

SAARLAND UNIVERSITY
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
DEPARTMENT OF MATHEMATICS

ALGEBRAIC NUMBER THEORY I & II

LECTURE NOTES

WINTER 2017 / 2018
SUMMER 2018



PROF. DR. GABRIELA WEITZE-SCHMITHÜSEN

Table of contents

I. Algebraic number theory I	4
1. Small prefix	5
1.1. Motivation	5
1.2. The ring $\mathbb{Z}[\zeta]$	7
1.3. First case of Fermat in case of $\mathbb{Z}[\zeta]$ is a UFD (unique factorization domain)	12
2. Ring of integers	15
2.1. Integral ring extensions	15
2.2. Integral closures in field extensions	17
2.3. Ideals	26
2.4. Lattices and Minkowski	31
2.5. Minkowski theory	32
2.6. The class number	36
2.7. The theorem of Dirichlet	39
2.8. Prime ideals in \mathcal{O}_K	42
2.9. Hilbert's theorem of ramification	52
2.10. Cyclotomic Fields	56
3. Fermat's theorem for regular primes	62
3.1. The proof using a lemma of Kummer	62
4. Geometric aspects	67
4.1. Localisation	67
4.2. Affine Schemes (Perspective	74
4.2.1. Classical affine varieties	74
4.3. Affine schemes and Dedekind domains	81
4.4. A „general“example	82
II. Algebraische Zahlentheorie II	84
5. Bewertungstheorie	85
5.1. Normierte Körper	85
5.2. Archimedische und nicht-archimedische Normen	88
5.3. Komplettierung	93
5.4. Projektive Limiten	97

5.5. Ganze p -adische Zahlen	100
5.6. Endliche Erweiterung von vollständigen Körpern	104
5.7. Lokale Körper	110

Part I.

Algebraic number theory I

1. Small prefix

Recall:

- L numberfield : $\iff L$ is a finite extension of \mathbb{Q}
In particular: L/\mathbb{Q} is separable $\Rightarrow L/\mathbb{Q}$ is primitive, i.e. $L = \mathbb{Q}(\alpha), \mathbb{Q}[X] \ni f_\alpha =$ minimal polynomial of α over \mathbb{Q} and $[L : \mathbb{Q}] = \deg(f_\alpha)$.
- $\mathcal{O} := \{\alpha \in L \mid f_\alpha \in \mathbb{Z}[X]\}$ is called *ring of integers* (generalization of $\mathbb{Z} \subseteq \mathbb{Q}$).
 \mathcal{O} is an integral domain.
- Goal: study the ring \mathcal{O}
- Questions:
 1. What is \mathcal{O}^\times ? What is its structure?
 2. What are the prime ideals of \mathcal{O} ?
 3. Do we have a unique prime factorization, i.e. is \mathcal{O} a UFD?

1.1. Motivation

Problem 1.1.1 (Fermat's conjecture, ~ 1640). Show that the equation $x^n + y^n = z^n$ has no nontrivial integer solutions, i.e. solutions (x, y, z) with $x, y, z \in \mathbb{Z} \setminus \{0\}$ for $n \geq 3$.

History:

- 1770: Euler found solution for $n = 3$
- 1825: Dirichlet and Legendre using Germain
- Kummer showed it for many primes, he showed as well that his idea doesn't work for all $n \in \mathbb{N}_{>2}$
- Conjecture was proved by Wiles in 1997

Remark 1.1.2. i) If Fermat's is true for n , then also for nk for all $k \in \mathbb{N}$.

ii) It is sufficient to prove Fermat's conjecture for $n = 4$ and all odd primes.

Proof. i) Suppose (x, y, z) is a nontrivial solution of $x^{nk} + y^{nk} = z^{nk} \Rightarrow (x^k, y^k, z^k)$ is a nontrivial solution to $x^n + y^n = z^n$.

ii) Follows from i).

□

Proposition 1.1.3 ($n = 2$). Suppose $x, y, z \in \mathbb{Z}$, $\gcd(x, y, z) = 1$

- i) x, y, z are pairwise coprime if $x^2 + y^2 = z^2$
- ii) $x^2 + y^2 = z^2 \Rightarrow$ either x or y is even
- iii) $x^2 + y^2 = z^2 \iff \exists r, s \in \mathbb{N}_0, \gcd(r, s) = 1$ s.t. $x = \pm 2rs, y = \pm(r^2 - s^2), z = \pm(r^2 + s^2)$.

Proof. i) clear \checkmark

ii) One of x, y, z has to be even, since $odd + odd \neq odd$. Suppose z is even. Then look at equation mod 4, this gives a contradiction. By i) only one of x and y is even.

iii) „ \Leftarrow “: calculation

„ \Rightarrow “: Wlog. assume $x, y, z \in \mathbb{N}_0$, x even, y, z odd:

$$\begin{aligned} \Rightarrow x = 2u, z + y = 2v, z - y = 2w, \gcd(w, v) = 1 (y, z \text{ are coprime}), x^2 + y^2 = z^2 \\ \Rightarrow 4u^2 = x^2 = z^2 - y^2 = (z - y)(z + y) = 4wv \Rightarrow u^2 = vw \\ \xRightarrow{\gcd(v, w)=1} v = r^2, w = s^2 \Rightarrow z = v + w = r^2 + s^2, y = v - w = r^2 - s^2 \\ \text{and } x = 2u = 2\sqrt{vw} = 2rs \end{aligned}$$

□

Remark. $(x, y, z) \in \mathbb{Z}^3$ with $x^2 + y^2 = z^2$ are called *pythagorean triples*.

Proposition 1.1.4 ($n = 4$). The equation $x^4 + y^4 = z^2$ (and $x^4 + y^4 = z^4$) have no nontrivial integer solutions.

Proof. Suppose $x, y, z \in \mathbb{Z}$ with $x^4 + y^4 = z^2, xyz \neq 0$. Wlog $x, y, z > 0, x, y, z$ coprime, $x = 2\tilde{x}$ for some $\tilde{x} \in \mathbb{N}$. Choose z minimal with this conditions.

$$\begin{aligned} \text{Prop. 1.2} \Rightarrow \exists r, s \in \mathbb{N} \text{ s.t. } x^2 = 2rs, y^2 = r^2 - s^2, z = r^2 + s^2 \text{ and } \gcd(r, s) = 1 \\ \Rightarrow y^2 + s^2 = r^2 \text{ with } y, s, r \text{ coprime.} \end{aligned}$$

$$\text{Prop. 1.2} \Rightarrow \exists a, b \in \mathbb{N} \text{ s.t. } s = 2ab, y = a^2 - b^2, r = a^2 + b^2 \text{ and } \gcd(a, b) = 1.$$

$$\text{plug in} \Rightarrow x^2 = 4ab(a^2 + b^2)$$

$$\Rightarrow \tilde{x}^2 = ab(a^2 + b^2) \text{ and } a, b, a^2 + b^2 \text{ pairwise coprime}$$

As in proof of Prop. 1.2 (they are coprime but a square number)

$$\begin{aligned} \Rightarrow \exists c, d, e \in \mathbb{N} \text{ s.t. } a = c^2, b = d^2, a^2 + b^2 = e^2 \\ \Rightarrow c^4 + d^4 = a^2 + b^2 = e^2 \text{ and } e \leq a^2 + b^2 = r < z \end{aligned}$$

!since z was chosen to be minimal.

□

From now on: $n = p$ odd prime.

Idea 1.1.5 (by Germain). Distinguish 2 cases in Fermat's problem:

1. „First case“: x, y, z with p does not divide xyz .
2. „Second case“: exactly one of x, y, z is divided by p .

Some approach:

- Use primitive p -th root of unity $\zeta = \zeta_p$.
- Reminder: $X^p - 1 = (X - 1)(X - \zeta) \dots (X - \zeta^{p-1})$
- Setting $\tilde{y} = -y$ we get:

$$\begin{aligned}
 x^p + y^p &= x^p - \tilde{y}^p = \tilde{y}^p \left(\left(\frac{x}{\tilde{y}} \right)^p - 1 \right) \\
 &= \tilde{y}^p \left(\frac{x}{\tilde{y}} - 1 \right) \left(\frac{x}{\tilde{y}} - \zeta \right) \dots \left(\frac{x}{\tilde{y}} - \zeta^{p-1} \right) \\
 &= (x - \tilde{y})(x - \tilde{y}\zeta) \dots (x - \tilde{y}\zeta^{p-1}) \\
 &= (x + y)(x + y\zeta) \dots (x + y\zeta^{p-1})
 \end{aligned}$$

Lemma 1.1.6. For $x, y, z \in \mathbb{Z}$ we have $x^p + y^p = z^p \iff (x+y)(x+y\zeta) \dots (x+y\zeta^{p-1}) = z^p$

Idea: Look at prime divisors in $\mathbb{Z}[\zeta]$.

Problem: Would be good to have unique prime factorization. This will not be true in general.

1.2. The ring $\mathbb{Z}[\zeta]$

Suppose ζ is a primitive n -th root of unity

Reminder 1.2.1. i) $\mathbb{Q}(\zeta)/\mathbb{Q}$ is algebraic extension of degree $[\mathbb{Q}(\zeta) : \mathbb{Q}] = \varphi(n)$

ii) $\mathbb{Q}(\zeta)/\mathbb{Q}$ is a Galois extension. In particular:

$$\text{Hom}(\mathbb{Q}(\zeta)/\mathbb{Q}) \cong \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) = \{\sigma_i \text{ with } \sigma_i(\zeta) = \zeta^i \mid i \in (\mathbb{Z}/n\mathbb{Z})^\times\} \cong (\mathbb{Z}/n\mathbb{Z})^\times$$

iii) Consider the norm map $\mathcal{N} : \mathbb{Q}(\zeta) \rightarrow \mathbb{Q}$, $\alpha \mapsto \det(\gamma \mapsto \alpha\gamma)$. We have for $\alpha = r(\zeta)$ ($r \in \mathbb{Q}[X]$ polynomial) with min. polynomial $f_\alpha = X^k + c_{k-1}X^{k-1} + \dots + c_0$:

- If we have $\mathbb{Q}(\alpha) = \mathbb{Q}(\zeta)$, then $\mathcal{N}(\alpha) = (-1)^{\varphi(n)} c_0$
- $\mathcal{N}(\alpha) = \prod_{\sigma \in \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})} \sigma(\alpha) = \prod_{i \in (\mathbb{Z}/n\mathbb{Z})^\times} r(\zeta^i)$
- $\alpha \in \mathbb{Q} \Rightarrow \mathcal{N}(\alpha) = \alpha^{\varphi(n)}$

iv) $X^{n-1} + X^{n-2} + \dots + 1 = \frac{X^n - 1}{X - 1} = (X - \zeta)(X - \zeta^2) \dots (X - \zeta^{n-1})$
 $\xrightarrow{X=1} n = (1 - \zeta)(1 - \zeta^2) \dots (1 - \zeta^{n-1})$

Reminder 1.2.2 (and preview). i) $\mathcal{O} := \mathbb{Z}[\zeta] := \{r(\zeta) \mid r \in \mathbb{Z}[X]\}$

ii) $\mathbb{Z}[\zeta] = \{\alpha \in \mathbb{Q}(\zeta) \mid f_\alpha \in \mathbb{Z}[X]\}$ (proof later)

iii) $\mathbb{Z}[\zeta]$ is a free \mathbb{Z} -module with basis $\{1, \zeta, \dots, \zeta^{d-1}\}$ with $d = \varphi(n)$ (proof later)

iv) $\alpha \in \mathbb{Z}[\zeta] \Rightarrow \mathcal{N}(\alpha) \in \mathbb{Z}$ (proof later)

v) $\{\alpha \in \mathcal{O} \mid |\alpha| = 1\}$ is finite (proof later)

Reminder 1.2.3. Suppose R is an integral domain:

i) $\alpha \in R$ is *irreducible* : \iff If $\alpha = \alpha_1 \alpha_2$ with $\alpha_i \in R$, then $\alpha_1 \in R^\times$ or $\alpha_2 \in R^\times$

ii) $\alpha, \alpha' \in R$ are *associated to each other* : $\iff \exists \varepsilon \in R^\times : \alpha = \varepsilon \alpha'$

iii) R is called *factorial* : \iff each $\alpha \in R, \alpha \neq 0$ can be written in a unique way as $\alpha = \varepsilon \pi_1 \cdot \dots \cdot \pi_r$ with π_i irreducible up to multiplication with $\varepsilon \in R^\times$

iv) $\alpha_1, \alpha_2 \in R$ are called *coprime* : \iff If $\alpha' \in R$ with $\exists \beta_1, \beta_2 \in R : \alpha_1 = \alpha' \beta_1, \alpha_2 = \alpha' \beta_2$ then $\alpha' \in R^\times$.

Remark (and correction). 1. Recall: L/\mathbb{Q} field extensions:

$$\mathcal{O} := \{\alpha \in L \mid f_\alpha \in \mathbb{Z}[X]\}$$

!! Here: f_α is by definition monic, i.e. leading coefficient is 1.

Remark: $\mathcal{O} = \{\alpha \in L \mid \exists f \in \mathbb{Z}[X] \text{ with } f \text{ monic and } f(\alpha) = 0\}$

„ \subseteq “: clear

„ \supseteq “: Lemma of Gauss

2. Recall: Definition of field norm for L/K finite field extension How is norm defined?

$\mathcal{N} : L \rightarrow K$ defined as follows:

Suppose $\alpha \in L \Rightarrow \varphi_\alpha : \beta \mapsto \alpha\beta$ is linear map over K . Then:

$$\mathcal{N}_{L/K}(\alpha) := \det(\phi_\alpha)$$

Properties:

a) If $L = K(\alpha)$ and $X^n + c_{n-1}X^{n-1} + \dots + c_0$ is a minimal polynomial of α over K , then $\mathcal{N}_{L/K}(\alpha) = (-1)^n c_0$.

b) $\mathcal{N}_{L/K}(\alpha) = (\prod_{i=1}^r \sigma_i(\alpha))^q$ with $\text{Hom}_K(L, \bar{K}) = \{\sigma_1, \dots, \sigma_r\}$ and $q = \text{inseparable degree, i.e. } [L : K] = [L : K]_s \cdot q$.

c) $\alpha \in K \Rightarrow \mathcal{N}_{L/K}(\alpha) = \alpha^d$ with $d = [L : K]$ (see Bosch „Algebra“ 4.7).

General reference: NEUKIRCH

This chapter: BOREVICH + SHAFEREVICH Chapter 3.1.

Recall: Goal: prove for p prime and odd

$$x^p + y^p = z^p$$

has no non-trivial solutions. Last time:

$$x^p + y^p = z^p = (x + y)(x + y\zeta)(x + y\zeta^2) \dots (x + y\zeta^{p-1}) \in \mathbb{Z}[\zeta]$$

From now on: p odd prime, $\zeta = e^{\frac{2\pi i}{p}}$ primitive p -th root of unity $\mathcal{O} = \mathbb{Z}[\zeta]$.

Proposition 1.2.4. *For the group of units \mathcal{O}^\times of $\mathcal{O} = \mathbb{Z}[\zeta]$ we have:*

$$\mathcal{O}^\times = \{\alpha \in \mathcal{O} \mid \mathcal{N}(\alpha) = \pm 1\}$$

Notation: $\mathcal{N} = \mathcal{N}_{\mathbb{Q}(\zeta)/\mathbb{Q}}$ in this chapter.

Proof. „ \subseteq “: “ $\alpha \in \mathcal{O}^\times \Rightarrow \exists \beta \in \mathcal{O}$ with $\alpha\beta = 1 \Rightarrow 1 = \mathcal{N}(\alpha\beta) \stackrel{!}{=} \underbrace{\mathcal{N}(\alpha)}_{\in \mathbb{Z}} \underbrace{\mathcal{N}(\beta)}_{\in \mathbb{Z} \text{ by 2.2 v}} \Rightarrow \text{claim}$ “

„ \supseteq “: Suppose $\alpha \in \mathcal{O}$ with $\mathcal{N}(\alpha) = \pm 1$.

$$\Rightarrow \pm 1 = \mathcal{N}(\alpha) = \prod_{\sigma \in \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})} \sigma(\alpha)$$

Note: $\alpha = a_0 + a_1\zeta + \dots + a_{p-2}\zeta^{p-2} \in \mathbb{Z}[\zeta]$

$$\Rightarrow \sigma(\alpha) = a_0 + a_1\zeta^i + \dots + a_{p-2}\zeta^{i(p-2)} \text{ for some } i \in \{1, \dots, p-1\} \Rightarrow \sigma(\alpha) \in \mathbb{Z}[\zeta]$$

$$\Rightarrow \alpha \text{ is a divisor of 1 in } \mathbb{Z}[\zeta] \Rightarrow \alpha \in \mathcal{O}^\times. \quad \square$$

Lemma 1.2.5. *i) $\mathcal{N}(1 - \zeta^s) = p$ for $s \in \mathbb{Z}$ with $s \not\equiv 0 \pmod{p}$*

ii) $1 - \zeta$ is irreducible in $\mathcal{O} = \mathbb{Z}[\zeta]$.

iii) $p = \varepsilon \cdot (1 - \zeta)^{p-1}$ with some $\varepsilon \in \mathcal{O}^\times$.

Proof. i) 2.1. iv) $\Rightarrow p = (1 - \zeta)(1 - \zeta^2) \dots (1 - \zeta^{p-1})$

$$2.1. \text{ iii) } \Rightarrow \mathcal{N}(1 - \zeta^s) = \prod_{\sigma \in \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})} \sigma(1 - \zeta^s) = \prod_{i=1}^{p-1} (1 - \zeta^{si}) = \prod_{j=1}^{p-1} (1 - \zeta^j) = p$$

ii) We obtain from i) that $1 - \zeta \notin \mathcal{O}^\times$. Suppose $1 - \zeta = \alpha\beta$ with $\alpha, \beta \in \mathcal{O}$

$$\Rightarrow p = \mathcal{N}(1 - \zeta) = \mathcal{N}(\alpha)\mathcal{N}(\beta) \Rightarrow \mathcal{N}(\alpha) = \pm 1 \text{ or } \mathcal{N}(\beta) = \pm 1 \xrightarrow{\text{Prop 2.4}} \alpha \in \mathcal{O}^\times \text{ or } \beta \in \mathcal{O}^\times.$$

iii) Use: $1 - \zeta^s = (1 - \zeta) \underbrace{(1 + \zeta + \zeta^2 + \dots + \zeta^{s-1})}_{\varepsilon_s} = (1 - \zeta)\varepsilon_s$

$$\Rightarrow p = \mathcal{N}(1 - \zeta^s) = \underbrace{\mathcal{N}(1 - \zeta)}_{=p} \cdot \mathcal{N}(\varepsilon_s) \Rightarrow \mathcal{N}(\varepsilon_s) = 1 \Rightarrow \varepsilon_s \in \mathcal{O}^\times$$

$$\text{Hence } p = \prod_{s=1}^{p-1} (1 - \zeta^s) = \prod_{s=1}^{p-1} \underbrace{\varepsilon_s}_{\in \mathcal{O}^\times} (1 - \zeta) = (1 - \zeta)^{p-1} \underbrace{\prod_{s=1}^{p-1} \varepsilon_s}_{\in \mathcal{O}^\times}$$

□

Notation: $\varepsilon_s = 1 + \zeta + \dots + \zeta^s$.

Lemma 1.2.6. *i) $a \in \mathbb{Z}$ with $1 - \zeta$ divides a in $\mathcal{O} \Rightarrow p$ divides a .*

ii) An n -th root of unity lies in $\mathbb{Q}(\zeta) \iff n$ divides $2p$.

Proof. i) $a = (1 - \zeta)\beta$ with $\beta \in \mathcal{O} \Rightarrow a^{p-1} = \mathcal{N}(a) = p\mathcal{N}(\beta) \xrightarrow{(\mathcal{N}(\beta) \in \mathbb{Z})} p \text{ divides } a$.

ii) „ \Leftarrow “: $-1 \in \mathbb{Q}(\zeta)$ and thus $e^{\frac{2\pi i}{2p}} \in \mathbb{Q}(\zeta)$

„ \Rightarrow “: Consider $H := \{\omega \in \mathbb{Q}(\zeta) \mid \omega \text{ is a root of unity}\}$

- a) $H \subseteq \mathbb{Z}[\zeta]$: Suppose $\omega \in H \Rightarrow \omega^n - 1 = 0$ for some $n \in \mathbb{N} \Rightarrow f_\omega$ is a divisor of $X^n - 1 \Rightarrow f_\omega \in \mathbb{Z}[X] \xrightarrow{2.2ii)} \omega \in \mathbb{Z}[\zeta]$.
- b) $\tilde{\omega}$ some conjugate of $\omega \Rightarrow \tilde{\omega}$ is a root of $X^n - 1 \Rightarrow |\tilde{\omega}| = 1 \xrightarrow{2.2v)} H$ is finite $\Rightarrow H$ is a cyclic subgroup of $\mathbb{Q}(\zeta)^\times$.
 Choose some generator ω_0 of H and denote $m := \text{ord}(\omega_0)$. Since $\zeta \in H$ and $\text{ord}(\zeta) = p \Rightarrow p$ divides m . Decompose $m = p^s \cdot m'$ with $s \geq 1$ and $\gcd(m', p) = 1$. Consider the field extensions chain:

$$\mathbb{Q} \subseteq \mathbb{Q}(\omega_0) \subseteq \mathbb{Q}(\zeta)$$

with degrees $[\mathbb{Q}(\zeta) : \mathbb{Q}] = p - 1 = \varphi(p)$ and $[\mathbb{Q}(\omega_0) : \mathbb{Q}] = \varphi(m) = p^{s-1}(p-1)\varphi(m') \leq p-1 \Rightarrow s = 1$ and $\varphi(m') = 1$ and thus $m' = 1, 2 \Rightarrow \text{ord}(\omega_0) \leq 2p$. □

Notation 1.2.7.

1. L/K field extension, $\alpha \in L, \bar{K}$ given algebraic closure. The elements $\sigma(\alpha)$ with $\sigma \in \text{Hom}_K(L, \bar{K})$ are called *conjugates of α* . In particular: L/K normal \Rightarrow conjugates live in L .
2. R ring, I ideal in R , $p : R \rightarrow R/I$ canonical projection. For $\alpha, \beta \in R$ we denote $\alpha \equiv \beta \pmod{I} : \iff p(\alpha) = p(\beta)$.
 If $I = \langle q \rangle$ is a principal ideal, we denote $\alpha \equiv \beta \pmod{q} : \iff \alpha \equiv \beta \pmod{\langle q \rangle}$

Example 1.2.8. Consider $\mathbb{Q}(\zeta)/\mathbb{Q}$ with $\zeta^p = 1, R = \mathcal{O} = \mathbb{Z}[\zeta], \alpha = a_0 + a_1\zeta + a_2\zeta^2 + \cdots + a_{p-2}\zeta^{p-2}$

- i) The conjugates of α are: $\alpha_h = a_0 + a_1\zeta^h + a_2\zeta^{2h} + \cdots + a_{p-2}\zeta^{h(p-2)}$ with $h \in \{1, \dots, p-1\}$.
- ii) Consider $\lambda = 1 - \zeta$ and $I = \langle \lambda \rangle$.
 $1 \equiv \zeta \pmod{\lambda}$ and $\alpha \equiv a_0 + a_1 + \cdots + a_{p-2} \pmod{\lambda} (\in \mathbb{Z})$.
- iii) $\alpha^p \equiv a_0^p + (a_1\zeta)^p + \cdots + (a_{p-2}\zeta^{p-2})^p = \underbrace{a_0^p + a_1^p + \cdots + a_{p-1}^p}_{\in \mathbb{Z}} \pmod{p}$

Theorem 1 (Kummer's Lemma). *If $\varepsilon \in \mathbb{Z}[\zeta]$ is a unit, i.e. $\varepsilon \in \mathbb{Z}[\zeta]^\times$,*

$$\frac{\varepsilon}{\bar{\varepsilon}} = \zeta^a \quad \text{for some } a \in \mathbb{Z}$$

Here $\bar{\varepsilon} = \tau(\varepsilon)$, where τ is the complex conjugation.

Recall: $\tau \in \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$.

Proof. Denote $\varepsilon = a_0 + a_1\zeta + \cdots + a_{p-2}\zeta^{p-2} = r(\zeta)$ with $r(X) = \sum_{i=0}^{p-2} a_i X^i \in \mathbb{Z}[X]$.
Observe:

1. $\varepsilon \in \mathcal{O}^\times \Rightarrow \exists \varepsilon' \in \mathcal{O} \text{ s.t. } \varepsilon \varepsilon' = 1 \Rightarrow \bar{\varepsilon} \bar{\varepsilon}' = 1 \Rightarrow \bar{\varepsilon} \in \mathcal{O}^\times$
2. $\mu := \frac{\varepsilon}{\bar{\varepsilon}} = \frac{r(\zeta)}{r(\bar{\zeta})}$ and the conjugate μ_k of μ is $\frac{r(\zeta^k)}{r(\bar{\zeta}^k)} = \frac{r(\zeta^k)}{r(\zeta^k)}$. In particular $|\mu_k| = 1$.
 It follows that $\mu_k \in \{\alpha \in \mathcal{O}^\times \mid |\alpha| = 1\}$ which is by 2.2. v) a finite subgroup of $\mathbb{Q}(\zeta)^\times \Rightarrow \mu$ is a root of unity
 Lemma 2.6 $\Rightarrow \mu = \pm \zeta^a$ for some $a \in \mathbb{Z}$.
Claim: $\mu = \zeta^a$
Proof of claim: suppose $\mu = -\zeta^a$, i.e. $\varepsilon = -\bar{\varepsilon} \zeta^a$ (\star)
Idea: calculation mod $\lambda = 1 - \zeta$ $\varepsilon = a_0 + a_1 \zeta + \dots + a_{p-2} \zeta^{p-2}$
 Ex. 2.8.ii) $\Rightarrow \varepsilon \equiv \underbrace{a_0 + a_1 + \dots + a_{p-2}}_{=: M \in \mathbb{Z}} \equiv \bar{\varepsilon} \pmod{\lambda}$
 $(\star) \Rightarrow \varepsilon \equiv -\bar{\varepsilon} \pmod{\lambda} \Rightarrow M \equiv -M \pmod{\lambda} \Rightarrow 2M \equiv 0 \pmod{\lambda} \xrightarrow{\text{Lemma 2.6 i)}} p \text{ divides } 2M \text{ in } \mathbb{Z} \xrightarrow{p \text{ odd}} p \text{ divides } M$
 $\Rightarrow \lambda = 1 - \zeta \text{ divides } M \text{ in } \mathcal{O} \text{ by Lemma 2.5.}$
 $\Rightarrow \varepsilon \equiv \bar{\varepsilon} \equiv M \equiv 0 \pmod{\lambda = 1 - \zeta} \Rightarrow \text{Contradiction to } \varepsilon \text{ is unit and } 1 - \zeta \text{ is irreducible}$

□

Corollary 1.2.9. $\varepsilon \text{ unit in } \mathbb{Z}[\zeta] \Rightarrow \varepsilon = r \zeta^s \text{ with some } r \in \mathbb{R}, s \in \mathbb{Z}.$

Proof. Prop 2.9 $\Rightarrow \exists a \in \mathbb{Z}, \varepsilon = \zeta^a \cdot \bar{\varepsilon}$.

Choose $s \in \mathbb{Z}$ with $2s \equiv a \pmod{p}$

$\Rightarrow \frac{\varepsilon}{\zeta^s} = \zeta^s \cdot \bar{\varepsilon} = \frac{\bar{\varepsilon}}{\zeta^{-s}} = \frac{\bar{\varepsilon}}{\zeta^s} = r \in \mathbb{R} \text{ and } \varepsilon = r \cdot \zeta^s.$

□

Lemma 1.2.10. Suppose $x, y, m, n \in \mathbb{Z}$ with $m \not\equiv n \pmod{p}$. $x + y \zeta^n$ and $x + y \zeta^m$ are relatively prime $\iff (x \text{ and } y \text{ are relatively prime}) \text{ and } (x + y \text{ not divisible by } p)$

Proof. „ \Rightarrow “:

- $d \mid x \text{ and } d \mid y \Rightarrow d \mid x + \zeta^n y \text{ and } d \mid x + \zeta^m y \nmid$
- „ $p \mid x + y$ “ Recall: $p = \varepsilon(1 - \zeta)^{p-1}$ with $\varepsilon \in \mathcal{O}^\times$
 $\Rightarrow x + \zeta^m y = \underbrace{x + y}_{\text{divisible by } p} + y \cdot \underbrace{(\zeta^m - 1)}_{(\zeta - 1)(1 + \zeta + \zeta^2 + \dots + \zeta^{m-1})} \equiv 0 \pmod{1 - \zeta}$
 same way $x + \zeta^n y \equiv 0 \pmod{1 - \zeta} \nmid$

„ \Leftarrow “: Idea: show: $\exists \alpha_0, \beta_0 \in \mathcal{O}$ with:

$$1 = \alpha_0(x + \zeta^m y) + \beta_0(x + \zeta^n y)$$

Consider: $A := \{\alpha(x + \zeta^m y) + \beta(x + \zeta^n y) \mid \alpha, \beta \in \mathcal{O}\}$

A is an ideal in \mathcal{O} . We have:

1. $(x + \zeta^m y) - (x + \zeta^n y) = \zeta^m(1 - \zeta^{n-m})y = \underbrace{\zeta^n \varepsilon_{n-m}}_{\in \mathcal{O}^\times} (1 - \zeta)y \Rightarrow (1 - \zeta)y \in A$

2. $\zeta^n(x + \zeta^m y) - \zeta^m(x + \zeta^n y) = (\zeta^n - \zeta^m)x = \zeta^n \cdot (1 - \zeta^{n-m})x = \underbrace{\zeta^n \varepsilon_{m-n}}_{\in \mathcal{O}^\times} \cdot (1 - \zeta)x \Rightarrow (1 - \zeta)x \in A.$
3. $\gcd(x, y) = 1 \Rightarrow \exists a, b \in \mathbb{Z} \text{ with } 1 = ax + by \Rightarrow (1 - \zeta)xa + (1 - \zeta)yb = 1 - \zeta \xrightarrow{1. \& 2.} 1 - \zeta \in A$
4. $x + y = \underbrace{x + \zeta^n y}_{\in A} + \underbrace{(1 - \zeta^n)y}_{\in A} \in A$
5. $\gcd(p, x + y) = 1 \Rightarrow \exists \bar{a}, \bar{b} \in \mathbb{Z} : 1 = \underbrace{\bar{a}p}_{\in A} + \underbrace{\bar{b}(x + y)}_{\in A} \in A.$
 \Rightarrow Hence $x + \zeta^n y$ and $x + \zeta^m y$ are coprime.

□

Remark 1.2.11. Suppose $\alpha = a_0 + a_1\zeta + \dots + a_{p-1}\zeta^{p-1} \in \mathcal{O}$ with $a_i \in \mathbb{Z}$ and at least one $a_j \neq 0$.

If $n \in \mathbb{Z}$ with n divides α in \mathcal{O} , then n divides all a_i

Proof. Recall from 2.2 (preview): $1, \zeta, \zeta^2, \dots, \zeta^{p-2}$ is a basis of \mathcal{O} .

Furthermore: $1 + \zeta + \dots + \zeta^{p-1} = 0$

$\Rightarrow \{1, \zeta, \dots, \zeta^{p-1}\} \setminus \{\zeta^j\}$ is a basis \Rightarrow claim.

□

1.3. First case of Fermat in case of $\mathbb{Z}[\zeta]$ is a UFD (unique factorization domain)

Reference: BOREVICH + SHAFEREVIC + WASHINGTON Chapter 1

As before: p odd prime, $\zeta = e^{\frac{2\pi i}{p}}$ p -th root of unity.

Theorem 2. Suppose that $\mathbb{Z}[\zeta]$ is a UFD, then $x^p + y^p = z^p$ has no non-trivial solutions (x, y, z) , such that neither x, y nor z is divisible by p .

Theorem 3 ($p = 3$). Suppose $x, y, z \in \mathbb{Z}$ with $x^3 + y^3 = z^3 \pmod{9} \Rightarrow 3$ divides x, y or z .

Proof. Recall: Little Fermat's theorem $x^p \equiv x, y^p \equiv y, z^p \equiv z \pmod{p}$.

$$\begin{aligned}
 x^3 + y^3 &= z^3 \pmod{3} \Rightarrow x + y \equiv z \pmod{3} \\
 &\Rightarrow z = x + y + 3u \text{ with } u \in \mathbb{Z} \\
 \Rightarrow \underline{x^3 + y^3} &\equiv (x + y + 3u)^3 \equiv \underline{x^3 + y^3} + 3xy^2 + 3x^2y \pmod{9} \\
 &\Rightarrow 0 \equiv xy^3 + x^2y \equiv xy(x + y) \equiv xyz \pmod{3} \\
 &\Rightarrow x, y \text{ or } z \text{ is divisible by } 3
 \end{aligned}$$

□

Lemma 1.3.1. *Let $p \geq 5$. Suppose $x, y, z \in \mathbb{Z}$ with $x^p + y^p = z^p$. If $x \equiv y \equiv -z \pmod{p}$, then $p|z$.*

Proof. $z \equiv z^p = x^p + y^p \equiv -2z^p \equiv -2z \pmod{p} \Rightarrow 3z \equiv 0 \pmod{p} \xrightarrow{p \neq 3} p|z$. \square

Remark 1.3.2. It follows from Lemma 3.2 that in the first case of Fermat we may assume for $p \geq 5$ that $x \not\equiv y \pmod{p}$ because we can replace $x^p + y^p = z^p$ by $x^p + (-z)^p = (-y)^p$ and $x \not\equiv -z \pmod{p}$.

of Thm. 1. $p = 3 \Rightarrow$ claim follows from Prop 3.1.

Now: $p \geq 5$. Suppose $x, y, z \in \mathbb{Z}$ with p divides neither x, y nor z , x, y, z are pairwise coprime and $x \not\equiv y \pmod{p}$. Suppose $z^p = x^p + y^p = (x + y)(x + \zeta y) \dots (x + \zeta^{p-1}y)$.

Apply Lemma 2.11:

- $\gcd(x, y) = 1$ ✓
- Little Fermat $\Rightarrow x + y \equiv x^p + y^p \equiv z^p \not\equiv 0 \pmod{p}$

$\xrightarrow{2.11} x + y, x + \zeta y, \dots, x + \zeta^{p-1}y$ are pairwise coprime.

$\xrightarrow{\mathbb{Z}[\zeta] \text{ UFD}} \text{„}x + \zeta^i y \text{ have to be } p\text{-power“}$ More precisely: $x + \zeta y = \varepsilon \alpha^p$ with $\varepsilon \in \mathcal{O}^\times, \alpha \in \mathcal{O}$, since they are coprime factors of a p -th power.

1. Cor. 2.10 $\Rightarrow \varepsilon = r\zeta^s$ with $r \in \mathbb{R}, s \in \mathbb{Z}$
2. Example 2.8. iii) $\Rightarrow \exists a \in \mathbb{Z}$ with $\alpha^p \equiv a \pmod{p}$.

$$\begin{aligned} x + \zeta y &= r\zeta^s \alpha^p \equiv r\zeta^s a \pmod{p} \\ x + \zeta^{-1}y &= \overline{x + \zeta y} \equiv r\zeta^{-s} a \pmod{p} \\ \Rightarrow \zeta^{-s}(x + \zeta y) &\equiv ra \equiv \zeta^s(x + \zeta^{-1}y) \pmod{p} \\ \Rightarrow \underbrace{x + \zeta y - \zeta^{2s}x - \zeta^{2s-1}y}_{=x \cdot 1 + y\zeta - x\zeta^{2s} - y\zeta^{2s-1}} &\equiv 0 \pmod{p} \end{aligned}$$

Idea: Use Rem. 2.12

Case 1: $1, \zeta, \zeta^{2s-1}, \zeta^{2s}$ are distinct $\xrightarrow{p \geq 5, \text{ Rem 2.12}} p|x$ and $p|y$. Contradiction to first case.

\square

Recall: $L = \mathbb{Q}(\zeta)$, $\mathcal{O} = \mathbb{Z}[\zeta]$, where ζ is a p -th root of unity

Last time:

- (1) $a_1 1 + a_2 \zeta + \dots + a_p \zeta^{p-1} = \alpha$ and at least one $a_j = 0$

If α is divided by $n \in \mathbb{Z}$ then all the a_i are divided by n .

- (2) $x + y\zeta - x\zeta^{2s} - y\zeta^{2s-1} \equiv 0 \pmod{p}$

Continuation of proof of Theorem 1. “Case 2” $1, \zeta, \dots, \zeta^{2s}$ are not distinct.

Observe: $1 \neq \zeta$ and $\zeta^{2s-1} \neq \zeta^{2s}$

“Case 2A” $1 = \zeta^{2s} (\Leftrightarrow p|s)$.

(2) implies $y\zeta - y\zeta^{2s-1} \equiv 0 \pmod{p}$ such that Remark 2.12 yields the contradiction $p|y$.

“Case 2B” $1 = \zeta^{2s-1} (\Leftrightarrow \zeta = \zeta^{2s})$.

(2) implies $(x - y)1 + (y - x)\zeta \equiv 0 \pmod{p}$ such that Remark 2.12 yields $p|y - x$, which contradicts the assumption $x \not\equiv y \pmod{p}$.

“Case 2C” $\zeta = \zeta^{2s-1}$.

(2) implies $x - x\zeta^2 \equiv 0 \pmod{p}$ such that Remark 2.12 yields the contradiction $p|x$. \square

Questions:

(1) Under which assumption is \mathcal{O} a UFD?

(2) What can we do if \mathcal{O} is not a UFD?

→ Idea of Kummer: “calculate with ideals”

Prospect: Theorem (Montgomery, Uchida, 1971)

$\mathbb{Z}[\zeta]$ is a UFD if and only if $p \leq 19$, p prime.

Preview: From Kummer’s idea we obtain a better criterion for p called **regular**, which ensures that Fermat’s conjecture holds for p .

Conjecture. *There are infinitely many regular primes.*

2. Ring of integers

In this chapter, all rings are assumed to be commutative with 1.

2.1. Integral ring extensions

Definition 2.1.1 (“ganze Ringerweiterungen”). Let $A \subset B$ be a ring extension.

- (i) $b \in B$ is **integral** over A if there exists a monic polynomial $f(X) = X^n + a_{n-1}X^{n-1} + \dots + a_0 \in A[X]$ with $f(b) = 0$.
- (ii) B is **integral** over A if all $b \in B$ are integral over A .

Proposition 2.1.2. Let $A \subset B$ be a ring extension and $b_1, \dots, b_n \in B$. Then b_1, \dots, b_n are integral over A if and only if

$$A[b_1, \dots, b_n] = \{f(b_1, \dots, b_n) \mid f \in A[X_1, \dots, X_n]\}$$

is a finitely generated A -module.

Reminder 2.1.3 (“Adjunkte”). Let R be a ring and $A \in R^{n \times n}$

- (i) $A^\# = (a_{i,j}^\#)$ with $a_{i,j}^\# = (-1)^{i+j} \det(A_{j,i})$, where $A_{j,i}$ is obtained from A by deleting the j -th row and i -th column of A .
- (ii) We have $AA^\# = A^\#A = \det(A)I$. In particular, $Ax = 0$ implies $A^\#Ax = 0$ such that $\det(A)x = 0$.

Proof of Proposition 1.2. “ \Rightarrow ” If $n = 1$ and b is integral over A , then there is an $f \in A[X]$ with f monic such that $f(b) = 0$. Let $g \in A[X]$ be arbitrary. Then

$$g(X) = q(X)f(X) + r(X)$$

with $q, r \in A[X]$ and $\deg r < \deg f = d$. Hence $g(b) = r(b)$ with $\deg r < d$. Thus $\{1, b, \dots, b^{d-1}\}$ generate $A[b]$ as an A -module. The case $n \geq 2$ follows by induction.

“ \Leftarrow ” $A[b_1, \dots, b_n]$ is finitely generated as an A -module by w_1, \dots, w_r . If $b \in A[b_1, \dots, b_n]$ then

$$bw_i = \sum_{j=1}^r a_{j,i} w_j$$

such that

$$(bI - (a_{i,j}))w = 0.$$

Thus, $\det(bI - (a_{i,j}))w = 0$ and hence

$$\det(bI - (a_{i,j}))w_i = 0$$

for all $i = 1, \dots, r$. If we now use that

$$1 = c_1w_1 + \dots + c_rw_r$$

we can infer $\det(bI - (a_{i,j}))1 = 0$. Consider

$$M = bI - (a_{i,j}) = \begin{pmatrix} b - a_{1,1} & -a_{1,2} & \cdots & -a_{1,r} \\ -a_{2,1} & b - a_{2,2} & \cdots & -a_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{r,1} & -a_{r,2} & \cdots & b - a_{r,r} \end{pmatrix}.$$

By the Leibniz formula we have

$$\det(M) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n m_{\sigma(i),i}$$

which is a polynomial over b with leading coefficient 1. Hence b is integral over A . \square

Corollary 2.1.4 (And Definition). *(i) If $A \subset B$ is an extension of rings then*

$$\overline{A} = \{b \in B \mid b \text{ is integral over } A\}$$

*is a ring. It is called the **integral closure** of A in B . If $\overline{A} = A$ then A is called **integrally closed** in B .*

(ii) We have transitivity, that is to say, if A, B, C are rings with $A \subset B \subset C$ such that C is integral over B and B is integral over A then C is integral over A .

(iii) The integral closure of A in B is integrally closed, i.e., $\overline{\overline{A}} = \overline{A}$.

Proof. “(i)” If $b_1, b_2 \in \overline{A}$ then $A[b_1], A[b_2]$ are finitely generated A -modules. Hence $A[b_1, b_2]$ is a finitely generated A -module. Thus, by Proposition 1.3, $b_1 + b_2$ and b_1b_2 are integral, i.e., elements of \overline{A} .

“(ii)” If $c \in C$ then c is integral over B and hence there is a monic polynomial $f = X^n + b_{n-1}X^{n-1} + \dots + b_0 \in B[X]$ with $f(c) = 0$. This shows that c is integral over $R = A[b_1, \dots, b_{n-1}]$ such that Proposition 1.3 shows that $R[c]$ is a finitely generated R -module. Furthermore, b_0, \dots, b_{n-1} are integral over A such that another application of Proposition 1.3 shows that R is a finitely generated A -module. Hence, $R[c]$ is a finitely generated A module such that c is integral over A by Proposition 1.3.

“(iii)” Follows from (ii). \square

Definition 2.1.5 (“ganzer Abschluss und normaler Ring”). If A is an integral domain we call its integral closure \overline{A} in $K = \text{Quot}(A)$ the **normalization** or the **integral closure** of A . We say A is **integrally closed** if A is integrally closed in K .

Remark 2.1.6. If A is a UFD then A is integrally closed.

Proof. Suppose $b = \frac{a}{a'} \in \text{Quot}(A)$ with $\gcd(a, a') = 1$ is integral over A . Then there exist $a_0, \dots, a_{n-1} \in A$ with

$$\left(\frac{a}{a'}\right)^n + a_{n-1} \left(\frac{a}{a'}\right)^{n-1} + a_{n-2} \left(\frac{a}{a'}\right)^{n-2} + \dots + a_0 = 0$$

such that

$$a^n + a_{n-1}a'a^{n-1} + a_{n-2}a'^2a^{n-2} + \dots + a_0a'^n = 0.$$

Let $a' = \varepsilon \pi_1 \cdots \pi_r$ be the prime factorization of a' with $\varepsilon \in A^\times$ and π_1, \dots, π_r primes. Since $\pi_i | a'$ the above equation shows that actually $\pi_i | a^n$. But this implies $\pi_i | a$ which is a contradiction to $\gcd(a, a') = 1$. Hence we have $a' = \varepsilon \in A^\times$ such that $b \in A$. \square

2.2. Integral closures in field extensions

Setting:

- A is an integral domain.
- A is integrally closed.
- $K = \text{Quot}(A)$.
- L/K is a finite field extension with $\overline{A}_K = A \subset K = \text{Quot}(A) \hookrightarrow L \supset B = \overline{A}_L$.
- B is the integral closure of A in L . Observe: $B \cap K = A$

Remark 2.2.1. (i) B is integrally closed in L .

(ii) If $\beta \in L$ then there are $b \in B$ and $a \in A \setminus \{0\}$ such that $\beta = \frac{b}{a}$.

In particular, $L = \text{Quot}(B)$.

(iii) For $\beta \in L$ we have $\beta \in B$ if and only if $f_\beta \in A[X]$, where f_β is the minimal polynomial of β over K .

Proof. “(i)” Follows from the transitivity in Corollary 1.4.

“(ii)” Choose $a \in A$ with $a^n f_\beta(X) = a^n X^n + a^{n-1} c_{n-1} X^{n-1} + \dots + c_0 \in A[X]$. Then we have

$$a^n \beta^n + c_{n-1} a^{n-1} \beta^{n-1} + \dots + c_0 = 0$$

and hence

$$(a\beta)^n + c_{n-1} (a\beta)^{n-1} + \dots + c_0 = 0$$

such that $a\beta$ is integral over A . Consequently, $b = a\beta \in B$ and $\beta = \frac{b}{a}$.

“(iii)” “ \Leftarrow ” Obvious. “ \Rightarrow ” Let β be a zero of $g(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in A[X]$. Then f_β divides g . If β_1, \dots, β_n are the zeros of f_β in \overline{K} then they are also zeros of g and thus integral over A . Hence the coefficients of f_β are integral over A and are elements of K such that $f_\beta \in A[X]$ as claimed. \square

Reminder 2.2.2 (Trace, Norm). Let $K \subseteq L$ be a finite field extension. For α in L consider the map $T_\alpha : \beta \mapsto \alpha\beta$. The following holds

i) $\text{Tr}_{L/K}(\alpha) = \text{Tr}(T_\alpha)$ and $\mathcal{N}_{L/K}(\alpha) = \det(T_\alpha)$,

ii) If $L = K(\alpha)$ and $f_\alpha(X) = X^n + \sum_{i=0}^{n-1} a_i X^i$ then

$$\text{Tr}_{L/K}(\alpha) = -a_{n-1} \text{ and } \mathcal{N}_{L/K}(\alpha) = (-1)^n \cdot a_0,$$

iii) Since $T_{\alpha+\beta} = T_\alpha + T_\beta$ and $T_{\alpha\beta} = T_\alpha \circ T_\beta$, we conclude that

$$\text{Tr}_{L/K} : (L, +) \rightarrow (K, +) \text{ and } \mathcal{N}_{L/K} : (L^*, \cdot) \rightarrow (K^*, \cdot)$$

are group homomorphisms,

iv) Suppose $K \subseteq L$ is a separable field extension with $L = K(\alpha)$. Further assume $\text{Hom}_K(L, \overline{K}) = \{\sigma_1, \dots, \sigma_n\}$. Then the following holds

- $f_\alpha = \prod_{i=1}^n (X - \sigma_i(\alpha))$,
- $\text{Tr}_{L/K}(\alpha) = \sum_{i=1}^n \sigma_i(\alpha)$,
- $\mathcal{N}_{L/K}(\alpha) = \prod_{i=1}^n \sigma_i(\alpha)$,

v) Trace and norm are transitive, i.e., for field extensions $K \subseteq L \subseteq M$ it holds

- $\mathcal{N}_{L/K} \circ \mathcal{N}_{M/L} = \mathcal{N}_{M/K}$,
- $\text{Tr}_{L/K} \circ \text{Tr}_{M/L} = \text{Tr}_{M/K}$.

Definition 2.2.3 (Discriminant). Let $K \subseteq L$ be a separable field extension and let $\alpha_1, \dots, \alpha_n$ be a K -basis of L . Further let $\text{Hom}_K(L, \overline{K}) = \{\sigma_1, \dots, \sigma_n\}$. Consider the matrix

$$A := \begin{pmatrix} \sigma_1(\alpha_1) & \sigma_1(\alpha_2) & \cdots & \sigma_1(\alpha_n) \\ \sigma_2(\alpha_1) & \sigma_2(\alpha_2) & \cdots & \sigma_2(\alpha_n) \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_n(\alpha_1) & \sigma_n(\alpha_2) & \cdots & \sigma_n(\alpha_n) \end{pmatrix} = (\sigma_i(\alpha_j))_{i,j} \in L^{n \times n}.$$

We call $d(\alpha_1, \dots, \alpha_n) := \det(A^2)$ the **discriminant** of L over K with respect to the basis $\alpha_1, \dots, \alpha_n$.

Remark 2.2.4. In the situation of Definition (2.2.3) the following holds.

- i) Consider the matrix $B = (\text{Tr}_{L/K}(\alpha_i \alpha_j))_{i,j}$ in $K^{n \times n}$. Then the discriminant is given by $d(\alpha_1, \dots, \alpha_n) = \det(B)$. In particular, the discriminant $d(\alpha_1, \dots, \alpha_n)$ lies in K .
- ii) Suppose we have Θ in L such that $1, \Theta, \dots, \Theta^{n-1}$ forms a basis of L . Then the following equality holds

$$d(1, \Theta, \dots, \Theta^{n-1}) = \prod_{1 \leq i < j \leq n} (\Theta_i - \Theta_j)^2.$$

Here Θ_i denotes $\sigma_i(\Theta)$. If $L = K(\Theta)$ then $d(1, \Theta, \dots, \Theta^{n-1})$ coincides with the discriminant of the minimal polynomial f_Θ . Note that we use the notion of discriminants for polynomials here.

