Timetabling Ferries

1 Introduction

TBW

2 Literature review

TBD

3 The problem

The problem we are dealing with is how to determine the timetables and the vessels' itineraries in a public transport ferry system, where both the *strategic* and *tactical* decisions are exogenous. This means that the stations in the system, as well as the lines and their headways, are all given and are constraints of the system. On the other hand, the exact departure of every sailing of every line can be decided, as long as the required headways are fulfilled. These headways are not constant, reflecting that most ferry systems offer different frequencies during and off-peak periods. The speed of the vessels and the berthing time in every station are also exogenous.

Moreover, there is a fleet of vessels (whose size and composition is exogenous), and we need to determine which vessel is going to execute each sailing, subject to a number of constraints. Besides obvious constraints, such as every sailing must be executed and vessels cannot do two things at the same time, there are: capacity constraints in the wharves, charging constraints for the vessels, and constraints related to crew pauses. The details of the problem inputs and required outputs are as follows.

The problem is characterized by:

- A timespan where the system is being operated. We assume a discretisation of this whole interval, and denote the set of periods by $\mathcal{T} = \{1, \ldots, T\}$. E.g., every period could last 5 minutes, so if we are modelling 10 hours of operation, we would have $T = 120$.

- The set of *stations*. In this context, a station is a physical place with many wharves, so that more than one line could use each station. It is analogous to a metro or train station, that might have several platforms. For each station S, we know how many wharves it has, which we denote by P_S. For each wharf w in the system, we denote by C_w its capacity, i.e., the maximum number of vessels that can be using it at the same time. We denote by $S(w)$ the station corresponding to the wharf w.

  For example, the station Circular Quay in Sydney has 6 wharves, although just 4 of them are used in the public transport system. Each of these wharves has two berths, one at each side. This is the main station in the system. Most stations in Sydney only have one wharf, with capacity 1 or 2.

  For each pair of stations S_1, S_2, we denote by $\xi(S_1, S_2)$ how long does it take to navigate between them.

- The lines: Each line l has a route (i.e. a sequence of stations to be visited), headways (detailed in a different bullet point below), and a set of times when the first sailing of that line can begin. We denote by R_l the route of line l, where R_l is a vector containing the sequence of stations. Throughout the paper, we use the notation $S \in R_l$ to describe that S is one element of the vector, i.e., that the line stops there.

We remark that if a line is bidirectional, we consider each direction as a different line for the whole of this paper.

For every station visited by the line, we know which wharves could be used. If line l visits station S (i.e., if S is part of the vector R_l), we denote by $\mathcal{C}(l, S)$ the set of wharves in S that can be used. While in some stations any wharf can be used, in others there will be geographical limits, e.g., a line going eastwards might need to depart from a wharf located in the east section of a station.

It is noteworthy that:

- For every station that is not the last one, the line must always visit the same wharf. That is to say, part of the modelling problem is to decide one wharf that will be used in every sailing.
- For its last station, a different wharf can be used in every sailing[1].

The rationale behind that distinction relates to boarding passengers. It is easier for the users if they can learn from which wharf their line departs, so they do not need to check every time. In the last station, there are no new passengers boarding, which is why this restriction can be lifted.

- Given a line l and a station S visited by it, we know for how long a wharf will be occupied. We denote this dwelling time by $dw(l, S)$, which is expressed in number of time periods. This time includes any buffer when subsequent vessels cannot berth yet. E.g., if the vessel needs to stop for 5 minutes for passengers to board and alight, and there are 5 extra minutes where no other vessel can berth due to safety reasons, then dw will represent 10 minutes (i.e., two time periods if every period lasts for 5 minutes).

- Headways are typically divisors of 60 (so that the timetable is clockface), and repeated during a period (peak, off-peak, etc). For example, one sailing every 30 minutes before the morning peak, every 20 during the peak, every 30 in the afternoon, every 20 in the evening peak, and every 60 after that. One possible way to describe this is as a list $H_l = \{h_{l,1}, \cdots, h_{l,n_l-1}\}$, where n_l is the number of sailings. In the ILP formulation we shall use a slightly more general description, considering that this list could depend on the departure time of the first sailing. In any case, this information is taken as exogenous.

