Analysis of Trusses

Basic formulation and examples

Ricardo A. Fernandes
ricardoaf@lccv.ufal.br

Advisor: Adeildo S. Ramos Jr.

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References

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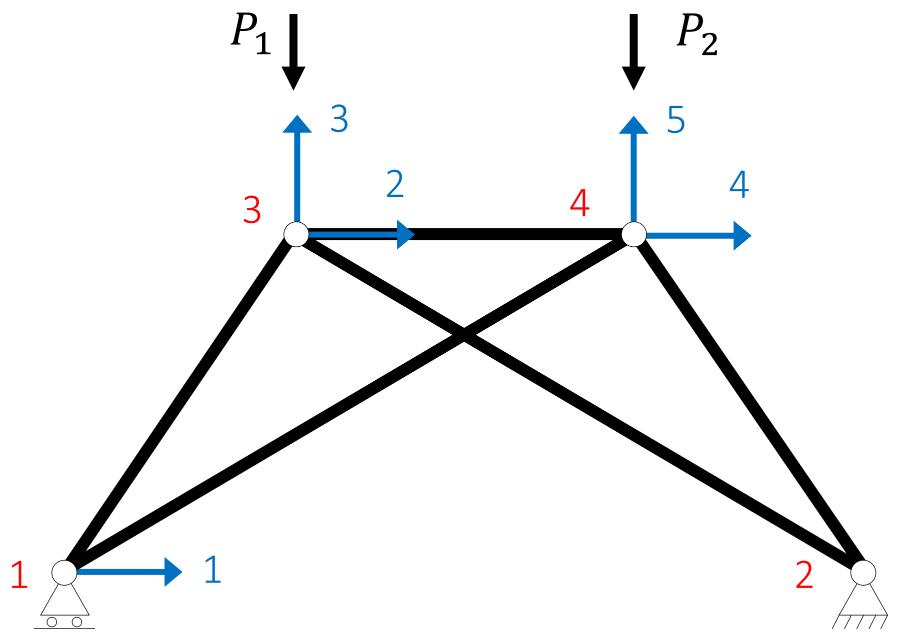
Coordinate Systems

Given the following isostatic truss:

- 4 nodes, 5 bars and 5 free DOFs

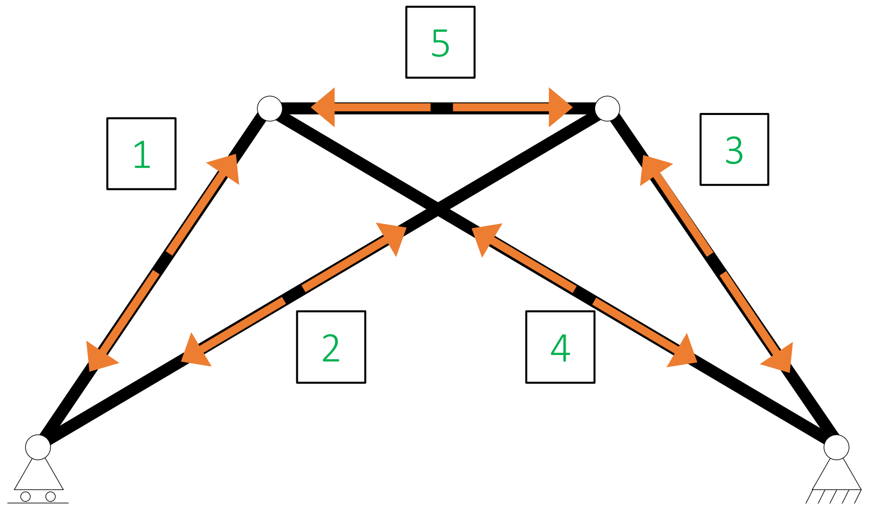


Global coordinate system

- Displacements: $\boldsymbol{u} = [u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5]^T$
- External forces: $\boldsymbol{f} = [0 \quad 0 \quad -P_1 \quad 0 \quad -P_2]^T$

Internal coordinate system

- Elongations: $\boldsymbol{\delta} = [\delta_1 \quad \delta_2 \quad \delta_3 \quad \delta_4 \quad \delta_5]^T$
- Internal forces: $\boldsymbol{n} = [n_1 \quad n_2 \quad n_3 \quad n_4 \quad n_5]^T$



