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Very High Order Finite Volume Approximation for the 1D Biharmonic Operator

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Outline

Biharmonic Operator
Formulation

$$
\begin{aligned}
-\mu\psi^{(4)} &= f && \text{in } \Omega =]x_{\frac{1}{2}}, x_{I+\frac{1}{2}}[\\
\psi &= \psi_{\text{lf},0} && \text{at } x = x_{\frac{1}{2}} \\
\psi^{(1)} &= \psi_{\text{lf},1} && \text{at } x = x_{\frac{1}{2}} \\
\psi &= \psi_{\text{rg},0} && \text{at } x = x_{I+\frac{1}{2}} \\
\psi^{(1)} &= \psi_{\text{rg},1} && \text{at } x = x_{I+\frac{1}{2}}
\end{aligned}
$$

where,

- if μ is constant $\rightarrow -\mu\psi^{(4)} = f$ in $\Omega =]x_{\frac{1}{2}}, x_{I+\frac{1}{2}}[$

- Integrating the equation in the cells of the mesh c_i, $i = 1, \ldots, I$

$$-\mu\psi^{(4)} = f \Rightarrow -\int_{c_i} \mu\psi^{(4)} \mathrm{d}x = \int_{c_i} f \mathrm{d}x \Leftrightarrow$$

$$-(\underbrace{\mu\psi^{(3)}|_{x_{i+\frac{1}{2}}}}_{\mathcal{T}_{i+\frac{1}{2}}} - \underbrace{\mu\psi^{(3)}|_{x_{i-\frac{1}{2}}}}_{\mathcal{T}_{i-\frac{1}{2}}}) = \int_{c_i} f \mathrm{d}x \Leftrightarrow$$

$$\boxed{-(\mathcal{T}_{i+\frac{1}{2}} - \mathcal{T}_{i-\frac{1}{2}}) = h_i f_i, \quad i = 1, \ldots, I}$$

- Goal: approximate $\psi^{(3)}\left(x_{i+\frac{1}{2}}\right)$, $i = 0, \ldots, I$
- PRO method (**P**olynomial **R**econstruction **O**perator)

- Conservation of $\psi(x_{\frac{1}{2}}) = \psi_{\mathsf{lf},0}$

$$\psi_{\mathsf{d},\frac{1}{2}}(x) = \sum_{\alpha=0}^{\mathsf{d}} \widehat{\mathcal{R}}_{\frac{1}{2},\alpha}(x - x_{\frac{1}{2}})^{\alpha}$$

- Coefficients $\widehat{\mathcal{R}}_{\frac{1}{2}} = (\widehat{\mathcal{R}}_{\frac{1}{2},0}, \ldots, \widehat{\mathcal{R}}_{\frac{1}{2},\mathsf{d}})^T$ is the solution of the constrained linear least squares problem

$$\boxed{\begin{aligned} \min_{\widehat{\mathcal{R}}_{\frac{1}{2},0},\ldots,\widehat{\mathcal{R}}_{\frac{1}{2},\mathsf{d}}} \quad & \sum_{j\in\widehat{S}_{\frac{1}{2}}} \omega_j \left[\frac{1}{h_j}\int_{c_j} \psi_{\mathsf{d},\frac{1}{2}}(x)\mathsf{d}x - \psi_j\right]^2 \\ \text{s.t.} \quad & \psi_{\mathsf{d},\frac{1}{2}}(x_{\frac{1}{2}}) = \psi_{\mathsf{lf},0} \end{aligned}} \quad \Leftrightarrow$$

$$\boxed{\begin{aligned} \min_{\widehat{\mathcal{R}}_{\frac{1}{2},0},\ldots,\widehat{\mathcal{R}}_{\frac{1}{2},\mathsf{d}}} \quad & \|A_{\frac{1}{2}}\widehat{\mathcal{R}}_{\frac{1}{2}} - B_{\frac{1}{2}}\|_2^2 \\ \text{s.t.} \quad & C_{\frac{1}{2}}\widehat{\mathcal{R}}_{\frac{1}{2}} = D_{\frac{1}{2}} \end{aligned}}$$

