In this tests we consider:

- $\psi(x) = x^4$
- $\psi_{\mathrm{l}} = 0$
- $\psi_{\mathrm{r}} = 1$
- $\psi_{\mathrm{ll}} = 0$
- $\psi_{\mathrm{rr}} = 4$
- $g(x) = -24$

Table 1: Numerical results of PRO1 scheme.

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	3.33E−03	—	2.51E−03	—	2.51E−03	—	2.51E−03	—
	40	4.31E−04	2.95	3.21E−04	2.97	3.21E−04	2.97	3.21E−04	2.97
	80	5.46E−05	2.98	4.04E−05	2.99	4.04E−05	2.99	4.04E−05	2.99
	160	6.86E−06	2.99	5.07E−06	2.99	5.07E−06	2.99	5.07E−06	2.99
	320	8.59E−07	3.00	6.35E−07	3.00	6.35E−07	3.00	6.35E−07	3.00
	640	1.08E−07	3.00	7.85E−08	3.02	7.78E−08	3.03	7.92E−08	3.00
$\mathbb{P}_5(6)$	20	9.04E−15	—	8.46E−14	—	1.06E−14	—	7.08E−14	—
	40	1.90E−13	↑	6.23E−14	0.44	9.63E−13	↑	1.23E−13	↑
	80	9.53E−13	↑	6.88E−12	↑	5.70E−13	0.76	3.56E−12	↑
	160	9.30E−12	↑	1.39E−11	↑	3.07E−11	↑	3.49E−11	↑
	320	4.35E−11	↑	6.27E−11	↑	1.61E−10	↑	4.95E−11	↑
	640	1.12E−09	↑	1.88E−09	↑	5.79E−09	↑	7.56E−10	↑

Table 2: Numerical results of PRO1 scheme (consistency).

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	8.23E−01	—	8.35E−01	—	8.35E−01	—	8.35E−01	—
	40	4.12E−01	1.00	4.17E−01	1.00	4.17E−01	1.00	4.17E−01	1.00
	80	2.06E−01	1.00	2.09E−01	1.00	2.09E−01	1.00	2.09E−01	1.00
	160	1.03E−01	1.00	1.04E−01	1.00	1.04E−01	1.00	1.04E−01	1.00
	320	5.15E−02	1.00	5.22E−02	1.00	5.22E−02	1.00	5.22E−02	1.00
	640	2.57E−02	1.00	2.61E−02	1.00	2.61E−02	1.00	2.61E−02	1.00
$\mathbb{P}_5(6)$	20	7.52E−11	—	3.74E−11	—	2.38E−11	—	5.41E−11	—
	40	1.47E−10	↑	4.74E−10	↑	5.89E−10	↑	9.67E−10	↑
	80	1.16E−09	↑	2.85E−09	↑	2.67E−09	↑	4.02E−09	↑
	160	1.30E−08	↑	1.01E−08	↑	1.72E−08	↑	1.40E−08	↑
	320	3.97E−08	↑	2.42E−08	↑	5.45E−08	↑	3.54E−07	↑
	640	2.91E−07	↑	5.06E−07	↑	2.68E−07	↑	4.88E−07	↑