- A set of vessels. For each vessel v, we know its starting station S_v and which lines can it serve $li(v)$. This last definition reflects that some lines might require a particular type of vessel, either in terms of capacity or speed.

- The set of wharves where it is possible for a wharf to wait W^1, the set of wharves where it is possible to recharge W^2, and the set of wharves where it is possible to do a crew pause W^3. We assume that every berth within a wharf shares the same characteristics. Note that these wharves are not necessarily disjoint. Moreover, some of these wharves might also be part of the set $\mathcal{C}(l, S)$ for some line. In other words, every wharf in the system has up to 4 possible ways to be used (by a line, to wait, to recharge, and to do a crew pause).

- A rate r_v at which the battery of vessel v discharges whenever it is rebalancing, and a rate r_v^+ at which it gets charged when it is in a charging wharf. There is a plugging/unplugging time p_c spent at the beginning and at the end of the charging process where the vessel is not charging. The discharging rate per line is also known, but they might be line-dependent. We also know, for every vessel v, its initial charge $Q_{v,0}$. We normalise the charge levels, so they are always within $[0, 1]$.

- We assume that every vessel has the same crew during the whole day, and we need to fulfill some rules regulating the crew breaks: The minimum number of breaks n_c, their duration D_c, and the maximum time separation between two consecutive breaks T_c.

We need to output three objects:

1. The *timetable*: At which times will each line depart from each wharf. We remark that, for every line l, it suffices to provide:

   (a) The time at which the first sailing of the line departs, denoted FS_l. With this information, we know exactly at which time will every other sailing depart, as the headways are exogenous.

   (b) For every station S visited by the line, with the exception of the last one, the wharf $w \in \mathcal{C}(l, S)$ that will be used by the line.

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[1]Both this intuitive formulation and the ILP studied in the next section could be easily extended to a more general scenario where it is not just the last station, but an arbitrary subset, that the line can change the wharf in every sailing.

(c) For every sailing s, and for the last station S in R_l, a wharf in $w_s \in \mathcal{C}(l, S)$.

(d) For every sailing s and for every wharf w visited in the sailing, a berth $b_{l,s,w}$ within the wharf to be utilised.

2. An *itinerary* for each vessel: A list of tasks that the vessel needs to do consecutively from the beginning of day, where a task can be i) execute a sailing, ii) rebalance from a given berth in a wharf to a given berth in a different wharf, iii) wait for a given time at a given berth in a wharf, iv) recharge for a given time at a given berth in a wharf, or v) do a crew break at a given berth in a wharf.

An itinerary is feasible if

(a) The finishing points and times of every task coincide exactly with the starting point and time of the next one.

(b) The first task begins exactly at the start of the timespan, and the last task finishes exactly at the end of the timespan.

(c) The first task of v begins in S_v.

(d) For every sailing as described in the timetable, there is exactly one vessel associated to it. Said vessel must be able to serve the corresponding line.

(e) The charge of the vessel is always positive.

(f) Crew pauses are scheduled according to the rules described above.

3. For every wharf and every berth within the wharf, a list showing which vessel (if any) is using it for every time period. This list is feasible if

(a) The number of vessels never exceeds 1.

(b) Every task associated to a vessel that uses that berth is reflected in the list.

4   ILP Model

4.1   Additional inputs

In order to propose an ILP formulation, we need to introduce some additional notation. Let us first introduce some extra inputs.

- $\mathcal{D}(l) \subset \mathcal{T}$ is the set of times where line l can execute its first sailing. If the first sailing starts at time $d \in \mathcal{D}(l)$, then for every sailing $s \in \{1, \ldots, n_l\}$ of line l, we denote by $h(s, d)$ the departure time of s. For instance, if the headway was constant throughout and equal to h_l, then $h(s, d) = d + h_l(s - 1)$.

- $\mathcal{B}$ denotes the set of wharves where a vessel can *wait*, i.e., where they can spend any number of periods without moving. In this formulation, every wharf with a charger belongs in $\mathcal{B}$, so that if a vessel is waiting in a wharf with charger, we assume it is being recharged.

