In this tests we consider:

- $\psi(x) = \sin(\pi x)$
- $\psi_{\mathrm{l}} = 0$
- $\psi_{\mathrm{ll}} = \pi$
- $\psi_{\mathrm{r}} = 0$
- $\psi_{\mathrm{rr}} = -\pi$
- $g(x) = -\pi^4 \sin(\pi x)$
- the different PRO schemes are:
  - PRO1 — weak ($R = A^{\dagger}B$) and degree d
  - PRO2 — strong (constrained least squares) and degree d

Table 1: Numerical results of PRO1 scheme.

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	5.37E−03	—	4.42E−03	—	4.42E−03	—	4.42E−03	—
	40	7.55E−04	2.83	6.90E−04	2.68	6.90E−04	2.68	6.90E−04	2.68
	80	1.51E−04	2.32	1.47E−04	2.24	1.47E−04	2.24	1.47E−04	2.24
	160	3.53E−05	2.09	3.50E−05	2.07	3.50E−05	2.07	3.50E−05	2.07
	320	8.67E−06	2.02	8.65E−06	2.02	8.65E−06	2.02	8.65E−06	2.02
	640	2.16E−06	2.01	2.16E−06	2.00	2.16E−06	2.00	2.16E−06	2.00
$\mathbb{P}_5(6)$	20	2.68E−05	—	2.24E−05	—	2.24E−05	—	2.24E−05	—
	40	3.73E−07	6.17	4.59E−07	5.61	4.59E−07	5.61	4.59E−07	5.61
	80	5.88E−08	2.66	5.41E−08	3.08	5.41E−08	3.08	5.41E−08	3.08
	160	4.19E−09	3.81	3.79E−09	3.84	3.73E−09	3.86	3.73E−09	3.86
	320	4.12E−10	3.34	2.39E−10	3.99	1.63E−10	4.52	2.26E−10	4.04
	640	7.13E−10	↑	1.37E−09	↑	5.09E−09	↑	6.06E−09	↑

Table 2: Numerical results of PRO2 scheme.

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	5.37E−03	—	4.42E−03	—	4.42E−03	—	4.42E−03	—
	40	7.55E−04	2.83	6.90E−04	2.68	6.90E−04	2.68	6.90E−04	2.68
	80	1.51E−04	2.32	1.47E−04	2.24	1.47E−04	2.24	1.47E−04	2.24
	160	3.53E−05	2.09	3.50E−05	2.07	3.50E−05	2.07	3.50E−05	2.07
	320	8.67E−06	2.02	8.65E−06	2.02	8.65E−06	2.02	8.65E−06	2.02
	640	2.16E−06	2.01	2.16E−06	2.01	2.16E−06	2.00	2.16E−06	2.01
$\mathbb{P}_5(6)$	20	2.68E−05	—	2.24E−05	—	2.24E−05	—	2.24E−05	—
	40	3.73E−07	6.17	4.59E−07	5.61	4.59E−07	5.61	4.59E−07	5.61
	80	5.88E−08	2.66	5.41E−08	3.08	5.41E−08	3.08	5.41E−08	3.08
	160	4.11E−09	3.84	3.75E−09	3.85	3.77E−09	3.85	3.75E−09	3.85
	320	2.99E−10	3.78	2.60E−10	3.85	4.78E−10	2.98	2.61E−10	3.85
	640	2.17E−09	↑	1.24E−08	↑	1.93E−08	↑	1.24E−08	↑

In this tests we consider:

- $\psi(x) = \sin(2\pi x)$
- $\psi_{\mathrm{l}} = 0$
- $\psi_{\mathrm{ll}} = 2\pi$
- $\psi_{\mathrm{r}} = 0$
- $\psi_{\mathrm{rr}} = 2\pi$
- $g(x) = -16\pi^4 \sin(2\pi x)$
- the different PRO schemes are:
  - PRO1 — weak ($R = A^\dagger B$) and degree d
  - PRO2 — strong (constrained least squares) and degree d

Table 3: Numerical results of PRO1 scheme.

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	6.05E−02	—	4.96E−02	—	4.96E−02	—	4.96E−02	—
	40	6.90E−03	3.13	6.47E−03	2.94	6.47E−03	2.94	6.47E−03	2.94
	80	1.24E−03	2.47	1.23E−03	2.40	1.23E−03	2.40	1.23E−03	2.40
	160	2.82E−04	2.14	2.82E−04	2.12	2.82E−04	2.12	2.82E−04	2.12
	320	6.89E−05	2.03	6.88E−05	2.03	6.88E−05	2.03	6.88E−05	2.03
	640	1.71E−05	2.01	1.71E−05	2.01	1.71E−05	2.01	1.71E−05	2.01
$\mathbb{P}_5(6)$	20	3.65E−03	—	1.85E−03	—	1.85E−03	—	1.85E−03	—
	40	1.55E−05	7.88	1.95E−05	6.57	1.95E−05	6.57	1.95E−05	6.57
	80	1.54E−06	3.34	1.76E−06	3.47	1.76E−06	3.47	1.76E−06	3.47
	160	1.29E−07	3.57	1.20E−07	3.87	1.20E−07	3.87	1.20E−07	3.87
	320	8.81E−09	3.87	7.72E−09	3.96	7.78E−09	3.95	7.80E−09	3.95
	640	4.03E−09	1.13	3.61E−09	1.10	3.36E−09	1.21	5.43E−09	0.52

Table 4: Numerical results of PRO2 scheme.