Proof. We begin by proving statement i). One computes

$$\det(A)^2 = \det(A^t) \cdot \det(A) = \det(A^t \cdot A).$$

The following calculation proves the claim

$$\begin{aligned} A^t \cdot A &= (\sigma_j(\alpha_i))_{i,j} \cdot (\sigma_k(\alpha_\ell))_{k,\ell} \\ &= \left(\sum_{j=1}^n \sigma_j(\alpha_i) \cdot \sigma_j(\alpha_\ell) \right)_{i,\ell} \\ &= \left(\sum_{j=1}^n \sigma_j(\alpha_i \cdot \alpha_\ell) \right)_{i,\ell} \\ &= (\text{Tr}_{L/K}(\alpha_i \cdot \alpha_\ell))_{i,\ell} \\ &= B. \end{aligned}$$

For statement ii), we will compute the determinant of the following Vandermonde matrix

$$\det(A) = \det \begin{pmatrix} 1 & \Theta_1 & \dots & \Theta_1^{n-1} \\ 1 & \Theta_2 & \dots & \Theta_2^{n-1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & \Theta_n & \dots & \Theta_n^{n-1} \end{pmatrix} =: V_n(\Theta_1, \dots, \Theta_n).$$

By induction, we prove that $V_n(\Theta_1, \dots, \Theta_n)$ is nonzero and that the following equality holds

$$V_n(\Theta_1, \dots, \Theta_n) = \prod_{1 \leq i < j \leq n} (\Theta_j - \Theta_i).$$

For $n = 2$, we have

$$\det(A) = \det \begin{pmatrix} 1 & \Theta_1 \\ 1 & \Theta_2 \end{pmatrix} = \Theta_2 - \Theta_1 \neq 0.$$

Hence the claim holds for $n = 2$. Now we assume that the claim holds for a $n \in \mathbb{N}_{\geq 2}$. We want to prove that viewed as polynomials in Z the following equality holds

$$V_{n+1}(\Theta_1, \dots, \Theta_n, Z) = V_n(\Theta_1, \dots, \Theta_n) \cdot \prod_{i=1}^n (Z - \Theta_i). \quad (2.1)$$

This implies that

$$V_n(\Theta_1, \dots, \Theta_{n+1}) = V_n(\Theta_1, \dots, \Theta_n) \cdot \prod_{i=1}^n (\Theta_{n+1} - \Theta_i) = \prod_{1 \leq i < j \leq n} (\Theta_j - \Theta_i).$$

To show equality (2.1), recall that

$$V_{n+1}(\Theta_1, \dots, \Theta_n, Z) = \det \begin{pmatrix} 1 & \Theta_1 & \cdots & \Theta_1^n \\ 1 & \Theta_2 & \cdots & \Theta_2^n \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \Theta_n(\alpha_2) & \cdots & \Theta_n^n \\ 1 & Z & \cdots & Z^n \end{pmatrix}.$$

One sees that the polynomials on both sides of equality (2.1) have degree n . Moreover, $\{\Theta_1, \dots, \Theta_n\}$ is the set of zeros for both polynomials. Since the leading coefficient in both cases is $V_n(\Theta_1, \dots, \Theta_n)$, the polynomials are equal. This proves the claim. \square

Example 2.2.5. Consider $L = \mathbb{Q}(\sqrt{D})$ for a square free integer D different from 0 and 1. Then the following holds

- $\mathfrak{B}_1 = \{1, \sqrt{D}\}$ is a \mathbb{Q} -basis of L .
- Define $\sigma_2 : L \rightarrow \overline{\mathbb{Q}}, a + b\sqrt{D} \mapsto a - b\sqrt{D}$. Then we have

$$\text{Hom}_{\mathbb{Q}}(L, \overline{\mathbb{Q}}) = \{\sigma_1 = \text{id}, \sigma_2\}.$$

- $\text{Tr}_{L/\mathbb{Q}}(a + b\sqrt{D}) = a + b\sqrt{D} + a - b\sqrt{D} = 2a$.
- $\mathcal{N}_{L/\mathbb{Q}}(a + b\sqrt{D}) = (a + b\sqrt{D}) \cdot (a - b\sqrt{D}) = a^2 - b^2 \cdot D$.
- $d(\mathfrak{B}_1) = \det \begin{pmatrix} 1 & \sqrt{D} \\ 1 & -\sqrt{D} \end{pmatrix}^2 = (-2\sqrt{D})^2 = 4D$.
- We have

$$(\alpha_i \alpha_j)_{i,j} = \begin{pmatrix} 1 & \sqrt{D} \\ \sqrt{D} & D \end{pmatrix}.$$

Hence we compute

$$\det((\text{Tr}(\alpha_i \alpha_j))_{i,j}) = \det \begin{pmatrix} 2 & 0 \\ 0 & 2D \end{pmatrix} = 4D.$$

- Consider the \mathbb{Q} -basis of L given by $\mathfrak{B}_2 = \{1 + \sqrt{D}, 1 - \sqrt{D}\}$. Computing the discriminant for this basis yields

$$d(1 + \sqrt{D}, 1 - \sqrt{D}) = \det \begin{pmatrix} 1 + \sqrt{D} & 1 - \sqrt{D} \\ 1 - \sqrt{D} & 1 + \sqrt{D} \end{pmatrix}^2 = 16D.$$

Hence we see that the discriminant depends on the basis we choose.

Proposition 2.2.6. *Let $K \subseteq L$ be a separable field extension.*

i) *The bilinear map*

$$h : L^2 \rightarrow K, (x, y) \mapsto \text{Tr}_{L/K}(xy)$$

is non degenerate, i.e., $h(x, y) = 0$ for all $y \in L$ implies that $x = 0$.

ii) *If $\alpha_1, \dots, \alpha_n$ forms a basis of L/K then $d(\alpha_1, \dots, \alpha_n) \neq 0$.*

Proof. For statement i), we choose a primitive element Θ . Then $1, \Theta, \dots, \Theta^{n-1}$ is a K -basis of L . Let B be the matrix representation of h with respect to this basis. We find

$$\begin{aligned} \det(B) &\stackrel{(2.4) \text{ i)}}{=} d(1, \Theta, \dots, \Theta^{n-1}) \\ &\stackrel{(2.4) \text{ ii)}}{=} \prod_{1 \leq i < j \leq n} (\Theta_i - \Theta_j)^2 \neq 0. \end{aligned}$$

Here Θ_i denotes $\sigma_i(\Theta)$. This shows that h is non degenerate. We now prove statement ii). Observe that the matrix $M = (\text{Tr}_{L/K}(\alpha_i \alpha_j))_{i,j}$ is the matrix representation of h with respect to $\alpha_1, \dots, \alpha_n$. By Remark (2.4), we conclude

$$d(\alpha_1, \dots, \alpha_n) = \det(M).$$

Now, i) implies that $\det(M)$ is nonzero. □

Remark 2.2.7. Let $A \subseteq B$ be an integral ring extension with $B \subseteq L$ and $A = B \cap K \subseteq K$. Assuming that $\text{Hom}_K(L, \overline{K}) = \{\text{id} = \sigma_1, \dots, \sigma_n\}$ the following holds

- i) If $x \in B$ then $\sigma_i(x) \in B$ for all $1 \leq i \leq n$.
- ii) For all $x \in B$ the trace $\text{Tr}_{L/K}(x)$ and the norm $\mathcal{N}_{L/K}(x)$ lie in A .
- iii) Let $x \in B$. Then x lies in B^* if and only if the norm $\mathcal{N}_{L/K}(x)$ lie in A^* .

Proof. We start by proving i). Let x in B . By Remark (2.1), we have that the minimal polynomial f_x lies in $A[X]$. Since $\sigma(x)$ is also a zero of f_x , it is contained in B . This shows i). Now, statement ii) follows from i), Remark (2.2) iv) and the fact that $A = B \cap K$. For iii), assume that x is a unit in B , i.e., we find y in B with $xy = 1$. Hence

$$\mathcal{N}_{L/K}(x) \cdot \mathcal{N}_{L/K}(y) = \mathcal{N}_{L/K}(xy) = 1.$$

Using ii), we deduce that $\mathcal{N}_{L|K}(x)$ lies in A^* . This proves one direction. For the other direction, assume that $\mathcal{N}_{L|K}(x)$ lies in A^* , i.e., we find $a \in A$ with

$$\begin{aligned} 1 &= a \cdot \mathcal{N}_{L|K}(x) \\ &= a \cdot \prod_{i=1}^n \sigma_i(x) \\ &= a \cdot x \cdot \underbrace{\prod_{i=2}^n \sigma_i(x)}_{\in B, \text{ by i)}}. \end{aligned}$$

Hence x lies in B^* . This proves iii). \square

Proposition 2.2.8. Suppose $\alpha_1, \dots, \alpha_n \in B$ forms a K -basis of L . Let d denote the discriminant $d(\alpha_1, \dots, \alpha_n) \in A$. Then $d \cdot B$ is contained in $A\alpha_1 + \dots + A\alpha_n$.

Proof. Suppose $\alpha = \sum_{j=1}^n c_j \alpha_j \in B$ for $c_i \in K$. We want to solve for (c_1, \dots, c_n) . Applying the trace to the equalities

$$\alpha_i \alpha = \sum_{j=1}^n c_j \alpha_i \alpha_j, \quad 1 \leq i \leq n,$$

we obtain

$$\text{Tr}_{L/K}(\alpha_i \alpha) = \sum_{j=1}^n c_j \text{Tr}_{L/K}(\alpha_i \alpha_j), \quad 1 \leq i \leq n.$$

Hence $x = (c_1, \dots, c_n)$ is the solution of the linear system $Mx = y$, where

$$M = ((\text{Tr}_{L/K}(\alpha_i \alpha_j)))_{i,j} \in A^{n \times n}, \quad y = (\text{Tr}_{L/K}(\alpha_i \alpha))_i \in A^n.$$

By Remark (1.3), we have

$$\det(M) \cdot x = M^\# Mx = M^\# y \in A^n.$$

Using Remark (2.4), we know $\det(M) = d(\alpha_1, \dots, \alpha_n) =: d$. We conclude that dc_i lies in A for $1 \leq i \leq n$, which proves the claim. \square

Definition 2.2.9 (Ganzheitsbasis). Suppose $\omega_1, \dots, \omega_n \in B$ forms a basis of B over A , i.e., every $\alpha \in B$ can be written in a unique way as an A -linear combination $\sum_{i=1}^n c_i \omega_i$. Then $\omega_1, \dots, \omega_n$ is called an **integral basis** of B over A .

Example 2.2.10. Same situation as in Ex. 2.5. $\mathcal{B}_1 = \{1, \sqrt{D}\} \subseteq B$. Consider:

$$\begin{aligned} \alpha &= \frac{1}{2}(1 + \sqrt{D}) \Rightarrow 2\alpha = 1 + \sqrt{D} \\ \Rightarrow (2\alpha - 1)^2 &= D \Rightarrow 4\alpha^2 - 4\alpha + 1 = D \\ \Rightarrow f_\alpha(X) &= X^2 - X + \frac{1-D}{4} \end{aligned}$$

Hence if $D \equiv 1 \pmod{4} \Rightarrow \alpha \in B$ and \mathcal{B}_1 is not an integral basis.

Proposition 2.2.11. *Let $D \in \mathbb{Z}$, D square-free, $D \neq 0, 1$, $B :=$ integral closure of \mathbb{Z} in $\mathbb{Q}(\sqrt{D}) = L$.*

- i) $D \equiv 2, 3 \pmod{4} \Rightarrow \{1, \sqrt{D}\}$ is an integral basis of B/\mathbb{Z} in particular $B = \mathbb{Z}[\sqrt{D}]$.
- ii) $D \equiv 1 \pmod{4} \Rightarrow \{1, \frac{1}{2}(\sqrt{D}+1)\}$ is an integral basis of B/\mathbb{Z} . and $B = \mathbb{Z}[\frac{1}{2}(1+\sqrt{D})]$.

Proof. Consider $\alpha = a + b\sqrt{D} \in \mathbb{Q}(\sqrt{D})$ with $a, b \in \mathbb{Q}$.

$$\Rightarrow f_\alpha = X^2 - 2aX + a^2 - b^2D.$$

Rem 2.1: $\alpha \in B \iff f_\alpha \in \mathbb{Z}[X] \iff 2a \in \mathbb{Z} \text{ and } a^2 - b^2D \in \mathbb{Z}$.

- (1) Show: $\alpha \in B \Rightarrow 2b \in \mathbb{Z}$.

$$\alpha \in B \Rightarrow 4a^2 - 4b^2D = 4z \text{ with } z \in \mathbb{Z}. \text{ Write } b = \frac{p}{q} \text{ with } p, q \in \mathbb{Z}, \gcd(p, q) = 1$$

$$\Rightarrow 4p^2D = ((2a)^2 - 4z)q^2 \quad (\star)$$

$$\Rightarrow q = 1 \text{ or } 2.$$

- (2) Show: $q = 2 \Rightarrow D \equiv 1 \pmod{4}$

$$(\star) \Rightarrow p^2D = (2a)^2 - 4z \equiv (2a)^2 \pmod{4}$$

$$p \text{ is odd, hence } p^2 \equiv 1 \pmod{4} \Rightarrow (2a)^2 \text{ is odd (i.e. } a = \frac{2n-1}{2} \in \mathbb{Q})$$

$$\Rightarrow (2a)^2 \equiv 1 \pmod{4} \Rightarrow D \equiv 1 \pmod{4}.$$

- (3) It follows from (2) if $D \equiv 1 \pmod{4}$:

$$\alpha \in B \iff \alpha = a + b\sqrt{D} \text{ or } \alpha = \frac{1}{2}(a + b\sqrt{D}) \text{ with } a, b \in \mathbb{Z}. \text{ Hence we obtain:}$$

$$B = \begin{cases} \mathbb{Z}[\sqrt{D}] & , \text{ if } D \equiv 2, 3 \pmod{4} \\ \mathbb{Z}[\frac{1}{2}(1 + \sqrt{D})] & , \text{ if } D \equiv 1 \pmod{4} \end{cases}$$

For the second case observe that $\frac{a}{2} + \frac{b}{2}\sqrt{D} = \frac{a-b}{2} + \frac{b}{2}(1 + \sqrt{D}) \in \mathbb{Z}[\frac{1}{2}(1 + \sqrt{D})]$.

This implies the claim. □

Proposition 2.2.12. *Suppose L/K separable and A is a principal ideal domain. Let $M \neq 0$ be a finitely generated B -submodule of $L \Rightarrow M$ is a free A -module. In particular: B is a free A -module of rank $n := [L : K]$.*

Reminder 2.2.13. Suppose A is a principal ideal domain and M_0 is a finitely generated free A -module.

- i) Any submodule M of M_0 is free.

- ii) $\text{rank}(M_0) \geq \text{rank}(M)$

of Prop 2.12. Let $\mu_1, \dots, \mu_r \in M \subseteq L$ be generators of M as B -module and let $\alpha_1, \dots, \alpha_n$ be a basis of L/K in B and $d := d(\alpha_1, \dots, \alpha_n) \in A$.

Recall: $L = \{\frac{b}{a} \mid b \in B, a \in A \setminus \{0\}\}$.

- (1) Prop 2.7 $\Rightarrow dB \subseteq A\alpha_1 + \dots + A\alpha_n$

$$(2) \exists a \in A : a\mu_1, \dots, a\mu_r \in B$$

Hence: $daM \subseteq dB \subseteq A\alpha_1 + \dots + A\alpha_n =: M_0$

(M_0 is a free A -module, since $\alpha_1, \dots, \alpha_n$ are basis of L/K).

Reminder 2.13 $\Rightarrow adM$ is a free A -module $\Rightarrow M$ is a free A -module.

Furthermore: $\text{rank}(M) = \text{rank}(adM) \stackrel{\text{Rem. 2.13}}{\leq} \text{rank}(M_0) = n$.

Suppose that $M = B$. So far we got that B is a free A -module and $\text{rank}(B) \leq n$.

Show: $\text{rank}(B) \geq n$.

Let μ_1, \dots, μ_r be a basis of B as A -module. By $L = \{\frac{b}{a} \mid b \in B, a \in A \setminus \{0\}\}$ we have that μ_1, \dots, μ_r generate L over K . \square

Hence: if A is a principal ideal domain, then B has always an integral basis.

Proposition 2.2.14. *Suppose we are in the following situation:*

- L/K and L'/K are Galois extensions of degree n and m in some field E
- A a subring of K such that $K = \text{Quot}(A)$ and B and B' are the integral closures of A in L and L' .
- $\{\omega_1, \dots, \omega_n\}$ and $\{\omega'_1, \dots, \omega'_m\}$ are integral basis for B/A and B'/A .
- $d := d(\omega_1, \dots, \omega_n)$ and $d' := d(\omega'_1, \dots, \omega'_m) \in A$ with d and d' are coprime in A , i.e. $\exists x, x' \in A$ with $1 = dx + d'x'$.
- $K = L \cap L'$

Then we have: $\{\omega_i \omega'_j \mid i \in \{1, \dots, n\}, j \in \{1, \dots, m\}\}$ is an integral basis and its discriminant is $d^m (d')^n$.

Proof. Recall: $L \cap L' = K \Rightarrow [LL' : K] = nm$ and $\{\omega_i \omega'_j\}$ is a basis of the field extension LL'/K .

$\text{Gal}(L/K) = \{\sigma_1, \dots, \sigma_n\}$ and $\text{Gal}(L'/K) = \{\sigma'_1, \dots, \sigma'_m\}$

\Rightarrow obtain unique lifts $\hat{\sigma}_i \in \text{Gal}(LL'/L')$ and $\hat{\sigma}'_j \in \text{Gal}(LL'/L)$ and $\text{Gal}(LL'/K) = \{\hat{\sigma}_i \hat{\sigma}'_j \mid i \in \{1, \dots, n\}, j \in \{1, \dots, m\}\}$.

Consider: $\alpha \in \tilde{B} :=$ integral closure of A in LL' .

Write $\alpha = \sum_{i,j} \alpha_{i,j} \omega_i \omega'_j = \sum_j \beta_j \omega'_j$ with $\alpha_{i,j} \in K$ and $\beta_j = \sum_i \alpha_{i,j} \omega_i \in L$.

$\Rightarrow \hat{\sigma}'_i(\alpha) = \sum_j \beta_j \hat{\sigma}'_i(\omega'_j)$, since $\hat{\sigma}'_i \in \text{Gal}(LL'/L)$.

\Rightarrow We have a linear system:

$$a = Tb \text{ with } a = \begin{pmatrix} \hat{\sigma}'_1(\alpha) \\ \vdots \\ \hat{\sigma}'_m(\alpha) \end{pmatrix} \in \tilde{B}^m, \quad b = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} \in L^m, \quad T = (\hat{\sigma}'_i(\omega'_j))_{(i,j)} \in \tilde{B}^{m \times m}$$

Observe: $\det(T)^2 = d'$

$$\begin{aligned}
 &\Rightarrow \det(T)b = T^\# T b = T^\# a \in \tilde{B}^m && \Rightarrow d'b \in \tilde{B}^m \\
 &\Rightarrow \forall j : d'\beta_j = \sum_i d'\alpha_{i,j}\omega_i \in \tilde{B} \cap L = B \\
 &\Rightarrow d'\alpha_{i,j} \in A, \text{ since } \{\omega_1, \dots, \omega_n\} \text{ is an integral basis.} \\
 &\Rightarrow d\alpha_{i,j} \in A \text{ in the same way} \\
 &\Rightarrow \alpha_{i,j} = (x'd' + xd)\alpha_{i,j} = x'd'\alpha_{i,j} + xd\alpha_{i,j} \in A.
 \end{aligned}$$

Hence: $\{\omega_i\omega'_j \mid (i,j) \in \{(1,1), \dots, (n,m)\}\}$ is an integral basis of \tilde{B}/A .

For calculating the discriminant consider the matrix $M = (\hat{\sigma}_k \circ \hat{\sigma}'_l(\omega_i\omega'_j))_{(k,l),(i,j)} = (\hat{\sigma}_k(\omega_i)\hat{\sigma}'_l(\omega'_j))$.

Consider $Q = (\hat{\sigma}_k(\omega_i))$

$$\Rightarrow M = \begin{pmatrix} Q & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & Q \end{pmatrix} \cdot \begin{pmatrix} I \cdot \hat{\sigma}'_1(\omega'_1) & \dots & I \cdot \hat{\sigma}'_1(\omega'_1) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ I \cdot \hat{\sigma}'_m(\omega'_m) & \dots & I \cdot \hat{\sigma}'_m(\omega'_m) \end{pmatrix}$$

Observe:

$$(1) \det(Q)^2 = d(\omega_1, \omega_n) = d$$

$$(2) \text{ The second matrix can be transformed by switching rows and columns to } \begin{pmatrix} Q' & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & Q' \end{pmatrix}$$

with $Q' = (\sigma'_l(\omega'_j))$ and $\det(Q') = d'$

$$\Rightarrow \det(M)^2 = \det(Q)^{2m} \cdot \det(Q')^{2n} = d^m d'^n. \quad \square$$

Remark 2.2.15 (and Definition). Suppose $K = \mathbb{Q}$, $A = \mathbb{Z}$, L a number field and $B = \mathcal{O}_k$.

(i) There is always an integral basis w_1, \dots, w_n .

(ii) The **discriminant** $d_k = d_k(\mathcal{O}_k) = d(w_1, \dots, w_n)$ does not depend on the choice of integral basis.

Proof. “(i)” Proposition 2.12 “(ii)” Let w'_1, \dots, w'_n be another integral basis. Then there exists a base change matrix $T \in \text{GL}_n(\mathbb{Z})$ with

$$\begin{pmatrix} w'_1 \\ \vdots \\ w'_n \end{pmatrix} = T \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}.$$

Hence

$$\begin{pmatrix} \sigma(w'_1) \\ \vdots \\ \sigma(w'_n) \end{pmatrix} = T \begin{pmatrix} \sigma(w_1) \\ \vdots \\ \sigma(w_n) \end{pmatrix}.$$

such that

$$d(w'_1, \dots, w'_n) = \underbrace{\det T}_{\in \{1, -1\}}^2 d(w_1, \dots, w_n) = d_k.$$

□

Example 2.2.16. Let $L = \mathbb{Q}(\sqrt{D})$ with $D \in \mathbb{Z}$ square-free. By Proposition 2.14 we have:

- (i) $\mathcal{O}_k = \mathbb{Z}[\sqrt{D}]$ and $\{1, \sqrt{D}\}$ is an integral basis for $D \equiv 2, 3 \pmod{4}$ and $d_k = 4D$.
- (ii) $\mathcal{O}_k = \mathbb{Z}\left[\frac{1+\sqrt{D}}{2}\right]$ and $\left\{1, \frac{1+\sqrt{D}}{2}\right\}$ is an integral basis for $D \equiv 1 \pmod{4}$ and $d_k = D$.

In particular, this holds for $D = -1$, i.e., the Gaussian integers $\mathbb{Z}[i]$.

2.3. Ideals

Let R be a commutative ring with 1.

Problem: \mathcal{O}_k is not a UFD in many cases, e.g. in $\mathbb{Z}[\sqrt{-5}]$ we have

$$(1 + \sqrt{-5})(1 - \sqrt{-5}) = 1 + 5 = 6 = 2 \cdot 3,$$

that is, two different ways to factor 6 in irreducible elements.

Idea:

- (1) Maybe we have too few elements, i.e.,

$$1 + \sqrt{-5} = p_1 p_2, 1 - \sqrt{-5} = p_3 p_4 \text{ and } 2 = p_2 p_3, 3 = p_1 p_4$$

for some primes p_i .

- (2) An element is determined (up to units) by the set of elements it divides, e.g.

$$p_i \longleftrightarrow \{x \in \mathcal{O}_k; p_i | x\} = p_i \mathcal{O}_k \text{ (this is an ideal).}$$

Notation 2.3.1. Let $I, J \subset R$ be ideals. We define

- $I + J = \{a + b; a \in I, b \in J\}$,
- $IJ = \{\sum_i a_i b_i; a_i \in I, b_i \in J\}$.

Definition 2.3.2 (and Reminder). Let $I \subsetneq R$ be an ideal.

- (a) I is called **prime** if for all $a, b \in R$ with $ab \in I$ we already have $a \in I$ or $b \in I$.
 \Leftrightarrow For all ideals $A, B \subset R$ with $AB \subset I$ we have $A \subset I$ or $B \subset I$.
- (b) I is called **maximal** if for any ideal $I \subset J \subset R$ we have $J = I$ or $J = R$.
 $\Leftrightarrow R/I$ is a field.
- (c) R is called **Noetherian** if every ascending chain of ideals

$$I_1 \subset I_2 \subset \dots$$

becomes stationary, i.e., if there is an $N \in \mathbb{N}$ such that $I_n = I_N$ for all $n \geq N$.

\Leftrightarrow Every ideal in R is finitely generated.

- (d) R is called a **Dedekind domain** if
- R is an integral domain,
 - R is integrally closed,
 - R is Noetherian, and
 - every prime ideal in R is maximal.

Proposition 2.3.3. *If \mathcal{O} is the integral closure of \mathbb{Z} in a number field then \mathcal{O} is a Dedekind domain.*

Proof. It is clear that \mathcal{O} is an integral domain and integrally closed. Furthermore, by Proposition 2.12 each \mathbb{Z} -submodule is finitely generated as a \mathbb{Z} -module, thus also as an \mathcal{O} -module. Hence \mathcal{O} is Noetherian.

Now, let $I \subset \mathcal{O}$ be a prime ideal. Then $I \cap \mathbb{Z} \subset \mathbb{Z}$ is a prime ideal such that $\mathbb{Z}/(I \cap \mathbb{Z}) = \mathbb{F}_p$. Using $\mathcal{O} = \mathbb{Z}[w_1, \dots, w_n]$ we conclude

$$\mathcal{O}/I = \mathbb{Z}/(I \cap \mathbb{Z})[w'_1, \dots, w'_n] = \mathbb{F}_p[w'_1, \dots, w'_n] = \mathbb{F}_p(w'_1, \dots, w'_n),$$

where $w'_i \equiv w_i \pmod{I}$. Thus \mathcal{O}/I is a field and hence I maximal. □

From now on: Let \mathcal{O} denote a Dedekind domain.

Theorem 4. *Every ideal $0 \neq I \subset \mathcal{O}$ has a unique factorization*

$$I = P_1 \cdots P_n$$

into prime ideals $P_i \subset \mathcal{O}$.

Lemma 2.3.4. *For every ideal $0 \neq I \subset \mathcal{O}$ there exist nonzero prime ideals $P_i \subset \mathcal{O}$ such that*

$$P_1 \cdots P_n \subset I.$$

Proof. Set $M = \{0 \neq I \subset \mathcal{O} \text{ ideal; } I \text{ does not have such } P_i\}$ and suppose $M \neq \emptyset$. Then M is partially ordered by inclusion and since \mathcal{O} is Noetherian, every chain in M has an upper bound. Thus, the Lemma of Zorn yields a maximal element $I_0 \in M$. Since I_0 cannot be prime there are $a, b \in \mathcal{O}$ such that $ab \in I_0$ but $a, b \notin I_0$. Consider the ideals $I_1 = (a) + I_0$ and $I_2 = (b) + I_0$ which satisfy $I_0 \subsetneq I_1$, $I_0 \subsetneq I_2$ and $I_1 I_2 \subset I_0$. Since I_0 is a maximal ideal in M , we have $I_{1,2} \notin M$ hence we find prime ideals $P_1, \dots, P_n, P'_1, \dots, P'_m \subset \mathcal{O}$ with

$$P_1 \dots P_n \subset I_1 \text{ and } P'_1 \dots P'_m \subset I_2.$$

Finally, we conclude $P_1 \dots P_n P'_1 \dots P'_m = I_1 I_2 \subset I_0 \Rightarrow I_0 \notin M \nRightarrow M = \emptyset$. \square

Lemma 2.3.5. Let $0 \neq P \subset \mathcal{O}$ be a prime ideal, $I \subset \mathcal{O}$ an ideal and $K = \text{Quot}(\mathcal{O})$. Then:

$$(i) \ P^{-1} := \{x \in K; xP \subset \mathcal{O}\} \supsetneq \mathcal{O}$$

$$(ii) \ I \subsetneq P^{-1}I := \{\sum_i a_i x_i; a_i \in I, x_i \in P^{-1}\}$$

Proof. “(i)” Let $0 \neq a \in P$, $P_1 \dots P_n \subset (a) \subset P$ as in Lemma 3.5 with n minimal.

Claim: Without loss of generality we can assume that $P_1 = P$.

Proof of the claim: Since $P_1 \dots P_n \subset P$ and P is prime, there is an index i such that $P_i \subset P$, by reindexing we may assume that $i = 1$. However, we assumed \mathcal{O} to be Dedekind, hence P_1 is a maximal ideal in \mathcal{O} . Thus, $P_1 \subset P \subsetneq \mathcal{O}$ implies that $P_1 = P$ as claimed.

Now, since n was chosen minimal we have $P_2 \dots P_n \not\subset (a)$, i.e., there exists an element $b \in (a) \setminus P_2 \dots P_n$. On the one hand we thus have

$$a^{-1}b \notin \mathcal{O}$$

and on the other hand $bP \subset (a)$ such that $a^{-1}bP \subset \mathcal{O}$ and hence

$$a^{-1}b \in P^{-1}.$$

Both of this together shows that $P^{-1} \supsetneq \mathcal{O}$.

“(ii)” Assume there is an ideal $I \subset \mathcal{O}$ such that $P^{-1}I \subset I$. Let $\{\alpha_1, \dots, \alpha_n\} \subset I$ be a generating set and choose $x \in P^{-1} \setminus \mathcal{O}$. Then,

$$x\alpha_i = \sum_j a_{ij}\alpha_j$$

for some $a_{ij} \in \mathcal{O}$. Consider the matrix $A = xE_n - (a_{ij})_{i,j}$, which satisfies

$$A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = 0.$$

Since $A^\# A = \det A$ we conclude $\det A = 0$ such that x is a zero of the monic polynomial $\det(XE_n - (a_{ij})_{i,j})$ over \mathcal{O} . But since \mathcal{O} is integrally closed this implies $x \in \mathcal{O}$, a contradiction. \square

Proof of Theorem 3.4. Existence of a factorization: Let

$$M = \{0 \neq I \subset \mathcal{O} \text{ ideal}; I \text{ has no factorization}\}$$

and assume that $M \neq \emptyset$. As in Lemma 3.5, let $I_0 \in M$ be a maximal element and let $P \supset I_0$ be a maximal ideal containing I_0 . Since I_0 is not prime we have $I_0 \neq P$ such that by Lemma 3.6,

$$I_0 \subsetneq P^{-1}I_0 \subset P^{-1}P = \mathcal{O}.$$

Note that $I_0 = I_0\mathcal{O} = I_0P^{-1}P$ and $I_0 \neq P$ imply $P^{-1}I_0 \subsetneq \mathcal{O}$. Since I_0 was maximal in M we thus have $P^{-1}I_0 \notin M$, i.e., there are prime ideals $P_1, \dots, P_n \subset \mathcal{O}$ with $P^{-1}I = P_1 \cdots P_n$. This leads to the contradiction $I = PP_1 \cdots P_n$.

Uniqueness of the factorization: Suppose that

$$I = P_1 \cdots P_n = Q_1 \cdots Q_m$$

are two prime factorizations. Then $P_1 \supset I = Q_1 \cdots Q_m$, hence without loss of generality we can assume that $Q_1 \subset P_1$. Since \mathcal{O} is Dedekind we conclude $Q_1 = P_1$ such that

$$P_2 \cdots P_n = P_1^{-1}I = Q_2 \cdots Q_m.$$

The claim follows by induction. □

Definition 2.3.6. We call two ideals $0 \neq I, J \subset \mathcal{O}$ **coprime** $:\Leftrightarrow I + J = \mathcal{O}$. For example, one could take two distinct prime ideals in a Dedekind ring.

Remark 2.3.7. Let $P_1, \dots, P_n \subset \mathcal{O}$ be pairwise coprime. Then P_1 and $P_2 \cdots P_n$ are coprime and we have $\prod_{i=1}^n P_i = \bigcap_{i=1}^n P_i$.

Proof. Induction on n : The case $n = 2$ is clear. Let $n > 2$. Since P_1 and P_2 are coprime, $\exists p_1 \in P_1, p_2 \in P_2$, such that we can write $1 = p_1 + p_2$. By induction hypothesis, $\exists p'_1 \in P_1, p \in P_3 \cdots P_n$, such that $1 = p'_1 + p$. It follows

$$1 = p_1 + p_2 \cdot (p'_1 + p) = \underbrace{p_1 + p_2 p'_1}_{\in P_1} + \underbrace{p_2 p}_{\in P_2 \cdots P_n},$$

which yields the first claim.

For the second claim, first note that $\prod P_i \subset \bigcap P_i$ is clear.

For the converse, let $a \in \bigcap P_i$, which of course implies that $a \in P_i$ for all i . As above, we write $1 = p_1 + p$, $p_1 \in P_1, p \in P_2 \cdots P_n$. We get $a = ap_1 + ap$, which implies that $a \in aP_1 + P_1 \cdot \prod_{i=2}^n P_i$ for all i and by induction hypothesis, we get $a \in \prod P_i$. □

Theorem 5 (Chinese Remainder Theorem). *Let $P_1, \dots, P_n \subset \mathcal{O}$ be pairwise coprime ideals, $I = \bigcap_{i=1}^n P_i$. Then we have*

$$\mathcal{O}/I \cong \bigoplus_{i=1}^n \mathcal{O}/P_i$$

Proof. Consider the map

$$\phi : \mathcal{O} \longrightarrow \bigoplus_i \mathcal{O}/P_i, \quad a \mapsto \bigoplus_i a \pmod{P_i}.$$

Obviously, $\ker(\phi) = I$. It remains to show, that ϕ is surjective. Let first $n = 2$: For $p_1 \in P_1, p_2 \in P_2$ let $1 = p_1 + p_2$ and for any $a_1, a_2 \in \mathcal{O}$ write $a = a_2 p_1 + a_1 p_2$. Then $\phi(a) = a_1 \oplus a_2 \in \mathcal{O}/P_1 \oplus \mathcal{O}/P_2$.

In general, by 3.8, we know that $\exists y_i \in \mathcal{O}$ with $y_i \equiv 1 \pmod{P_i}$ and $y_i \equiv 0 \pmod{\bigcap_{j \neq i} P_j}$. Hence the element $a = \sum_{i=1}^n a_i y_i$ is mapped to $\bigoplus_{i=1}^n a_i \pmod{P_i}$ \square

Definition 2.3.8. A **fractional ideal** of K is a finitely generated \mathcal{O} -module $0 \neq I$ of K . Since \mathcal{O} is noetherian, this is equivalent to: $\exists c \in \mathcal{O}$, such that $c \cdot I \subset \mathcal{O}$ is an ideal (since every submodule of \mathcal{O} is finitely generated). The product of two fractional ideals is denoted in the same way as introduced in 3.3. Ideals in \mathcal{O} are called **integral ideals**.

Theorem 6. *The fractional ideals of K , together with the product, form an abelian group, which we denote by \mathcal{J}_K .*

Proof. Commutativity and associativity are clear. The unit in \mathcal{J}_K is given by \mathcal{O} . We define $I^{-1} := \{x \in K \mid x \cdot I \subset \mathcal{O}\}$ and show, that this defines an inverse for all $I \in \mathcal{J}_K$.

For a prime ideal $P \subset \mathcal{O}$, we have already seen in 3.4 that $P^{-1}P = \mathcal{O}$ and for an integral ideal $I = P_1 \cdots P_n$, we have $J = P_1^{-1} \cdots P_n^{-1}$ as an inverse:

$J \subset I^{-1}$ is clear. For the converse, let $x \in I^{-1}$, we then have $x \cdot IJ \subset \mathcal{O}$, with $x \cdot I \subset \mathcal{O}$ and $IJ = \mathcal{O}$, therefore $x \cdot 1 \in J$ and $I^{-1} \subset J$ follows.

Let now I be fractional. Then $\exists c \in \mathcal{O}$, such that cI is integral. But then $(cI)^{-1} = c^{-1}I^{-1}$ and hence $II^{-1} = (cI)(c^{-1}I^{-1}) = \mathcal{O}$ \square

Corollary 2.3.9. *Every fractional ideal I has a unique factorization $I = \prod P_i^{n_i}$, with $n_i \in \mathbb{Z}$, $P_i \subset \mathcal{O}$ distinct prime ideals and only finitely many $n_i \neq 0$. In particular, \mathcal{J}_K is a free abelian group on the prime ideals of \mathcal{O} .*

Proof. By 3.11, every element $I \in \mathcal{J}_K$ can be written as $I = AB^{-1}$ for some integral ideals $A, B \subset \mathcal{O}$. Therefore, by 3.4, we get $I = \prod P_i^{n_i}$ and by multiplying denominators, we see that this presentation is unique. \square

Definition 2.3.10. The principle ideals generate a subgroup \mathcal{P}_K of \mathcal{J}_K . We call the quotient group $\text{Cl}_K := \mathcal{J}_K/\mathcal{P}_K$ the **ideal class group**. We have an exact sequence of groups

$$1 \longrightarrow \mathcal{O}^\times \longrightarrow K^\times \xrightarrow{a \mapsto a\mathcal{O}} \mathcal{J}_K \longrightarrow \text{Cl}_K \longrightarrow 1.$$

2.4. Lattices and Minkowski

Definition 2.4.1. Let V be an n -dimensional \mathbb{R} -vector space. A **lattice** $\Lambda \subset V$ is a subgroup of the form $\mathbb{Z}v_1 + \dots + \mathbb{Z}v_m$, where v_1, \dots, v_m are linearly independent over V . We call (v_1, \dots, v_m) a **basis** of Λ and $\phi := \{x_1v_1 + \dots + x_mv_m \mid x_i \in [0, 1)\}$ a **fundamental domain** of Λ . We call Λ **complete**, if $n = m$.

CAUTION: For many people, lattices are always complete!

Example 2.4.2. (a) $\mathbb{Z} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbb{Z} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \subset \mathbb{R}^2$ is a complete lattice

(b) $\mathbb{Z} + \mathbb{Z}\sqrt{2} \subset \mathbb{R}$ is not a lattice, since 1 and $\sqrt{2}$ are not linearly independent.

(c) $\mathbb{Z} \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \subset \mathbb{R}^2$ is a non-complete lattice.

Proposition 2.4.3. A subgroup $\Lambda \subset V$ is a lattice $\Leftrightarrow \Lambda$ is a discrete subgroup of V .

Proof. " \Rightarrow ": Take $\{\lambda + x_1v_1 + \dots + x_nv_n + \text{rest of basis} \mid |x_n| < 1\}$ as a neighbourhood for $\lambda \in \Lambda$.

" \Leftarrow ": Let $V_0 = \langle \Lambda \rangle_{\mathbb{R}}$. Then we can choose a basis v_1, \dots, v_m of V_0 in Λ , such that $\Lambda_0 := \mathbb{Z}v_1 + \dots + \mathbb{Z}v_m$ is a lattice in V_0 .

Claim: The index $[\Lambda : \Lambda_0]$ is finite.

Proof of the claim: Since Λ_0 is complete, $V = \bigsqcup_{\lambda \in \Lambda_0} \phi_0 + \lambda$. Since Λ is discrete and ϕ_0 bounded, $\Lambda \cap \phi_0$ is finite. Hence we have only finitely many residue classes $\lambda + \Lambda_0$ of Λ and therefore $[\Lambda : \Lambda_0] =: d < \infty$.

From this follows, that $\Lambda \subset \frac{1}{d}\Lambda_0 = \mathbb{Z}(\frac{1}{d}v_1) + \dots + \mathbb{Z}(\frac{1}{d}v_m)$. Therefore, Λ has a \mathbb{Z} -basis $w_1 = v_1n_1, \dots, w_r = v_rn_r$ for some $n_i \in \frac{1}{d}\mathbb{N}$ and since Λ spans V_0 , we get $r = m$ and they are linearly independent. \square

Let $\Gamma = v_1\mathbb{Z} + \dots + v_n\mathbb{Z} \subset \mathbb{R}^n$ be a complete lattice. We define

$$\text{vol } \Gamma = \text{vol } \phi = |\det(v_1, \dots, v_n)|.$$

Note that this definition is independent of the chosen basis since for a transformation

$$A(v_1, \dots, v_n) = (v'_1, \dots, v'_n)$$

between two bases we have $\det A = \pm 1$.

Theorem 7 (Minkowski). Let $X \subset \mathbb{R}^n$ be a convex, symmetric central (i.e., $x \in X$ implies $-x \in X$) subset and let $\Gamma \subset \mathbb{R}^n$ be a complete lattice. If

$$\text{vol } X > 2^n \text{vol } \Gamma$$

then there exists some $\gamma \in \Gamma \setminus \{0\}$ such that $\gamma \in X$.

Proof. Claim: It suffices to show that there are $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$, such that

$$\left(\frac{1}{2}X + \gamma_1\right) \cap \left(\frac{1}{2}X + \gamma_2\right) \neq \emptyset.$$

Proof of claim: Let $x = \frac{1}{2}x_1 + \gamma_1 = \frac{1}{2}x_2 + \gamma_2$ with some $x_1, x_2 \in X$. Then

$$y = \frac{1}{2}(x_1 - x_2) = \gamma_2 - \gamma_1 \in \Gamma \setminus \{0\}$$

with $y \in X$ since X is symmetrical central.

Now let us assume that the family $\left(\frac{1}{2}X + \gamma\right)_{\gamma \in \Gamma}$ is pairwise disjoint. Then

$$\left(\left[\frac{1}{2}X + \gamma\right] \cap \phi\right)_{\gamma \in \Gamma}$$

also consists of pairwise disjoint sets such that we obtain the contradiction

$$\begin{aligned} \text{vol } \Gamma = \text{vol } \phi &\geq \sum_{\gamma \in \Gamma} \text{vol} \left(\left[\frac{1}{2}X + \gamma\right] \cap \phi \right) = \sum_{\gamma \in \Gamma} \text{vol} \left(\frac{1}{2}X \cap [\phi - \gamma] \right) \\ &= \text{vol} \left(\frac{1}{2}X \right) = \frac{1}{2^n} \text{vol } X. \end{aligned}$$

□

2.5. Minkowski theory

Let $[K : \mathbb{Q}] = n$ be a field extension, $\tau_i: K \hookrightarrow \mathbb{C}$ different embeddings and consider the embedding

$$j: K \hookrightarrow K_{\mathbb{C}} = \prod_{\tau_i} \mathbb{C}, \quad a \mapsto (\tau_1(a), \dots, \tau_n(a)).$$

Define a hermitian scalar product on $K_{\mathbb{C}}$ by

$$\langle (x_{\tau_i}), (y_{\tau_i}) \rangle = \sum_{\tau_i} x_{\tau_i} \overline{y_{\tau_i}}$$

and consider the complex conjugation $F \in \text{Gal}(\mathbb{C}/\mathbb{R})$ given by $\mathbb{C} \rightarrow \mathbb{C}, z \mapsto \bar{z}$. Let

$$F(\tau) = \bar{\tau}: a \mapsto \overline{\tau(a)}$$

and extend it to $K_{\mathbb{C}}$ by

$$F: K_{\mathbb{C}} \rightarrow K_{\mathbb{C}}, (x_{\tau}) \mapsto (\bar{x}_{\bar{\tau}}).$$

Example. Let $D > 0$ be square-free. Consider

$$\mathbb{Q}(\sqrt{D}) \hookrightarrow \mathbb{Q}(\sqrt{D})_{\mathbb{C}} = \prod_{\tau_i} \mathbb{C}$$

with

$$\tau_1(a + b\sqrt{D}) = a + b\sqrt{D} \quad \text{and} \quad \tau_2(a + b\sqrt{D}) = a - b\sqrt{D}.$$

Then

$$j(a + b\sqrt{D}) = (a + b\sqrt{D}, a - b\sqrt{D})$$

and $F(\tau_1) = \tau_1, F(\tau_2) = \tau_2$ such that

$$F(x_{\tau_1}, x_{\tau_1}) = (\bar{x}_{\tau_1}, \bar{x}_{\tau_2}).$$

Remark. • $F(\langle x, y \rangle) = \langle F(x), F(y) \rangle$

• $\text{Tr}: K_{\mathbb{C}} \rightarrow \mathbb{C}, (x_{\tau}) \mapsto \sum_{\tau} x_{\tau}$ such that $(\text{Tr} \circ j)(a) = \text{Tr}_{K/\mathbb{Q}}(a)$

Now define the F -invariant \mathbb{R} -vector space

$$K_{\mathbb{R}} = K_{\mathbb{C}}^+ = \{x \in K_{\mathbb{C}} \mid F(x) = x\} = \{x \in K_{\mathbb{C}} \mid x_{\bar{\tau}} = \overline{x_{\tau}} \text{ for all } \tau\}.$$

Since $\bar{\tau}(a) = \overline{\tau(a)}$ for all $a \in K$ and all τ , we have $j(K) \subset K_{\mathbb{R}}$. We call $K_{\mathbb{R}}$ the **Minkowski space** and $\langle \cdot, \cdot \rangle|_{K_{\mathbb{R}}}$ the **canonical metric**.

Remark. Note that $j: K \rightarrow K_{\mathbb{R}} \cong K \otimes_{\mathbb{Q}} \mathbb{R}$, where the isomorphism is given by $a \otimes x \mapsto j(a)x$ for $x \in \mathbb{R}$.

Explicit description of $K_{\mathbb{R}}$: Let $n = r + 2s$, where r and s are the number of embeddings

$$\varphi_1, \dots, \varphi_r: K \hookrightarrow \mathbb{R}$$

and

$$\sigma_1, \overline{\sigma_1}, \dots, \sigma_s, \overline{\sigma_s}: K \hookrightarrow \mathbb{C},$$

respectively. Notice that $F(\varphi_i) = \varphi_i$ and $F(\sigma_j) = \overline{\sigma_j}$. Then elements of $K_{\mathbb{C}}$ are of the form

$$x = (x_{\varphi(1)}, \dots, x_{\varphi(r)}, x_{\sigma_1}, x_{\overline{\sigma_1}}, \dots, x_{\sigma_s}, x_{\overline{\sigma_s}})$$

with

$$F(x) = (\overline{x_{\varphi_1}}, \dots, \overline{x_{\varphi_r}}, \overline{x_{\sigma_1}}, \overline{x_{\sigma_1}}, \dots, \overline{x_{\sigma_s}}, \overline{x_{\sigma_s}}).$$

Hence we have

$$K_{\mathbb{R}} = \{x \in K_{\mathbb{C}} \mid x_{\varphi_i} \in \mathbb{R}, x_{\overline{\sigma_j}} = \overline{x_{\sigma_j}}\}.$$

Proposition 2.5.1. *The map*

$$f: K_{\mathbb{R}} \xrightarrow{\cong} \mathbb{R}^{r+2s} = \prod_{\tau} \mathbb{R},$$

$$x \mapsto (x_{\varphi_1}, \dots, x_{\varphi_r}, \operatorname{Re} x_{\sigma_1}, \operatorname{Im} x_{\sigma_1}, \dots, \operatorname{Re} x_{\sigma_s}, \operatorname{Im} x_{\sigma_s}).$$

is an isomorphism. It transforms the canonical metric into the scalar product

$$\langle x, y \rangle = \sum_{\tau} \alpha_{\tau} x_{\tau} y_{\tau},$$

where

$$\alpha_{\tau} = \begin{cases} 1, & \tau = \varphi_i \text{ for some } i, \\ 2, & \tau = \sigma_j \text{ for some } j. \end{cases}$$

Proof. Obviously, f is an isomorphism. For $x = (x_{\tau}), y = (y_{\tau}) \in K_{\mathbb{R}}$ we have

$$\begin{aligned} \langle x, y \rangle|_{K_{\mathbb{R}}} &= \sum_{\tau} x_{\tau} \overline{y_{\tau}} \\ &= \sum_{\varphi_i} x_{\varphi_i} y_{\varphi_i} + \sum_{\sigma_j} x_{\sigma_j} \overline{y_{\sigma_j}} + \sum_{\overline{\sigma_j}} \overline{(x_{\sigma_j} y_{\sigma_j})} \\ &= \dots = (f(x), f(y)). \end{aligned}$$

□

Remark. • The canonical metric induces a volume $\operatorname{vol}_{\text{can}}$ on $K_{\mathbb{R}}$ and thus on \mathbb{R}^{r+2s} .

