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1 01-01 tests

Example 1.1. In this tests we consider:

- $\psi(x) = x^4$
- $\psi_{\mathrm{l}} = 0$
- $\psi_{\mathrm{r}} = 1$
- $\psi_{\mathrm{ll}} = 0$
- $\psi_{\mathrm{rr}} = 4$
- $g(x) = -24$

Table 1: Numerical results of PRO1 scheme to the example 1.1.

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$
$\mathbb{P}_3(4)$	20	3.33E−03	—	2.51E−03	—	2.51E−03	—	2.51E−03	—
	40	4.31E−04	2.95	3.21E−04	2.97	3.21E−04	2.97	3.21E−04	2.97
	80	5.46E−05	2.98	4.04E−05	2.99	4.04E−05	2.99	4.04E−05	2.99
	160	6.86E−06	2.99	5.07E−06	2.99	5.07E−06	2.99	5.07E−06	2.99
$\mathbb{P}_5(6)$	20	6.36E−15	—	3.12E−14	—	4.36E−15	—	1.90E−14	—
	40	5.08E−14	↑	1.73E−13	↑	6.78E−14	↑	2.22E−13	↑
	80	8.44E−13	↑	3.31E−13	↑	8.35E−14	↑	3.54E−13	↑
	160	6.92E−12	↑	3.51E−12	↑	2.70E−12	↑	3.71E−12	↑
$\mathbb{P}_7(8)$	20	4.52E−13	—	1.16E−13	—	7.32E−14	—	2.85E−14	—
	40	5.41E−13	↑	1.41E−13	↑	1.33E−13	↑	6.85E−13	↑
	80	2.98E−12	↑	2.01E−12	↑	1.99E−12	↑	1.63E−12	↑
	160	5.16E−11	↑	1.09E−11	↑	1.87E−11	↑	2.59E−11	↑

Example 1.2. In this tests we consider:

- $\psi(x) = \exp(x)$
- $\psi_{\mathrm{l}} = 1$
- $\psi_{\mathrm{r}} = e$
- $\psi_{\mathrm{ll}} = 1$
- $\psi_{\mathrm{rr}} = e$
- $g(x) = -\exp(x)$

Table 2: Numerical results of PRO1 scheme to the example 1.2.

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	2.60E−04	—	2.07E−04	—	2.07E−04	—	2.06E−04	—
	40	3.35E−05	2.95	2.65E−05	2.96	2.65E−05	2.96	2.65E−05	2.96
	80	4.14E−06	3.02	3.27E−06	3.02	3.27E−06	3.02	3.27E−06	3.02
	160	4.90E−07	3.08	3.82E−07	3.10	3.82E−07	3.10	3.82E−07	3.10
$\mathbb{P}_5(6)$	20	1.78E−07	—	1.48E−07	—	1.48E−07	—	1.48E−07	—
	40	5.36E−09	5.05	4.46E−09	5.06	4.46E−09	5.06	4.46E−09	5.06
	80	1.57E−10	5.09	1.38E−10	5.01	1.37E−10	5.02	1.41E−10	4.99
	160	8.64E−11	0.86	5.63E−11	1.30	3.58E−11	1.94	1.65E−11	3.09
$\mathbb{P}_7(8)$	20	6.73E−10	—	5.08E−10	—	5.09E−10	—	5.09E−10	—
	40	1.03E−11	6.03	4.40E−12	6.85	4.39E−12	6.86	2.08E−12	7.94
	80	5.93E−12	0.80	1.93E−11	↑	1.83E−12	1.26	2.72E−12	↑
	160	6.87E−11	↑	2.07E−10	↑	1.12E−10	↑	1.33E−10	↑

Table 3: Numerical results of PRO1 scheme to the example 1.2.

