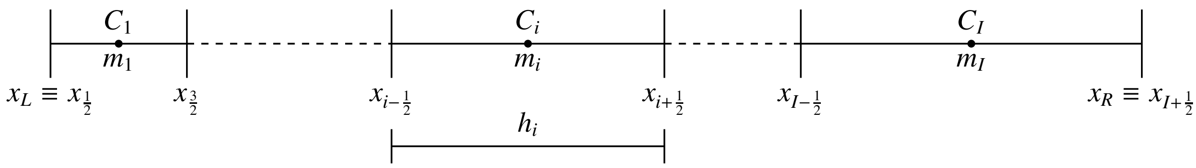
1 Formulation and Discretization of the Problem of Pure Diffusion

- Formulation:

$$\begin{aligned} -(\kappa\phi')' &= s && \text{in } \Omega =]x_{\frac{1}{2}}, x_{I+\frac{1}{2}}[\\ \phi &= \phi_{\mathrm{lf},0} && \text{at } x = x_{\frac{1}{2}} \\ -\kappa\phi' &= \phi_{\mathrm{rg},1} && \text{at } x = x_{I+\frac{1}{2}} \end{aligned}$$

- Mesh:



- Discretization:

$$\begin{aligned} &-(\kappa\phi')' = s \\ &\int_{C_i} (-(\kappa\phi')')\,\mathrm{d}x = \int_{C_i} s\,\mathrm{d}x \\ &-(\kappa\phi')\Big|_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = \int_{C_i} s\,\mathrm{d}x \\ &-\left(\mathbb{F}_{\mathrm{D},i+\frac{1}{2}} - \mathbb{F}_{\mathrm{D},i-\frac{1}{2}}\right) = \mathbb{S}_i, \quad i = 1,\dots,I \end{aligned}$$

2 Reconstructions and Matrices

- At $x_{\frac{1}{2}}$:

$$\phi_{\mathrm{d},\frac{1}{2}}(x) = \sum_{\alpha=0}^{\mathrm{d}} \mathcal{R}_{\frac{1}{2},\alpha}(x - x_{\frac{1}{2}})^{\alpha}$$

$\widehat{\mathcal{R}}_{\frac{1}{2}} = \left(\widehat{\mathcal{R}}_{\frac{1}{2},0}, \dots, \widehat{\mathcal{R}}_{\frac{1}{2},\mathrm{d}}\right)^{\mathrm{T}}$ is the solution of the constrained linear least squares problem

$$\boxed{\begin{aligned} &\min \quad \sum_{j \in S_{\frac{1}{2}}} \omega_j \left[\frac{1}{h_j}\int_{C_j} \phi_{\mathrm{d},\frac{1}{2}}(x)\,\mathrm{d}x - \phi_j\right]^2 \\ &\text{s.t.} \quad \phi_{\mathrm{d},\frac{1}{2}}(x_{\frac{1}{2}}) = \phi_{\mathrm{lf},0} \end{aligned}} \quad \Leftrightarrow \quad \boxed{\begin{aligned} &\min \quad \|A_{\frac{1}{2}}\mathcal{R}_{\frac{1}{2}} - B_{\frac{1}{2}}\|_2^2 \\ &\text{s.t.} \quad C_{\frac{1}{2}}\mathcal{R}_{\frac{1}{2}} = D_{\frac{1}{2}} \end{aligned}}$$