- $\mathcal{B}_c$ denotes the set of wharves where vessels can do a crew pause. *Formally, the elements in $\mathcal{B}_c$ are copies of the original wharves, meaning that whenever we cup elements from $\mathcal{B}$ and $\mathcal{B}_c$, it is a disjoint union.*

A crucial concept in our formulation are the *tasks*. A task is any activity that can be executed by a vessel. Rebalancing is not considered as a task per se, but is implicit if the ending point of a task does not coincide with the starting point of the next one. Formally:

- The set of tasks $\mathcal{J}$ is defined as $\mathcal{J} = \mathcal{L} \cup \mathcal{B} \cup \mathcal{B}_+ \cup \mathcal{B}_c$. The set $\mathcal{B}_+$ contains one copy of every wharf with a charger. If w is a wharf with a charger and its copy in $\mathcal{B}$ is selected, it means that it is the first or last period charging, meaning that part of that period will be lost plugging or unplugging; if the copy in $\mathcal{B}_+$ is selected, it means that the vessel is being charged during the whole period. For a wharf $w \in \mathcal{B}_+$, we denote by $\varphi(w)$ its copy in $\mathcal{B}$.

That is to say, at every time a vessel is either told to execute the sailing of a line, to wait at a wharf (potentially being recharged), or to do a crew pause at a given wharf.

- We denote by $\mu(j)$ the duration, in units of time periods, of task j. Note that every line has a predetermined duration, and a crew pause lasts D_c. Waiting (charging or not), on the other hand, is variable. Therefore, for $j \in \mathcal{B} \cup \mathcal{B}_+$, we assume that $\mu(j)$ is the minimum number of periods (e.g. $\mu(j) = 1$): it is possible to wait for more than $\mu(j)$ by repeating the same task several consecutive times.

- For each pair of tasks j_1, j_2, we denote by $\xi(j_1, j_2)$ the number of time units required to go from the ending point of j_1 to the starting point of j_2. That is to say, if a vessel is told to execute these two tasks consecutively, $\xi(j_1, j_2)$ represents the rebalancing time.

- To consider the starting point of each vessel. For each $v \in \mathcal{V}$ and each $j \in \mathcal{J}$, we denote by $\xi^0(v, j)$ the time period in which vessel v would execute task j, if that was the first task the v performs. Typically, $\xi^0(v, j)$ represents the time required to travel from the starting position of v to the wharf where j begins. If the starting point of every vessel can be decided, then $\xi^0 = 0$.

- We denote by $\mathcal{C}(j)$ all the wharves that could be used given task j. Note that if j is a line, then $\mathcal{C}(j) = \cup_{S \in R_l} \mathcal{C}(l, S)$. If j is a wharf (either in $\mathcal{B}$ or $\mathcal{B}_c$), then $\mathcal{C}(j) = \{j\}$.

- If a vessel v begins a sailing s in time t_0, we can know in advance exactly which stations are going to be used by v and when. We denote by $\Delta(j, w)$ the set of times t such that the wharf w will be occupied in time $t_0 + t$ because of j if it selects $w \in \mathcal{C}(j)$. If j is a line, as we are assuming that the wharf is known, then we exclude the last station from Δ. This will be tackled in the next bullet point.

  Note that if $j \in \mathcal{B} \cup \mathcal{B}_+ \cup \mathcal{B}_c$, then $w = j$ and $\Delta(j, j) = \{0, ..., \mu(j) - 1\}$.

  Formally, for any wharf $w \notin \mathcal{C}(j)$, we define $\Delta(j, w) = \emptyset$.

  If j is a line, Δ is typically given by the dwelling times. For example, consider that it takes a to arrive to station s since the start of a sailing of line l, and the dwelling time is d_w. Then, $\Delta(l, w)$ must contain at least $(w, a), \ldots, (w, a + d_w - 1)$. This abstract formulation permits including extra safety time, e.g., a few time units before a and after $a + d_w - 1$ to ensure that no vessels will be doing maneuvers at the same time in the same place (a *safety buffer*).