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$		$\omega = 1\|3,0.1$		$\omega = 1\|3,0.01$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	2.60E−04	—	2.07E−04	—	2.07E−04	—	2.06E−04	—	2.06E−04	—	2.06E−04	—
	40	3.35E−05	2.95	2.65E−05	2.96	2.65E−05	2.96	2.65E−05	2.96	2.65E−05	2.96	2.65E−05	2.96
	80	4.14E−06	3.02	3.27E−06	3.02	3.27E−06	3.02	3.27E−06	3.02	3.27E−06	3.02	3.27E−06	3.02
	160	4.90E−07	3.08	3.82E−07	3.10	3.82E−07	3.10	3.82E−07	3.10	3.82E−07	3.10	3.82E−07	3.10
$\mathbb{P}_5(6)$	20	1.78E−07	—	1.48E−07	—	1.48E−07	—	1.48E−07	—	1.48E−07	—	1.48E−07	—
	40	5.36E−09	5.05	4.46E−09	5.06	4.46E−09	5.06	4.46E−09	5.06	4.46E−09	5.06	4.45E−09	5.06
	80	1.54E−10	5.12	1.38E−10	5.01	1.37E−10	5.02	1.41E−10	4.99	1.43E−10	4.96	1.38E−10	5.02
	160	2.18E−11	2.82	5.63E−11	1.30	3.58E−11	1.94	1.65E−11	3.09	3.42E−11	2.07	2.85E−11	2.27
$\mathbb{P}_7(8)$	20	6.75E−10	—	5.08E−10	—	5.09E−10	—	5.09E−10	—	5.09E−10	—	5.09E−10	—
	40	8.06E−12	6.39	4.40E−12	6.85	4.39E−12	6.86	2.08E−12	7.94	2.24E−12	7.83	5.71E−12	6.48
	80	1.12E−12	2.85	1.93E−11	↑	1.83E−12	1.26	2.72E−12	↑	4.55E−12	↑	2.49E−11	↑
	160	1.37E−10	↑	2.07E−10	↑	1.12E−10	↑	1.33E−10	↑	1.78E−10	↑	2.04E−10	↑

Example 1.3. In this tests we consider:

- $\psi(x) = -\exp(x) + x^3(3-e) + x^2(2e-5) + x + 1$
- $\psi_{\mathrm{l}} = 0$
- $\psi_{\mathrm{r}} = 0$
- $\psi_{\mathrm{ll}} = 0$
- $\psi_{\mathrm{rr}} = 0$
- $g(x) = \exp(x)$

Table 4: Numerical results of PRO1 scheme to the example 1.3.

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	2.60E−04	—	2.07E−04	—	2.07E−04	—	2.06E−04	—
	40	3.35E−05	2.95	2.65E−05	2.96	2.65E−05	2.96	2.65E−05	2.96
	80	4.14E−06	3.02	3.27E−06	3.02	3.27E−06	3.02	3.27E−06	3.02
	160	4.90E−07	3.08	3.82E−07	3.10	3.82E−07	3.10	3.82E−07	3.10
$\mathbb{P}_5(6)$	20	1.78E−07	—	1.48E−07	—	1.48E−07	—	1.48E−07	—
	40	5.36E−09	5.05	4.46E−09	5.06	4.46E−09	5.06	4.46E−09	5.06
	80	1.55E−10	5.11	1.41E−10	4.98	1.41E−10	4.98	1.41E−10	4.98
	160	5.30E−12	4.88	4.82E−12	4.87	4.87E−12	4.85	4.73E−12	4.89
$\mathbb{P}_7(8)$	20	6.74E−10	—	5.09E−10	—	5.09E−10	—	5.09E−10	—
	40	8.64E−12	6.29	4.11E−12	6.95	4.11E−12	6.95	4.11E−12	6.95
	80	6.83E−14	6.98	2.37E−14	7.44	3.61E−14	6.83	6.97E−14	5.88
	160	1.57E−13	↑	2.08E−13	↑	1.84E−13	↑	3.50E−13	↑

2 02-02 tests

Example 2.1. In this tests we consider:

- $\psi(x) = x^4$
- $\psi_{\mathrm{l}} = 0$
- $\psi_{\mathrm{r}} = 1$
- $M_{\mathrm{l}} = 0$
- $M_{\mathrm{r}} = -12$
- $g(x) = -24$

Table 5: Numerical results of PRO1 scheme to the example 2.1.

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	2.26E−02	—	1.81E−02	—	1.81E−02	—	1.81E−02	—
	40	5.45E−03	2.05	4.40E−03	2.04	4.40E−03	2.04	4.40E−03	2.04
	80	1.33E−03	2.04	1.08E−03	2.03	1.08E−03	2.03	1.08E−03	2.03
	160	3.26E−04	2.02	2.68E−04	2.01	2.68E−04	2.01	2.68E−04	2.01
$\mathbb{P}_5(6)$	20	1.73E−13	—	1.53E−13	—	2.36E−14	—	1.51E−14	—
	40	4.75E−13	↑	1.65E−13	↑	5.11E−13	↑	8.54E−13	↑
	80	1.45E−12	↑	8.49E−13	↑	1.61E−12	↑	2.39E−12	↑
	160	2.05E−11	↑	8.50E−12	↑	2.44E−12	↑	3.71E−11	↑
$\mathbb{P}_7(8)$	20	8.20E−13	—	9.55E−13	—	1.50E−13	—	1.97E−12	—
	40	2.12E−12	↑	1.37E−12	↑	6.82E−12	↑	2.23E−12	↑
	80	2.83E−11	↑	2.66E−11	↑	1.86E−11	↑	3.90E−11	↑
	160	1.16E−10	↑	3.15E−10	↑	1.64E−11	0.18	4.77E−10	↑

Example 2.2. In this tests we consider:

- $\psi(x) = \exp(x)$
- $\psi_{\mathrm{l}} = 1$
- $\psi_{\mathrm{r}} = e$
- $M_{\mathrm{l}} = -1$
- $M_{\mathrm{r}} = -e$
- $g(x) = -\exp(x)$

Table 6: Numerical results of PRO1 scheme to the example 2.2.

