?$$

Let $x \in K^\times \Rightarrow \underbrace{x \cdot \mathcal{O}}_{\text{fractional ideal}} = \hat{\mathcal{P}}_1^{e_1} \cdot \dots \cdot \hat{\mathcal{P}}_n^{e_n} = \prod \hat{\mathcal{P}} \in \text{Spec}(\mathcal{O}) \hat{\mathcal{P}}^e(\star)$ with $\hat{\mathcal{P}}_1, \dots, \hat{\mathcal{P}}_n$ are prime ideals in \mathcal{O} and $e_1, \dots, e_n \in \mathbb{Z}$.

Claim: $v_{\hat{\mathcal{P}}_0}(x) = e_{\hat{\mathcal{P}}_0}$. Proof: Consider the localisation $\mathcal{O}_{\hat{\mathcal{P}}_0}$:

For $\hat{\mathcal{P}} \neq \hat{\mathcal{P}}_0$ we have $\hat{\mathcal{P}} \cdot \mathcal{O}_{\hat{\mathcal{P}}_0} = \mathcal{O}_{\hat{\mathcal{P}}_0}$.

$\Rightarrow \underbrace{x \cdot \mathcal{O}}_{\text{fractional ideal for } \mathcal{O}_{\hat{\mathcal{P}}_0}} = \hat{\mathcal{P}}_0^{e_{\hat{\mathcal{P}}_0}} \cdot \mathcal{O}_{\hat{\mathcal{P}}_0} \stackrel{(!)}{=} m^{e_{\hat{\mathcal{P}}_0}}$, where m is the maximal ideal of $\mathcal{O}_{\hat{\mathcal{P}}_0} \Rightarrow v_{\hat{\mathcal{P}}_0} = e_{\hat{\mathcal{P}}_0}$.

Observe: $v_{\hat{\mathcal{P}}_0}|_{\mathbb{Q}} = e \cdot v_p$ where e is the ramification index of $\hat{\mathcal{P}}_0$.

Example 4.1.16 (Why „local ring“?). $\mathcal{O} = \mathbb{C}[X] \Rightarrow$

- \mathcal{O} is noetherian ✓
- \mathcal{O} is a UFD $\Rightarrow \mathcal{O}$ is integrally closed ✓
- \mathcal{O} is a PID \Rightarrow every prime ideal $\neq 0$ is maximal

$\Rightarrow \mathcal{O}$ is a Dedekind domain.

\Rightarrow

- $K := \text{Quot}(\mathcal{O}) = \mathbb{C}(C)$
- $\text{Spec}(\mathcal{O}) = \{(X_z) \mid z \in \mathbb{C}\} \cup \{(0)\}$
- $\mathcal{O}_{(X-z)} = \left\{ \frac{f}{g} \in \mathbb{C}(X) \mid f, g \in \mathbb{C}[X], g \notin (X-z) \right\}$
 $= \left\{ \frac{f}{g} \mid f, g \in \mathbb{C}[X], g(z) \neq 0 \right\}$
 $= \left\{ \frac{f}{g} \mid \frac{f}{g} \text{ is defined in } z \right\}$
 is a discrete valuation ring, in particular it is a local ring
- maximal ideal $m_{(X-z)} = \left\{ \frac{f}{g} \in \mathbb{C}(X) \mid f, g \in \mathbb{C}[X] \text{ with } f(z) = 0, g(z) \neq 0 \right\} = \left\{ \frac{f}{g} \in \mathbb{C}(X) \mid \text{s.t. } \frac{f}{g} \text{ has a zero in } z \right\}$ The corresponding discrete valuation $v : \mathbb{C}(X) \rightarrow \mathbb{Z} \cup \{\infty\}$ is induced by $v : \mathbb{C}[X] \setminus \{0\} \rightarrow \mathbb{Z}, f \mapsto \text{ord}_z(f) = \text{order of zero of } f \text{ in } z = \max\{k \in \mathbb{N}_0 \mid (X-z)^k \text{ divides } f\}$.