- Given a line l, we denote by $\mathcal{A}(l)$ its last station. In order to account for the wharves used in that station, let us denote by $F(l)$ the number of time periods between the start of a sailing of that line and the moment in which the vessel arrives to the last station. Moreover, we denote by $\mu_F(l)$ the time that a wharf in the last station is occupied. In other words, if a vessel starts a sailing of line l at period t, then it will be using some wharf in its last station between $t + F(l), \ldots, t + F(l) + \mu_F(l) - 1$. Again, μ_F includes any potential safety buffer.

Battery-wise, as our formulation is based on tasks, it is useful to introduce a notation that links the changes in battery levels with the tasks. Concretely, for each vessel v, and $j \in \mathcal{J}$, and each $t \in \{0, 1, \ldots, \mu(j) - 1\}$, we consider a coefficient $q_{v,j,t} \in \mathbb{R}$. $q_{v,j,t}$ represents the battery change of the vessel v, t times unit after it starts task j, compared to $t - 1$. Note that:

1. If the vessel is charging in the middle of the process (i.e., if $j \in \mathcal{B}_+$), then $\forall t, q_{v,j,t} = r_v^+$.

2. If the vessel is in its first or last period of charging (i.e., if $j \in mB$, specifically in a wharf with charger), then $\forall t, q_{v,j,t} = \varepsilon r_v^+$, where $\varepsilon \in (0, 1)$ represents the percentage of time being charged.

3. If j represents waiting (i.e., if $j \in \mathcal{B}$, specifically a wharf where it is not possible to charge), then $\forall t, q_{v,j,t} = 0$.

4. If the vessel is executing a sailing, then $q_{v,j,t}$ will be a sequence of $-r_j$ when the vessel is moving, and 0 during the intermediate stops. The parameter $r_{v,j}$ represents the battery consumption when executing a task, that could be vessel and task-dependent (e.g., some lines are served with greater speeds).

Note that if a vessel is rebalancing, it is being discharged but not executing any task. This will be captured in the constraints, specifically by noting that i) any time where the vessel is not executing any task will imply that the vessel is being rebalanced, and thus ii) any time where the vessel is not executing any task it is being discharged at rate r_v.

4.2 Auxiliary notation

Let us now define some auxiliary notation that do not require any extra inputs, but is constructed through the inputs we have already introduced above:

- The set of all sailings in the system is denoted $\mathcal{Z}$. For a given sailing $s \in \mathcal{Z}$, the notation $\ell(s)$ denotes its line.

- For every $j \in \mathcal{J}$ and $t \in \mathcal{T}$ we denote by $\phi(j, t)$ a set of time periods. Concretely, if $t' \in \phi(j, t)$, and a vessel starts executing task j at time t', then it will still be executing j at time t. Formally, $\phi(j, t) = \{t' \in \mathcal{T} : 0 \leq t - t' < \mu(j)\} = \{\max(1, t - \mu(j) + 1), ..., t\}$ (both definitions are equivalent).

- For each $j \in \mathcal{J}$, let $f(j) = T + 1 - \mu(j)$. This is, $f(j)$ is the last time in which task j can be scheduled to start, so that it is finished before the end of the day.

- For each $l \in \mathcal{L}$, we define $G(l) \subseteq \mathcal{T}$ as the set of valid times in which a sailing of line l can be started. The formal definition is:
$$G(l) = \bigcup_{d \in \mathcal{D}(l)} \bigcup_{s=1,...,n_l} \{h(s, d)\}$$

We extend this definition to tasks in general: if j is waiting or a pause crew, then $G(j) = \mathcal{T}$.

- For each task j and vessel v, we define $H(v, j)$ as the set of times in which vessel v can start task j, considering the restrictions in terms of i) its initial position (it cannot start j if it does not have enough time since the beginning of the timespan), and ii) it needs enough time to complete the task before the end of the timespan. Formally, $H(v, j) = \{t \in G(j) : \xi^0(v, l) \leq t \leq f(j)\}$.