So,

$$
\begin{aligned}
&\omega_j\left[\frac{1}{h_j}\int_{C_j}\phi_{\mathrm{d},\frac{1}{2}}\,\mathrm{d}x-\phi_j\right]=0\\
\Leftrightarrow&\omega_j\left[\frac{1}{h_j}\int_{C_j}\phi_{\mathrm{d},\frac{1}{2}}\,\mathrm{d}x\right]=\omega_j\phi_j\\
\Leftrightarrow&\omega_j\left[\frac{1}{h_j}\int_{C_j}\left(\sum_{\alpha=0}^{\mathrm{d}}\mathcal{R}_{\frac{1}{2},\alpha}(x-x_{\mathrm{lf}})^{\alpha}\right)\mathrm{d}x\right]=\omega_j\phi_j\\
\Leftrightarrow&\omega_j\left[\frac{1}{h_j}\int_{C_j}\left(\mathcal{R}_{\frac{1}{2},0}(x-x_{\mathrm{lf}})^{0}+\mathcal{R}_{\frac{1}{2},1}(x-x_{\mathrm{lf}})^{1}+\cdots+\mathcal{R}_{\frac{1}{2},\mathrm{d}}(x-x_{\mathrm{lf}})^{\mathrm{d}}\right)\mathrm{d}x\right]=\omega_j\phi_j\\
\Leftrightarrow&\omega_j\left[\mathcal{R}_{\frac{1}{2},0}\underbrace{\left(\frac{1}{h_j}\int_{C_j}(x-x_{\mathrm{lf}})^{0}\,\mathrm{d}x\right)}_{A}+\mathcal{R}_{\frac{1}{2},1}\left(\frac{1}{h_j}\int_{C_j}(x-x_{\mathrm{lf}})^{1}\,\mathrm{d}x\right)+\cdots+\mathcal{R}_{\frac{1}{2},\mathrm{d}}\left(\frac{1}{h_j}\int_{C_j}(x-x_{\mathrm{lf}})^{\mathrm{d}}\,\mathrm{d}x\right)\right]=\underbrace{\omega_j\phi_j}_{B}
\end{aligned}
$$

and,

$$
\begin{aligned}
&\phi_{\mathrm{d},\frac{1}{2}}(x_{\mathrm{lf}})=\phi_{\mathrm{lf},0}\\
\Leftrightarrow&\sum_{\alpha=0}^{\mathrm{d}}\mathcal{R}_{\frac{1}{2},\alpha}(x_{\mathrm{lf}}-x_{\mathrm{lf}})^{\alpha}=\phi_{\mathrm{lf},0}\\
\Leftrightarrow&\mathcal{R}_{\frac{1}{2},0}\underbrace{(x_{\mathrm{lf}}-x_{\mathrm{lf}})^{0}}_{C}+\mathcal{R}_{\frac{1}{2},1}(x_{\mathrm{lf}}-x_{\mathrm{lf}})^{1}+\cdots+\mathcal{R}_{\frac{1}{2},\mathrm{d}}(x_{\mathrm{lf}}-x_{\mathrm{lf}})^{\mathrm{d}}=\underbrace{\phi_{\mathrm{lf},0}}_{D}
\end{aligned}
$$

Then, the matrices are:

$$
\begin{aligned}
A_{\frac{1}{2}}&=[A_{\frac{1}{2},j\alpha}]\in\mathcal{M}_{n_S\times(\mathrm{d}+1)}(\mathbb{R})=\omega_j\frac{1}{h_j}\int_{C_j}(x-x_{\frac{1}{2}})^{\alpha}\,\mathrm{d}x\\
B_{\frac{1}{2}}&=[B_{\frac{1}{2},j}]\in\mathcal{M}_{n_S\times 1}(\mathbb{R})=\omega_j\phi_j\\
C_{\frac{1}{2}}&=[C_{\frac{1}{2},j\alpha}]\in\mathcal{M}_{1\times(\mathrm{d}+1)}(\mathbb{R})=\begin{cases}1 & \text{if}\quad \alpha=0\\ 0 & \text{if}\quad \alpha=1,\ldots,\mathrm{d}\end{cases}\\
D_{\frac{1}{2}}&=[D_{\frac{1}{2},j}]\in\mathcal{M}_{1\times 1}(\mathbb{R})=\phi_{\mathrm{lf},0}
\end{aligned}
$$

- At $x_{\frac{3}{2}},\ldots,x_{I-\frac{1}{2}}$:

$$\phi_{\mathrm{d},i+\frac{1}{2}}(x)=\sum_{\alpha=0}^{\mathrm{d}}\mathcal{R}_{i+\frac{1}{2},\alpha}(x-x_{i+\frac{1}{2}})^{\alpha},$$