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	6.05E−02	—	4.96E−02	—	4.96E−02	—	4.96E−02	—
	40	6.90E−03	3.13	6.47E−03	2.94	6.47E−03	2.94	6.47E−03	2.94
	80	1.24E−03	2.47	1.23E−03	2.40	1.23E−03	2.40	1.23E−03	2.40
	160	2.82E−04	2.14	2.82E−04	2.12	2.82E−04	2.12	2.82E−04	2.12
	320	6.89E−05	2.03	6.88E−05	2.03	6.88E−05	2.03	6.88E−05	2.03
	640	1.71E−05	2.01	1.71E−05	2.01	1.71E−05	2.01	1.71E−05	2.01
$\mathbb{P}_5(6)$	20	3.65E−03	—	1.85E−03	—	1.85E−03	—	1.85E−03	—
	40	1.55E−05	7.88	1.95E−05	6.57	1.95E−05	6.57	1.95E−05	6.57
	80	1.54E−06	3.34	1.76E−06	3.47	1.76E−06	3.47	1.76E−06	3.47
	160	1.29E−07	3.57	1.20E−07	3.87	1.20E−07	3.87	1.20E−07	3.87
	320	8.58E−09	3.91	7.74E−09	3.96	7.89E−09	3.93	7.74E−09	3.96
	640	1.23E−09	2.81	6.66E−09	0.22	2.20E−09	1.84	3.87E−09	1.00

In this tests we consider:

- $\psi(x) = \sin(3\pi x)$
- $\psi_{\mathrm{l}} = 0$
- $\psi_{\mathrm{ll}} = 3\pi$
- $\psi_{\mathrm{r}} = 0$
- $\psi_{\mathrm{rr}} = -3\pi$
- $g(x) = -81\pi^4 \sin(3\pi x)$
- the different PRO schemes are:
  - PRO1 — weak ($R = A^\dagger B$) and degree d
  - PRO2 — strong (constrained least squares) and degree d

Table 5: Numerical results of PRO1 scheme.

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	4.35E−01	—	3.01E−01	—	3.01E−01	—	3.01E−01	—
	40	3.00E−02	3.86	4.55E−02	2.73	4.55E−02	2.73	4.55E−02	2.73
	80	1.64E−02	0.88	1.74E−02	1.39	1.74E−02	1.39	1.74E−02	1.39
	160	4.66E−03	1.81	4.72E−03	1.88	4.72E−03	1.88	4.72E−03	1.88
	320	1.20E−03	1.96	1.21E−03	1.97	1.21E−03	1.97	1.21E−03	1.97
	640	3.03E−04	1.99	3.03E−04	1.99	3.03E−04	1.99	3.03E−04	1.99
$\mathbb{P}_5(6)$	20	8.55E−02	—	7.29E−02	—	7.29E−02	—	7.29E−02	—
	40	2.78E−03	4.94	2.31E−03	4.98	2.31E−03	4.98	2.31E−03	4.98
	80	1.06E−04	4.71	9.56E−05	4.59	9.56E−05	4.59	9.56E−05	4.59
	160	5.75E−06	4.21	5.19E−06	4.20	5.19E−06	4.20	5.19E−06	4.20
	320	3.43E−07	4.07	3.12E−07	4.06	3.11E−07	4.06	3.12E−07	4.06
	640	2.34E−08	3.87	1.74E−08	4.17	2.09E−08	3.90	2.98E−08	3.38

Table 6: Numerical results of PRO2 scheme.

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	4.35E−01	—	3.01E−01	—	3.01E−01	—	3.01E−01	—
	40	3.00E−02	3.86	4.55E−02	2.73	4.55E−02	2.73	4.55E−02	2.73
	80	1.64E−02	0.88	1.74E−02	1.39	1.74E−02	1.39	1.74E−02	1.39
	160	4.66E−03	1.81	4.72E−03	1.88	4.72E−03	1.88	4.72E−03	1.88
	320	1.20E−03	1.96	1.21E−03	1.97	1.21E−03	1.97	1.21E−03	1.97
	640	3.03E−04	1.99	3.03E−04	1.99	3.03E−04	1.99	3.03E−04	1.99
$\mathbb{P}_5(6)$	20	8.55E−02	—	7.29E−02	—	7.29E−02	—	7.29E−02	—
	40	2.78E−03	4.94	2.31E−03	4.98	2.31E−03	4.98	2.31E−03	4.98
	80	1.06E−04	4.71	9.56E−05	4.59	9.56E−05	4.59	9.56E−05	4.59
	160	5.75E−06	4.21	5.19E−06	4.20	5.19E−06	4.20	5.19E−06	4.20
	320	3.43E−07	4.07	3.12E−07	4.06	3.12E−07	4.06	3.12E−07	4.06
	640	2.13E−08	4.01	1.37E−08	4.51	1.37E−08	4.51	1.37E−08	4.51