Macro III - Problem Set 1

August 2024

Deadline: Wednesday, Aug 21

1 Solving Nonlinear Equations - Bisection Method (1 pt)

Consider a Solow model of economic growth in which the steady-state equilibrium equation is characterised by:

$$sf(k) = (n + g + \delta)k, \tag{1.1}$$

where s is the saving rate, $f(k) = k^\alpha$ is the production function, n is the population growth rate, g is the technological progress rate, and δ is the depreciation rate.

(a) Derive the analytical steady-state capital per worker. Given the parameter values $s = 0.3$, $\alpha = 0.3$, $n = 0.01$, $g = 0.02$ and $\delta = 0.05$, compute the analytical solution for k^*. This solution will serve as a benchmark for the numerical method.

(b) Implement the bisection method to find the numerical solution for k^* given the same parameter values. Make sure to specify your initial interval for the root search. Discuss your choice for this interval.

(c) Compare the numerical and analytical solutions. What is the relative error between these solutions? How many iterations does it take for the bisection method to converge to the analytical solution?

(e) Now, let's test the robustness of the bisection method. Alter the saving rate s to 0.1, 0.2, 0.4, and 0.5. For each value of s, calculate the analytical solution, the numerical solution (via the bisection method), the relative error between these two solutions, and the computation time of the numerical method. Discuss your findings.

2 Solving Nonlinear Equations - Newton's Method in Optimal Control Problem with Non-linear Wage (2 pts)

Consider a firm that maximises its profits over a certain time period. The firm's profit is given by

$$\Pi_t = A_t l_t^\alpha - l_t w_t(l_t), \tag{2.1}$$

where A_t is the total factor productivity, and the wage rate is given by a non-linear function $w_t(l_t) = w_0 e^{\eta l_t}$, where w_0 is the base wage and η is the elasticity of the wage with respect to labor. The firm chooses l_t to maximise its profits over time.

(a) Derive the firm's first order condition with respect to labor, l_t. Can you analytically find the optimal labor demand l_t? Discuss why.

(b) Implement Newton's method to find the numerical solution for l^* given $A = 1$, $\alpha = 0.4$, $w_0 = 5$, and $\eta = 0.1$. How many iterations does it take to converge? Try different initial guesses.

(c) Perform a sensitivity analysis to check how changes in the parameters α, A, w_0, and η affect the numerical solution for l^*. Interpret the economic implications of these results.

(d) Now use the bisection method to find the solution. Discuss the advantages and disadvantages of using Newton's method instead of the bisection method for solving this problem. Which one was faster?

3 Solving Nonlinear Equations - Secant Method in a Three-Period Consumption-Saving Problem (2 pts)

Consider a simple three-period consumption-saving model where a consumer receives income y in period 1 and must allocate her consumption over three periods (c_1, c_2, and c_3). The consumer can save and earn r as interest. The consumer's preferences are given by the utility function $U = u(c_1) + \beta u(c_2) + \beta^2 u(c_3)$, where $u(c) = \frac{c^{1-\theta}}{1-\theta}$, $\theta > 0$ is the relative risk aversion parameter, and $0 < \beta < 1$ is the discount factor. The consumer maximises utility subject to the budget constraints in each period.

(a) Write down the consumer's problem. What are the choice variables, and what are the constraints she faces?

(b) Write down the system of equations that characterise the optimal consumption-saving decision.

(c) Can you solve this system of equations analytically to find the optimal consumption levels c_1, c_2, and c_3? Discuss.

(d) Implement the secant method to find the numerical solutions for c_1, c_2, and c_3 given $y = 100$, $r = 0.05$, $\theta = 2$, and $\beta = 0.95$.

(e) Perform counterfactual exercises to understand how changes in the parameters y, r, θ, and β affect the optimal consumption-saving decisions. Interpret the economic implications of these results.

(f) Calibration exercise: try to find an interest rate value r such that $c_1 = c_3$. Can you write a code for that?