- For each task j and time t, we define $\mathcal{F}(j, t) \subseteq \mathcal{J}$ as the set of tasks that can be performed-and-finished after task j, if j started at time t. Formally, if $t > f(j)$, then $\mathcal{F}(j, t) = \emptyset$ (not even j would have time to be finished, let alone a subsequent task). Otherwise, $\mathcal{F}(j, t) = \{j' \in \mathcal{J} : f(j') \geq t + \mu(j) + \xi(j, j')\}$.

- For each wharf w and time t, we define $E(w, t)$ as the pairs (j, t') such that, if a vessel starts task j in time t', and $w \in \mathcal{C}(j)$, then the vessel will be using a wharf in the station $S(w)$ in time t. Formally, $E(w, t) = \{(j, t') \in \mathcal{J} \times \mathcal{T} : t - t' \in \Delta(j, w)\}$.

4.3   Variables

We define the following variables:

- $x_{l,d} \in \{0, 1\}$ for each $l \in \mathcal{L}$, and each $d \in \mathcal{D}(l)$. This is a binary variable, which is 1 in case the first departure of line l starts at time t.

- $y_{v,j,t} \in \{0, 1\}$ for each $v \in \mathcal{V}$, $j \in \mathcal{J}$, $t \in \mathcal{T}$. A binary variable, which is 1 in case vessel v executes task j starting in time t.

- $Q_{v,t} \in [0, 1]$ for each $v \in \mathcal{V}, t \in \mathcal{T}$. This variable represents the level of charge of vessel v at time t.

- $z_{w,j} \in \{0, 1\}$ for each task j and each $w \in \mathcal{C}(j)$. It is a binary variable, taking the value 1 if task j will use wharf w. These variables are used for the stations that are not the last one.

- $Z_{l,w,t} \in \{0, 1\}$ for each line $l \in \mathcal{L}$, for each wharf $w \in \mathcal{A}(l)$, and for each $t \in \mathcal{T}$. It takes the value 1 if any sailing of line l starts using wharf w (belonging to its last station) at time t. Similarly, we define $Z'_{l,w,t} \in \{0, 1\}$ if the wharf is being used during time t. Note that they refer to the last station of line l. As will be shown later, variables Z' are just auxiliary and not a *true* decision variable, as they will get fully defined through the variables Z. However, the writing of the problem becomes simpler if we use them.

4.4   Constraints

4.4.1   Linking constraints

These are the constraints that connect the different variables, ensuring that all sailings are executed in time, among others. We explain in detail each of them below.

$$\sum_{d \in \mathcal{D}(l)} x_{l,d} = 1 \qquad\qquad \forall l \in \mathcal{L}, \qquad (1a)$$

$$\sum_{v \in \mathcal{V}} y_{v,\ell(s),h(s,d)} = x_{\ell(s),d} \qquad\qquad \forall s \in \mathcal{Z}, \forall d \in \mathcal{D}(\ell(s)), \qquad (1b)$$

$$y_{v,j,t} = 0 \qquad\qquad \forall v \in \mathcal{V}, \forall j \in \mathcal{J}, \forall t \in \mathcal{T} \setminus H(v,j), \qquad (1c)$$

$$y_{v,j,t} = 0 \qquad\qquad \forall t \in \mathcal{T}, v \in \mathcal{V}, j \in \mathcal{L} \setminus li(v) \qquad (1d)$$

$$\sum_{j \in \mathcal{J}} y_{v,j,\xi^0(v,j)} = 1 \qquad\qquad \forall v \in \mathcal{V}, \qquad (1e)$$

$$\sum_{j \in \mathcal{J}} \sum_{t' \in \phi(j,t)} y_{v,j,t'} \leq 1 \qquad\qquad \forall v \in \mathcal{V}, \forall t \in \mathcal{T}, \qquad (1f)$$

$$\sum_{w \in \mathcal{C}(l,S)} z_{l,w} = 1 \qquad\qquad \forall l \in \mathcal{L}, S \in R_l \setminus \mathcal{A}(l), \qquad (1g)$$

