# Lecture notes

These lecture notes are not guaranteed to be 100% correct and they are possibly not exactly identical to the lecture. However, in case you find any errors, please feel free to send a mail to Andy: awiese@dii.uchile.cl

- pivot of Simplex algorithm
  - given basis $B$, compute b.f.s. $x = (x_B, x_N)$
  - compute reduced costs $\bar{c}$. If $\bar{c} \geq 0$ then $x$ is optimal, STOP. Otherwise select variable $x_j$ with $\bar{c}_j < 0$
  - compute direction $d$. If $d \geq 0$ then OPT$=-\infty$, STOP.
  - compute $\theta^*$ and determine index $\ell$ such that $y_{B(\ell)} = 0$ where $y = x + \theta^* d^j$
  - form new basis by replacing $x_{B(\ell)}$ with $x_j$

**Theorem 1.** *Assume $P \neq \emptyset$, every basic feasible solution is nondegenerate, and that the algorithm is initialized with a basic feasible solution. Then it terminates after a finite number of iterations. At termination, there are the following two possibilities:*

- *We have an optimal basis $B$ and an associated basic feasible solution which is optimal.*
- *We have found a vector $d$ satisfying $Ad = 0$, $d \geq 0$, and $c^T d < 0$, and thus LP is unbounded (i.e., $OPT = -\infty$).*
- *Proof:*
- *If algorithm terminates in step 2 then we are optimal because $\bar{c} \geq 0$.*
- *If algorithm terminates in step 3 then we are at basic feasible solution $x$ and found a direction $d^j \geq 0$ with*
  - *with $Ad^j = 0$*
  - *$\Rightarrow x + \theta d^j \in P$ for all $\theta > 0$*
  - *$c^T d^j = \bar{c}_j < 0$*
  - *cost of $x + \theta d^j$ is $c^T(x + \theta d^j) = c^T x + \theta \underbrace{c^T d^j}_{<0}$*
  - *$\Rightarrow$ LP is unbounded*
- *Termination:*
- *non-degeneracy: always $\theta^* > 0$*
- *cost at new point $x + \theta d^j$ is $c^T(x + \theta d^j) = c^T x + \theta \underbrace{c^T d^j}_{<0} < c^T x$*
- *$\Rightarrow$ in each pivot the objective value strictly decreases*
- *$\Rightarrow$ no vertex is visited twice*
- *only finite number of vertices $\Rightarrow$ only finite number of iterations*

# 1 Degenerate problems

- stepsize $\theta^*$ can be zero.
  - then new solution is same as old one *(but with different basis)* $\Rightarrow$ *no progress*
  - *(happens if* $d_i^j < 0$ *but basic variable* $x_i = 0$*)*
- simplex algorithm can cycle and stay forever in the same solution
- **idea: use smarter pivoting rule!**
- (*pivoting rule tells me what decision to take if I have a choice*
- degrees of freedom
  - select variable to enter the basis (among those with negative $\bar{c}_j$)
  - select variable to leave basis (if more than one option)

- Bland's rule
  1. find smallest $j$ for which $\bar{c}_j < 0$ and let $x_j$ enter the basis
  2. out of all variables $x_i$ tied for leaving basis choose variable $x_i$ with smallest $i$
  - finite number of iterations guaranteed

- There are also other pivoting rules that guarantee finite number of iterations

# 2 Find initial basic feasible solution

- Simplex requires basic feasible solution to start
- how to find? $\rightsquigarrow$ with another LP

| given problem | Generate new (artificial) problem |
|---|---|
| minimize $c^T x$ | minimize $\sum_{i=1}^m y_i$ |
| s.t. $Ax = b$ | s.t. $Ax + y = b$ |
| $x \geq 0$ | $x \geq 0$ |
| assume w.l.o.g. that $b \geq 0$ (can be achieved by multiplying rows by $-1$) | $y \geq 0$ |

- if $x$ feasible solution to given problem $\Rightarrow$ $(x, 0)$ is feasible sol. to new problem with zero cost
- $\Rightarrow$ optimal solution to new problem has cost zero
- if optimal solution $(x^*, y^*)$ to new problem has non-zero cost $\Rightarrow$ given problem is infeasible
- Phase I
  - solve artificial problem: start vertex: $x = 0$, $y = b$ is basic feasible solution
  - initial basis consists of all $y$-variables
  - let $(x, y)$ optimal solution to new problem
    - if $y \neq 0$ then given problem is infeasible
    - if $y = 0$ then $x$ is feasible solution to given problem
  - let $(x, 0)$ be optimal basic feasible solution to new problem with basis $B$

- if $B$ contains only $x$-variables $\Rightarrow x$ is basic solution with basis $B$ for given problem
- assume $B$ contains only $k < m$ $x$-variables
- in $(x, 0)$ only the variables $x_{B(1)}, ..., x_{B(k)}$ can be non-zero and $\sum_{i=1}^{k} A_{B(i)} x_{B(i)} = b$
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- columns $A_{B(1)}, ..., A_{B(k)}$ are linearly independent
- we can find indices $B'(k+1), ..., B'(m)$ such that $A_{B(1)}, ..., A_{B(k)}, A_{B'(k+1)}, ..., A_{B'(m)}$ are linearly independent
- $B' = (B(1), ..., B(k), B'(k+1), ..., B'(m))$ is basis for $x$ in given problem

- Phase II
  - solve given problem
  - start with $x$ and basis $B'$

# 3 Simplex running time

- each iteration in polytime
  - compute $A_B^{-1}$: $O(m^3)$ steps
  - compute reduced costs $\bar{c}$: $O(nm)$ (recall that $\bar{c} = c^T - c_B^T A_B^{-1} A$)
  - compute direction $d$ and index $\ell$: $O(m^2)$ ($d_B^j = -A_B^{-1} A_j$)
  - compute new point and new basis: $O(n)$
  - overall: $O(m^3 + nm)$ *(recall that $m \leq n$)*
- number of iterations: depends on pivoting rule, can be exponential (=slow)
- no rule known that guarantees polynomial running time
- but: Simplex is very fast in practise
- Unknown: is there path between any two vertices of length $poly(n, m)$ ?
- Hirsch conjecture: always exists path of length $m - n$.
- False: [Santos 2010]$\exists$ polytope in $\mathbb{R}^{43}$ with $m = 86$ where two vertices are 44 steps apart.