4.2 Affine Schemes (Perspective)

Idea: Link geometric objects to algebraic objects

<u>Geometry</u>	<u>Algebra</u>
-----------------	----------------

affine varieties V	$\text{Spec}(R)$
----------------------	------------------

+	+
functions on V	R

4.2.1 Classical affine varieties

Definition 4.2.1. Let k be a field and denote $\mathbb{A}^n(k) = k^n$ („affine space“)

$V \subseteq \mathbb{A}^n(k)$ is an affine variety : $\iff \exists S \subseteq k[X_1, \dots, X_n]$ s.t. $V = V(S) = \{z \in \mathbb{A}^n(k) \mid \forall f \in S : f(z) = 0\}$.

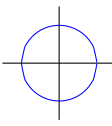
Remark 4.2.2. i) $S_1 \subseteq S_2 \Rightarrow V(S_1) \supseteq V(S_2)$

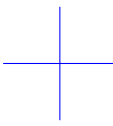
ii) Let (S) be the ideal generated by S in $k[X_1, \dots, X_n]$, then $V(S) = V((S))$.

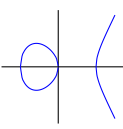
iii) Denote for $f_1, \dots, f_r : V(f_1, \dots, f_r) = V(\{f_1, \dots, f_r\})$. Every affine variety V is equal to $V(f_1, \dots, f_r)$ for finitely many polynomial f_1, \dots, f_r since $k[X_1, \dots, X_n]$ is noetherian.

Example 4.2.3. i) $S = \emptyset \Rightarrow V(S) = \mathbb{A}^n(k)$

ii) $S = k[X_1, \dots, X_n] \Rightarrow V(S) = V(1) = \emptyset$

iii) $V(X^2 + Y^2 - 1) =$ 

iv) $V(X \cdot Y) =$ 

v) $V(Y^2 - X^3 + X) (\Leftrightarrow Y^2 = X^3 - X = X(X - 1)(X + 1)) =$ 

vi) $a, b \in k \Rightarrow V(X - a, Y - b) = \{(a, b)\}$

Example 4.2.4. What are the affine varieties in $\mathbb{A}^1(k)$?

- Rem 2.2 ii) \Rightarrow sufficient to consider ideals I in $k[X]$
- Recall: Every ideal is a principal ideal, hence $I = (f)$ with $f \in k[X]$.

$$\Rightarrow V(I) = \begin{cases} \mathbb{A}^1(k), & \text{if } f = 0 \iff \deg(f) = -\infty \\ \emptyset, & \text{if } \deg(f) = 0 \\ \{z_1, \dots, z_k \mid z_i \text{ zero of } f\}, & \text{if } \deg(f) \geq 1. \end{cases}$$

Classical goal: Study geometry of affine varieties.

Idea: Consider „good“ classes of functions on them.

Consider: (1) $k[X_1, \dots, X_n]$ as set of *regular functions* on $\mathbb{A}^n(k) = k^n$ and

(2) $k(X_1, \dots, X_n)$ as set of *rational functions* on $\mathbb{A}^n(k)$.

Observe: $f_1, f_2 \in k[X_1, \dots, X_n], V \subseteq \mathbb{A}^n(k), f_1 \equiv f_2 \text{ on } V \iff f_1 - f_2 \equiv 0 \text{ on } V$.

Definition 4.2.5. i) $I(V) := \{f \in k[X_1, \dots, X_n] \mid \forall z \in V : f(z) = 0\}$ is called vanishing ideal of V

ii) $A(V) := k[V] := k[X_1, \dots, X_n]/I(V)$ is called the k -algebra of regular functions of V

Example 4.2.6. i) $V = \emptyset \Rightarrow I(V) = k[X_1, \dots, X_n] \Rightarrow k[V] = 0$

ii) $V = \{z\} \subseteq \mathbb{A}^1(k) \Rightarrow I(V) = (X - z) \Rightarrow k[V] = k[X]/(X - z)$

iii) $V = \mathbb{A}^1(k), k \text{ infinite} \Rightarrow I(V) = (0) \Rightarrow k[V] \cong k[X]$

iv) $V = \mathbb{A}^1(k), k = \mathbb{F}_p \text{ finite} \Rightarrow I(V) = (X \cdot (X - 1) \cdot \dots \cdot (X - (p - 1)))$