$$\sum_{w \in \mathcal{C}(l,\mathcal{A}(l))} Z_{l,w,t} = \sum_{v \in \mathcal{V}} y_{v,l,t-F(l)} \qquad\qquad \forall l \in \mathcal{L}, \forall t \geq F(l), \qquad (1h)$$

$$Z'_{l,w,t} = \sum_{k=0}^{\mu_F(l)-1} Z_{l,w,t-k} \qquad \forall t \in \mathcal{T}, \forall l \in \mathcal{L}, w \in \mathcal{C}(l,\mathcal{A}(l)), t' = 0, \ldots, \mu_F(l) - 1, \qquad (1i)$$

$$y_{v,w,t} \leq y_{v,w,t-1} + y_{v,\varphi(w),t-1} \qquad\qquad \forall w \in \mathcal{B}_+, v \in \mathcal{V}, t \geq 1, \qquad (1j)$$

$$y_{v,w,t} \leq y_{v,w,t+1} + y_{v,\varphi(w),t+1} \qquad\qquad \forall w \in \mathcal{B}_+, v \in \mathcal{V}, t \leq T - 1, \qquad (1k)$$

$$\sum_{j' \in \mathcal{F}(j,t)} y_{v,j',t+\mu(j)+\xi(j,j')} \geq y_{v,j,t} \qquad\qquad \forall (j,t) \in \mathcal{J} \times \mathcal{T}, \mathcal{F}(j,t) \neq \emptyset, \forall v \in \mathcal{V} \qquad (2)$$

$$y_{v,j,t} + y_{v,j',t'} \leq 1 \qquad \forall j \in \mathcal{J}, \forall t \in \mathcal{T}, \forall j' \in \mathcal{F}(j,t), \forall t' \in \mathcal{T}, \mu(j) \leq t' - t < \mu(j) + \xi(j,j'). \qquad (3)$$

Constraints (1) impose that sailings are assigned to time periods, that they are consistent with line departures and that vessels are assigned to tasks in a feasible way.

Specifically, note the following:

- Constraints (1a) forces that each line has exactly one initial departure.

- Constraints (1b) forces that for each sailing, its line has exactly one vessel assigned to that task in the time period associated to that sailing in the chosen departure of the line, and that the same line has no vessel assigned in other time periods.

- Constraints (1c) forces that vessels cannot be assigned to start tasks in time periods where it is not feasible. This might happen because it is a time before the time it takes for the vessel to arrive to start the task, it is a time too late so the task cannot be completed in time, or because it is a line task (sailing), and there is no departure that would make the task start at that time. While some of these situations would already be prevented with the previous constraints, including these ones will hasten the computation.

- Constraints (1d) ensures that a vessel can only be assigned to the appropriate lines (given by the set $li(v)$).

- Constraints (1e) forces that each vessel starts the day with one assigned task and in the correct time.

- Constraints (1f) forces that for each vessel and each time period, that vessel is executing at most one task at the time.

- (1g) forces that for each line, and every station that is not the last one, we select exactly one wharf.

- (1h) forces that exactly one wharf is selected in the last station.

- (1i) forces that if a vessel arrives at a last station of a line during time t, it will continue using a wharf there during μ_F. We remark that the right hand side can have maximum one positive value, as the same vessel cannot execute the same line within short temporal distance. Moreover, this equation shows that the variables Z' are fully defined via Z. In other words, one could not define Z', and just replace them by the right hand side of Eq. (1i) in the only other time they appear, namely in Eq. (4) below that describes the capacity constraints.

- (1j)-(1k) ensure that in order to select a task in $\mathcal{B}_+$, i.e., to leverage a full period charging a vessel (without the need to plug or unplug), the previous and next period must also be charging in the same wharf (potentially plugging or unplugging).

- Constraint (2) is equivalent to write

$$\forall (j,t) \in \mathcal{J} \times \mathcal{T}, \mathcal{F}(j,t) \neq \emptyset, \forall v \in \mathcal{V} \left(y_{v,j,t} = 1 \implies \sum_{j' \in \mathcal{F}(j,t)} y_{v,j',t+\mu(j)+\xi(j,j')} = 1 \right).$$

  This forces that after performing a task j, another task j' must immediately follow the first time period possible after executing j and traveling from the j task's ending wharf to the j' task's starting wharf.