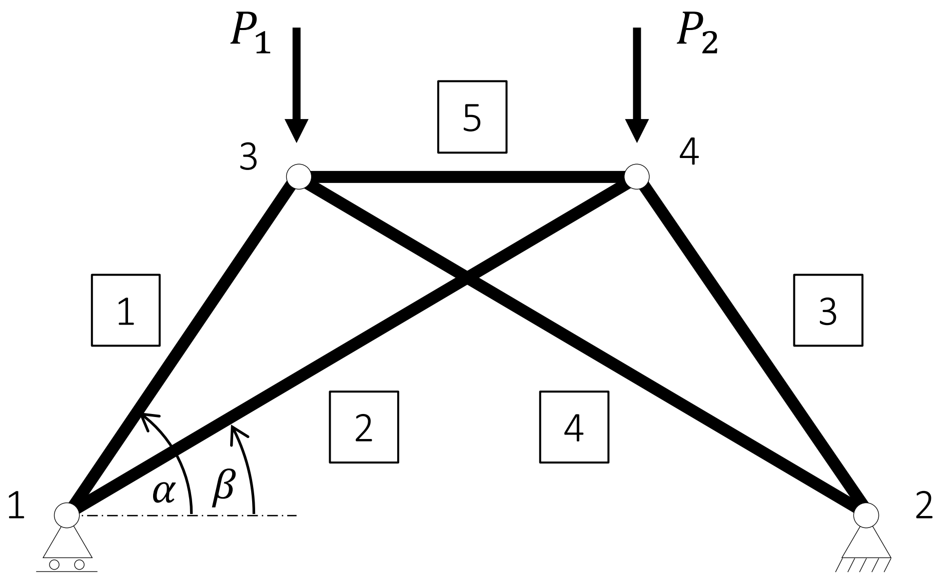
Global/Internal transformations

$\boldsymbol{B}^T\ \boldsymbol{n} = \boldsymbol{f}$ — $\boldsymbol{B}^T$ is the force equilibrium matrix

$\boldsymbol{\delta} = \boldsymbol{B}\ \boldsymbol{u}$ — $\boldsymbol{B}$ is the incidence matrix

Force Equilibrium Matrix

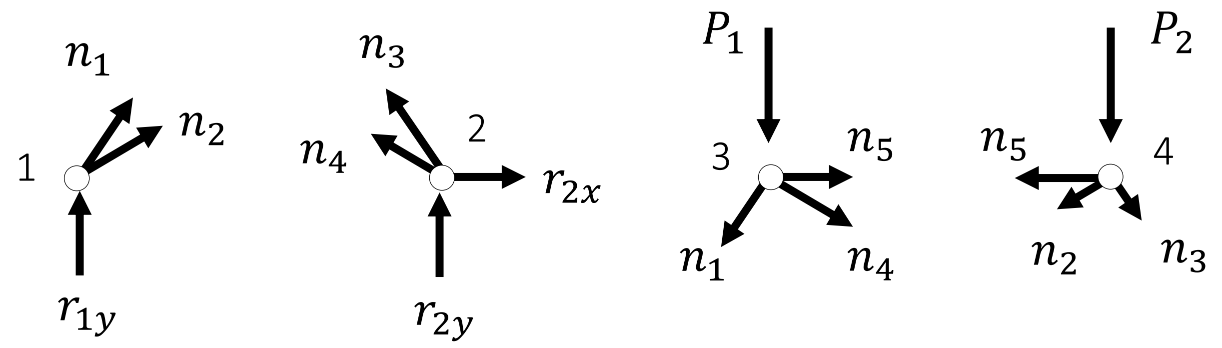
Example



$$c_\alpha = \cos\alpha \qquad c_\beta = \cos\beta$$

$$s_\alpha = \sin\alpha \qquad s_\beta = \sin\beta$$

Performing nodal equilibrium



$$\begin{bmatrix} -c_\alpha & -c_\beta & 0 & 0 & 0 & 0 & 0 & 0 \\ -s_\alpha & -s_\beta & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & c_\alpha & c_\beta & 0 & 0 & -1 & 0 \\ 0 & 0 & -s_\alpha & -s_\beta & 0 & 0 & 0 & -1 \\ c_\alpha & 0 & 0 & -c_\beta & -1 & 0 & 0 & 0 \\ s_\alpha & 0 & 0 & s_\beta & 0 & 0 & 0 & 0 \\ 0 & c_\beta & -c_\alpha & 0 & 1 & 0 & 0 & 0 \\ 0 & s_\beta & s_\alpha & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ r_{1y} \\ r_{2x} \\ r_{2y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -P_1 \\ 0 \\ -P_2 \end{bmatrix}$$

Matrix is <u>square</u> and <u>invertible</u>: **system is statically determinate**