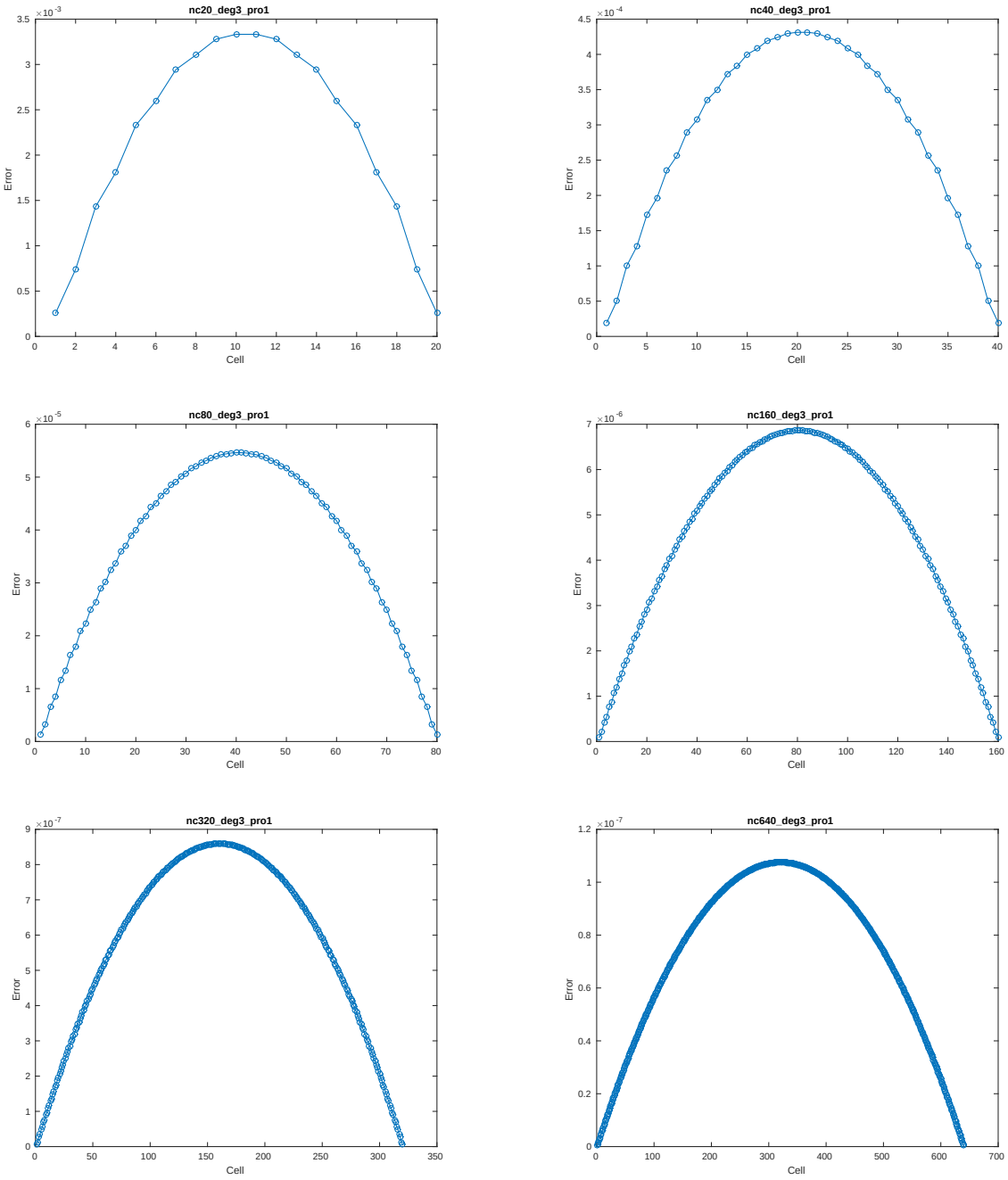

Figure 1: $\omega = 1|1, 1$, d=3

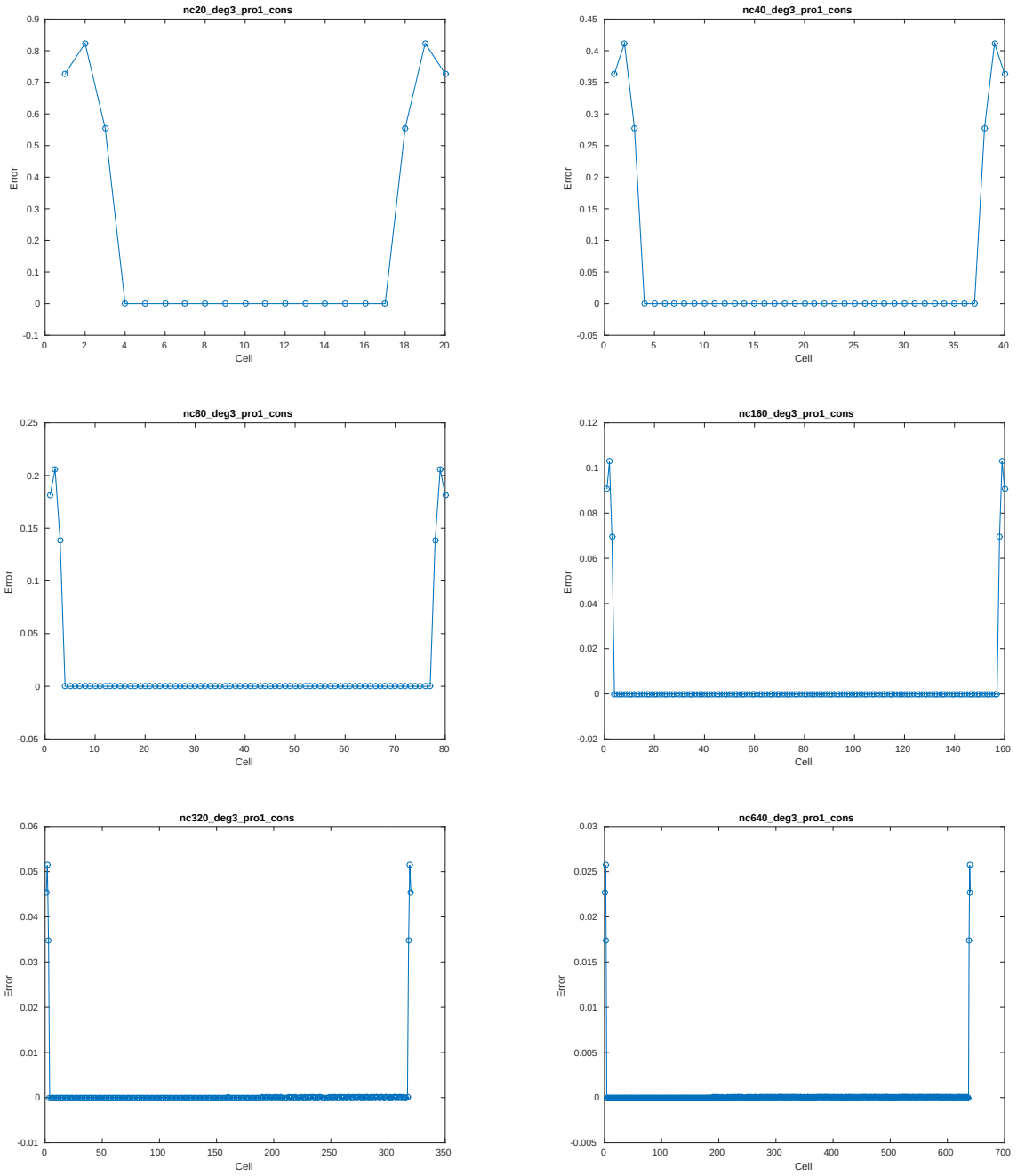

Figure 2: $\omega = 1|1, 1$, d=3 (consistency)

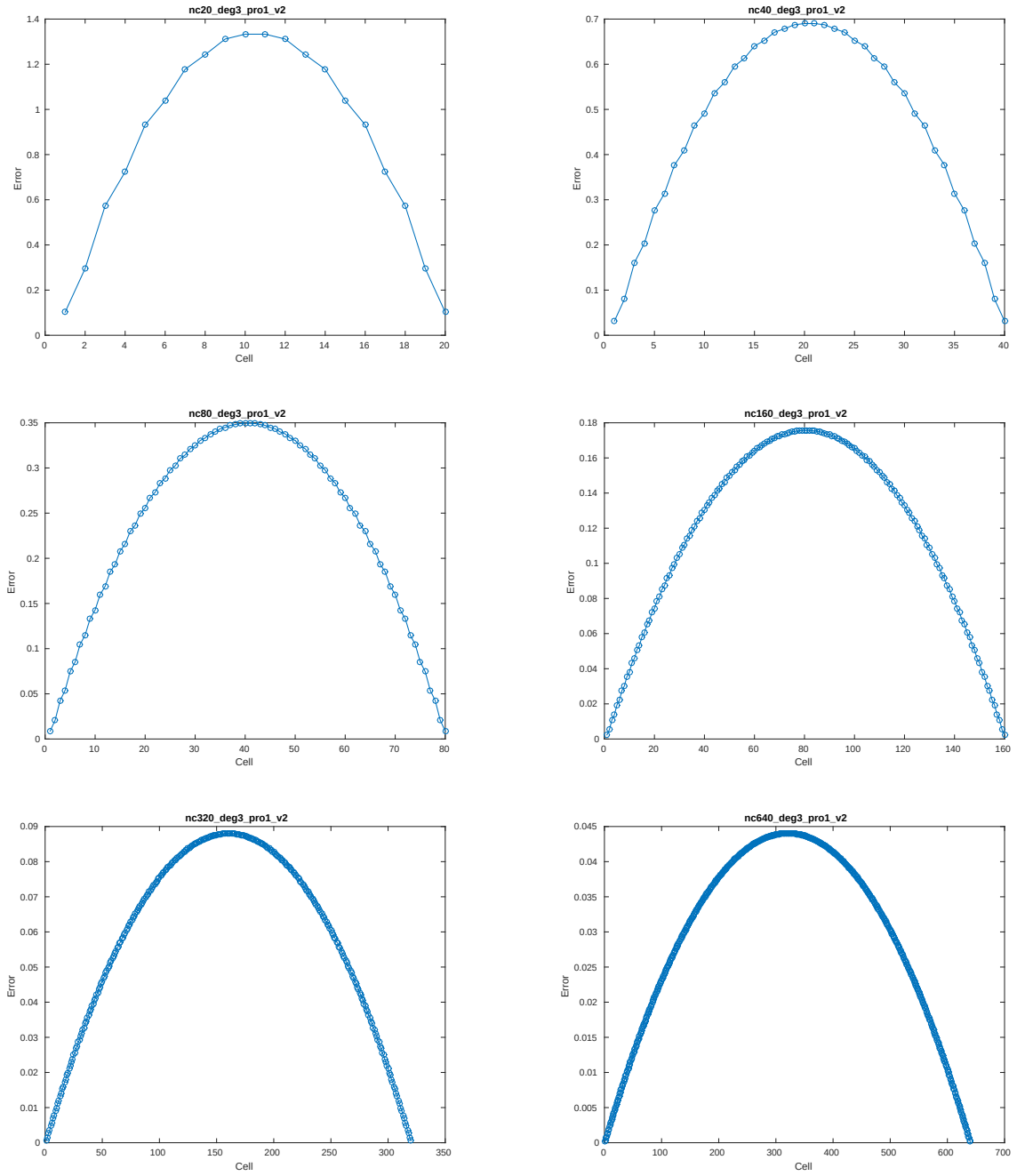

Figure 3: $\omega = 1|1,1,$ d=3 $\left(\frac{x-\overline{x}}{h^2}\right)$

In this tests we consider:

- $\psi(x) = \exp(x)$
- $\psi_{\mathrm{l}} = 1$
- $\psi_{\mathrm{r}} = e$
- $\psi_{\mathrm{ll}} = 1$
- $\psi_{\mathrm{rr}} = e$
- $g(x) = -\exp(x)$

Table 3: Numerical results of PRO1 scheme.

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	2.60E−04	—	2.07E−04	—	2.07E−04	—	2.06E−04	—
	40	3.35E−05	2.95	2.65E−05	2.96	2.65E−05	2.96	2.65E−05	2.96
	80	4.14E−06	3.02	3.27E−06	3.02	3.27E−06	3.02	3.27E−06	3.02
	160	4.90E−07	3.08	3.82E−07	3.10	3.82E−07	3.10	3.82E−07	3.10
	320	5.37E−08	3.19	4.08E−08	3.23	4.07E−08	3.23	4.07E−08	3.23
	640	4.89E−09	3.46	3.84E−09	3.41	3.68E−09	3.47	3.36E−09	3.60
$\mathbb{P}_5(6)$	20	1.78E−07	—	1.48E−07	—	1.48E−07	—	1.48E−07	—
	40	5.36E−09	5.05	4.46E−09	5.06	4.46E−09	5.06	4.46E−09	5.06
	80	1.56E−10	5.11	1.41E−10	4.98	1.40E−10	4.99	1.38E−10	5.01
	160	3.10E−12	5.65	5.96E−12	4.57	2.20E−12	5.99	1.21E−11	3.51
	320	3.84E−11	↑	5.59E−11	↑	1.97E−10	↑	7.13E−11	↑
	640	4.06E−10	↑	1.22E−09	↑	1.36E−09	↑	6.03E−10	↑

Table 4: Numerical results of PRO1 scheme (consistency).