$\widetilde{\mathcal{R}}_{i+\frac{1}{2}}=\left(\widetilde{\mathcal{R}}_{i+\frac{1}{2},0},\ldots,\widetilde{\mathcal{R}}_{i+\frac{1}{2},\mathrm{d}}\right)^{\mathrm{T}}$ is the solution of the constrained linear least squares problem

$$\boxed{\min\quad\sum_{j\in S_{i+\frac{1}{2}}}\omega_j\left[\frac{1}{h_j}\int_{C_j}\phi_{\mathrm{d},i+\frac{1}{2}}(x)\,\mathrm{d}x-\phi_j\right]^2}$$

So,

$$
\begin{aligned}
&\omega_j\left[\frac{1}{h_j}\int_{C_j}\phi_{\mathrm{d},i+\frac{1}{2}}\,\mathrm{d}x-\phi_j\right]=0\\
\Leftrightarrow&\omega_j\left[\frac{1}{h_j}\int_{C_j}\phi_{\mathrm{d},i+\frac{1}{2}}\,\mathrm{d}x\right]=\omega_j\phi_j\\
\Leftrightarrow&\omega_j\left[\frac{1}{h_j}\int_{C_j}\left(\sum_{\alpha=0}^{\mathrm{d}}\mathcal{R}_{i+\frac{1}{2},\alpha}(x-x_{i+\frac{1}{2}})^{\alpha}\right)\mathrm{d}x\right]=\omega_j\phi_j\\
\Leftrightarrow&\omega_j\left[\frac{1}{h_j}\int_{C_j}\left(\mathcal{R}_{i+\frac{1}{2},0}(x-x_{i+\frac{1}{2}})^{0}+\mathcal{R}_{i+\frac{1}{2},1}(x-x_{i+\frac{1}{2}})^{1}+\cdots+\mathcal{R}_{i+\frac{1}{2},\mathrm{d}}(x-x_{i+\frac{1}{2}})^{\mathrm{d}}\right)\mathrm{d}x\right]=\omega_j\phi_j\\
\Leftrightarrow&\omega_j\left[\mathcal{R}_{i+\frac{1}{2},0}\underbrace{\left(\frac{1}{h_j}\int_{C_j}(x-x_{i+\frac{1}{2}})^{0}\,\mathrm{d}x\right)}_{A}+\mathcal{R}_{i+\frac{1}{2},1}\left(\frac{1}{h_j}\int_{C_j}(x-x_{i+\frac{1}{2}})^{1}\,\mathrm{d}x\right)+\cdots+\mathcal{R}_{i+\frac{1}{2},\mathrm{d}}\left(\frac{1}{h_j}\int_{C_j}(x-x_{i+\frac{1}{2}})^{\mathrm{d}}\,\mathrm{d}x\right)\right]=\underbrace{\omega_j\phi_j}_{B}
\end{aligned}
$$

Then, the matrices are:

$$
\begin{aligned}
A_{i+\frac{1}{2}}&=[A_{i+\frac{1}{2},j\alpha}]\in\mathcal{M}_{n_S\times(\mathrm{d}+1)}(\mathbb{R})=\omega_j\frac{1}{h_j}\int_{C_j}(x-x_{i+\frac{1}{2}})^{\alpha}\,\mathrm{d}x\\
B_{i+\frac{1}{2}}&=[B_{i+\frac{1}{2},j}]\in\mathcal{M}_{n_S\times 1}(\mathbb{R})=\omega_j\phi_j
\end{aligned}
$$

- At $x_{I+\frac{1}{2}}$:

$$\phi_{\mathrm{d},I+\frac{1}{2}}=\sum_{\alpha=0}^{\mathrm{d}}\mathcal{R}_{I+\frac{1}{2},\alpha}(x-x_{\mathrm{rg}})^{\alpha}$$

$\widehat{\mathcal{R}}_{I+\frac{1}{2}}=\left(\widehat{\mathcal{R}}_{I+\frac{1}{2},0},\ldots,\widehat{\mathcal{R}}_{I+\frac{1}{2},\mathrm{d}}\right)^{\mathrm{T}}$ is the solution of the constrained linear least squares problem