Force Equilibrium Matrix

Example

Performing nodal equilibrium

- Each row (equation) is associated with a DOF
- Each column is associated with an unknown

$$\begin{bmatrix} \boldsymbol{B} & \boldsymbol{B}_{rn} \\ \boldsymbol{B}_{rr} & \boldsymbol{B}_{rr} \end{bmatrix}^T \begin{bmatrix} \boldsymbol{n} \\ \boldsymbol{r} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{bmatrix}$$

$$\begin{bmatrix} -c_\alpha & -c_\beta & 0 & 0 & 0 & 0 & 0 & 0 \\ -s_\alpha & -s_\beta & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & c_\alpha & c_\beta & 0 & 0 & -1 & 0 \\ 0 & 0 & -s_\alpha & -s_\beta & 0 & 0 & 0 & -1 \\ c_\alpha & 0 & 0 & -c_\beta & -1 & 0 & 0 & 0 \\ s_\alpha & 0 & 0 & s_\beta & 0 & 0 & 0 & 0 \\ 0 & c_\beta & -c_\alpha & 0 & 1 & 0 & 0 & 0 \\ 0 & s_\beta & s_\alpha & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ r_{1y} \\ r_{2x} \\ r_{2y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -P_1 \\ 0 \\ -P_2 \end{bmatrix}$$

Solving for internal forces

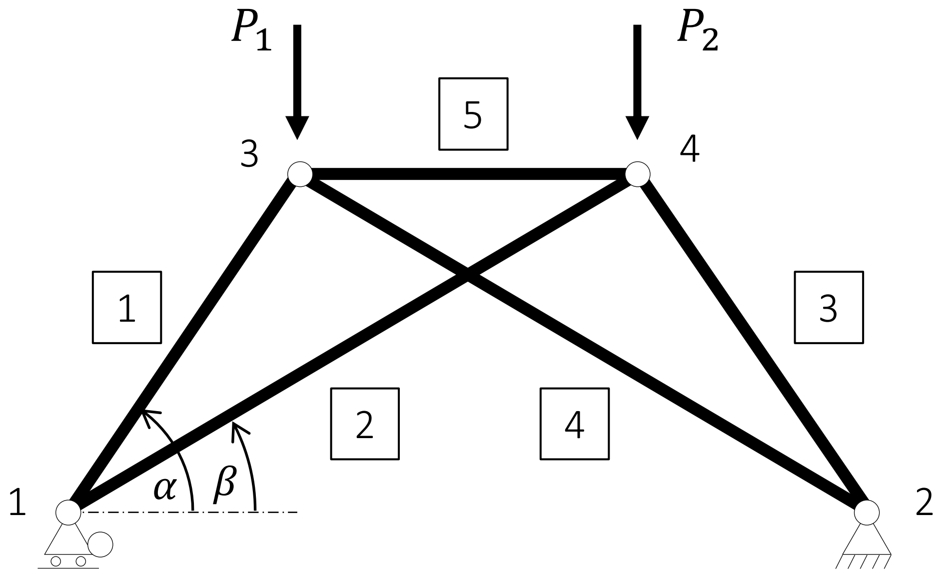
- Keep only the free DOFs

$$\boxed{\boldsymbol{B}^T \boldsymbol{n} = \boldsymbol{f}}$$

$$\begin{bmatrix} -c_\alpha & -c_\beta & 0 & 0 & 0 \\ c_\alpha & 0 & 0 & -c_\beta & -1 \\ s_\alpha & 0 & 0 & s_\beta & 0 \\ 0 & c_\beta & -c_\alpha & 0 & 1 \\ 0 & s_\beta & s_\alpha & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -P_1 \\ 0 \\ -P_2 \end{bmatrix}$$