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	8.75E−02	—	9.24E−02	—	9.24E−02	—	9.24E−02	—
	40	4.52E−02	0.95	4.58E−02	1.01	4.58E−02	1.01	4.58E−02	1.01
	80	2.29E−02	0.98	2.33E−02	0.98	2.33E−02	0.98	2.33E−02	0.98
	160	1.16E−02	0.99	1.17E−02	0.99	1.17E−02	0.99	1.17E−02	0.99
	320	5.81E−03	0.99	5.89E−03	0.99	5.89E−03	0.99	5.89E−03	0.99
	640	2.91E−03	1.00	2.95E−03	1.00	2.95E−03	1.00	2.95E−03	1.00
$\mathbb{P}_5(6)$	20	4.48E−04	—	4.68E−04	—	4.68E−04	—	4.68E−04	—
	40	5.84E−05	2.94	6.11E−05	2.94	6.11E−05	2.94	6.11E−05	2.94
	80	7.46E−06	2.97	7.81E−06	2.97	7.81E−06	2.97	7.81E−06	2.97
	160	9.30E−07	3.00	9.77E−07	3.00	9.72E−07	3.01	9.89E−07	2.98
	320	2.49E−07	1.90	2.02E−07	2.27	2.36E−07	2.04	1.48E−07	2.74
	640	2.09E−06	↑	2.59E−06	↑	2.18E−06	↑	2.03E−06	↑

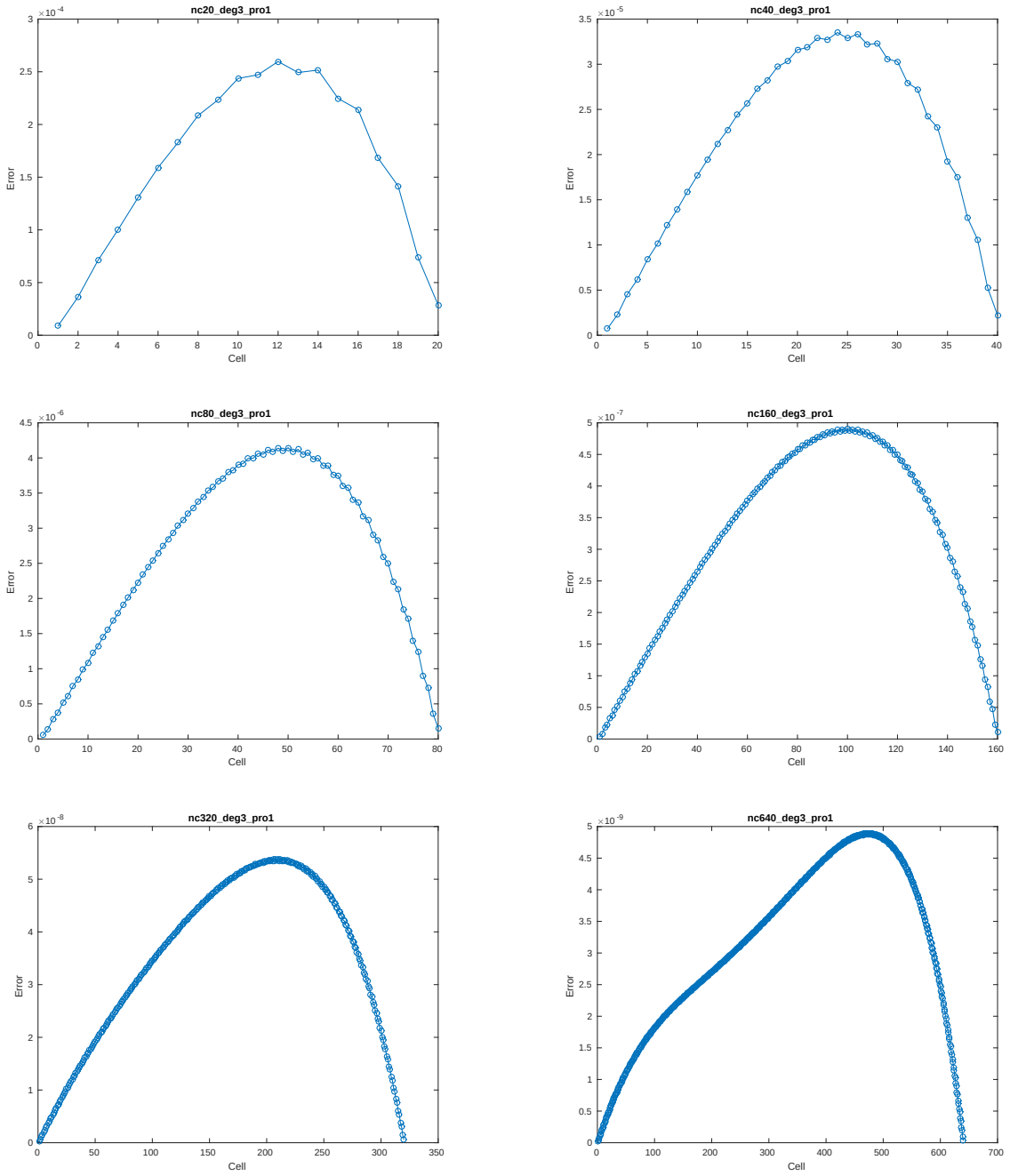

Figure 4: $\omega = 1|1, 1$, d=3

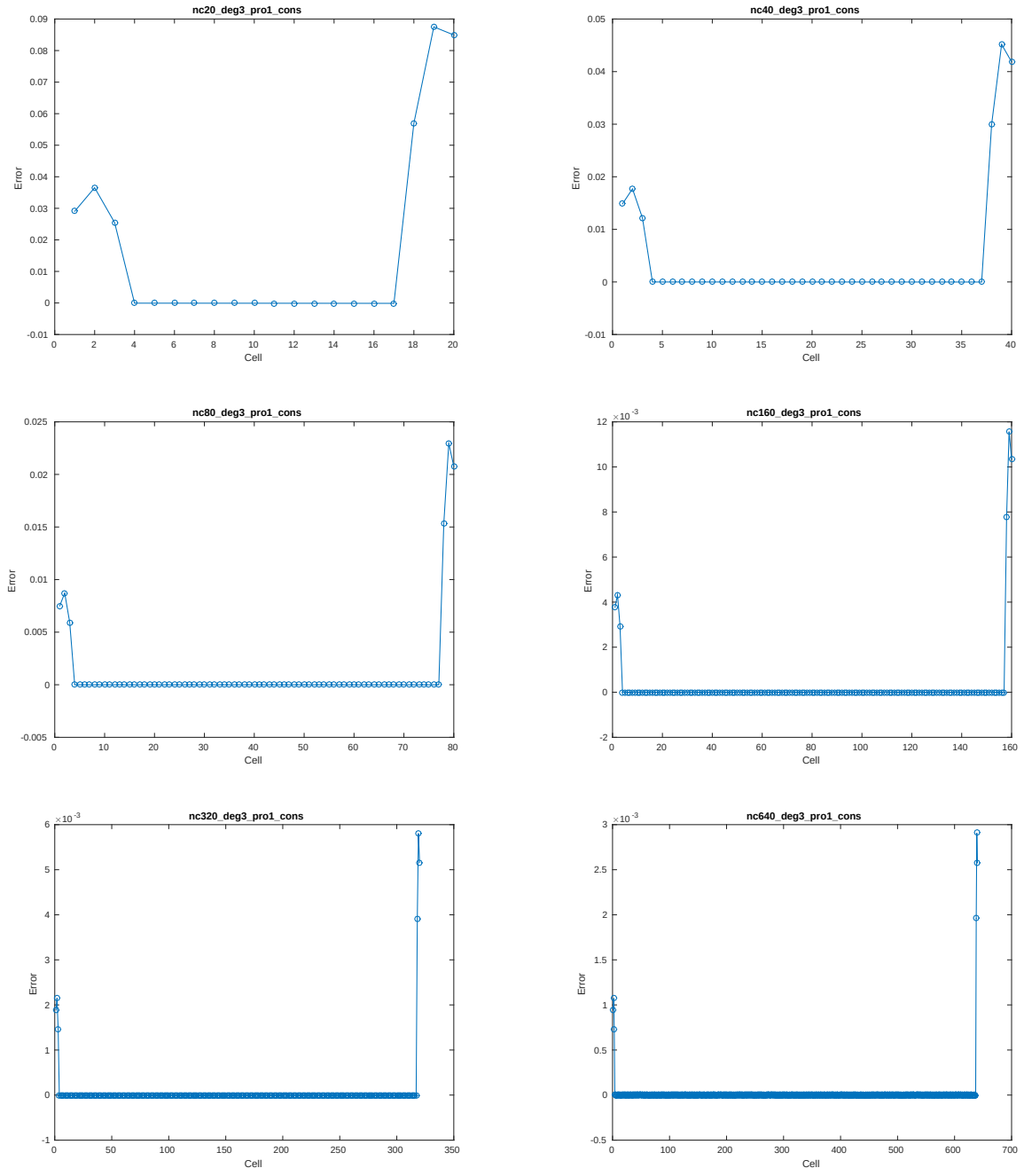

Figure 5: $\omega = 1|1, 1$, d=3 (consistency)

