Brilliant sequence of umbrellas

- The goal is to find a sequence of k numbers between 1 to N s.t. the sequence is increasing and adjacent gcds are increasing

- First thing to note is the specific constraints on k which is $\lceil \frac{2}{3}\sqrt{n} \rceil \leq k$?

- How can we even approach the problem may we focuss on just the gcds & ignore the n constraint

- In a pefect solution we find a sequence $a_1 < a_2 < a_3 \ldots < a_k$ s.t. $\gcd(a_1, a_2) = 1 < \gcd(a_2, a_3) = 2 < \gcd(a_3, a_4) = 3 \ldots$ & $a_i \leq N$

- But this seems quite hard maybe impossible? Lets denote the gcd sequence as $b_1 < b_2 < b_3 \ldots b_{k-1}$ where $b_i = \gcd(a_i, a_{i+1})$

- if $b = [1, 2, 3, 4, \ldots]$ then $a_2 = 2 \cdot 3 \cdot x_1$ $a_3 = 3 \cdot 4 \cdot x_2$ $a_4 = 4 \cdot 5 \cdot x_3$ but then $\gcd(a_2, a_3) \geq 6$ so this persect sequence isn't possible. But it raises a problem if we have a gcd sequence then all adjacent values are coprime

- How can we guarantee coprime numbers? what if we pick only primes & 1 so $[1, 2, 3, 5, 7 \ldots]$ then we try to generate the output sequence we end up with $1, 2, 6, 15, 35 \ldots$ so values are increasing & adjacent gcds are increasing so we are done? well no

- If $N = 10^{12}$ then we need $\lceil \frac{2}{3}\sqrt{10^{12}} \rceil = \frac{2}{3}10^6$ primes as an estimate of primes less than x $\pi(x) = \frac{x}{\ln(x)}$ for $\frac{x}{\ln x} \geq \frac{2}{3} \cdot 10^6$ x will have to be $\sim 10^7$ so our final number $\sim 10^{14}$ slightly too high

- Another possible gcd sequence could be all odd numbers $[1, 3, 5, 7 \ldots]$ a little closer but still no after some trial and error we come to the gcd sequence $[1, 2, 3, 5, 7, 8, 9, 11]$ which gives us an original sequence $[1, 2, 6, 15, 35, 56, 72, 99 \ldots]$

- How do we know gcds are increasing if all adjacent vaues in the assumed gcd sequence are coprime since the sequence only contains either b_i, b_i+1 since one number is even and one odd $\gcd(b, b+1) = 1$ or b_i, b_i+2 where both are odd $\gcd(b_i, b_i+2) = 1$

- What is the expected max value? If we look at differences we have $\overbrace{+1\ +1\ +2\ +2}^{6}$ $+1\ +1\ +2\ +2$ we add 6 every 4 numbers so our max value

$$\left(\frac{x}{4} \cdot 6\right)^2 \geq N \Rightarrow x = \sqrt{N} \cdot \frac{4}{6} = \sqrt{N} \cdot \frac{2}{3}$$ which is where that weird lower bound comes from