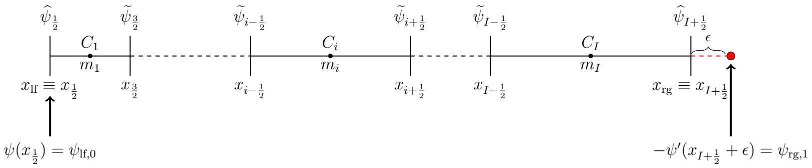
Pure Diffusion | Formulation

$$
\begin{cases}
-\psi'' = s & \text{in} \quad \Omega =]x_{\text{lf}}, x_{\text{rg}} + \epsilon[\\
\psi = \psi_{\text{lf},0} & \text{on} \quad x = x_{\text{lf}} \\
-\psi' = \psi_{\text{rg},1} & \text{on} \quad x = x_{\text{rg}} + \epsilon
\end{cases}
$$

Mesh



- C_i — cell i
- I — number of cells
- $x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}$ — boundary points of cell i
- h_i — length of cell i
- m_i — centroid of cell i

Polynomial Reconstructions | Inner Vertices

$$\psi_{i+\frac{1}{2},\mathrm{d}}(x) = \sum_{\alpha=0}^{\mathrm{d}} \mathcal{R}_{i+\frac{1}{2},\alpha}(x - x_{i+\frac{1}{2}})^{\alpha}$$

$$\min_{\mathcal{R}_{i+\frac{1}{2},0},\ldots,\mathcal{R}_{i+\frac{1}{2},\mathrm{d}}} \quad \sum_{j\in\widehat{S}_{i+\frac{1}{2}}} \omega_j \left[\frac{1}{h_j}\int_{c_j} \psi_{i+\frac{1}{2},\mathrm{d}}(x)\mathrm{d}x - \psi_j\right]^2$$

This will be needed to approximate $\mathbf{F}_{i+\frac{1}{2}} \approx \mathcal{F}_{i+\frac{1}{2}} = \widetilde{\psi}'_{i+\frac{1}{2}}(x_{i+\frac{1}{2}})$

Polynomial Reconstructions | Left Boundary

$$\psi_{\frac{1}{2},\mathrm{d}}(x) = \sum_{\alpha=0}^{\mathrm{d}} \mathcal{R}_{\frac{1}{2},\alpha}(x - x_{\mathrm{lf}})^{\alpha}$$

$$\min_{\mathcal{R}_{\frac{1}{2},0},\ldots,\mathcal{R}_{\frac{1}{2},\mathrm{d}}} \quad \sum_{j\in\widehat{S}_{\frac{1}{2}}} \omega_j \left[\frac{1}{h_j}\int_{c_j} \psi_{\frac{1}{2},\mathrm{d}}(x)\mathrm{d}x - \psi_j\right]^2$$

$$\text{s.t.} \quad \psi_{\frac{1}{2},\mathrm{d}}(x_{\mathrm{lf}}) = \psi_{\mathrm{lf},0}$$

This will be needed to approximate $\mathbf{F}_{\frac{1}{2}} \approx \mathcal{F}_{\frac{1}{2}} = \psi'_{\frac{1}{2}}(x_{\mathrm{lf}})$

Polynomial Reconstructions | Right Boundary

$$\psi_{I+\frac{1}{2},\mathsf{d}}(x) = \sum_{\alpha=0}^{\mathsf{d}} \mathcal{R}_{I+\frac{1}{2},\alpha}(x - x_{\mathsf{rg}})^{\alpha}$$

$$\begin{aligned} \min_{\mathcal{R}_{I+\frac{1}{2},0},\dots,\mathcal{R}_{I+\frac{1}{2},\mathsf{d}}} \quad & \sum_{j \in \widehat{S}_{I+\frac{1}{2}}} \omega_j \left[\frac{1}{h_j} \int_{c_j} \psi_{I+\frac{1}{2},\mathsf{d}}(x)\mathrm{d}x - \psi_j \right]^2 \\ \text{s.t.} \quad & -\psi'_{I+\frac{1}{2},\mathsf{d}}(x_{\mathsf{rg}} + \epsilon) = \psi_{\mathsf{rg},1} \end{aligned}$$