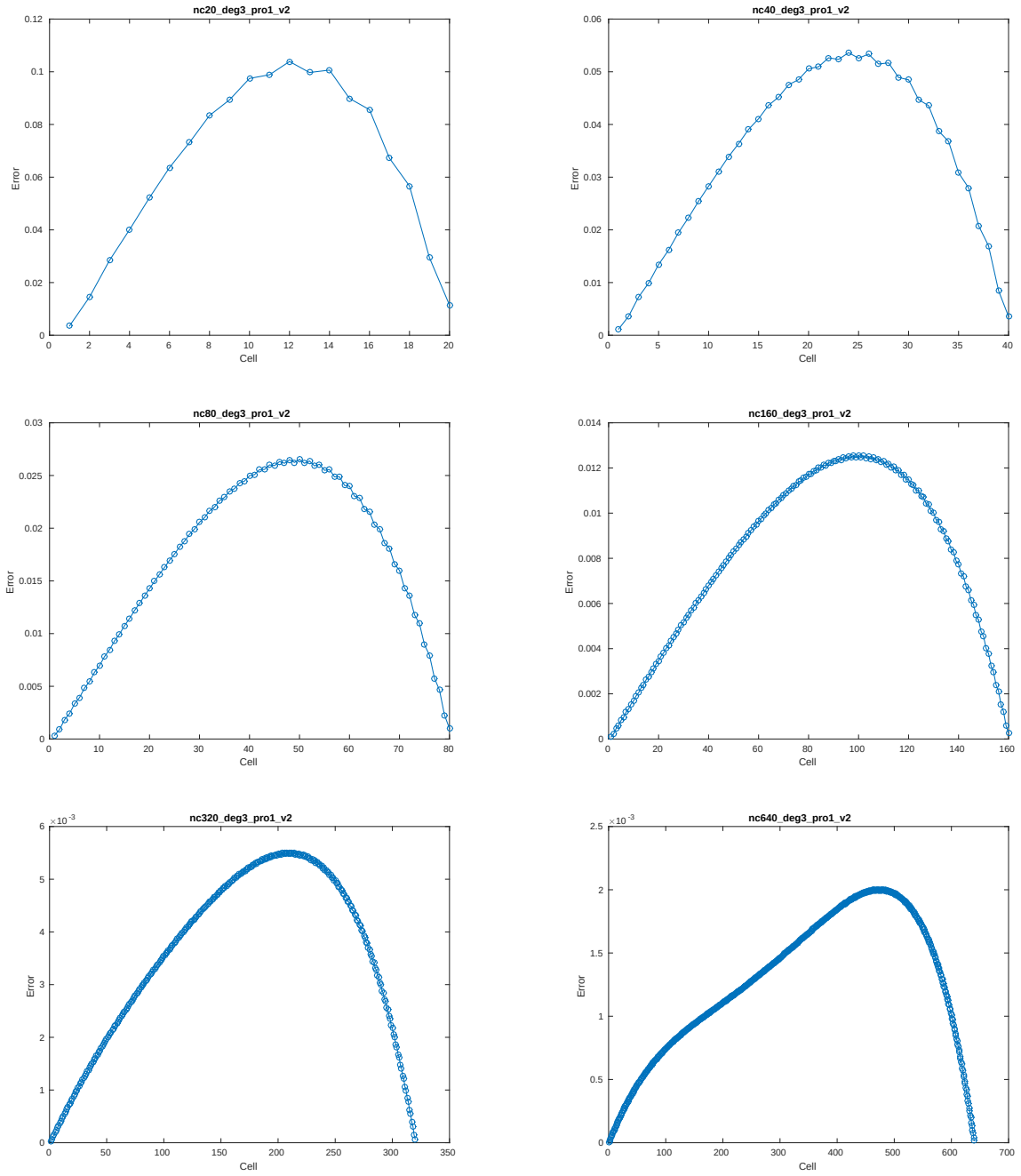

Figure 6: $\omega = 1|1,1$, d=3 $\left(\frac{x-\overline{x}}{h^2}\right)$

In this tests we consider:

- $\psi(x) = \sin(\pi x)$
- $\psi_{\mathrm{l}} = 0$
- $\psi_{\mathrm{ll}} = \pi$
- $\psi_{\mathrm{r}} = 0$
- $\psi_{\mathrm{rr}} = -\pi$
- $g(x) = -\pi^4 \sin(\pi x)$

Table 5: Numerical results of PRO1 scheme.

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	5.37E−03	—	4.42E−03	—	4.42E−03	—	4.42E−03	—
	40	7.55E−04	2.83	6.90E−04	2.68	6.90E−04	2.68	6.90E−04	2.68
	80	1.51E−04	2.32	1.47E−04	2.24	1.47E−04	2.24	1.47E−04	2.24
	160	3.53E−05	2.09	3.50E−05	2.07	3.50E−05	2.07	3.50E−05	2.07
	320	8.67E−06	2.02	8.65E−06	2.02	8.65E−06	2.02	8.65E−06	2.02
	640	2.14E−06	2.02	2.15E−06	2.01	2.16E−06	2.00	2.15E−06	2.01
$\mathbb{P}_5(6)$	20	2.68E−05	—	2.24E−05	—	2.24E−05	—	2.24E−05	—
	40	3.73E−07	6.17	4.59E−07	5.61	4.59E−07	5.61	4.59E−07	5.61
	80	5.88E−08	2.66	5.41E−08	3.08	5.42E−08	3.08	5.41E−08	3.08
	160	4.11E−09	3.84	3.93E−09	3.78	3.55E−09	3.93	3.75E−09	3.85
	320	4.63E−10	3.15	1.62E−09	1.28	2.64E−09	0.43	3.06E−10	3.61
	640	4.31E−08	↑	3.45E−09	↑	8.41E−09	↑	1.32E−08	↑

Table 6: Numerical results of PRO1 scheme (consistency).

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	6.97E−01	—	6.79E−01	—	6.79E−01	—	6.79E−01	—
	40	1.74E−01	2.00	1.70E−01	2.00	1.70E−01	2.00	1.70E−01	2.00
	80	4.34E−02	2.00	4.25E−02	2.00	4.25E−02	2.00	4.25E−02	2.00
	160	1.08E−02	2.00	1.06E−02	2.00	1.06E−02	2.00	1.06E−02	2.00
	320	2.71E−03	2.00	2.65E−03	2.00	2.65E−03	2.00	2.65E−03	2.00
	640	6.78E−04	2.00	6.64E−04	2.00	6.64E−04	2.00	6.64E−04	2.00
$\mathbb{P}_5(6)$	20	5.18E−02	—	4.91E−02	—	4.91E−02	—	4.91E−02	—
	40	3.36E−03	3.95	3.18E−03	3.95	3.18E−03	3.95	3.18E−03	3.95
	80	2.12E−04	3.99	2.00E−04	3.99	2.00E−04	3.99	2.00E−04	3.99
	160	1.33E−05	4.00	1.26E−05	4.00	1.26E−05	4.00	1.25E−05	4.00
	320	8.50E−07	3.96	7.78E−07	4.01	8.18E−07	3.94	9.69E−07	3.69
	640	5.31E−07	0.68	1.04E−06	↑	5.68E−07	0.52	1.13E−06	↑