- If we denote the Lebesgue measure on \mathbb{R}^{r+2s} by $\operatorname{vol}_{\text{Leb}}$ then, for $X \subset K_{\mathbb{R}}$,

$$2^s \operatorname{vol}_{\text{Leb}} f(X) = \operatorname{vol}_{\text{can}} X.$$

- We will thus consider $K \supset U \xrightarrow{j} j(U) \xrightarrow{f} \mathbb{R}^{r+2s}$.

Example. Let $e_j = (0, \dots, 1, \dots, 0)$. Note that we have $\langle e_{\varphi_i}, e_{\varphi_i} \rangle = 1$ and $\langle e_{\sigma_j}, e_{\varphi_j} \rangle = 2$, such that $\langle \frac{e_{\sigma_j}}{\sqrt{2}}, \frac{e_{\sigma_j}}{\sqrt{2}} \rangle = 1$. Hence

$$\left\{ e_{\varphi_1}, \dots, e_{\varphi_r}, \frac{e_{\sigma_1}}{\sqrt{2}}, \frac{e_{\overline{\sigma_1}}}{\sqrt{2}}, \dots \right\}$$

is an orthonormal basis. Using the correspondence

$$X \subset K_{\mathbb{R}} \leftrightarrow f(X) \subset \mathbb{R}^{r+2s}$$

we thus define

$$\operatorname{vol}_{\text{can}} X = \operatorname{vol}_{\text{can}} f(X) = 2^s \operatorname{vol}_{\text{Leb}} f(X).$$

Proposition 2.5.2. *If $I \neq 0$ is an \mathcal{O}_k -ideal then $\Gamma = j(I)$ is a complete lattice in $K_{\mathbb{R}}$. Its fundamental domain has volume*

$$\text{vol } \Gamma = \text{vol } \phi = \sqrt{|d_k|} \cdot [\mathcal{O}_k : I].$$

Proof. Choose α_i such that $I = \alpha_1\mathbb{Z} + \cdots + \alpha_n\mathbb{Z}$. Then $\Gamma = j(I) = j(\alpha_1)\mathbb{Z} + \cdots + j(\alpha_n)\mathbb{Z}$. Define

$$A = (j(\alpha_1), \dots, j(\alpha_n))^T = \begin{pmatrix} \tau_1(\alpha_1) & \cdots & \tau_n(\alpha_1) \\ \vdots & \ddots & \vdots \\ \tau_1(\alpha_n) & \cdots & \tau_n(\alpha_n) \end{pmatrix}$$

such that

$$\text{vol } \phi = |\det A| = \sqrt{|d_k|} \cdot [\mathcal{O}_k : I].$$

Furthermore,

$$d(I) = d(\alpha_1, \dots, \alpha_n) = |\det A|^2 = d(\mathcal{O}_k) \cdot [\mathcal{O}_k : I]^2,$$

with $[\mathcal{O}_k : I] = |\det M|$ for the change of basis M from \mathcal{O}_k to I . \square

Theorem 8. *Let $I \neq 0$ be an ideal in \mathcal{O}_k . Let $(c_\tau)_\tau$ be a collection of real number such that $c_\tau > 0$, $c_\tau = c_{\bar{\tau}}$ and*

$$\prod_{\tau} c_\tau > \left(\frac{2}{\pi}\right)^s \sqrt{|d_k|} \cdot [\mathcal{O}_k : a].$$

Then there exists $a \in I \setminus \{0\}$ such that

$$|\tau(a)| < c_\tau$$

for all $\tau \in \text{Hom}(K, \mathbb{C})$.

Proof. Consider the convex, central symmetric set

$$X = \{(x_\tau) \in K_{\mathbb{R}} \mid |x_\tau| < c_\tau \text{ for all } \tau\}$$

and let $f: K_{\mathbb{R}} \rightarrow \mathbb{R}^n$, $n = r + 2s$, as in Proposition 5.1. Notice that for $x \in X$ we have $f(x) = (x_{\varphi_1}, \dots, x_{\varphi_r}, a_1, b_1, \dots, a_s, b_s)$ with $|x_{\varphi_i}| < c_{\varphi_i}$ and $a_j^2 + b_j^2 < c_{\sigma_j}^2$. Hence

$$\text{vol}_{\text{can}} X = 2^s \text{vol}_{\text{Leb}} f(X) = 2^s \left(\prod_{i=1}^r 2c_{\varphi_i} \right) \left(\prod_{j=1}^s \pi c_{\sigma_j}^2 \right) = 2^{r+s} \pi^s \prod_{\tau} c_\tau,$$

and thus, by Proposition 5.2,

$$\begin{aligned} 2^n \text{vol } \Gamma &= 2^{r+2s} \sqrt{|d_k|} \cdot [\mathcal{O}_k : I] \\ &= 2^{r+s} \pi^s \left[\left(\frac{2}{\pi}\right)^s \sqrt{|d_k|} \cdot [\mathcal{O}_k : a] \right] \\ &< 2^{r+s} \pi^s \prod_{\tau} c_\tau \\ &= \text{vol}_{\text{can}} X. \end{aligned}$$

Consequently, by Minkowski's theorem, there exists $j(a) \in \Gamma \setminus \{0\}$ with $j(a) \in X$ and $|\tau(a)| < c_\tau$ for all τ . \square

Multiplicative Minkowsky theory

Define

$$j: K^* \hookrightarrow K_{\mathbb{C}}^* = \prod_{\tau} \mathbb{C}^*, a \mapsto (\tau(a))_{\tau}$$

and

$$\mathcal{N}: K_{\mathbb{C}}^* \rightarrow \mathbb{C}^*, (x_{\tau}) \mapsto \prod_{\tau} x_{\tau}.$$

Denote the composition of these maps by $\mathcal{N}_{K/\mathbb{Q}} = \mathcal{N} \circ j$. Furthermore, consider

$$l: \mathbb{C}^* \rightarrow \mathbb{R}, z \mapsto \log |z|$$

and its extension

$$l: K_{\mathbb{C}}^* \rightarrow \prod_{\tau} \mathbb{R}, (x_{\tau}) \mapsto (\log |x_{\tau_1}|, \dots, \log |x_{\tau_n}|).$$

All in all, we have

$$\begin{array}{ccccc} K^* & \xrightarrow{j} & K_{\mathbb{C}}^* & \xrightarrow{l} & \prod_{\tau} \mathbb{R} \\ \mathcal{N}_{K/\mathbb{Q}} \downarrow & & \downarrow \mathcal{N} & & \downarrow \text{Tr} \\ \mathbb{Q}^* & \xrightarrow{i} & \mathbb{C}^* & \xrightarrow{l} & \mathbb{R} \end{array}$$

with

$$\left[\prod_{\tau} \mathbb{R} \right]^+ = \prod_{\varphi_i} \mathbb{R} \times \prod_{\sigma_j} [\mathbb{R} \times \mathbb{R}]^+ \xrightarrow{\cong} \mathbb{R}^{r+s},$$

where the isomorphism is given by

$$(x_{\varphi_1}, \dots, x_{\varphi_r}, x_{\sigma_1}, x_{\overline{\sigma_1}}, \dots, x_{\sigma_s}, x_{\overline{\sigma_s}}) \mapsto (x_{\varphi_1}, \dots, x_{\varphi_r}, 2x_{\sigma_1}, \dots, 2x_{\sigma_s}),$$

and we have

$$K_{\mathbb{R}} \rightarrow \mathbb{R}^{r+s}, (x_{\tau}) \mapsto (\log |x_{\varphi_1}|, \dots, \log |x_{\varphi_r}|, \log |x_{\sigma_1}|^2, \dots, \log |x_{\sigma_s}|^2).$$

2.6. The class number

Let $n = [K : \mathbb{Q}]$, denote by J_K the group of fractional ideals of K , by P_k its subgroup of principal ideals and by $\text{Cl}_k = J_k/P_k$ the ideal class group. Define the **absolute norm** of an ideal $I \subset \mathcal{O}_k$ by

$$n(I) = [\mathcal{O}_k : I].$$

For $I = (\alpha)$, we have $n(I) = \mathcal{N}_{K/\mathbb{Q}}(\alpha)$. If $\mathcal{O}_k = w_1\mathbb{Z} + \dots + w_n\mathbb{Z}$ and $I = \alpha w_1\mathbb{Z} + \dots + \alpha w_n\mathbb{Z}$ we have

$$\alpha w_i = \sum_j a_{ij} w_j$$

for some matrix $A = (a_{ij})$ such that $\mathcal{N}_{K/\mathbb{Q}}(\alpha) = |\det A| = [\mathcal{O}_k : I]$.

Proposition 2.6.1. If $I = P_1^{\nu_1} \cdots P_r^{\nu_r}$ then $n(I) = n(P_1)^{\nu_1} \cdots n(P_r)^{\nu_r}$.

Proof. By the Chinese remainder theorem,

$$\mathcal{O}_k/I \cong (\mathcal{O}_k/P_1^{\nu_1}) \oplus \cdots \oplus (\mathcal{O}_k/P_r^{\nu_r}),$$

such that

$$n(I) = [\mathcal{O}_k : I] = \prod_j [\mathcal{O}_k : P_j^{\nu_j}] = \prod_j n(P_j)^{\nu_j}.$$

Claim: $P \supsetneq P^2 \supsetneq \cdots \supsetneq P^\nu$ and P^i/P^{i+1} is a (\mathcal{O}_k/P) -vector space of dimension 1

Proof of Claim: Let $a \in P^i/P^{i+1}$. Then we have

$$P^i \supset J = (a) + P^{i+1} \supsetneq P^{i+1}$$

and

$$\mathcal{O}_k \supset J' = JP^{-i} \supsetneq P = P^{i+1}P^{-i}.$$

Since $J'|P$ we have $J = P^i$ and thus $[a] \in P^i/P^{i+1}$ is a basis.

Now, the Claim yields

$$n(P^\nu) = [\mathcal{O}_k : P^\nu] = [\mathcal{O}_k : P] [P : P^2] \cdots [P^{\nu-1} : P^\nu] n(P)^\nu.$$

□

In particular, for integral ideals I, J we have $n(IJ) = n(I)n(J)$ such that we can extend n to J_k by

$$n: J_k \rightarrow \mathbb{R}_+^*, I = P_1^{\nu_1} \cdots P_r^{\nu_r} \mapsto n(I) = n(P_1)^{\nu_1} \cdots n(P_r)^{\nu_r}.$$

Reminder 2.6.2. \mathcal{J}_K = group of fractional ideals = abelian group generated by all prime ideals

\mathcal{P}_K = group of all principal fractional ideals.

$\text{Cl}_K := \mathcal{J}_K/\mathcal{P}_K$

\Rightarrow obtain following exact sequence:

$$1 \rightarrow \underbrace{\mathcal{O}_K^\times}_{\text{How big?}} \rightarrow K^\times \rightarrow \mathcal{J}_K \rightarrow \underbrace{\text{Cl}_K}_{\text{How big?}} \rightarrow 1$$

$$a \mapsto (a) = a\mathcal{O}_K$$

Last Time: \mathfrak{a} ideal in $\mathcal{O}_K, \mathfrak{a} \neq 0$.

- $\mathcal{N}(\mathfrak{a}) = (\mathcal{O}_K : \mathfrak{a})$ absolute norm.

In particular: $\mathcal{N}((a)) := |\mathcal{N}_{K/\mathbb{Q}}(a)|$.

- $\mathfrak{a} = \mathcal{P}_1^{\nu_1} \cdots \mathcal{P}_r^{\nu_r}$ decomposition into primes
 $\Rightarrow \mathcal{N}(\mathfrak{a}) = \mathcal{N}(\mathcal{P}_1)^{\nu_1} \cdots \mathcal{N}(\mathcal{P}_r)^{\nu_r}$

In particular: $\mathcal{N}(\mathfrak{a}_1\mathfrak{a}_2) = \mathcal{N}(\mathfrak{a}_1)\mathcal{N}(\mathfrak{a}_2)$.

- Hence \mathcal{N} can be extended to fractional ideals: $\mathcal{N} : \mathcal{J}_K \rightarrow \mathbb{R}_+^\times$.

Goal: Show that Cl_K is finite.

Idea:

- Find in each integral ideal \mathfrak{a} an element $a \neq 0$ of norm bounded by $\mathcal{N}(\mathfrak{a})$.
- Show: For $M > 0$ there are only finitely many integral ideals \mathfrak{a} with $\mathcal{N}(\mathfrak{a}) \leq M$.
- Show: Each class $[\mathfrak{a}] \in \text{Cl}_K$ contains an integral ideal \mathfrak{a}_1 s.t. $\mathcal{N}(\mathfrak{a}_1) \leq M_0 = \left(\frac{2}{\pi}\right)^s \sqrt{|d_K|}$.
Recall: s = number of not-real embeddings of K into \mathbb{C} .

Lemma 2.6.3. *Suppose: $\mathfrak{a} \neq 0$ is an integral ideal $\Rightarrow \exists a \in \mathfrak{a}, a \neq 0$ s.t. $|\mathcal{N}_{K/\mathbb{Q}}(a)| \leq \left(\frac{2}{\pi}\right)^s \sqrt{|d_K|} \mathcal{N}(\mathfrak{a})$.*

Proof. $M_0 := \left(\frac{2}{\pi}\right)^s \sqrt{|d_K|}$

Idea: Use „Thm. 5.3“

given: $c_\tau \in \mathbb{R}_{>0} (\tau \in \text{Hom}(K, \mathbb{C}))$ with $c_\tau = c_{\bar{\tau}}$ and $\prod_\tau c_\tau > M_0 \mathcal{N}(\mathfrak{a})$

$\Rightarrow \exists a \in \mathfrak{a}, a \neq 0$ with $|\tau(a)| < c_\tau$ for all τ .

For each $\varepsilon > 0$ choose a sequence $c_\tau \in \mathbb{R}_{>0}$ with $c_\tau = c_{\bar{\tau}}$ and $\prod_\tau c_\tau = M_0 \mathcal{N}(\mathfrak{a}) + \varepsilon$

$\xRightarrow{\text{Thm 5.3}}$ Find $a_\varepsilon \neq 0$ in \mathfrak{a} with

$$|\mathcal{N}_{K/\mathbb{Q}}(a)| = \prod_\tau |\tau(a)| < M_0 \mathcal{N}(\mathfrak{a}) + \varepsilon$$

Since left side is integer, we obtain: $\exists a \neq 0$ in \mathfrak{a} with $|\mathcal{N}_{K/\mathbb{Q}}(a)| \leq M_0 \mathcal{N}(\mathfrak{a})$. \square

Lemma 2.6.4. *Let $M \in \mathbb{R}_{>0}$. There are only finitely many integral ideals \mathfrak{a} with $\mathcal{N}(\mathfrak{a}) \leq M$.*

Proof. (1) Consider first only prime ideals $\mathcal{P} \neq 0$: Suppose $\mathcal{N}(\mathcal{P}) \leq M$

Recall: $\mathcal{P} \cap \mathbb{Z} = p\mathbb{Z}$ with p prime number (Prop. 3.3)

\Rightarrow obtain embedding $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathcal{O}_K/\mathcal{P} \Rightarrow \mathcal{N}(\mathcal{P}) = (\mathcal{O}_K : \mathcal{P}) = \#\mathcal{O}_K/\mathcal{P} = p^f$

Hence: $p^f \leq M$. In particular P is bounded.

Furthermore: There are only finitely many prime ideals \mathcal{P} with $\mathcal{P} \cap \mathbb{Z} = p\mathbb{Z}$.

Since $\mathcal{P} \cap \mathbb{Z} = p\mathbb{Z} \Rightarrow p \in \mathcal{P} \Rightarrow (p) \subseteq \mathcal{P}$ But there are only finitely many prime ideals in \mathcal{O}_K which divide (p) .

(2) Suppose now \mathfrak{a} is an arbitrary integral ideal, $\mathfrak{a} \neq 0$:

$\Rightarrow \mathfrak{a} = \mathcal{P}_1^{\nu_1} \cdots \mathcal{P}_r^{\nu_r}$ with \mathcal{P}_i prime ideal and $\nu_i \in \mathbb{N}$ and $\mathcal{N}(\mathfrak{a}) = \mathcal{N}(\mathcal{P}_1)^{\nu_1} \cdots \mathcal{N}(\mathcal{P}_r)^{\nu_r}$.

Now the claim follows from (1). \square

Theorem 9 (Finiteness of Cl_K). *The ideal class group of $\text{Cl}_K = \mathcal{J}_K/\mathcal{P}_K$ is finite.*

Proof. Let $M_0 := \left(\frac{2}{\pi}\right)^s \sqrt{|d_K|}$

Show that each class $[\mathfrak{a}] \in \text{Cl}_K$ contains an integral ideal \mathfrak{a}_1 with $\mathcal{N}(\mathfrak{a}_1) \leq M_0$. Then the

claim follows from Lemma 6.3.

Let $[a] \in \text{Cl}_K$. Choose $\gamma \in \mathcal{O}_K, \gamma \neq 0$ with γa^{-1} is integral.

$$\begin{aligned} \text{Lemma 6.2} &\Rightarrow \exists b \in \mathfrak{b} := \gamma a^{-1} \text{ with } b \neq 0 \text{ and } |\mathcal{N}_{K/\mathbb{Q}}(b)| \leq M_0 \mathcal{N}(\mathfrak{b}) \\ &\Rightarrow \mathcal{N}((b)\mathfrak{b}^{-1}) = \mathcal{N}((b))\mathcal{N}(\mathfrak{b}^{-1}) \leq M_0 \end{aligned}$$

Observe: The factorial ideal $(b)\mathfrak{b}^{-1} = (b)\gamma^{-1}a \in [a]$, hence $a_1 := b\gamma^{-1}a$ does the job. a_1 is an integral ideal, since $(b) \subseteq \gamma a^{-1}$ \square

Definition 2.6.5 („Klassenzahl“). $h_K := \# \text{Cl}_K := (\mathcal{J}_K : \mathcal{P}_K)$ is called the class number of K .

Proposition 2.6.6. Suppose R is a Dedekind domain.

R is a UFD $\iff R$ is a PID (principal ideal domain).

Proof. „ \Leftarrow “: true for general domains.

„ \Rightarrow “: Suppose R is a UFD.

Step 1: Every prime ideal is principal.

Let \mathcal{P} be a prime ideal, $\mathcal{P} \neq 0$. Choose $a \in \mathcal{P}, a \neq 0$. Let $a = p_1 \cdots p_n$ be its prime factor decomposition. \mathcal{P} prime $\Rightarrow p_i \in \mathcal{P}$ for one of the i 's $\Rightarrow \mathcal{P} \supseteq (p_i) \Rightarrow \mathcal{P} = (p_i)$, since (p_i) is a prime ideal and R is a Dedekind domain.

Step 2: \mathfrak{a} arbitrary ideal.

$\Rightarrow \mathfrak{a} = \mathcal{P}_1 \cdots \mathcal{P}_n$ is a product of prime ideals

$\Rightarrow \mathfrak{a}$ is principal, since each \mathcal{P}_i is. \square

Corollary 2.6.7. We have for a number field K :

$h_K = 1 \iff \mathcal{O}_K$ is a principal domain $\iff \mathcal{O}_K$ is a UFD.

2.7. The theorem of Dirichlet

Goal: Describe \mathcal{O}_K^\times

Recall:

- $\mathcal{O}^\times = \{\varepsilon \in \mathcal{O}_K \mid \mathcal{N}_{K/\mathbb{Q}}(\varepsilon) = \pm 1\}$
- $\mu(K) := \{x \in \mathcal{O}_K \mid \exists n \in \mathbb{N} \text{ with } x^n = 1\} \subseteq \mathcal{O}_K^\times$ is a finite subgroup.

Idea: Use multiplicative Minkowsky theory:

- $\text{Hom}(K, \mathbb{C}) = \{\tau_1, \dots, \tau_r, \tau_{r+1}, \overline{\tau_{r+1}}, \tau_{r+s}, \overline{\tau_{r+s}}\}$
- $j : K^\times \hookrightarrow K_\mathbb{R}^\times = \{x \in \prod_\tau \mathbb{C}^\times \mid x_{\bar{\tau}} = \overline{x_\tau}\}, a \mapsto (\tau(a))_\tau$
- $l : K_\mathbb{R}^\times \rightarrow [\prod_\tau \mathbb{R}]^+ := \{z \in \prod_\tau \mathbb{R} \mid z_{\bar{\tau}} = z_\tau\}, x = (x_\tau) \mapsto (\log |x_\tau|)_\tau$

\Rightarrow commutative diagramm:

$$\begin{array}{ccccc}
 \mathcal{O}_K^\times & & S & & H \\
 \cap & & \cap & & \cap \\
 K^\times & \xrightarrow{j} & K_{\mathbb{R}}^\times & \xrightarrow{l} & [\prod_{\tau} \mathbb{R}]^+ \\
 \downarrow \mathcal{N}_{K/\mathbb{Q}} & & \downarrow \mathcal{N} & & \downarrow \text{Tr} \\
 \mathbb{Q}^\times & \longrightarrow & \mathbb{R} & \xrightarrow{\log|\cdot|} & \mathbb{R}
 \end{array}$$

with $\mathcal{N}(x) = \prod_{\tau} x_{\tau}$, $\text{Tr}(z) = \sum_{\tau} z_{\tau}$.

Consider the three groups:

- (1) $\mathcal{O}_K^\times = \{\varepsilon \in \mathcal{O}_K \mid \mathcal{N}_{K/\mathbb{Q}}(\varepsilon) = \pm 1\}$
- (2) $S := \{x \in K_{\mathbb{R}}^\times \mid \mathcal{N}(x) = \pm 1\}$ „Norm 1 hyper surface“
- (3) $H := \{z \in [\prod_{\tau} \mathbb{R}]^+ \mid \text{Tr}(z) = 0\}$ „Trace 0 hypersurface“

\Rightarrow Morphisms restrict to

$$\mathcal{O}_K^\times \xrightarrow{j} S \xrightarrow{l} H.$$

Define $\Gamma := l \circ j(\mathcal{O}_K^\times) = \text{image of } l \circ j$.

Recall from additive Minkowski-Theory: $j(\mathcal{O}_K)$ is a complete lattice in $K_{\mathbb{R}}$

Proposition 2.7.1. *The sequence*

$$1 \rightarrow \mu(K) \rightarrow \mathcal{O}_K^\times \xrightarrow{l \circ j} \Gamma \rightarrow 1$$

is an exact sequence.

Proof. $\lambda := l \circ j$

We have to show: $\ker(\lambda) = \mu(K)$.

Observe: $a \in \ker(\lambda) \iff \forall \tau \in \text{Hom}(K, \mathbb{C}) : \log |\tau(a)| = 0 \iff |\tau(a)| = 1$

Hence: $\ker(\lambda) = \{a \in \mathcal{O}^\times \mid |\tau(a)| = 1\}$.

„ \supseteq “: \checkmark

„ \subseteq “: $j(\ker(\lambda))$ is bounded as subset of $K_{\mathbb{R}}^\times$. Furthermore: $j(\ker(\lambda)) \subseteq j(\mathcal{O})$ which is a lattice in $K_{\mathbb{R}} \Rightarrow j(\ker(\lambda))$ is finite and thus also $\ker(\lambda)$.

Altogether: $\ker(\lambda)$ is a finite subgroup of $K^\times \Rightarrow$ every element in $\ker(\lambda)$ has finite order \Rightarrow every element is a root of unity. \square

Goal: Describe Γ

Recall: $\alpha, \alpha' \in \mathcal{O}_K$ are associated : $\iff \exists \varepsilon \in \mathcal{O}_K^\times$ s.t. $\alpha' = \alpha \cdot \varepsilon$.

Proposition 2.7.2. *Let $a \in \mathbb{Z}$. There are at most $(\mathcal{O}_K : a\mathcal{O}_K) = \mathcal{N}((a))$ elements $\alpha \in \mathcal{O}_K$ up to associates with $\mathcal{N}_{K/\mathbb{Q}}(\alpha) = \pm a$.*

Proof. Suppose w.l.o.g.: $a > 1$.

Consider the cosets of \mathcal{O}_K modulo the subgroup $a\mathcal{O}_K$. Show that each coset contains at most one such α up to associates.

Suppose: $\alpha \in \mathcal{O}$ with $\mathcal{N}_{K/\mathbb{Q}}(\alpha) = \pm a$ and suppose $\beta = \alpha + a\gamma$ with $\gamma \in \mathcal{O}_K$ also satisfies $\mathcal{N}_{K/\mathbb{Q}}(\beta) = \pm a \Rightarrow \frac{\beta}{\alpha} = 1 \pm \frac{\mathcal{N}_{K/\mathbb{Q}}(\alpha)}{\alpha} \gamma$.

Recall: $\frac{\mathcal{N}(\alpha)}{\alpha} \in \mathcal{O}_K \Rightarrow \frac{\beta}{\alpha} \in \mathcal{O}_K$.

Obtain in the same way $\frac{\alpha}{\beta} \in \mathcal{O}_K$. Hence $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ are in $\mathcal{O}_K^\times \Rightarrow \alpha$ and β are associated. \square

Lemma 2.7.3. *Let V be an \mathbb{R} -vector space of dimension n , Γ a lattice in V .*

Γ is complete $\iff \exists M \subseteq V$ with M bounded s.t. $\bigcup_{\gamma \in \Gamma} M + \gamma = V$.

Proof. „ \Rightarrow “: $\Gamma = \mathbb{Z}v_1 + \dots + \mathbb{Z}v_n \Rightarrow M := \phi := \{r_1v_1 + \dots + r_nv_n \mid 0 \leq r_i < 1\}$ does it.

„ \Leftarrow “: Consider: $V_0 := \mathbb{R}$ -vector space generated by Γ . Have to show: $V_0 = V$.

Let $v \in V$. Consider the sequence kv ($k \in \mathbb{N}$).

Precondition $\Rightarrow \forall k \exists a_k \in M$ and $\gamma_k \in \Gamma$ with $kv = a_k + \gamma_k$

M bounded $\Rightarrow \frac{1}{k}a_k \rightarrow 0 \Rightarrow v = \lim_{k \rightarrow \infty} \frac{1}{k}a_k + \frac{1}{k}\gamma_k = \lim_{k \rightarrow \infty} \frac{1}{k}\gamma_k \Rightarrow v \in V_0$, since V_0 is closed. \square

Theorem 10. *The group Γ is a complete lattice in $H = \{x \in [\prod_\tau \mathbb{R}]^+ \mid \text{Tr}(x) = 0\} \cong \mathbb{R}^{r+s-1}$. Hence Γ is isomorphic to \mathbb{Z}^{r+s-1} .*

Proof. Step 1: Show that Γ is a lattice, i.e. show that Γ is a discrete subgroup of H .

More precisely: show that $\forall c > 0$:

$$\Gamma \cap \{(z_\tau)_\tau \in \prod_\tau \mathbb{R} \mid |z_\tau| \leq c\} =: Q_c$$

is finite.

Observe: $l^{-1}(Q_c) = \{(x_\tau)_\tau \in \prod_\tau \mathbb{C}^\times \mid e^{-c} \leq |x_\tau| \leq e^c\}$ since $l((x_\tau)_\tau) = (\log|x_\tau|)_\tau$.

$\Rightarrow l^{-1}(Q_c) \cap j(\mathcal{O}_K^\times)$ is finite, since $j(\mathcal{O}_K)$ is a lattice in $K_\mathbb{R}$. This shows the claim.

Step 2: Show that Γ is complete.

Idea: Use Lemma 7.3.

Hence: find $M \subseteq H$ as required in the lemma.

Equivalently: find $T \subseteq S$, s.t. $S = \bigcup_{\varepsilon \in \mathcal{O}_K^\times} T \cdot j(\varepsilon)$ and T is bounded.

Then we have for $M := l(T) : H = \bigcup_{\varepsilon \in \mathcal{O}_K^\times} M + l(j(\varepsilon)) = \bigcup_{\gamma \in \Gamma} M + \gamma$.

Furthermore: T bounded $\Rightarrow \exists C > 0 : \forall x \in T : \forall \tau : |x_\tau| < C$.

Since $\prod_\tau |x_\tau| = 1 \Rightarrow \exists c > 0 : \forall x \in T : \forall \tau : |x_\tau| > c \Rightarrow M = l(T)$ is bounded in H .

Step 3: Definition of T

- Choose sequence (c_τ) with $c_\tau > 0$, $c_{\bar{\tau}} = c_\tau$ and $C := \prod c_\tau > M_0 = (\frac{2}{\pi})^s \sqrt{d_K}$ and define $X := \{(x_\tau)_\tau \mid |x_\tau| < c_\tau\}$.
- Choose $\alpha_1, \dots, \alpha_N \in \mathcal{O}_K$ s.t. each $\alpha \in \mathcal{O}_K, \alpha \neq 0$ with $|\mathcal{N}_{K/\mathbb{Q}}(\alpha)| \leq C$ is associated to one α_i (by Prop 7.2. possible).

Define $T := S \cap \bigcup_{i=1}^n X \cdot j(\alpha_i)^{-1}$.

Step 4: T does the job:

- (1) X is bounded $\Rightarrow Xj(\alpha_i)^{-1}$ is bounded $\Rightarrow T$ is bounded.
- (2) Observe: $y = (y_\tau) \in S \Rightarrow Xy = \{(x_\tau) \in K_{\mathbb{R}} \mid |x_\tau| < c'_\tau\}$ with $c'_\tau = c_\tau \cdot |y_\tau|$
 $\Rightarrow c'_\tau = c'_\tau$ and $\prod_\tau c'_\tau = \prod_\tau c_\tau \underbrace{\prod_{y \in S} |y_\tau|}_{=1} = C$.
 $\Rightarrow \exists \alpha \in \mathcal{O}_K$ with $|\tau(\alpha)| < c'_\tau \forall \tau \Rightarrow j(\alpha) \in Xy$
- (3) Show that: $S = \bigcup_{\varepsilon \in \mathcal{O}_K^\times} Tj(\varepsilon)$
 Suppose $y \in S \stackrel{(2)}{\Rightarrow} \exists \alpha \in \mathcal{O}_K \setminus \{0\}$ with $j(\alpha) \in Xy^{-1} \Rightarrow j(\alpha) = xy^{-1}$ for some $x \in X$.
 Furthermore: $|\mathcal{N}_{K/\mathbb{Q}}(\alpha)| = |\mathcal{N}(xy^{-1})| = |\mathcal{N}(x)| < \prod_\tau c_\tau = C$.
 $\Rightarrow \alpha$ is associated to some α_i , hence $\alpha_i = \varepsilon \alpha$ with $\varepsilon \in \mathcal{O}_K^\times$.
 $\Rightarrow y = xj(\alpha)^{-1} = xj(\alpha_i^{-1}\varepsilon)$.
 Finally: y and $j(\varepsilon) \in S \Rightarrow xj(\alpha_i)^{-1} \in S \cap Xj(\alpha_i)^{-1} \subseteq T \Rightarrow y \in Tj(\varepsilon)$.

□

Corollary 2.7.4. $\mathcal{O}_K^\times \cong \mathbb{Z}^{r+s-1} \times \mu(K)$.

Proof. We have the exact sequence

$$1 \rightarrow \mu(K) \rightarrow \mathcal{O}_K^\times \xrightarrow{l} \Gamma \cong \mathbb{Z}^{r+s-1} \rightarrow 1$$

Fix a basis v_1, \dots, v_t ($t := r + s - 1$) of Γ and preimages $\varepsilon_1, \dots, \varepsilon_t$ in \mathcal{O}_K^\times .

Let $A := \langle \varepsilon_1, \dots, \varepsilon_t \rangle \subseteq \mathcal{O}_K^\times$.

Then $\lambda|_A$ is an isomorphism and thus $A \cap \mu(K) = \{1\}$. In particular every $\alpha \in \mathcal{O}_K^\times$ decomposes in a unique way as $\alpha = \nu \cdot \mu$ with $\nu \in A$ and $\mu \in \mu(K)$. □

2.8. Prime ideals in \mathcal{O}_K

Question: Describe the prime ideals in \mathcal{O}_K that "live above a prime ideal $\mathfrak{p} \subset \mathbb{Z}$ ".

Consider the following, more general situation: Let

- \mathcal{O} be a Dedekind domain,
- $K = \text{Quot}(\mathcal{O})$,
- $L \mid K$ a finite and separable field extension,
- $\hat{\mathcal{O}}$ the integral closure of \mathcal{O} in L .

Definition 2.8.1. In the setting above, we say that a prime ideal $\hat{\mathfrak{p}} \in \hat{\mathcal{O}}$ lies above a prime ideal $\mathfrak{p} \in \mathcal{O} : \Leftrightarrow \hat{\mathfrak{p}} \cap \mathcal{O} = \mathfrak{p}$.

Proposition 2.8.2. $\hat{\mathcal{O}}$ is a Dedekind domain.

Proof. (1) $\hat{\mathcal{O}}$ is an integral domain and is integrally closed (see **Remark 2.1**).

- (2) We show, that every prime ideal $0 \neq \hat{\mathfrak{p}} \in \hat{\mathcal{O}}$ is maximal: We know that $\mathfrak{p} := \hat{\mathfrak{p}} \cap \mathcal{O}$ is a prime ideal in \mathcal{O} .

(Claim:) $\mathfrak{p} \neq 0$. Choose $0 \neq x \in \hat{\mathfrak{p}}$. Since $\hat{\mathcal{O}}$ is integrally closed, $\exists a_0, \dots, a_{n-1}$, such that

$$x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0.$$

We may assume that the equation is minimal, i.e. $a_0 \neq 0$. Then we have

$$0 \neq a_0 = -a_1x - \dots - a_{n-1}x^{n-1} - x^n \in \hat{\mathfrak{p}} \cap \mathcal{O} = \mathfrak{p}.$$

Since \mathfrak{p} is a prime ideal of \mathcal{O} , it is also maximal, i.e. \mathcal{O}/\mathfrak{p} is a field. Hence $\hat{\mathcal{O}}/\hat{\mathfrak{p}}$ is a finite extension of \mathcal{O}/\mathfrak{p} as an \mathcal{O}/\mathfrak{p} -algebra. Therefore \mathcal{O}/\mathfrak{p} a field $\Rightarrow \hat{\mathcal{O}}/\hat{\mathfrak{p}}$ is a field $\Rightarrow \hat{\mathfrak{p}}$ is a maximal ideal.

- (3) $\hat{\mathcal{O}}$ is Noetherian: Choose a basis $\alpha_1, \dots, \alpha_n$ of $L \mid K$ with $\alpha_1, \dots, \alpha_n \in \hat{\mathcal{O}}$. Let $d := d(\alpha_1, \dots, \alpha_n) \neq 0$ (**Proposition 2.6**). Recall that $d \cdot \hat{\mathcal{O}} \subset \mathcal{O}\alpha_1 + \dots + \mathcal{O}\alpha_n$ (**Proposition 2.8**) and that therefore $\hat{\mathcal{O}} \subset \mathcal{O}\frac{\alpha_1}{d} + \dots + \mathcal{O}\frac{\alpha_n}{d}$. Hence every ideal $I \subset \hat{\mathcal{O}}$ can be regarded as a submodule of the \mathcal{O} -module $\mathcal{O}\frac{\alpha_1}{d} + \dots + \mathcal{O}\frac{\alpha_n}{d}$. But since this module is finitely generated and \mathcal{O} is Noetherian, I must be finitely generated as well.

□

Proposition 2.8.3. *Let $\mathfrak{p} \subset \mathcal{O}$ be a prime ideal. Then $\mathfrak{p} \cdot \hat{\mathcal{O}} \subsetneq \hat{\mathcal{O}}$.*

Proof. We may assume $\mathfrak{p} \neq 0$.

- (1) Choose $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$. Then we can write $\pi \cdot \mathcal{O} = \mathfrak{p} \cdot \mathfrak{u}$ with $\mathfrak{p}, \mathfrak{u}$ coprime, i.e. $\mathcal{O} = \mathfrak{p} + \mathfrak{u} \Rightarrow \exists s \in \mathfrak{u}, t \in \mathfrak{p} : 1 = s + t$. In particular, $s \notin \mathfrak{p}$ since $1 \notin \mathfrak{p}$ and $s \cdot \mathfrak{p} \subset \mathfrak{u} \cdot \mathfrak{p} = \pi \cdot \mathcal{O}$.
- (2) Suppose $\mathfrak{p}\hat{\mathcal{O}} = \hat{\mathcal{O}}$. Then $s \cdot \hat{\mathcal{O}} = s\mathfrak{p}\hat{\mathcal{O}} \subset \pi\hat{\mathcal{O}} \Rightarrow s = \pi x$ with some $x \in \hat{\mathcal{O}} \cap K = \mathcal{O} \Rightarrow s \in \pi\mathcal{O} \subset \mathfrak{p}$, a contradiction.

□

Remark 2.8.4. Let $\mathfrak{p} \neq 0$ be a prime ideal in \mathcal{O} . Then:

- (i) $\mathfrak{p} \cdot \hat{\mathcal{O}} = \mathfrak{p}_1^{e_1} \cdot \dots \cdot \mathfrak{p}_r^{e_r}$ with $e_1, \dots, e_r \in \mathbb{N}$ and $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ prime ideals in $\hat{\mathcal{O}}$.
- (ii) A prime ideal $\hat{\mathfrak{p}}$ in $\hat{\mathcal{O}}$ satisfies: $\hat{\mathfrak{p}} \cap \mathcal{O} = \mathfrak{p} \Leftrightarrow \hat{\mathfrak{p}} = \mathfrak{p}_i$ for some i .

Proof. (i) follows from **Proposition 8.2+8.3**.

- (ii) " \Leftarrow ": $\mathfrak{p}\hat{\mathcal{O}} = \mathfrak{p}_1^{e_1} \cdot \dots \cdot \mathfrak{p}_r^{e_r} \Rightarrow \mathfrak{p}\mathcal{O} \subset \mathfrak{p}_i \Rightarrow \mathfrak{p} \subset \mathfrak{p}_i \cap \mathcal{O}$. We have $\mathfrak{p}_i \cap \mathcal{O} \neq 0$, $1 \notin \mathfrak{p}_i \cap \mathcal{O}$ and \mathfrak{p} is maximal, hence $\mathfrak{p} = \mathfrak{p}_i \cap \mathcal{O}$.
- " \Rightarrow ": $\hat{\mathfrak{p}} \cap \mathcal{O} = \mathfrak{p} \Rightarrow \mathfrak{p}\hat{\mathcal{O}} \subset \hat{\mathfrak{p}} \Rightarrow \hat{\mathfrak{p}}$ divides $\mathfrak{p}\hat{\mathcal{O}}$.

□

Definition 2.8.5. Let $0 \neq \mathfrak{p}$ be a prime ideal in \mathcal{O} and $\mathfrak{p}\hat{\mathcal{O}} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$ the decomposition into prime ideals.

- (i) e_i is called **ramification index** of \mathfrak{p}_i .
 \mathfrak{p}_i is called **unramified** $:\Leftrightarrow e_i = 1$.
 \mathfrak{p} is called **unramified**, if all \mathfrak{p}_i are unramified.
 \mathfrak{p} is called **totally ramified** $:\Leftrightarrow r = 1$.
- (ii) $f_i := \dim_K \hat{\mathcal{O}}/\mathfrak{p}_i$ with $K := \mathcal{O}/\mathfrak{p}$ is called **local degree** or **relative degree** of \mathfrak{p}_i .

Theorem 11. In the situation of **Definition 8.5**, we have the fundamental equation:

$$\sum_{i=1}^r e_i \cdot f_i = n \quad \text{with } n = [L : K]$$

Proof. We can write

$$\hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}} = \bigoplus_{i=1}^r \hat{\mathcal{O}}/\mathfrak{p}_i^{e_i}$$

by the Chinese Remainder Theorem. Let $k = \mathcal{O}/\mathfrak{p}$

Step 1: We show, that $\dim_k \hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}} = n$. Choose a basis $\bar{\omega}_1, \dots, \bar{\omega}_m$ of $\hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}}$ over k and choose preimages $\omega_1, \dots, \omega_m$ in $\hat{\mathcal{O}}$. We will show, that $\omega_1, \dots, \omega_m$ is a basis of $L \mid K$, i.e $m = n$, from which the claim follows.

- (1) Suppose $\omega_1, \dots, \omega_m$ are linearly dependant, i.e $\exists \alpha_1, \dots, \alpha_m \in K$, not all zero and such that

$$\alpha_1 \omega_1 + \cdots + \alpha_m \omega_m = 0. \quad (*)$$

Since $K = \text{Quot}(\mathcal{O})$, we may choose $\alpha_1, \dots, \alpha_m \in \mathcal{O}$, since we can just clear denominators. Consider the ideal $\mathfrak{u} := \langle \alpha_1, \dots, \alpha_m \rangle \subset \mathcal{O}$. $\mathfrak{p} \neq 0 \Rightarrow \mathfrak{u}^{-1}\mathfrak{p} \subsetneq \mathfrak{u}^{-1}$. Choose some $\alpha \in \mathfrak{u}^{-1} \setminus \mathfrak{u}^{-1}\mathfrak{p} \Rightarrow \alpha \cdot \mathfrak{u} \not\subseteq \mathfrak{p} \Rightarrow \alpha\alpha_1, \dots, \alpha\alpha_m \in \mathcal{O}$, but not all lie in \mathfrak{p} .

$\xRightarrow{(*)} \alpha\alpha_1\omega_1 + \cdots + \alpha\alpha_m\omega_m = 0 \pmod{\mathfrak{p}}$ with at least one of the $\alpha\alpha_i \notin \mathfrak{p}$. Hence $\alpha\alpha_1\bar{\omega}_1 + \cdots + \alpha\alpha_m\bar{\omega}_m = 0$ with at least one $\alpha\alpha_i \neq 0$, which contradicts the assumption that $\bar{\omega}_1, \dots, \bar{\omega}_m$ is a basis.

- (2) Consider $M := \mathcal{O}\omega_1 + \cdots + \mathcal{O}\omega_m$ and $N := \hat{\mathcal{O}}/M$. Since $\hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}} = K\bar{\omega}_1 + \cdots + K\bar{\omega}_m$, we have $\hat{\mathcal{O}} = M + \mathfrak{p}\hat{\mathcal{O}} \xrightarrow{\text{mod } M} N = \mathfrak{p}N$. The proof of **Proposition 8.2** implies, that $\hat{\mathcal{O}}$ and N are finitely generated as \mathcal{O} -modules. Choose generators $\bar{\alpha}_1, \dots, \bar{\alpha}_s$ of N . $N = \mathfrak{p}N \Rightarrow \exists \alpha_{i,j} \in \mathfrak{p}$ with $\bar{\alpha}_i = \sum_{j=1}^s \alpha_{i,j} \bar{\alpha}_j$. Consider $A = (\alpha_{i,j})_{i,j=1}^s - I$. Then

$$A \cdot \begin{pmatrix} \bar{\alpha}_1 \\ \vdots \\ \bar{\alpha}_s \end{pmatrix} = 0.$$

Furthermore, $d := \det(A) = (-1)^s \pmod{\mathfrak{p}} \Rightarrow d \neq 0$. We now see

$$0 = A^\# A \begin{pmatrix} \bar{\alpha}_1 \\ \vdots \\ \bar{\alpha}_s \end{pmatrix} = d \cdot \begin{pmatrix} \bar{\alpha}_1 \\ \vdots \\ \bar{\alpha}_s \end{pmatrix} \Rightarrow d \cdot N = 0,$$

hence $d \cdot \hat{\mathcal{O}} \subset M = \mathcal{O}\omega_1 + \dots \mathcal{O}\omega_m$. Now, for some $\beta \in L$, we have $\beta = d \underbrace{\beta'}_{\in L} = d \cdot \frac{b}{a} = \frac{1}{a}db$, with $b \in \hat{\mathcal{O}}$ and $a \in \mathcal{O}$. Hence $\beta \in K\omega_1 + \dots + K\omega_m \Rightarrow m = n$ and $\omega_1, \dots, \omega_m$ generate $L \mid K$.

Step 2: We show, that $\dim_K \hat{\mathcal{O}}/\mathfrak{p}_i^{e_i} = e_i f_i$. Consider the chain

$$\hat{\mathcal{O}}/\mathfrak{p}_i^{e_i} \supsetneq \mathfrak{p}_i/\mathfrak{p}_i^{e_i} \supsetneq \dots \supsetneq \mathfrak{p}_i^{e_i-1}/\mathfrak{p}_i^{e_i} \supsetneq 0$$

as a chain of K -vectorspaces. Choose an $\alpha \in \mathfrak{p}_i^j \setminus \mathfrak{p}_i^{j+1}$ and consider the homomorphism

$$\begin{aligned} \hat{\mathcal{O}} &\longrightarrow \mathfrak{p}_i^j/\mathfrak{p}_i^{j+1} \\ a &\longmapsto \alpha \cdot a, \end{aligned}$$

which is surjective with kernel \mathfrak{p}_i (since \mathfrak{p}_i^{j+1} is coprime to $\alpha\hat{\mathcal{O}}$). Therefore $\mathfrak{p}_i^j/\mathfrak{p}_i^{j+1} \cong \hat{\mathcal{O}}/\mathfrak{p}_i$ and we have

$$\dim_K \hat{\mathcal{O}}/\mathfrak{p}_i^{e_i} = \sum_{j=0}^{e_i-1} \dim_K \mathfrak{p}_i^j/\mathfrak{p}_i^{j+1} = e_i \cdot f_i$$

□

Next, we will examine the example of the Gaussian integers $\mathbb{Z}[i]$. By **Proposition 2.10**, $\mathbb{Z}[i]$ is the ring of integers $\hat{\mathcal{O}}$ of the field extension $\mathbb{Q}[i] \mid \mathbb{Q}$.

Reminder 2.8.6. (i) $\mathbb{Z}[i]$ is an euclidean ring $\Rightarrow \mathbb{Z}[i]$ is a PID $\Rightarrow \mathbb{Z}[i]$ is an UFD

(ii) In particular, all prime ideals $\mathfrak{p} = \langle \pi \rangle$ with π prime.

Remark 2.8.7. Let R be a domain, $a, b \in R$. Then $\langle a \rangle = \langle b \rangle \Leftrightarrow a$ and b are associated.

Proof. " \Rightarrow ": $\langle a \rangle = \langle b \rangle \Rightarrow \exists r, r' \in R : b = ra$ and $a = r'b \Rightarrow b = rr'b \Rightarrow (1 - rr')b = 0 \xrightarrow{R \text{ domain}} r, r' \in R^\times$.

" \Leftarrow ": $a = \epsilon b$ with $\epsilon \in R^\times \Rightarrow b = \epsilon^{-1}a \Rightarrow \langle a \rangle = \langle b \rangle$. \square

Remark 2.8.8. For $L = \mathbb{Q}[i]$ and $K = \mathbb{Q}$, we have

- (i) $\text{Gal } L | K = \{\text{id}, (a + bi \mapsto a - bi)\}$
- (ii) $\mathcal{N}_{L|K}(a + bi) = (a + bi) \cdot (a - bi) = a^2 + b^2$.
- (iii) Since $\mathbb{Z}[i]$ is a UFD, an element is prime \Leftrightarrow it is irreducible.
- (iv) $\mathbb{Z}[i]^\times = \{\alpha \in \mathbb{Z}[i] \mid \mathcal{N}_{L|K}(\alpha) = 1\} = \{1, -1, i, -i\}$.
- (v) For $\alpha = a + bi$, its associated elements are $-a - bi, ai - b, -ai + b$.

Proposition 2.8.9 (Theorem of Wilson). *Let $p \in \mathbb{Z}$ be a prime number. Then:*

- (i) $(p - 1)! \equiv -1 \pmod{p}$.
- (ii) If $p = 4n + 1$ with $n \in \mathbb{N}$, then $(2n)!^2 \equiv -1 \pmod{p}$.

Proof. (i) Since the statement is obvious for $p = 2$, let $p > 2$. Consider $X^{p-1} - 1 \in \mathbb{Z}/p\mathbb{Z}[x]$. Then $1, \dots, p - 1$ are all zeroes and

$$X^{p-1} - 1 = (x - 1) \cdot (x - 2) \cdot \dots \cdot (x - (p - 1)) \in \mathbb{Z}/p\mathbb{Z}[X].$$

When we look at the constant term, we see that $-1 = (-1)^{p-1} \cdot (p - 1)! = (p - 1)!$

- (ii) $(-1) \equiv (p-1)! \equiv (4n)! = 1 \cdot 2 \cdot \dots \cdot 2n \cdot (p-1) \cdot \dots \cdot (p-2n) \equiv (2n)! \cdot (-1)^{2n} \cdot (2n)! \equiv (2n)!^2 \pmod{p}$.

\square

Proposition 2.8.10. *If p is a prime in \mathbb{Z} with $p \equiv 1 \pmod{4}$, then p is not a prime in $\mathbb{Z}[i]$.*

Proof. Write $p = 4n + 1$. By the Theorem of Wilson, we have $X^2 \equiv -1 \pmod{p}$ for $x = (2n)!$. Then $p \mid X^2 + 1 = (x + i)(x - i) \in \mathbb{Z}[i]$, but $\frac{x \pm i}{p} \notin \mathbb{Z}[i]$. \square

Proposition 2.8.11. *Each prime element $\pi \in \mathbb{Z}[i]$ is associated to one of the following prime elements of $\mathbb{Z}[i]$:*

- (1) $\pi = 1 + i$.
- (2) $\pi = a + bi$, with $a^2 + b^2 = p$ prime in \mathbb{Z} and $p \equiv 1 \pmod{4}$.
- (3) $\pi = p$ prime in \mathbb{Z} and $p \equiv 3 \pmod{4}$.