Force Equilibrium Matrix

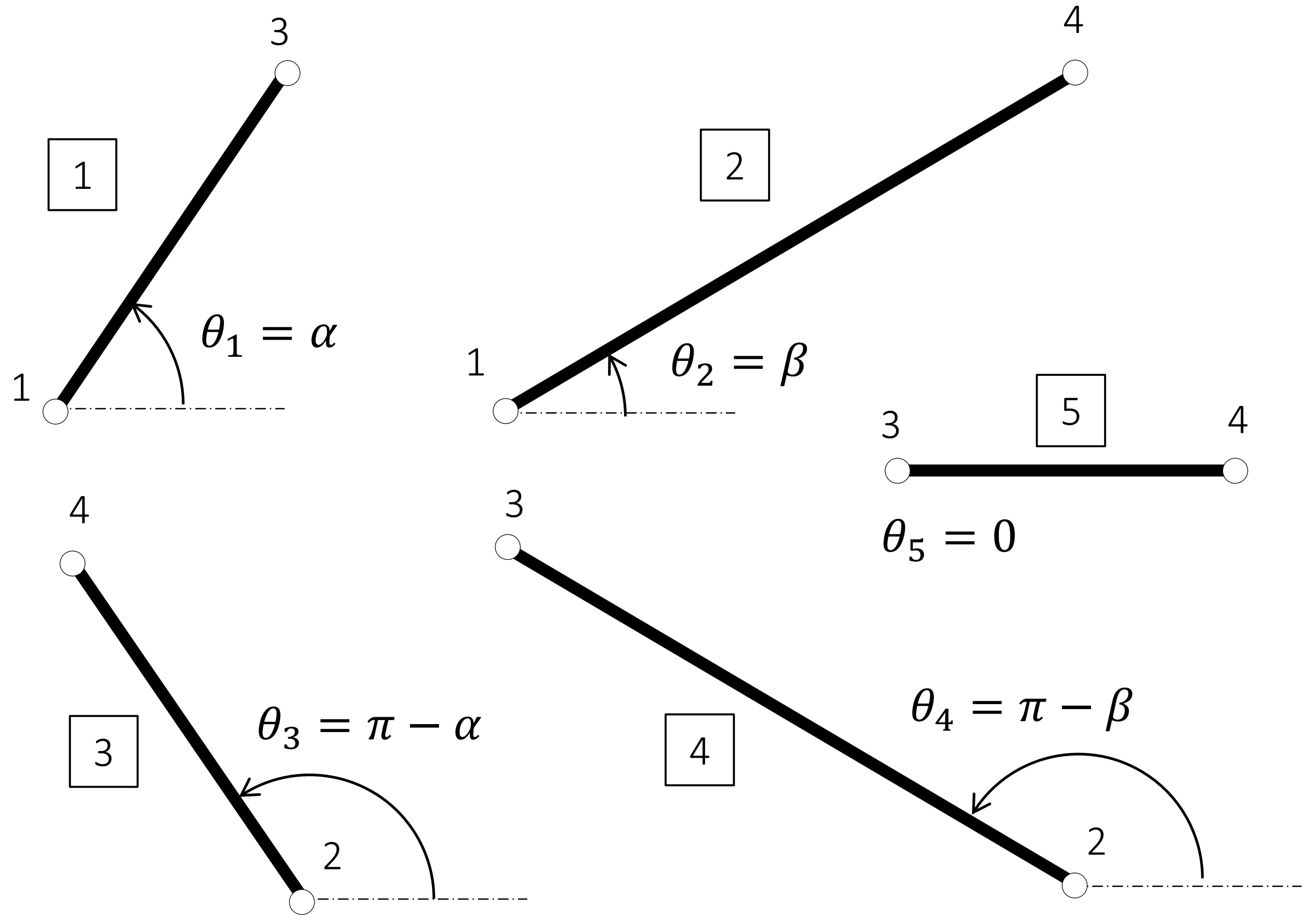
Example



Given connectivity of the bars

$$\mathbf{BAR} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \\ 2 & 4 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$$

We can define directional cosines for each bar



Force Equilibrium Matrix

Example

Calculating directional unit vectors

For each bar

$$\theta_1 = \alpha \Rightarrow \hat{d}_1 = [c_\alpha \quad s_\alpha]^T$$

$$\theta_2 = \beta \Rightarrow \hat{d}_2 = [c_\beta \quad s_\beta]^T$$

$$\theta_3 = \pi - \alpha \Rightarrow \hat{d}_3 = [-c_\alpha \quad s_\alpha]^T$$

$$\theta_4 = \pi - \beta \Rightarrow \hat{d}_4 = [-c_\beta \quad s_\beta]^T$$

$$\theta_5 = 0 \Rightarrow \hat{d}_5 = [1 \quad 0]^T$$

$$\boxed{\boldsymbol{q}_e = \begin{bmatrix} -\hat{\boldsymbol{d}}_e \\ \hat{\boldsymbol{d}}_e \end{bmatrix}}$$

A vector $\boldsymbol{q}_e$ for each element e can be used to assemble $\boldsymbol{B}^T$

For example, let $e = 2$

$$\mathbf{BAR} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \\ 2 & 4 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \Rightarrow \mathbf{DOF} = \begin{bmatrix} 1 & 2 & 5 & 6 \\ 1 & 2 & 7 & 8 \\ 3 & 4 & 7 & 8 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 \end{bmatrix} \qquad \boldsymbol{q}_{e=2} = \begin{bmatrix} -c_\beta \\ -s_\beta \\ c_\beta \\ s_\beta \end{bmatrix}$$