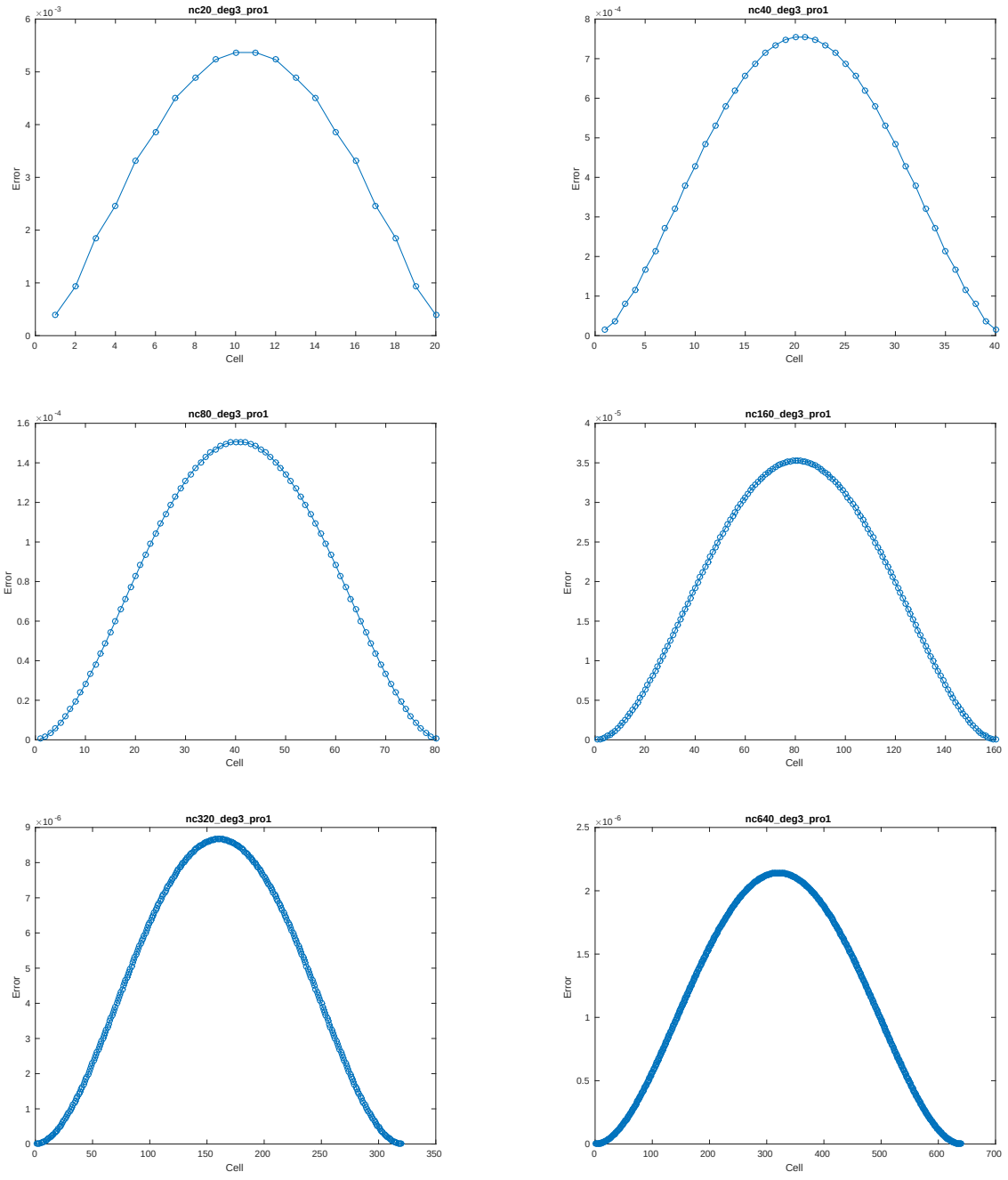

Figure 7: $\omega = 1|1, 1$, d=3

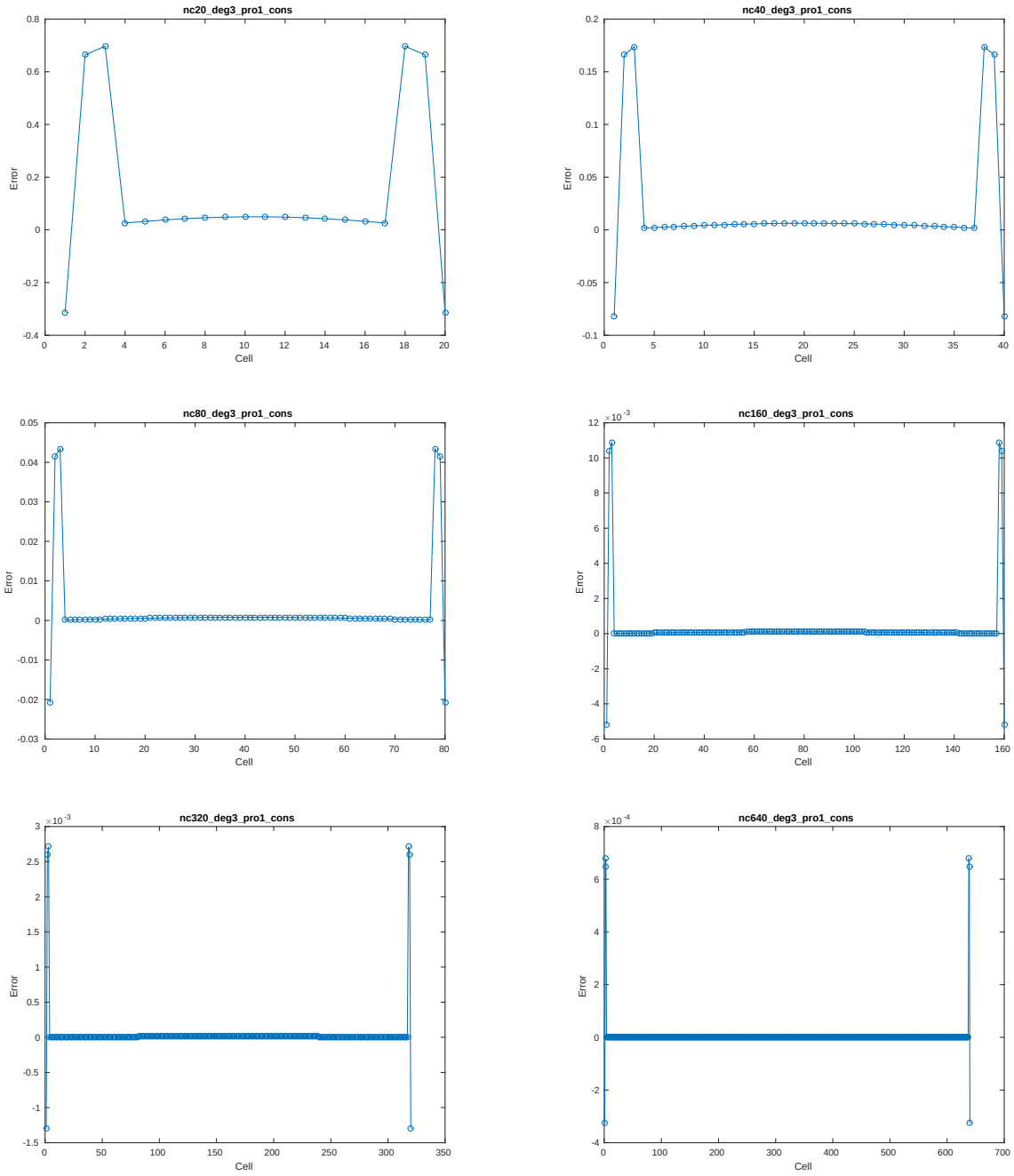

Figure 8: $\omega = 1|1, 1$, d=3 (consistency)

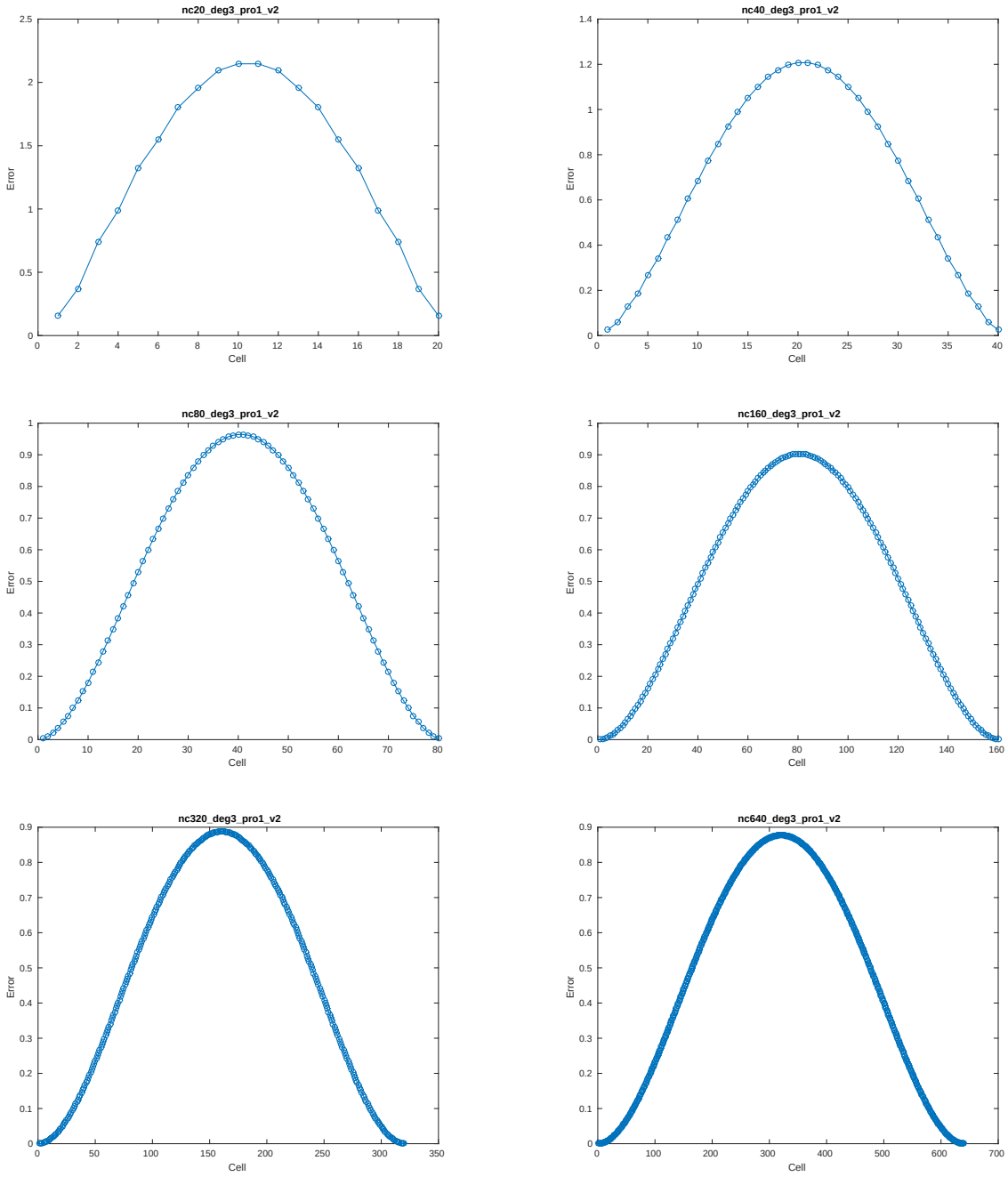

Figure 9: $\omega = 1|1,1$, d=3 $\left(\frac{x-\overline{x}}{h^2}\right)$

In this tests we consider:

- $\psi(x) = -\exp(x) - (\mathrm{e}-3)x^3 - (5-2\mathrm{e})x^2 + x + 1$
- $\psi_{\mathrm{l}} = 0$
- $\psi_{\mathrm{ll}} = 0$
- $\psi_{\mathrm{r}} = 0$
- $\psi_{\mathrm{rr}} = 0$
- $g(x) = \exp(x)$

Table 7: Numerical results of PRO1 scheme.