- Constraints (3) forces that whenever a vessel v starts a task j at time period t, then the same vessel v cannot perform a following task j' in the time periods that would follow the end of execution of task j, while traveling from j's ending wharf to j''s starting wharf.

4.4.2 Capacity constraints

For every wharf, the capacity constraints need the consider the tasks where the wharf is defined a priori, and the sailings that could choose it in the last station. This is captured in Eq. (4).

$$\sum_{(j,t') \in E(w,t)} \sum_{v \in \mathcal{V}} y_{v,j,t'} \cdot z_{w,j} + \sum_l Z'_{l,w,t} \leq C_w \qquad \forall w \in \mathcal{W}, t \in \mathcal{T}. \tag{4}$$

Note that we are not assigning specific berths, but just imposing that the capacity of the wharf will not be exceeded. However, it is easy to see that this suffices. Indeed, every time a vessel arrives to a wharf, Eq. (4) guarantees that at least one berth will be available. The vessel can use that berth, and stay there as long as it needs to, again thanks to Eq. (4).

4.4.3 Batteries

To ensure that the vessel always has battery, we impose that:

$$Q_{v,t} \geq 0 \qquad \forall v \in \mathcal{V}, t \in \mathcal{T}, \tag{5a}$$

$$Q_{v,t} \leq 1 \qquad \forall v \in \mathcal{V}, t \in \mathcal{T}, \tag{5b}$$

$$Q_{v,t-1} + \sum_{j \in \mathcal{J}} \sum_{t' \in \phi(j,t)} q_{v,j,t-t'} y_{v,j,t'} - r_v \left(1 - \sum_{j \in \mathcal{J}} \sum_{t' \in \phi(j,t)} y_{v,j,t'} \right) \geq Q_{v,t} \qquad \forall v \in \mathcal{V}, t \in \mathcal{T} \tag{5c}$$

The first two equations here just force that $Q_{v,t}$ is always where it has to be. Note that, for safety reasons, one might require vessels to have a certain minimum battery level. This can still be captured by this formulation, just by defining that minimum threshold as the zero level of the batteries.

Eq. (5c) says the $Q_{v,t}$ is updated from its previous state. If the vessel is executing a task, the second term is activated with a change equal to $q_{j,t'-t}$. If the vessel is not executing a task, then it is moving between tasks and therefore it loses r. Note that we use $\geq$, rather than $=$, which is because if the vessel is being charged, it might happen that the battery gets full in the middle of the time period, case in which $Q_{v,t}$ gets stuck at 1 and thus it becomes lower than the left hand side.

4.4.4 Crew pauses

We impose

$$\sum_{t \in T, j \in \mathcal{B}_c} y_{v,j,t} \geq n_c \qquad\qquad \forall v \in \mathcal{V} \qquad (6a)$$

$$\sum_{t'=1}^{T_c-1} \sum_{j \in \mathcal{B}_c} y_{v,j,t+t'} \geq 1 \qquad\qquad \forall v \in \mathcal{V} \qquad (6b)$$

Eq. (6a) guarantees the minimum number of pauses, and Eq. (6b) ensures that the distance between two consecutive pauses is no greater than $T_c - 1$ (which is equivalent to guaranteing that the vessel cannot spend T_c time periods without starting a crew pause). Let us remark that the duration D_c of the crew pause is captured by its duration $\mu(j)$.

4.5 Objective functions

Several objective functions can be studied in this context. In this paper, we consider two: Fleet actually utilised and rebalancing time.

4.5.1 Fleet size

While the fleet is assumed to an exogenous input, one might wonder whether it is possible to use a smaller fleet. To do this, we include a new decision variable $\psi_v \in \{0, 1\}$, representing whether vessel v executed the sailing of any line. Then, the objective function would be:

$$\min \sum_{v \in \mathcal{V}} \psi_v \qquad\qquad (7)$$

And we would need to include the additional set of constraints:

$$\psi_v \geq \frac{1}{M} \sum_{l,t} y_{v,l,t} \qquad\qquad (8)$$

Where M is a large number ensuring that the left hand side in Eq. (8) is always lower than 1. E.g., one can use $M = |\mathcal{T}|$.