$$
\boxed{\begin{array}{ll}\min & \displaystyle\sum_{j\in S_{I+\frac{1}{2}}}\omega_j\left[\frac{1}{h_j}\int_{C_j}\phi_{\mathrm{d},I+\frac{1}{2}}(x)\,\mathrm{d}x-\phi_j\right]^2\\ \text{s.t.} & -\kappa(x_{I+\frac{1}{2}}+\epsilon)\phi_{\mathrm{d},I+\frac{1}{2}}(x_{I+\frac{1}{2}}+\epsilon)=\phi_{\mathrm{rg},1}\end{array}}
\quad\Leftrightarrow\quad
\boxed{\begin{array}{ll}\min & \|A_{I+\frac{1}{2}}\mathcal{R}_{I+\frac{1}{2}}-B_{I+\frac{1}{2}}\|_2^2\\ \text{s.t.} & C_{I+\frac{1}{2}}\mathcal{R}_{I+\frac{1}{2}}=D_{I+\frac{1}{2}}\end{array}}
$$

So,

$$
\begin{aligned}
&\omega_j\left[\frac{1}{h_j}\int_{C_j}\phi_{\mathrm{d},I+\frac{1}{2}}\,\mathrm{d}x-\phi_j\right]=0\\
\Leftrightarrow&\omega_j\left[\frac{1}{h_j}\int_{C_j}\phi_{\mathrm{d},I+\frac{1}{2}}\,\mathrm{d}x\right]=\omega_j\phi_j\\
\Leftrightarrow&\omega_j\left[\frac{1}{h_j}\int_{C_j}\left(\sum_{\alpha=0}^{\mathrm{d}}\mathcal{R}_{I+\frac{1}{2},\alpha}(x-x_{\mathrm{rg}})^{\alpha}\right)\mathrm{d}x\right]=\omega_j\phi_j\\
\Leftrightarrow&\omega_j\left[\frac{1}{h_j}\int_{C_j}\left(\mathcal{R}_{I+\frac{1}{2},0}(x-x_{\mathrm{rg}})^{0}+\mathcal{R}_{I+\frac{1}{2},1}(x-x_{\mathrm{rg}})^{1}+\cdots+\mathcal{R}_{I+\frac{1}{2},\mathrm{d}}(x-x_{\mathrm{rg}})^{\mathrm{d}}\right)\mathrm{d}x\right]=\omega_j\phi_j\\
\Leftrightarrow&\omega_j\left[\mathcal{R}_{I+\frac{1}{2},0}\underbrace{\left(\frac{1}{h_j}\int_{C_j}(x-x_{\mathrm{rg}})^{0}\,\mathrm{d}x\right)}_{A}+\mathcal{R}_{I+\frac{1}{2},1}\left(\frac{1}{h_j}\int_{C_j}(x-x_{\mathrm{rg}})^{1}\,\mathrm{d}x\right)+\cdots+\mathcal{R}_{I+\frac{1}{2},\mathrm{d}}\left(\frac{1}{h_j}\int_{C_j}(x-x_{\mathrm{rg}})^{\mathrm{d}}\,\mathrm{d}x\right)\right]=\underbrace{\omega_j\phi_j}_{B}
\end{aligned}
$$

and,

$$
\begin{aligned}
& -\kappa(x_{\mathrm{rg}}+\epsilon)\phi_{\mathrm{d},I+\frac{1}{2}}(x_{\mathrm{rg}}+\epsilon)=\phi_{\mathrm{lf},1} \\
\Leftrightarrow & \sum_{\alpha=0}^{\mathrm{d}} \mathcal{R}_{I+\frac{1}{2},\alpha}\,\alpha(x_{\mathrm{rg}}+\epsilon-x_{\mathrm{rg}})^{\alpha-1}=-\frac{\phi_{\mathrm{lf},1}}{\kappa(x_{\mathrm{rg}}+\epsilon)} \\
\Leftrightarrow & \mathcal{R}_{I+\frac{1}{2},0}\times 0+\mathcal{R}_{I+\frac{1}{2},1}\times 1\times(\epsilon)^0+\mathcal{R}_{I+\frac{1}{2},2}\times 2\times(\epsilon)^1+\cdots+\mathcal{R}_{I+\frac{1}{2},\mathrm{d}}\times\underbrace{\mathrm{d}\times(\epsilon)^{\mathrm{d}-1}}_{C}=\underbrace{-\frac{\phi_{\mathrm{lf},1}}{\kappa(x_{\mathrm{rg}}+\epsilon)}}_{D}
\end{aligned}
$$