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	2.60E−04	—	2.07E−04	—	2.07E−04	—	2.06E−04	—
	40	3.35E−05	2.95	2.65E−05	2.96	2.65E−05	2.96	2.65E−05	2.96
	80	4.14E−06	3.02	3.27E−06	3.02	3.27E−06	3.02	3.27E−06	3.02
	160	4.90E−07	3.08	3.82E−07	3.10	3.82E−07	3.10	3.82E−07	3.10
	320	5.40E−08	3.18	4.03E−08	3.25	4.01E−08	3.25	4.11E−08	3.22
	640	1.07E−08	2.34	7.36E−09	2.45	6.71E−09	2.58	1.09E−08	1.91
$\mathbb{P}_5(6)$	20	1.78E−07	—	1.48E−07	—	1.48E−07	—	1.48E−07	—
	40	5.36E−09	5.05	4.46E−09	5.06	4.45E−09	5.06	4.45E−09	5.06
	80	1.54E−10	5.12	1.57E−10	4.82	1.58E−10	4.82	1.40E−10	4.99
	160	3.50E−11	2.14	1.11E−10	0.51	1.69E−10	↑	4.62E−10	↑
	320	1.77E−09	↑	1.18E−09	↑	4.14E−09	↑	1.60E−10	1.53
	640	1.09E−08	↑	2.60E−08	↑	3.89E−08	↑	1.19E−08	↑

Table 8: Numerical results of PRO1 scheme (consistency).

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	8.75E−02	—	9.24E−02	—	9.24E−02	—	9.24E−02	—
	40	4.52E−02	0.95	4.58E−02	1.01	4.58E−02	1.01	4.58E−02	1.01
	80	2.29E−02	0.98	2.33E−02	0.98	2.33E−02	0.98	2.33E−02	0.98
	160	1.16E−02	0.99	1.17E−02	0.99	1.17E−02	0.99	1.17E−02	0.99
	320	5.81E−03	0.99	5.89E−03	0.99	5.89E−03	0.99	5.89E−03	0.99
	640	2.91E−03	1.00	2.95E−03	1.00	2.95E−03	1.00	2.95E−03	1.00
$\mathbb{P}_5(6)$	20	4.48E−04	—	4.68E−04	—	4.68E−04	—	4.68E−04	—
	40	5.84E−05	2.94	6.11E−05	2.94	6.11E−05	2.94	6.11E−05	2.94
	80	7.45E−06	2.97	7.81E−06	2.97	7.81E−06	2.97	7.81E−06	2.97
	160	9.18E−07	3.02	9.64E−07	3.02	9.64E−07	3.02	9.64E−07	3.02
	320	1.25E−07	2.87	1.63E−07	2.57	1.63E−07	2.57	1.63E−07	2.57
	640	8.45E−07	↑	7.96E−07	↑	7.96E−07	↑	7.96E−07	↑