Bar

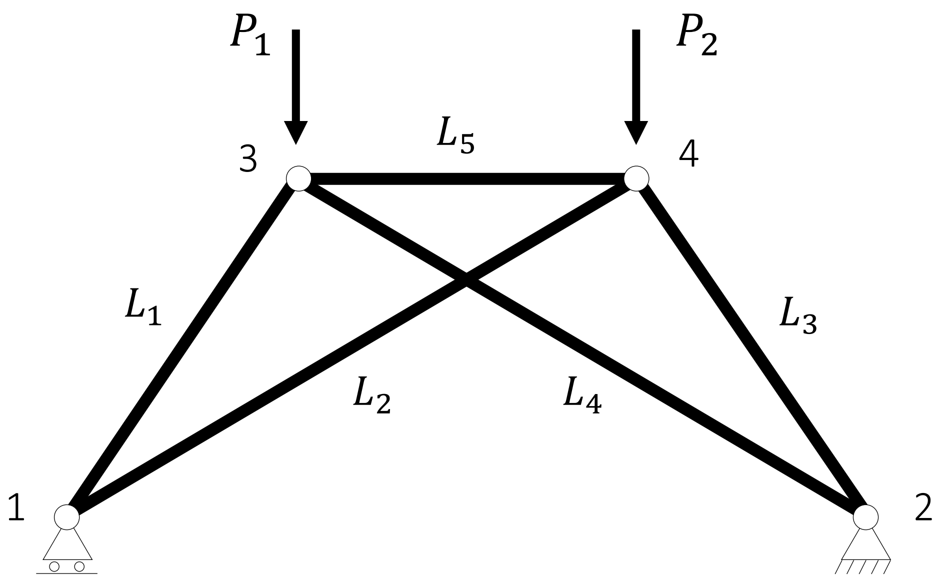
$$\begin{bmatrix} \boldsymbol{B} & \boldsymbol{B}_{rn} \\ \boldsymbol{B}_{rr} & \boldsymbol{B}_{rr} \end{bmatrix}^T = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & & & \end{array} \\ \begin{bmatrix} -c_\alpha & -c_\beta & 0 & 0 & 0 & 0 & 0 & 0 \\ -s_\alpha & -s_\beta & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & c_\alpha & c_\beta & 0 & 0 & -1 & 0 \\ 0 & 0 & -s_\alpha & -s_\beta & 0 & 0 & 0 & -1 \\ c_\alpha & 0 & 0 & -c_\beta & -1 & 0 & 0 & 0 \\ s_\alpha & 0 & 0 & s_\beta & 0 & 0 & 0 & 0 \\ 0 & c_\beta & -c_\alpha & 0 & 1 & 0 & 0 & 0 \\ 0 & s_\beta & s_\alpha & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \text{DOF} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

Automatic Assembly of B^T

Example using Julia

Given

- List of nodal coordinates
- List of supported DOFs
- List of bar connectivity



$$\mathbf{NODE} = \begin{bmatrix} 0 & 0 \\ 3 & 0 \\ 1 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\mathbf{SUPP} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{BAR} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \\ 2 & 4 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$$

Automatic Assembly of BT

In [1]:
```
using LinearAlgebra
```

Given nodal coords and bar connectivity

In [2]:
```
# Nodal coords: [xᵢ, yᵢ] = NODE[i,:]
NODE = [0 0; 3 0; 1 2; 2 2]

# Support DOF: 0 for free, 1 for fix
SUPP = [0 1; 1 1; 0 0; 0 0]

# Bar connectivity: barₑ have nodes ELEM[e,:]
BAR = [1 3; 1 4; 2 4; 2 3; 3 4];
```

Pre process data

In [3]:
```
# Get number of DOFs
n_dof = 2 * size(NODE, 1)
println("Number of DOFs : ", n_dof)

# Define DOFs of each bar
DOF = [2BAR[:, 1].-1 2BAR[:, 1] 2BAR[:, 2].-1 2BAR[:, 2]]
println("DOFs of bars : ", DOF)

# Define free DOFs
free = setdiff(1:n_dof, findall(SUPP'[:] .== 1))
println("Free DOFs : ", free)
```