4 Approximation Methods - Interpolation - Consumption Function (1.5 pts)

Consider a piecewise quadratic consumption function $C(y)$ on the domain $y \in [0, 10]$ where y is the disposable income:

$$C(y) = \begin{cases} 0.8y & \text{for } 0 \leq y < 5 \\ 4 + 0.2(y - 5)^2 & \text{for } 5 \leq y \leq 10 \end{cases}$$

This function represents a hypothetical scenario where consumers spend a certain fraction of their income if it's less than a certain threshold (in this case, 5), and behave differently when the income is above this threshold.

(a) Approximate $C(y)$ with a cubic spline, using $n = 5$ equally spaced nodes. Plot the function along with your approximation and calculate the root mean squared error (RMSE) of your approximation over a fine grid, with an interval of 0.001.

(b) Approximate $C(y)$ with a cubic spline, using $n = 10$ equally spaced nodes. Plot this new approximation along with both the actual function and your approximation from (a) and calculate the RMSE of your approximation over a fine grid, with an interval of 0.001. Discuss the reasons for the differences in your answers.

(c) Conduct a sensitivity analysis on the number of nodes used for the approximation. Specifically, calculate and plot the RMSE as you increase the number of nodes from $n = 5$ to $n = 50$. Describe the relationship you observe between the number of nodes and the RMSE.

(d) Based on your result from part (c), what would you consider a "good" number of nodes to balance the accuracy of approximation and computational efficiency?

5 Estimating π (1 pt)

In this question, you will use a Monte Carlo procedure to estimate the value of π. Pick a square with a perimeter of $8r$ and inscribe a circumference with a $2r$ diameter on it. We can estimate π by randomly drawing points inside the square and counting how many of them lie inside the circle.

(a) Describe in detail the algorithm you intend to use to estimate π.

(b) Plot the square, the inscribed circumference, and a random draw of 100 points. What is the value of your estimated π with these 100 points?

(c) Replicate your algorithm 20 times, using different amounts of random draws ($n = 5000, 10000, 15000, \ldots, 100000$). Plot the estimates of π in the vertical axis as a function of the number of random draws (n) in the horizontal axis.

6   Two-period model (2.5 pts)

Consider the following two-period ($t = 0, 1$) standard model economy. Preferences are over consumption c and hours worked h and are represented by

$$U = \sum_{t=0}^{1} \beta^t \left[u\left(c_t\right) - \chi v\left(h_t\right) \right], \quad \beta \in (0, 1),$$

and $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$ and $v(h) = \frac{h^{1+\eta}-1}{1+\eta}$ with $(\sigma, \eta) \in \mathbb{R}^2_{++}$. On the production side, there is a competitive firm, with access to a technology to produce the consumption good:

$$Y_t = A_t L_t^{\alpha}, A_t > 0 \text{ and } \alpha \in (0, 1),$$

where Y_t is output, L_t denotes labor input and A_t is a productivity factor. Suppose that there is an asset market where agents can trade one-period bonds among themselves.

(a) Build the problem of each agent and the market clearing condition. How many endogenous variables this economy has?

(b) Define a competitive equilibrium for this economy. Be precise about the problem of each agent, each firm, and market clearing conditions.

(c) Derive the first-order conditions of the problem of each agent and each firm and go as far as possible in writing the market clearing equilibrium conditions.

(d) Now, let $A_0 = 1, A_1 = 1, \alpha = 0.6, \sigma = 2, \eta = 5, \beta = 0.98^{25}$ (subjective discount rate of 2% per year, but a model period is 25 years) and $\chi = 1$. Find all allocations (consumption, labor, and assets) and prices (w_0, w_1 and r) for this economy. You probably need to do this numerically - explain how you did it.

(e) Let A_0 decrease from 1 to 0.90 and keep the value of all other parameters as before. Redo all your calculations and comment on your results.

(f) Let A_1 decrease from 1 to 0.90 and keep the value of all other parameters as before. Redo all your calculations and comment on your results.

(g) Now, let $\sigma = 1.5$ and then redo (c)-(e). Comment on your results.