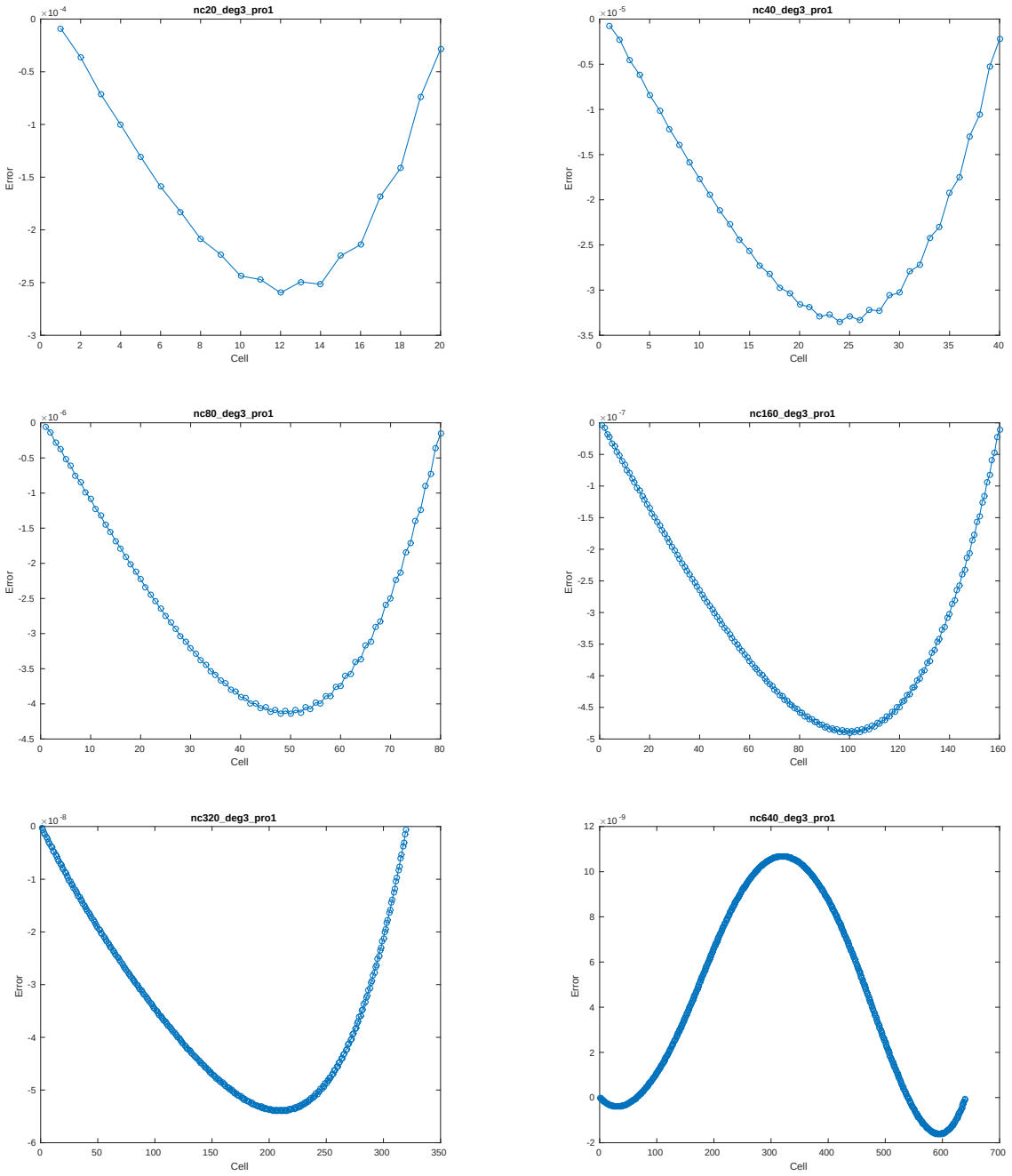

Figure 10: $\omega = 1|1, 1$, d=3

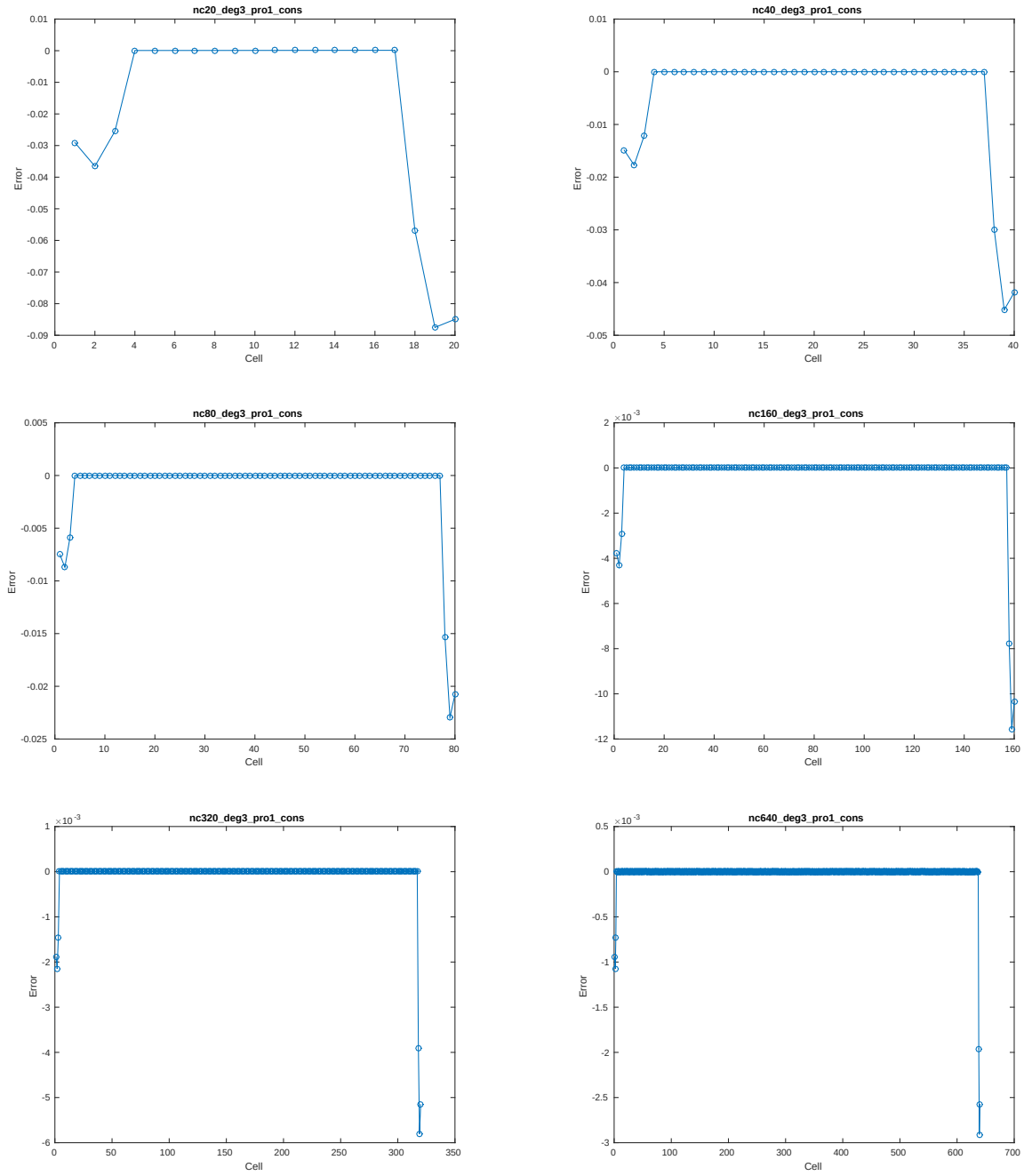

Figure 11: $\omega = 1|1, 1$, d=3 (consistency)

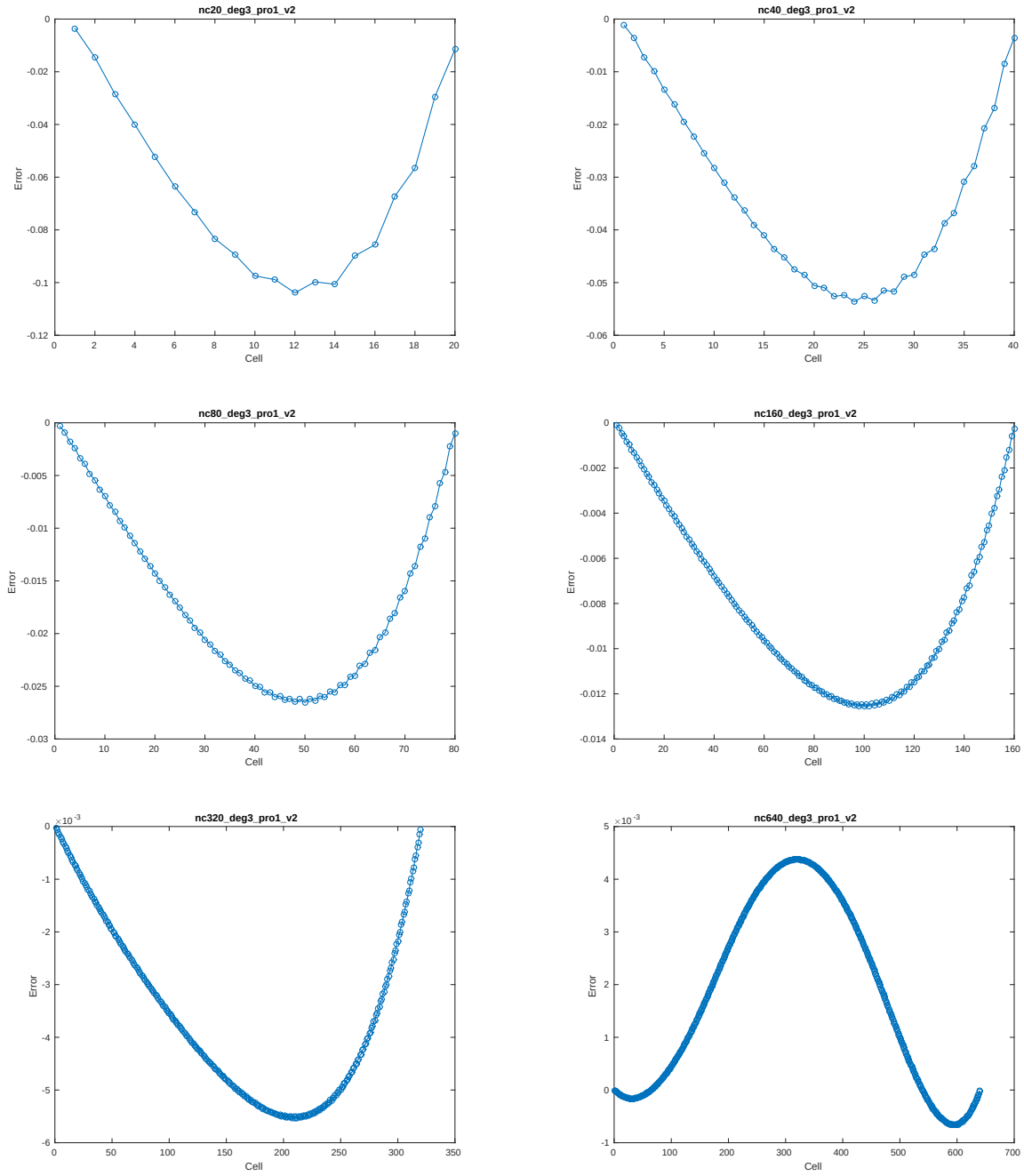

Figure 12: $\omega = 1|1,1$, d=3 $\left(\frac{x-\overline{x}}{h^2}\right)$

In this tests we consider:

- $\psi(x) = -\exp(x) - (\mathrm{e}-3)x^3 - (5-2\mathrm{e})x^2 + 1$
- $\psi_{\mathrm{l}} = 0$
- $\psi_{\mathrm{ll}} = -1$
- $\psi_{\mathrm{r}} = -1$
- $\psi_{\mathrm{rr}} = -1$
- $g(x) = \exp(x)$

Table 9: Numerical results of PRO1 scheme.

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	2.60E−04	—	2.07E−04	—	2.07E−04	—	2.06E−04	—
	40	3.35E−05	2.95	2.65E−05	2.96	2.65E−05	2.96	2.65E−05	2.96
	80	4.14E−06	3.02	3.27E−06	3.02	3.27E−06	3.02	3.27E−06	3.02
	160	4.90E−07	3.08	3.82E−07	3.10	3.82E−07	3.10	3.82E−07	3.10
	320	5.37E−08	3.19	4.08E−08	3.23	4.08E−08	3.23	4.08E−08	3.23
	640	5.07E−09	3.41	3.70E−09	3.46	3.70E−09	3.46	3.70E−09	3.46
$\mathbb{P}_5(6)$	20	1.78E−07	—	1.48E−07	—	1.48E−07	—	1.48E−07	—
	40	5.36E−09	5.05	4.46E−09	5.06	4.46E−09	5.06	4.46E−09	5.06
	80	1.55E−10	5.11	1.41E−10	4.98	1.41E−10	4.98	1.41E−10	4.98
	160	5.35E−12	4.86	5.51E−12	4.68	4.38E−12	5.01	4.88E−12	4.85
	320	1.16E−12	2.21	5.27E−12	0.06	6.54E−12	↑	4.27E−12	0.19
	640	1.67E−11	↑	8.64E−12	↑	1.03E−11	↑	1.87E−12	1.19

Table 10: Numerical results of PRO1 scheme (consistency).