The matrices are:

$$
\begin{aligned}
A_{I+\frac{1}{2}} &= [A_{I+\frac{1}{2},j\alpha}]\in\mathcal{M}_{n_S\times(\mathrm{d}+1)}(\mathbb{R})=\omega_j\frac{1}{h_j}\int_{C_j}(x-x_{I+\frac{1}{2}})^{\alpha}\,\mathrm{d}x \\
B_{I+\frac{1}{2}} &= [B_{I+\frac{1}{2},j}]\in\mathcal{M}_{n_S\times 1}(\mathbb{R})=\omega_j\phi_j \\
C_{I+\frac{1}{2}} &= [C_{I+\frac{1}{2},j\alpha}]\in\mathcal{M}_{1\times(\mathrm{d}+1)}(\mathbb{R})=\begin{cases}0 & \text{if} \quad \alpha=0 \\ 1 & \text{if} \quad \alpha=1 \\ \alpha\epsilon^{\alpha-1} & \text{if} \quad \alpha=2,\ldots,d\end{cases} \\
D_{I+\frac{1}{2}} &= [D_{I+\frac{1}{2},j}]\in\mathcal{M}_{1\times 1}(\mathbb{R})=-\frac{\phi_{\mathrm{rg},1}}{\kappa(x_{I+\frac{1}{2}}+\epsilon)}
\end{aligned}
$$

3 Results

3.1 Type II - Dirichlet + Neumann

In this tests we will consider:

- $\overline{\Omega} = [0, 1+\epsilon]$
- $\psi(x) = \exp(x)$
- $\psi(0) = 1$
- $\varphi_{\mathrm{n}2} = -\exp(1+\epsilon)$
- reconstructions of degree d and $d+1$

With $\epsilon = 0$:

degree d

	I	E_∞	O_∞
$\mathbb{P}_1$	10	1.74E−02	—
	20	4.13E−03	2.07
	30	1.80E−03	2.04
	40	1.00E−03	2.03
$\mathbb{P}_3$	10	3.62E−05	—
	20	2.29E−06	3.99
	30	4.54E−07	3.99
	40	1.44E−07	3.99
$\mathbb{P}_5$	10	1.05E−07	—
	20	2.12E−09	5.62
	30	2.01E−10	5.81
	40	3.71E−11	5.88

With $\epsilon = h^2$:

degree d

	I	E_∞	O_∞
$\mathbb{P}_1$	10	1.51E−02	—
	20	3.95E−03	1.93
	30	1.78E−03	1.96
	40	1.01E−03	1.97
$\mathbb{P}_3$	10	1.45E−04	—
	20	1.00E−05	3.86
	30	2.05E−06	3.92
	40	6.58E−07	3.95
$\mathbb{P}_5$	10	6.38E−07	—
	20	1.04E−08	5.94
	30	9.23E−10	5.97
	40	1.66E−10	5.97

degree $d+1$

	I	E_∞	O_∞
$\mathbb{P}_1$	10	9.52E−03	—
	20	2.51E−03	1.92
	30	1.14E−03	1.96
	40	6.45E−04	1.97
$\mathbb{P}_3$	10	9.73E−05	—
	20	6.69E−06	3.86
	30	1.36E−06	3.92
	40	4.38E−07	3.95
$\mathbb{P}_5$	10	1.30E−06	—
	20	2.31E−08	5.82
	30	2.11E−09	5.90
	40	3.83E−10	5.93

With $\epsilon = \frac{h}{2}$:

degree d

	I	E_∞	O_∞
$\mathbb{P}_1$	10	1.15E−01	—
	20	6.30E−02	0.87
	30	4.31E−02	0.93
	40	3.28E−02	0.96
$\mathbb{P}_3$	10	1.21E−03	—
	20	1.66E−04	2.87
	30	5.05E−05	2.93
	40	2.16E−05	2.95
$\mathbb{P}_5$	10	6.56E−06	—
	20	2.31E−07	4.83
	30	3.17E−08	4.90
	40	7.67E−09	4.93

degree $d+1$

	I	E_∞	O_∞
$\mathbb{P}_1$	10	9.10E−03	—
	20	2.28E−03	2.00
	30	1.02E−03	2.00
	40	5.72E−04	2.00
$\mathbb{P}_3$	10	4.61E−04	—
	20	6.66E−05	2.79
	30	2.07E−05	2.88
	40	8.94E−06	2.92
$\mathbb{P}_5$	10	9.72E−06	—
	20	3.63E−07	4.74
	30	5.06E−08	4.86
	40	1.24E−08	4.90