Remark 4.2.7. Suppose $V \subseteq \mathbb{A}^n(k)$

i) $I(V)$ is a radical ideal, i.e. $f^e \in I(V) \Rightarrow f \in I(V)$

- ii) $V(I(V)) \supseteq V$ and $I(V(I)) \supseteq I$.
- iii) $\bar{V} := V(I(V))$ is the smallest affine variety containing V .
In particular, if V is already an affine variety, then $V(I(V)) = V$.
- iv) For affine varieties $V_1 = V(I_1)$ and $V_2 = V(I_2) : V_1 \subseteq V_2 \iff I(V_1) \supseteq I(V_2)$. In particular $V_1 = V_2 \iff I(V_1) = I(V_2)$
- v) $I_1 \subseteq I_2 \Rightarrow V(I_1) \supseteq V(I_2)$

Proof. i) ✓

ii) ✓

iii) Consider affine variety $V(J) \supseteq V \Rightarrow J \subseteq I(V) \Rightarrow V(J) \supseteq V(I(V))$

iv) „ \Rightarrow “: Suppose $f \in I(V_2) \Rightarrow \forall x \in V_1 \subseteq V_2 : f(x) = 0 \Rightarrow f \in I(V_2)$.

„ \Leftarrow “: $I_2 \stackrel{ii)}{\subseteq} I(V(I_2)) = I(V_2) \subseteq I(V_1)$ Hence: $x \in V_1, f \in I_2 \Rightarrow f(x) = 0$. Thus $x \in V_2 = V(I_2)$.

v) ✓

□

Hilberts Nullstellensatz (without proof)

k is algebraically closed, I ideal in $k[X_1, \dots, X_n]$.

Then $I(V(I)) = \sqrt{I} := \{f \in k[X_1, \dots, X_n] \mid \exists l \in \mathbb{N} \text{ with } f^l \in I\}$.

Idea: Define a topology on $\mathbb{A}^n(k)$.

Remark and Definition 4.2.8. The affine varieties define the closed sets of a topology called Zariski topology.

Proof. • $\mathbb{A}^n(k) = V((0))$ and $\emptyset = V((1))$.

• $V_1 = V(I_1)$ and $V_2 = V(I_2)$ affine varieties $\Rightarrow V_1 \cup V_2 \stackrel{!)}{=} V(I_1 \cdot I_2) \stackrel{!)}{=} V(I_1 \cap I_2)$

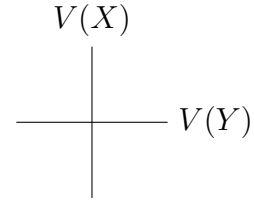
• $V_j = V(I_j)$ family of affine varieties with $j \in J \Rightarrow \bigcap_{j \in J} V_j \stackrel{!)}{=} V(\sum_{j \in J} I_j)$.

□

Example 4.2.9. In $\mathbb{A}^1(k)$ a subset is closed \iff it is finite or \emptyset or $\mathbb{A}^1(k)$.

Definition 4.2.10. A topological space X is called irreducible : $\iff X = A \cup B$ with A, B closed implies $X = A$ or $X = B$.

Otherwise X is called reducible



Example 4.2.11. $V = V(X \cdot Y) = V(X) \cup V(Y)$ is reducible.

Proposition 4.2.12. *An affine variety V is irreducible $\iff I(V)$ is a prime ideal.*

Proof. „ \Rightarrow “: Consider $f, g \in k[X_1, \dots, X_n]$ with $f, g \in I(V)$.

Suppose: $f \notin I(V)$. Show that $g \in I(V)$.

$f \notin I(V) \Rightarrow \exists f(x) \neq 0 \Rightarrow V \not\subseteq V(f)(\star)$

Observe: $V \subseteq V(f \cdot g) = V(f) \cup V(g) \Rightarrow V = \underbrace{(V(f) \cap V)}_{\text{closed}} \cup \underbrace{(V(g) \cap V)}_{\text{closed}}$

$\xRightarrow{(\star)} V(g) \cap V = V \Rightarrow V \subseteq V(g) \Rightarrow g \in I(V)$.

„ \Leftarrow “: Suppose $V = V_1 \cup V_2$ with $V_1 = V(I_1)$ and $V_2 = V(I_2)$.