This will be needed to approximate $\mathbf{F}_{I+\frac{1}{2}} \approx \mathcal{F}_{I+\frac{1}{2}} = \widehat{\psi}'_{I+\frac{1}{2}}(x_{\mathsf{rg}})$

Tests

In this test we will consider:

- unstructured mesh
- $\overline{\Omega} = [0, 1 + \epsilon]$
- $\psi(x) = \exp(x)$
- $\psi(0) = 1$
- $\varphi_{\text{n2}} = -\exp(1 + \epsilon)$

Tests | $\epsilon = 0$ | d and $d+1$

	I	$\mathrm{E}_{0,I}(E_\infty)$	$\mathrm{E}_{0,I}(O_\infty)$
$\mathbb{P}_1$	10	2.18E−02	—
	20	5.50E−03	1.99
	30	2.56E−03	1.88
	40	1.60E−03	1.64
$\mathbb{P}_2$	10	4.28E−03	—
	20	2.05E−03	1.06
	30	1.70E−03	0.45
	40	4.96E−04	4.29
$\mathbb{P}_3$	10	4.11E−05	—
	20	2.29E−06	4.16
	30	7.21E−07	2.85
	40	1.76E−07	4.91
$\mathbb{P}_4$	10	1.07E−05	—
	20	8.77E−07	3.61
	30	2.26E−07	3.34
	40	6.37E−08	4.40
$\mathbb{P}_5$	10	7.38E−07	—
	20	1.01E−08	6.20
	30	1.05E−09	5.58
	40	2.04E−10	5.69

	I	$\mathrm{E}_{0,I}(E_\infty)$	$\mathrm{E}_{0,I}(O_\infty)$
$\mathbb{P}_1$	10	2.18E−02	—
	20	5.50E−03	1.99
	30	2.56E−03	1.88
	40	1.60E−03	1.64
$\mathbb{P}_2$	10	4.28E−03	—
	20	2.05E−03	1.06
	30	1.70E−03	0.45
	40	4.96E−04	4.29
$\mathbb{P}_3$	10	4.11E−05	—
	20	2.29E−06	4.16
	30	7.21E−07	2.85
	40	1.76E−07	4.91
$\mathbb{P}_4$	10	1.07E−05	—
	20	8.77E−07	3.61
	30	2.26E−07	3.34
	40	6.37E−08	4.40
$\mathbb{P}_5$	10	7.38E−07	—
	20	1.01E−08	6.20
	30	1.05E−09	5.58
	40	2.04E−10	5.69

Tests | $\epsilon = \frac{h}{2}$ | d and $d+1$

	I	$\mathrm{E}_{0,I}(E_\infty)$	$\mathrm{E}_{0,I}(O_\infty)$
$\mathbb{P}_1$	10	1.22E−01	—
	20	6.44E−02	0.92
	30	4.37E−02	0.95
	40	3.30E−02	0.98
$\mathbb{P}_2$	10	4.58E−03	—
	20	2.19E−03	1.07
	30	1.83E−03	0.44
	40	1.46E−04	8.79
$\mathbb{P}_3$	10	6.19E−05	—
	20	5.52E−06	3.49
	30	7.02E−07	5.09
	40	2.41E−07	3.72
$\mathbb{P}_4$	10	6.79E−06	—
	20	3.29E−07	4.37
	30	5.89E−08	4.25
	40	1.03E−08	6.07
$\mathbb{P}_5$	10	5.66E−07	—
	20	8.38E−09	6.08
	30	9.61E−10	5.34
	40	1.87E−10	5.68

	I	$\mathrm{E}_{0,I}(E_\infty)$	$\mathrm{E}_{0,I}(O_\infty)$
$\mathbb{P}_1$	10	2.02E−02	—
	20	5.28E−03	1.94
	30	2.45E−03	1.89
	40	1.46E−03	1.80
$\mathbb{P}_2$	10	3.29E−03	—
	20	2.15E−03	0.61
	30	1.82E−03	0.40
	40	1.30E−04	9.19
$\mathbb{P}_3$	10	4.21E−05	—
	20	2.34E−06	4.17
	30	7.18E−07	2.92
	40	1.77E−07	4.87
$\mathbb{P}_4$	10	5.24E−06	—
	20	2.05E−07	4.68
	30	6.60E−08	2.79
	40	1.22E−08	5.86
$\mathbb{P}_5$	10	5.45E−07	—
	20	8.16E−09	6.06
	30	9.52E−10	5.30
	40	1.86E−10	5.68

