Universidade Federal de Alagoas - UFAL
Centro de Tecnologia - CTEC
Curso de Engenharia Civil

Mecânica dos Sólidos 3 - ECIV051D (2020.2)

Exercícios:
Deslocamentos em vigas isostáticas usando a equação diferencial da elástica
Encontro Assíncrono
(Versão atualizada 23/07/2021)

Monitores:
Hugo Vinícius F. Azevedo
Milton Mateus G. Santos
Ricardo A. Fernandes

Professor/Supervisor:
Adeildo S. Ramos Jr.

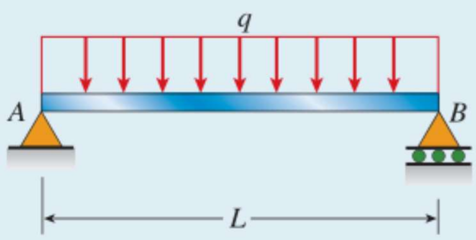
Maceió/AL, 16/07/2021

Exemplo 9.1

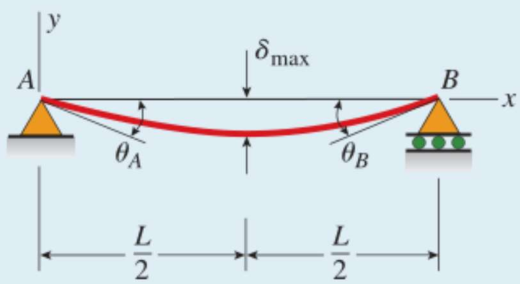
Referência: Gere & Goodno (2013). Mechanics of Materials, 8th Edition. Cengage Learning.

Determine the equation of the deflection curve for a simple beam *AB* supporting a uniform load of intensity *q* acting throughout the span of the beam (Fig. 9-8a).

Also, determine the maximum deflection δ_{max} at the midpoint of the beam and the angles of rotation θ_A and θ_B at the supports (Fig. 9-8b). (*Note:* The beam has length *L* and constant flexural rigidity *EI*.)



(a)



(b)

$restart:$

Reações de apoio

$$assign\left(solve\left(\left\{R_A + R_B - q\cdot L = 0,\ R_B\cdot L - (q\cdot L)\cdot\frac{L}{2} = 0\right\},\ \{R_A, R_B\}\right)\right):$$

$$R_A, R_B = \frac{q\,L}{2}, \frac{q\,L}{2}$$

Momento fletor

$$M := unapply\left(solve\left(M - R_A\cdot x + (q\cdot x)\cdot\frac{x}{2} = 0, M\right), x\right):$$

$$M(x) = \frac{1}{2}\,q\,L\,x - \frac{1}{2}\,q\,x^2$$

Equação da curva de deflexão

$$diff\,(v(x), x\$2) = \frac{d^2}{dx^2}\,v(x)$$

$$v := unapply\left(rhs\left(dsolve\left(\left\{diff\,(v(x), x\$2) = \frac{M(x)}{EI},\ v(0) = 0,\ v(L) = 0\right\}, v(x)\right)\right), x\right):$$

$$v(x) = \frac{q\,L\,x^3}{12\,EI} - \frac{q\,x^4}{24\,EI} - \frac{q\,L^3\,x}{24\,EI}$$

Deflexão máxima

$$solve\left(\mathrm{D}[1](v)\left(x_{\max}\right) = 0, x_{\max}\right) = \frac{L}{2}, \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) L, \left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) L$$

$$evalf(\%) = 0.5000000000\,L, 1.366025404\,L, -0.3660254040\,L$$

$$\delta_{\max} := -v\left(\frac{L}{2}\right) = \frac{5}{384}\,\frac{q\,L^4}{EI}$$

Ângulo de rotação nas extremidades

$$\theta_A := -\mathrm{D}[1](v)(0) = \frac{1}{24}\,\frac{q\,L^3}{EI}$$

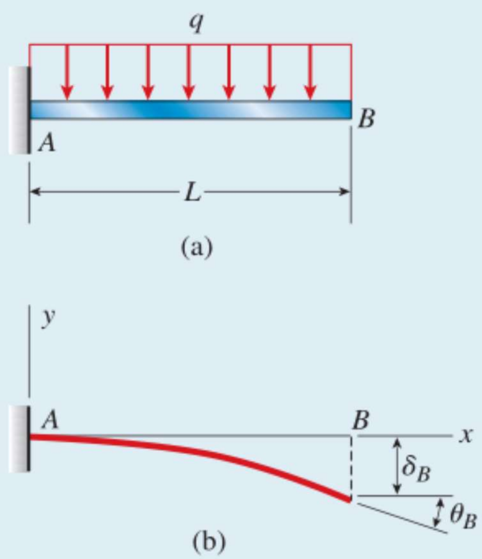
$$\theta_B := \mathrm{D}[1](v)(L) = \frac{1}{24}\,\frac{q\,L^3}{EI}$$

Exemplo 9.2

Referência: Gere & Goodno (2013). Mechanics of Materials, 8th Edition. Cengage Learning.

