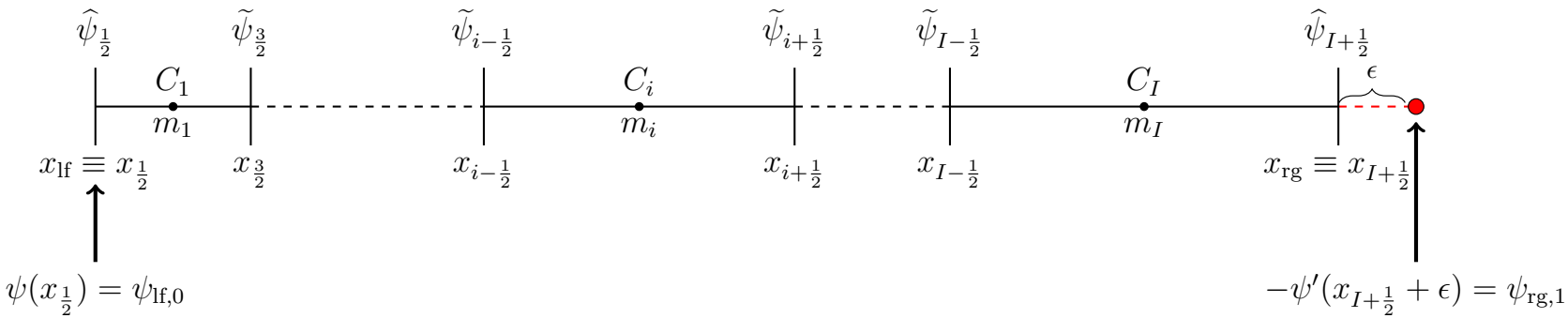
Pure Diffusion | Formulation

$$\begin{cases} -\psi'' = s & \text{in} \quad \Omega =]x_{\text{lf}}, x_{\text{rg}} + \epsilon[\\ \psi = \psi_{\text{lf},0} & \text{on} \quad x = x_{\text{lf}} \\ -\psi' = \psi_{\text{rg},1} & \text{on} \quad x = x_{\text{rg}} + \epsilon \end{cases}$$

Mesh



- C_i — cell i
- I — number of cells
- $x_{i-\frac{1}{2}}$, $x_{i+\frac{1}{2}}$ — boundary points of cell i
- h_i — length of cell i
- m_i — centroid of cell i

Polynomial Reconstructions | Inner Vertices

$$\psi_{i+\frac{1}{2},\mathrm{d}}(x) = \sum_{\alpha=0}^{\mathrm{d}} \mathcal{R}_{i+\frac{1}{2},\alpha}(x - x_{i+\frac{1}{2}})^{\alpha}$$

$$\min_{\mathcal{R}_{i+\frac{1}{2},0},\dots,\mathcal{R}_{i+\frac{1}{2},\mathrm{d}}} \quad \sum_{j \in \widehat{S}_{i+\frac{1}{2}}} \omega_j \left[\frac{1}{h_j} \int_{c_j} \psi_{i+\frac{1}{2},\mathrm{d}}(x)\mathrm{d}x - \psi_j \right]^2$$

This will be needed to approximate $\mathbf{F}_{i+\frac{1}{2}} \approx \mathcal{F}_{i+\frac{1}{2}} = \widetilde{\psi}'_{i+\frac{1}{2}}(x_{i+\frac{1}{2}})$

Polynomial Reconstructions | Left Boundary

$$\psi_{\frac{1}{2},\mathrm{d}}(x) = \sum_{\alpha=0}^{\mathrm{d}} \mathcal{R}_{\frac{1}{2},\alpha}(x - x_{\mathrm{lf}})^{\alpha}$$

$$\min_{\mathcal{R}_{\frac{1}{2},0},\ldots,\mathcal{R}_{\frac{1}{2},\mathrm{d}}} \quad \sum_{j \in \widehat{S}_{\frac{1}{2}}} \omega_j \left[\frac{1}{h_j} \int_{c_j} \psi_{\frac{1}{2},\mathrm{d}}(x)\mathrm{d}x - \psi_j \right]^2$$

$$\text{s.t.} \quad \psi_{\frac{1}{2},\mathrm{d}}(x_{\mathrm{lf}}) = \psi_{\mathrm{lf},0}$$

This will be needed to approximate $\mathbf{F}_{\frac{1}{2}} \approx \mathcal{F}_{\frac{1}{2}} = \psi'_{\frac{1}{2}}(x_{\mathrm{lf}})$

Polynomial Reconstructions | Right Boundary

$$\psi_{I+\frac{1}{2},\mathrm{d}}(x) = \sum_{\alpha=0}^{\mathrm{d}} \mathcal{R}_{I+\frac{1}{2},\alpha}(x - x_{\mathrm{rg}})^{\alpha}$$

$$\min_{\mathcal{R}_{I+\frac{1}{2},0},\ldots,\mathcal{R}_{I+\frac{1}{2},\mathrm{d}}} \quad \sum_{j \in \widehat{S}_{I+\frac{1}{2}}} \omega_j \left[\frac{1}{h_j} \int_{c_j} \psi_{I+\frac{1}{2},\mathrm{d}}(x)\mathrm{d}x - \psi_j \right]^2$$

$$\text{s.t.} \quad -\psi'_{I+\frac{1}{2},\mathrm{d}}(x_{\mathrm{rg}} + \epsilon) = \psi_{\mathrm{rg},1}$$