Tests | $\epsilon = h$ | d and $d+1$

	I	$\mathrm{E}_{0,I}(E_\infty)$	$\mathrm{E}_{0,I}(O_\infty)$
$\mathbb{P}_1$	10	2.66E−01	—
	20	1.34E−01	0.98
	30	9.01E−02	0.99
	40	6.75E−02	1.00
$\mathbb{P}_2$	10	5.97E−03	—
	20	2.26E−03	1.40
	30	1.84E−03	0.51
	40	1.90E−04	7.89
$\mathbb{P}_3$	10	8.01E−05	—
	20	6.90E−06	3.54
	30	7.87E−07	5.35
	40	2.99E−07	3.37
$\mathbb{P}_4$	10	8.09E−06	—
	20	3.70E−07	4.45
	30	5.78E−08	4.58
	40	1.09E−08	5.80
$\mathbb{P}_5$	10	5.69E−07	—
	20	8.42E−09	6.08
	30	9.63E−10	5.35
	40	1.88E−10	5.69

	I	$\mathrm{E}_{0,I}(E_\infty)$	$\mathrm{E}_{0,I}(O_\infty)$
$\mathbb{P}_1$	10	2.02E−02	—
	20	5.27E−03	1.93
	30	2.45E−03	1.89
	40	1.46E−03	1.80
$\mathbb{P}_2$	10	3.33E−03	—
	20	2.15E−03	0.63
	30	1.83E−03	0.41
	40	1.31E−04	9.15
$\mathbb{P}_3$	10	4.22E−05	—
	20	2.35E−06	4.17
	30	7.18E−07	2.92
	40	1.77E−07	4.87
$\mathbb{P}_4$	10	5.23E−06	—
	20	2.07E−07	4.66
	30	6.60E−08	2.81
	40	1.24E−08	5.82
$\mathbb{P}_5$	10	5.45E−07	—
	20	8.16E−09	6.06
	30	9.52E−10	5.30
	40	1.86E−10	5.68

Tests | $\epsilon = h^2$ | d and $d+1$

	I	$\mathrm{E}_{0,I}(E_\infty)$	$\mathrm{E}_{0,I}(O_\infty)$
$\mathbb{P}_1$	10	1.94E−02	—
	20	5.19E−03	1.90
	30	2.43E−03	1.87
	40	1.45E−03	1.78
$\mathbb{P}_2$	10	3.10E−03	—
	20	1.95E−03	0.67
	30	1.74E−03	0.27
	40	3.56E−04	5.53
$\mathbb{P}_3$	10	4.11E−05	—
	20	3.47E−06	3.56
	30	7.10E−07	3.91
	40	1.74E−07	4.90
$\mathbb{P}_4$	10	7.07E−06	—
	20	4.68E−07	3.92
	30	9.43E−08	3.95
	40	2.54E−08	4.55
$\mathbb{P}_5$	10	5.80E−07	—
	20	8.49E−09	6.09
	30	9.69E−10	5.35
	40	1.89E−10	5.69

	I	$\mathrm{E}_{0,I}(E_\infty)$	$\mathrm{E}_{0,I}(O_\infty)$
$\mathbb{P}_1$	10	2.06E−02	—
	20	5.34E−03	1.95
	30	2.46E−03	1.91
	40	1.47E−03	1.79
$\mathbb{P}_2$	10	3.09E−03	—
	20	2.05E−03	0.60
	30	1.79E−03	0.33
	40	1.81E−04	7.97
$\mathbb{P}_3$	10	4.20E−05	—
	20	2.34E−06	4.17
	30	7.18E−07	2.91
	40	1.77E−07	4.88
$\mathbb{P}_4$	10	5.34E−06	—
	20	2.27E−07	4.55
	30	5.74E−08	3.39
	40	1.42E−08	4.85
$\mathbb{P}_5$	10	5.53E−07	—
	20	8.23E−09	6.07
	30	9.56E−10	5.31
	40	1.87E−10	5.68