Determine the equation of the deflection curve for a cantilever beam *AB* subjected to a uniform load of intensity *q* (Fig. 9-10a).

Also, determine the angle of rotation θ_B and the deflection δ_B at the free end (Fig. 9-10b). (*Note:* The beam has length *L* and constant flexural rigidity *EI*.)



restart :

Reações de apoio

$$assign\left(solve\left(\left\{R_A - q\cdot L = 0, M_A - (q\cdot L)\cdot\frac{L}{2} = 0\right\}, \{R_A, M_A\}\right)\right):$$

$$R_A, M_A = q\,L, \frac{q\,L^2}{2}$$

Momento fletor

$$M := unapply\left(solve\left(M + M_A - R_A\cdot x + (q\cdot x)\cdot\frac{x}{2} = 0, M\right), x\right):$$

$$M(x) = -\frac{1}{2}\,q\,L^2 + q\,L\,x - \frac{1}{2}\,q\,x^2$$

Equação da curva de deflexão

$$v := unapply\left(rhs\left(dsolve\left(\left\{diff(v(x), x\$2) = \frac{M(x)}{EI}, v(0) = 0, \mathrm{D}[1](v)(0) = 0\right\}, v(x)\right)\right), x\right):$$

$$v(x) = -\frac{q\,(L-x)^4}{24\,EI} - \frac{q\,L^3\,x}{6\,EI} + \frac{q\,L^4}{24\,EI}$$

Ângulo de rotação e deflexão na extermidade livre

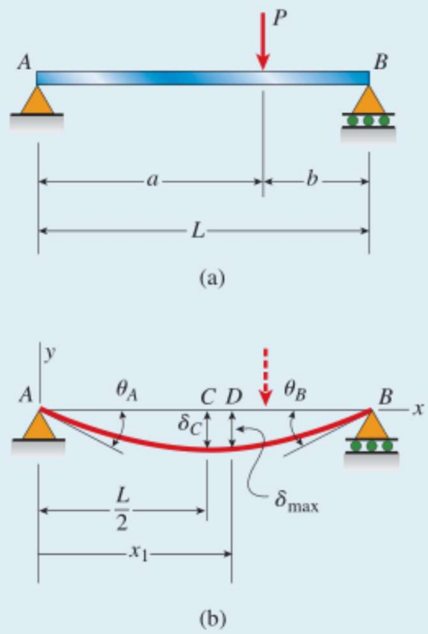
$$\theta_B := -D[1](v)(L) = \frac{1}{6}\ \frac{q\,L^3}{EI}$$

$$\delta_B := -v(L) = \frac{1}{8}\ \frac{q\,L^4}{EI}$$

Exemplo 9.3
Referência: Gere & Goodno (2013). Mechanics of Materials, 8th Edition. Cengage Learning.

A simple beam *AB* supports a concentrated load *P* acting at distances *a* and *b* from the left-hand and right-hand supports, respectively (Fig. 9-12a).

Determine the equations of the deflection curve, the angles of rotation θ_A and θ_B at the supports, the maximum deflection δ_{max}, and the deflection δ_C at the midpoint *C* of the beam (Fig. 9-12b). (*Note:* The beam has length *L* and constant flexural rigidity *EI*.)