This will be needed to approximate $\mathbf{F}_{I+\frac{1}{2}} \approx \mathcal{F}_{I+\frac{1}{2}} = \widehat{\psi}'_{I+\frac{1}{2}}(x_{\mathrm{rg}})$

Tests

In this test we will consider:

- $\overline{\Omega} = [0, 1 + \epsilon]$
- $\psi(x) = \exp(x)$
- $\psi(0) = 1$
- $\varphi_{n2} = -\exp(1 + \epsilon)$

Tests | $\epsilon = 0$

	I	$\mathrm{E}_{0,I}(E_\infty)$	$\mathrm{E}_{0,I}(O_\infty)$
$\mathbb{P}_1$	10	1.09E−02	—
	20	2.68E−03	2.02
	30	1.19E−03	2.01
	40	6.67E−04	2.01
$\mathbb{P}_2$	10	4.93E−03	—
	20	1.38E−03	1.83
	30	8.01E−04	1.35
	40	3.88E−04	2.52
$\mathbb{P}_3$	10	2.99E−05	—
	20	1.93E−06	3.95
	30	3.86E−07	3.97
	40	1.23E−07	3.98
$\mathbb{P}_4$	10	1.15E−05	—
	20	1.20E−06	3.26
	30	2.00E−07	4.42
	40	6.69E−08	3.80
$\mathbb{P}_5$	10	9.53E−08	—
	20	2.00E−09	5.58
	30	1.91E−10	5.79
	40	3.53E−11	5.86

Tests | $\epsilon = h$

	I	$\mathrm{E}_{0,I}(E_\infty)$	$\mathrm{E}_{0,I}(O_\infty)$
$\mathbb{P}_1$	10	2.61E−01	—
	20	1.33E−01	0.97
	30	8.94E−02	0.98
	40	6.73E−02	0.99
$\mathbb{P}_2$	10	9.72E−03	—
	20	1.38E−03	2.81
	30	5.72E−04	2.17
	40	2.01E−04	3.64
$\mathbb{P}_3$	10	1.78E−04	—
	20	1.30E−05	3.78
	30	2.70E−06	3.87
	40	8.78E−07	3.91
$\mathbb{P}_4$	10	1.90E−05	—
	20	1.03E−06	4.20
	30	1.05E−07	5.62
	40	2.63E−08	4.82
$\mathbb{P}_5$	10	4.00E−07	—
	20	7.42E−09	5.75
	30	6.91E−10	5.86
	40	1.27E−10	5.90

Tests | $\epsilon = h$ | $d + 1$

	I	$\mathrm{E}_{0,I}(E_\infty)$	$\mathrm{E}_{0,I}(O_\infty)$
$\mathbb{P}_1$	10	8.99E−03	—
	20	2.27E−03	1.98
	30	1.01E−03	1.99
	40	5.71E−04	2.00
$\mathbb{P}_2$	10	2.42E−03	—
	20	4.02E−04	2.59
	30	2.55E−04	1.13
	40	6.98E−05	4.50
$\mathbb{P}_3$	10	1.90E−05	—
	20	1.13E−06	4.07
	30	2.21E−07	4.04
	40	6.93E−08	4.02
$\mathbb{P}_4$	10	5.93E−06	—
	20	5.85E−07	3.34
	30	3.34E−08	7.06
	40	7.62E−09	5.13
$\mathbb{P}_5$	10	1.62E−07	—
	20	2.96E−09	5.78
	30	2.75E−10	5.86
	40	5.03E−11	5.90

Tests | $\epsilon = 2h$

	I	$\mathrm{E}_{0,I}(E_\infty)$	$\mathrm{E}_{0,I}(O_\infty)$
$\mathbb{P}_1$	10	5.61E−01	—
	20	2.76E−01	1.02
	30	1.83E−01	1.01
	40	1.37E−01	1.01
$\mathbb{P}_2$	10	1.68E−02	—
	20	2.23E−03	2.91
	30	7.88E−04	2.57
	40	3.04E−04	3.31
$\mathbb{P}_3$	10	2.66E−04	—
	20	1.82E−05	3.87
	30	3.74E−06	3.90
	40	1.21E−06	3.93
$\mathbb{P}_4$	10	2.45E−05	—
	20	1.22E−06	4.33
	30	1.32E−07	5.48
	40	3.30E−08	4.82
$\mathbb{P}_5$	10	5.28E−07	—
	20	9.72E−09	5.76
	30	9.01E−10	5.86
	40	1.65E−10	5.90