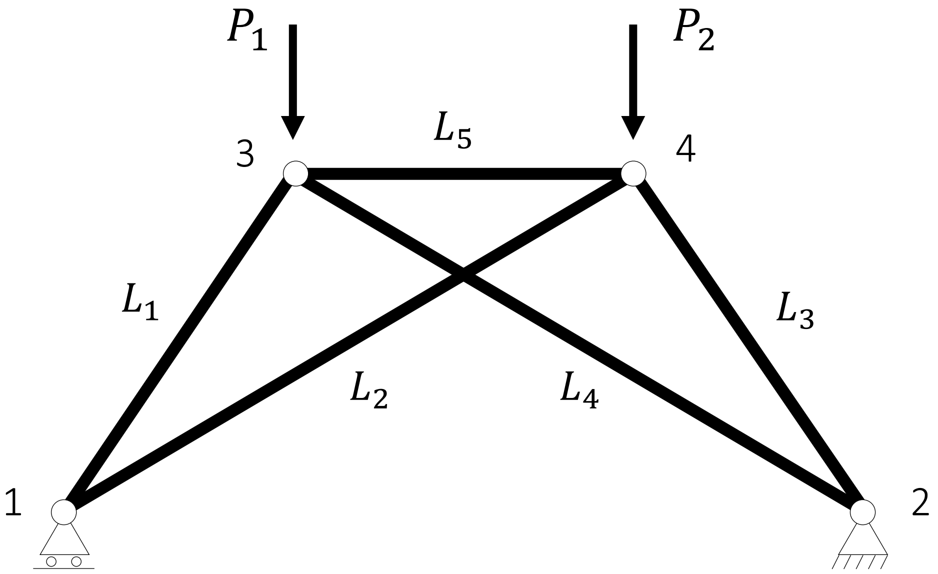
```
Number of DOFs : 8
DOFs of bars : [1 2 5 6; 1 2 7 8; 3 4 7 8; 3 4 5 6; 5 6 7 8]
Free DOFs : [1, 5, 6, 7, 8]
```

Automatic Assembly of B^T

Example using Julia

Given

- List of nodal coordinates
- List of supported DOFs
- List of bar connectivity



$$\mathbf{NODE} = \begin{bmatrix} 0 & 0 \\ 3 & 0 \\ 1 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\mathbf{SUPP} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{BAR} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \\ 2 & 4 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$$

Calculate length of bars

In [4]:
```
# Define directional vectors of each bar
d(e) = NODE[BAR[e, 2], :] - NODE[BAR[e, 1], :]

# Calculate and show length of bars
n_bar = size(BAR, 1);
L = [norm(d(e)) for e in 1:n_bar]
[println("L$e : $(L[e])") for e in 1:n_bar];
```

```
L1 : 2.23606797749979
L2 : 2.8284271247461903
L3 : 2.23606797749979
L4 : 2.8284271247461903
L5 : 1.0
```

Calculate directional unit vectors of bars

In [5]:
```
# Calculate and show directional unit vector of each bar
d̲ = [d(e) / L[e] for e in 1:n_bar]
[println("d̲$e : $(d̲[e])") for e in 1:n_bar];
```

```
d̲1 : [0.4472135954999579, 0.8944271909999159]
d̲2 : [0.7071067811865475, 0.7071067811865475]
d̲3 : [-0.4472135954999579, 0.8944271909999159]
d̲4 : [-0.7071067811865475, 0.7071067811865475]
d̲5 : [1.0, 0.0]
```

Assembly Force Equilibrium Matrix, BT

In [6]:
```
# Init and fill BT matrix
BT = zeros(n_dof, n_bar)
for e in 1:n_bar
    BT[DOF[e, :], e] = [-d̲[e]; d̲[e]]
end

# Remove fixed DOFs
BT = BT[free, :]
```

Out[6]:
```
5×5 Array{Float64,2}:
 -0.447214  -0.707107   0.0        0.0        0.0
  0.447214   0.0        0.0       -0.707107  -1.0
  0.894427   0.0        0.0        0.707107  -0.0
  0.0        0.707107  -0.447214   0.0        1.0
  0.0        0.707107   0.894427   0.0        0.0
```

Statical Determinacy

Given the force equilibrium relationship

$$\underset{g\,\mathrm{x}\,b}{\boldsymbol{B}^T}\ \underset{b\,\mathrm{x}\,1}{\boldsymbol{n}} = \underset{g\,\mathrm{x}\,1}{\boldsymbol{f}}$$

g: Number of free DOFs

b: Number of bars

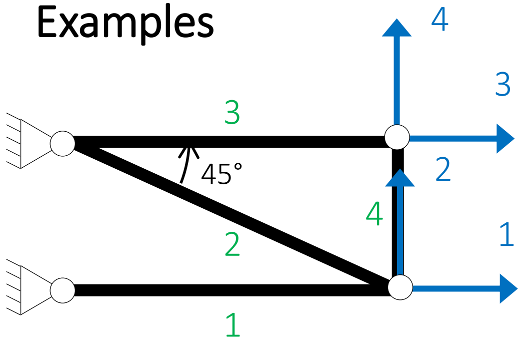
We can also evaluate matrix $\boldsymbol{B}^T$ rank, r