Hence: $V = V(I_1) \cup V(I_2) = V(I_1 I_2)$ and thus $I_1 \cdot I_2 \stackrel{2.7}{\subseteq} I(V(I_1 \cdot I_2)) = I(V)$.

Suppose: $V_1 \neq V \Rightarrow \exists x \in V : f \in I_1$ with $f(x) \neq 0$.

Hence $f \notin I(V) \forall g \in I_2 : f \cdot g \in I_1 I_2 \subseteq I(V) \xRightarrow{I(V) \text{ prime}} g \in I(V)$.

Hence $I_2 \subseteq I(V) \Rightarrow V_2 = V(I_2) \supseteq V(I(V)) \supseteq V \Rightarrow V_2 = V$. □

Remark 4.2.13. An affine variety V is irreducible $\iff k[V] = k[X_1, \dots, X_n]/I(V)$ is an integral domain.

From now on k is always algebraically closed and all affine varieties are irreducible.

Example 4.2.14. $V = \mathbb{A}^1(k), I(V) = (0), k[V] = k[X]$

Definition 4.2.15. Let $U \subseteq V$ be an open subset of V . Define

$$\mathcal{O}(U) := \{\varphi : U \rightarrow k \mid \forall z \in U \exists \text{ open neighbourhood } U_z \ni z \stackrel{\text{open}}{\subseteq} V, \exists f, g \in k[V] \\ \text{with } \forall x \in U_z : g(x) \neq 0 \text{ and } \varphi(x) = \frac{f(x)}{g(x)}\}$$

as the set of regular functions on U .

Remark. \mathcal{O} defines a sheaf on V .

Definition 4.2.16. For $z \in V$ we define the local ring \mathcal{O}_z as follows:

$$\mathcal{O}_z := \{(U, f) \mid U \text{ open neighbourhood of } z, f \in \mathcal{O}(U)\} / \sim$$

with $(U_1, f_1) \sim (U_2, f_2) \iff f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$.

Remark. • \mathcal{O}_z is the stalk of the sheaf \mathcal{O} in z .

• $\mathcal{O}_z = k[V]_{m_z}$ here $m_z = \{f \in k[V] \mid f(z) = 0\}$

Remark 4.2.17. m_z as defined above is a maximal ideal in $k[V]$.

Proof. $\varphi_u : k[V] = k[X_1, \dots, X_n]/I(V) \rightarrow k$, $f \mapsto f(z)$

is a k -algebra homomorphism, which is surjective.

Hence $m_z = \ker(\varphi_z)$ is a maximal ideal. □

Remark 4.2.18. In particular \mathcal{O}_z is a local ring.

Definition 4.2.19. The field of rational functions $\text{Rat}(V)$:

$$\text{Rat}(V) := \{(U, f) \mid U \text{ open in } V, f \in \mathcal{O}(U)\} / \sim$$

with $(U_1, f_1) \sim (U_2, f_2) \iff f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$.

Remark (without proof). $\text{Rat}(V) = \text{Quot}(k[V])$

Conclusion 4.2.20. *Still assume k is algebraically closed, V affine variety. Then we have the following correspondences:*

$$\begin{aligned} \text{closed subsets of } V &\xleftrightarrow{1:1} \text{ radical ideals in } k[V] \\ V' &\mapsto I(V') \end{aligned}$$

$$\begin{aligned} \text{irreducible closed subsets of } V &\xleftrightarrow{1:1} \text{ prime ideals} \\ \text{points} &\xleftrightarrow{1:1} \text{ maximal ideals in } k[V] \\ x &\mapsto I(\{x\}) = m_x \end{aligned}$$

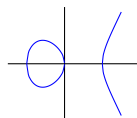
Furthermore we have:

- V_1, V_2 closed subsets of V : $V_1 \subseteq V_2 \iff I(V_1) \supseteq I(V_2)$
In particular: $x \in V_1 \iff m_x \supseteq I(V_1)$.
- \mathfrak{a} ideal in $k[V]$:
 $V(\mathfrak{a}) = \{z \in V \mid \forall f \in \mathfrak{a} : f(z) = 0\}$
 $= \{z \in V \mid \forall f \in \mathfrak{a} : f \in m_z\} = \{z \in V \mid \mathfrak{a} \subseteq m_z\}$
- $S \subseteq V$: $I(S) = \{f \in k[V] \mid \forall s \in S : f(s) = 0\} = \bigcap_{s \in S} m_s$.

