Bayesian Econometrics
Estimating US Yield Curve using Dynamic factor Models

April 10, 2025

The term structure of interest rates, known as the yield curve, illustrates the connection between the remaining time until debt securities mature and the yield they offer. Yield curves serve multiple practical purposes, such as pricing different fixed-income securities. They are closely monitored by both market participants and policymakers, as they can provide valuable insights into the market's sentiment regarding the future direction of the policy rate and the overall macroeconomic outlook.

The Nelson-Siegel model, introduced in [2], interpolates the yield curve (in terms of spot rates) by the following function:

$$s(\tau) = \beta_0 + \beta_1 \cdot \left\{ \frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right\} + \beta_2 \cdot \left[\left\{ \frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right\} - e^{-\lambda\tau} \right], \tag{1}$$

where $s(\tau)$ is the spot rate at any given time to maturity τ, and β_0, β_1, β_2, and λ are model parameters. This parametric model captures stylized shapes observed in yield curves, including monotonic, humped, and S-shaped curves. Parameters β_0, β_1, and β_2 are interpreted as the level, slope and curvature factors, respectively, as they control the long, short and medium segments of the curve. The decay parameter λ_1 determines the exponential decay rate (in monthe to maturity) of the slope and curvature factors, in addition to control the location of the hump (or trough) associated with the curvature factor.

Let $y_t(\tau)$ denote the weekly yield on date t at maturity τ. The yield is typically model by the static model

$$y_t(\tau) = s(\tau) + \eta_t(\tau), \quad \eta_t(\tau) \sim NID(0, \sigma_\tau^2). \tag{2}$$

Diebold, Ruddebusch and Aruoba [1] replace the static parameters by a dynamic specification:

$$\begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_N) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1-\mathrm{e}^{-\tau_1\lambda}}{\tau_1\lambda} & \frac{1-\mathrm{e}^{-\tau_1\lambda}}{\tau_1\lambda} - \mathrm{e}^{-\tau_1\lambda} \\ 1 & \frac{1-\mathrm{e}^{-\tau_2\lambda}}{\tau_2\lambda} & \frac{1-\mathrm{e}^{-\tau_2\lambda}}{\tau_2\lambda} - \mathrm{e}^{-\tau_2\lambda} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-\mathrm{e}^{-\tau_N\lambda}}{\tau_N\lambda} & \frac{1-\mathrm{e}^{-\tau_N\lambda}}{\tau_N\lambda} - \mathrm{e}^{-\tau_N\lambda} \end{pmatrix} \begin{pmatrix} \beta_{0t} \\ \beta_{1t} \\ \beta_{2t} \end{pmatrix} + \begin{pmatrix} \varepsilon_t(\tau_1) \\ \varepsilon_t(\tau_2) \\ \vdots \\ \varepsilon_t(\tau_N) \end{pmatrix}, \tag{3}$$

where the dynamics of $(\beta_1, \beta_2, \beta_3)'$ follows a vector autoregressive process of first order:

$$\begin{pmatrix} \beta_{1t} - \mu_1 \\ \beta_{2t} - \mu_2 \\ \beta_{3t} - \mu_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \beta_{1,t-1} - \mu_1 \\ \beta_{2,t-1} - \mu_2 \\ \beta_{3,t-1} - \mu_3 \end{pmatrix} + \begin{pmatrix} \eta_{1t} \\ \eta_{2t} \\ \eta_{3t} \end{pmatrix}. \tag{4}$$

In an obvious matrix notation, this state-space system is written as a factor model:

$$\begin{aligned} (\beta_{t+1} - \mu) &= A\,(\beta_t - \mu) + \eta_{t+1} \\ y_t &= \Lambda \beta_t + \varepsilon_t \end{aligned} \tag{5}$$

with

$$\begin{aligned} &\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim NID\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix}\right], \\ &\mathrm{E}\,(\beta_0 \eta_t') = 0, \\ &\mathrm{E}\,(\beta_0 \varepsilon_t') = 0. \end{aligned} \tag{6}$$

We will suppose Q is dense but H is a diagonal matrix.

We construct the term structure of interest rates by taking weekly closing prices zero coupon yields released by Bloomberg. We consider the following fixed maturities (τ): 3, 6, 9, 12, 15, 18, 21, 24, 30, 36, 48, 60, 72, 84, 96, 108, 120, 180, 240 and 360 months. The data covers twenty years, from first week of 2002 until last week of 2021.

1. Identify the model parameters and propose a prior distribution for them.
2. Propose a MCMC sampler for the model parameters.
3. Implement the sampler and show its diagnostics.
4. Plot the posterior density of the elements of the autoregressive matrix A.
5. Plot a predictive curve with respective predictive intervals for maturities from 3 to 360 months.

OBS.: Implement the sampling algorithm exactly as described in item 2. Do not use packages such as Stan, JAGS, or similar probabilistic modeling frameworks. However, you may utilize packages for data manipulation, plotting, MCMC diagnostics, as well as Kalman filtering or forward filtering backward smoothing (FFBS).

References

[1] Francis X. Diebold and Glenn D. Ruddebusch and S. Boragˇan Aruoba. The Macroeconomy and the Yield Curve. *Journal of Econometrics*, 131(1-2):309–338, 2006.

[2] Charles R. Nelson and Andrew F. Siegel. Parsimonious modeling of yield curve. *The Journal of Business*, 60(4):473–489, 1987.