4.5.2 Rebalancing time

One proxy for the operational costs is the amount time the vessels are moving, as this relates to both energy costs, and to more frequent maintenance. A vessel can be moving either because of executing a sailing or rebalancing between consecutive tasks that do not coincide in space. The former is a fixed cost, as the number of sailings is exogenous. Therefore, here we minimise the latter, i.e., the total rebalancing time spent by each vessel.

Let us remark that this quantity has already been computed in Eq. (5c), where we multiply by r a quantity representing whether a vessel v is being rebalanced during period t. So the objective function here results from adding that expression for every v and t, i.e.

$$\min \sum_{v \in \mathcal{V}, t \in mT} \left(1 - \sum_{j \in \mathcal{J}} \sum_{t' \in \phi(j,t)} y_{v,j,t'} \right) \qquad\qquad (9)$$

Equivalently, one can maximise the time that a vessel is not being rebalanced, thus

$$\max \sum_{v \in \mathcal{V}, t \in mT} \sum_{j \in \mathcal{J}} \sum_{t' \in \phi(j,t)} y_{v,j,t'} \qquad (10)$$

4.6 Re-building the timetable, itineraries, and berths utilisation

4.6.1 Timetables

The timetables are produced directly by observing the variables $x_{l,d}$ and $z_{w,l}$. Indeed, for every line l we have exactly one $x_{l,d} = 1$ (Eq. 1a), which will indicate the first departure of the line. This first departure, in turn, fully characterises at which time will the line depart from every station. Finally, for those stations that have more than one available wharf, there will be exactly one $z_{w,l} = 1$ (Eq. 1g), indicating which wharf will be used.

4.6.2 Vessels' itinerary

The itinerary of every vessel is mostly given by the variables $y_{v,j,t}$. These variables indicate the vessels exactly which task to start (if any) at any given time. Two tasks j_1, j_2 will be executed consecutively if there are times t_1, t_2, with $t_1 < t_2$ so that $y_{v,j_1,t_1} = y_{v,j_2,t_2} = 1$ and $y_{v,j',t'} = 0 \forall j \in \mathcal{J}, \forall t' \in \{t_1 + 1, \ldots, t_2 - 1\}$. Thanks to our constraints, we know that:

- There is one task correctly assigned to begin the day with (Eq. 1e).

- If there is a need to rebalance the vessel between two consecutive tasks, the right time will be allocated for this (Eq. 2).

- The vessel will not execute two tasks at the same time, nor will it execute a task while rebalancing (Eqs. 1f,3).

- The vessel will always have battery (Eq. 5a).

- Crew pauses will be programmed according to the rules (Eqs. 6a,6b).

- There is one vessel executing every sailing and in the right time (Eq. 1b).

- The vessel has tasks till the end of the day (Eq. 2).

Note that, when executing a line l, the variables $y_{v,l,t}$ do not indicate which wharves to use. As described in section 4.6.1, for stations that are not the last one, this is given by the variables $z_{w,j}$. For the last station, Eq. (1h) ensures that there is exactly one wharf selected per sailing. Finally, which berth to use is decided as explained in section 4.4.2.

4.6.3 Wharves' utilisation

The list of vessels using wharf w at a given t is given directly by considering:

- Every v, j, t' such that $y_{v,j,t'} = 1, z_{w,j} = 1$, and $t - t' \in \Delta(j, w)$. These include all the vessels waiting, charging, doing a crew pause, or executing a line if w is not in its last station.

- Every line l such that $Z'_{l,w,t} = 1$. In this case, thanks to Eqs. (1h)-(1i), we know that there is exactly one vessel v using wharf w as the last stop of line l, given by the right hand side of Eq. (1h).

The capacity constraints ensure that the sum of the two bullet points above does not violate the wharf's capacity.

Acknowledgments

EV Maritime data on electric charging info.