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	1.68E−03	—	1.40E−03	—	1.40E−03	—	1.40E−03	—
	40	4.13E−04	2.03	3.47E−04	2.02	3.47E−04	2.02	3.47E−04	2.02
	80	1.01E−04	2.03	8.61E−05	2.01	8.61E−05	2.01	8.61E−05	2.01
	160	2.50E−05	2.02	2.14E−05	2.01	2.14E−05	2.01	2.14E−05	2.01
$\mathbb{P}_5(6)$	20	3.14E−06	—	2.79E−06	—	2.79E−06	—	2.79E−06	—
	40	2.40E−07	3.71	1.79E−07	3.96	1.79E−07	3.96	1.79E−07	3.96
	80	1.30E−08	4.21	1.13E−08	3.98	1.13E−08	3.98	1.13E−08	3.98
	160	9.45E−10	3.78	1.07E−09	3.41	6.87E−10	4.04	1.09E−09	3.37
$\mathbb{P}_7(8)$	20	1.47E−08	—	1.36E−08	—	1.36E−08	—	1.36E−08	—
	40	6.35E−10	4.53	2.12E−10	6.01	2.13E−10	6.00	2.26E−10	5.91
	80	8.54E−11	2.89	1.47E−10	0.53	2.60E−11	3.04	2.19E−11	3.37
	160	1.28E−09	↑	9.86E−11	0.58	8.54E−10	↑	1.30E−09	↑

Example 2.3. In this tests we consider:

- $\psi(x) = -\exp(x) + \left(\frac{e-1}{6}\right) x^3 + \frac{x^2}{2} + \left(\frac{5e-8}{6}\right) x + 1$
- $\psi_{\mathrm{l}} = 0$
- $\psi_{\mathrm{r}} = 0$
- $M_{\mathrm{l}} = 0$
- $M_{\mathrm{r}} = 0$
- $g(x) = \exp(x)$

Table 7: Numerical results of PRO1 scheme to the example 2.3.

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	1.68E−03	—	1.40E−03	—	1.40E−03	—	1.40E−03	—
	40	4.13E−04	2.03	3.47E−04	2.02	3.47E−04	2.02	3.47E−04	2.02
	80	1.01E−04	2.03	8.61E−05	2.01	8.61E−05	2.01	8.61E−05	2.01
	160	2.50E−05	2.02	2.14E−05	2.01	2.14E−05	2.01	2.14E−05	2.01
$\mathbb{P}_5(6)$	20	3.14E−06	—	2.79E−06	—	2.79E−06	—	2.79E−06	—
	40	2.40E−07	3.71	1.79E−07	3.96	1.79E−07	3.96	1.79E−07	3.96
	80	1.30E−08	4.21	1.13E−08	3.98	1.13E−08	3.98	1.13E−08	3.98
	160	9.98E−10	3.70	7.20E−10	3.98	7.15E−10	3.99	7.20E−10	3.98
$\mathbb{P}_7(8)$	20	1.47E−08	—	1.36E−08	—	1.36E−08	—	1.36E−08	—
	40	4.53E−10	5.02	2.20E−10	5.95	2.20E−10	5.95	2.20E−10	5.95
	80	5.87E−12	6.27	3.45E−12	6.00	3.93E−12	5.81	3.32E−12	6.05
	160	6.63E−12	↑	1.59E−11	↑	2.49E−12	0.66	8.74E−12	↑

3 01-23 tests

Example 3.1. In this tests we consider:

- $\psi(x) = x^4$
- $\psi_{\mathrm{l}} = 0$
- $\psi_{\mathrm{ll}} = 0$
- $M_{\mathrm{r}} = -12$
- $G = -24$
- $g(x) = 24$

Table 8: Numerical results of PRO1 scheme to the example 3.1.