Proof. We proof the proposition in 3 steps.

Step 1: If π is as in (1) or (2), then π is prime. Suppose $\pi = \alpha\beta$. Then $p = \mathcal{N}(\pi) = \mathcal{N}(\alpha) \cdot \mathcal{N}(\beta) \in \mathbb{Z}$, so either $\mathcal{N}(\alpha) = 1$ or $\mathcal{N}(\beta) = 1$, i.e. α or β is a unit.

Step 2: If π is as in (3), then π is a prime in \mathbb{Z} . Suppose $\pi = \alpha\beta \in \mathbb{Z}[i]$. Then $p^2 = \mathcal{N}(\pi) = \mathcal{N}(\alpha) \cdot \mathcal{N}(\beta)$. If $\alpha, \beta \notin \mathbb{Z}[i]^\times$, then $\mathcal{N}(\alpha) = \mathcal{N}(\beta) = p$. Write $\alpha = a + bi$. Then $p = \mathcal{N}(\alpha) = a^2 + b^2 \not\equiv 3 \pmod{4}$, since it is always $a^2 + b^2 \equiv 0, 1 \pmod{4}$, a contradiction.

Step 3: We have now shown, that the elements (1) – (3) are prime. Let now $\pi_0 \in \mathbb{Z}[i]$ be a prime element. We will show, that π_0 is associated to one of the three elements above. Look at $\mathcal{N}(\pi_0) = p_1 \cdots p_r$ with p_1, \dots, p_r primes in \mathbb{Z} . Since π_0 is prime, it divides $p := p_i$, $1 \leq i \leq r \Rightarrow \mathcal{N}(\pi_0)$ divides $\mathcal{N}(p) = p^2$, i.e. $\mathcal{N}(\pi_0) = p$ or p^2 .

Case 1: $\mathcal{N}(\pi_0) = p$. If $p = 2$, then $\pi_0 \in \{1 + i, 1 - i, -1 + i, -1 - i\}$, i.e. π_0 is associated to $1 + i$. If $p > 2$, then $p = \mathcal{N}(\pi_0) = a^2 + b^2 \equiv 1 \pmod{4} \Rightarrow \pi_0$ is associated to an element as in (2).

Case 2: $\mathcal{N}(\pi_0) = p^2 \Rightarrow \pi_0 | p^2 \Rightarrow \pi_0 | p \Rightarrow \frac{p}{\pi_0} \in \mathbb{Z}[i]$ and $\mathcal{N}(\frac{p}{\pi_0}) = \frac{\mathcal{N}(p)}{\mathcal{N}(\pi_0)} = \frac{p^2}{p^2} = 1$, i.e. $\frac{p}{\pi_0}$ is a unit, hence π_0 is associated to p . By **Proposition 8.10**, $p \not\equiv 1 \pmod{4}$. Also $p \neq 2$, since $2 = (1 + i)(1 - i)$ is not prime in $\mathbb{Z}[i]$. Hence $p \equiv 3 \pmod{4}$ and π_0 is associated to an element as in (3).

□

Corollary 2.8.12 (Fermat). (i) If p is prime then $p = a^2 + b^2 \Leftrightarrow p \not\equiv 3 \pmod{4}$

(ii) $\forall n \in \mathbb{N} : n = a^2 + b^2 \Leftrightarrow \nu_p(n)$ is even for all primes $p \equiv 3 \pmod{4}$ ($\nu_p(n)$ = exponent of p in prime factorization of n over \mathbb{Z}).

Proof. (i) " \Rightarrow ": Same as in Step 2 of 8.11

" \Leftarrow ": If $p = 2$, then $2 = 1 + 1$. If $p \equiv 1 \pmod{4}$, then by **Proposition 8.10**, $p = \alpha\beta \in \mathbb{Z}[i]$ with $\mathcal{N}(\alpha) = \mathcal{N}(\beta) = p$. Write $\alpha = a + bi$ and get $p = \mathcal{N}(\alpha) = a^2 + b^2$.

(ii) " \Rightarrow ": $n = a^2 + b^2 \Rightarrow n = \mathcal{N}(\alpha)$ with $\alpha = a + bi \in \mathbb{Z}[i]$. Write $\alpha = \epsilon \cdot \pi_1 \cdots \pi_r \cdot \pi_{r+1} \cdots \pi_{r+s}$ with π_1, \dots, π_r as in (3) and $\pi_{r+1}, \dots, \pi_{r+s}$ as in (1) or (2). Then $\mathcal{N}(\alpha) = \prod_{i=1}^r \mathcal{N}(\pi_i) = p_1^2 \cdots p_r^2 \cdot p_{r+1} \cdots p_{r+s}$ with $p_1, \dots, p_r \equiv 3 \pmod{4}$ and $p_{r+1}, \dots, p_{r+s} \not\equiv 3 \pmod{4}$.

" \Leftarrow ": $n = p_1^2 \cdots p_r^2 \cdot p_{r+1} \cdots p_{r+s}$ as above. By (i), $p_j \not\equiv 3 \pmod{4}$ and hence $p_j = a_j^2 + b_j^2$ for $r+1 \leq j \leq r+s$. Define $\alpha := p_1 \cdots p_r \cdot (a_{r+1} + ib_{r+1}) \cdots (a_{r+s} + ib_{r+s})$. Then $\mathcal{N}(\alpha) = n$.

□

Corollary 2.8.13. The prime ideals \mathfrak{p}_i in $\mathbb{Z}[i]$ that lie over a prime ideal $\mathfrak{p} = \langle p \rangle$ in \mathbb{Z} are obtained as follows:

(i) $p = 2 \Rightarrow \langle 2 \rangle \mathbb{Z}[i] = \langle 1 + i \rangle \langle 1 - i \rangle = \langle 1 + i \rangle^2$. Hence $r = 1$, $e_1 = 2$, $f_1 = 1$.

(ii) $p \equiv 1 \pmod{4} \xrightarrow{p=a^2+b^2} \langle p \rangle \mathbb{Z}[i] = \langle a+bi \rangle \langle a-bi \rangle$. Hence $r = 2$, $e_1 = e_2 = 1$, $f_1 = f_2 = 1$.

(iii) $p \equiv 3 \pmod{4} \Rightarrow \langle p \rangle \mathbb{Z}[i]$ is a prime ideal. Hence $r = 1$, $e_1 = 1$, $f_1 = 2$.

□

GOAL: Describe prime ideals explicitly for all simple extensions $L = K[\Theta]$ with $\Theta \in \hat{\mathcal{O}}$.

Caution: Before, we had $\mathbb{Z}[i] = \hat{\mathcal{O}}$. In general, we might have $\hat{\mathcal{O}}' := \mathcal{O}[\Theta] \subsetneq \hat{\mathcal{O}}$.

Idea: Take the largest ideal of $\hat{\mathcal{O}}$ which also lies in $\hat{\mathcal{O}}'$.

Definition 2.8.14. The set $\mathcal{F} := \{ \alpha \in \hat{\mathcal{O}} \mid \alpha \hat{\mathcal{O}} \subset \hat{\mathcal{O}}' \}$ is called **conductor**.

Example 2.8.15. If $\hat{\mathcal{O}} = \mathbb{Z}[i]$ and $\Theta = i$, then $\hat{\mathcal{O}}' = \mathcal{O}[\Theta] \Rightarrow \mathcal{F} = \hat{\mathcal{O}}$.

Proposition 2.8.16. In the situation above, let $f(X) := f_{\Theta}(X)$ be the minimal polynomial of Θ . Let \mathfrak{p} be a prime ideal in \mathcal{O} and $K := \mathcal{O}/\mathfrak{p}$. Consider the image \bar{f} of f in $K[X]$ and let $\bar{f} = \bar{f}_1^{e_1} \cdots \bar{f}_r^{e_r}$ be the prime factorization in $K[X]$. Choose preimages $f_1, \dots, f_r \in \mathcal{O}[X]$. Then:

If \mathfrak{p} is coprime to \mathcal{F} , i.e. $\mathfrak{p} + \mathcal{F} \cap \mathcal{O} = \mathcal{O}$, then the ideals in $\hat{\mathcal{O}}$ which lie over \mathfrak{p} are given as follows: $\mathfrak{p}_i := \mathfrak{p}\hat{\mathcal{O}} + f_i(\Theta)\hat{\mathcal{O}}$, $1 \leq i \leq r$ and the local degree of \mathfrak{p}_i is equal to $\deg(\bar{f}_i)$.

Proposition 2.8.17. Let R and S be rings and $\varphi: R \rightarrow S$ a ring homomorphism.

(i) If \mathfrak{q} is a prime ideal in S then $\varphi^{-1}(\mathfrak{q})$ is a prime ideal in R .

(ii) If φ is surjective and \mathfrak{p} is a prime ideal in R with $\ker \varphi \subset \mathfrak{p}$ then $\varphi(\mathfrak{p})$ is a prime ideal in S .

Proof. “(i)” Preimages of ideals are ideals. Suppose $ab \in \varphi^{-1}(\mathfrak{q})$. Then $\varphi(a)\varphi(b) \in \mathfrak{q}$ such that, without loss of generality, $\varphi(a) \in \mathfrak{q}$ and hence $a \in \varphi^{-1}(\mathfrak{q})$.

“(ii)” Images of ideals under surjective homomorphisms are ideals. Let $\bar{a}\bar{b} \in \varphi(\mathfrak{p})$. Since φ is surjective there are $a, b \in R$ with $\varphi(a) = \bar{a}$, $\varphi(b) = \bar{b}$ and there is $c \in \mathfrak{p}$ with $\varphi(c) = \bar{a}\bar{b}$. Hence

$$ab - c \in \ker \varphi \subset \mathfrak{p}$$

such that $ab \in \mathfrak{p}$. We may assume that $a \in \mathfrak{p}$ and conclude $\bar{a} = \varphi(a) \in \varphi(\mathfrak{p})$. □

Definition 2.8.18. In the situation of Proposition 2.8.17 we define:

(i) $\text{Spec}(R) = \{ \mathfrak{p} \subset R \mid \mathfrak{p} \text{ is a prime ideal} \}$

(ii) $\text{Spec}_S(R) = \{ \mathfrak{p} \subset \text{Spec}(R) \mid \mathfrak{p} \supset \ker \varphi \}$

Corollary 2.8.19. In the situation of Proposition 2.8.17 we have:

(i) If $\varphi: R \rightarrow S$ is a homomorphism of rings then φ induces a map

$$\varphi^*: \text{Spec}(S) \rightarrow \text{Spec}_S(R), \mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p}).$$

(ii) If φ is surjective then φ^* is a bijection with inverse map

$$\varphi_*: \operatorname{Spec}_S(R) \rightarrow \operatorname{Spec}(S), \mathfrak{p} \mapsto \varphi(\mathfrak{p}).$$

Reminder 2.8.20. For $a \in \mathbb{Z}$ and p prime in \mathbb{Z} the **Legendre symbol** is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 0, & p \text{ divides } a, \\ 1, & \text{there is an } x \in \mathbb{Z}/p\mathbb{Z} \text{ such that } x^2 \equiv a \pmod{p}, \\ -1, & \text{else.} \end{cases}$$

Example 2.8.21. Apply Proposition 8.15 for quadratic number fields, D square-free:

$$\begin{array}{ccccc} \hat{\mathcal{O}} & = & \mathbb{Z}[\theta] & \subset & \mathbb{Q}(\sqrt{D}) \\ & & \uparrow & & \uparrow \\ \mathcal{O} & = & \mathbb{Z} & \subset & \mathbb{Q} \end{array}$$

Reminder 2.8.22. If $D \not\equiv 1 \pmod{4}$ then we can choose $\theta = \sqrt{D}$ and obtain $f = f_\theta = X^2 - D$ and $d(f_\theta) = 4D$.

If $D \equiv 1 \pmod{4}$ then we can choose $\theta = \frac{1}{2}(1 + \sqrt{D})$ and obtain $f = f_\theta = X^2 - X - \frac{D-1}{4}$ and $d(f_\theta) = D$.

Consider $p \in \mathbb{Z}$ prime and define $\bar{f} = \bar{f}_\theta$ as the image of f in $\mathbb{Z}/p\mathbb{Z}[X]$.

Observe: \bar{f} has two equal zeroes in $\mathbb{Z}/p\mathbb{Z}$ iff $d(f) = 0$ in $\mathbb{Z}/p\mathbb{Z}$ iff

$$\begin{cases} \left(\frac{4D}{p}\right) = 0, & D \not\equiv 1 \pmod{4}, \\ \left(\frac{D}{p}\right) = 0, & D \equiv 1 \pmod{4}. \end{cases}$$

\bar{f} has two different zeroes in $\mathbb{Z}/p\mathbb{Z}$ iff $d(f)$ is a non-zero square in $\mathbb{Z}/p\mathbb{Z}$ iff

$$\begin{cases} \left(\frac{4D}{p}\right) = 1, & D \not\equiv 1 \pmod{4}, \\ \left(\frac{D}{p}\right) = 1, & D \equiv 1 \pmod{4} \end{cases} \Leftrightarrow \left(\frac{D}{p}\right) = 1.$$

\bar{f} has no zeroes in $\mathbb{Z}/p\mathbb{Z}$ iff $\left(\frac{D}{p}\right) = -1$.

Proposition 8.15 then implies in the first case that $p\hat{\mathcal{O}} = \hat{\mathcal{P}}_1^2$ with

$$\mathcal{P}_1 = \begin{cases} p\hat{\mathcal{O}} + \theta\hat{\mathcal{O}}, & D \not\equiv 1 \pmod{4}, \\ p\hat{\mathcal{O}} + \left(\theta - \frac{1}{2}\right)\hat{\mathcal{O}}, & D \equiv 1 \pmod{4}, \end{cases}$$

In the second case we obtain $p\hat{\mathcal{O}} = \hat{\mathcal{P}}_1\hat{\mathcal{P}}_2$ with $\hat{\mathcal{P}}_{1,2} = p\hat{\mathcal{O}} + (\theta \pm x)\hat{\mathcal{O}}$, where $x^2 \equiv D \pmod{p}$.

In the third case $p\hat{\mathcal{P}}$ is a prime ideal.

Example. Let $D \not\equiv 1 \pmod{p}$, $\left(\frac{4D}{p}\right) = 0$ and $p \neq 2$. Consider the map $\pi: \hat{\mathcal{O}} \rightarrow \hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}}$ with $\hat{\mathcal{O}} = \mathbb{Z}[\sqrt{D}]$ and $\mathfrak{p}\hat{\mathcal{O}} = \{a + b\sqrt{D} \mid p|a \text{ and } p|b\}$ and thus

$$\begin{aligned} \hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}} &\cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}[\sqrt{D}] \cong (\mathbb{Z}/p\mathbb{Z}[X])/(X^2 - D), \\ \theta &\leftrightarrow (0, \sqrt{D}) \leftrightarrow \bar{X}. \end{aligned}$$

We have

$$\hat{\mathcal{P}}_1 = \pi^{-1}((\bar{\theta})) = \{a + b\sqrt{D} \mid p \text{ divides } a\}.$$

Example. Let $D \not\equiv 1 \pmod{p}$ and $\left(\frac{4D}{p}\right) = 1$. Then there exists $x \in \mathbb{Z}$ with $x^2 \equiv D \pmod{p}$ and $p \nmid x$. Here, $\pi: \hat{\mathcal{O}} \rightarrow \hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}}$ is the map

$$\begin{aligned} \mathbb{Z}[\sqrt{D}] &\rightarrow \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}[\sqrt{D}] \\ &\cong (\mathbb{Z}/p\mathbb{Z}[X])/(X - x)(X + x) \\ &\cong (\mathbb{Z}/p\mathbb{Z}[X])/(X - x) \oplus (\mathbb{Z}/p\mathbb{Z}[X])/(X + x) \end{aligned}$$

given by

$$a + b\sqrt{D} \mapsto \bar{a} + \bar{b}\sqrt{D} \cong \bar{a} + \bar{b}X \cong (\bar{a} + \bar{b}x, \bar{a} - \bar{b}x).$$

Recall that $\bar{f}(X) = (X - x)(X + x) = \bar{f}_1\bar{f}_2$ with $\bar{f}_1, \bar{f}_2 \in \mathbb{Z}[X]$ and

$$f_1(\theta) = \theta - x = \sqrt{D} - x = -x + \sqrt{D}$$

with $\pi(f_1(\theta)) \leftrightarrow (0, -2\bar{x})$. Observe that for $\bar{x} \in \mathbb{F}_p^x$ we have the correspondence

$$(\pi(f_1(\theta))) \leftrightarrow \mathcal{O} \oplus (\mathbb{Z}/p\mathbb{Z}[X])/(X + p) \cong \mathcal{O} \oplus \mathbb{Z}/p\mathbb{Z}$$

and hence $\hat{\mathcal{P}}_1 = \pi^{-1}(\mathcal{O} \oplus \mathbb{Z}/p\mathbb{Z})$.

Proof of Prop. 8.16. Consider the map $\pi: \hat{\mathcal{O}} \rightarrow \hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}}$. By Corollary 8.19 we have a bijection

$$\{\hat{\mathcal{P}} \mid \hat{\mathcal{P}} \text{ prime ideal in } \hat{\mathcal{O}} \text{ with } \hat{\mathcal{P}} \cap \mathcal{O} = \mathfrak{p}\} \leftrightarrow \{\mathfrak{q} \mid \mathfrak{q} \text{ prime ideal in } \hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}}\}.$$

We show:

$$\hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}} \cong \hat{\mathcal{O}}'/\mathfrak{p}\hat{\mathcal{O}}' \cong k[X]/(\bar{f}),$$

where $k = \hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}}$ and $\hat{\mathcal{O}}' = \mathcal{O}[\theta]$.

Step 1: $\hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}} \cong \hat{\mathcal{O}}'/\mathfrak{p}\hat{\mathcal{O}}'$

Consider the homomorphism $\varphi: \hat{\mathcal{O}}' \rightarrow \hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}}$ induced by the inclusion $\hat{\mathcal{O}}' \hookrightarrow \hat{\mathcal{O}}$.

“(1)” φ is surjective: If $\mathfrak{p} + (\mathbb{F} \cap \mathcal{O}) = \mathcal{O}$ then $\mathfrak{p}\hat{\mathcal{O}} + \mathbb{F} = \hat{\mathcal{O}}$ and hence $\mathfrak{p}\hat{\mathcal{O}} + \hat{\mathcal{O}}' = \hat{\mathcal{O}}$ (multiply both sides of first equation with $\hat{\mathcal{O}}$).

“(2)” $\ker \varphi = \mathfrak{p}\hat{\mathcal{O}}'$: “ \supset ” Clear. “ \subset ” We have $\ker \varphi = \hat{\mathcal{O}}' \cap \mathfrak{p}\hat{\mathcal{O}}$. Use $\mathfrak{p} + (\mathbb{F} \cap \mathcal{O}) = \mathcal{O}$ and write $1 = p + a$ with $p \in \mathfrak{p}$ and $a \in \mathbb{F} \cap \mathcal{O}$. For $x \in \hat{\mathcal{O}}' \cap \mathfrak{p}\hat{\mathcal{O}}$ we have:

$$x = 1 \cdot x = (p + a)x = px + ax \in \mathfrak{p}\hat{\mathcal{O}}'.$$

Step 2: $\hat{\mathcal{O}}'/\mathfrak{p}\hat{\mathcal{O}}' \cong k[X]/(\bar{f})$

Recall that $\hat{\mathcal{O}}' = \mathcal{O}[\theta] \cong \mathcal{O}[X]/(f)$. Consider $\Psi: \mathcal{O}[X] \rightarrow k[X]/(\bar{f})$, which is surjective. It holds that $\ker \Psi = (\mathfrak{p}, f)$ and hence Ψ induces an isomorphism $\hat{\mathcal{O}}'/\mathfrak{p}\hat{\mathcal{O}}' \rightarrow k[X]/(\bar{f})$.

Step 3: Consider now $R = k[X]/(\bar{f})$ and determine $\text{Spec}(R)$.

“(1)” Recall the prime decomposition $\bar{f} = \bar{f}_1^{e_1} \cdots \bar{f}_r^{e_r}$ in $k[X]$ and consider the projection $k[X] \twoheadrightarrow k[X]/(\bar{f})$. By Corollary 8.19 we have the correspondence

$$\text{Spec}(R) \leftrightarrow \{\mathfrak{p} \text{ prime ideal in } k[X] \mid \bar{f} \in \mathfrak{p}\}$$

and hence $\text{Spec}(R) = \{(\bar{f}_i) \mid i = 1, \dots, r\}$.

“(2)” Notice that

$$R/(\bar{f}_i) = (k[X]/(\bar{f})) / (\bar{f}_i) \cong k[X]/(\bar{f}_i)$$

is a k -vector space of dimension $\deg(\bar{f}_i)$ such that

$$[R/(\bar{f}_i) : k] = \deg(\bar{f}_i).$$

“(3)” In R we have

$$\bigcap_{i=1}^r (\bar{f}_i)^{e_i} = (\bar{f}) = 0.$$

Step 4: Use the isomorphism

$$k[X]/(\bar{f}) \rightarrow \hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}}, g \mapsto g(\theta)$$

and obtain from Step 3 with $\mathcal{P}_i = (f_i(\theta))$ that:

$$(i) \quad \text{Spec}(\hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}}) = \{\mathcal{P}_i \mid i = 1, \dots, r\}$$

$$(ii) \quad \left[(\hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}}) / \mathcal{P}_i : k \right] = \deg(\bar{f}_i)$$

$$(iii) \quad \bigcap_{i=1}^r \mathcal{P}_i^{e_i} = 0$$

Step 5: Take preimages in $\hat{\mathcal{O}}$ via $\hat{\mathcal{O}} \rightarrow \hat{\mathcal{O}}/\mathfrak{p}\hat{\mathcal{O}}$ and observe that (iii) implies $\bigcap_{i=1}^r \hat{\mathcal{P}}_i^{e_i} \subset \mathfrak{p}\hat{\mathcal{O}}$ such that $\mathfrak{p}\hat{\mathcal{O}}$ divides $\bigcap_{i=1}^r \hat{\mathcal{P}}_i^{e_i}$. Furthermore,

$$[L : K] = n = \deg(f) = \sum_{i=1}^r e_i f_i$$

such that by Theorem 11, $\mathfrak{p}\hat{\mathcal{O}} = \prod_{i=1}^r \hat{\mathcal{P}}_i^{e_i}$. □

$$\begin{array}{ccccccc}
& & \hat{\mathcal{P}} & \subseteq & \hat{\mathcal{O}} & \subseteq & L \\
& & \uparrow & & \uparrow & & \uparrow \\
\hat{\mathcal{O}}\mathcal{P} = \hat{\mathcal{P}}_1^{e_1} \cdots \hat{\mathcal{P}}_r^{e_r} & & \mathcal{P} & \subseteq & \mathcal{O} & \subseteq & K
\end{array}$$

Proposition 2.8.23. *There are only finitely many prime ideals $\hat{\mathcal{P}}$ in $\hat{\mathcal{O}}$ which are ramified over $\mathcal{P} = \hat{\mathcal{P}} \cap \mathcal{O}$.*

Proof. Choose primitive element θ of $L|K$ in $\hat{\mathcal{O}}$. Let $f_\theta \in \mathcal{O}[X]$ be the minimal polynomial of θ and $d := \text{discr}(f_\theta) = \text{discr}(1, \theta, \dots, \theta^{n-1}) = \prod_{i < j} (\theta_i - \theta_j)^2 \in \mathcal{O}$.

Here θ_i, θ_j are the zeroes of f_θ in the algebraic closure.

Claim: If \mathcal{P} is a prime ideal in \mathcal{O} s.t.

- \mathcal{P} is coprime to (d) and
- \mathcal{P} is coprime to $\mathbb{F} \cap \mathcal{O}$

then \mathcal{P} is unramified, i.e. all $\hat{\mathcal{P}}$ lying above \mathcal{P} are unramified.

From the claim we obtain that there are only finitely many \mathcal{P} which allow ramification.

Proof of the claim: Write $\hat{\mathcal{O}}\mathcal{P} = \hat{\mathcal{P}}_1^{e_1} \cdots \hat{\mathcal{P}}_r^{e_r}$. Consider $\bar{f}_\theta \in \mathcal{O}/\mathcal{P}[X]$. As in Prop. 8.15

$$\bar{f}_\theta = \bar{f}_1^{e_1} \cdots \bar{f}_r^{e_r} \quad (\star)$$

a prime decomposition. (d) and \mathcal{P} are coprime $\Rightarrow \bar{d} = \text{image of } d \text{ in } \mathcal{O}/\mathcal{P} \neq 0 \Rightarrow \bar{f}_\theta$ has only single zeroes in an algebraic closure of $\mathcal{O}/\mathcal{P} \xrightarrow{(\star)} e_1 = \cdots = e_r = 1$ \square

Definition 2.8.24.

- \mathcal{P} is said to split completely or to be totally split : $\iff e_i = f_i = 1 \ \forall i \in \underline{r}$.
- \mathcal{P} is said to be indecomposed, nonsplit or totally ramified : $\iff r = 1$.

2.9. Hilbert's theorem of ramification

Idea: Consider Galois extensions $L|K \rightarrow$ life becomes much nicer.

Same setting as in 8. Suppose further that $L|K$ normal and consider $G = \text{Gal}(L|K)$.

Remark 2.9.1. i) $\hat{\mathcal{P}}$ prime ideals in $\hat{\mathcal{O}}$ with $\mathcal{P} := \hat{\mathcal{P}} \cap \mathcal{O}$. For $\sigma \in \text{Gal}(L|K)$ we have $\sigma(\hat{\mathcal{P}})$ is a prime ideal in $\hat{\mathcal{O}}$ above \mathcal{P} .

ii) $\text{Gal}(L|K)$ acts transitively on the set of prime ideals $\hat{\mathcal{P}}$ in $\hat{\mathcal{O}}$ over \mathcal{P} .

Proof. i) Recall from Rem 2.1 iii) that $\sigma(\hat{\mathcal{O}}) = \hat{\mathcal{O}}$
 $\Rightarrow \sigma(\hat{\mathcal{P}})$ is again a prime ideal in $\hat{\mathcal{O}}$.
 $\sigma(\hat{\mathcal{P}}) \cap \mathcal{O} = \sigma(\hat{\mathcal{P}} \cap \mathcal{O}) = \sigma(\mathcal{P}) = \mathcal{P}$
 $\Rightarrow \sigma(\hat{\mathcal{P}})$ lies above \mathcal{P} .

- ii) follows from i) that we have such an action. Let $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}'$ be prime ideals above $\mathcal{P} = \hat{\mathcal{P}} \cap \mathcal{O} = \hat{\mathcal{P}}' \cap \mathcal{O}$. Assume that $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}'$ are not in the same G -orbit. Hence $\hat{\mathcal{P}}'$ and $\sigma(\hat{\mathcal{P}})$ are coprime for each $\sigma \in G$.
 $\Rightarrow \hat{\mathcal{P}}'$ is coprime to $\sigma_1(\hat{\mathcal{P}}) \cdot \dots \cdot \sigma_n(\hat{\mathcal{P}})$, where $G = \{\sigma_1, \dots, \sigma_n\}$.
 CRT $\Rightarrow \exists x \in \hat{\mathcal{O}}$ with $x \equiv 0 \pmod{\hat{\mathcal{P}}'}$ and $x \equiv 1 \pmod{\sigma(\hat{\mathcal{P}})}$ for all $\sigma \in G$.
 In particular: $\mathcal{N}_{L|K}(x) = \prod_{\sigma \in G} \sigma(x) \in \hat{\mathcal{P}}' \cap \mathcal{O} = \mathcal{P}$
 Also: $\forall \sigma \in G : x \notin \sigma(\hat{\mathcal{P}}) \Rightarrow \forall \sigma \in G : \sigma(x) \notin \mathcal{P}$
 $\Rightarrow \mathcal{N}_{L|K}(x) = \prod_{\sigma \in G} \sigma(x) \notin \hat{\mathcal{P}} \cap \mathcal{O} = \mathcal{P} \nmid$.

□

Definition 2.9.2. Let $\hat{\mathcal{P}}$ be a prime ideal of $\hat{\mathcal{O}}$ above \mathcal{P} .

- i) $G_{\hat{\mathcal{P}}} := \text{Stab}_G(\hat{\mathcal{P}}) = \{\sigma \in G \mid \sigma(\hat{\mathcal{P}}) = \hat{\mathcal{P}}\}$ is called decomposition group („Zerlegungsgruppe“)
 ii) $Z_{\hat{\mathcal{P}}} := \{x \in L \mid \sigma(x) = x \text{ for all } \sigma \in G_{\hat{\mathcal{P}}}\}$ is called decomposition field („Zerlegungskörper“)

Remark 2.9.3. Let $\hat{\mathcal{P}}_0$ be a prime ideal which lies above \mathcal{P} .

- i) $G/G_{\hat{\mathcal{P}}_0} := \{gG_{\hat{\mathcal{P}}_0} \mid g \in G\} \xleftrightarrow{1:1} \{\hat{\mathcal{P}} \mid \hat{\mathcal{P}} \text{ lies above } \mathcal{P}\}$
 ii) $G_{\hat{\mathcal{P}}_0} = \{1\} \iff [G : G_{\hat{\mathcal{P}}_0}] = [L : K] = n \iff \mathcal{P} \text{ is totally split} \iff Z_{\hat{\mathcal{P}}_0} = L$ ($r = [G : G_{\hat{\mathcal{P}}_0}]$)
 iii) $G_{\hat{\mathcal{P}}_0} = G \iff [G : G_{\hat{\mathcal{P}}_0}] = 1 \iff \mathcal{P} \text{ is nonsplit} \iff Z_{\hat{\mathcal{P}}_0} = K$
 iv) $G_{\sigma(\hat{\mathcal{P}}_0)} = \sigma \circ G_{\hat{\mathcal{P}}_0} \circ \sigma^{-1}$

Proof. Follows from Prop 9.1 + definitions + group actions. □

Remark 2.9.4. Suppose $\mathcal{P}\hat{\mathcal{O}} = \hat{\mathcal{P}}_1^{e_1} \cdot \dots \cdot \hat{\mathcal{P}}_r^{e_r}$ with local degrees $f_i = [\hat{\mathcal{O}}/\hat{\mathcal{P}}_i : \mathcal{O}/\mathcal{P}]$. Then $e_1 = \dots = e_r$ and $f_1 = \dots = f_r$.

Proof. Prop. 9.1 $\Rightarrow \exists \sigma_i \in G$ s.t. $\sigma_i(\hat{\mathcal{P}}_1) = \hat{\mathcal{P}}_i$
 $\Rightarrow \hat{\mathcal{O}}/\hat{\mathcal{P}}_1 \cong \hat{\mathcal{O}}/\hat{\mathcal{P}}_i$, $a \pmod{\hat{\mathcal{P}}_1} \mapsto \sigma_i(a) \pmod{\hat{\mathcal{P}}_i}$ as $k = \mathcal{O}/\mathcal{P}$ -vectorspaces $\Rightarrow f_1 = f_i$ and $\hat{\mathcal{P}}_i^k \supseteq \mathcal{P}\hat{\mathcal{O}} \iff \hat{\mathcal{P}}_i^k = (\sigma_i(\hat{\mathcal{P}}_1))^k \supseteq \mathcal{P}\hat{\mathcal{O}} = \sigma_i(\mathcal{P}\hat{\mathcal{O}}) \Rightarrow e_i = e_1$. □

Consider the field extensions $K \subseteq Z_{\hat{\mathcal{P}}} \subseteq L$. We have:

$$\begin{array}{ccccccc}
 & & \hat{\mathcal{P}} & \subseteq & \hat{\mathcal{O}} & \subseteq & L \\
 & & \mid & & \mid & & \mid \\
 \hat{\mathcal{P}}_Z & := & \hat{\mathcal{P}} \cap Z_{\hat{\mathcal{P}}} & \subseteq & \hat{\mathcal{O}} \cap Z_{\hat{\mathcal{P}}} & \subseteq & Z_{\hat{\mathcal{P}}} \\
 & & \mid & & \mid & & \mid \\
 & & \mathcal{P} & \subseteq & \mathcal{O} & \subseteq & K
 \end{array}$$

Observe $\hat{\mathcal{O}} \cap Z_{\hat{\mathcal{P}}}$ is the integral closure of \mathcal{O} in $Z_{\hat{\mathcal{P}}}$.

Proposition 2.9.5. Suppose $\mathcal{P}\hat{\mathcal{O}} = (\prod_{\sigma} \sigma(\hat{\mathcal{P}}))^e$ with local degree f .

- i) $\hat{\mathcal{P}}_Z$ is non-split in $\hat{\mathcal{O}}$, i.e. $\hat{\mathcal{P}}$ is the only prime ideal above $\hat{\mathcal{P}}_Z$.
- ii) $\hat{\mathcal{P}}/\hat{\mathcal{P}}_Z$ has ramification index e and local degree f .
- iii) $\hat{\mathcal{P}}_Z/\mathcal{P}$ has ramification index 1 and local degree 1, i.e. $\hat{\mathcal{P}}_Z/\mathcal{P}$ is totally split.

Proof. i) $Z_{\hat{\mathcal{P}}} = L^{G_{\hat{\mathcal{P}}}} \Rightarrow \text{Gal}(L/Z_{\hat{\mathcal{P}}}) = G_{\hat{\mathcal{P}}}$. Now statement follows from 9.3 iii)

ii)+iii) Let $e' = \text{ramification index of } \hat{\mathcal{P}}/\hat{\mathcal{P}}_Z$ and $e'' = \text{ramification index of } \hat{\mathcal{P}}_Z/\mathcal{P}$

Let $f' = \text{local degree of } \hat{\mathcal{P}}/\hat{\mathcal{P}}_Z$ and $f'' = \text{local degree of } \hat{\mathcal{P}}_Z/\mathcal{P}$.

Hence: $\hat{\mathcal{P}}_Z\hat{\mathcal{O}} = \hat{\mathcal{P}}^{e'}$ and $\mathcal{P}(\hat{\mathcal{O}} \cap Z_{\hat{\mathcal{P}}}) = \hat{\mathcal{P}}_Z^{e''} \cdot \dots \Rightarrow \mathcal{P}\hat{\mathcal{O}} = (\hat{\mathcal{P}}^{e'})^{e''} \cdot \dots$

$\Rightarrow e = e' \cdot e'' \quad (\star).$

Also we have for the field extensions

$$\hat{\mathcal{O}}/\hat{\mathcal{P}} \supseteq \underbrace{\hat{\mathcal{O}} \cap Z_{\hat{\mathcal{P}}}/\hat{\mathcal{P}}_Z}_{f'} \supseteq \underbrace{\mathcal{O}/\mathcal{P}}_{f''}$$

$\Rightarrow f = f' \cdot f'' \quad (\star\star).$

Thm. 11 \Rightarrow 1) For $L|K : n = [L : K] = e \cdot f \cdot r$ with $r = [G : G_{\hat{\mathcal{P}}}] \quad (n = |G|).$

2) For $L|Z_{\hat{\mathcal{P}}} : |G_{\hat{\mathcal{P}}}| = \frac{n}{r} \stackrel{\text{Thm. 11}}{=} e' \cdot f' \cdot \underbrace{r'}_{=1(\text{by i})} \stackrel{1)}{=} e \cdot f \Rightarrow e' = e, f' = f$ and

$e'' = 1 = f'' \Rightarrow \text{Claim.}$

□

Definition 2.9.6. In our general setting we call $\kappa(\hat{\mathcal{P}}) := \hat{\mathcal{O}}/\hat{\mathcal{P}}$ the residue class field („Restklassenkörper“).

Remark 2.9.7. Prop 9.5 iii) $\Rightarrow [\kappa(\hat{\mathcal{P}}_Z) : \kappa(\mathcal{P})] = 1$ hence, $\kappa(\hat{\mathcal{P}}_Z) = \kappa(\mathcal{P}) = \mathcal{O}/\mathcal{P} =: k$.

Proposition 2.9.8. If $\hat{\mathcal{P}}/\mathcal{P}$ is non-split, i.e. $\hat{\mathcal{P}}$ is the only prime ideal over \mathcal{P} , then we obtain the following surjective group homomorphism: $\varphi : G = \text{Gal}(L/K) \rightarrow \text{Aut}(\kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P}))$.

Proof. Step 1: φ is well-defined:

Since $\hat{\mathcal{P}}/\mathcal{P}$ is totally split, we have $\sigma(\hat{\mathcal{P}}) = \hat{\mathcal{P}}$. Therefore $\sigma \in \text{Gal}(L/K)$ induces an automorphism of $\kappa(\hat{\mathcal{P}})$.

Step 2: $\kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P})$ is a normal extension:

Denote $k := \kappa(\mathcal{P})$ and $\kappa := \kappa(\hat{\mathcal{P}})$. Consider $\bar{\theta} \in \kappa$ and let $\bar{g} \in k[X]$ be its minimal polynomial over k . Have to show that \bar{g} decomposes into linear factors over κ . Let θ be a preimage of $\bar{\theta}$ in $\hat{\mathcal{O}}$ and $f \in \mathcal{O}[X]$ its minimal polynomial $\Rightarrow f(\bar{\theta}) = 0$. Let \bar{f} be the image of f in $k[X]$, hence $\bar{f}(\bar{\theta}) = 0$ and thus \bar{g} divides \bar{f} .

Furthermore: L/K is normal $\Rightarrow f$ decomposes into linear factors over $L \Rightarrow$ also over $\hat{\mathcal{O}}$, since Galois-Automorphisms preserve $\hat{\mathcal{O}} \Rightarrow \bar{f}$ decomposes into linear factors over $\kappa = \hat{\mathcal{O}}/\mathcal{P} \Rightarrow \bar{g}$ does so.

Step 3: φ is surjective:

Let $\bar{\sigma} \in \text{Aut}(\kappa/k)$. Consider the field extension: $k \subseteq E \subseteq \underbrace{\kappa}_{\text{purely inseparable} \Rightarrow \text{Aut}(\kappa/E)=\{1\}}$ (\star)

with E is the maximal separable field extension.

$\Rightarrow \exists \bar{\theta} \in E$ with $E = k(\bar{\theta})$ and $\theta \in \hat{\mathcal{O}}$ a preimage. Let again $\bar{g} \in k[X]$ be the minimal polynomial of $\bar{\theta}$ and f, \bar{f} as in Step 2.

$\Rightarrow \bar{\sigma}(\bar{\theta})$ is a zero of \bar{g} , hence $(X - \bar{\sigma}(\bar{\theta}))$ divides \bar{g} and hence \bar{f} since \bar{g}, f and \bar{f} decompose into linear factors.

$\Rightarrow \exists \theta' \in \hat{\mathcal{O}}$ with $\theta' \bmod \hat{\mathcal{P}} = \bar{\sigma}(\bar{\theta})$ and θ' is a zero of f (there is a linear factor $(X - \theta')$ of f which is sent to the factor $(X - \bar{\sigma}(\bar{\theta}))$ of \bar{f})

$\Rightarrow \exists \sigma \in \text{Gal}(L/K)$ with $\sigma(\theta) = \theta'$ and thus $\sigma(\theta) \equiv \theta' \equiv \bar{\sigma}(\bar{\theta}) \bmod \hat{\mathcal{P}}$.

$\Rightarrow \varphi(\sigma)|_E = \bar{\sigma}|_E \xrightarrow{(\star)} \varphi(\sigma) = \bar{\sigma}$ □

Remark 2.9.9. Observe that for Step 2 we did not need that $\hat{\mathcal{P}}/\mathcal{P}$ is non-split. Hence we have in the general situation of this section:

$$L/K \text{ normal} \Rightarrow \kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P}) \text{ is normal.}$$

Proposition 2.9.10. *In general, we obtain the following surjective group homomorphism:*

$$G_{\hat{\mathcal{P}}} \rightarrow \text{Aut}(\kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P})), \quad \sigma \mapsto (a \bmod \hat{\mathcal{P}} \mapsto \sigma(a) \bmod \hat{\mathcal{P}})$$

Proof. Idea: Consider $K \subseteq Z_{\hat{\mathcal{P}}} \subseteq \underbrace{L}_{\text{non-split}}$. Remark 9.7 $\Rightarrow \kappa(\hat{\mathcal{P}}_Z) = k := \kappa(\mathcal{P})$

Lemma 9.8 $\Rightarrow \underbrace{\text{Gal}(L/Z_{\hat{\mathcal{P}}})}_{=G_{\hat{\mathcal{P}}}} \twoheadrightarrow \underbrace{\text{Aut}(\kappa(\hat{\mathcal{P}})/\kappa(\hat{\mathcal{P}}_Z))}_{\text{Aut}(\kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P}))} \Rightarrow \text{Claim.}$ □

Definition 2.9.11 („Trägheitsgruppe“/„Trägheitskörper“). Let $\varphi : G_{\hat{\mathcal{P}}} \twoheadrightarrow \text{Gal}(\kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P}))$ be the surjective group homomorphism from Prop. 9.10.

i) $I_{\hat{\mathcal{P}}} := \ker(\varphi)$ is called inertia group.

ii) $T_{\hat{\mathcal{P}}} := \{x \in L \mid \sigma(x) = x \forall \sigma \in I_{\hat{\mathcal{P}}}\}$ is called inertia field.

Remark 2.9.12. i) We obtain the following chain of field extensions:

$$K \subseteq Z_{\hat{\mathcal{P}}} \subseteq T_{\hat{\mathcal{P}}} \subseteq L$$

ii) We have the following short exact sequence:

$$1 \rightarrow I_{\hat{\mathcal{P}}} \rightarrow G_{\hat{\mathcal{P}}} \rightarrow \text{Aut}(\kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P})) \rightarrow 1$$

Proposition 2.9.13. *In the situation of 9.12 we have:*

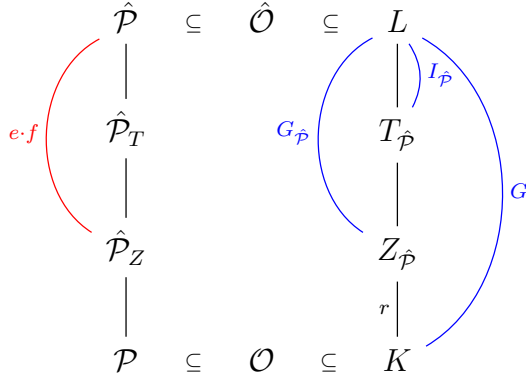
i) $T_{\hat{\mathcal{P}}}/Z_{\hat{\mathcal{P}}}$ is normal and $\text{Gal}(T_{\hat{\mathcal{P}}}/Z_{\hat{\mathcal{P}}}) \cong \text{Aut}(\kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P}))$.

Furthermore: $\text{Gal}(L/T_{\hat{\mathcal{P}}}) \cong I_{\hat{\mathcal{P}}}$.

ii) If $\kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P})$ is separable, then: $\#I_{\hat{\mathcal{P}}} = [L : T_{\hat{\mathcal{P}}}] = e$ and $[G_{\hat{\mathcal{P}}} : I_{\hat{\mathcal{P}}}] = [T_{\hat{\mathcal{P}}} : Z_{\hat{\mathcal{P}}}] = f$

iii) If $\kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P})$ is separable and $\hat{\mathcal{P}}_T := \hat{\mathcal{P}} \cap T_{\hat{\mathcal{P}}}$, then we have

- The ramification index of $\hat{\mathcal{P}}$ over $\hat{\mathcal{P}}_T$ is e and the local degree is 1.
- The ramification index of $\hat{\mathcal{P}}_T$ over $\hat{\mathcal{P}}_Z$ is 1 and the local degree is f .



Proof. i) • $I_{\hat{\mathcal{P}}}$ is normal in $G_{\hat{\mathcal{P}}}$.

- $\text{Gal}(T_{\hat{\mathcal{P}}}/Z_{\hat{\mathcal{P}}}) \cong G_{\hat{\mathcal{P}}}/I_{\hat{\mathcal{P}}} \stackrel{\text{Rem 9.12}}{\cong} \text{Aut}(\kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P}))$
- $T_{\hat{\mathcal{P}}}$ is the fixed field of $I_{\hat{\mathcal{P}}}$

$$\text{ii) } \kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P}) \text{ is separable} \Rightarrow \# \text{Aut}(\kappa(\hat{\mathcal{P}})/\kappa(\mathcal{P})) = \underbrace{[\kappa(\hat{\mathcal{P}}) : \kappa(\mathcal{P})]}_{\hat{\mathcal{O}}/\hat{\mathcal{P}}} \stackrel{9.12}{=} \underbrace{\#G_{\hat{\mathcal{P}}}}_{e.f} / \#I_{\hat{\mathcal{P}}} = f$$

iii) We will show below that $\kappa(\hat{\mathcal{P}}_T) = \kappa(\hat{\mathcal{P}})$. This implies:

- local degree of $\hat{\mathcal{P}}_T/\hat{\mathcal{P}}$ is 1
- ramification index of $\hat{\mathcal{P}}_T/\hat{\mathcal{P}}$ is e since $[L/T_{\hat{\mathcal{P}}}] = \#I_{\hat{\mathcal{P}}} = e$
- multiplicativity of e and $f \Rightarrow \text{rest } \checkmark$

Show that $\kappa(\hat{\mathcal{P}}_T) = \kappa(\hat{\mathcal{P}})$:

Use Lemma 9.8 \Rightarrow Obtain surjective group homomorphism

$$I_{\hat{\mathcal{P}}} = \text{Gal}(L/T_{\hat{\mathcal{P}}}) \xrightarrow{\varphi} \text{Aut}(\kappa(\hat{\mathcal{P}})/\kappa(\hat{\mathcal{P}}_T))$$

By definition of $I_{\hat{\mathcal{P}}}$ the image of this homomorphism is trivial.

$$\Rightarrow \text{Aut}(\kappa(\hat{\mathcal{P}})/\kappa(\hat{\mathcal{P}}_T)) = \{1\} \stackrel{\text{normal} + \text{separable}}{\implies} [\kappa(\hat{\mathcal{P}}) : \kappa(\hat{\mathcal{P}}_T)] = 1.$$

□

2.10. Cyclotomic Fields

In this section, we have

- $\zeta = \zeta_n$ = primitive n -th root of unity

- $L = \mathbb{Q}(\zeta)$
- \mathcal{O} = ring of integers in L
- $d = \varphi(n) = [L : \mathbb{Q}]$.

GOAL:

- (1) Show, that $\mathcal{O} = \mathbb{Z}[\zeta]$
- (2) Describe the prime ideals in \mathcal{O}

Lemma 2.10.1. Suppose $n = l^k$ with l prime and hence $d = \varphi(n) = l^k - l^{k-1} = l^{k-1}(l-1)$.

- The minimal polynomial $\phi(X)$ of ζ is $\phi(X) = X^{(l-1)l^{k-1}} + X^{(l-2)l^{k-1}} + \dots + X^{l^{k-1}} + 1$.
- We have $l = \prod_{g \in (\mathbb{Z}/n\mathbb{Z})^\times} (1 - \zeta^g)$.
- $1 - \zeta^g = \epsilon_g(1 - \zeta)$ with $\epsilon_g \in \mathcal{O}^\times$ for $g \not\equiv 0 \pmod{l}$.
- $l = \epsilon(1 - \zeta)^d$ with $\epsilon \in \mathcal{O}^\times$.
- $\mathcal{N}_{L/\mathbb{Q}}(1 - \zeta) = l$.

Proof. (i)

$$\begin{aligned} \phi(x) &= \prod_{g \in (\mathbb{Z}/n\mathbb{Z})^\times} (X - \zeta^g) = \frac{\prod_{g \in (\mathbb{Z}/n\mathbb{Z})} (X - \zeta^g)}{\prod_{g \in (\mathbb{Z}/l^{k-1}\mathbb{Z})} (X - \zeta^{gl})} = \frac{X^{l^k} - 1}{X^{l^{k-1}} - 1} \\ &= X^{(l-1)l^{k-1}} + X^{(l-2)l^{k-1}} + \dots + X^{l^{k-1}} + 1 \end{aligned}$$

(ii) Follows from (i) with $X = 1$.

(iii) Observe

$$\epsilon_g := \frac{1 - \zeta^g}{1 - \zeta} = 1 + \zeta + \dots + \zeta^{g-1} \in \mathcal{O}$$

and

$$\frac{1}{\epsilon_g} = \frac{1 - \zeta}{1 - \zeta^g}$$

Since $g \not\equiv 0 \pmod{l}$, we can choose some $g' \in \mathbb{Z}$ with $gg' \equiv 1 \pmod{l^k}$. Hence

$$\frac{1}{\epsilon_g} = \frac{1 - \zeta^{gg'}}{1 - \zeta^g} = 1 + \zeta^g + \dots + (\zeta^g)^{g'-1} \in \mathcal{O}.$$

(iv) Follows from (ii) and (iii) with $\epsilon := \prod_{g \in (\mathbb{Z}/n\mathbb{Z})^\times} \epsilon_g$.