Tests | $\epsilon = 2h$ | $d + 1$

	I	$\mathrm{E}_{0,I}(E_\infty)$	$\mathrm{E}_{0,I}(O_\infty)$
$\mathbb{P}_1$	10	8.89E−03	—
	20	2.27E−03	1.97
	30	1.01E−03	1.99
	40	5.71E−04	1.99
$\mathbb{P}_2$	10	2.47E−03	—
	20	4.11E−04	2.59
	30	2.62E−04	1.11
	40	7.24E−05	4.47
$\mathbb{P}_3$	10	1.88E−05	—
	20	1.13E−06	4.06
	30	2.20E−07	4.03
	40	6.93E−08	4.02
$\mathbb{P}_4$	10	6.21E−06	—
	20	5.76E−07	3.43
	30	3.39E−08	6.99
	40	7.81E−09	5.11
$\mathbb{P}_5$	10	1.75E−07	—
	20	3.19E−09	5.78
	30	2.96E−10	5.87
	40	5.41E−11	5.90

Tests | $\epsilon = 3h$

	I	$\mathrm{E}_{0,I}(E_\infty)$	$\mathrm{E}_{0,I}(O_\infty)$
$\mathbb{P}_1$	10	8.93E−01	—
	20	4.26E−01	1.07
	30	2.80E−01	1.04
	40	2.08E−01	1.03
$\mathbb{P}_2$	10	2.33E−02	—
	20	3.01E−03	2.95
	30	9.98E−04	2.72
	40	3.98E−04	3.19
$\mathbb{P}_3$	10	3.55E−04	—
	20	2.33E−05	3.93
	30	4.77E−06	3.91
	40	1.53E−06	3.94
$\mathbb{P}_4$	10	3.15E−05	—
	20	1.42E−06	4.48
	30	1.60E−07	5.38
	40	4.00E−08	4.81
$\mathbb{P}_5$	10	6.51E−07	—
	20	1.19E−08	5.77
	30	1.10E−09	5.87
	40	2.01E−10	5.91

Tests | $\epsilon = 3h$ | $d + 1$

	I	$\mathrm{E}_{0,I}(E_\infty)$	$\mathrm{E}_{0,I}(O_\infty)$
$\mathbb{P}_1$	10	8.81E−03	—
	20	2.26E−03	1.96
	30	1.01E−03	1.99
	40	5.70E−04	1.99
$\mathbb{P}_2$	10	2.49E−03	—
	20	4.15E−04	2.59
	30	2.65E−04	1.10
	40	7.35E−05	4.46
$\mathbb{P}_3$	10	1.88E−05	—
	20	1.13E−06	4.05
	30	2.23E−07	3.99
	40	7.25E−08	3.91
$\mathbb{P}_4$	10	6.44E−06	—
	20	5.74E−07	3.49
	30	3.45E−08	6.93
	40	7.97E−09	5.09
$\mathbb{P}_5$	10	1.83E−07	—
	20	3.32E−09	5.78
	30	3.07E−10	5.87
	40	5.62E−11	5.91

Tests | $\epsilon = h^2$

	I	$\mathrm{E}_{0,I}(E_\infty)$	$\mathrm{E}_{0,I}(O_\infty)$
$\mathbb{P}_1$	10	1.51E−02	—
	20	3.95E−03	1.93
	30	1.78E−03	1.96
	40	1.01E−03	1.97
$\mathbb{P}_2$	10	2.56E−03	—
	20	9.50E−04	1.43
	30	6.20E−04	1.05
	40	2.93E−04	2.60
$\mathbb{P}_3$	10	6.39E−05	—
	20	4.51E−06	3.82
	30	9.26E−07	3.91
	40	2.99E−07	3.93
$\mathbb{P}_4$	10	1.34E−05	—
	20	9.97E−07	3.75
	30	1.29E−07	5.05
	40	3.95E−08	4.11
$\mathbb{P}_5$	10	2.40E−07	—
	20	4.23E−09	5.83
	30	3.86E−10	5.90
	40	7.02E−11	5.93

Tests | $\epsilon = h^2$ | $d + 1$

	I	$\mathrm{E}_{0,I}(E_\infty)$	$\mathrm{E}_{0,I}(O_\infty)$
$\mathbb{P}_1$	10	9.29E−03	—
	20	2.31E−03	2.01
	30	1.02E−03	2.01
	40	5.75E−04	2.00
$\mathbb{P}_2$	10	2.89E−03	—
	20	7.20E−04	2.00
	30	4.91E−04	0.95
	40	1.97E−04	3.17
$\mathbb{P}_3$	10	1.94E−05	—
	20	1.15E−06	4.08
	30	2.22E−07	4.04
	40	6.98E−08	4.03
$\mathbb{P}_4$	10	6.57E−06	—
	20	7.40E−07	3.15
	30	6.79E−08	5.89
	40	2.03E−08	4.20
$\mathbb{P}_5$	10	1.27E−07	—
	20	2.18E−09	5.86
	30	1.98E−10	5.92
	40	3.57E−11	5.94