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	7.32E−02	—	6.80E−02	—	6.80E−02	—	6.80E−02	—
	40	1.96E−02	1.90	1.78E−02	1.93	1.78E−02	1.93	1.78E−02	1.93
	80	5.03E−03	1.96	4.57E−03	1.97	4.57E−03	1.97	4.57E−03	1.97
	160	1.27E−03	1.98	1.15E−03	1.98	1.15E−03	1.98	1.15E−03	1.98
$\mathbb{P}_5(6)$	20	5.09E−13	—	4.73E−13	—	7.95E−13	—	4.41E−13	—
	40	1.26E−13	2.01	3.44E−12	↑	1.48E−11	↑	1.81E−11	↑
	80	1.01E−10	↑	1.31E−10	↑	4.82E−11	↑	2.18E−10	↑
	160	4.00E−10	↑	7.20E−10	↑	1.49E−09	↑	5.40E−10	↑
$\mathbb{P}_7(8)$	20	2.11E−11	—	3.95E−12	—	3.43E−12	—	1.48E−11	—
	40	1.33E−10	↑	5.76E−11	↑	5.98E−11	↑	3.84E−11	↑
	80	6.34E−11	1.07	6.60E−11	↑	9.44E−11	↑	2.71E−10	↑
	160	1.22E−09	↑	5.10E−09	↑	8.10E−10	↑	6.20E−09	↑

Example 3.2. In this tests we consider:

- $\psi(x) = \exp(x)$
- $\psi_{\mathrm{l}} = 1$
- $\psi_{\mathrm{ll}} = 1$
- $M_{\mathrm{r}} = -\mathrm{e}$
- $G = -\mathrm{e}$
- $g(x) = -\exp(x)$

Table 9: Numerical results of PRO1 scheme to the example 3.2.

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	7.69E−03	—	5.82E−01	—	5.82E−01	—	5.82E−01	—
	40	2.06E−03	1.90	5.99E−01	↑	5.99E−01	↑	5.99E−01	↑
	80	5.28E−04	1.96	6.09E−01	↑	6.09E−01	↑	6.09E−01	↑
	160	1.33E−04	1.98	6.14E−01	↑	6.14E−01	↑	6.14E−01	↑
$\mathbb{P}_5(6)$	20	1.79E−05	—	5.75E−01	—	5.75E−01	—	5.75E−01	—
	40	1.22E−06	3.88	5.97E−01	↑	5.97E−01	↑	5.97E−01	↑
	80	9.00E−08	3.76	6.08E−01	↑	6.08E−01	↑	6.08E−01	↑
	160	4.67E−09	4.27	6.14E−01	↑	6.14E−01	↑	6.14E−01	↑
$\mathbb{P}_7(8)$	20	1.16E−07	—	5.75E−01	—	5.75E−01	—	5.75E−01	—
	40	2.58E−09	5.49	5.97E−01	↑	5.97E−01	↑	5.97E−01	↑
	80	1.53E−09	0.75	6.08E−01	↑	6.08E−01	↑	6.08E−01	↑
	160	1.55E−09	↑	6.14E−01	↑	6.14E−01	↑	6.14E−01	↑

Example 3.3. In this tests we consider:

- $\psi(x) = -\exp(x) + x^3\left(\frac{e-1}{6}\right) + \frac{x^2}{2} + x + 1$
- $\psi_{\mathrm{l}} = 0$
- $\psi_{\mathrm{ll}} = 0$
- $M_{\mathrm{r}} = 0$
- $G = 1$
- $g(x) = \exp(x)$

Table 10: Numerical results of PRO1 scheme to the example 3.3.

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	7.69E−03	—	6.96E−03	—	6.96E−03	—	6.96E−03	—
	40	2.06E−03	1.90	1.85E−03	1.92	1.85E−03	1.92	1.85E−03	1.92
	80	5.28E−04	1.96	4.75E−04	1.96	4.75E−04	1.96	4.75E−04	1.96
	160	1.33E−04	1.98	1.20E−04	1.98	1.20E−04	1.98	1.20E−04	1.98
$\mathbb{P}_5(6)$	20	1.79E−05	—	1.42E−05	—	1.42E−05	—	1.42E−05	—
	40	1.22E−06	3.88	9.97E−07	3.84	9.97E−07	3.84	9.97E−07	3.84
	80	8.92E−08	3.77	6.58E−08	3.92	6.58E−08	3.92	6.58E−08	3.92
	160	5.36E−09	4.06	4.26E−09	3.95	4.24E−09	3.96	4.16E−09	3.98
$\mathbb{P}_7(8)$	20	1.16E−07	—	6.43E−08	—	6.43E−08	—	6.43E−08	—
	40	2.06E−09	5.82	1.15E−09	5.80	1.15E−09	5.80	1.16E−09	5.79
	80	7.63E−11	4.75	7.66E−11	3.91	3.28E−11	5.14	5.16E−11	4.49
	160	3.12E−10	↑	3.99E−10	↑	3.42E−10	↑	1.08E−10	↑