(v) Follows from (ii). □

Proposition 2.10.2. *Suppose again that $n = l^k$ with l prime. Set $\lambda := 1 - \zeta$. Then*

(i) $\Pi := (\lambda)$ is a prime ideal of local degree 1.

(ii) $l \cdot \mathcal{O} = \Pi^d$. In particular, $l\mathcal{O}$ is non-split.

Proof. 10.1 (iv) $\Rightarrow l\mathcal{O} = (\lambda)^d$. Let $l\mathcal{O} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$ be the decomposition into prime ideals. By Theorem 11, $d = e_1 f_1 + \cdots + e_r f_r$, where f_i = local degree of \mathfrak{p}_i , hence the above is already the prime decomposition and the local degree is 1. □

Remark 2.10.3. 10.1 and 10.2 generalize Lemma I.25.

Proposition 2.10.4. *Let $n = l^k$, l prime. The basis $1, \zeta, \zeta^2, \dots, \zeta^{d-1}$ of $\mathbb{Q}(\zeta)|\mathbb{Q}$ has the discriminant $d(1, \zeta, \dots, \zeta^{d-1}) = (-1)^a l^s$ with $s = l^{k-1}(kl - k - 1)$ and $a \in \{0, 1\}$.*

Proof. Step 1: Show $d(1, \dots, \zeta^{d-1}) = \pm \mathcal{N}(\phi'(\zeta))$.

Let $\zeta = \zeta_1, \zeta_2, \dots, \zeta_d$ be the conjugates of ζ .

$$\text{Remark 2.4} \Rightarrow d(1, \dots, \zeta^{d-1}) = d(\phi) = \prod_{1 \leq i < j \leq d} (\zeta_i - \zeta_j) = \pm \prod_{\substack{i,j=1 \\ i \neq j}}^d (\zeta_i - \zeta_j).$$

Observe

$$\phi(X) = \prod_{i=1}^d (X - \zeta_i) \Rightarrow \phi'(X) = \sum_{m=1}^d \prod_{\substack{i=1 \\ i \neq m}}^d (X - \zeta_i)$$

and therefore

$$\phi'(\zeta_j) = \prod_{\substack{i=1 \\ i \neq j}}^d (\zeta_j - \zeta_i).$$

Hence we have $d(1, \dots, \zeta^{d-1}) = \pm \prod_{j=1}^d \phi'(\zeta_j) = \pm \mathcal{N}(\phi'(\zeta))$.

Step 2: Calculate $\mathcal{N}(\phi'(\zeta))$ partially.

Observe: $(X^{l^{k-1}} - 1)\phi(X) = X^{l^k} - 1$. Differentiating yields $(X^{l^{k-1}} - 1)\phi'(X) + \phi(X)(\dots) = l^k X^{l^k-1}$. Plugging in $X = \zeta$ gives $(\zeta^{l^{k-1}} - 1)\phi'(\zeta) = l^k \zeta^{l^k-1} = l^k \zeta^{-1}$. Set $\xi := \zeta^{l^{k-1}}$. Then ξ is a root of unity of order l and we have $\mathcal{N}(\phi'(\zeta)) = \frac{(l^k)^d}{\mathcal{N}(\xi-1)}$.

Step 3: Calculate $\mathcal{N}(\xi - 1)$.

Lemma 10.1 $\Rightarrow \mathcal{N}_{\mathbb{Q}(\xi)|\mathbb{Q}}(\xi - 1) = l$. Hence $\mathcal{N}_{\mathbb{L}|\mathbb{Q}}(\xi - 1) = (\mathcal{N}_{\mathbb{Q}(\xi)|\mathbb{Q}}(\xi - 1))^{l^{k-1}} = l^{l^{k-1}}$.

Now combining all 3 steps yields: $d(1, \dots, \zeta^{d-1}) = \pm \frac{l^{kd}}{l^{l^{k-1}}} = \pm l^s$. □

Proposition 2.10.5. *Let n be some natural number. Then $1, \zeta, \dots, \zeta^{d-1}$ is an integral basis of \mathcal{O} .*

Proof. Step 1: Show the claim for $n = l^k$ with l prime.

- (1) Proposition 2.7 $\Rightarrow \pm l^s = d(1, \dots, \zeta^{d-1}) \Rightarrow l^s \cdot \mathcal{O} \subset \mathbb{Z} + \dots + \mathbb{Z}\zeta^{d-1} = \mathbb{Z}[\zeta] \subset \mathcal{O}$.
- (2) Consider $\lambda := (1 - \zeta)$. Proposition 10.2 \Rightarrow local degree of (λ) is 1 $\Rightarrow \mathcal{O}/(\lambda) = \mathbb{Z}/(l)$
 $\Rightarrow \mathcal{O} = \mathbb{Z} + \lambda\mathcal{O}$ (every element of $\mathcal{O} \bmod (\lambda)$ has an representant in \mathbb{Z})
 $\Rightarrow \mathcal{O} = \mathbb{Z}[\zeta] + \lambda\mathcal{O} \quad (*)$.
 Multiplying with λ yields $\lambda\mathcal{O} = \lambda\mathbb{Z}[\zeta] + \lambda^2\mathcal{O} \xrightarrow{(*)} \mathcal{O} = \mathbb{Z}[\zeta] + \lambda^2\mathcal{O} \Rightarrow \dots$
 $\Rightarrow \mathcal{O} = \mathbb{Z}[\zeta] + \lambda^t\mathcal{O} \quad \forall t \geq 1$.
- (3) Plug in $t = s\varphi(l^k)$ and by Proposition 10.2 $l\mathcal{O} = \lambda^{\varphi(l^k)}\mathcal{O}$:
 $\mathcal{O} = \mathbb{Z}[\zeta] + \lambda^{s\varphi(l^k)}\mathcal{O} = \mathbb{Z}[\zeta] + l^s\mathcal{O} = \mathbb{Z}[\zeta]$.

Step 2: Generalize to arbitrary $n = l_1^{k_1} \cdot \dots \cdot l_r^{k_r}$.

Consider $\zeta_i := \zeta^{n_i}$ with $n_i := \frac{n}{l_i^{k_i}}$, a primitive $l_i^{k_i}$ -th root of unity. Then $\text{ord}(\zeta_1), \dots, \text{ord}(\zeta_r)$ are relatively prime. Hence:

- (1) $\mathbb{Q}(\zeta) = \mathbb{Q}(\zeta_1) \cdot \dots \cdot \mathbb{Q}(\zeta_r)$.
- (2) $\mathbb{Q}(\zeta_1) \cdot \dots \cdot \mathbb{Q}(\zeta_{i-1}) \cap \mathbb{Q}(\zeta_i) = \mathbb{Q}$.
- (3) Apply Proposition 2.13 to $\mathbb{Q}(\zeta_1) \cdot \dots \cdot \mathbb{Q}(\zeta_r)$ successively. We obtain, that

$$\{\zeta_1^{j_1}, \dots, \zeta_r^{j_r} \mid 0 \leq j_i \leq d_i - 1\}$$

with $d_i = \varphi(l_i^{k_i})$ is an integral basis of $\mathbb{Q}(\zeta_1, \dots, \zeta_r) = \mathbb{Q}(\zeta)$.

Hence $\mathcal{O} = \mathbb{Z}[\zeta] = \mathbb{Z} + \mathbb{Z}\zeta + \dots + \mathbb{Z}\zeta^{d-1}$, since all ζ_i 's are powers of ζ .

□

Lemma 2.10.6. *Let p be a prime which does not divide n . Then we have in $\mathcal{O} = \mathbb{Z}[\zeta]$:*

$$p\mathcal{O} = \hat{\mathcal{P}}_1 \cdot \dots \cdot \hat{\mathcal{P}}_r$$

with $\hat{\mathcal{P}}_i$ different prime ideals in \mathcal{O} and the local degree of each $\hat{\mathcal{P}}_i$ is $f = \min(\{k \in \mathbb{N} \mid p^k \equiv 1 \pmod{n}\})$.

Proof. Idea: Use Proposition 8.15.

Observe: Since $\mathcal{O} = \mathbb{Z}[\zeta]$, Proposition 8.15 can be applied to all prime ideals of \mathcal{O} .

- Consider $f(X) = \phi_n(X)$.
- Take the image $h(X) := f(\bar{X}) \in \mathbb{F}_p[X]$ and decompose it as $h(X) = h_1^{e_1} \cdot \dots \cdot h_r^{e_r}$ into irreducible factors over \mathbb{F}_p .

Then we have: $p\mathcal{O} = \mathfrak{p}_1^{e_1} \cdot \dots \cdot \mathfrak{p}_r^{e_r}$ with prime ideals \mathfrak{p}_i of local degree $f_i := \deg h_i$.

Step 1: Show $e_1 = \dots = e_r = 1$.

Consider $q(X) := X^n - 1 \in \mathbb{F}_p[X]$. Since $p \nmid n$, $q'(X) = nX^{n-1}$ and q have no common zeroes in $\mathbb{F}_p \Rightarrow q(X)$ has no multiple zeroes in $\mathbb{F}_p \Rightarrow$ The same must be true for $h(x) \Rightarrow e_1 = \dots = e_r = 1$.

Step 2: Show: $f_1 = f_2 = \dots = f_r = k_0 := \min\{k \mid p^k \equiv 1 \pmod{n}\}$

Recall: $f(X) = \phi_n(X)$, $h(X) := \text{image in } \mathbb{F}_p[X] = h_1^{l_1}(X) \cdot \dots \cdot h_r^{l_r}(X)$

Consider the field $L := \mathbb{F}_{p^{k_0}}$ with p^{k_0} elements as field extension of \mathbb{F}_p . Write $p^{k_0} - 1 = nw$ with $w \in \mathbb{N}$.

Observe: $L^\times = \langle a \rangle$ with $\text{ord}(a) = nw \Rightarrow \bar{\zeta} = a^w$ is a primitive n -th root of unity and h decomposes into linear factors over L .

Furthermore: $L = \mathbb{F}_p(\bar{\zeta})$ by minimality of k_0 , since $\#\mathbb{F}_p[\bar{\zeta}] = p^M$ for some M and $\text{ord}(\bar{\zeta}) = n$ divides $p^M - 1 \Rightarrow k_0 = M$.

Let $\bar{f}_1(X)$ be the minimal polynomial of $\bar{\zeta}$ over $\mathbb{F}_p \Rightarrow$

- \bar{f}_1 is an irreducible divisor of $h(X) \Rightarrow \text{w.l.o.g. } \bar{f}_1 = h_1$
- $f_1 = \deg(h_1) = \deg(\bar{f}_1) = [L : \mathbb{F}_p] = k_0 \Rightarrow f_1 = k_0$

□

Proposition 2.10.7 (CHARACTERISATION OF PRIME IDEALS). *Let $n = p_1^{k_1} \cdot \dots \cdot p_r^{k_r}$ be the prime decomposition of n and p some arbitrary prime number.*

Then $p\mathcal{O} = (\hat{\mathcal{P}}_1 \cdot \dots \cdot \hat{\mathcal{P}}_r)^{e_p}$ with $e_p = \varphi(p^{k_p})$ is the factorisation into prime ideals and each prime ideal $\hat{\mathcal{P}}_i$ is of local degree $f_p := \min\{k \in \mathbb{N} \mid p^k \equiv 1 \pmod{\frac{n}{p^{k_p}}}\}$

Proof. Again: Use Prop. 8.15 which applies to all prime ideals in \mathcal{O}

$\Rightarrow \phi_n(X) \in \mathbb{Z}[X]$ min. polynomial of $\zeta \Rightarrow \bar{\phi}_n(X) \in \mathbb{F}_p[X]$ image in $\mathbb{F}_p[X]$.

Denote $n = mp^a$ with $\gcd(p, m) = 1$, i.e. $a = k_p$.

Remember $U_m^\times = \{\text{primitive } m\text{-th roots of unity}\} \cong ((\mathbb{Z}/m\mathbb{Z})^\times, \cdot)$ ($\zeta^k \leftrightarrow k$).

Use the isomorphism:

$$\begin{aligned} U_m^\times \times U_{p^a}^\times &\rightarrow U_n, (\xi, \eta) \mapsto \xi \cdot \eta \\ \Rightarrow \phi_n(X) &= \prod_{g \in (\mathbb{Z}/n\mathbb{Z})^\times} (X - \zeta^g) = \prod_{\substack{\xi \in U_m^\times, \\ \eta \in U_{p^a}^\times}} (X - \xi\eta) \end{aligned}$$

Step 1: Show that $\phi_n(X) \equiv \phi_m(X)^{\varphi(p^a)} \pmod{p}$

(1) Observe: $X^{p^a} - 1 \equiv (X - 1)^{p^a} \pmod{p}$. For prime ideal $\hat{\mathcal{P}}$ over (p) :

$$X^{p^a} - 1 \equiv (X - 1)^{p^a} \pmod{\hat{\mathcal{P}}}$$

Let $\eta_1, \dots, \eta_{\varphi(p^a)}$ be the primitive p^a -th roots of unity.

$$0 = \eta_j^{p^a} - 1 \equiv (\eta_j - 1)^{p^a} \pmod{\hat{\mathcal{P}}} \Rightarrow \eta_j \equiv 1 \pmod{\hat{\mathcal{P}}}.$$

(2)

$$\begin{aligned}\phi_n(X) &= \prod_{\substack{\xi \in U_m^\times, \\ \eta \in U_{p^a}^\times}} (X - \xi\eta) = \prod_{g \in (\mathbb{Z}/m\mathbb{Z})^\times} (X - \xi)^{\varphi(p^a)} = \phi_m^{\varphi(p^a)} \pmod{\hat{\mathcal{P}}} \\ \Rightarrow \phi_n(X) &\equiv \phi_m(X)^{\varphi(p^a)} \pmod{p}\end{aligned}$$

Step 2: Use Lemma 10.5:

Proof of Lemma 10.5 \Rightarrow exponents of $\phi_m(X) \pmod{p}$ are all 1 \Rightarrow all exponents of $\phi_n(X) \pmod{p}$ are $\varphi(p^a)$. The local degree of the prime factors are by Lemma 10.5 $f = \min\{k \in \mathbb{N} \mid p^k \equiv 1 \pmod{\underbrace{m}_{=n/p^a}}\}$. \square

Corollary 2.10.8. *i) p is ramified in $\mathbb{Q}(\zeta) \iff n \equiv 0 \pmod{p}$ and we have not $p = 2 = \gcd(4, n)$.*

ii) $p \neq 2$. Then p is totally split $\iff p \equiv 1 \pmod{n}$.

Proof. i) Prop. 10.6 $\Rightarrow p$ is unramified $\iff e = 1 \xleftrightarrow{\text{Prop 10.6}} \varphi(p^{k_p}) = 1 \iff k_p = 0$ or $p^{k_p} - p^{k_p-1} = p^{k_p-1}(p - 1) = 1 \iff k_p = 0$ or $(p = 2 \text{ and } 2 = \gcd(4, n))$.

ii) $p \neq 2 : e = 1 \iff k_p = 0 \iff p \nmid n$
 $f = 1 \iff \min\{k \mid p^k \equiv 1 \pmod{\frac{n}{p^k}}\} = 1 \iff p \equiv 1 \pmod{n}$. \square

Remark 2.10.9. We have now in particular proved I.2.2.

3. Fermat's theorem for regular primes

3.1. The proof using a lemma of Kummer

Setting: K -number field, \mathcal{O} = ring of integers

Recall: \mathcal{J}_K := group of fractional ideals, \mathcal{P}_K = subgroup of principal ideals, $\text{Cl}_K = \mathcal{J}_K / \mathcal{P}_K$, $h_K = \# \text{Cl}_K$

Definition 3.1.1. A prime $p \in \mathbb{N}$ is regular : $\iff h_K$ is not divisible by p where $K = \mathbb{Q}(\zeta_p)$.

Remark 3.1.2. Suppose p regular. Then we have for each ideal I in \mathcal{O} = ring of integers in K :

If I^p is a principal ideal, then I is a principal ideal.

Proof. $p \nmid h_K \Rightarrow$ No element of Cl_K has order p . □

Recall: (Lemma I.2.11) $x, y \in \mathbb{Z}$, $\gcd(x, y) = 1$, $x + y \not\equiv 0 \pmod{p}$
 $\Rightarrow x + \zeta^i y$ and $x + \zeta^j y$ are coprime, if $i \not\equiv j \pmod{p}$.

Theorem 12. If p is a regular prime, then Fermat's theorem holds, i.e.

$$x^p + y^p = z^p \text{ in } \mathbb{Z} \Rightarrow xyz = 0.$$

Recall:

$$(1) \quad x^p + y^p = (x + y)(x + \zeta y) \cdots (x + \zeta^{p-1} y) \text{ in } \mathbb{Z}[\zeta].$$

$$(2) \quad \lambda = 1 - \zeta \text{ is prime in } \mathcal{O} = \mathbb{Z}[\zeta]$$

$$(3) \quad 1 - \zeta \sim 1 - \zeta^g \text{ for all } g \not\equiv 0 \pmod{p}$$

Lemma 3.1.3. Suppose that $x, y \in \mathcal{O}$ with x, y are coprime and p does not divide y .

Then we have: either the ideals $(x + \zeta^i y)$ (with $i \in \{0, \dots, p-1\}$) are relatively prime or they all have $(1 - \zeta)$ as a common factor and the ideals $(\frac{x + \zeta^i y}{1 - \zeta})$ (with $i \in \{0, \dots, p-1\}$) are relatively prime.

Proof. Use from the proof of Lemma I.2.11: Let $0 \leq j < i \leq p-1$. $A := (x + \zeta \cdot y, x + \zeta^j \cdot y) \Rightarrow$

$$(1) \quad (1 - \zeta) \cdot y \in A$$

$$(2) (1 - \zeta) \cdot x \in A$$

$$(3) 1 - \zeta \in A \text{ and thus } p \in A$$

$$(4) x + y \in A$$

Suppose q is a prime ideal with $q|(x + \zeta^i \cdot y)$ and $q|(x + \zeta^j \cdot y)$.

Hence $q \supseteq A \stackrel{(3)}{\ni} 1 - \zeta \stackrel{1-\zeta \text{ prime}}{\implies} q = (1 - \zeta)$.

Hence $q = (1 - \zeta)$ is the only prime ideal which possibly divides $(x + \zeta^i \cdot y), (x + \zeta^j \cdot y)$.

Show: If $q = (1 - \zeta)$ divides $(x + \zeta^i \cdot y)$, then it divides $(x + \zeta^{i+1} \cdot y)$.

This follows from the following calculation: $x + \zeta^{i+1} \cdot y = x + \zeta^i \cdot y + \zeta^i(\zeta - 1) \cdot y$

Finally show: If $(1 - \zeta)$ divides $x + \zeta^i \cdot y$, then the $(\frac{x+\zeta^i \cdot y}{1-\zeta})$ and $(\frac{x+\zeta^j \cdot y}{1-\zeta})$ are coprime for $0 \leq j < i \leq p-1$.

Recall: $p \nmid y \Rightarrow 1 - \zeta \nmid y$

Proof: $x + \zeta^i \cdot y - (x + \zeta^j \cdot y) = \zeta^j \cdot y \underbrace{(\zeta^{i-j} - 1)}_{\sim (\zeta-1)} \Rightarrow \frac{x+\zeta^i \cdot y}{1-\zeta} - \frac{x+\zeta^j \cdot y}{1-\zeta} \sim y$.

But $(1 - \zeta) \nmid y \Rightarrow$ Claim. □

Proposition 3.1.4 (“First Case”). Suppose p is a regular prime with $p \geq 5$ such that $x^p + y^p = z^p$ and $p \nmid xyz$ with $x, y, z \in \mathbb{Z}$. Then $xyz = 0$.

Proof. Without loss of generality we may assume that x, y, z are coprime. Proceed as in the proof of Theorem 1:

- $z^p = x^p + y^p = (x + y)(x + \zeta y) \cdots (x + \zeta^{p-1}y)$
- Since $p \nmid z$ we have $x + y \equiv x^p + y^p = z^p \equiv z \not\equiv 0 \pmod p$ by little Fermat’s theorem such that $p \nmid x + y$.
- Lemma 2.11 implies that $(x + y), (x + \zeta y), \dots, (x + \zeta^{p-1}y)$ are pairwise coprime such that the first bullet point together with the regularity of p and Remark 1.2 yields $(x + \zeta^i y) = (\alpha_i)^p$ for some $\alpha_i \in \mathcal{O}$. Thus $x + \zeta^i y = \varepsilon_i \alpha_i^p$ with $\varepsilon_i \in \mathcal{O}^\times$.

Now continue as in the proof of Theorem 1. □

Recall (Example 1.2.8). If $\alpha \in \mathcal{O}$ then $\alpha = a_0 + a_1 \zeta + \cdots + a_{p-2} \zeta^{p-2}$ such that

$$\alpha^p \equiv \underbrace{a_0^p + a_1^p + \cdots + a_{p-2}^p}_{=a \in \mathbb{Z}} \pmod p.$$

Lemma 3.1.5 (Kummer’s Lemma II). Suppose p is a regular prime. If $u \in \mathcal{O}^\times$ such that $u \equiv a \pmod p$ for some $a \in \mathbb{Z}$ then there is an $\alpha \in \mathcal{O}^\times$ such that $u = \alpha^p$.

The proof is hard and needs more theory.

Remark 3.1.6. $1, 1 - \zeta, \dots, (1 - \zeta)^{p-2}$ is an integral basis of $\mathcal{O} = \mathbb{Z}[\zeta]$.

Proof. $1, \zeta, \dots, \zeta^{p-2}$ is an integral basis by Proposition 2.10.4. Furthermore,

$$\zeta^i = (1 - (1 - \zeta))^i = \sum_{k=0}^i \binom{k}{i} (-1)^{i-k} (1 - \zeta)^{i-k}$$

and $1 - \zeta$ has minimal polynomial of degree lesser equal than $p - 1$. \square

Lemma 3.1.7. *If $\alpha \in \mathcal{O} \setminus (1 - \zeta)$ then there exist $a \in \mathbb{Z}$ and $l \in \mathbb{N}_0$ such that*

$$\zeta^l \alpha \equiv a \pmod{(1 - \zeta)^2}.$$

Proof. We do the proof in multiple steps:

(1) Since $1, 1 - \zeta, \dots, (1 - \zeta)^{p-2}$ is an integral basis of \mathcal{O} we have

$$\alpha \equiv a_0 1 + a_1 (1 - \zeta) \pmod{(1 - \zeta)^2}$$

with $a_0, a_1 \in \mathbb{Z}$.

(2) Since $1 - \zeta \nmid \alpha$ we have $1 - \zeta \nmid a_0$ such that $p \nmid a_0$ and hence there is $l \in \mathbb{Z}$ with $a_0 l \equiv a_1 \pmod{p}$.

(3) Since $\zeta = 1 - (1 - \zeta)$ we have

$$\zeta^l \equiv 1 - l(1 - \zeta) \pmod{(1 - \zeta)^2}.$$

(4) By (1), (2) and (3) we conclude

$$\begin{aligned} \zeta^l \alpha &\equiv (1 - l(1 - \zeta)) (a_0 + a_1 (1 - \zeta)) \\ &\equiv a_0 + (a_1 - la_0) (1 - \zeta) \\ &\equiv a_0 \pmod{(1 - \zeta)^2}. \end{aligned}$$

\square

Proposition 3.1.8 (“Second case”). *Suppose p is a regular prime with $p \geq 5$ such that $x^p + y^p = z^p$ and $p \mid xyz$ with $x, y, z \in \mathbb{Z}$. Then $xyz = 0$.*

Proof. Without loss of generality x, y, z are pairwise coprime. By changing the role of x, y and z and possibly replacing x by $-x$, y by $-y$ and z by $-z$ we can furthermore assume that $p \nmid z$, $p \nmid x$ and $p \nmid y$. Then, by 2.10.1,

$$z = p^m z_0 = \varepsilon (1 - \zeta)^{(p-1)m} z_0$$

with $z_0 \in \mathbb{Z}$, $m \geq 1$, $\gcd(z_0, p) = 1$ and $\varepsilon \in \mathcal{O}^\times$ such that

$$x^p + y^p = \varepsilon^p (1 - \zeta)^{(p-1)mp} z_0^p.$$

By assumption:

- x, y and z_0 are pairwise coprime since x, y and z are pairwise coprime.
- $1 - \zeta$ and z_0 are coprime since p and z are coprime.
- x and $1 - \zeta$ are coprime since $p \nmid x$. The same holds for y and $1 - \zeta$.

Hence the following Lemma 1.9 yields $xyz_0 = 0$ such that $xyz = 0$ as claimed. \square

Lemma 3.1.9. *Suppose p is a regular prime with $p \geq 5$, $x, y, z_0 \in \mathcal{O}$, $\varepsilon \in \mathcal{O}^\times$ and $x, y, z_0, 1 - \zeta$ are pairwise coprime. If $x^p + y^p = \varepsilon(1 - \zeta)^{kp} z_0^p$ with $k \in \mathbb{N}$, then $xyz_0 = 0$.*

Proof. Assume that there are x, y, z_0 as in the lemma with $xyz_0 \neq 0$. We may assume that k is minimal.

“**Step 1:**” Show that $(1 - \zeta)^2 | x + y$.

(1) By assumption we have

$$\varepsilon(1 - \zeta)^{kp} z_0^p = (x + y)(x + \zeta y) \cdots (x + \zeta^{p-1} y) \quad (*)$$

such that, since $1 - \zeta$ is prime, there is $i \in \{0, \dots, p-1\}$ with $1 - \zeta | x + \zeta^i y$. Hence $1 - \zeta$ divides all $x + \zeta^i y$ by Lemma 1.3, in particular $x + y$.

(2) By Lemma 1.7 there are $a, b \in \mathbb{Z}$ and $l, j \in \mathbb{N}_0$ such that

$$\zeta^l x \equiv a \pmod{(1 - \zeta)^2} \quad \text{and} \quad \zeta^j y \equiv b \pmod{(1 - \zeta)^2}.$$

(3) We may replace x by $x\zeta^l$ and y by $y\zeta^j$ and thus can assume that $x \equiv a, y \equiv b \pmod{(1 - \zeta)^2}$ with $a, b \in \mathbb{Z}$.

(4) $1 - \zeta | x + y$ implies $1 - \zeta | a + b$ such that $(1 - \zeta)^{p-1} | a + b$ (since $a + b \in \mathbb{Z}$ we have also $p | a + b$) and hence $(1 - \zeta)^2 | x + y$. In particular, $k \geq 2$.

“**Step 2:**” Show that $(1 - \zeta)^{(k-1)p+1} | x + y$.

Since the quotients $\frac{x + \zeta^i y}{1 - \zeta}$ are pairwise coprime, all “extra powers” of $1 - \zeta$ have to divide $x + y$. Thus,

$$(1 - \zeta)^{kp-(p-1)} | x + y.$$

Furthermore:

$$1 - \zeta \nmid \frac{x + y}{(1 - \zeta)^{kp-(p-1)}}$$

“**Step 3:**” Show that $\frac{x + \zeta^i y}{1 - \zeta}$ is associated to a p -power.

From (*) we obtain

$$((1 - \zeta)^{k-1} z_0)^p = \prod_{i=0}^{p-1} \left(\frac{x + \zeta^i y}{1 - \zeta} \right).$$

Since the ideals on the right side are pairwise coprime, $\left(\frac{x+\zeta^i y}{1-\zeta}\right)$ is a p -th power. Thus Remark 1.2 yields

$$\frac{x + \zeta^i y}{1 - \zeta} = \varepsilon_i \alpha_i^p$$

with $\alpha_i \in \mathcal{O}$ and $\varepsilon \in \mathcal{O}^\times$. Furthermore, the α_i are pairwise coprime.

“Step 4:” Find $\varepsilon', \eta \in \mathcal{O}^\times$ and $\beta \in \mathcal{O}$ with $\varepsilon'(1 - \zeta)^{(k-1)p} \beta^p = -\alpha_1^p + \eta \alpha_{-1}^p$.

By Step 2, $(1 - \zeta)^{k-1}$ divides α_0 . More precisely, $\alpha_0 = (1 - \zeta)^{k-1} \beta$ with $\beta \in \mathcal{O}$ and $1 - \zeta, \beta$ coprime. Do some ugly calculation:

$$y = \frac{x + y - (x + \zeta y)}{1 - \zeta} = \varepsilon_0 \alpha_0^p - \varepsilon_1 \alpha_1^p = \varepsilon_0 (1 - \zeta)^{(k-1)p} \beta^p - \varepsilon_1 \alpha_1^p \quad (\text{A})$$

$$y = \frac{(x + \zeta^{-1} y) - (x + y)}{\zeta^{-1}(1 - \zeta)} = \zeta \varepsilon_{-1} \alpha_{-1}^p - \zeta \varepsilon_0 \alpha_0^p = \zeta \varepsilon_{-1} \alpha_{-1}^p - \zeta \varepsilon_0 (1 - \zeta)^{(k-1)p} \beta^p \quad (\text{B})$$

Then (B) – (A) yields

$$0 = \zeta \varepsilon_{-1} \alpha_{-1}^p + \varepsilon_1 \alpha_1^p + \varepsilon_0 (1 - \zeta)^{p(k-1)} \beta^p (-\zeta - 1).$$

Now define

$$\varepsilon' = \frac{(1 + \zeta) \varepsilon_0}{-\varepsilon_1} \quad \text{and} \quad \eta = \frac{\zeta \varepsilon_{-1}}{-\varepsilon_1}$$

to obtain

$$\varepsilon' (1 - \zeta)^{p(k-1)} \beta^p = \eta \alpha_{-1}^p - \alpha_1^p. \quad (**)$$

“Step 5:” Show that η is a p -th power.

By (**) we have $0 \equiv \eta \alpha_{-1}^p - \alpha_1^p \pmod{p}$ such that Example 1.2.8 ascertains the existence of $a_{-1}, a_1 \in \mathbb{Z}$ with $\alpha_{-1}^p \equiv a_{-1}, \alpha_1^p \equiv a_1 \pmod{p}$.

“Step 6:” Find a smaller solution to (\star) :

$$x' := \alpha_{-1}, y' := v \eta_1, z_0 := \beta.$$

With $(\star\star) : \varepsilon' (1 - \zeta)^{p(k-1)} \cdot z_0^p = y'^p + x'^p$ is a smaller solution, a contradiction. \square

4. Geometric aspects

4.1. Localisation

Recall: Here all rings are commutative with 1.

Reminder 4.1.1. (i) Let R be a ring and $S \subseteq R \setminus \{0\}$ be a multiplicative system, i.e.

- (1) $a, b \in S \Rightarrow a \cdot b \in S$ and
- (2) $1 \in S$.

$$R \cdot S^{-1} := \{(a, s) \mid a \in R, s \in S\} / \sim$$

with $(a, s) \sim (a', s')$ if there is $t \in S : t(as' - a's) = 0$.

Denote $\frac{a}{s} := [(a, s)] / \sim$ equivalence class of (a, s) .

RS^{-1} becomes a ring with

$$\begin{aligned} \frac{a_1}{s_1} + \frac{a_2}{s_2} &= \frac{a_1 s_2 + a_2 s_1}{s_1 s_2}, \\ \frac{a_1}{s_1} \cdot \frac{a_2}{s_2} &= \frac{a_1 a_2}{s_1 s_2} \end{aligned}$$

RS^{-1} is called localisation of R by S .

- (ii) The map

$$j_S : R \rightarrow RS^{-1}, \quad r \mapsto \frac{r}{1}$$

is a ring homomorphism with $j_S(S) \subseteq (RS^{-1})^\times$. $\ker(j_S) = \{r \in R \mid \exists a \in S \text{ with } ar = 0\}$. In particular: R is an integral domain $\Rightarrow j_S$ is an embedding and $\frac{a}{b} = \frac{a'}{b'}$ is equivalent to $ab' = a'b$.

Furthermore: R is an integral domain $\Rightarrow RS^{-1} \subseteq \text{Quot}(R)$, $\frac{a}{b} \mapsto \frac{a}{b}$.

- (iii) Localisation has the following universal property: $f : R \rightarrow R'$ a ring homomorphism with $f(S) \subseteq (R')^\times$ then there exists a unique ringhomomorphism $g : RS^{-1} \rightarrow R'$ with $f = g \circ j_S$

$$\begin{array}{ccc} R & \xrightarrow{j_S} & RS^{-1} \\ & \searrow f & \swarrow \exists! g \\ & R' & \end{array}$$

Example 4.1.2. (i) R integral domain, $S = R \setminus 0 \Rightarrow RS^{-1} = \text{Quot}(R)$

- (ii) p prime ideal in R , $S := R \setminus p \Rightarrow R_p := RS^{-1}$.

Proposition 4.1.3 (Description of prime ideals in localisations). *We have the following bijection:*

$$\begin{aligned} \{p \in \operatorname{Spec}(R) \mid p \subseteq R \setminus S\} &\leftrightarrow \{q \in \operatorname{Spec}(RS^{-1})\} \\ \phi : p &\mapsto pS^{-1} = \left\{ \frac{a}{s} \mid a \in p, s \in S \right\} \\ j_S^{-1}(q) &\leftarrow q : \psi \end{aligned}$$

Proof. (1) $\frac{a}{s} = \frac{a'}{s'}$, then $a \in p \iff a' \in p$:

Suppose $a \in p, a' \in R, s, s' \in S$ and $\frac{a}{s} = \frac{a'}{s'} \Rightarrow \exists t \in S : \underbrace{t}_{\notin p} (as' - a's) = 0 \in p$

So $as' - a's \in p$, hence $a's \in p$ and $a' \in p$.

(2) ϕ is well defined, i.e. pS^{-1} is a prime ideal: clear.

(3) ψ is well-defined by Prop. II.8.16.

(4) $\psi \circ \phi(p) = j_S^{-1}(pS^{-1}) = p$:
 $r \in j_S^{-1}(pS^{-1}) \iff j_S(r) \in pS^{-1} \iff \frac{r}{1} \in pS^{-1} \iff r \in p$

(5) $\phi \circ \psi(q) = \psi(j_S^{-1}(q)) = j_S^{-1}(q)S^{-1} = q$:
 $\frac{r}{s} \in j_S^{-1}(q)S^{-1} \iff r \in j_S^{-1}(q) \iff j_S(r) \in q \iff \frac{r}{1} \in q \iff \frac{r}{s} \in q$

□

Definition 4.1.4 (and Prop., lokaler Ring). A ring is a local ring if R has one of the following equivalent properties:

- (i) R has a unique maximal ideal m .
- (ii) $R \setminus R^\times$ is an ideal.
- (iii) $\forall x \in R : x \in R^\times$ or $1 - x \in R^\times$.

In particular we have: If R is a local ring then $m = R \setminus R^\times$ is the unique maximal ideal of R .

Proof. (i) \Rightarrow (ii) : Show that $R = R^\times \cup m$:

(1) $R = R^\times \cup m : a \in R \setminus m$. Hence (a) is not contained in m . So $(a) = R$ and hence $a \in R^\times$.

(2) $R^\times \cap m = \emptyset : a \in R^\times$, so $a \notin m$ since $m \neq R = (a)$. It follows that $m = R \setminus R^\times$ and thus $R \setminus R^\times$ is an ideal.

(ii) \Rightarrow (iii) : Suppose x and $1 - x \in R \setminus R^\times$. Hence $1 = x + (1 - x) \in R \setminus R^\times$.

(iii) \Rightarrow (i) : Suppose that m and m' are two different maximal ideals. Let $a \in m' \setminus m$. Since m is maximal we have $(m, a) = R \Rightarrow \exists b \in m, r \in R$ with $1 = b + ra$. We know $ra \in m'$, hence $ra \notin R^\times$ and by assumption (iii) $\Rightarrow b = 1 - ra \in R^\times$ to $b \in m$. □

Proposition 4.1.5 (localisations by prime ideals are local). *Let R be a ring and $p \in \text{Spec}(R)$. Then R_p is a local ring with maximal ideal pS^{-1} where $S = R \setminus p$.*

Proof. We show that $R_p = R_p^\times \cup pS^{-1}$. Hence $R_p \setminus R_p^\times = pS^{-1}$ is an ideal. Thus R_p is a local ring.

$$(1) R_p = pS^{-1} \cup R_p^\times :$$

Let $a \in R, s \in S = R \setminus p$. Suppose $\frac{a}{s} \notin pS^{-1}$, i.e. $a \notin p$. So $\frac{s}{a} \in R_p$ and $\frac{a}{s} \frac{s}{a} = 1$. Hence $\frac{a}{s} \in R_p^\times$.

$$(2) pS^{-1} \cap R_p^\times = \emptyset :$$

Suppose that $\frac{a}{s} \in R_p^\times$ (with $a \in R, s \in S$) $\Rightarrow \exists a' \in R, s' \in S : \frac{a}{s} \frac{a'}{s'} = 1 \Rightarrow \exists t \in S$ with $t(aa' - ss') = 0 \in p$. Since $t \notin p$ we have $aa' - \underbrace{ss'}_{\notin p} \in p$, so $aa' \notin p$. Since $a \notin p$

it follows $\frac{a}{s} \notin pS^{-1}$.

□

Proposition 4.1.6 (being Dedekind is stable under localisation). *Let \mathcal{O} be a Dedekind domain, $S \subseteq \mathcal{O} \setminus \{0\}$ multiplicative system, then $\mathcal{O}S^{-1}$ is a Dedekind domain.*

Proof. \mathcal{O} is an integral domain, so $\mathcal{O} \subseteq \mathcal{O}S^{-1} \subseteq \text{Quot}(\mathcal{O})$.

$$(1) \mathcal{O}S^{-1} \text{ is an integral domain, since } \mathcal{O}S^{-1} \subseteq \text{Quot}(\mathcal{O}).$$

$$(2) \text{ Show that } \mathcal{O}S^{-1} \text{ is Noetherian, i.e. each ideal is finitely generated:}$$

Let q be an ideal in $\mathcal{O}S^{-1}$ and $p := j_S^{-1}(q)$.

Prop 1.3 says that $q = pS^{-1}$. \mathcal{O} is a Dedekind domain, hence p is finitely generated i.e. $p = (a_1, \dots, a_n) \Rightarrow q = pS^{-1} = (\frac{a_1}{1}, \dots, \frac{a_n}{1})$ is finitely generated.

$$(3) \text{ Show that } \mathcal{O}S^{-1} \text{ is integrally closed:}$$

Suppose $x \in \text{Quot}(\mathcal{O}S^{-1}) = \text{Quot}(\mathcal{O})$ with $x^n + \frac{r_{n-1}}{s_{n-1}}x^{n-1} + \dots + \frac{r_0}{s_0} = 0$ and $r_0, \dots, r_{n-1} \in \mathcal{O}, s_0, \dots, s_{n-1} \in S$.

Let $s := s_0 \cdot \dots \cdot s_{n-1} \in S$, then

$$(sx)^n + \underbrace{s \frac{r_{n-1}}{s_{n-1}}}_{\in \mathcal{O}} (sx)^{n-1} + \dots + \underbrace{s^n \frac{r_0}{s_0}}_{\in \mathcal{O}} = 0$$

$\Rightarrow sx$ is integral over \mathcal{O} and $\hat{x} = sx \in \mathcal{O}$, since \mathcal{O} is integrally closed.

$\Rightarrow x = \frac{\hat{x}}{s} \in \mathcal{O}S^{-1}$. Thus $\mathcal{O}S^{-1}$ is integrally closed.

$$(4) \text{ Prop 1.3 implies that every prime ideal } q \neq 0 \text{ in } \mathcal{O}S^{-1} \text{ is maximal.}$$

□

Definition 4.1.7 („diskreter Bewertungsring“). A ring is called discrete valuation ring (DVR) if

- R is a principal ideal domain and
- R has a (unique) maximal ideal $\mathfrak{m} = (\pi) \neq 0$.

In particular

- R is an integral domain
- R is not a field.

Remark 4.1.8. Let R be a DVR with maximal ideal $\mathfrak{m} = (\pi)$.

- (i) π is prime and any prime π' is associated to π .
- (ii) Any $r \in R \setminus \{0\}$ can be written as $r = \varepsilon \pi^k$ with $\varepsilon \in R^\times$ and $k \in \mathbb{N}$ depending only on r .

Proof. “(i)” Since R is a PID it is also a UFD. Since π is prime, if $\pi = ab$ with $a, b \in R$ then $(\pi) \subset (a)$ such that $(a) = R$ or $(a) = (\pi)$. Hence $a \in R^\times$ or $a = \varepsilon \pi$ with $\varepsilon \in R^\times$. Thus one of a, b must be a unit such that π is prime as a irreducible element of a UFD.

If π' is another prime, then $(\pi') \subset (\pi)$ and we can write $\pi' = a\pi$ with $a \in R$. Since π' is prime, a must be a unit and $\pi' \sim \pi$ follows as claimed.

“(ii)” Since r has a unique prime factorization (up to a unit) the claim follows from (i). \square

Proposition 4.1.9. R is a DVR if and only if R is a local Dedekind domain and not a field.

Proof. “ \Rightarrow ” R is local since it is a DVR, noetherian since it is a PID and integrally closed since it is a UFD. Furthermore, by Remark 1.8 every prime ideal is maximal. Also, since for the maximal ideal $\mathfrak{m} = (\pi)$ we have $\mathfrak{m} \neq 0$, R is not a field.

“ \Leftarrow ” R has a unique maximal ideal \mathfrak{m} since it is a local ring and $\mathfrak{m} \neq 0$ since R is not a field. We need to show that R is a PID:

- (1) Show that \mathfrak{m} is a principal ideal:

Since R is a Dedekind domain it holds that $\mathfrak{m} \neq \mathfrak{m}^2$. Let $\pi \in \mathfrak{m} \setminus \mathfrak{m}^2$ and observe that \mathfrak{m} is the only non-zero prime ideal. Thus, $(\pi) = \mathfrak{m}^k$ and $k = 1$ since $\pi \notin \mathfrak{m}^2$.

- (2) Any ideal \mathfrak{a} is a principal ideal since $\mathfrak{a} = \mathfrak{m}^k = (\pi^k)$.

\square

Definition 4.1.10. Let K be a field. A function $v: K \rightarrow \mathbb{Z} \cup \{\infty\}$ is called **discrete valuation** if for all $x, y \in K$ the following conditions hold:

- (i) $v(xy) = v(x) + v(y)$
- (ii) $v(x + y) \geq \min \{v(x), v(y)\}$

(iii) $v(x) = \infty$ if and only if $x = 0$

(iv) $v \not\equiv 0$ and $v \not\equiv \infty$

Example 4.1.11. Let $p \in \mathbb{Z}$ be prime and $K = \mathbb{Q}$. Define $v_p: \mathbb{Q} \rightarrow \mathbb{Z} \cup \{\infty\}$ by:

(1) If $z \in \mathbb{Z} \setminus \{0\}$ with $z = p^k b$, where $\gcd(p, b) = 1$, then $v_p(z) = k$.

(2) If $x \in \mathbb{Q} \setminus \{0\}$ with $x = \frac{a}{b}$, where $a, b \in \mathbb{Z}$, then $v_p(x) = v_p(a) - v_p(b)$.

Then v_p is a discrete valuation.

Proposition 4.1.12. (i) Let $v: K \rightarrow \mathbb{Z} \cup \{\infty\}$ be a discrete valuation. Then:

- $v(1) = v(-1) = 0$
- $v\left(\frac{a}{b}\right) = v(a) - v(b)$
- $\mathcal{O}_K = \{x \in K; v(x) \geq 0\}$ is a ring with units $\mathcal{O}_K^\times = \{x \in K; v(x) = 0\}$

(ii) The ring \mathcal{O}_K from (i) is a DVR.

(iii) Conversely, if R is a discrete valuation ring, then there exists a discrete valuation $v: K \rightarrow \mathbb{Z} \cup \{\infty\}$ with $K = \text{Quot}(R)$ such that $R = \mathcal{O}_K$ for this valuation.

Proof. “(i)” We have

$$v(1) = v(1 \cdot 1) = v(1) + v(1)$$

such that $v(1) = 0$. Furthermore,

$$0 = v(1) = v((-1)(-1)) = 2v(-1)$$

and hence also $v(-1) = 0$. Next,

$$v(a) = v\left(\frac{a}{b} \cdot b\right) = v\left(\frac{a}{b}\right) + v(b)$$

such that $v\left(\frac{a}{b}\right) = v(a) - v(b)$. Now, if $v(x) = 0$ then $v\left(\frac{1}{x}\right) = v(1) - v(x) = -v(x) = 0$ and hence $\frac{1}{x} \in \mathcal{O}_K$, i.e., $x \in \mathcal{O}_K^\times$. Finally, if $x \in \mathcal{O}_K^\times$ then there is a $y \in \mathcal{O}_K$ with $xy = 1$ such that

$$0 = v(1) = v(xy) = \underbrace{v(x)}_{\geq 0} + \underbrace{v(y)}_{\geq 0}$$

and thus $v(x) = 0$.

“(ii)”

- \mathcal{O}_K is an integral domain since $\mathcal{O}_K \subset K$.
- Show that \mathcal{O}_K is a PID:

Let \mathfrak{a} be an ideal in \mathcal{O}_K . Choose $a \in \mathfrak{a}$ with $v(a)$ minimal. For $b \in \mathfrak{a}$ we have $v(b) \geq v(a)$ and hence $v\left(\frac{b}{a}\right) \geq 0$ such that $\frac{b}{a} \in \mathcal{O}_K$. Thus, $b = \frac{b}{a} \cdot a$ with $\frac{b}{a} \in \mathcal{O}_K$ such that $b \in (a)$ and hence $\mathfrak{a} = (a)$.

- Show that \mathcal{O}_K has a unique maximal ideal:

Define $\mathfrak{m} = \{a \in \mathcal{O}_K; v(a) > 0\}$. Observe that \mathfrak{m} is an ideal in \mathcal{O}_K and $\mathfrak{m} = \mathcal{O}_K \setminus \mathcal{O}_K^\times$ such that \mathfrak{m} is a unique maximal ideal in \mathcal{O}_K .

- \mathcal{O}_K is not a field since the valuation V is not allowed to be 0 everywhere on K^\times .

“(iii)”

- (1) Use Remark 1.8 to define $v: R \setminus \{0\} \rightarrow \mathbb{N}_0$, $r = \varepsilon \pi^k \mapsto k$. Observe that $v(ab) = v(a) + v(b)$, $v(a+b) \geq \min\{v(a), v(b)\}$ and define $v(0) = \infty$.
- (2) Define $v: K = \text{Quot}(R) \rightarrow \mathbb{Z}$ by $v\left(\frac{a}{b}\right) = v(a) - v(b)$ if $a \neq 0$ and $v(0) = \infty$ to obtain a discrete valuation v .
- (3) By definition we have $R \subset \mathcal{O}_K$. Show that $\mathcal{O}_K \subset R$:

Let $\frac{a}{b} \neq 0$ be in \mathcal{O}_K , i.e., $v\left(\frac{a}{b}\right) \geq 0$. Then we have $v(a) \geq v(b)$. Let $\mathfrak{m} = (\pi) \neq 0$ be the unique maximal ideal of R . Then $a = \varepsilon_1 \pi^{k_1}$ and $b = \varepsilon_2 \pi^{k_2}$ with $\varepsilon_1, \varepsilon_2 \in R^\times$, $k_1, k_2 \in \mathbb{N}_0$ and $k_1 \geq k_2$ such that

$$\frac{a}{b} = \frac{\varepsilon_1}{\varepsilon_2} \pi^{k_1 - k_2} \in R.$$

□

Proposition 4.1.13. *Let R be an integral domain. Recall that for $\mathfrak{p} \in \text{Spec } R$ we have $R \subset R_{\mathfrak{p}} \subset \text{Quot}(R)$. In this situation,*

$$R = \bigcap_{\mathfrak{p} \in \text{Spec } R} R_{\mathfrak{p}}.$$

Proof. Let $\frac{a}{b} \in \bigcap_{\mathfrak{p} \in \text{Spec } R} R_{\mathfrak{p}}$. Consider $\mathfrak{a} = \{x \in R; xa \in bR\}$ and observe that \mathfrak{a} is an ideal. We show that $a \notin \mathfrak{p}$ for any $\mathfrak{p} \in \text{Spec } R$:

Since $\frac{a}{b} \in R_{\mathfrak{p}}$ there are $c \in R, s \in R \setminus \mathfrak{p}$ with $\frac{a}{b} = \frac{c}{s}$. Then we have $as = cb$, which implies $s \in \mathfrak{a}$ and $s \notin \mathfrak{p}$.

