Escola de Economia de São Paulo - Fundação Getulio Vargas

Course: Econometria 2

Instructor: Vitor Possebom

Problem Set 2 - Total = 200 points

Question 1 (Dynamic Multipliers of Nonstationary Processes - 40 points)

For any $s \in \{0\} \cup \mathbb{N}$, the dynamic multiplier s-periods ahead for a stochastic process $\{Y_t\}$ is given by

$$\frac{\partial Y_{t+s}}{\partial \epsilon_t}$$

and it captures the consequences for Y_{t+s} if ϵ_t were to increase by one unit with ϵ's for all other dates unaffected.

1. *Let $\{Y_t\}$ be a MA(1) process with a deterministic time trend, i.e.,*

   $$Y_t = \alpha + \delta \cdot t + \epsilon_t + \theta \cdot \epsilon_{t-1}$$

   where $\{\epsilon_t\}$ is a white noise process.

   What is the dynamic multiplier s-periods ahead for this stochastic process and any $s \in \{0\} \cup \mathbb{N}$? (5 points for $s = 0$; 5 points for $s = 1$ and 5 points for $s > 2$ — 15 points in total)

2. *Let $\{Y_t\}$ be a I(1) process with a drift, i.e.,*

   $$Y_t = \delta + Y_{t-1} + \epsilon_t$$

   where $\{\epsilon_t\}$ is a white noise process.

   What is the dynamic multiplier s-periods ahead for this stochastic process and any $s \in \{0\} \cup \mathbb{N}$? (15 points for $s \in \{0\} \cup \mathbb{N}$)

3. *Given the results above, explain why unit root processes are considered to have infinite memory. (10 points)*

Question 2 (Deterministic Time Trends - 40 points)

In class, we saw a result that is very surprising in my opinion: for a deterministic time trend model

$$\frac{\hat{\delta}_T - \delta_0}{\left\{ s_T^2 \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} \left(\sum_{t=1}^{T} X_t X_t' \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}^{1/2}} \xrightarrow{d} N(0,1)$$

even though the rate of convergence of $\hat{\delta}_T$ is not $\sqrt{T}$.

We will use a Monte Carlo simulation to deepen our understanding of this result.

Consider the deterministic time trend model

$$Y_t = \alpha + \delta \cdot t + \epsilon_t$$

where $\{\epsilon_t\}$ is i.i.d.

We will simulate this model with sample size $T = 10,000$, $\alpha = 0$ and $\delta = 1$. The number of Monte Carlo repetitions will be $M = 10,000$.

For each Monte Carlo repetition, we will test "$H_0 : \delta = 1$" at the 10% significance level using the usual OLS t-test. We will use this simulation to compute the rejection rate of this test.

1. *Impose that ϵ_t follows a t-distribution with 5 degrees of freedom. What is the test's rejection rate? (15 points)*

2. *Impose that ϵ_t follows a t-distribution with 1 degree of freedom. What is the test's rejection rate? (15 points)*

3. *Are the rejection rates in items 1 and 2 very different from each other? If so, what features of our model can explain this difference? If not, why are they similar? (10 points)*

Question 3 (Asymptotic Properties of an AR(1) with a unit root - 50 points)

1. ***No Constant Term or Time Trend in the Regression, True Process is a Random Walk***

   True Process: $Y_t = \rho \cdot Y_{t-1} + \epsilon_t$ *where* $\rho = 1$, $\{\epsilon_t\}$ *is i.i.d. with mean zero and variance* σ^2.

   Estimating Equation: $Y_t = \rho \cdot Y_{t-1} + \epsilon_t$

   Prove that:

   (a) $T \cdot (\hat{\rho}_T - 1) \xrightarrow{d} \dfrac{(1/2) \cdot \left\{ [W(1)]^2 - 1 \right\}}{\int_0^1 [W(r)]^2 \, dr}$ *(10 points)*

   (b) $\dfrac{\hat{\rho}_T - 1}{\hat{\sigma}_{\rho_T}} \xrightarrow{d} \dfrac{(1/2) \cdot \left\{ [W(1)]^2 - 1 \right\}}{\left\{ \int_0^1 [W(r)]^2 \, dr \right\}^{1/2}}$ *(10 points)*

2. ***Constant Term but No Time Trend in the Regression, True Process is a Random Walk***

   True Process: $Y_t = \alpha + \rho \cdot Y_{t-1} + \epsilon_t$ *where* $\alpha = 0, \rho = 1$, $\{\epsilon_t\}$ *is i.i.d. with mean zero and variance* σ^2.

   Estimating Equation: $Y_t = \alpha + \rho \cdot Y_{t-1} + \epsilon_t$

   Prove that:

   (a) $T \cdot (\hat{\rho}_T - 1) \xrightarrow{d} \dfrac{(1/2) \cdot \left\{ [W(1)]^2 - 1 \right\} - W(1) \cdot \int_0^1 W(r) \, dr}{\int_0^1 [W(r)]^2 \, dr - \left[\int_0^1 W(r) \, dr \right]^2}$ *(10 points)*

   (b) $\dfrac{\hat{\rho}_T - 1}{\hat{\sigma}_{\rho_T}} \xrightarrow{d} \dfrac{(1/2) \cdot \left\{ [W(1)]^2 - 1 \right\} - W(1) \cdot \int_0^1 W(r) \, dr}{\left\{ \int_0^1 [W(r)]^2 \, dr - \left[\int_0^1 W(r) \, dr \right]^2 \right\}^{1/2}}$ *(10 points)*

3. ***Constant Term but No Time Trend in the Regression, True Process is a Random Walk with Drift***

   True Process: $Y_t = \alpha + \rho \cdot Y_{t-1} + \epsilon_t$ *where* $\alpha \neq 0, \rho = 1$, $\{\epsilon_t\}$ *is i.i.d. with mean zero and variance* σ^2.

   Estimating Equation: $Y_t = \alpha + \rho \cdot Y_{t-1} + \epsilon_t$

Prove that (10 points):

$$\begin{bmatrix} T^{1/2}\left(\hat{\alpha}_T - \alpha\right) \\ T^{3/2}\left(\hat{\rho}_T - 1\right) \end{bmatrix} \xrightarrow{d} N\left(0, \sigma^2 \cdot \begin{bmatrix} 1 & \alpha/2 \\ \alpha/2 & \alpha^2/3 \end{bmatrix}^{-1}\right)$$

Question 4 (Augmented Dickey-Fuller Test in Practice - 70 points)

In the file `corn-production-land-us.csv`*, you can find the time series for corn production (tonnes) in the U.S.. Using the data between 1950 and 2021, test whether this stochastic process has a unit root or not.*