$restart:$
$L := a + b:$

Reações de apoio
$assign(solve(\{R_A + R_B - P = 0,\ -P \cdot a + R_B \cdot L = 0\},\ \{R_A, R_B\})):$

$$R_A, R_B = \frac{P\,b}{a+b}, \frac{P\,a}{a+b}$$

Momento fletor
$M_{AP} := unapply(solve(M - R_A \cdot x = 0, M), x):$
$M_{PB} := unapply(solve(-M + R_B \cdot (L - x) = 0, M), x):$

$$M_{AP}(x), M_{PB}(x) = \frac{P\,b\,x}{a+b}, \frac{P\,a\,(a+b-x)}{a+b}$$

Equação da curva de deflexão
$bc := v_{AP}(0) = 0,\ v_{PB}(L) = 0,\ v_{AP}(a) = v_{PB}(a),\ \mathrm{D}[1](v_{AP})(a) = \mathrm{D}[1](v_{PB})(a):$

$$res := dsolve\left(\left\{diff(v_{AP}(x), x\$2) = \frac{M_{AP}(x)}{EI},\ diff(v_{PB}(x), x\$2) = \frac{M_{PB}(x)}{EI},\ bc\right\},\ \{v_{AP}(x), v_{PB}(x)\}\right):$$

$v_{AP} := unapply(simplify(rhs(res[1])), x):$
$v_{PB} := unapply(simplify(rhs(res[2])), x):$

$$v_{AP}(x) = -\frac{P\,b\,x\,(a^2+2\,a\,b-x^2)}{6\,(a+b)\,EI}$$

$$v_{PB}(x) = \frac{(a^2-2\,a\,x-2\,x\,b+x^2)\,(a+b-x)\,P\,a}{6\,(a+b)\,EI}$$

$$simplify\left(v_{AP}(x) - \left(-\frac{P\cdot b\cdot x}{6\cdot L\cdot EI}\cdot(L^2-b^2-x^2)\right)\right) = 0$$

$$simplify\left(v_{PB}(x) - \left(-\frac{P\cdot b\cdot x}{6\cdot L\cdot EI}\cdot(L^2-b^2-x^2) - \frac{P\cdot(x-a)^3}{6\cdot EI}\right)\right) = 0$$

Ângulos de rotação nos suportes

$$\theta_A := -\mathrm{D}[1](v_{AP})(0) = \frac{1}{6}\,\frac{P\,b\,(a^2+2\,a\,b)}{(a+b)\,EI}$$

$$\theta_B := simplify(\mathrm{D}[1](v_{PB})(L)) = \frac{1}{6}\,\frac{b\,(2\,a+b)\,P\,a}{(a+b)\,EI}$$

$$simplify\left(\theta_A - \left(\frac{P\cdot a\cdot b\cdot(L+b)}{6\cdot L\cdot EI}\right)\right) = 0$$

$$simplify\left(\theta_B - \left(\frac{P\cdot a\cdot b\cdot(L+a)}{6\cdot L\cdot EI}\right)\right) = 0$$

Deflexão máxima

Considerando $a \geq b$

$$solve(\mathrm{D}[1](v_{AP})(x_{\max}) = 0, x_{\max}) = \frac{\sqrt{3\,a^2+6\,a\,b}}{3}, -\frac{\sqrt{3\,a^2+6\,a\,b}}{3}$$

$$x_{\max} := \%[1] = \frac{1}{3}\sqrt{3\,a^2+6\,a\,b}$$

$$\delta_{\max} := simplify(-v_{AP}(x_{\max})) = \frac{1}{27}\,\frac{P\,b\sqrt{3}\,\sqrt{a\,(a+2\,b)}\,a\,(a+2\,b)}{(a+b)\,EI}$$

$$simplify\left(\delta_{\max} - \left(\frac{P\cdot b\cdot(L^2-b^2)^{\frac{3}{2}}}{9\cdot\sqrt{3}\cdot L\cdot EI}\right)\right) = 0$$

Deflexão do ponto médio da viga

Considerando $a \geq b$

$$\delta_C := simplify\left(-v_{AP}\left(\frac{L}{2}\right)\right) = \frac{1}{48}\,\frac{(3\,a^2+6\,a\,b-b^2)\,P\,b}{EI}$$

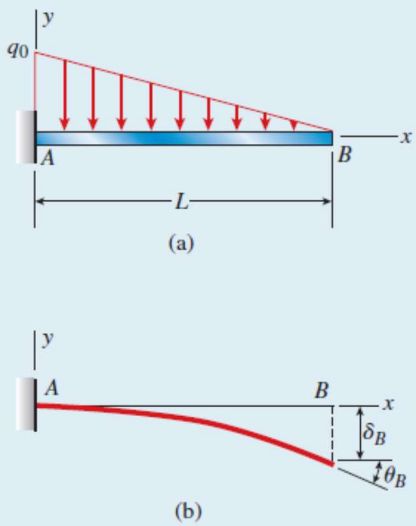
$$simplify\left(\delta_C - \left(\frac{P\cdot b\cdot(3\cdot L^2-4\cdot b^2)}{48\cdot EI}\right)\right) = 0$$

Exemplo 9.4

Referência: Gere & Goodno (2013). Mechanics of Materials, 8th Edition. Cengage Learning.