It follows that $\mathfrak{a} = R$, i.e., $1 \in \mathfrak{a}$ such that $1 \cdot a \in bR$ and thus $\frac{a}{b} \in R$. □

Theorem 13. *Let \mathcal{O} be a noetherian integral domain. Then \mathcal{O} is a Dedekind domain if and only if $\mathcal{O}_{\mathfrak{p}}$ is a DVR for all $\mathfrak{p} \in \text{Spec}(\mathcal{O}) \setminus \{0\}$.*

Proof. “ \Rightarrow ” By Proposition 1.6, $\mathcal{O}_{\mathfrak{p}}$ is a Dedekind domain and by Proposition 1.5, $\mathcal{O}_{\mathfrak{p}}$ is local. Since $\mathfrak{p} \neq 0$ we have $\mathfrak{p}S^{-1} \neq 0$ and hence $\mathcal{O}_{\mathfrak{p}}$ is not a field. Hence, $\mathcal{O}_{\mathfrak{p}}$ is a DVR by Proposition 1.9.

“ \Leftarrow ” By Proposition 1.11 we have $\mathcal{O} = \bigcap_{\mathfrak{p} \in \text{Spec } \mathcal{O}} \mathcal{O}_{\mathfrak{p}}$. Furthermore, Proposition 1.9 implies that $\mathcal{O}_{\mathfrak{p}}$ is integrally closed for any $\mathfrak{p} \in \text{Spec}(\mathcal{O})$ and hence the same holds true for \mathcal{O} .

Show: Every prime ideal $\mathfrak{p} \neq 0$ in \mathcal{O} is maximal.

Consider $\mathfrak{p} \subset \mathfrak{m}$ for some maximal ideal \mathfrak{m} . Consider the localisation $\mathcal{O}_{\mathfrak{m}} = \mathcal{O}S^{-1}$ with $S = \mathcal{O} \setminus \mathfrak{m}$. Then we obtain $\mathfrak{p}S^{-1} \subset \mathfrak{m}S^{-1}$ and $\mathfrak{p}S^{-1}, \mathfrak{m}S^{-1}$ both are prime ideals by Proposition 1.3. Since $\mathcal{O}_{\mathfrak{m}}$ is a DVR, Proposition 1.9 implies that $\mathcal{O}_{\mathfrak{m}}$ is a Dedekind domain, whence $\mathfrak{p}S^{-1} = \mathfrak{m}S^{-1}$, such that by Proposition 1.3 we may finally conclude $\mathfrak{p} = \mathfrak{m}$. \square

Reminder: $v_p : \mathbb{Q} \rightarrow \mathbb{Z} \cup \{\infty\}$, $\mathbb{Z} \ni z = p^k \cdot b \mapsto k$

Proposition 4.1.14. *The discrete valuations in Ex. 1.10 are up to scaling the only discrete valuations on \mathbb{Q} .*

Proof. Let $v : \mathbb{Q} \rightarrow \mathbb{Z} \cup \{\infty\}$ be a discrete valuation. Observe for $n \in \mathbb{N}$ we have $v(n) = v(1 + \dots + 1) \geq v(1) = 0$. For $z \in \mathbb{Z}$ we have $v(z) \geq 0$, since $v(-1) = 0$.

1) Show that $\exists p$ prime with $v(p) < 0$:

Suppose $v(p) = 0$ for all primes p .

$\Rightarrow v(n) = 0$ for all $n \in \mathbb{Z}$.

$\Rightarrow v(x) = 0$ for all $x \in \mathbb{Q} \setminus \{0\}$

2) Observe $\mathfrak{a} := \{a \in \mathbb{Z} \mid v(a) > 0\}$ is an ideal in \mathbb{Z} . Use that $v(z) \geq 0$ for all $z \in \mathbb{Z}$.

3) Let p be prime with $v(p) > 0$. Such a prime exists by 1). Observe $\mathfrak{a} = (p)$, since $p \in \mathfrak{a}$ and (p) is maximal in \mathbb{Z} . Let $c := v(p) > 0$. Denote $z \in \mathbb{Z}$ as $z = p^k \cdot b$ with $\gcd(b, p) = 1$. $\Rightarrow v(z) = k \cdot v(p) + v(b) = k \cdot c$ since $v(b) = 0, b \notin (p) = \mathfrak{a}$.

4) Obtain the result for $x \in \mathbb{Q}^\times$. \square

Example 4.1.15. Let K/\mathbb{Q} be a number field, \mathcal{O} its ring of integers, $\hat{\mathcal{P}}_0$ a prime ideal in \mathcal{O} above (p) .

Thm. 12 $\Rightarrow \mathcal{O}_{\hat{\mathcal{P}}_0}$ is a discrete valuation ring.

What is the corresponding discrete valuation on $K = \text{Quot}(\mathcal{O}_{\hat{\mathcal{P}}_0}) = \text{Quot}(\mathcal{O})$

$$v_{\hat{\mathcal{P}}_0} : K \rightarrow \mathbb{Z} \cup \{\infty\}?$$

Let $x \in K^\times \Rightarrow \underbrace{x \cdot \mathcal{O}}_{\text{fractional ideal}} = \hat{\mathcal{P}}_1^{e_1} \cdot \dots \cdot \hat{\mathcal{P}}_n^{e_n} = \prod \hat{\mathcal{P}} \in \text{Spec}(\mathcal{O}) \hat{\mathcal{P}}^e(\star)$ with $\hat{\mathcal{P}}_1, \dots, \hat{\mathcal{P}}_n$ are prime ideals in \mathcal{O} and $e_1, \dots, e_n \in \mathbb{Z}$.

Claim: $v_{\hat{\mathcal{P}}_0}(x) = e_{\hat{\mathcal{P}}_0}$. Proof: Consider the localisation $\mathcal{O}_{\hat{\mathcal{P}}_0}$:

For $\hat{\mathcal{P}} \neq \hat{\mathcal{P}}_0$ we have $\hat{\mathcal{P}} \cdot \mathcal{O}_{\hat{\mathcal{P}}_0} = \mathcal{O}_{\hat{\mathcal{P}}_0}$.

$\Rightarrow \underbrace{x \cdot \mathcal{O}}_{\text{fractional ideal for } \mathcal{O}_{\hat{\mathcal{P}}_0}} = \hat{\mathcal{P}}_0^{e_{\hat{\mathcal{P}}_0}} \cdot \mathcal{O}_{\hat{\mathcal{P}}_0} \stackrel{(!)}{=} m^{e_{\hat{\mathcal{P}}_0}}$, where m is the maximal ideal of $\mathcal{O}_{\hat{\mathcal{P}}_0} \Rightarrow v_{\hat{\mathcal{P}}_0} = e_{\hat{\mathcal{P}}_0}$.

Observe: $v_{\hat{\mathcal{P}}_0}|_{\mathbb{Q}} = e \cdot v_p$ where e is the ramification index of $\hat{\mathcal{P}}_0$.

Example 4.1.16 (Why „local ring“?). $\mathcal{O} = \mathbb{C}[X] \Rightarrow$

- \mathcal{O} is noetherian ✓
- \mathcal{O} is a UFD $\Rightarrow \mathcal{O}$ is integrally closed ✓
- \mathcal{O} is a PID \Rightarrow every prime ideal $\neq 0$ is maximal

$\Rightarrow \mathcal{O}$ is a Dedekind domain.

\Rightarrow

- $K := \text{Quot}(\mathcal{O}) = \mathbb{C}(C)$
- $\text{Spec}(\mathcal{O}) = \{(X_z) \mid z \in \mathbb{C}\} \cup \{(0)\}$
- $\mathcal{O}_{(X-z)} = \left\{ \frac{f}{g} \in \mathbb{C}(X) \mid f, g \in \mathbb{C}[X], g \notin (X-z) \right\}$
 $= \left\{ \frac{f}{g} \mid f, g \in \mathbb{C}[X], g(z) \neq 0 \right\}$
 $= \left\{ \frac{f}{g} \mid \frac{f}{g} \text{ is defined in } z \right\}$
 is a discrete valuation ring, in particular it is a local ring
- maximal ideal $m_{(X-z)} = \left\{ \frac{f}{g} \in \mathbb{C}(X) \mid f, g \in \mathbb{C}[X] \text{ with } f(z) = 0, g(z) \neq 0 \right\} = \left\{ \frac{f}{g} \in \mathbb{C}(X) \mid \text{s.t. } \frac{f}{g} \text{ has a zero in } z \right\}$ The corresponding discrete valuation $v : \mathbb{C}(X) \rightarrow \mathbb{Z} \cup \{\infty\}$ is induced by $v : \mathbb{C}[X] \setminus \{0\} \rightarrow \mathbb{Z}, f \mapsto \text{ord}_z(f) = \text{order of zero of } f \text{ in } z = \max\{k \in \mathbb{N}_0 \mid (X-z)^k \text{ divides } f\}$.

4.2. Affine Schemes (Perspective)

Idea: Link geometric objects to algebraic objects

<u>Geometry</u>	<u>Algebra</u>
-----------------	----------------

affine varieties V	$\text{Spec}(R)$
----------------------	------------------

+	+
functions on V	R

4.2.1. Classical affine varieties

Definition 4.2.1. Let k be a field and denote $\mathbb{A}^n(k) = k^n$ („affine space“)

$V \subseteq \mathbb{A}^n(k)$ is an affine variety : $\iff \exists S \subseteq k[X_1, \dots, X_n]$ s.t. $V = V(S) = \{z \in \mathbb{A}^n(k) \mid \forall f \in S : f(z) = 0\}$.

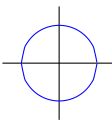
Remark 4.2.2. i) $S_1 \subseteq S_2 \Rightarrow V(S_1) \supseteq V(S_2)$

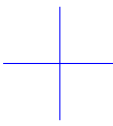
ii) Let (S) be the ideal generated by S in $k[X_1, \dots, X_n]$, then $V(S) = V((S))$.

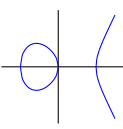
iii) Denote for $f_1, \dots, f_r : V(f_1, \dots, f_r) = V(\{f_1, \dots, f_r\})$. Every affine variety V is equal to $V(f_1, \dots, f_r)$ for finitely many polynomial f_1, \dots, f_r since $k[X_1, \dots, X_n]$ is noetherian.

Example 4.2.3. i) $S = \emptyset \Rightarrow V(S) = \mathbb{A}^n(k)$

ii) $S = k[X_1, \dots, X_n] \Rightarrow V(S) = V(1) = \emptyset$

iii) $V(X^2 + Y^2 - 1) =$ 

iv) $V(X \cdot Y) =$ 

v) $V(Y^2 - X^3 + X) (\Leftrightarrow Y^2 = X^3 - X = X(X-1)(X+1)) =$ 

vi) $a, b \in k \Rightarrow V(X - a, Y - b) = \{(a, b)\}$

Example 4.2.4. What are the affine varieties in $\mathbb{A}^1(k)$?

- Rem 2.2 ii) \Rightarrow sufficient to consider ideals I in $k[X]$
- Recall: Every ideal is a principal ideal, hence $I = (f)$ with $f \in k[X]$.

$$\Rightarrow V(I) = \begin{cases} \mathbb{A}^1(k), & \text{if } f = 0 \iff \deg(f) = -\infty \\ \emptyset, & \text{if } \deg(f) = 0 \\ \{z_1, \dots, z_k \mid z_i \text{ zero of } f\}, & \text{if } \deg(f) \geq 1. \end{cases}$$

Classical goal: Study geometry of affine varieties.

Idea: Consider „good“ classes of functions on them.

Consider: (1) $k[X_1, \dots, X_n]$ as set of *regular functions* on $\mathbb{A}^n(k) = k^n$ and

(2) $k(X_1, \dots, X_n)$ as set of *rational functions* on $\mathbb{A}^n(k)$.

Observe: $f_1, f_2 \in k[X_1, \dots, X_n], V \subseteq \mathbb{A}^n(k), f_1 \equiv f_2 \text{ on } V \iff f_1 - f_2 \equiv 0 \text{ on } V$.

Definition 4.2.5. i) $I(V) := \{f \in k[X_1, \dots, X_n] \mid \forall z \in V : f(z) = 0\}$ is called vanishing ideal of V

ii) $A(V) := k[V] := k[X_1, \dots, X_n]/I(V)$ is called the k -algebra of regular functions of V

Example 4.2.6. i) $V = \emptyset \Rightarrow I(V) = k[X_1, \dots, X_n] \Rightarrow k[V] = 0$

ii) $V = \{z\} \subseteq \mathbb{A}^1(k) \Rightarrow I(V) = (X - z) \Rightarrow k[V] = k[X]/(X - z)$

iii) $V = \mathbb{A}^1(k), k \text{ infinite} \Rightarrow I(V) = (0) \Rightarrow k[V] \cong k[X]$

iv) $V = \mathbb{A}^1(k), k = \mathbb{F}_p \text{ finite} \Rightarrow I(V) = (X \cdot (X - 1) \cdot \dots \cdot (X - (p - 1)))$

Remark 4.2.7. Suppose $V \subseteq \mathbb{A}^n(k)$

i) $I(V)$ is a radical ideal, i.e. $f^e \in I(V) \Rightarrow f \in I(V)$

- ii) $V(I(V)) \supseteq V$ and $I(V(I)) \supseteq I$.
- iii) $\bar{V} := V(I(V))$ is the smallest affine variety containing V .
In particular, if V is already an affine variety, then $V(I(V)) = V$.
- iv) For affine varieties $V_1 = V(I_1)$ and $V_2 = V(I_2) : V_1 \subseteq V_2 \iff I(V_1) \supseteq I(V_2)$. In particular $V_1 = V_2 \iff I(V_1) = I(V_2)$
- v) $I_1 \subseteq I_2 \Rightarrow V(I_1) \supseteq V(I_2)$

Proof. i) ✓

ii) ✓

iii) Consider affine variety $V(J) \supseteq V \Rightarrow J \subseteq I(V) \Rightarrow V(J) \supseteq V(I(V))$

iv) „ \Rightarrow “: Suppose $f \in I(V_2) \Rightarrow \forall x \in V_1 \subseteq V_2 : f(x) = 0 \Rightarrow f \in I(V_2)$.

„ \Leftarrow “: $I_2 \stackrel{ii)}{\subseteq} I(V(I_2)) = I(V_2) \subseteq I(V_1)$ Hence: $x \in V_1, f \in I_2 \Rightarrow f(x) = 0$. Thus $x \in V_2 = V(I_2)$.

v) ✓

□

Hilberts Nullstellensatz (without proof)

k is algebraically closed, I ideal in $k[X_1, \dots, X_n]$.

Then $I(V(I)) = \sqrt{I} := \{f \in k[X_1, \dots, X_n] \mid \exists l \in \mathbb{N} \text{ with } f^l \in I\}$.

Idea: Define a topology on $\mathbb{A}^n(k)$.

Remark and Definition 4.2.8. *The affine varieties define the closed sets of a topology called Zariski topology.*

Proof. • $\mathbb{A}^n(k) = V((0))$ and $\emptyset = V((1))$.

• $V_1 = V(I_1)$ and $V_2 = V(I_2)$ affine varieties $\Rightarrow V_1 \cup V_2 \stackrel{!)}{=} V(I_1 \cdot I_2) \stackrel{!)}{=} V(I_1 \cap I_2)$

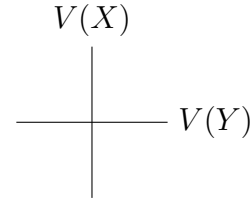
• $V_j = V(I_j)$ family of affine varieties with $j \in J \Rightarrow \bigcap_{j \in J} V_j \stackrel{!)}{=} V(\sum_{j \in J} I_j)$.

□

Example 4.2.9. In $\mathbb{A}^1(k)$ a subset is closed \iff it is finite or \emptyset or $\mathbb{A}^1(k)$.

Definition 4.2.10. A topological space X is called irreducible : $\iff X = A \cup B$ with A, B closed implies $X = A$ or $X = B$.

Otherwise X is called reducible



Example 4.2.11. $V = V(X \cdot Y) = V(X) \cup V(Y)$ is reducible.

Proposition 4.2.12. *An affine variety V is irreducible $\iff I(V)$ is a prime ideal.*

Proof. „ \Rightarrow “: Consider $f, g \in k[X_1, \dots, X_n]$ with $f, g \in I(V)$.

Suppose: $f \notin I(V)$. Show that $g \in I(V)$.

$f \notin I(V) \Rightarrow \exists f(x) \neq 0 \Rightarrow V \not\subseteq V(f)(\star)$

Observe: $V \subseteq V(f \cdot g) = V(f) \cup V(g) \Rightarrow V = \underbrace{(V(f) \cap V)}_{\text{closed}} \cup \underbrace{(V(g) \cap V)}_{\text{closed}}$

$\xRightarrow{(\star)} V(g) \cap V = V \Rightarrow V \subseteq V(g) \Rightarrow g \in I(V)$.

„ \Leftarrow “: Suppose $V = V_1 \cup V_2$ with $V_1 = V(I_1)$ and $V_2 = V(I_2)$.

Hence: $V = V(I_1) \cup V(I_2) = V(I_1 I_2)$ and thus $I_1 \cdot I_2 \stackrel{2.7}{\subseteq} I(V(I_1 \cdot I_2)) = I(V)$.

Suppose: $V_1 \neq V \Rightarrow \exists x \in V : f \in I_1$ with $f(x) \neq 0$.

Hence $f \notin I(V) \forall g \in I_2 : f \cdot g \in I_1 I_2 \subseteq I(V) \xRightarrow{I(V) \text{ prime}} g \in I(V)$.

Hence $I_2 \subseteq I(V) \Rightarrow V_2 = V(I_2) \supseteq V(I(V)) \supseteq V \Rightarrow V_2 = V$. \square

Remark 4.2.13. An affine variety V is irreducible $\iff k[V] = k[X_1, \dots, X_n]/I(V)$ is an integral domain.

From now on k is always algebraically closed and all affine varieties are irreducible.

Example 4.2.14. $V = \mathbb{A}^1(k), I(V) = (0), k[V] = k[X]$

Definition 4.2.15. Let $U \subseteq V$ be an open subset of V . Define

$$\mathcal{O}(U) := \{\varphi : U \rightarrow k \mid \forall z \in U \exists \text{ open neighbourhood } U_z \ni z \stackrel{\text{open}}{\subseteq} V, \exists f, g \in k[V] \\ \text{with } \forall x \in U_z : g(x) \neq 0 \text{ and } \varphi(x) = \frac{f(x)}{g(x)}\}$$

as the set of regular functions on U .

Remark. \mathcal{O} defines a sheaf on V .

Definition 4.2.16. For $z \in V$ we define the local ring \mathcal{O}_z as follows:

$$\mathcal{O}_z := \{(U, f) \mid U \text{ open neighbourhood of } z, f \in \mathcal{O}(U)\} / \sim$$

with $(U_1, f_1) \sim (U_2, f_2) \iff f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$.

Remark. • \mathcal{O}_z is the stalk of the sheaf \mathcal{O} in z .

• $\mathcal{O}_z = k[V]_{m_z}$ here $m_z = \{f \in k[V] \mid f(z) = 0\}$

Remark 4.2.17. m_z as defined above is a maximal ideal in $k[V]$.

Proof. $\varphi_u : k[V] = k[X_1, \dots, X_n]/I(V) \rightarrow k$, $f \mapsto f(z)$ is a k -algebra homomorphism, which is surjective. Hence $m_z = \ker(\varphi_z)$ is a maximal ideal. \square

Remark 4.2.18. In particular \mathcal{O}_z is a local ring.

Definition 4.2.19. The field of rational functions $\text{Rat}(V)$:

$$\text{Rat}(V) := \{(U, f) \mid U \text{ open in } V, f \in \mathcal{O}(U)\} / \sim$$

with $(U_1, f_1) \sim (U_2, f_2) \iff f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$.

Remark (without proof). $\text{Rat}(V) = \text{Quot}(k[V])$

Conclusion 4.2.20. *Still assume k is algebraically closed, V affine variety. Then we have the following correspondences:*

$$\begin{aligned} \text{closed subsets of } V &\xleftrightarrow{1:1} \text{radical ideals in } k[V] \\ V' &\mapsto I(V') \end{aligned}$$

$$\begin{aligned} \text{irreducible closed subsets of } V &\xleftrightarrow{1:1} \text{prime ideals} \\ \text{points} &\xleftrightarrow{1:1} \text{maximal ideals in } k[V] \\ x &\mapsto I(\{x\}) = m_x \end{aligned}$$

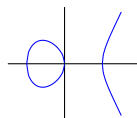
Furthermore we have:

- V_1, V_2 closed subsets of V : $V_1 \subseteq V_2 \iff I(V_1) \supseteq I(V_2)$
In particular: $x \in V_1 \iff m_x \supseteq I(V_1)$.
- \mathfrak{a} ideal in $k[V]$:
 $V(\mathfrak{a}) = \{z \in V \mid \forall f \in \mathfrak{a} : f(z) = 0\}$
 $= \{z \in V \mid \forall f \in \mathfrak{a} : f \in m_z\} = \{z \in V \mid \mathfrak{a} \subseteq m_z\}$
- $S \subseteq V : I(S) = \{f \in k[V] \mid \forall s \in S : f(s) = 0\} = \bigcap_{s \in S} m_s$.

Example 4.2.21. Let $k = \mathbb{C}$, $I = (Y^2 - X(X-1)(X-\lambda))$ with $\lambda \in \mathbb{C}$ and $V = V(I)$. Observe that I is a prime ideal since f_λ is irreducible (use Eisenstein criterion). In particular, $I = I(V)$. Hence V is irreducible and

$$K[V] = k[X, Y]/I.$$

Graph for $\lambda = -1$:



Definition 4.2.22. (i) Let X be a topological space. The **Krull dimension** $\dim(X)$ of X is defined as

$$\dim(X) = \sup\{n \in \mathbb{N} \mid \text{there are closed, irreducible subsets } X_i \subset X \text{ with } \emptyset \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n\}.$$

(ii) Let R be an arbitrary commutative noetherian ring. Then

$$\dim(R) = \sup\{n \in \mathbb{N} \mid \text{there are } \mathfrak{p}_i \in \operatorname{Spec} R \text{ with } \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n\}.$$

is called the **dimension** of R .

Remark 4.2.23. By Conclusion 2.20 we have $\dim V = \dim k[V]$.

Fact 4.2.24. $\dim k[X_1, \dots, X_n] = n$ and hence $\dim \mathbb{A}^n(k) = n$.

Observe. $(X_1, \dots, X_n) \supsetneq (X_1, \dots, X_{n-1}) \supsetneq \cdots \supsetneq (X_1) \supsetneq (0)$ is a chain of prime ideals of length n , hence $\dim k[X_1, \dots, X_n] \geq n$.

But you have to do some work to show that there is not a longer one.

Example 4.2.25. Let $V = V((f_\lambda))$ with $f_\lambda = Y^2 - X(X-1)(X-\lambda)$ and observe that $\mathfrak{p}_0 = (X, Y) \supsetneq \mathfrak{p}_1 = (f_\lambda) = I$ in $k[X, Y]$ induces $\bar{\mathfrak{p}}_0 \supsetneq \bar{\mathfrak{p}}_1$ in $k[V] = k[X, Y]/I$, where $\bar{\mathfrak{p}}_i$ is the image of \mathfrak{p}_i in the quotient. Hence $\dim k[V] \geq 1$.

But any chain of prime ideals $\bar{\mathfrak{p}}_0 \supsetneq \bar{\mathfrak{p}}_1 \supsetneq \cdots \supsetneq \bar{\mathfrak{p}}_n = (0)$ of length n in $k[V]$ induces a chain of prime ideals $\mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n \supsetneq (0)$ of length $N+1$, where \mathfrak{p}_i is the preimage of $\bar{\mathfrak{p}}_i$. Hence, by Fact 2.24, $\dim k[V] = 1$.

Definition 4.2.26. An **affine curve** over k is an irreducible affine variety V over k of dimension 1.

Definition 4.2.27. Let $V \subset \mathbb{A}^n(k)$ be an irreducible affine variety of dimension d with $V = V(f_1, \dots, f_r)$. For $z \in V$ consider the Jacobian matrix

$$J = \left(\frac{\partial f_i}{\partial x_j}(z) \right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(z) & \cdots & \frac{\partial f_1}{\partial x_n}(z) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial x_1}(z) & \cdots & \frac{\partial f_r}{\partial x_n}(z) \end{pmatrix}.$$

In this case,

- $z \in V$ is called **singular** if $\operatorname{rank} J < n - d$,
- $z \in V$ is called **non-singular** or **regular** if $\operatorname{rank} J = n - d$,
- V is called **regular** if all points in V are regular points.

Fact 4.2.28 (without proof). We always have $\operatorname{rank} J \leq n - d$.

Example 4.2.29. For $V = V((f_\lambda))$ with

$$f_\lambda = Y^2 - X(X-1)(X-\lambda) = Y^2 - (X^3 - (1+\lambda)X^2 + \lambda X)$$

we have

$$J(x, y) = (d - 3x^2 + 2(1+\lambda)x - \lambda - 2y).$$

Hence $(x, y) \in V$ is singular iff $\text{rank } J(x, y) < 2 - 1 = 1$ iff $\text{rank } J(x, y) = 0$ such that (x, y) has to satisfy the following relations:

$$y^2 = x(x-1)(x-\lambda) \tag{1}$$

$$-3x^2 + 2(1+\lambda)x - \lambda = 0 \tag{2}$$

$$2y = 0 \tag{3}$$

Now, (3) implies $y = 0$ and hence $x \in \{0, 1, \lambda\}$ by (1).

Conclusion:

- $\lambda \notin \{0, 1\}$: V is a regular curve
- $\lambda = 0$: $(0, 0)$ is a singularity
- $\lambda = 1$: $(1, 0)$ is a singularity

Fact 4.2.30 (without proof). Let V be an affine curve. Then x is a regular point of V if and only if $\mathcal{O}_x = k[V]_{\mathfrak{m}_x}$ is a discrete valuation ring.

Remark 4.2.31. Let V be an affine variety. Then V is a regular affine curve if and only if $k[V]$ is a Dedekind domain.

Proof. “ \Rightarrow ”

- $k[V] = k[X_1, \dots, X_n]/I$ is noetherian.
- $k[V]$ is a domain since I is a prime ideal. This is because V is irreducible by the definition of a curve.
- By Theorem 12 and Fact 2.30, $k[V]$ is a Dedekind domain.

“ \Leftarrow ”

- Since $k[V]$ is a domain, V is irreducible.
- Every prime ideal is maximal, hence $\dim k[V] = \dim V = 1$.
- By Theorem 12 and Fact 2.30, V is regular.

□

4.3. Affine schemes and Dedekind domains

Conclusion 2.20 motivates the following definition:

Definition 4.3.1. Let R be a ring and $X = \operatorname{Spec} R$.

- (i) For an ideal \mathfrak{a} in R define its **vanishing locus** by $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{a} \subset \mathfrak{p}\}$.
- (ii) For a subset $S \subset X$ define its **vanishing ideal** by $I(S) = \bigcap_{\mathfrak{p} \in S} \mathfrak{p}$.

Remark and Definition 4.3.2. Suppose $R = k[X_1, \dots, X_n]$

Note: In contrast to the geometric situation we consider all irreducible subsets of $\mathbb{A}^n(k)$ as points in $\operatorname{Spec} R$.

The maximal ideals in $\operatorname{Spec} R$ are sometimes called **geometric points** or **closed points**.

Remark 4.3.3. The sets $V(\mathfrak{a})$ with \mathfrak{a} an ideal in R form the closed subsets of a topology on $\operatorname{Spec} R$ called the **Zariski topology**.

Proof. Check the axioms, similarly to Remark 2.8. □

Remark 4.3.4. If $S \subset \operatorname{Spec} R$ then $\overline{S} = V(\mathfrak{a})$ with $\mathfrak{a} = \bigcap_{\mathfrak{p} \in S} \mathfrak{p}$. In particular,

$$\overline{\{\mathfrak{p}\}} = V(\mathfrak{p}) = \{\mathfrak{a} \in \operatorname{Spec} R \mid \mathfrak{a} \supset \mathfrak{p}\}.$$

Hence $\{\mathfrak{p}\}$ is closed if and only if \mathfrak{p} is maximal.

Question. How can we understand the elements of R as functions on $\operatorname{Spec} R$?

Example 4.3.5. Let V be an affine variety and $R = k[V]$. If $f \in R$ then

$$f: V \rightarrow k, z \mapsto f(z).$$

Recall. The evaluation map $\Phi_z: k[V] \rightarrow k, f \mapsto f(z)$ defines an isomorphism $k[V]/\mathfrak{m}_z \cong k$ with $f(z) = f \bmod \mathfrak{m}_z$.

Remark 4.3.6. R now an arbitrary ring.

The elements of R define functions on $\operatorname{Spec}(R)$ as follows:

$$f: p \mapsto f \bmod p \in R_p/p =: \underbrace{\kappa(p)}_{\text{residue field of } p}$$

Example 4.3.7. $R = \mathbb{Z}, X = \operatorname{Spec}(\mathbb{Z}) = \{(p) \mid p \text{ prime}\} \cup \{(0)\}$. Observe:

- The (p) are closed points.
- $\{(0)\} = \operatorname{Spec}(\mathbb{Z})$

Definition 4.3.8. Let $U \subseteq \operatorname{Spec}(R)$ be an open subset. Define

$$\mathcal{O}(U) := \{s = (s_p)_{p \in U} \in \prod_{p \in U} R_p \mid \forall p \exists U_p \text{ open nbhd. of } p \text{ and } f, g \in R \text{ with :}$$

$$\forall q \in U_p : \underbrace{g(q)}_{=g \bmod q} \neq 0 \text{ and } s_q = \frac{f}{g} \text{ in } R_q\}$$

$$\underbrace{\neq 0}_{\iff g \not\equiv 0 \bmod q}$$

Observe: \mathcal{O} is a sheaf.

Definition 4.3.9. The pair $(\operatorname{Spec} R, \mathcal{O})$ is called an affine scheme.

It is called noetherian $\iff R$ is noetherian.

Its dimension is the dimension of R .

Example 4.3.10. i) $R = k[V]$ for some affine variety V with k is algebraically closed
 $\Rightarrow V \hookrightarrow \operatorname{Spec}(R)$ as set of closed points.

ii) $R = K$ field $\Rightarrow \operatorname{Spec}(K) = \{(0)\} : \mathcal{O}(\{0\}) = K$
 $\underline{!} \operatorname{Spec}(K)$ as affine scheme depends on K .

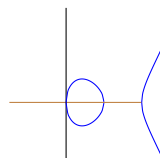
iii) R discrete valuation ring with maximal ideal $m \Rightarrow \operatorname{Spec}(R) = \{(0), m\}$
 Closed subsets: $\emptyset, \{m\}, \operatorname{Spec}(R)$
 Open subsets: $\operatorname{Spec}(R), \{0\}, \emptyset$
 $\mathcal{O}(\{0\}) = \operatorname{Quot}(R), \mathcal{O}(\operatorname{Spec}(R)) = R_m = R$

Definition 4.3.11. For a one-dimensional (!) noetherian domain R we say that the associated affine scheme $(\operatorname{Spec} R, \mathcal{O})$ is regular : \iff All local rings R_p (p prime in $R, p \neq (0)$) are discrete valuation rings.

Observation 4.3.12. For a domain R we have: $(\operatorname{Spec} R, \mathcal{O})$ is a noetherian, one-dimensional, regular affine scheme $\iff R$ is a Dedekind domain.

4.4. A „general“example

Example 4.4.1. Consider $V_1 := V(f_\lambda) \subseteq \mathbb{A}^2(\mathbb{C})$ with $f_\lambda = Y^2 - X(X-1)(X-\lambda)$ and $\lambda \in \mathbb{C} \setminus \{0, 1\}, V_2 : \mathbb{A}^1(\mathbb{C}), f : V_1 \rightarrow V_2, (x, y) \mapsto x$



Remark 4.4.2. i) $x \in \mathbb{A}^1(\mathbb{C})$ has two preimages $\iff x \notin \{0, 1, \lambda\}$

ii) $x \in \mathbb{A}^1(\mathbb{C})$ has one preimage $\iff x \in \{0, 1, \lambda\}$, namely $(x, 0)$.

Remark 4.4.3. Recall: $\mathcal{O}_1 := \mathbb{C}[V_1] = \mathbb{C}[X, Y]/(f_\lambda), \mathcal{O}_2 := \mathbb{C}[V_2] = \mathbb{C}[X]$
 The map f induces:

- i) a morphism of k -algebras: $f^* : \mathbb{C}[V_2] \rightarrow \mathbb{C}[V_1]$, $h \mapsto h \circ f$
- ii) a morphism of fields: $f^* : \text{Rat}(V_2) = \text{Quot}(\mathcal{O}_2) \hookrightarrow \text{Rat}(V_1) = \text{Quot}(\mathcal{O}_1)$ induced by f^* in i).

Conclusion 4.4.4. Hence we have the following setting:

$$\begin{array}{ccc}
 \mathcal{O}_1 & \hookrightarrow & \text{Quot}(\mathcal{O}_1) =: K_1 \\
 \uparrow f^* & & \uparrow f^* \\
 \mathcal{O}_2 & \hookrightarrow & \text{Quot}(\mathcal{O}_2) =: K_2
 \end{array}
 \quad
 \begin{array}{c}
 \mathbb{C}(X) \\
 \uparrow f^*
 \end{array}
 \quad
 \begin{array}{ccc}
 V_1 = \mathbb{C} & & \\
 \downarrow f & & \\
 V_2 = \mathbb{A}^1 & \xrightarrow{h} & \mathbb{C}
 \end{array}$$

- \mathcal{O}_1 and \mathcal{O}_2 are Dedekind domains
- $\text{Quot}(\mathcal{O}_2) = \text{Quot}(\mathbb{C}[X]) = \mathbb{C}(X)$
 $\text{Quot}(\mathcal{O}_1) = \text{Quot}(\mathbb{C}[X, Y]/(f_\lambda)) = \mathbb{C}(X)(\alpha)$ with α has minimal polynomial $Y^2 - X(X-1)(X-\lambda) \in \mathbb{C}(X)[Y]$
- $[K_1 : K_2] = 2$

Recall: points in V_1 resp. $V_2 \leftrightarrow$ maximal ideals in \mathcal{O}_1 resp. \mathcal{O}_2
 $f(a) = b \iff f^*(m_b) \subseteq m_a \iff m_a$ lies above m_b .

$$\begin{array}{ccccc}
 \mathbb{C}[X][\alpha] = \mathbb{C}[X, Y]/(f_\lambda) & = & \mathcal{O}_1 & \hookrightarrow & \text{Quot}(\mathcal{O}_1) =: K_1 = \mathbb{C}(X)(\alpha) \\
 & & \uparrow f^* & & \uparrow f^* \\
 (X-b) \subseteq \mathbb{C}[X] & = & \mathcal{O}_2 & \hookrightarrow & \text{Quot}(\mathcal{O}_2) =: K_2 = \mathbb{C}(X)
 \end{array}$$

Remark 4.4.5. Consider the prime ideal $(X-b)$ for $b \in \mathbb{C}$.

Apply Prop. II.8.15: Observe $\mathcal{O}_1 = \mathcal{O}_2[\alpha]$, $f_\alpha(Y) = Y^2 - X(X-1)(X-\lambda) \in \mathcal{O}_2[Y]$

Consider image $\bar{f}_\alpha \in \mathcal{O}_2/(X-b)[Y] \xrightarrow{\text{evaluation map}} \mathbb{C}[Y]$

Hence $\bar{f}_\alpha = Y^2 - b(b-1)(b-\lambda) \in \mathbb{C}[Y] = (Y - \sqrt{b(b-1)(b-\lambda)})(Y + \sqrt{b(b-1)(b-\lambda)})$.

Prop. II.8.15. \Rightarrow :

- If $b \notin \{0, 1, \lambda\}$: $\mathfrak{o}\mathcal{O}_1 = \hat{\mathcal{P}}_1 \cdot \hat{\mathcal{P}}_2$.
Hence we have two prime ideals $\hat{\mathcal{P}}_1$ and $\hat{\mathcal{P}}_2$ lying above $p := (X-b)$ and the ramification indices are $e_1 = e_2 = 1$.
Degree formula: $2 = e_1 f_1 + e_2 f_2 \Rightarrow f_1 = f_2 = 1$.
- If $b \in \{0, 1, \lambda\}$: $p\mathcal{O}_1 = \hat{\mathcal{P}}^2$.
Hence we have one prime ideal $\hat{\mathcal{P}}$ over p with ramification index 2 and $f = 1$.

Remark 4.4.6. Why is f in the example = 1?

Recall: $f = \dim_{\mathcal{O}_2/p} \mathcal{O}_1/\hat{\mathcal{P}}$ Here: $\mathcal{O}_2 = k[V_2]$, p = maximal ideal corresp. to a point b . $\Rightarrow \mathcal{O}_2/p \cong \mathbb{C}$. Same way: $\mathcal{O}_1/\hat{\mathcal{P}} \cong \mathbb{C}$.

Part II.

Algebraische Zahlentheorie II

5. Bewertungstheorie

5.1. Normierte Körper

Definition 5.1.1. Eine **Norm** auf einem Körper K ist eine Funktion $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$ mit:

- (i) $|x| \geq 0$ und $|x| = 0 \Leftrightarrow x = 0$
- (ii) $|xy| = |x| |y|$
- (iii) $|x + y| \leq |x| + |y|$

Remark 5.1.2. (i) $|1| = |-1| = 1$ und $|-x| = |x|$

- (ii) $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$ mit $|0| = 0$ und $|x| = 1$ für $x \neq 0$ ist eine Norm und heißt **triviale Norm**.

Ab jetzt: K immer ein Körper

Remark 5.1.3. Jede Norm definiert eine Metrik auf K durch $d(x, y) = |x - y|$.

Remark 5.1.4. Jede Metrik auf einer Menge X definiert eine Topologie auf X durch:

$$U \subset X \text{ offen} \Leftrightarrow \text{für alle } x \in U \text{ existiert } \varepsilon > 0 \text{ mit } K_\varepsilon(x) \subset U$$

Hier: $K_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$

Recall. Für eine Menge X heißt $T \subset \mathcal{P}(X)$ **Topologie** falls

- (i) $\emptyset, X \in T$,
- (ii) $U, V \in T$ impliziert $U \cap V \in T$,
- (iii) $U_i \in T$ für alle $i \in I$ impliziert $\bigcup_{i \in I} U_i \in T$.

Remark 5.1.5. Seien X eine Menge, d eine Metrik auf X und T die induzierte Topologie. Dann hängt Konvergenz von Punktfolgen nur von der Topologie ab.

Genauer: Für eine Folge von Punkten $x_n \in X$ und $x \in X$ gilt:

$$x_n \xrightarrow{n \rightarrow \infty} x \Leftrightarrow \text{Für alle } U \in T \text{ mit } x \in U \text{ ein } N \in \mathbb{N} \text{ existiert mit } x_n \in U \text{ für alle } n \geq N.$$

Proof. “ \Rightarrow ” Zu $U \in T$ existiert $r > 0$ mit $K_r(x) \subset U$. Wähle $N \in \mathbb{N}$ mit $d(x_n, x) < r$ für alle $n \geq N$. Dann ist $x_n \in K_r(x) \subset U$ für alle $n \geq N$.

“ \Leftarrow ” Sei $\varepsilon > 0$. Wähle $U = K_\varepsilon(x)$. Da U offen ist, existiert $N \in \mathbb{N}$ mit $x_n \in U$ für alle $n \geq N$. Also ist $d(x_n, x) < \varepsilon$ für alle $n \geq N$. \square

Definition 5.1.6. Zwei Normen $|\cdot|_1$ und $|\cdot|_2$ auf K heißen **äquivalent**, $|\cdot|_1 \sim |\cdot|_2$, wenn sie die selbe Topologie auf K induzieren.

Proposition 5.1.7. $|\cdot|_1$ und $|\cdot|_2$ sind äquivalent genau dann, wenn ein $s > 0$ existiert mit $|\cdot|_2 = |\cdot|_1^s$.

Lemma 5.1.8. Falls $|\cdot|_1$ und $|\cdot|_2$ äquivalent sind, dann gilt:

$$|x|_1 < 1 \Leftrightarrow |x|_2 < 1$$

Proof. Seien d_1 und d_2 die zugehörigen Metriken.

$$|x|_1 < 1 \Leftrightarrow |x|_1^n \xrightarrow{n \rightarrow \infty} 0 \Leftrightarrow x^n \xrightarrow{n \rightarrow \infty}_{d_1} 0 \Leftrightarrow x^n \xrightarrow{n \rightarrow \infty}_{d_2} 0 \Leftrightarrow |x|_2 < 1$$

□

Lemma 5.1.9. Sei $|\cdot|_1$ eine nicht-triviale Metrik. Dann gilt:

Es gibt $s \in \mathbb{R}$ mit $|\cdot|_2 = |\cdot|_1^s$ genau dann, wenn für alle $x, y \in K^\times$ gilt, dass

$$\log |x|_1 \log |y|_2 = \log |x|_2 \log |y|_1.$$

In diesem Fall ist $s = \frac{\log |x|_2}{\log |x|_1}$, falls $x \in K^\times$ mit $|x|_1 \neq 1$.

Proof. “ \Rightarrow ” $\log |x|_1 \log |y|_2 = s \log |x|_1 \log |y|_1 = \log |x|_2 \log |y|_1$

“ \Leftarrow ” Wähle $x \in K^\times$ mit $|x|_1 \neq 1$ und setze $s = \frac{\log |x|_2}{\log |x|_1}$. Dann gilt für alle $y \in K^\times$:

$$\log |y|_2 = \frac{\log |x|_2}{\log |x|_1} \log |y|_1 = s \log |y|_1$$

□

Proof of Proposition 5.1.7. “ \Leftarrow ” Die Kreisscheiben um x mit Radius ε bzgl. $|\cdot|_1$ sind genau die Kreisscheiben um x mit Radius ε^s bzgl. $|\cdot|_2$. Daher erhalten wir die selben Topologien.

“ \Rightarrow ” Ohne Einschränkung sei $|\cdot|_1$ nicht-trivial. Wähle $x \in K^\times$ mit $|x| \neq 1$. Sei nun $y \in K^\times$ und definiere $\alpha = \frac{\log |y|_1}{\log |x|_1}$ so, dass $|y|_1 = |x|_1^\alpha$.

(1) Zeige: $|y|_2 \leq |x|_2^\alpha$

Wähle Folge $\frac{m_i}{n_i} \in \mathbb{Q}$ mit $\frac{m_i}{n_i} \xrightarrow{i} \alpha$ und $\frac{m_i}{n_i} \geq \alpha$ für alle i . Dann gilt:

$$\begin{aligned} |y|_1 &= |x|_1^\alpha \leq |x|_1^{\frac{m_i}{n_i}} \Leftrightarrow |y|_1^{n_i} \leq |x|_1^{m_i} \\ &\Leftrightarrow \left| \frac{y^{n_i}}{x^{m_i}} \right|_1 \leq 1 \\ &\Leftrightarrow \left| \frac{y^{n_i}}{x^{m_i}} \right|_2 \leq 1 \\ &\Leftrightarrow |y|_2 \leq |x|_2^{\frac{m_i}{n_i}} \end{aligned}$$

Mit $i \rightarrow \infty$ erhalte $|y_2| \leq |x|_2^\alpha$.

(2) Erhalte analog $|y|_2 \geq |x|_2^\alpha$ durch Folge $\frac{m_i}{n_i} \in \mathbb{Q}$ mit $\frac{m_i}{n_i} \xrightarrow{i} \alpha$ und $\frac{m_i}{n_i} \leq \alpha$ für alle i .

Insgesamt: $|y|_2 = |x|_2^\alpha$

Es gilt also

$$\log |y|_2 = \alpha \log |x|_2 = \frac{\log |y|_1}{\log |x|_1} \log |x|_2.$$

Verwende Lemma 5.1.9 und erhalte für alle $y, y' \in K^\times$, dass

$$\log |y|_1 \log |y'|_2 = \log |y|_1 \frac{\log |y'|_1}{\log |x|_1} \log |x|_2 = \log |y|_2 \log |y'|_1.$$

Lemma 5.1.9 liefert nun $|\cdot|_2 = |\cdot|_1^s$ mit $s = \frac{\log |x|_2}{\log |x|_1} > 0$ nach Lemma 5.1.8. \square

Remark 5.1.10. Für beliebige $s > 0$ gilt **nicht**: $|\cdot|_1$ Norm $\Rightarrow |\cdot|_2^s$ Norm

Example 5.1.11. Die **Betrags-Norm** auf \mathbb{Q} ist definiert durch

$$|\cdot| : \mathbb{Q} \rightarrow \mathbb{R}_{\geq 0}, \quad q \mapsto |q| = \begin{cases} q, & q \geq 0, \\ -q, & q \leq 0. \end{cases}$$

Recall. $v_p : \mathbb{Q} \rightarrow \mathbb{Z} \cup \{\infty\}$ p-adische Bewertung:

- Für $z \in \mathbb{Z}$ gilt: $v_p(z) = h \Leftrightarrow z = ap^h$ mit $\text{ggT}(a, p) = 1$
- Für $q = \frac{x}{y} \in \mathbb{Q}^\times$ mit $x, y \in \mathbb{Z}$ gilt: $v_p(q) = v_p(x) - v_p(y)$
- $v_p(0) = \infty$

Es gilt: v_p ist **diskrete Bewertung**:

- (i) $v_p(x) = \infty \Leftrightarrow x = 0$
- (ii) $v_p(xy) = v_p(x) + v_p(y)$
- (iii) $v_p(x + y) \geq \min\{v_p(x), v_p(y)\}$

Remark 5.1.12. Definiere $|\cdot|_p : \mathbb{Q} \rightarrow \mathbb{R}_{\geq 0}$ durch $|0|_p = 0$ und $|x|_p = p^{-v_p(x)}$ für $x \neq 0$. Dann ist $|\cdot|_p$ eine Norm, die **p-adische Norm**.

Proof. (i) und (ii) in Definition 5.1.1 folgen aus (i) und (ii) in Beispiel 5.1.12. (iii) in Definition 5.1.1 folgt aus (iii) in Beispiel 5.1.12: Für $x, y \neq 0$ gilt:

$$\begin{aligned} |x + y|_p &= p^{-v_p(x+y)} \\ &\leq p^{-\min\{v_p(x), v_p(y)\}} \\ &= p^{\max\{-v_p(x), -v_p(y)\}} \\ &= \max\{|x|_p, |y|_p\} \\ &\leq |x|_p + |y|_p \end{aligned}$$

□

Remark 5.1.13. (i) Es gilt sogar die **verschärfte Dreiecksungleichung**:

$$|x + y|_p \leq \max\{|x|_p, |y|_p\} \text{ für alle } x, y \in K$$

(ii) Für $z \in \mathbb{Z}$ gilt

$$|z|_p = |1 + \dots + 1|_p \leq |1|_p = 1.$$

(iii) Die Norm $|\cdot|$ auf \mathbb{Q} definiert durch $|x| = q^{-v_p(x)}$ mit beliebigem $q > 1$ ist äquivalent zu $|\cdot|_p$.

Example 5.1.14. (i) $|1|_p = |2|_p = |11|_p = 1$

(ii) $|5|_p = \frac{1}{5}, |100|_p = \frac{1}{25}$

(iii) $|\frac{1}{5}|_p = 5, |\frac{1}{25}|_p = 25$

5.2. Archimedische und nicht-archimedische Normen

Ab jetzt: Betrachte nur nicht-triviale Normen!

Definition 5.2.1. Eine Norm $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$ heißt **nicht-archimedisch** falls ein $C \in \mathbb{R}_{\geq 0}$ existiert mit $|n| \leq C$ für alle $n \in \mathbb{N}$. Andernfalls heißt $|\cdot|$ **archimedisch**.

Hierbei: $n = 1 + \dots + 1 \in K$

Proposition 5.2.2. Eine Norm $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$ ist nicht-archimedisch genau dann, wenn die verschärfte Dreiecksungleichung gilt, d.h. für alle $x, y \in K$:

$$|x + y| \leq \max\{|x|, |y|\}$$

Proof. “ \Leftarrow ” $|n| \leq \max\{|1|, \dots, |1|\} = |1| = 1$

“ \Rightarrow ” Sei $C \in \mathbb{R}_{\geq 0}$ mit $|n| \leq C$ für alle $n \in \mathbb{N}$. Für $x, y \in K$ können wir ohne Einschränkung annehmen, dass $\max\{|x|, |y|\} = |x|$ gilt und somit

$$|x|^j |y|^{n-j} \leq |x|^n$$

für alle $j \in \mathbb{N}_0$ und daher

$$\begin{aligned} |x + y|^n &= |(x + y)^n| = \left| \sum_{j=0}^n \binom{n}{j} x^j y^{n-j} \right| \\ &\leq \sum_{j=0}^n \binom{n}{j} |x|^j |y|^{n-j} \leq \sum_{j=0}^n C |x|^n = (n+1)C |x|^n \end{aligned}$$

Es folgt $|x + y| \leq \sqrt[n]{n+1} \sqrt[n]{C} |x|$ so, dass $|x + y| \leq |x| = \max\{|x|, |y|\}$. □

Remark 5.2.3. Ist $|\cdot|$ nicht-archimedisch, so gilt:

$$|x| \neq |y| \Rightarrow |x + y| = \max\{|x|, |y|\}$$

Proof. Ohne Einschränkung sei $|y| < |x|$.

“ \leq ” Gilt nach der verschärften Dreiecksungleichung.