Polynomial Reconstructions
Left Boundary (ii)

- Where the matrices,

$$A_{\frac{1}{2}} = [A_{\frac{1}{2},j\alpha}] \in \mathcal{M}_{n_S\times(\mathsf{d}+1)}(\mathbb{R}), A_{\frac{1}{2},j\alpha} = \omega_j \frac{1}{h_j}\int_{c_j}(x-x_{\frac{1}{2}})^{\alpha-1}\mathsf{d}x$$
$$B_{\frac{1}{2}} = [B_{\frac{1}{2},j}] \in \mathcal{M}_{n_S\times 1}(\mathbb{R}), B_{\frac{1}{2},j} = \omega_j\psi_j$$
$$C_{\frac{1}{2}} = [C_{\frac{1}{2},j\alpha}] \in \mathcal{M}_{1\times(\mathsf{d}+1)}(\mathbb{R}), C_{\frac{1}{2},j\alpha} = \begin{cases}1 & \text{if } \alpha=1\\ 0 & \text{if } \alpha=2,\ldots,\mathsf{d}+1\end{cases}$$
$$D_{\frac{1}{2}} = [D_{\frac{1}{2},j}] \in \mathcal{M}_{1\times 1}(\mathbb{R}), D_{\frac{1}{2},j} = \psi_{\mathsf{lf},0}$$

- The procedure for the right boundary is similar

- Conservation of ψ_1 and “strong” conservation of $\psi^{(1)}(x_{\frac{1}{2}}) = \psi_{\mathsf{lf},1}$

$$\psi_{\mathsf{d},1}(x) = \sum_{\alpha=0}^{\mathsf{d}} \widehat{\mathcal{R}}_{1,\alpha}(x - m_1)^{\alpha}$$

- Coefficients $\widehat{\mathcal{R}}_1 = (\widehat{\mathcal{R}}_{1,0}, \ldots, \widehat{\mathcal{R}}_{1,\mathsf{d}})^T$ is the solution of the constrained linear least squares problem

$$\boxed{\begin{aligned} \min_{\widehat{\mathcal{R}}_{1,0},\ldots,\widehat{\mathcal{R}}_{1,\mathsf{d}}} \quad & \sum_{j\in\widehat{S}_1} \omega_j \left[\frac{1}{h_j} \int_{c_j} \psi_{\mathsf{d},1}(x)\mathsf{d}x - \psi_j \right]^2 \\ \text{s.t.} \quad & \frac{1}{h_1} \int_{c_1} \psi_{\mathsf{d},1}(x)\mathrm{d}x = \psi_1 \\ & \psi^{(1)}_{\mathsf{d},1}(x_{\frac{1}{2}}) = \psi_{\mathsf{lf},1} \end{aligned}} \quad \Leftrightarrow$$

$$\boxed{\begin{aligned} \min_{\widehat{\mathcal{R}}_{1,0},\ldots,\widehat{\mathcal{R}}_{1,\mathsf{d}}} \quad & \|A_1 \widehat{\mathcal{R}}_1 - B_1\|_2^2 \\ \text{s.t.} \quad & C_1 \widehat{\mathcal{R}}_1 = D_1 \end{aligned}}$$

- Where the matrices,

$$A_1 = [A_{1,j\alpha}] \in \mathcal{M}_{n_S\times(\mathsf{d}+1)}(\mathbb{R}), A_{1,j\alpha} = \omega_j \frac{1}{h_j}\int_{c_j}(x-m_1)^{\alpha-1}\mathsf{d}x$$

$$B_1 = [B_{1,j}] \in \mathcal{M}_{n_S\times 1}(\mathbb{R}), B_{1,j} = \omega_j\psi_j$$

$$C_1 = [C_{1,j\alpha}] \in \mathcal{M}_{2\times(\mathsf{d}+1)}(\mathbb{R}), C_{1,j\alpha} = \begin{cases} \frac{1}{h_1}\int_{c_1}(x-m_1)^{\alpha-1}\mathsf{d}x & \text{if } j=1 \\ 0 & \text{if } j=2 \text{ and } \alpha=1 \\ (\alpha-1)(x_{\frac{1}{2}}-m_1)^{\alpha-2} & \text{if } j=2 \text{ and } \alpha=2,\dots,\mathsf{d}+1 \end{cases}$$

$$D_1 = [D_{1,j}] \in \mathcal{M}_{2\times 1}(\mathbb{R}), D_{1,j} = \begin{cases} \psi_1 & \text{if } j=1 \\ \psi_{\mathsf{lf},1} & \text{if } j=2 \end{cases}$$