Example 4.2.21. Let $k = \mathbb{C}$, $I = (Y^2 - X(X-1)(X-\lambda))$ with $\lambda \in \mathbb{C}$ and $V = V(I)$. Observe that I is a prime ideal since f_λ is irreducible (use Eisenstein criterion). In particular, $I = I(V)$. Hence V is irreducible and

$$K[V] = k[X, Y]/I.$$

Graph for $\lambda = -1$:



Definition 4.2.22. (i) Let X be a topological space. The **Krull dimension** $\dim(X)$ of X is defined as

$$\dim(X) = \sup\{n \in \mathbb{N} \mid \text{there are closed, irreducible subsets } X_i \subset X \text{ with } \emptyset \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n\}.$$

(ii) Let R be an arbitrary commutative noetherian ring. Then

$$\dim(R) = \sup\{n \in \mathbb{N} \mid \text{there are } \mathfrak{p}_i \in \operatorname{Spec} R \text{ with } \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n\}.$$

is called the **dimension** of R .

Remark 4.2.23. By Conclusion 2.20 we have $\dim V = \dim k[V]$.

Fact 4.2.24. $\dim k[X_1, \dots, X_n] = n$ and hence $\dim \mathbb{A}^n(k) = n$.

Observe. $(X_1, \dots, X_n) \supsetneq (X_1, \dots, X_{n-1}) \supsetneq \cdots \supsetneq (X_1) \supsetneq (0)$ is a chain of prime ideals of length n , hence $\dim k[X_1, \dots, X_n] \geq n$.

But you have to do some work to show that there is not a longer one.

Example 4.2.25. Let $V = V((f_\lambda))$ with $f_\lambda = Y^2 - X(X-1)(X-\lambda)$ and observe that $\mathfrak{p}_0 = (X, Y) \supsetneq \mathfrak{p}_1 = (f_\lambda) = I$ in $k[X, Y]$ induces $\bar{\mathfrak{p}}_0 \supsetneq \bar{\mathfrak{p}}_1$ in $k[V] = k[X, Y]/I$, where $\bar{\mathfrak{p}}_i$ is the image of \mathfrak{p}_i in the quotient. Hence $\dim k[V] \geq 1$.

But any chain of prime ideals $\bar{\mathfrak{p}}_0 \supsetneq \bar{\mathfrak{p}}_1 \supsetneq \cdots \supsetneq \bar{\mathfrak{p}}_n = (0)$ of length n in $k[V]$ induces a chain of prime ideals $\mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n \supsetneq (0)$ of length $N+1$, where \mathfrak{p}_i is the preimage of $\bar{\mathfrak{p}}_i$. Hence, by Fact 2.24, $\dim k[V] = 1$.

Definition 4.2.26. An **affine curve** over k is an irreducible affine variety V over k of dimension 1.

Definition 4.2.27. Let $V \subset \mathbb{A}^n(k)$ be an irreducible affine variety of dimension d with $V = V(f_1, \dots, f_r)$. For $z \in V$ consider the Jacobian matrix

$$J = \left(\frac{\partial f_i}{\partial x_j}(z) \right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(z) & \cdots & \frac{\partial f_1}{\partial x_n}(z) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial x_1}(z) & \cdots & \frac{\partial f_r}{\partial x_n}(z) \end{pmatrix}.$$

In this case,

- $z \in V$ is called **singular** if $\operatorname{rank} J < n - d$,
- $z \in V$ is called **non-singular** or **regular** if $\operatorname{rank} J = n - d$,
- V is called **regular** if all points in V are regular points.

Fact 4.2.28 (without proof). We always have $\operatorname{rank} J \leq n - d$.