“ \geq ” $|y| < |x| = |x + y - y| \leq \max\{|x + y|, |y|\} = |x + y|$ □

Definition 5.2.4. Wir betrachten eine Abbildung $v: K \rightarrow \mathbb{R} \cup \{\infty\}$.

(i) v heißt **Bewertung**, falls für alle $x, y \in K$ die folgenden Bedingungen erfüllt sind:

$$(i) \quad v(x) = \infty \Leftrightarrow x = 0$$

$$(ii) \quad v(xy) = v(x) + v(y)$$

$$(iii) \quad v(x + y) \geq \min\{v(x), v(y)\}$$

(ii) v heißt **diskrete Bewertung**, falls v eine Bewertung ist und $v(K) \cap \mathbb{R}_{>0}$ ein Minimum s besitzt. Dies ist genau dann der Fall, wenn $v(K) = s\mathbb{Z} \cup \{\infty\}$.

(iii) Zwei Bewertungen $v_1, v_2: K \rightarrow \mathbb{R}_{>0}$ heißen **äquivalent**, falls ein $r \in \mathbb{R} \setminus \{0\}$ existiert mit $v_1(x) = rv_2(x)$ für alle $x \in K^\times$.

Die Bewertung $v_{\text{trivial}}: K \rightarrow \mathbb{R} \cup \{\infty\}$ mit $v_{\text{trivial}}(x) = 0$ für alle $x \in K^\times$ heißt **triviale Bewertung**.

Proposition 5.2.5. Die Bewertungen auf K entsprechen bijektiv den nicht-archimedischen Normen, wobei die triviale Bewertung der trivialen Norm entspricht und äquivalente Bewertungen äquivalenten Normen entsprechen.

Proof. Seien $q > 1$,

$$M_1 = \{v: K \rightarrow \mathbb{R} \cup \{\infty\} \mid v \text{ ist Bewertung}\}$$

und

$$M_2 = \{|\cdot|: K \rightarrow \mathbb{R} \mid |\cdot| \text{ ist Norm}\}.$$

Definiere weiterhin

$$f = f_q: M_1 \rightarrow M_2, : v \mapsto |\cdot| \text{ mit } |x| = q^{-v(x)} \text{ für } x \in K^\times$$

sowie

$$g = g_q: M_2 \rightarrow M_1, : |\cdot| \mapsto v \text{ mit } v(x) = -\log_q |x| \text{ für } x \in K^\times.$$

Es gilt: f, g sind invers zueinander und bilden trivial (äquivalent) auf trivial (äquivalent) ab. □

Remark 5.2.6. Im Beweis von Proposition 5.2.5 spielt das gewählte $q > 1$ bis auf Äquivalenz keine Rolle.

Theorem 1 (Satz von Ostrowski). Jede nicht-triviale Norm $\|\cdot\|$ auf \mathbb{Q} ist äquivalent zu $|\cdot|_\infty$ oder zu $|\cdot|_p$.

Proof. **Fall 1:** $\|\cdot\|$ ist archimedisch

Schreibe $|\cdot| = |\cdot|_\infty$. Da $\|\cdot\|$ archimedisch ist, gibt es $n_0 \in \mathbb{N}_{>1}$ mit $\|n_0\| > 1$.

Zeige: Für alle $m, n \in \mathbb{N}$ gilt: $\log \|m\| \log |n| = \log |m| \log \|n\|$

Dann folgt die Behauptung aus Lemma 5.1.9.

Für $n = 1$ oder $m = 1$ ist obige Gleichung offensichtlich erfüllt. Schreibe $m = a_r n^r + \dots + a_0$ mit $a_i \in \{0, \dots, n-1\}$ und $a_r \neq 0$. Insbesondere gelten:

$$(A) \ 0 \leq r \leq \frac{\log m}{\log n} \quad (B) \ \|a_i\| = \underbrace{\|1 + \dots + 1\|}_{a_i\text{-mal}} \leq n$$

Fall A: $\|n\| \geq 1$:

$$\begin{aligned} \|m\| &= \|a_0 + \dots + a_r n^r\| \leq \sum_{i=0}^r \|a_i\| \|n\|^i \\ &\stackrel{(B)}{\leq} \sum_{i=0}^r n \|n\|^i = (r+1)n \|n\|^r \\ &\leq \left(\frac{\log m}{\log n} + 1 \right) n \|n\|^{\frac{\log m}{\log n}} \end{aligned}$$

Es folgt für alle $k \in \mathbb{N}$, dass

$$\|m^k\| \leq \left(\frac{k \log m}{\log n} + 1 \right) n \left(\|n\|^{\frac{\log m}{\log n}} \right)^k$$

und somit

$$\|m\| \leq \left(\frac{k \log m}{\log n} + 1 \right)^{\frac{1}{k}} n^{\frac{1}{k}} \|n\|^{\frac{\log m}{\log n}},$$

also $\|m\| \leq \|n\|^{\frac{\log m}{\log n}}$.

Fall B: $\|n\| < 1$:

Analog wie oben erhalte $\|m\| \leq \left(\frac{\log m}{\log n} + 1 \right) n$ und damit auch $\|m^k\| \leq \left(\frac{k \log m}{\log n} + 1 \right) n$ für alle $k \in \mathbb{N}$. Es folgt

$$\|m\| \leq \left(\frac{k \log m}{\log n} + 1 \right)^{\frac{1}{k}} n^{\frac{1}{k}}$$

so, dass der Grenzübergang $k \rightarrow \infty$ schließlich $\|m\| \leq 1$ liefert.

Insgesamt: $\|m\| \leq \max \left\{ \|n\|^{\frac{\log m}{\log n}}, 1 \right\}$

Wähle jetzt $m = n_0$, d.h. $\|m\| = \|n_0\| > 1$, so, dass obige Abschätzung $\|m\| \leq \|n\|^{\frac{\log m}{\log n}}$ liefert. Damit gilt insbesondere $\|n\| > 1$ für alle $n \in \mathbb{N}_{>1}$.

Also tritt Fall B nicht auf und für $n, m \in \mathbb{N}_{>1}$ gilt: $\|m\| \leq \|n\|^{\frac{\log m}{\log n}}$. Durch Vertauschen der Rollen von m und n erhalten wir ebenso $\|n\| \leq \|m\|^{\frac{\log n}{\log m}}$ und damit

$$\|m\| \leq \|n\|^{\frac{\log m}{\log n}} \leq \|m\|$$

so, dass

$$\|m\| = \|n\|^{\frac{\log m}{\log n}}.$$

Es folgt

$$\log \|m\| = \frac{\log |m|}{\log |n|} \log \|n\|$$

und Lemma 5.1.9 liefert schließlich $\|\cdot\| = |\cdot|^s$ mit

$$s = \frac{\log \|n_o\|}{\log |n_o|} > 0.$$

Dies zeigt die Äquivalenz von $\|\cdot\|$ und $|\cdot|$.

Fall 2: $\|\cdot\|$ ist nicht-archimedisch

AZT1 [IV, Prop. 1.3]: Die einzigen diskreten Bewertungen auf \mathbb{Q} sind die p -adischen Bewertungen (bis auf Skalierung). Im Beweis wird **nicht** verwendet, dass v diskret ist. Also folgt die Behauptung aus Proposition 5.2.5 \square

Remark 5.2.7. Sei $v: K \rightarrow \mathbb{R} \cup \{\infty\}$ eine Bewertung und $\mathcal{O} = \{x \in K \mid v(x) \geq 0\}$. Dann gilt für alle $x \in K^\times$, dass $x \in \mathcal{O}$ oder $\frac{1}{x} \in \mathcal{O}$.

Definition 5.2.8. Sei R ein Integritätsbereich und $K = \text{Quot}(R)$. Dann nennen wir R **Bewertungsring**, falls für alle $x \in K$ gilt, dass $x \in R$ oder $\frac{1}{x} \in R$.

Proposition 5.2.9. Sei R ein Bewertungsring. Dann gilt:

- (i) $R^\times = \{x \in R \mid \frac{1}{x} \in R\}$
- (ii) $\mathfrak{m} = \{x \in R \mid \frac{1}{x} \notin R\}$ ist das einzige maximale Ideal von R , insbesondere ist R lokal.
- (iii) R ist ganz abgeschlossen.

Proof. “(i)” Klar. “(ii)” Es genügt zu zeigen, dass \mathfrak{m} ein Ideal ist. Dann folgt die Behauptung aus $\mathfrak{m} = R \setminus R^\times$.

- (a) Seien $r \in R \setminus \{0\}$ und $x \in \mathfrak{m} \setminus \{0\}$. Angenommen, $rx \notin \mathfrak{m}$. Dann gilt $rx \notin R$, denn sonst $rx \in R^\times$ und damit auch $x \in R^\times$. Da R ein Bewertungsring ist, folgt $\frac{1}{rx} \in R$ und damit

$$\frac{1}{x} = r \frac{1}{rx} \in R$$

so, dass $x \in R^\times$ im Widerspruch zur Annahme.

- (b) Seien $x, y \in \mathfrak{m} \setminus \{0\}$. Ohne Einschränkung ist $\frac{x}{y} \in R$, da R ein Bewertungsring ist. Mit (a) folgt dann

$$x + y = \left(1 + \frac{x}{y}\right) y \in \mathfrak{m}.$$

“(iii)” Sei $x \in K = \text{Quot } R$ ganz über R . Dann existieren $a_0, \dots, a_{n-1} \in R$ mit

$$x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0,$$

also

$$x^n = -a_{n-1}x^{n-1} - \dots - a_0. \quad (\star)$$

Angenommen, $x \notin R$. Dann ist $\frac{1}{x} \in R$ und mit (\star) folgt der Widerspruch

$$x = \frac{1}{x^{n-1}} x^n = -a_{n-1} - a_{n-2} \frac{1}{x} - \dots - a_0 \frac{1}{x^{n-1}} \in R.$$

□

Corollary 5.2.10. Sei $v: K \rightarrow \mathbb{R} \cup \{\infty\}$ eine Bewertung mit zugehöriger Norm $|\cdot|$. Dann ist

$$\mathcal{O} = \{x \in K \mid v(x) \geq 0\} = \{x \in K \mid |x| \leq 1\}$$

ein Bewertungsring mit Einheiten

$$\mathcal{O}^\times = \{x \in K \mid v(x) = 0\} = \{x \in K \mid |x| = 1\}$$

und eindeutigem maximalem Ideal

$$\mathfrak{m} = \{x \in K \mid v(x) > 0\} = \{x \in K \mid |x| < 1\}.$$

Example 5.2.11. Seien k ein Körper und

$$k\{X\} = \left\{ a_0 X^{l_0} + a_1 X^{l_1} + \dots \mid a_i \in k, l_i \in \mathbb{R}, l_0 < l_1 < \dots, \lim_{i \rightarrow \infty} l_i = \infty \right\}$$

mit Addition und Multiplikation definiert analog zu formalen Laurent-Reihen. Dann ist $k\{X\}$ ein Körper und $v: K\{X\} \rightarrow \mathbb{R} \cup \{\infty\}$ mit $0 \mapsto \infty$,

$$0 \neq z = a_0 X^{l_0} + a_1 X^{l_1} + \dots \mapsto l_0,$$

wobei $a_0 \neq 0$, ist eine Bewertung, die nicht diskret ist.

5.3. Kompletterung

Sei K stets ein Körper.

Definition 5.3.1. Sei (X, d) ein metrischer Raum.

- (i) Eine Folge $(a_n)_n$ von Punkten in X heißt **Cauchy-Folge**, falls für alle $\varepsilon > 0$ ein $N \in \mathbb{N}$ existiert mit $d(x_n, x_m) < \varepsilon$ für alle $n, m \geq N$.
- (ii) (X, d) heißt **vollständig**, falls jede Cauchy-Folge in (X, d) konvergiert.
- (iii) \hat{X}, \hat{d} heißt **Kompletterung** von (X, d) , falls
 - (\hat{X}, \hat{d}) ist vollständig,
 - es existiert eine isometrische Einbettung $\varphi: X \hookrightarrow \hat{X}$ mit $\varphi(X) \subset \hat{X}$ dicht.

Fact 5.3.2. Jeder metrische Raum besitzt eine **eindeutige** metrische Kompletterung.

Theorem 2. (i) Jeder Körper $(K, |\cdot|)$ besitzt eine **Vervollständigung** $(\hat{K}, \|\cdot\|, i)$, d.h.

- $(\hat{K}, \|\cdot\|)$ ist ein normierter Körper,
 - $i: K \rightarrow \hat{K}$ ist eine Körpererweiterung, die die Norm erhält, d.h. $\|i(x)\| = |x|$ für alle $x \in K$,
 - \hat{K} ist die metrische Kompletterung von K bezüglich den von den Normen induzierten Metriken.
- (ii) Die Vervollständigung in (i) ist eindeutig wie folgt: Sind $(\hat{K}_1, \|\cdot\|_1, i_1), (\hat{K}_2, \|\cdot\|_2, i_2)$ Vervollständigungen von $(K, |\cdot|)$, dann gibt es einen eindeutigen isometrischen Körperisomorphismus $h: \hat{K}_1 \rightarrow \hat{K}_2$ mit $h \circ i_1 = i_2$.

Fact 5.3.3. \mathbb{R} ist die Vervollständigung von \mathbb{Q} bezüglich $|\cdot|_\infty$.

Proposition 5.3.4. (i) $(\mathbb{Q}, |\cdot|_p)$ ist nicht vollständig für $p \neq 2$.

(ii) Ist $(K, |\cdot|)$ eine nicht-archimedische Norm, so gilt:

$$(a_n)_n \text{ ist Cauchy-Folge in } K \Leftrightarrow |a_{n+1} - a_n| \xrightarrow{n \rightarrow \infty} 0$$

Proof. “(ii)” “ \Rightarrow ” Klar. “ \Leftarrow ” Ohne Einschränkung sei $m < n$. Es gilt

$$|a_m - a_n| = |a_m - a_{m-1} + a_{m-1} - \cdots + a_{n-1} - a_n| \leq \max\{|a_m - a_{m-1}|, \dots, |a_{n-1} - a_n|\}$$

und damit folgt die Behauptung.

“(i)” **Beweisidee für $p \neq 2$:** Wähle $a \in \mathbb{Z}$ mit

- (1) $p \nmid a$,
- (2) a kein Quadrat,

$$(3) \ a \equiv b^2 \pmod{p}.$$

Konstruiere Folge $(b_n)_n$ mit

$$b_n \equiv b_{n-1} \pmod{p^n} \quad \text{und} \quad b_n^2 \equiv a \pmod{p^{n+1}}.$$

Dann gilt

$$|b_n - b_{n-1}|_p \leq \frac{1}{p^n} \xrightarrow{n \rightarrow \infty} 0,$$

also ist $(b_n)_n$ eine Cauchy-Folge. Angenommen $b_n \rightarrow b_\infty \in \mathbb{Q}$, dann auch $b_n^2 \rightarrow b_\infty^2$. Andererseits gilt $b_n^2 \rightarrow a$, also $a = b_\infty^2$, aber a ist kein Quadrat.

Existenz einer solchen Folge (b_n) folgt aus Lemma 5.3.6. □

Example 5.3.5. $1, 1+p, 1+p+p^2, 1+p+p^2+p^3, \dots$ ist eine Cauchy-Folge. Man kann zeigen, dass diese Folge in \mathbb{Q} bezüglich $|\cdot|_p$ konvergiert.

Lemma 5.3.6. Seien $a, b \in \mathbb{Z}, n \in \mathbb{N}, p \neq 2$ prim mit $p \nmid a$ und $a \equiv b^2 \pmod{p^n}$. Dann existiert ein $x \in \mathbb{Z}$ mit $x \equiv b \pmod{p^n}$ und $x^2 \equiv a \pmod{p^{n+1}}$.

Proof. Nach Voraussetzung existiert ein $c \in \mathbb{Z}$ mit $a = b^2 + cp^n$. Wir suchen nun ein x mit $x = b + dp^n$ und $x^2 \equiv a \pmod{p^{n+1}}$. Es gilt

$$(b + dp^n)^2 = b^2 + 2bdp^n + d^2p^{2n} = a - cp^n + 2bdp^n + d^2p^{2n} \equiv a + p^n(-c + 2bd) \pmod{p^{n+1}}.$$

Wähle also d so, dass $-c + 2bd \equiv 0 \pmod{p}$. Dies ist möglich, da $b \not\equiv 0 \pmod{p}$ und $p \neq 2$. Dann leistet $x = b + dp^n$ das Gewünschte. □

Beweisidee von Satz 2. “(i)” Konstruiere \hat{K} wie folgt:

- (1) $\mathcal{C} = \{(a_n)_n \mid a_n \in K, (a_n)_n \text{ Cauchy-Folge}\}$ mit komponentenweiser Addition und Multiplikation ist ein Ring.
- (2) Sei $\mathcal{N} = \{(a_n)_n \in \mathcal{C} \mid |a_n| \rightarrow 0\}$ die Menge aller Nullfolgen. Dann ist \mathcal{N} ein Ideal in \mathcal{C} , nach Lemma 3.7 sogar maximal.
- (3) Definiere $\hat{K} = \mathcal{C}/\mathcal{N}$ und $i: K \hookrightarrow \hat{K}, z \mapsto (z, z, \dots)$.
- (4) Definiere $\|(a_n)_n\| = \lim_{n \rightarrow \infty} |a_n|$.

□

Lemma 5.3.7. \mathcal{N} wie oben definiert ist maximal.

Proof. Angenommen, es gibt ein Ideal I mit $\mathcal{C} \supsetneq I \supsetneq \mathcal{N}$. Wähle $b = (b_n) \in I \setminus \mathcal{N}$, d.h. b ist Cauchy-Folge, aber keine Nullfolge.

Erinnerung:

- (1) (b_n) Cauchy-Folge $\Rightarrow (|b_n|)$ Cauchy-Folge

(2) (b_n) Nullfolge $\Leftrightarrow (|b_n|)$ Nullfolge

Also existieren $\lambda > 0$ und $N \in \mathbb{N}$ so, dass $|b_n| > \lambda$ für alle $n \geq N$. Definiere $b' = (b'_n)$ mit

$$b'_n = \begin{cases} 1, & n < N, \\ b_n, & n \geq N. \end{cases}$$

Also $b \in \mathcal{C}^\times$ und $a = b - b' \in \mathcal{N} \subset I$ so, dass $b' \in I$ und daher $I = \mathcal{C}$. \square

Lemma 5.3.8. Sei $a = (a_n) \in \mathcal{C}$. Dann existiert $\|a\| = \lim |a_n|$, ist invariant unter Addition von Nullfolgen und $a \mapsto \|a\|$ definiert eine Norm auf \hat{K} .

Proof. Ist $a = (a_n)$ eine Cauchy-Folge, so auch $(|a_n|)$, also existiert $\lim |a_n|$, da \mathbb{R} vollständig ist. Ist weiter (c_n) eine Nullfolge so gilt

$$\|a\| \stackrel{n}{\leftarrow} |a_n| - |c_n| \leq |a_n + c_n| \leq |a_n| + |c_n| \stackrel{n}{\rightarrow} \|a\|$$

so, dass $\lim |a_n + c_n| = \|a\|$. Schließlich gilt

$$\|a\| = 0 \Leftrightarrow \lim |a_n| = 0 \Leftrightarrow (a_n) \text{ Nullfolge}$$

und die restlichen Normeigenschaften bleiben bei Limesbildung erhalten. \square

Lemma 5.3.9. $i(K)$ liegt dicht in \hat{K} .

Proof. Sei $a = (a_n) \in \hat{K}$ und definiere die Folge $a^{(n)} \in \hat{K}$ durch

$$a^{(n)} = i(a_n) = (a_n, a_n, \dots).$$

Dann gilt $a^{(n)} \xrightarrow{n} a$, denn (a_n) ist eine Cauchy-Folge, also existiert zu allen $\varepsilon > 0$ ein $N \in \mathbb{N}$ mit $|a_n - a_m| < \varepsilon$ für alle $n, m \geq N$ so, dass

$$\|a - a^{(n)}\| = \lim_{j \rightarrow \infty} |a_j - a_n| < \varepsilon$$

für alle $n \geq N$ gilt. \square

Lemma 5.3.10. \hat{K} ist vollständig.

Proof. Sei $(a^{(n)})$ eine Cauchy-Folge in \hat{K} . Nach Lemma 5.3.8 existiert für alle $n \in \mathbb{N}$ ein $x_n \in K$ mit $\|i(x_n) - a^{(n)}\| < \frac{1}{n}$, das heißt

$$\lim_{j \rightarrow \infty} |x_n - a_j^{(n)}| < \frac{1}{n}.$$

Definiere $x = (x_n)$ und zeige:

- (1) x ist Cauchy-Folge in K und somit $x \in \hat{K}$.
- (2) $a^{(n)} \xrightarrow{n} x$ in \hat{K} .

“(1)” Sei $\varepsilon > 0$. Dann existiert N_1 mit $\|a^{(n)} - a^{(m)}\| < \frac{\varepsilon}{2}$ für alle $n, m \geq N_1$. Wähle N_1 so groß, dass auch

$$\frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2}$$

für alle $n, m \geq N_1$ gilt. Dann folgt

$$\begin{aligned} |x_n - x_m| &= \|i(x_n) - i(x_m)\| \leq \|i(x_n) - a^{(n)}\| + \|a^{(n)} - a^{(m)}\| + \|a^{(m)} - i(x_m)\| \\ &< \frac{1}{n} + \frac{\varepsilon}{2} + \frac{1}{m} < \varepsilon. \end{aligned}$$

“(2)” Sei $\varepsilon > 0$. Dann existiert N_1 mit $|x_n - x_m| < \frac{\varepsilon}{2}$ für alle $n, m \geq N_1$. Wähle $N_2 \geq N_1$ mit $\frac{1}{N_2} < \frac{\varepsilon}{2}$. Dann gilt

$$\|a^{(n)} - x\| = \lim_{j \rightarrow \infty} |a_j^{(n)} - x_j| \leq \lim_{j \rightarrow \infty} (|a_j^{(n)} - x_n| + |x_n - x_j|) \leq \frac{1}{n} + \frac{\varepsilon}{2} < \varepsilon$$

für alle $n \geq N_2$, also ist \hat{K} vollständig. □

Lemma 5.3.11. In der Situation von Theorem 2 (ii) gilt: Definiere $h: \hat{K}_1 \rightarrow \hat{K}_2$ durch:

$$a = \lim_{n \rightarrow \infty} i_1(a_n) \text{ mit } a_n \in K \mapsto h(a) = \lim_{n \rightarrow \infty} i_2(a_n)$$

Proof. Nachrechnen. □

Beweis von Theorem 2. Folgt aus Lemma 3.7-3.11. □

Remark 5.3.12. Es sei $(\hat{K}, \|\cdot\|)$ die Vervollständigung von $(K, |\cdot|)$ für eine nicht-archimedische Norm $|\cdot|$ und

$$\hat{v}: \hat{K} \rightarrow \mathbb{R} \cup \{\infty\}, x \mapsto -\log \|x\|$$

die von $\|\cdot\|$ induzierte Bewertung. Dann gilt:

(i) Für $x = (c_n) \in \hat{K}$ gilt $\hat{v}(x) = \lim_{n \rightarrow \infty} v(c_n)$, wobei $v = -\log |\cdot|$.

(ii) Für $x \neq 0$ wird $v(c_n)$ stationär.

Proof. “(i)” Ergibt sich aus den Definitionen. “(ii)” Für große n wird $\|x - c_n\|$ klein, also wird $\hat{v}(x - c_n)$ groß. Nach Bemerkung 2.3 gilt daher

$$\hat{v}(c_n) = \hat{v}(c_n - x + x) = \min\{\hat{v}(c_n - x), \hat{v}(x)\} = \hat{v}(x).$$

Also wird die Folge $(v(c_n))$ stationär. □

Definition 5.3.13. Die Vervollständigung von \mathbb{Q} bezüglich $|\cdot|_p$ heißt der **Körper der p -adischen Zahlen** \mathbb{Q}_p . Bezeichne die von $|\cdot|_p$ auf \mathbb{Q}_p induzierte Norm ebenfalls mit $|\cdot|_p$ und definiere den zugehörigen Bewertungsring durch

$$\mathbb{Z}_p = \{q \in \mathbb{Q}_p \mid \|q\|_p \leq 1\}.$$

Wir nennen \mathbb{Z}_p den **Ring der ganzen p -adischen Zahlen**.

5.4. Projektive Limiten

Sei \mathcal{C} eine Kategorie. Bezeichne für zwei Objekte $X, Y \in \mathcal{C}$ mit $\text{Mor}(X, Y)$ die Menge der Morphismen von X nach Y .

Definition 5.4.1. Ein **inverses System** besteht aus

- (i) einer partiell geordneten Menge (I, \leq) ,
- (ii) einem Objekt $A_i \in \mathcal{C}$ für jedes $i \in I$,
- (iii) einem Morphismus $f_{ij}: A_j \rightarrow A_i$ für jedes $(i, j) \in I \times I$ mit $i \leq j$ mit folgenden Verträglichkeitseigenschaften:
 - $f_{ii} = \text{id}_{A_i}$
 - $f_{ik} = f_{ij} \circ f_{jk}$, falls $i \leq j \leq k$

Example 5.4.2. Sei $\mathcal{C} = \mathbf{Groups}$.

- (i) Sei $I = \mathbb{N}$ mit $n \leq m$ falls $n|m$. Dann ist $A_n = \mathbb{Z}/n\mathbb{Z}$ und für $n \leq m$ ist

$$f_{nm}: \mathbb{Z}/m\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$$

die natürliche Projektion.

- (ii) Allgemeiner sei G eine beliebige Gruppe und

$$I = \{N \mid N \text{ Normalteiler von } G \text{ von endlichem Index}\}$$

mit $N_q \leq N_2$ falls $N_2 \subset N_1$. Dann ist $A_N = G/N$ und

$$f_{NM}: G/M \rightarrow G/N$$

die natürliche Projektion.

Definition 5.4.3. Der **projektive Limes** oder **inverse Limes** eines inversen Systems wie in Definition 5.4.1 besteht aus

- (i) einem Objekt A ,
- (ii) einem Morphismus $\Pi_i: A \rightarrow A_i$ für jedes $i \in I$,

so, dass

- (A) $f_{ij} \circ \Pi_j = \Pi_i$ für jedes $(i, j) \in I \times I$ mit $i \leq j$,
- (B) für jedes Objekt B mit einer Familie $\{\Psi_i: B \rightarrow A_i \mid i \in I\}$ die ebenfalls (A) erfüllen gibt es einen eindeutigen Morphismus $u: B \rightarrow A$ so, dass $\Pi_i \circ u = \Psi_i$ für alle $i \in I$ gilt.

Der projektive Limes $(A, \{\Pi_i\}_{i \in I})$ wird mit $\varprojlim_{i \in I} A_i$ bezeichnet.

Remark 5.4.4. Falls ein projektiver Limes wie in Definition 5.4.3 definiert existiert, dann ist er eindeutig bis auf eindeutigen Isomorphismus, das heißt: Ist $(A', \{\Pi'_i\}_{i \in I})$ ebenfalls ein projektiver Limes, dann gibt es einen eindeutigen Isomorphismus $u: A' \rightarrow A$ mit $\Pi_i \circ u = \Pi'_i$ für alle $i \in I$.

Proof. Wie immer. □

Proposition 5.4.5. *In der Kategorie **Groups** gibt es für jedes inverse System einen projektiven Limes.*

Proof. Verwende die Notation aus Definition 5.4.1. Definiere

$$A = \left\{ a = (a_i)_{i \in I} \in \prod_{i \in I} A_i \mid f_{ij}(a_j) = a_i \text{ für alle } i \leq j \right\}$$

und $\Pi_i: A \rightarrow A_i, a \mapsto a_i$. Mit komponentenweiser Verknüpfung wird $\prod_{i \in I} A_i$ zu einer Gruppe und A ist eine Untergruppe. Zeige nun, dass A projektiver Limes ist. Offensichtlich gilt Eigenschaft (A), zu (B) definiere

$$u: B \rightarrow A, b \mapsto (\Psi_i(b_i))_{i \in I}.$$

□

Example 5.4.6. (i) Für das inverse System in Beispiel 5.4.2 (i) erhalten wir als projektiven Limes

$$\hat{\mathbb{Z}} = \{(a_n)_{n \in \mathbb{N}} \mid a_n \in \mathbb{Z}/n\mathbb{Z}, n \text{ teilt } m \Rightarrow a_m \equiv a_n \pmod{n}\}$$

mit komponentenweiser Addition.

(ii) Für Beispiel 5.4.2 (ii) erhalten wir

$$\hat{G} = \{(g_N)_N \mid \begin{array}{l} g_N \text{ Normalteiler von } G \text{ mit endlichem Index} \\ g_N \in G/N, \\ M \subset N \Rightarrow g_N \pmod{N} = g_M \pmod{M} \end{array}\}.$$

\hat{G} heißt **proendliche Komplettierung** von G .

Remark 5.4.7. Analog erhält man projektive Limiten unter anderem in folgenden Kategorien:

Mengen, Ringe, R -Moduln, k -Algebren

Definition 5.4.8. Eine **topologische Gruppe** besteht aus

- (1) einer Gruppe G und
- (2) einer Topologie T auf G so, dass

- (A) die Gruppenabbildung $G \times G \rightarrow G, (a, b) \mapsto ab$ stetig ist und
 (B) die Inversenabbildung $G \rightarrow G, g \mapsto g^{-1}$ stetig ist.

Ein **Morphismus** φ zwischen topologischen Gruppen ist ein stetiger Gruppenhomomorphismus.

Remark 5.4.9. Topologische Gruppen bilden mit Morphismen eine Kategorie, **TopGroups**.

Definition 5.4.10. Sei I eine Indexmenge und $(X_i, T_i)_{i \in I}$ eine Familie topologischer Räume. Sei $X = \prod_{i \in I} X_i$ das kartesische Produkt und $p_i: X \rightarrow X_i, (x_k) \mapsto x_i$. Dann ist die **Produkttopologie** T die kleinste Topologie auf X für die alle p_i stetig sind. Das heißt

$$\mathcal{B} = \{p_i^{-1}(U) \mid i \in I, U \subset X_i \text{ offen}\}$$

bildet eine Subbasis von T .

Reminder 5.4.11. • $\mathcal{B} \subset T$ heißt **Subbasis**, falls jede offene Menge $U \in T$ eine beliebige Vereinigung von Mengen, die endliche Schnitte von Mengen in \mathcal{B} sind, ist.

- Für $x \in X$ heißt $\mathcal{B}_x \subset T$ **Umgebungsbasis** von x , falls:
 - (1) Für alle $U \in \mathcal{B}_x$ gilt $x \in U$.
 - (2) Für jedes $V \in T$ mit $x \in V$ existiert ein $U \in \mathcal{B}_x$ mit $U \subset V$.

Remark 5.4.12. Sei G eine topologische Gruppe mit neutralem Element 1.

- (1) Jedes $g \in G$ definiert einen Homöomorphismus (**kein** Automorphismus)

$$\Phi_g: G \rightarrow G, x \mapsto gx$$

mit Umkehrabbildung $\Phi_{g^{-1}}$.

- (2) $\mathcal{B}_{(1)}$ ist Umgebungsbasis von 1 genau dann, wenn

$$g\mathcal{B}_{(1)} = \{gU \mid U \in \mathcal{B}_{(1)}\}$$

eine Umgebungsbasis von g ist für jedes $g \in G$.

- (3) Ist G zusammenhängend so gilt

$$G = \bigcup_{n \in \mathbb{N}} \mathcal{B}_{(1)}^n.$$

Remark 5.4.13. In der Kategorie **TopSpaces** und in der Kategorie **TopGroups** existieren jeweils projektive Limiten.

Proof. Analog zu Proposition 5.4.5 betrachte auf $\prod_{i \in I} A_i$ die Produkttopologie und auf A die Spurtopologie davon. \square

Remark 5.4.14. Die proendliche Kompletterung \hat{G} einer Gruppe G wird zu einer topologischen Gruppe wie folgt:

- (1) Versee die Quotienten G/N mit der diskreten Topologie $\mathcal{P}(G/N)$.
- (2) Bilde den projektiven Limes in der Kategorie der topologischen Gruppen **TopGroups**.

Definition 5.4.15. Eine topologische Gruppe G heißt **proendlich**, falls G isomorph zum projektiven Limes von endlichen Gruppen, jeweils versehen mit der diskreten Topologie, ist.

Remark 5.4.16. Analog definiert man **topologische Ringe** als Ringe $(R, +, \cdot)$ mit $(R, +)$ topologische Gruppe und $\cdot: R \times R \rightarrow R$ stetig und sieht, dass auch für diese projektive Limiten existieren.

5.5. Ganze p -adische Zahlen

Ziel: Zeige, dass

$$\mathbb{Z}_p \cong \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/p^n \mathbb{Z},$$

beziehungsweise allgemeiner, dass

$$\hat{\mathcal{O}} \cong \varprojlim_{n \in \mathbb{N}} \mathcal{O}/\mathfrak{p}^n.$$

Sei ab jetzt:

- K ein Körper, v eine diskrete Bewertung auf K und $|\cdot| = q^{-v(\cdot)}$ die zugehörige Norm für ein $q > 1$
- \hat{K} die Vervollständigung von K bezüglich $|\cdot|$, \hat{v} die zugehörige Bewertung und $|\cdot| = q^{-\hat{v}(\cdot)}$ die zugehörige Norm
- \mathcal{O} der Bewertungsring zu v mit maximalem Ideal $\mathfrak{p} = \Pi \cdot \mathcal{O}$ mit $v(\Pi) = 1$
- $\hat{\mathcal{O}}$ der Bewertungsring zu \hat{v} mit maximalem Ideal $\hat{\mathfrak{p}} = \Pi \cdot \hat{\mathcal{O}}$

Remark 5.5.1. Da v diskrete Bewertung ist gilt $v(K) = \hat{v}(\hat{K})$, denn für $c_n \xrightarrow{n \rightarrow \infty} x$ gilt $\hat{v}(x) = \lim_{n \rightarrow \infty} v(c_n)$.

Ohne Einschränkung sei ab jetzt $v(k) = \hat{v}(\hat{K}) = \mathbb{Z} \cup \{\infty\}$.

Proposition 5.5.2. Für alle $n \in \mathbb{N}_0$ gilt $\hat{\mathcal{O}}/\hat{\mathfrak{p}}^n \cong \mathcal{O}/\mathfrak{p}^n$.

Proof. Für $x \in \mathcal{O}$ gilt:

$$x \in \mathfrak{p}^n \Leftrightarrow v(x) \geq n \Leftrightarrow \hat{v}(x) \geq n \Leftrightarrow x \in \hat{\mathfrak{p}}^n$$

Also steigt $i: \mathcal{O} \hookrightarrow \hat{\mathcal{O}}$ ab zu einem injektiven Morphismus $\mathcal{O}/\mathfrak{p}^n \hookrightarrow \hat{\mathcal{O}}/\hat{\mathfrak{p}}^n$. Dieser ist surjektiv, denn zu $x \in \hat{\mathcal{O}}$ existiert eine Folge (c_k) in K mit $c_k \rightarrow x$. Da v eine diskrete Bewertung ist, gilt ohne Einschränkung $c_k \in \mathcal{O}$, also ist für k groß genug $|x - c_k| < q^{-n}$ und damit $v(x - c_k) > n$. Also $c_k \equiv x \pmod{\mathfrak{p}^n}$. Folglich wird $c_k \pmod{\mathfrak{p}^n}$ abgebildet auf $x \pmod{\hat{\mathfrak{p}}^n}$. \square

Proposition 5.5.3. *Es gilt $\hat{\mathcal{O}} \cong \varprojlim_{n \in \mathbb{N}} \mathcal{O}/\mathfrak{p}^n$ in der Kategorie **TopRings** von topologischen Ringen. Hierbei erhält $\hat{\mathcal{O}}$ die von der Norm $|\cdot|$ induzierte Topologie und die Ringe $\mathcal{O}/\mathfrak{p}^n$ die diskrete Topologie.*

Proof. $p_{m,n}^r: \mathcal{O}/\mathfrak{p}^n \rightarrow \mathcal{O}/\mathfrak{p}^m$ sei die natürliche Projektion für $m \leq n$ und

$$h_n: \mathcal{L} = \varprojlim_{n \in \mathbb{N}} \mathcal{O}/\mathfrak{p}^n \rightarrow \mathcal{O}/\mathfrak{p}^n$$

die Projektionsabbildung des Limes.

(1) Definiere eine Abbildung $\Phi: \hat{\mathcal{O}} \rightarrow \mathcal{L}$:

Verwende den Isomorphismus $\alpha_n: \hat{\mathcal{O}}/\hat{\mathfrak{p}}^n \rightarrow \mathcal{O}/\mathfrak{p}^n$ aus Proposition 5.5.2 und definiere

$$\Phi_n: \hat{\mathcal{O}} \rightarrow \mathcal{O}/\mathfrak{p}^n, x \mapsto \alpha_n(x \bmod \hat{\mathfrak{p}}^n).$$

Dann ist Φ_n ein Ringhomomorphismus und die $\{\Phi_n\}$ passen zusammen mit den Projektionen $p_{m,n}$.

Zeige noch: Φ_n ist stetig

Sei $U \subset \mathcal{O}/\mathfrak{p}^n$ und $V = \Phi_n^{-1}(U)$, Es gilt $\ker \Phi_n = \hat{\mathfrak{p}}^n$. Für $x \in V$ und $\varepsilon_n = q^{-n}$ ist dann

$$\begin{aligned} K_{\varepsilon_n}(x) &= \{y \in \hat{\mathcal{O}} \mid |y - x| < \varepsilon_n\} \\ &= \{y \in \hat{\mathcal{O}} \mid \hat{v}(y - x) > n\} \\ &= x + \hat{\mathfrak{p}}^{n+1}. \end{aligned}$$

Folglich gilt $\Phi(n)(K_{\varepsilon_n}(x)) = \{x\}$ und daher $K_{\varepsilon_n}(x) \subset V$. Also ist V offen. Nach der Definition von projektivem Limes induziert $\{\Phi_n\}$ also einen stetigen Ringhomomorphismus $\Phi: \hat{\mathcal{O}} \rightarrow \mathcal{L}$.

(2) Φ ist injektiv:

$$\begin{aligned} \Phi(x) = 0 &\Leftrightarrow \Phi_n(x) = 0 \text{ für alle } n \in \mathbb{N} \\ &\Leftrightarrow x \bmod \hat{\mathfrak{p}}^n \text{ für alle } n \in \mathbb{N} \\ &\Leftrightarrow x = 0 \end{aligned}$$

(3) Φ ist surjektiv:

Sei $s = (\overline{s_n}) \in \mathcal{L}$, also $p_{n,m}^r(\overline{s_m}) = \overline{s_n}$ für $n \leq m$. Wähle für jedes $\overline{s_n}$ ein Urbild $s_n \in \mathcal{O}$. Es gilt also $s_m - s_n \in \mathfrak{p}^n$ für $m \geq n$. Also ist (s_n) eine Cauchy-Folge und es existiert $x = \lim_n s_n \in \hat{\mathcal{O}}$. Dann gilt $\Phi(x) = s$, denn:

Es gilt für alle $m \geq n$, dass $s_n \equiv s_m \bmod \mathfrak{p}^n$. Wegen $s_n \rightarrow x$ existiert k_0 mit $v(x - s_n) \geq n$. Wähle $k \geq k_0, n$, dann gilt

$$x \equiv s_k \equiv s_n \bmod \hat{\mathfrak{p}}^n$$

und damit $\Phi_n(x) = \overline{s_n}$ so, dass $\Phi(x) = s$.

(3) Φ ist offen, d.h. für alle $U \subset \hat{\mathcal{O}}$ offen ist $\Phi(U)$ offen:

Da $(\hat{\mathcal{O}}, +)$ eine topologische Gruppe ist, genügt es, dies für alle U einer Umgebungsbasis von \mathcal{O} zu überprüfen und

$$\mathcal{B} = \{K_{q^{-1}}(0) \mid n \in \mathbb{N}_0\} = \{\hat{\mathfrak{p}}^n \mid n \in \mathbb{N}\}$$

ist eine Umgebungsbasis von 0. Für $\hat{\mathfrak{p}}^n \in \mathcal{B}$ gilt:

$$\begin{aligned} \Phi(\hat{\mathfrak{p}}^n) &= \{(\overline{s_k}) \in \mathcal{L} \mid \overline{s_k} = 0 \text{ für } k \leq n\} \\ &= h_n^{-1}(\{0\}) \text{ offen in } \mathcal{L} \end{aligned}$$

Somit ist Φ offen. □

Remark 5.5.4. Die Mengen

$$U_n = \{0\} \times \cdots \times \{0\} \times \mathcal{O}/\mathfrak{p}^{n+1} \times \cdots$$

als offene Teilmengen von $\mathcal{L} = \varprojlim_{n \in \mathbb{N}} \mathcal{O}/\mathfrak{p}^n$ bilden eine Umgebungsbasis der 0.

Remark 5.5.5. Der Homomorphismus

$$\mathcal{O}^\times \rightarrow (\mathcal{O}/\mathfrak{p}^n)^\times, u \mapsto u \pmod{\mathfrak{p}^n}$$

hat als Kern $U^{(n)} = 1 + \mathfrak{p}^n$ und ist surjektiv. Somit ist $U^{(n)}$ eine Untergruppe von \mathcal{O}^\times und

$$\mathcal{O}^\times / U^{(n)} \cong (\mathcal{O}/\mathfrak{p}^n)^\times.$$

Corollary 5.5.6. *Als topologische Gruppen gilt*

$$\hat{\mathcal{O}}^\times = \varprojlim_{n \in \mathbb{N}} \mathcal{O}^\times / U^{(n)}.$$

Proof. Der Isomorphismus $\Phi: \hat{\mathcal{O}} \varprojlim_{n \in \mathbb{N}} \mathcal{O}/\mathfrak{p}^n$ schränkt sich ein zu einem Isomorphismus

$$\hat{\mathcal{O}}^\times \rightarrow \left(\varprojlim_{n \in \mathbb{N}} \mathcal{O}/\mathfrak{p}^n \right)^\times. \text{ Damit folgt}$$

$$\left(\varprojlim_{n \in \mathbb{N}} \mathcal{O}/\mathfrak{p}^n \right)^\times \cong \varprojlim_{n \in \mathbb{N}} (\mathcal{O}/\mathfrak{p}^n)^\times \cong \varprojlim_{n \in \mathbb{N}} \mathcal{O}^\times / U^{(n)}.$$

□

Corollary 5.5.7. $\mathbb{Z}_p \cong \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/p^n \mathbb{Z}$

Ziel: Wähle zu jedem $x \in \mathbb{Z}_p$ eine eindeutige Folge (s_n) in \mathbb{Z} mit $s_n \rightarrow x$ und zwar so, dass $v(s_{n+1} - s_n) > n$, d.h.

$$s_{n+1} = s_n + a_n p^n.$$

Damit hat s_{n+1} die Form

$$s_{n+1} = a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n.$$

Idee: Wenn wir $a_i \in \{0, 1, \dots, p-1\}$ fordern, dann geht das auf eindeutige Weise.

Proposition 5.5.8. Sei $R \subset \mathcal{O}$ ein Repräsentantensystem für $\kappa = \mathcal{O}/\mathfrak{p}$ mit $0 \in R$.

(i) Für alle $x \in \hat{\mathcal{O}}$ existiert eine eindeutige Folge $(a_n)_{n \in \mathbb{N}_0}$ in R mit

$$x \equiv s_n = a_0 + a_1 \Pi + a_2 \Pi^2 + \cdots + a_{n-1} \Pi^{n-1} \pmod{\hat{\mathfrak{p}}^n}$$

für alle $n \in \mathbb{N}$ und es gilt $s_n \rightarrow x$ bezüglich $|\cdot|$.

(ii) Für $x \in \hat{K}^\times$ existiert eine analoge Folge mit

$$s_n = a_m \Pi^m + a_{m+1} \Pi^{m+1} + \cdots + a_n \Pi^n$$

für $n \geq m$ so, dass $s_n \equiv x \pmod{\hat{\mathfrak{p}}^n}$ und $s_n \rightarrow x$. Hierbei ist $m = v(x)$.

Proof. “(i)” Nach Proposition 5.5.2 existiert ein eindeutiges $a_0 \in R$ mit $a_0 \equiv x \pmod{\hat{\mathfrak{p}}^n}$. Definiere a_n rekursiv wie folgt:

Es gilt bereits $x \equiv s_n = a_0 + a_1 \Pi + a_2 \Pi^2 + \cdots + a_{n-1} \Pi^{n-1} \pmod{\hat{\mathfrak{p}}^n}$. Dann ist

$$x - s_n = u_n \Pi^n$$

für ein $u_n \in \hat{\mathcal{O}}$. Nach Proposition 5.5.2 lässt sich u_n schreiben als

$$a_n + r \Pi$$

mit eindeutigem $a_n \in R$ und einem $r \in \hat{\mathcal{O}}$. Also

$$\begin{aligned} x &= a_0 + a_1 \Pi + a_2 \Pi^2 + \cdots + a_{n-1} \Pi^{n-1} + u_n \Pi^n \\ &= a_0 + a_1 \Pi + a_2 \Pi^2 + \cdots + a_{n-1} \Pi^{n-1} + a_n \Pi^n + r \Pi^{n+1}. \end{aligned}$$

Also leistet a_n das Gewünschte, d.h.

$$x \equiv a_0 + a_1 \Pi + a_2 \Pi^2 + \cdots + a_n \Pi^n = s_{n+1} \pmod{\hat{\mathfrak{p}}^{n+1}}.$$

Schließlich $v(x - s_n) \geq n + 1$ und damit $s_n \rightarrow x$.

“(ii)” Sei $x \in \hat{K}^\times$. Dann ist $x = u \Pi^m$ mit einem $u \in \mathcal{O}^\times$ und $m = v(x)$. Erhalte aus (i) eine Folge (a'_n) für u mit

$$a'_0 + a'_1 \Pi + \cdots + a'_n \Pi^n \rightarrow u.$$

Wegen $u \in \mathcal{O}^\times$ ist $a'_0 \neq 0$. Erhalte also

$$\Pi^m (a'_0 + a'_1 \Pi + \cdots + a'_n \Pi^n) \rightarrow x$$

und setze $a_n = a'_{n+m}$. Dann leistet (a_n) das Gewünschte. □

Example 5.5.9. Sei $K = \mathbb{Q}$ mit p -adischer Norm $|\cdot|_p$.

$$\hat{K} \quad \varprojlim \mathbb{Z}/p^n \mathbb{Z} \quad \{0, \dots, p-1\}^{N_0} \\ -1 \leftrightarrow (p-1, p-1, p-1, \dots) \leftrightarrow p-1 + (p-1)p + (p-1)p^2 + \dots$$

Corollary 5.5.10. *Es gilt*

$$\mathbb{Z}_p = \{a_0 + a_1 p + a_2 p^2 + \dots \mid a_i \in \{0, \dots, p-1\}\} \\ = \{0, \dots, p-1\}^{N_0}$$

und

$$\mathbb{Q}_p = \{a_m p^m + a_{m-1} p^{m-1} + a_{m-2} p^{m-2} + \dots \mid m \in \mathbb{Z}, a_i \in \{0, \dots, p-1\}, a_m \neq 0\} \\ = \text{Quot}(\mathbb{Z}_p).$$

Addition und Multiplikation ergibt sich wie bei Potenzreihen beziehungsweise Laurentreihen.

Example 5.5.11. Seien $K = k(T) = \text{Quot}(k[T])$ und

$$v: K \rightarrow \mathbb{Z} \cup \{\infty\}, \frac{p}{q} \mapsto \deg(p) - \deg(q).$$

Weiter sei $\mathcal{O} = k[T] \supset \mathfrak{p} = (T)$ und $\Pi = T$. Dann ist $\hat{\mathcal{O}} = k[[T]]$ der Ring der Potenzreihen und $\hat{K} = k((T))$ der Körper der Laurentreihen.

5.6. Endliche Erweiterung von vollständigen Körpern

Ziele

- (1) Archimedische Normen sind langweilig.
- (2) Vervollständigung bzgl. nicht archimedischer Normen „kann lokal“ \rightarrow Hensels Lemma.
- (3) Fortsetzen von vollständigen Körpern bezüglich nicht-archimedischer Norm auf algebraische Körper-Erweiterung.