Determine the equation of the deflection curve for a cantilever beam *AB* supporting a triangularly distributed load of maximum intensity q_0 (Fig. 9-14a).

Also, determine the deflection δ_B and angle of rotation θ_B at the free end (Fig. 9-14b). ~~Use the fourth-order differential equation of the deflection curve (the load equation).~~ (*Note:* The beam has length *L* and constant flexural rigidity *EI*.)



$restart:$

$q := unapply\left(q_0 - q_0 \cdot \frac{x}{L}, x\right):$

Reaçoes de apoio

$assign\left(solve\left(\left\{R_A - \frac{q_0 \cdot L}{2} = 0, M_A - \left(\frac{q_0 \cdot L}{2}\right) \cdot \frac{L}{3} = 0\right\}, \{R_A, M_A\}\right)\right):$

$R_A, M_A = \frac{q_0 L}{2}, \frac{q_0 L^2}{6}$

Momento fletor

$M := unapply\left(solve\left(M + M_A - R_A \cdot x + (q(x) \cdot x) \cdot \frac{x}{2} + \left((q_0 - q(x)) \cdot \frac{x}{2}\right) \cdot \frac{2 \cdot x}{3} = 0, M\right), x\right):$

$M(x) = -\frac{q_0 (L^3 - 3 L^2 x + 3 x^2 L - x^3)}{6 L}$

Equação da curva de deflexão

$v := unapply\left(rhs\left(dsolve\left(\left\{diff(v(x), x\$2) = \frac{M(x)}{EI}, v(0) = 0, D[1](v)(0) = 0\right\}, v(x)\right)\right), x\right):$

$v(x) = -\frac{q_0 (L - x)^5}{120 L EI} - \frac{q_0 L^3 x}{24 EI} + \frac{q_0 L^4}{120 EI}$

$$simplify\left(v(x) - \left(-\frac{q_0 \cdot x^2}{120 \cdot L \cdot EI} \cdot \left(10 \cdot L^3 - 10 \cdot L^2 \cdot x + 5 \cdot L \cdot x^2 - x^3\right)\right)\right) = 0$$

Deflexão e ângulo de rotação na extermidade livre

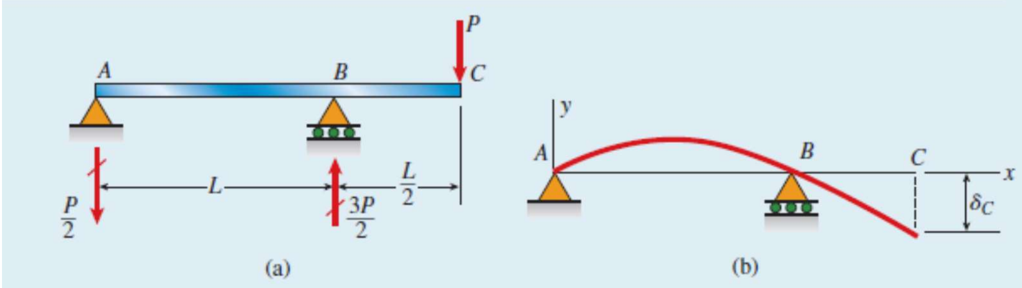
$$\delta_B := -v(L) = \frac{1}{30} \frac{q_0 L^4}{EI}$$

$$\theta_B := -\mathrm{D}[1](v)(L) = \frac{1}{24} \frac{q_0 L^3}{EI}$$

Exemplo 9.5
Referência: Gere & Goodno (2013). Mechanics of Materials, 8th Edition. Cengage Learning.

A simple beam *AB* with an overhang *BC* supports a concentrated load *P* at the end of the overhang (Fig. 9-15a). The main span of the beam has length *L* and the overhang has length *L*/2.