Example 4.2.29. For $V = V((f_\lambda))$ with

$$f_\lambda = Y^2 - X(X-1)(X-\lambda) = Y^2 - (X^3 - (1+\lambda)X^2 + \lambda X)$$

we have

$$J(x, y) = (d - 3x^2 + 2(1+\lambda)x - \lambda - 2y).$$

Hence $(x, y) \in V$ is singular iff $\text{rank } J(x, y) < 2 - 1 = 1$ iff $\text{rank } J(x, y) = 0$ such that (x, y) has to satisfy the following relations:

$$y^2 = x(x-1)(x-\lambda) \tag{1}$$

$$-3x^2 + 2(1+\lambda)x - \lambda = 0 \tag{2}$$

$$2y = 0 \tag{3}$$

Now, (3) implies $y = 0$ and hence $x \in \{0, 1, \lambda\}$ by (1).

Conclusion:

- $\lambda \notin \{0, 1\}$: V is a regular curve
- $\lambda = 0$: $(0, 0)$ is a singularity
- $\lambda = 1$: $(1, 0)$ is a singularity

Fact 4.2.30 (without proof). Let V be an affine curve. Then x is a regular point of V if and only if $\mathcal{O}_x = k[V]_{\mathfrak{m}_x}$ is a discrete valuation ring.

Remark 4.2.31. Let V be an affine variety. Then V is a regular affine curve if and only if $k[V]$ is a Dedekind domain.

Proof. “ \Rightarrow ”

- $k[V] = k[X_1, \dots, X_n]/I$ is noetherian.
- $k[V]$ is a domain since I is a prime ideal. This is because V is irreducible by the definition of a curve.
- By Theorem 12 and Fact 2.30, $k[V]$ is a Dedekind domain.

“ \Leftarrow ”

- Since $k[V]$ is a domain, V is irreducible.
- Every prime ideal is maximal, hence $\dim k[V] = \dim V = 1$.
- By Theorem 12 and Fact 2.30, V is regular.

□

4.3 Affine schemes and Dedekind domains

Conclusion 2.20 motivates the following definition:

Definition 4.3.1. Let R be a ring and $X = \operatorname{Spec} R$.

- (i) For an ideal \mathfrak{a} in R define its **vanishing locus** by $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{a} \subset \mathfrak{p}\}$.
- (ii) For a subset $S \subset X$ define its **vanishing ideal** by $I(S) = \bigcap_{\mathfrak{p} \in S} \mathfrak{p}$.

Remark and Definition 4.3.2. Suppose $R = k[X_1, \dots, X_n]$

Note: In contrast to the geometric situation we consider all irreducible subsets of $\mathbb{A}^n(k)$ as points in $\operatorname{Spec} R$.

The maximal ideals in $\operatorname{Spec} R$ are sometimes called **geometric points** or **closed points**.

Remark 4.3.3. The sets $V(\mathfrak{a})$ with \mathfrak{a} an ideal in R form the closed subsets of a topology on $\operatorname{Spec} R$ called the **Zariski topology**.

Proof. Check the axioms, similarly to Remark 2.8. □

Remark 4.3.4. If $S \subset \operatorname{Spec} R$ then $\overline{S} = V(\mathfrak{a})$ with $\mathfrak{a} = \bigcap_{\mathfrak{p} \in S} \mathfrak{p}$. In particular,

$$\overline{\{\mathfrak{p}\}} = V(\mathfrak{p}) = \{\mathfrak{a} \in \operatorname{Spec} R \mid \mathfrak{a} \supset \mathfrak{p}\}.$$

Hence $\{\mathfrak{p}\}$ is closed if and only if \mathfrak{p} is maximal.

Question. How can we understand the elements of R as functions on $\operatorname{Spec} R$?

Example 4.3.5. Let V be an affine variety and $R = k[V]$. If $f \in R$ then

$$f: V \rightarrow k, z \mapsto f(z).$$

Recall. The evaluation map $\Phi_z: k[V] \rightarrow k, f \mapsto f(z)$ defines an isomorphism $k[V]/\mathfrak{m}_z \cong k$ with $f(z) = f \bmod \mathfrak{m}_z$.

Remark 4.3.6. R now an arbitrary ring.