Lemma 5.6.1. *Seien $z_0 \in \mathbb{C}, \varepsilon > 0$ und $g(X) := X^2 - (z_0 + \bar{z}_0)X + z_0 \bar{z}_0 + \varepsilon \in \mathbb{R}[X]$. Die beiden Nullstellen z_1 und z_2 von g sind nicht reell, also $z_2 = \bar{z}_1$ und $|z_1| > |z_0|$.*

$$\text{Proof.} \quad (1) \quad z_{1/2} = \frac{z_0 + \bar{z}_0}{2} \pm \frac{1}{4} \sqrt{\underbrace{(z_0 + \bar{z}_0)^2 - 4z_0 \bar{z}_0 - 4\varepsilon}_{=: D}} \\ \Rightarrow D = \underbrace{(z_0 - \bar{z}_0)^2}_{2\text{Im}(z_0)^2} - 4\varepsilon = -4(\text{Im}(z_0))^2 - 4\varepsilon < 0$$

\Rightarrow erhalte zwei nicht-reele, zueinander komplex konjugierte Nullstellen.

$$(2) \text{ Vieta} \Rightarrow z_1 \bar{z}_1 = z_0 \bar{z}_0 + \varepsilon \Rightarrow |z_1|^2 = |z_0|^2 + \varepsilon > |z_0|^2$$

□

Theorem 3 (Satz von Ostrowski). *Jeder vollständig normierte Körper $(K, |\cdot|)$ mit archimedischer Norm $|\cdot|$ ist isomorph zu \mathbb{R} oder \mathbb{C} .*

Proof. Da $|\cdot|$ archimedisch ist, also $|n|$ beliebig groß wird für $n \in \mathbb{N}$, ist $\text{char}(K) = \infty$.
 \Rightarrow Wir haben Einbettung von $\mathbb{Q} \hookrightarrow K$. Satz 1 $\Rightarrow |\cdot|_{\mathbb{Q}} = |\cdot|_{\infty}$ mit der Betragsnorm $|\cdot|_{\infty}$.
 Da K vollständig ist, lässt sich die Einbettung $\mathbb{Q} \hookrightarrow K$ fortsetzen zu einer Einbettung $\mathbb{R} \hookrightarrow K$ und $|\cdot|_{\mathbb{R}} = |\cdot|_{\infty}$ (Betragsnorm).

Zeige nun: K ist algebraisch über \mathbb{R} , genauer: jedes $\alpha \in K$ hat Minimalpolynom vom Grad ≤ 2 .

Für $\alpha \in K$ definiere die Abbildung

$$f_{\alpha} : \mathbb{C} \rightarrow \mathbb{R}, \quad z \mapsto |\alpha^2 - (z + \bar{z})\alpha + z\bar{z}|$$

Zeige: f_{α} hat eine Nullstelle $z_0 \in \mathbb{C}$, dann ist

$$X^2 - (z_0 + \bar{z}_0)X + z_0\bar{z}_0 \in \mathbb{R}[X]$$

Vielfaches vom Minimalpolynom von α .

Schritt 1: Zeige: f_{α} nimmt auf \mathbb{C} ein Minimum an.

f_{α} ist stetig. Außerdem gilt

$$\begin{aligned} f_{\alpha}(z) &\geq |z\bar{z}| - |z + \bar{z}||\alpha| - |\alpha^2| \\ &\geq |z|^2 - 2|z||\alpha| - |\alpha^2| \rightarrow \infty \text{ für } |z| \rightarrow \infty \end{aligned}$$

\Rightarrow Erhalte Minimum m .

Schritt 2: Zeige $m = 0$:

Sei $S := \{z \in \mathbb{C} \mid f_{\alpha}(z) = m\} = f_{\alpha}^{-1}(\{m\}) \Rightarrow \exists z_0 \in S$ mit $|z_0| \geq |z|$ für alle $z \in S$.

Annahme: $m > 0$. Wähle $\varepsilon > 0$ mit $0 < \varepsilon < m$.

Einerseits: Sei $g(X) := X^2 - (z_0 + \bar{z}_0)X + z_0\bar{z}_0 + \varepsilon \in \mathbb{R}[X]$.

Lemma 6.1 $\Rightarrow g$ hat Nullstellen z_1, \bar{z}_1 mit $|z_1| > |z_0|$.

$\Rightarrow f_{\alpha}(z_1) > m$ wegen Maximalitätsbedingung in Def. von z_0 .

Andererseits: Sei $G(X) := (g(X) - \varepsilon)^n - (-\varepsilon)^n$ für $n \in \mathbb{N}$.

Es gilt $G(z_1) = (g(z_1) - \varepsilon)^n - (-\varepsilon)^n = 0$.

Seien $x_1 = z_1, x_2, \dots, x_{2n}$ die Nullstellen in \mathbb{C} von G . Das heißt:

$$\begin{aligned} G(X) &= \prod_{i=1}^{2n} (X - x_i) = \prod_{i=1}^{2n} (X - \bar{x}_i) \\ \Rightarrow G(X)^2 &= \prod_{i=1}^{2n} (X^2 - (x_i + \bar{x}_i)X + x_i\bar{x}_i) \end{aligned}$$

Es gilt dann $(\star) |G(\alpha)|^2 = \prod_{i=1}^{2n} f_{\alpha}(x_i) \geq f_{\alpha}(z_1) m^{2n-1}$ Außerdem $|G(\alpha)| \leq |\alpha^2 - (z_0 + \bar{z}_0)\alpha + z_0\bar{z}_0|^n + (-\varepsilon)^n = f_{\alpha}(z_0)^n + \varepsilon^n = m^n + \varepsilon^n (\star\star)$

$$\begin{aligned}
 (\star) + (\star\star) &\Rightarrow f_\alpha(z_1)m^{2n-1} \leq (m^n + \varepsilon^n)^2 \quad | : m^2n \\
 &\Longleftrightarrow \frac{f_\alpha(z_1)}{m} \leq \left(1 + \left(\frac{\varepsilon}{m}\right)^n\right)^2
 \end{aligned}$$

Für $n \rightarrow \infty$ erhalte: $f_\alpha(z_1) \leq mE$ zu „Einerseits“. \square

Ab jetzt: $(K, |\cdot|)$ ein Körper mit nicht archimedischer Norm. $\nu, \mathcal{O}, p, \mathcal{O}/p = \kappa$ Restklassenkörper wie immer.

Definition 5.6.2. Sei $f = a_n X^n + a_{n-1} X^{n-1} + \dots + a_0 \in \mathcal{O}[X]$.

1. $|f| := \max\{|a_0|, \dots, |a_n|\} \leq 1$
2. f heißt primitiv : $\Longleftrightarrow |f| = 1 \Longleftrightarrow f(X) \not\equiv 0 \pmod{p}$.

Theorem 4 (Lemma von Hensel¹). Sei K ein vollständiger Körper. Seien $f \in \mathcal{O}[X]$ ein primitives Polynom,

$$\text{proj} : \mathcal{O}[X] \rightarrow \kappa[X]$$

die natürliche Projektion und $\bar{f} := \text{proj}(f)$.

Dann gilt: Falls $\bar{f} = \bar{g}\bar{h}$ mit $\bar{g}, \bar{h} \in \kappa[X]$ teilerfremd, dann gibt es $g, h \in \mathcal{O}[X]$ mit:

- (1) $f(X) = g(X)h(X)$
- (2) $\text{proj}(g) = \bar{g}, \text{proj}(h) = \bar{h}$
- (3) $\deg(g) = \deg(\bar{g})$

Remark 5.6.3 (Vorbemerkung). $K[X]$ trägt die von $(K, |\cdot|)$ induzierte Topologie („Finale Topologie auf dem Raum der endlichen Folgen“). Insbesondere gilt für f_n Folge in $K[X]$ und $f \in K[X]$.

$f_n \rightarrow f (n \rightarrow \infty) \Longleftrightarrow$ Für die Koeffizienten $a_i^{(n)}$ von f_n gilt

$$\forall i : a_i^{(n)} \rightarrow a_i(n \rightarrow \infty)$$

wobei a_i die Koeffizienten von f sind. Wie in K gilt: $f_n \rightarrow f$ und $g_n \rightarrow g \Rightarrow f_n g_n \rightarrow f \cdot g, f_n + g_n \rightarrow f + g$

von Satz 4. Wähle $g_0, h_0 \in \mathcal{O}[X]$ mit $\text{proj}(g_0) = \bar{g}, \text{proj}(h_0) = \bar{h}, \deg(g_0) = \deg(\bar{g}) =: m, \deg(h_0) = \deg(\bar{h}) =: m'$.

Dann gilt insbesondere $m + m' = \deg(\bar{f}) \leq \deg(f) =: d$.

Verwende: (1) \bar{g}, \bar{h} sind teilerfremd $\Rightarrow \exists a, b \in \mathcal{O}[X]$ mit $1 \equiv ag_0 + bh_0 \pmod{p}$. Also $ag_0 + bh_0 - 1 \in p$.

(2) $f \equiv g_0 h_0 \pmod{p}$, also $f - g_0 h_0 \in p$. Wähle Π als Koeffizient von $ag_0 + bh_0 - 1, f - g_0 h_0$, sodass $\nu(\Pi)$ minimal ist.

$\Rightarrow \Pi \in P$ und $ag_0 + bh_0 - 1$ und $f - g_0 h_0$ sind beide durch Π teilbar.

Ansatz: $g := g_0 + s_1 \Pi + s_2^2 + \dots$

$h := h_0 + t_1 \Pi + t_2^2 + \dots$

¹Kurt Hensel, 1861-1941

$$(1) \quad g_n h_n \equiv f \pmod{\Pi^{n+1}}$$

$$(2) \quad \deg(g_n) = \deg(g_0) = m$$

Bestimme s_n und t_n rekursiv:

Es gilt: $\Pi | f - g_0 h_0 \Rightarrow f \equiv g_0 h_0 \pmod{\Pi}$

Außerdem: $\deg(g_0) = m$ und damit liegt der Leitkoeffizient nicht in p und ist somit eine Einheit.

Sei nun g_{n-1}, h_{n-1} wie gewünscht gegeben. Es gilt für $g_n = g_{n-1} + s_n \Pi^n$ und $h_n = h_{n-1} + t_n \Pi^n$:

$g_n h_n \equiv g_{n-1} h_{n-1} + (s_n h_{n-1} + t_n g_{n-1}) \Pi^n \pmod{\Pi^{n+1}}$ und $f - g_{n-1} h_{n-1} = f_n \Pi^n$. Also ist $f \equiv g_n h_n \pmod{\Pi^{n+1}} \iff g_{n-1} h_{n-1} + f_n \Pi^n \equiv g_{n-1} h_{n-1} + (s_n h_{n-1} + t_n g_{n-1}) \Pi^n \pmod{\Pi^{n+1}} \iff f_n \equiv s_n h_{n-1} + t_n g_{n-1} \pmod{\Pi}$. Verwende $f_n = f_n \cdot 1 \equiv f_n (a g_0 + b h_0) \pmod{\Pi}$ und $g_{n-1} \equiv g_0$ und $h_{n-1} \equiv h_0 \pmod{\Pi}$.

Also $(\star) \iff \underbrace{f_n a}_{\hat{t}_n} g_0 + \underbrace{f_n b}_{\hat{h}_n} h_0 \equiv s_n h_0 + t_n g_0 \pmod{\Pi}$. Definiere s_n als Rest von $f_n b$ beim

Teilen durch g_0 d.h. $f_n b = q g_0 s_n$ mit $q \in \mathcal{O}[X]$ und $\deg(s_n) < m$ und $t'_n := f_n a + q h_0$ bzw. das Polynom das aus t'_n entsteht, wenn alle durch Π teilbaren Koeffizienten von t'_n auf 0 gesetzt werden. Insgesamt: $g_n := g_{n-1} + s_n \Pi^n, h_n := h_{n-1} + t_n \Pi^n \equiv h_{n-1} + \hat{t}_n \Pi^n \pmod{\Pi^{n+1}}$ mit s_n, t_n wie oben gewählt. \Rightarrow

$$(1) \quad f \equiv g_n h_n \pmod{\Pi^{n+1}}$$

$$(2) \quad \deg(g_n) = \deg(g_{n-1}), \text{ da } \deg(s_n) < m \text{ und der Leitkoeffizient von } g_n = \text{Leitkoeff. von } g_{n-1}.$$

□

Example 5.6.4. $K = \mathbb{Q}_p, \mathcal{O} = \mathbb{Z}_p, \mathcal{P} = p\mathbb{Z}_p, \kappa = \mathbb{Z}_p/\mathcal{P} = \mathbb{F}_p, f(X) = X^{p-1} - 1 \in \mathcal{O}[X], \bar{f}(X) = X^{p-1} - 1 \in \mathbb{F}_p[X] = (X-1)(X-2)\cdots(X-(p-1))$

Lemma von Hensel $\Rightarrow f$ hat $p-1$ Nullstellen $\Rightarrow \mathbb{Z}_p$ enthält die $(p-1)$ -ten Einheitswurzeln.

Corollary 5.6.5. Seien $(K, |\cdot|)$ wieder vollständig + nicht-archimedisch, $f(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_0 \in K[X]$ irreduzibel. Dann gilt $|f| = \max\{|a_0|, |a_n|\}$. Insbesondere: f normiert und $a_0 \in \mathcal{O} \Rightarrow f \in \mathcal{O}[X]$.

Proof. Multipliziere f mit geeignetem $\alpha \in K \Rightarrow f \in \mathcal{O}[X]$ und $|f| = 1$.

Sei $r \in \{0, \dots, n\}$ minimal mit $|a_r| = 1 \Rightarrow f(X) \equiv a_n X^n + \cdots + a_r X^r \equiv X^r (a_n X^{n-r} + \cdots + a_r) \pmod{p}$.

Annahme: $0 < r < n$. Hensels Lemma $\Rightarrow f$ ist nicht irreduzibel \nexists . □

Lemma 5.6.6. Seien K vollständig + nicht-archimedisch, L/K endliche KE und $\hat{\mathcal{O}}$ der ganze Abschluss von \mathcal{O} in L .

$$\begin{array}{ccc} \hat{\mathcal{O}} & \hookrightarrow & L \\ | & & | \\ \mathcal{O} & \hookrightarrow & K \end{array}$$

Dann gilt:

$$\hat{\mathcal{O}} = \{\alpha \in L \mid N_{L/K}(\alpha) \in \mathcal{O}\}$$

Proof. „ \subseteq “ gilt auch für nicht vollständige Körper K [s. AZT 1, II.2.7].

„ \supseteq “ Folgt aus Kor. 6.5 wie folgt:

$\alpha \in L^\times$ mit $N_{L/K}(\alpha) \in \mathcal{O}$. Sei $f(X) = X^d + a_{d-1}X^{d-1} + \dots + a_0 \in K[X]$ Minimalpolynom von $\alpha \Rightarrow N_{K(\alpha)/K}(\alpha) = (-1)^d a_0 \in \mathcal{O}$.

$\Rightarrow N_{L/K}(\alpha)$ ist Potenz von $(-1)^d a_0$ und damit in \mathcal{O} .

$\stackrel{Kor.}{\Rightarrow} f(X) \in \mathcal{O}[X]$ und damit $\alpha \in \hat{\mathcal{O}}$. □

Example 5.6.7. $K = \mathbb{Q}, L = \mathbb{Q}(\sqrt{17}), f(X) = X^2 + \frac{1}{2}X - 1$ hat die Nullstellen $x_{1/2} = -\frac{1}{4} \pm \frac{1}{4}\sqrt{17}, \alpha = x_1 \Rightarrow N_{L/K}(\alpha) = -1$.

Aber: $\alpha \notin$ ganzem Abschluss von \mathbb{Z} in L .

Theorem 5 (FORTSETZUNG VON NORMEN). *Sei $(K, |\cdot|)$ vollständig. Dann hat $|\cdot|$ auf jeder algebraischen KE L/K eine eindeutige Norm-Fortsetzung.*

Falls L/K endlich, ist diese definiert durch $|\alpha| := \sqrt[n]{|N_{L/K}(\alpha)|}$ und ist vollständig, wobei $[L : K] = n$.

Proposition 5.6.8. *Seien:*

- $(K, |\cdot|)$ ein vollständig normierter Körper.
- V ein n -dimensionaler K -VR mit einer Norm $\|\cdot\|_1$
- $\{v_1, \dots, v_n\}$ Basis von V .

Dann ist $\phi : K^n \rightarrow V, (a_1, \dots, a_n) \mapsto a_1 v_1 + \dots + a_n v_n$ ein Homöomorphismus. Insbesondere ist damit V vollständig bezüglich der von $\|\cdot\|_1$ induzierten Topologie.

Proof. Definiere die Norm $\|\cdot\|_2$ auf V wie folgt: $\|a_1 v_1 + \dots + a_n v_n\| := \max\{|a_1|, \dots, |a_n|\}$. Zeige noch: $\|\cdot\|_1$ und $\|\cdot\|_2$ definieren dieselbe Topologie auf V . Dann folgt die Behauptung. Zeige dazu: $\exists \rho, \rho' > 0$ mit $\rho \|x\|_2 \leq \|x\|_1 \leq \rho' \|x\|_2 \forall x \in V$.

$$(1) \text{ Für } x = a_1 v_1 + \dots + a_n v_n \text{ gilt: } \|x\|_1 \leq |a_1| \|v_1\|_1 + \dots + |a_n| \|v_n\|_1 \leq \|x\|_2 \underbrace{(\|v_1\|_1 + \dots + \|v_n\|_1)}_{=: \rho'}$$

Existenz von ρ via vollständiger Induktion über n :

$$\begin{aligned} n = 1 : \|x\|_2 &= \|x_1 v_1\|_2 = |x_1| \\ \|x\|_1 &= |x_1| \|v_1\|_1 \Rightarrow \rho := \|v_1\|_1 \text{ tut es.} \end{aligned}$$

„ $n - 1 \rightarrow n$ “: $V_i := K v_1 \oplus K v_2 \oplus \dots \oplus K v_{i-1} \oplus K v_{i+1} \oplus \dots \oplus K v_n$ also: $V = V_i \oplus K v_i$

Induktionsvoraussetzung: $\Rightarrow (V_i, \|\cdot\|_1)$ ist vollständig $\Rightarrow V_i$ ist abgeschlossen in $(V, \|\cdot\|_1)$
 $\Rightarrow V_i + v_i$ ist abgeschlossen.

$$\Rightarrow \bigcup_{i=1}^n V_i + v_i \text{ ist abgeschlossen und enthält nicht } 0.$$

$$\Rightarrow \exists \rho > 0 \text{ mit } K_\rho(0) \cap \bigcup_{i=1}^n V_i + v_i = \emptyset$$

$$\Rightarrow \forall i \in \{1, \dots, n\} : \|w_i + v_i\|_1 \geq \rho \forall w_i \in V_i$$

Also: Für $x = a_1 v_1 + \dots + a_n v_n \neq 0$ gilt: Sei $r : |a_r| = \max\{|a_1|, \dots, |a_n|\} = \|x\|_2 > 0$

$$\Rightarrow \|a_r^{-1} x\|_1 = \left\| \underbrace{\frac{a_1}{a_r} v_1 + \dots + \frac{a_n}{a_r} v_n}_{\in v_r + V_r} \right\|_1 \geq \rho$$

$$\Rightarrow \|x\|_1 \geq |a_r| \rho = \rho \|x\|_2.$$

□

Corollary 5.6.9. In Satz 3 (Satz von Ostrowski) erhalte sogar Isomorphismus von normierten Körpern. Also $(K, |\cdot|)$ vollständiger Körper mit archimedischer Norm, dann:

$$(K, |\cdot|) \cong (\mathbb{R}, |\cdot|_\infty) \text{ oder } (\mathbb{C}, |\cdot|_\infty)$$

Proof. Sei $(K, |\cdot|)$ ein solcher Körper. Im Beweis von Satz 3: $\mathbb{R} \hookrightarrow K$ mit Einbettung erhält die Norm. K/\mathbb{R} ist \leq Grad 2 Erweiterung.

Prop 6.8+ Prop 1.7 \Rightarrow Behauptung. □

Lemma 5.6.10. Sei K ein Körper und $h : K \rightarrow \mathbb{R}_{\geq 0}$ Abbildung mit

$$h(\alpha) = 0 \iff \alpha = 0 \text{ und } \forall \alpha, \beta : h(\alpha\beta) = h(\alpha)h(\beta)$$

Dann sind äquivalent:

- h erfüllt die verschärfte Dreiecksungleichung $\forall \alpha, \beta : h(\alpha + \beta) \leq \max\{h(\alpha), h(\beta)\}$
- Es gilt: $\forall \alpha \in K : h(\alpha) \leq 1 \Rightarrow h(\alpha + 1) \leq 1$

Proof. „ \Rightarrow “: mit $\beta = 1$.

„ \Leftarrow “: $\alpha = \beta = 0$: ✓

$$h(\alpha) \leq h(\beta) \text{ und } \beta \neq 0 \Rightarrow h\left(\frac{\alpha}{\beta}\right) = \frac{h(\alpha)}{h(\beta)} \leq 1 \Rightarrow h(\alpha + \beta) = h(\beta)h\left(\frac{\alpha}{\beta} + 1\right) \stackrel{\text{Vor.}}{\leq} h(\beta) \cdot 1 \max\{h(\alpha), h(\beta)\}.$$

□

Beweis von Satz 5. (A) Falls $\|\cdot\|$ archimedisch, dann folgt Beh. aus:

- \mathbb{C} ist algebraischer Abschluss von \mathbb{R}
- Kor. 6.9
- $|z|_\infty = \sqrt{z\bar{z}} = \sqrt[n]{|\mathcal{N}_{\mathbb{C}/\mathbb{R}}(z)|}$ für $z \in \mathbb{C}$.

(B) Sei nun $\|\cdot\|$ nicht-archimedisch und L/K endlich. Zeige, dass $\alpha \mapsto \sqrt[n]{|\mathcal{N}_{L/K}(\alpha)|} =: h(\alpha)$ Norm ist.

$$(1) h(\alpha) = 0 = \mathcal{N}_{L/K}(\alpha) = 0 \iff \alpha = 0$$

$$(2) h(\alpha\beta) = h(\alpha)h(\beta)$$

(3) Seien $\mathcal{O} = \{\alpha \in K \mid \|\alpha\| \leq 1\}$ der BWR zu $\|\cdot\|$, und $\hat{\mathcal{O}}$ der ganze Abschluss von \mathcal{O} in L .

$$\text{Lemma 6.6.} \Rightarrow \hat{\mathcal{O}} = \{\alpha \in L \mid \mathcal{N}_{L/K}(\alpha) \in \mathcal{O}\} = \{\alpha \in L \mid h(\alpha) \leq 1\}.$$

Lemma 6.10. \Rightarrow die verschärfte Dreiecksungleichung gilt.

Prop. 6.8. \Rightarrow Jede Fortsetzung der Norm auf K nach L ist äquivalent zu $h \Rightarrow$ Eindeutigkeit + Vollständigkeit.

- (C) Sei nun $\|\cdot\|$ nicht-archimedisch und L/K algebraisch. Für $\alpha \in L$ wähle endlichen Zwischenkörper E von L/K . Definiere $|\alpha| := \sqrt[n]{\|\mathcal{N}_{E/K}(\alpha)\|}$, wobei $n := [E : K]$. Dies ist wohldefiniert, da für $\alpha \in F = E_1 \cap E_2$ mit $n_i := [E_i : F]$, $d := [F : K]$ gilt:

$$n_1 \cdot \sqrt[d]{\|\mathcal{N}_{E_1/K}(\alpha)\|} = n_1 \cdot \sqrt[d]{\|\mathcal{N}_{F/K}(\underbrace{\mathcal{N}_{E_1/F}(\alpha)}_{=\alpha^{n_1}})\|} = \sqrt[d]{\|\mathcal{N}_{F/K}(\alpha)\|} = n_2 \cdot \sqrt[d]{\|\mathcal{N}_{E_2/K}(\alpha)\|}$$

Außerdem ist das eine Norm, da die Bedingungen auf allen endlichen Zwischenkörpern erfüllt ist. □

Corollary 5.6.11. *In der Situation von Satz 5 gilt für nicht-archimedische Körper K und endliche KE L/K :*

- (i) *Die Bewertung auf K ist diskret \iff die Bewertung auf L ist diskret.*
- (ii) $\hat{\mathcal{O}} :=$ *der ganze Abschluss von \mathcal{O} in L*
 $= \{x \in L \mid \mathcal{N}_{L/K}(x) \in \mathcal{O}\}$
 $= \{x \in L \mid \|x\| \leq 1\} = \{x \in L \mid \hat{v}(x) \geq 0\}$

$$\begin{array}{ccccc} \mathfrak{p}\hat{\mathcal{O}} = \hat{\mathfrak{p}}^e & \subset & \hat{\mathcal{O}} & \subset & L \\ & \text{ganzer Abschluss} \uparrow & & & \uparrow \text{endlich} \\ \mathfrak{p} & \subset & \mathcal{O} & \subset & K \end{array}$$

Wir sind also im Setting von Algebraischer Zahlentheorie I:

\mathfrak{p} ist eindeutiges Primideal $\neq 0$ in \mathcal{O} . Dieses hat ein „Urbild“ $\hat{\mathfrak{p}}$ in $\hat{\mathcal{O}}$ und es gilt also $\mathfrak{p}\hat{\mathcal{O}} = \hat{\mathfrak{p}}^e$.

\mathfrak{p} ist also voll-verzweigt und der Trägheitsgrad bzw. lokale Grad ist $[\hat{\kappa} : \kappa]$ mit $\hat{\kappa} := \hat{\mathcal{O}}/\hat{\mathfrak{p}}$, $\kappa := \mathcal{O}/\mathfrak{p}$.

Remark 5.6.12. Seien L/K Körpererweiterung von normierten Körpern mit nicht-archimedischer Norm.

Notation: $\hat{v}, \hat{\mathcal{O}}, \hat{\mathfrak{p}}, \hat{\kappa}$ zu L wie immer, $v, \mathcal{O}, \mathfrak{p}, \kappa$ zu K wie immer.

L/K endlich $\Rightarrow [\hat{\kappa} : \kappa]$ endlich.

Proof. Seien $\bar{x}_1, \dots, \bar{x}_n \in \hat{\kappa} = \hat{\mathcal{O}}/\hat{\mathfrak{p}}$ linear unabhängig über κ . Wähle Urbilder $x_1, \dots, x_n \in \hat{\mathcal{O}} \subseteq L$ und zeige: diese sind linear unabhängig über K .

Seien $c_1, \dots, c_n \in K$ mit $c_1 x_1 + \dots + c_n x_n = 0$ nicht alle 0.

Durchmultiplizieren mit geeignetem $c \in K \Rightarrow c_1, \dots, c_n \in \mathcal{O}$ und ein $c_i \notin \mathfrak{p}$

$\Rightarrow \bar{c}_1 \bar{x}_1 + \dots + \bar{c}_n \bar{x}_n = 0$ mit $\bar{c}_i \neq 0$ in $\hat{\kappa}$. □

5.7. Lokale Körper

Definition 5.7.1. i) Ein Körper K heißt **globaler Körper** : \iff
 K ist Zahlkörper, d.h. endliche Erweiterung von \mathbb{Q} , oder

K ist endliche Erweiterung von $\mathbb{F}_q(T) = \text{Quot}(\mathbb{F}_q[T])$ mit $\mathbb{F}_q = \text{Körper mit } q \text{ Elementen}$.

ii) Ein Körper K heißt **lokaler Körper** : \Longleftrightarrow

K ist bezüglich der Norm von einer nicht-archimedischen Bewertung v vollständig,
 v ist diskret und der Restklassenkörper κ ist endlich.

Notation: Für K , Bewertung v , \mathcal{O} , $|\cdot| = r^{-v(\cdot)}$ mit $r > 1$, $\mathfrak{p}, \kappa = \mathcal{O}/\mathfrak{p}$

Proposition 5.7.2. Sei K lokaler Körper. Betrachte die von $|\cdot|$ induzierte Topologie auf K . Dann ist \mathcal{O} kompakt und K lokal kompakt.

Erinnerung 5.7.3. Sei (X, T) ein topologischer Raum der hausdorffsch ist. Dann ist (X, T) lokal-kompakt \Longleftrightarrow

$\forall x \in X \exists$ Umgebung C von x , die kompakt ist, d.h.

$\forall x \in X \exists U \in T, C$ kompakt mit $x \in U \subseteq C$.

\triangle Für nicht hausdorffsche Räume gibt es unterschiedliche Definitionen.

Proof. Erinnerung: Für die Kette $\mathcal{O} \supset \mathfrak{p} \supset \mathfrak{p}^2 \supset \mathfrak{p}^3 \supset \dots$ gilt:

$\mathfrak{p}^k/\mathfrak{p}^{k+1} \cong \mathcal{O}/\mathfrak{p} = \kappa$ endlich.

Aus \mathcal{O}/\mathfrak{p} endlich folgt, dass $\mathcal{O}/\mathfrak{p}^n$ endlich für alle $n \in \mathbb{N}$.

$\Rightarrow \prod_{n \in \mathbb{N}} \mathcal{O}/\mathfrak{p}^n$ ist kompakt (Satz von Tychonoff).

$\mathcal{O} \cong \varprojlim_n \mathcal{O}/\mathfrak{p}^n$ ist abgeschlossen in $\prod_n \mathcal{O}/\mathfrak{p}^n$ und damit ebenfalls kompakt.

Außerdem: \mathcal{O} ist offen in K , denn:

$x \in \mathcal{O} \iff |x| \leq 1 \iff v(x) \geq 0 \xrightarrow{v(K) = \mathbb{Z} \cup \{\infty\}} v(X) > -1 \iff |x| < r^{-1} \Rightarrow \mathcal{O}$ ist offene Kreisscheibe um 0.

$\Rightarrow a \in K$ ist also $a + \mathcal{O}$ eine offene und kompakte Umgebung und somit K lokal kompakt. \square

Remark 5.7.4. Sei L ein lokaler Körper mit Restklassenkörper κ mit q Elementen. Dann enthält L alle $(q-1)$ -ten Einheitswurzeln, d.h. $X^{q-1} - 1$ zerfällt über L in Linearfaktoren.

Proof. Mit Hensels Lemma genau wie in Bsp. 6.4. \square

Theorem 6 (BESTIMMUNG ALLER LOKALEN KÖRPER). L ist lokaler Körper \Longleftrightarrow L ist endliche Erweiterung von \mathbb{Q}_p oder $\mathbb{F}_p((T))$.

Lemma 5.7.5. Sei K vollständig nicht-archimedisch. Sei L/K Körpererweiterung normierter Körper mit diskreter Bewertung \hat{v} . Insbesondere $\hat{v}|_K = v$ soll nicht trivial sein. Falls $[\hat{\kappa} : \kappa]$ endlich ist, dann ist $\hat{\mathcal{O}}$ ein endlich erzeugter \mathcal{O} -Modul und damit $[L : K]$ endlich.

$$\begin{array}{ccccccc}
 (\hat{\Pi}) = \hat{\mathcal{P}} & & \hat{\kappa} & \longleftarrow & \hat{\mathcal{O}} & \subset & L & & \hat{v} \rightarrow \mathbb{Z} \\
 & & \uparrow & & \downarrow & & \downarrow & & \\
 (\Pi) = \mathfrak{p} & & \kappa & \longleftarrow & \mathcal{O} & \subset & K & & v \rightarrow e\mathbb{Z}
 \end{array}$$

Proof. Vorüberlegung: Normiere \hat{v} so, dass $\hat{v}(L^\times) = \mathbb{Z}$. Dann ist $v(K^\times)$ eine Untergruppe $e\mathbb{Z}$ von \mathbb{Z} . $\Rightarrow v(\hat{\Pi}) = 1$ und $v(\Pi) = e$ und damit $\Pi = \varepsilon \hat{\Pi}^e$ mit $\varepsilon \in \hat{\mathcal{O}}^\times \Rightarrow \hat{\mathcal{P}} = \mathfrak{p}\hat{\mathcal{O}}$.
Seien $\bar{w}_1, \dots, \bar{w}_f$ eine Basis von $\hat{\kappa}$ über κ und w_1, \dots, w_f Urbilder in $\hat{\mathcal{O}}$.

Ansatz: $M := \mathcal{O}w_1 + \mathcal{O}w_2 + \dots + \mathcal{O}w_f + \mathcal{O}w_1\hat{\Pi} + \dots + \mathcal{O}w_f\hat{\Pi} + \dots + \mathcal{O}w_1\hat{\Pi}^2 + \dots + \mathcal{O}w_1\hat{\Pi}^{e-1} + \dots + \mathcal{O}w_f\hat{\Pi}^{e-1}$.

Zeige: $M = \hat{\mathcal{O}}$.

Schritt 1: Zeige: $\hat{\mathcal{O}} = M + \mathfrak{p}\hat{\mathcal{O}}$

Sei $N := \mathcal{O}w_1 + \mathcal{O}w_2 + \dots + \mathcal{O}w_f$ und somit $M = N + N\hat{\Pi} + \dots + N\hat{\Pi}^{e-1}$.

Verwende: $\hat{\mathcal{O}} = N + \hat{\mathcal{O}}\hat{\Pi}$ wegen $\bar{w}_1, \dots, \bar{w}_f$ ist Basis von $\hat{\kappa}/\kappa$

$\Rightarrow \hat{\mathcal{O}} = N + \hat{\mathcal{O}}\hat{\Pi} = N + (N + \hat{\mathcal{O}})\hat{\Pi} = N + N\hat{\Pi} + \hat{\mathcal{O}}\hat{\Pi} = \dots = N + N\hat{\Pi} + \dots + N\hat{\Pi}^{e-1} + \hat{\mathcal{O}}\hat{\Pi}^e = M + \hat{\mathcal{O}}\mathfrak{p}$.

Schritt 2: Zeige $\hat{\mathcal{O}} = M$

Verwende: $\mathfrak{p}^k \cdot M \subseteq M$ für $k \in \mathbb{N}_0$, da $\mathfrak{p}^k \subseteq \mathcal{O}$.

$\hat{\mathcal{O}} = M + \mathfrak{p}\hat{\mathcal{O}} = M + \mathfrak{p}(M + \mathfrak{p}\hat{\mathcal{O}}) = M + \mathfrak{p}^2\hat{\mathcal{O}} = \dots = M + \mathfrak{p}^k\hat{\mathcal{O}}$ für alle $k \geq 1$

$\Rightarrow M$ liegt dicht in $\hat{\mathcal{O}}$.

$$\begin{array}{ccccccc} & & & & L & & \\ & & & & \uparrow & & \\ \mathcal{O}^n & \cong & M & \subset & M \otimes_{\mathcal{O}} K & =: & V \\ & & \uparrow & & \uparrow & & \\ & & \mathcal{O} & \subset & K & & \end{array}$$

$V := M \otimes_{\mathcal{O}} K$ (endlich dimensionaler K -Vektorraum) und Prop. 6.8. $\Rightarrow V$ ist homöomorph zu K^d für ein d .

$\Rightarrow M \cong \mathcal{O}^n$ und ist damit abgeschlossen in $\hat{\mathcal{O}} \Rightarrow M = \hat{\mathcal{O}}$.

Schritt 3: $[L : K]$ ist endlich: Übungsaufgabe. □

Beweis von Satz 6. „ \Leftarrow “ : Sei $K := \mathbb{Q}_p$ oder $K = \mathbb{F}_p((T))$ und L/K endliche Körpererweiterung.

Def. 3.12+Aufgabe 2 (Blatt 3) $\Rightarrow K$ ist vollständig $\xrightarrow{\text{Satz 5}} L$ ist vollständig.

Kor. 6.11 \Rightarrow Norm auf L gehört zu diskreter Bewertung. Bem. 6.12 $\Rightarrow [\hat{\kappa} : \kappa]$ ist endlich $\xrightarrow{\kappa \text{ endlich}} \hat{\kappa}$ endlich $\Rightarrow L$ ist lokaler Körper.

„ \Rightarrow “ : Sei L lokaler Körper mit diskreter Bewertung $\hat{v}, \hat{\mathcal{O}}, \hat{\mathcal{P}}, \hat{\kappa}$ wie immer.

Fall 1: $\text{char}(L) = 0 \Rightarrow$ Habe Einbettung $\mathbb{Q} \hookrightarrow L$.

Zeige: $\hat{v}|_{\mathbb{Q}}$ ist nicht trivial:

Sei $p := \text{char}(\hat{\kappa}) \Rightarrow \hat{\mathcal{O}} \ni \underbrace{1 + \dots + 1}_{p\text{-mal}} \in \hat{\mathcal{P}} \Rightarrow v(p) > 0$. Es folgt: $v|_{\mathbb{Q}} = v_p$.

\Rightarrow Erhalte Einbettung $\mathbb{Q}_p \hookrightarrow L$, da L vollständig.

Lemma 7.5 $\Rightarrow L/\mathbb{Q}_p$ ist endlich.

Fall 2: $\text{char}(L) \neq 0 \Rightarrow \text{char}() = \text{char}(\hat{\kappa}) =: p$.

- (1) Zeige: $\kappa \hookrightarrow L$
 $\kappa = \overline{\mathbb{F}_p}(\alpha)$ mit α hat separables Minimalpolynom $f_\alpha \in \mathbb{F}_p[X]$.
Hensels Lemma $\Rightarrow f_\alpha$ hat Nullstelle x_0 in L .
 $\Rightarrow \alpha \mapsto x_0$ induziert eine Einbettung $\iota : \hat{\kappa} = \mathbb{F}_p(\alpha) \hookrightarrow L$.
- (2) Zeige: $L \cong \kappa((T))$.
Prop. 5.8 \Rightarrow Ist $R \subseteq \hat{\mathcal{O}}$ ein Repräsentantensystem von $\hat{\mathcal{O}}/\hat{\mathcal{P}} = \hat{\kappa}$, dann lässt sich jedes Element aus L auf eindeutige Weise als Laurent-Reihe $\sum_{k=m}^{\infty} a_m \hat{\Pi}^m$ mit $a_m \in R$ beschreiben. Wähle $R := \text{Bild}(\iota) \Rightarrow$ Erhalte Isomorphismus $L \cong \hat{\kappa}((T))$.
- (3) $\hat{\kappa}((T))$ ist endliche Körpererweiterung von $\mathbb{F}_p((T)) \rightarrow$ siehe Blatt 3. Aufgabe 2.

□

ZIEL: Beschreibe K^\times für lokalen Körper K .

Sei ab jetzt stets K lokaler Körper.

Remark 5.7.6. Sei κ der Restklassenkörper von K mit $q = p^a$ Elementen.

- (1) μ_{q-1} = Gruppe der $(q-1)$ -ten Einheitswurzeln $\cong \mathbb{F}_q^\times \cong \mathbb{Z}/(q-1)\mathbb{Z}$.
- (2) $U^{(1)} = 1 + \mathfrak{p}$ und
- (3) $(\Pi) = \{\Pi^k \mid k \in \mathbb{Z}\} \cong \mathbb{Z}$ (Π ist Erzeuger von maximalen Ideal \mathfrak{p} in \mathcal{O})

sind Untegruppen von K^\times .

Proposition 5.7.7. $K^\times \cong (\Pi) \times \mu_{q-1} \times U^{(1)}$

Proof. (1) $x \in K^\times \Rightarrow x = \varepsilon \Pi^k$ mit $\varepsilon \in \mathcal{O}^\times$ und $k \in \mathbb{Z}$ auf eindeutige Weise \Rightarrow Erhalte Isomorphismus $(\Pi) \times \mathcal{O}^\times \rightarrow K^\times$, $(\Pi^k, \varepsilon) \mapsto \varepsilon \Pi^k$.

- (2) Sei $\varphi : \mathcal{O}^\times \rightarrow \kappa^\times$, $x \mapsto x \bmod \mathfrak{p}$.
Es ist $\text{Kern}(\varphi) = U^{(1)}$. Außerdem: $\alpha := \phi|_{\mu_{q-1}} : \mu_{q-1} \rightarrow \kappa^\times$ ist Isomorphismus
 \Rightarrow Erhalte Isomorphismus $\mu_{q-1} \times U^{(1)} \rightarrow \mathcal{O}^\times$, $(\omega, \alpha) \mapsto \omega \alpha$.

□

Notation 5.7.8. Das Bild von $x \in K^\times$ unter dem Isomorphismus aus Proposition 5.7.7 notieren wir als

$$(\Pi^{v(x)}, w(x), \langle x \rangle) \in (\Pi) \times \mu_{q-1} \times U^{(1)}.$$

Es gilt also $x = \Pi^{v(x)} \cdot w(x) \cdot \langle x \rangle$.

Achtung: $w(x)$ und $\langle x \rangle$ hängen von dem gewählten Π ab.

Remark 5.7.9. Der Isomorphismus

$$h : K^\times \rightarrow (\Pi) \times \mu_{q-1} \times U^{(1)}, : x \mapsto (\Pi^{v(x)}, w(x), \langle x \rangle)$$

ist ein Homöomorphismus, wobei $(\Pi) \cong \mathbb{Z}$ und $\mu_{q-1} \cong \mathbb{Z}/(q-1)\mathbb{Z}$ die diskrete Topologie erhalten und $U^{(1)}$ die Topologie von K erbt.

Proof. h ist stetig. Wegen $x_n \rightarrow x$ mit $x \neq 0$ gilt $v(x_n) = v(x)$ für n groß genug, genauso für $w(x_n) = w(x)$. Dann gilt:

$$\langle x_n \rangle = \frac{x_n}{\Pi^{v(x_n)} w(x_n)} = \frac{x_n}{\Pi^{v(x)} w(x)} \rightarrow \frac{x}{\Pi^{v(x)} w(x)} = \langle x \rangle.$$

Außerdem ist offensichtlich $h^{-1}: (\Pi^k, w, \alpha) \mapsto \Pi^k w \alpha$ stetig. \square

Definition 5.7.10. K heißt **p -adischer Zahlkörper** falls K eine endliche Erweiterung von \mathbb{Q}_p ist. Dies ist genau dann der Fall, wenn K ein lokaler Körper mit $\text{char } K = 0$ ist und $\text{char } \kappa = p$ für den Restklassenkörper κ gilt.

Remark 5.7.11. Sei K/\mathbb{Q}_p ein p -adischer Zahlkörper. Notiere wie immer $\hat{\mathcal{O}}, \hat{\mathfrak{p}}, \hat{\Pi}, \hat{\kappa}$ für K und $\mathcal{O}, \mathfrak{p}, \Pi, \kappa$ für \mathbb{Q}_p . Dann gilt $\mathfrak{p}\hat{\mathcal{O}} = \hat{\mathfrak{p}}^e$ mit $e \in \mathbb{N}$ und somit $\Pi\hat{\mathcal{O}} = \hat{\Pi}^e\hat{\mathcal{O}}$. Wählen wir nun \hat{v} als diskrete Bewertung auf K die normiert ist, das heißt $\hat{v}(K^\times) = \mathbb{Z}$ und $\hat{v}(\hat{\Pi}) = 1$, dann ist $\hat{v}(\mathbb{Q}_p^\times) = e\mathbb{Z}$ und $\hat{v}(\Pi) = e$. Somit gilt $\hat{v}|_{\mathbb{Q}_p} = ev_p$, wobei v_p die p -adische Bewertung auf \mathbb{Q}_p ist. Ab jetzt verwenden wir für p -adische Zahlkörper K dies als Notation. Insbesondere:

- \hat{v} ist normierte Bewertung auf K
- v_p ist die p -adische Bewertung auf \mathbb{Q}_p , beziehungsweise ihr Lift auf K

Proposition 5.7.12. Sei K ein p -adischer Zahlkörper und \hat{v} normiert.

(i) Wir erhalten einen wohldefinierten, stetigen Gruppenhomomorphismus

$$\log: U^{(1)} \rightarrow (K, +), 1 + x \mapsto x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

(ii) \log lässt sich auf eindeutige Weise zu einem stetigen Gruppenhomomorphismus

$$\log: K^\times \rightarrow (K, +)$$

fortsetzen so, dass $\log p = 0$ gilt.

Proof. “(i)” (1) Zeige: $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$ ist eine Cauchyfolge für $x \in \mathfrak{p}$.

Verwende:

- $x \in \hat{\mathfrak{p}}$ und damit $v_p(x) > 0$ so, dass $c = p^{v_p(x)} > 1$ und $v_p(x) = \frac{\log c}{\log p}$
- $p^{v_p(k)} \leq k$ für alle $k \in \mathbb{N}$, denn $k = up^{v_p(k)}$ mit $(p, u) = 1$, also $v_p(k) \leq \frac{\log k}{\log p}$

Damit:

$$v_p\left(\frac{x^k}{k}\right) = kv_p(x) - v_p(k) \geq k \frac{\log c}{\log p} - \frac{\log k}{\log p} = \frac{\log \frac{c^k}{k}}{\log p} \xrightarrow{k \rightarrow \infty} \infty$$

(2) Rechne nach, dass für die formalen Potenzreihen gilt

$$\log[(1+x)(1+y)] = \log(1+x) + \log(1+y).$$

(3) Zeige: \log ist stetig.

Allgemeiner: Sei K ein nicht-archimedischer Körper mit diskreter Bewertung v . Sei $\sum_{k \geq 0} a_k x^k$ eine Potenzreihe mit $a_k \in K$. Diese konvergiere auf einer Untergruppe \mathbb{D} von $(K, +)$. Dann ist

$$\mathbb{D} \rightarrow K, x \mapsto \sum_{k \geq 0} a_k x^k$$

stetig.

Proof. Ist $a = v(x - x_n) > 0$, so gilt $x = x_n + \Pi^a u$ mit $u \in \mathcal{O}$. Also $x^k = x_n^k + \Pi^a \cdot r$ so, dass

$$v(x^k - x_n^k) \geq v(x - x_n).$$

Sei nun (x_n) eine Folge mit $x_n \rightarrow x$. Dann gilt

$$\begin{aligned} v\left(\sum_{k \geq 0} a_k x^k - \sum_{k \geq 0} a_k x_n^k\right) &= v\left(\sum_{k \geq 0} a_k (x^k - x_n^k)\right) \\ &\geq \min\{v(a_k) + v(x^k - x_n^k) \mid k \geq 0\} \\ &\geq \min\{v(a_k) + v(x - x_n) \mid k \geq 0\} \\ &= \min\{v(a_k (x - x_n)) \mid k \geq 0\} \\ &\xrightarrow{n \rightarrow \infty} \infty, \end{aligned}$$

da $\sum_{k \geq 0} a_k (x - x_n)^k$ konvergent. □

“(ii)” Definiere für $x \in K^\times$ mit $x = \Pi^{v(x)} \cdot w(x) \cdot \langle x \rangle$ wie in Notation 5.7.8

$$\log x = \hat{v}(x) \log \hat{\Pi} + \log \langle x \rangle,$$

wobei

$$\log \hat{\Pi} = -\frac{1}{e} \log \langle p \rangle.$$

Dann gilt

$$\log p = e \log \hat{\Pi} + \log \langle p \rangle = 0.$$

Nach Konstruktion und Bemerkung 7.9 folgt, dass \log ein stetiger Gruppenhomomorphismus ist und \log auf $U^{(1)}$ fortsetzt.

Zeige noch: Die Fortsetzung ist eindeutig.

Sei $h: K^\times \rightarrow K$ eine Fortsetzung von $\log: U^{(1)} \rightarrow K$ mit $h(p) = 0$.

(1) Für $w \in \mu_{q-1}$ gilt $w^{q-1} = 1$, also

$$(q-1)h(w) = h(1) = 0.$$

(2) $h(\Pi) = -\log(\langle p \rangle)$ muss gelten damit $h(p) = 0$.

(3) Für $x \in K^\times$ mit $x = \hat{\Pi}^{v(x)} w(x) \langle x \rangle$ muss h wie oben definiert sein. □

Proposition 5.7.13. *Sei wiederum K/\mathbb{Q}_p ein p -adischer Zahlkörper mit $\hat{\mathcal{O}}, \hat{p}, \hat{\kappa}, \mathcal{O}, \mathfrak{p}, \kappa$ wie sonst und mit $\mathfrak{p}\hat{\mathcal{O}} = \hat{\mathfrak{p}}^e$. Seien $n > \frac{e}{p-1}$ und notiere $U^{(n)} = 1 + \hat{\mathfrak{p}}^n$. Dann erhalten wir zueinander inverse Isomorphismen*

$$\exp: \hat{\mathfrak{p}}^n \rightarrow U^{(n)} \quad \text{und} \quad \log: U^{(n)} \rightarrow \hat{\mathfrak{p}}^n,$$

wobei

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

und \log wie in Proposition 5.7.12.

Lemma 5.7.14. *Seien v_p die p -adische Bewertung auf \mathbb{Q} und $k \in \mathbb{N}$ mit $k > 1$. Dann gilt*

$$v_p(k) \leq \frac{k-1}{p-1}.$$

Proof. Für $v_p(k) = 0$ stimmt die Behauptung. Sei also $a = v_p(k) > 0$ und schreibe $k = p^a k_0$ mit $(p, k_0) = 1$. Dann gilt

$$\frac{v_p(k)}{k-1} = \frac{1}{p^a k_0 - 1} \leq \frac{a}{p^a - 1} = \frac{1}{p-1} \frac{a}{p^{a-1} + \cdots + p + 1} \leq \frac{1}{p-1}$$

und somit folgt die Behauptung. □

Lemma 5.7.15. *Sei $n \in \mathbb{N}$ mit $n = \sum_{i=0}^r a_i p^i$, wobei $0 \leq a_i < p$. Dann gilt*

$$v_p(n!) = \frac{1}{p-1} \sum_{i=0}^r a_i (p^i - 1) = \frac{1}{p-1} (n - t_n),$$

wobei $t_n = a_0 + a_1 + \cdots + a_r$.