- Number of linearly independent column vectors

Case	Order	Rank	Structural type
1	$g = b$	$r = g = b$	Isostatic
2	$g < b$	$r = g$	Hyperstatic
3	$g \leq b$	$r < g$	Mechanism or partly constrained
4	$g > b$	$r \leq b$	

Examples

Case 1

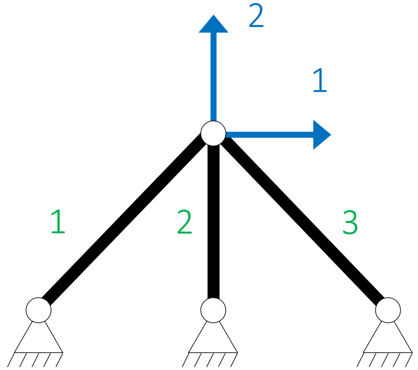


$g = b = 4$

$$\boldsymbol{B}^T = \begin{bmatrix} 1 & \sqrt{2}/2 & 0 & 0 \\ 0 & -\sqrt{2}/2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$r = 4$

Case 2



$g = 2 \qquad b = 3$

$$\boldsymbol{B}^T = \begin{bmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ -\sqrt{2}/2 & -1 & -\sqrt{2}/2 \end{bmatrix}$$

$r = 2$

Statical Determinacy

Given the force equilibrium relationship

$$\underset{g \times b}{\boldsymbol{B}^T} \; \underset{b \times 1}{\boldsymbol{n}} = \underset{g \times 1}{\boldsymbol{f}}$$

g: Number of free DOFs

b: Number of bars

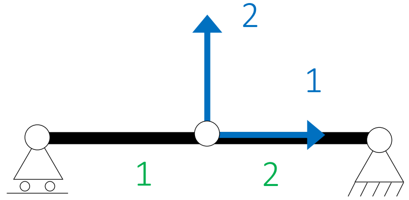
We can also evaluate matrix $\boldsymbol{B}^T$ rank, r

- Number of linearly independent column vectors

Case	Order	Rank	Structural type
1	$g = b$	$r = g = b$	Isostatic
2	$g < b$	$r = g$	Hyperstatic
3	$g \leq b$	$r < g$	Mechanism or partly constrained
4	$g > b$	$r \leq b$	Mechanism or partly constrained

Examples

Case 3

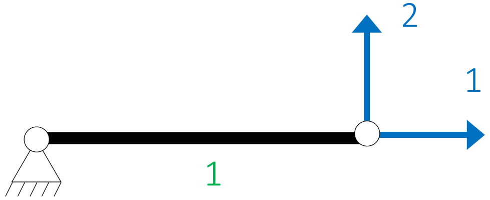


$g = b = 2$

$$\boldsymbol{B}^T = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$r = 1$

Case 4



$g = 2 \qquad b = 1$

$$\boldsymbol{B}^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$r = 1$

Stiffness Matrix

Strain-stress relationship

- Internal coordinate system
- Linear elastic material
- Young's modulus E_e

$$\boxed{\sigma_e = E_e \, \varepsilon_e}$$

Hooke Law

- Force-displacement relationship
- Bar stiffness k_e

$$\frac{n_e}{A_e} = E_e \frac{\delta_e}{L_e} \Rightarrow \quad n_e = \left[\frac{E_e A_e}{L_e}\right] \delta_e$$

$$\boxed{n_e = k_e \, \delta_e}$$

Generalizing for all bars

- $\boldsymbol{C}$ is a diagonal matrix with bar stiffness (internal stiffness matrix)

$$\boxed{\boldsymbol{n} = \boldsymbol{C} \, \boldsymbol{\delta}} \qquad \boldsymbol{C} = \begin{bmatrix} k_1 & 0 & \dots & 0 \\ 0 & k_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & k_{\mathrm{nbars}} \end{bmatrix}$$