The elements of R define functions on $\operatorname{Spec}(R)$ as follows:

$$f: p \mapsto f \bmod p \in R_p/p =: \underbrace{\kappa(p)}_{\text{residue field of } p}$$

Example 4.3.7. $R = \mathbb{Z}, X = \operatorname{Spec}(\mathbb{Z}) = \{(p) \mid p \text{ prime}\} \cup \{(0)\}$. Observe:

- The (p) are closed points.
- $\{(0)\} = \operatorname{Spec}(\mathbb{Z})$

Definition 4.3.8. Let $U \subseteq \operatorname{Spec}(R)$ be an open subset. Define

$$\mathcal{O}(U) := \{s = (s_p)_{p \in U} \in \prod_{p \in U} R_p \mid \forall p \exists U_p \text{ open nbhd. of } p \text{ and } f, g \in R \text{ with :}$$

$$\forall q \in U_p : \underbrace{g(q)}_{=g \bmod q} \neq 0 \text{ and } s_q = \frac{f}{g} \text{ in } R_q\}$$

$$\underbrace{\neq 0}_{\iff g \not\equiv 0 \bmod q}$$

Observe: \mathcal{O} is a sheaf.

Definition 4.3.9. The pair $(\operatorname{Spec} R, \mathcal{O})$ is called an affine scheme.

It is called noetherian $\iff R$ is noetherian.

Its dimension is the dimension of R .

Example 4.3.10. i) $R = k[V]$ for some affine variety V with k is algebraically closed
 $\Rightarrow V \hookrightarrow \operatorname{Spec}(R)$ as set of closed points.

ii) $R = K$ field $\Rightarrow \operatorname{Spec}(K) = \{(0)\} : \mathcal{O}(\{0\}) = K$
 $\underline{!} \operatorname{Spec}(K)$ as affine scheme depends on K .

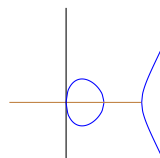
iii) R discrete valuation ring with maximal ideal $m \Rightarrow \operatorname{Spec}(R) = \{(0), m\}$
 Closed subsets: $\emptyset, \{m\}, \operatorname{Spec}(R)$
 Open subsets: $\operatorname{Spec}(R), \{0\}, \emptyset$
 $\mathcal{O}(\{0\}) = \operatorname{Quot}(R), \mathcal{O}(\operatorname{Spec}(R)) = R_m = R$

Definition 4.3.11. For a one-dimensional (!) noetherian domain R we say that the associated affine scheme $(\operatorname{Spec} R, \mathcal{O})$ is regular : \iff All local rings R_p (p prime in $R, p \neq (0)$) are discrete valuation rings.

Observation 4.3.12. For a domain R we have: $(\operatorname{Spec} R, \mathcal{O})$ is a noetherian, one-dimensional, regular affine scheme $\iff R$ is a Dedekind domain.

4.4 A „general“example

Example 4.4.1. Consider $V_1 := V(f_\lambda) \subseteq \mathbb{A}^2(\mathbb{C})$ with $f_\lambda = Y^2 - X(X-1)(X-\lambda)$ and $\lambda \in \mathbb{C} \setminus \{0, 1\}, V_2 : \mathbb{A}^1(\mathbb{C}), f : V_1 \rightarrow V_2, (x, y) \mapsto x$



Remark 4.4.2. i) $x \in \mathbb{A}^1(\mathbb{C})$ has two preimages $\iff x \notin \{0, 1, \lambda\}$

ii) $x \in \mathbb{A}^1(\mathbb{C})$ has one preimage $\iff x \in \{0, 1, \lambda\}$, namely $(x, 0)$.

Remark 4.4.3. Recall: $\mathcal{O}_1 := \mathbb{C}[V_1] = \mathbb{C}[X, Y]/(f_\lambda), \mathcal{O}_2 := \mathbb{C}[V_2] = \mathbb{C}[X]$

The map f induces:

- i) a morphism of k -algebras: $f^* : \mathbb{C}[V_2] \rightarrow \mathbb{C}[V_1]$, $h \mapsto h \circ f$
- ii) a morphism of fields: $f^* : \text{Rat}(V_2) = \text{Quot}(\mathcal{O}_2) \hookrightarrow \text{Rat}(V_1) = \text{Quot}(\mathcal{O}_1)$ induced by f^* in i).