With $\epsilon = h$:

degree d

	I	E_∞	O_∞
$\mathbb{P}_1$	10	2.61E−01	—
	20	1.33E−01	0.97
	30	8.94E−02	0.98
	40	6.73E−02	0.99
$\mathbb{P}_3$	10	3.62E−03	—
	20	4.87E−04	2.90
	30	1.48E−04	2.94
	40	6.31E−05	2.96
$\mathbb{P}_5$	10	2.77E−05	—
	20	9.67E−07	4.84
	30	1.32E−07	4.91
	40	3.20E−08	4.93

degree $d+1$

	I	E_∞	O_∞
$\mathbb{P}_1$	10	1.50E−02	—
	20	3.94E−03	1.92
	30	1.78E−03	1.96
	40	1.01E−03	1.97
$\mathbb{P}_3$	10	1.14E−03	—
	20	1.74E−04	2.72
	30	5.48E−05	2.84
	40	2.39E−05	2.89
$\mathbb{P}_5$	10	3.19E−05	—
	20	1.21E−06	4.73
	30	1.69E−07	4.85
	40	4.14E−08	4.89

3.2 Type I - Dirichlet + Dirichlet

In this tests we will consider:

- $\overline{\Omega} = [0, 1 + \epsilon]$
- $\psi(x) = \exp(x)$
- $\psi(0) = 1$
- $\psi(1 + \epsilon) = \exp(1 + \epsilon)$
- reconstructions of degree d and $d + 1$

With $\epsilon = 0$:

degree d

	I	E_∞	O_∞
$\mathbb{P}_1$	10	2.32E−02	—
	20	6.00E−03	1.95
	30	2.70E−03	1.97
	40	1.53E−03	1.98
$\mathbb{P}_3$	10	4.03E−05	—
	20	2.72E−06	3.89
	30	5.53E−07	3.93
	40	1.77E−07	3.95
$\mathbb{P}_5$	10	1.11E−07	—
	20	2.03E−09	5.78
	30	1.89E−10	5.86
	40	3.46E−11	5.90

With $\epsilon = h^2$:

degree d

	I	E_∞	O_∞
$\mathbb{P}_1$	10	2.24E−02	—
	20	5.91E−03	1.92
	30	2.67E−03	1.96
	40	1.52E−03	1.97
$\mathbb{P}_3$	10	3.65E−05	—
	20	2.60E−06	3.81
	30	5.36E−07	3.89
	40	1.73E−07	3.93
$\mathbb{P}_5$	10	1.21E−07	—
	20	2.12E−09	5.84
	30	1.94E−10	5.89
	40	3.53E−11	5.92

With $\epsilon = \frac{h}{2}$:

degree d

	I	E_∞	O_∞
$\mathbb{P}_1$	10	1.75E−02	—
	20	4.58E−03	1.93
	30	2.07E−03	1.96
	40	1.17E−03	1.97
$\mathbb{P}_3$	10	1.89E−05	—
	20	1.13E−06	4.06
	30	2.31E−07	3.91
	40	7.67E−08	3.84
$\mathbb{P}_5$	10	2.43E−07	—
	20	4.55E−09	5.74
	30	4.26E−10	5.84
	40	7.83E−11	5.89

With $\epsilon = h$:

degree d

	I	E_∞	O_∞
$\mathbb{P}_1$	10	9.40E−03	—
	20	2.32E−03	2.02
	30	1.03E−03	2.01
	40	5.77E−04	2.01
$\mathbb{P}_3$	10	1.11E−04	—
	20	8.39E−06	3.73
	30	1.76E−06	3.85
	40	5.75E−07	3.89
$\mathbb{P}_5$	10	8.84E−07	—
	20	1.71E−08	5.69
	30	1.61E−09	5.82
	40	2.97E−10	5.88