- The procedure for the last cell is similar

- Conservation of ψ_i

$$\psi_{\mathsf{d},i}(x) = \sum_{\alpha=0}^{\mathsf{d}} \widehat{\mathcal{R}}_{i,\alpha}(x - m_i)^{\alpha}$$

- Coefficients $\widehat{\mathcal{R}}_i = (\widehat{\mathcal{R}}_{i,0}, \ldots, \widehat{\mathcal{R}}_{i,\mathsf{d}})^T$ is the solution of the constrained linear least squares problem

$$\boxed{\begin{aligned} \min_{\widehat{\mathcal{R}}_{i,0},\ldots,\widehat{\mathcal{R}}_{i,\mathsf{d}}} \quad & \sum_{j \in \widehat{S}_i} \omega_j \left[\frac{1}{h_j} \int_{c_j} \psi_{\mathsf{d},i}(x) \mathsf{d}x - \psi_j \right]^2 \\ \text{s.t.} \quad & \frac{1}{h_i} \int_{c_i} \psi_{\mathsf{d},i}(x) \mathrm{d}x = \psi_i \end{aligned}}$$

- Where the matrices,

$$A_i = [A_{i,j\alpha}] \in \mathcal{M}_{n_S \times (\mathsf{d}+1)}(\mathbb{R}), A_{i,j\alpha} = \omega_j \frac{1}{h_j} \int_{c_j} (x - m_i)^{\alpha-1} \mathsf{d}x$$

$$B_i = [B_{i,j}] \in \mathcal{M}_{n_S \times 1}(\mathbb{R}), B_{i,j} = \omega_j \psi_j$$

$$C_i = [C_{i,j\alpha}] \in \mathcal{M}_{1 \times (\mathsf{d}+1)}(\mathbb{R}), C_{i,j\alpha} = \frac{1}{h_i} \int_{c_i} (x - m_i)^{\alpha-1} \mathsf{d}x$$

$$D_i = [D_{i,j}] \in \mathcal{M}_{1 \times 1}(\mathbb{R}), D_{i,j} = \psi_i$$

PRO Scheme

- Fluxes

$$\mathcal{T}_{i+\frac{1}{2}} = \begin{cases} \mu \widehat{\psi}^{(3)}_{\mathsf{d},\frac{1}{2}}(x_{\frac{1}{2}}) & \text{if } i = 0 \\ \mu \dfrac{\widehat{\psi}^{(3)}_{\mathsf{d},i}(x_{i+\frac{1}{2}}) + \widehat{\psi}^{(3)}_{\mathsf{d},i+1}(x_{i+\frac{1}{2}})}{2} & \text{if } i = 1,\ldots,I-1 \\ \mu \widehat{\psi}^{(3)}_{\mathsf{d},I+\frac{1}{2}}(x_{I+\frac{1}{2}}) & \text{if } i = I \end{cases}$$

Numerical Tests (i)

- $\psi(x) = \exp(x)$
- $\psi_l = 1$
- $\psi_{ll} = 1$
- $\psi_r = \exp(1)$
- $\psi_{rr} = \exp(1)$
- $f(x) = -\exp(x)$
- $\Omega \in [0, 1]$

Numerical Tests (ii)