Determine the equations of the deflection curve and the deflection δ_C at the end of the overhang (Fig. 9-15b). ~~Use the third-order differential equation of the deflection curve (the shear force equation).~~ (*Note:* The beam has constant flexural rigidity *EI*.)



restart :

Reações de apoio

$assign\left(solve\left(\left\{R_A + R_B - P = 0,\ R_B \cdot L - P \cdot \frac{3 \cdot L}{2} = 0\right\},\ \{R_A, R_B\}\right)\right)$:

$R_A, R_B = -\frac{P}{2}, \frac{3\,P}{2}$

Momento fletor

$M_{AB} := unapply\left(solve\left(M - R_A \cdot x = 0, M\right), x\right)$:

$M_{BC} := unapply\left(solve\left(-M - P \cdot \left(\frac{3 \cdot L}{2} - x\right) = 0, M\right), x\right)$:

$M_{AB}(x) = -\frac{P\,x}{2}$

$M_{BC}(x) = -\frac{P\,(3\,L - 2\,x)}{2}$

$simplify\left(M_{AB}(L) - M_{BC}(L)\right) = 0$

Equação da curva de deflexão

$bc := v_{AB}(0) = 0,\ v_{AB}(L) = 0,\ v_{BC}(L) = 0,\ \mathrm{D}[1]\left(v_{AB}\right)(L) = \mathrm{D}[1]\left(v_{BC}\right)(L)$:

$res := dsolve\left(\left\{diff\left(v_{AB}(x), x\$2\right) = \frac{M_{AB}(x)}{EI},\ diff\left(v_{BC}(x), x\$2\right) = \frac{M_{BC}(x)}{EI},\ bc\right\},\ \left\{v_{AB}(x), v_{BC}(x)\right\}\right)$

$$res := \left\{v_{AB}(x) = -\frac{P\,x^3}{12\,EI} + \frac{P\,L^2\,x}{12\,EI},\ v_{BC}(x) = \frac{-3\,L^3\,P + 10\,L^2\,P\,x - 9\,P\,L\,x^2 + 2\,P\,x^3}{12\,EI}\right\} \quad \textbf{(1)}$$

$v_{AB} := unapply(simplify(rhs(res[1])), x)$:

$v_{BC} := unapply(simplify(rhs(res[2])), x)$:

$$v_{AB}(x) = \frac{P\,x\,(L^2 - x^2)}{12\,EI}$$

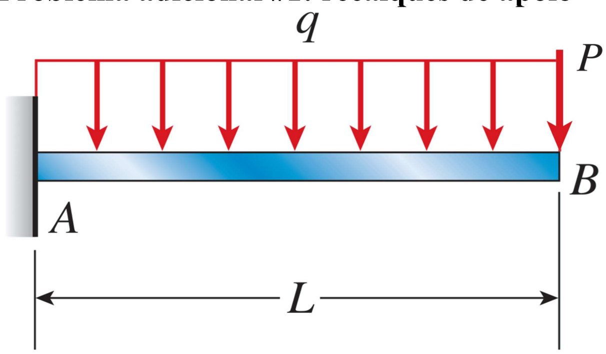
$$v_{BC}(x) = -\frac{(3\,L - x)\,P\,(L - 2\,x)\,(L - x)}{12\,EI}$$

$$simplify\left(v_{BC}(x) - \left(-\frac{P}{12 \cdot EI} \cdot \left(3 \cdot L^3 - 10 \cdot L^2 \cdot x + 9 \cdot L \cdot x^2 - 2 \cdot x^3\right)\right)\right) = 0$$

Deflexão na extremidade em balanço

$$\delta_C := -v_{BC}\left(\frac{3 \cdot L}{2}\right) = \frac{1}{8}\,\frac{P\,L^3}{EI}$$

Problema adicional #1: recalques de apoio



$restart:$

Reações de apoio

$$assign\left(solve\left(\left\{R_A - q\cdot L - P = 0, M_A - (q\cdot L)\cdot\frac{L}{2} - P\cdot L = 0\right\}, \{R_A, M_A\}\right)\right):$$

$$R_A, M_A = q\,L + P, \frac{1}{2}\,q\,L^2 + P\,L$$

Momento fletor

$$M := unapply\left(solve\left(M + M_A - R_A\cdot x + (q\cdot x)\cdot\frac{x}{2} = 0, M\right), x\right):$$

$$M(x) = -\frac{1}{2}\,q\,L^2 - P\,L + L\,q\,x + P\,x - \frac{1}{2}\,q\,x^2$$

Equação da curva de deflexão e rotação

$$bc := v(0) = v_0, \mathrm{D}[1](v)(0) = \theta_0:$$

$$v := unapply\left(expand\left(rhs\left(dsolve\left(\left\{diff(v(x), x\$2) = \frac{M(x)}{EI}, bc\right\}, v(x)\right)\right)\right), x\right):$$

$$\theta := unapply(expand(diff(v(x), x)), x):$$

$$v(x) = -\frac{q\,x^4}{24\,EI} + \frac{L\,x^3\,q}{6\,EI} + \frac{x^3\,P}{6\,EI} - \frac{L^2\,x^2\,q}{4\,EI} - \frac{L\,x^2\,P}{2\,EI} + \theta_0\,x + v_0$$

$$\theta(x) = -\frac{q\,x^3}{6\,EI} + \frac{L\,x^2\,q}{2\,EI} + \frac{x^2\,P}{2\,EI} - \frac{L^2\,x\,q}{2\,EI} - \frac{L\,x\,P}{EI} + \theta_0$$

