Bayesian Econometrics
Estimating US Yield Curve

March 20, 2025

The term structure of interest rates, known as the yield curve, illustrates the connection between the remaining time until debt securities mature and the yield they offer. Yield curves serve multiple practical purposes, such as pricing different fixed-income securities. They are closely monitored by both market participants and policymakers, as they can provide valuable insights into the market's sentiment regarding the future direction of the policy rate and the overall macroeconomic outlook.

The Nelson-Siegel model, introduced in [1], interpolates the yield curve (in terms of spot rates) by the following function:

$$s(\tau) = \beta_0 + \beta_1 \cdot \left\{ \frac{1 - e^{-\tau/\lambda}}{\tau/\lambda} \right\} + \beta_2 \cdot \left[\left\{ \frac{1 - e^{-\tau/\lambda}}{\tau\ /\lambda} \right\} - e^{-\tau/\lambda} \right], \tag{1}$$

where $s(\tau)$ is the spot rate at any given time to maturity τ, and β_0, β_1, β_2, and λ are model parameters. This parametric model captures stylized shapes observed in yield curves, including monotonic, humped, and S-shaped curves. Parameters β_0, β_1, and β_2 are interpreted as the level, slope and curvature factors, respectively, as they control the long, short and medium segments of the curve. The decay parameter λ determines the exponential decay rate (in months to maturity) of the slope and curvature factors, in addition to control the location of the hump (or trough) associated with the curvature factor.

We construct the term structure of interest rates by taking weekly closing prices zero coupon yields released by Bloomberg. We consider the following fixed maturities (τ): 3, 6, 12, 24, 30, 36, 48, 60, 72, 84, 96, 108, 180, 240 and 360 months. The data covers three years, from first week of 2019 until last week of 2021.

Let y_{it} denote the weekly yield on date t ($t = 1, ..., 157$) at maturity τ_i ($i = 1, ..., 20$), i.e., $t = 1$ is the 1st week of 2019, $t = 157$ is the last week of 2021, and $\tau_1 = 3$, $\tau_2 = 6$, ..., $\tau_{15} = 360$. The yield is modelled by

$$y_{it} = s(\tau_i) + u_{it}, \quad u_{it} \sim NID(0, \sigma^2). \tag{2}$$

1. propose prior distributions for the parameters β_0, β_1, β_2, λ explaining your choices and listing all references used;

2. write the likelihood function of your model;

3. propose (derive) an algorithm to estimate the parameters of the model (e.g, Gibbs sampler, Metropolis-Hastings, etc) showing all formulas required for implementation;

4. implement the proposed algorithm in your language of choice and present sampler diagnostics, using the data in *weekly yield* sheet;

5. construct a table with summary statistics for all parameters, including respective standard errors;

6. predict weekly yield on maturities 9, 15, 18, 21 and 120 months. Obtain a point estimate and 95% posterior predictive intervals;

7. compare your prediction with the actual weekly yield, present in the *forecast* sheet;

8. test whether the limiting spot rate β_0 is larger than 6;

9. Nelson and Siegel [1] proposed an extension of their model which considered two exponential decay factors, λ_1 and λ_2, for the slope and curvature:

$$s(\tau) = \beta_0 + \beta_1 \cdot \left\{ \frac{1 - e^{-\tau/\lambda_1}}{\tau/\lambda_1} \right\} + \beta_2 \cdot \left[\left\{ \frac{1 - e^{-\tau/\lambda_2}}{\tau/\lambda_2} \right\} - e^{-\tau/\lambda_2} \right]. \tag{3}$$

   Test whether the restricted model with $\lambda_1 = \lambda_2$ is more adequate than the *unrestricted* model. Note that you will have to propose a prior distribution and algorithm for both parameters as well.

OBS.:You have to implement the sampling algorithm as proposed in item 3. Do not use packages like Stan, JAGS, etc. However, you can use packages for data manipulation, plotting, MCMC evaluation, etc.

References

[1] Charles R. Nelson and Andrew F. Siegel. Parsimonious modeling of yield curve. *The Journal of Business*, 60(4):473–489, 1987.