Merging all equations

- $\boldsymbol{K}$ is the global stiffness matrix

$$\boldsymbol{f} = \boldsymbol{B}^T \boldsymbol{n} = \boldsymbol{B}^T (\boldsymbol{C} \, \boldsymbol{\delta}) = \boldsymbol{B}^T \boldsymbol{C} \, (\boldsymbol{B} \, \boldsymbol{u}) = \left(\boldsymbol{B}^T \, \boldsymbol{C} \, \boldsymbol{B}\right) \boldsymbol{u}$$

$$\boxed{\boldsymbol{f} = \boldsymbol{K} \, \boldsymbol{u}}$$

Flexibility Matrix

Internal Flexibility Matrix, $\boldsymbol{C}^{-1}$

- Since $\boldsymbol{C}$ is the internal stiffness matrix

$$\boxed{\boldsymbol{n} = \boldsymbol{C}\,\boldsymbol{\delta}} \qquad \boldsymbol{C} = \begin{bmatrix} k_1 & 0 & \dots & 0 \\ 0 & k_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & k_{\text{nbars}} \end{bmatrix}$$

- Multiplying both sides of equation by $\boldsymbol{C}^{-1}$

$$\boxed{\boldsymbol{\delta} = \boldsymbol{C}^{-1}\,\boldsymbol{n}} \qquad \boldsymbol{C}^{-1} = \begin{bmatrix} \frac{1}{k_1} & 0 & \dots & 0 \\ 0 & \frac{1}{k_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{k_{\text{nbars}}} \end{bmatrix}$$

Global Flexibility Matrix, $\boldsymbol{F}$

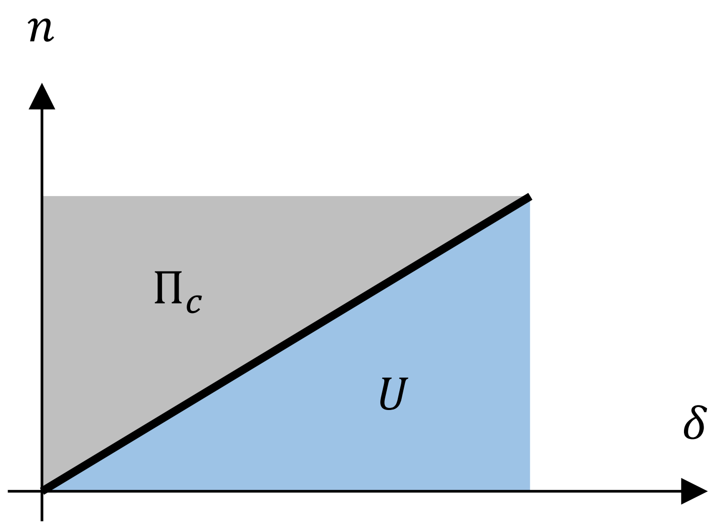
- The inverse of global stiffness matrix, $\boldsymbol{F} = \boldsymbol{K}^{-1}$

Remarks

- In some cases, structures may not be sufficiently constrained:
  - Global stiffness may not be invertible
    - Global flexibility matrix cannot be determined
  - Global flexibility matrix may not not invertible
    - Global stiffness matrix cannot be determined
  - Even then, internal coordinate system is consistent
    - Internal stiffness and flexibility matrices can be determined!

Energetic Analysis

Energetic Analysis



	Internal	Global
▪ Strain energy	$U = \frac{1}{2}\boldsymbol{\delta}^T \boldsymbol{C}\, \boldsymbol{\delta}$	$U = \frac{1}{2}\, \boldsymbol{u}^T\, \boldsymbol{K}\, \boldsymbol{u}$
▪ Complementary energy	$\Pi_c = \frac{1}{2}\boldsymbol{n}^T \boldsymbol{C}^{-1}\, \boldsymbol{n}$	$\Pi_c = \frac{1}{2}\, \boldsymbol{f}^T\, \boldsymbol{F}\, \boldsymbol{f}$
▪ Total Potential energy		$\Pi = U - \boldsymbol{f}^T \boldsymbol{u}$

- For linear elastic materials

$$\Pi_c = \frac{1}{2}\, \boldsymbol{f}^T\, \boldsymbol{K}^{-1}\, \boldsymbol{f} = \frac{1}{2}\, \boldsymbol{f}^T \boldsymbol{u} = \frac{1}{2}\, (\boldsymbol{K}\boldsymbol{u})^T \boldsymbol{u} = \frac{1}{2} \boldsymbol{u}^T \boldsymbol{K}\boldsymbol{u} = U$$

Questions? Comments?

This presentation
https://github.com/ricardoaf/conopt

New to Julia?
https://github.com/ricardoaf/juliafirststeps