	I	$\omega = 1\|1$				$\omega = 1\|3$			
		$\mathsf{E}_{1,0}$	$\mathsf{O}_{1,0}$	$\mathsf{E}_{\infty,0}$	$\mathsf{O}_{\infty,0}$	$\mathsf{E}_{1,0}$	$\mathsf{O}_{1,0}$	$\mathsf{E}_{\infty,0}$	$\mathsf{O}_{\infty,0}$
$\mathbb{P}_3$(d)	80	2.64E−06	—	4.14E−06	—	2.03E−06	—	3.27E−06	—
	160	3.14E−07	3.07	4.90E−07	3.08	2.38E−07	3.09	3.82E−07	3.10
	240	8.79E−08	3.14	1.36E−07	3.16	6.53E−08	3.19	1.05E−07	3.19
	320	3.48E−08	3.22	5.37E−08	3.24	2.52E−08	3.30	4.08E−08	3.28

Numerical Tests (ii)

	I	$\omega = 1\|1$ $\mathrm{E}_{1,0}$	$\mathrm{O}_{1,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\omega = 1\|3$ $\mathrm{E}_{1,0}$	$\mathrm{O}_{1,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3$(d)	80	2.64E−06	—	4.14E−06	—	2.03E−06	—	3.27E−06	—
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	320	3.48E−08	3.22	5.37E−08	3.24	2.52E−08	3.30	4.08E−08	3.28
$\mathbb{P}_3$(d+1)	80	5.39E−07	—	8.53E−07	—	3.39E−07	—	4.91E−07	—
	160	4.71E−08	3.52	7.67E−08	3.48	2.52E−08	3.75	3.66E−08	3.74
	240	8.15E−09	4.33	1.71E−08	3.70	3.24E−09	5.06	7.45E−09	3.93
	320	2.13E−09	4.66	5.75E−09	3.79	2.56E−09	0.82	5.69E−09	0.94

Numerical Tests (iii)

- $\psi(x) = -\exp(x) + (3 - \mathsf{e})x^3 + (2\mathsf{e} - 5)x^2 + x + 1$
- $\psi_l = 0$
- $\psi_{ll} = 0$
- $\psi_r = 0$
- $\psi_{rr} = 0$
- $f(x) = \exp(x)$
- $\Omega \in [0, 1]$

Numerical Tests (iv)

	I	$\omega = 1\|1$				$\omega = 1\|3$			
		$\mathsf{E}_{1,0}$	$\mathsf{O}_{1,0}$	$\mathsf{E}_{\infty,0}$	$\mathsf{O}_{\infty,0}$	$\mathsf{E}_{1,0}$	$\mathsf{O}_{1,0}$	$\mathsf{E}_{\infty,0}$	$\mathsf{O}_{\infty,0}$
$\mathbb{P}_3$(d)	80	2.64E−06	—	4.14E−06	—	2.03E−06	—	3.27E−06	—
	160	3.14E−07	3.07	4.90E−07	3.08	2.38E−07	3.09	3.82E−07	3.10
	240	8.79E−08	3.14	1.36E−07	3.16	6.53E−08	3.19	1.05E−07	3.19
	320	3.48E−08	3.22	5.37E−08	3.23	2.52E−08	3.30	4.08E−08	3.28

Numerical Tests (iv)

	I	$\omega = 1\|1$				$\omega = 1\|3$			
		$\mathsf{E}_{1,0}$	$\mathsf{O}_{1,0}$	$\mathsf{E}_{\infty,0}$	$\mathsf{O}_{\infty,0}$	$\mathsf{E}_{1,0}$	$\mathsf{O}_{1,0}$	$\mathsf{E}_{\infty,0}$	$\mathsf{O}_{\infty,0}$
$\mathbb{P}_3$(d)	80	2.64E−06	—	4.14E−06	—	2.03E−06	—	3.27E−06	—
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$\mathbb{P}_3$(d+1)	80	5.39E−07	—	8.53E−07	—	3.39E−07	—	4.91E−07	—
	160	4.71E−08	3.52	7.67E−08	3.48	2.52E−08	3.75	3.66E−08	3.74
	240	8.14E−09	4.33	1.71E−08	3.70	3.24E−09	5.06	7.45E−09	3.93
	320	2.14E−09	4.64	5.73E−09	3.81	2.55E−09	0.83	5.68E−09	0.94

Conclusions and Further Work