Avaliação de rotação e deflexão em $x = L$

$$expand(\theta(L)) = -\frac{q\,L^3}{6\,EI} - \frac{L^2\,P}{2\,EI} + \theta_0$$

$$expand(v(L)) = -\frac{q\,L^4}{8\,EI} - \frac{L^3\,P}{3\,EI} + \theta_0\,L + v_0$$

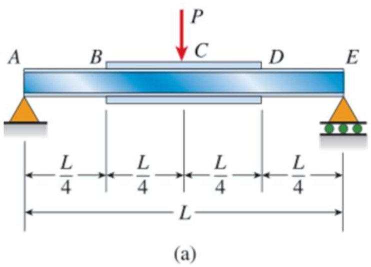
Problema adicional #2: Exemplo 9-13 (viga não prismática)
Referência: Gere & Goodno (2013). Mechanics of Materials, 8th Edition. Cengage Learning.

Example 9-13

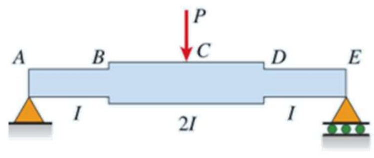
Example 9-13: Simple beam with two different moments of inertia

A beam *ABCDE* on simple supports is constructed from a wide-flange beam by welding cover plates over the middle half of the beam (Fig. 9-28a). The effect of the cover plates is to double the moment of inertia (Fig. 9-28b). A concentrated load *P* acts at the midpoint *C* of the beam.

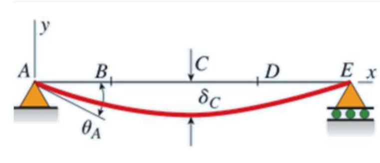
Determine the equations of the deflection curve, the angle of rotation θ_A at the left-hand support, and the deflection δ_C at the midpoint (Fig. 9-28c).



(a)



(b)



(c)

restart :

Reações de apoio

$$assign\left(solve\left(\left\{R_A + R_E - P = 0,\ -P\cdot\frac{L}{2} + R_E\cdot L = 0\right\},\ \{R_A, R_E\}\right)\right):$$

$$R_A, R_E = \frac{P}{2}, \frac{P}{2}$$

Momento fletor

$$M_{AC} := solve(M - R_A\cdot x = 0, M) = \frac{1}{2}\,P\,x$$

Equação da curva de deflexão

$$eqn := diff\left(v_{AB}(x), x\$2\right) = \frac{M_{AC}}{EI},\ diff\left(v_{BC}(x), x\$2\right) = \frac{M_{AC}}{2\cdot EI}:$$

$$bc := v_{AB}(0) = 0,\ \mathrm{D}[1](v_{BC})\left(\frac{L}{2}\right) = 0,\ v_{AB}\left(\frac{L}{4}\right) = v_{BC}\left(\frac{L}{4}\right),\ \mathrm{D}[1](v_{AB})\left(\frac{L}{4}\right) = \mathrm{D}[1](v_{BC})\left(\frac{L}{4}\right):$$

$$res := dsolve(\{eqn, bc\}, \{v_{AB}(x), v_{BC}(x)\}):$$

$$v_{AB} := unapply(simplify(rhs(res[1])), x):$$

$$v_{BC} := unapply(simplify(rhs(res[2])), x):$$

$$v_{AB}(x) = -\frac{P\,x\,(15\,L^2 - 32\,x^2)}{384\,EI}$$

$$v_{BC}(x) = -\frac{P\,(L^3 + 24\,L^2\,x - 32\,x^3)}{768\,EI}$$

Ângulo de rotação em A e deflexão em C

$$\theta_A := -\mathrm{D}[1](v_{AB})(0) = \frac{5}{128}\,\frac{P\,L^2}{EI}$$

$$\delta_C := -v_{BC}\left(\frac{L}{2}\right) = \frac{3}{256}\,\frac{P\,L^3}{EI}$$