Conclusion 4.4.4. Hence we have the following setting:

$$\begin{array}{ccc}
 \mathcal{O}_1 & \hookrightarrow & \text{Quot}(\mathcal{O}_1) =: K_1 \\
 \uparrow f^* & & \uparrow f^* \\
 \mathcal{O}_2 & \hookrightarrow & \text{Quot}(\mathcal{O}_2) =: K_2
 \end{array}
 \quad
 \begin{array}{c}
 \mathbb{C}(X) \\
 \uparrow f^*
 \end{array}
 \quad
 \begin{array}{ccc}
 V_1 = \mathbb{C} & & \\
 \downarrow f & & \\
 V_2 = \mathbb{A}^1 & \xrightarrow{h} & \mathbb{C}
 \end{array}$$

- \mathcal{O}_1 and \mathcal{O}_2 are Dedekind domains
- $\text{Quot}(\mathcal{O}_2) = \text{Quot}(\mathbb{C}[X]) = \mathbb{C}(X)$
 $\text{Quot}(\mathcal{O}_1) = \text{Quot}(\mathbb{C}[X, Y]/(f_\lambda)) = \mathbb{C}(X)(\alpha)$ with α has minimal polynomial $Y^2 - X(X-1)(X-\lambda) \in \mathbb{C}(X)[Y]$
- $[K_1 : K_2] = 2$

Recall: points in V_1 resp. $V_2 \leftrightarrow$ maximal ideals in \mathcal{O}_1 resp. \mathcal{O}_2
 $f(a) = b \iff f^*(m_b) \subseteq m_a \iff m_a$ lies above m_b .

$$\begin{array}{ccccc}
 \mathbb{C}[X][\alpha] = \mathbb{C}[X, Y]/(f_\lambda) & = & \mathcal{O}_1 & \hookrightarrow & \text{Quot}(\mathcal{O}_1) =: K_1 = \mathbb{C}(X)(\alpha) \\
 & & \uparrow f^* & & \uparrow f^* \\
 (X-b) \subseteq \mathbb{C}[X] & = & \mathcal{O}_2 & \hookrightarrow & \text{Quot}(\mathcal{O}_2) =: K_2 = \mathbb{C}(X)
 \end{array}$$

Remark 4.4.5. Consider the prime ideal $(X-b)$ for $b \in \mathbb{C}$.

Apply Prop. II.8.15: Observe $\mathcal{O}_1 = \mathcal{O}_2[\alpha]$, $f_\alpha(Y) = Y^2 - X(X-1)(X-\lambda) \in \mathcal{O}_2[Y]$

Consider image $\bar{f}_\alpha \in \mathcal{O}_2/(X-b)[Y] \xrightarrow{\text{evaluation map}} \mathbb{C}[Y]$

Hence $\bar{f}_\alpha = Y^2 - b(b-1)(b-\lambda) \in \mathbb{C}[Y] = (Y - \sqrt{b(b-1)(b-\lambda)})(Y + \sqrt{b(b-1)(b-\lambda)})$.

Prop. II.8.15. \Rightarrow :

- If $b \notin \{0, 1, \lambda\}$: $\mathfrak{o}\mathcal{O}_1 = \hat{\mathcal{P}}_1 \cdot \hat{\mathcal{P}}_2$.
Hence we have two prime ideals $\hat{\mathcal{P}}_1$ and $\hat{\mathcal{P}}_2$ lying above $p := (X-b)$ and the ramification indices are $e_1 = e_2 = 1$.
Degree formula: $2 = e_1 f_1 + e_2 f_2 \Rightarrow f_1 = f_2 = 1$.
- If $b \in \{0, 1, \lambda\}$: $p\mathcal{O}_1 = \hat{\mathcal{P}}^2$.
Hence we have one prime ideal $\hat{\mathcal{P}}$ over p with ramification index 2 and $f = 1$.

Remark 4.4.6. Why is f in the example = 1?

Recall: $f = \dim_{\mathcal{O}_2/\mathbb{P}} \mathcal{O}_1/\hat{\mathcal{P}}$ Here: $\mathcal{O}_2 = k[V_2]$, p = maximal ideal corresp. to a point b . $\Rightarrow \mathcal{O}_2/p \cong \mathbb{C}$. Same way: $\mathcal{O}_1/\hat{\mathcal{P}} \cong \mathbb{C}$.