		$\omega = 1\|1,1$		$\omega = 1\|3,1$		$\omega = 1\|3,3$		$\omega = 1\|3,10$	
	I	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$	$\mathrm{E}_{\infty,0}$	$\mathrm{O}_{\infty,0}$
$\mathbb{P}_3(4)$	20	8.75E−02	—	9.24E−02	—	9.24E−02	—	9.24E−02	—
	40	4.52E−02	0.95	4.58E−02	1.01	4.58E−02	1.01	4.58E−02	1.01
	80	2.29E−02	0.98	2.33E−02	0.98	2.33E−02	0.98	2.33E−02	0.98
	160	1.16E−02	0.99	1.17E−02	0.99	1.17E−02	0.99	1.17E−02	0.99
	320	5.81E−03	0.99	5.89E−03	0.99	5.89E−03	0.99	5.89E−03	0.99
	640	2.91E−03	1.00	2.95E−03	1.00	2.95E−03	1.00	2.95E−03	1.00
$\mathbb{P}_5(6)$	20	4.48E−04	—	4.68E−04	—	4.68E−04	—	4.68E−04	—
	40	5.84E−05	2.94	6.11E−05	2.94	6.11E−05	2.94	6.11E−05	2.94
	80	7.45E−06	2.97	7.81E−06	2.97	7.81E−06	2.97	7.81E−06	2.97
	160	9.15E−07	3.03	9.59E−07	3.02	9.59E−07	3.03	9.63E−07	3.02
	320	9.31E−08	3.30	1.27E−07	2.92	1.30E−07	2.88	7.93E−08	3.60
	640	9.00E−07	↑	9.67E−07	↑	9.19E−07	↑	6.80E−07	↑

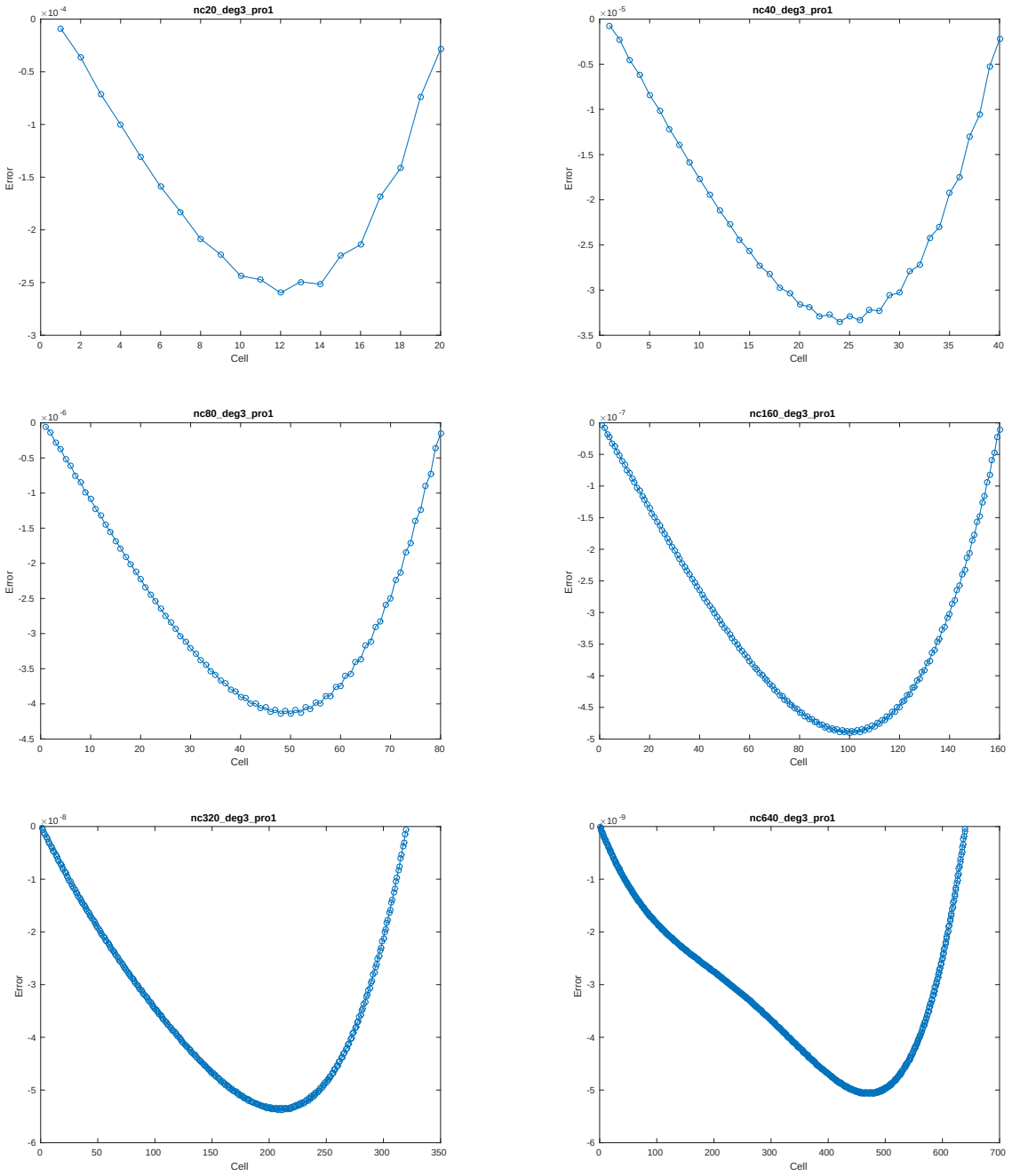

Figure 13: $\omega = 1|1, 1$, d=3

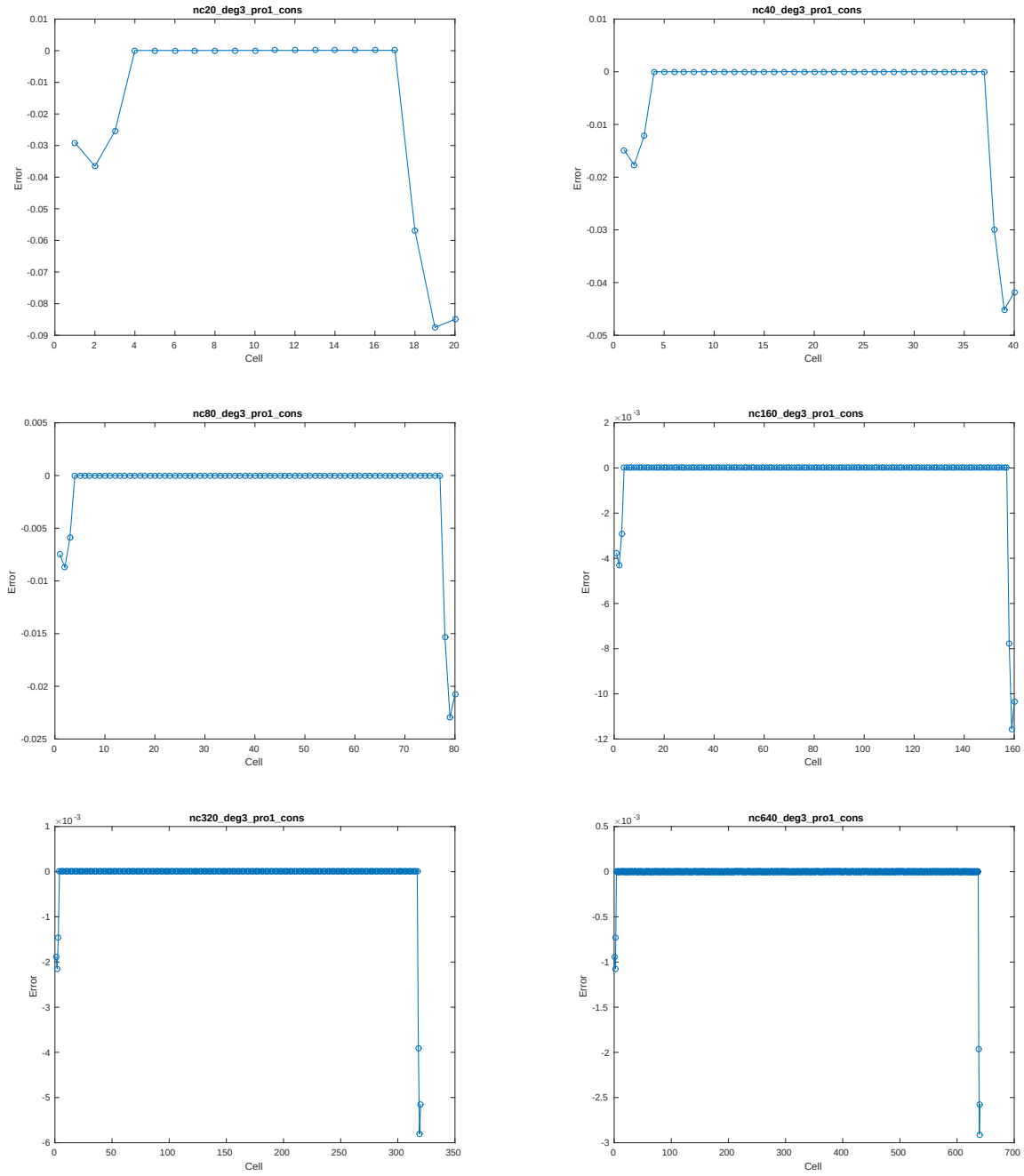

Figure 14: $\omega = 1|1, 1$, d=3 (consistency)

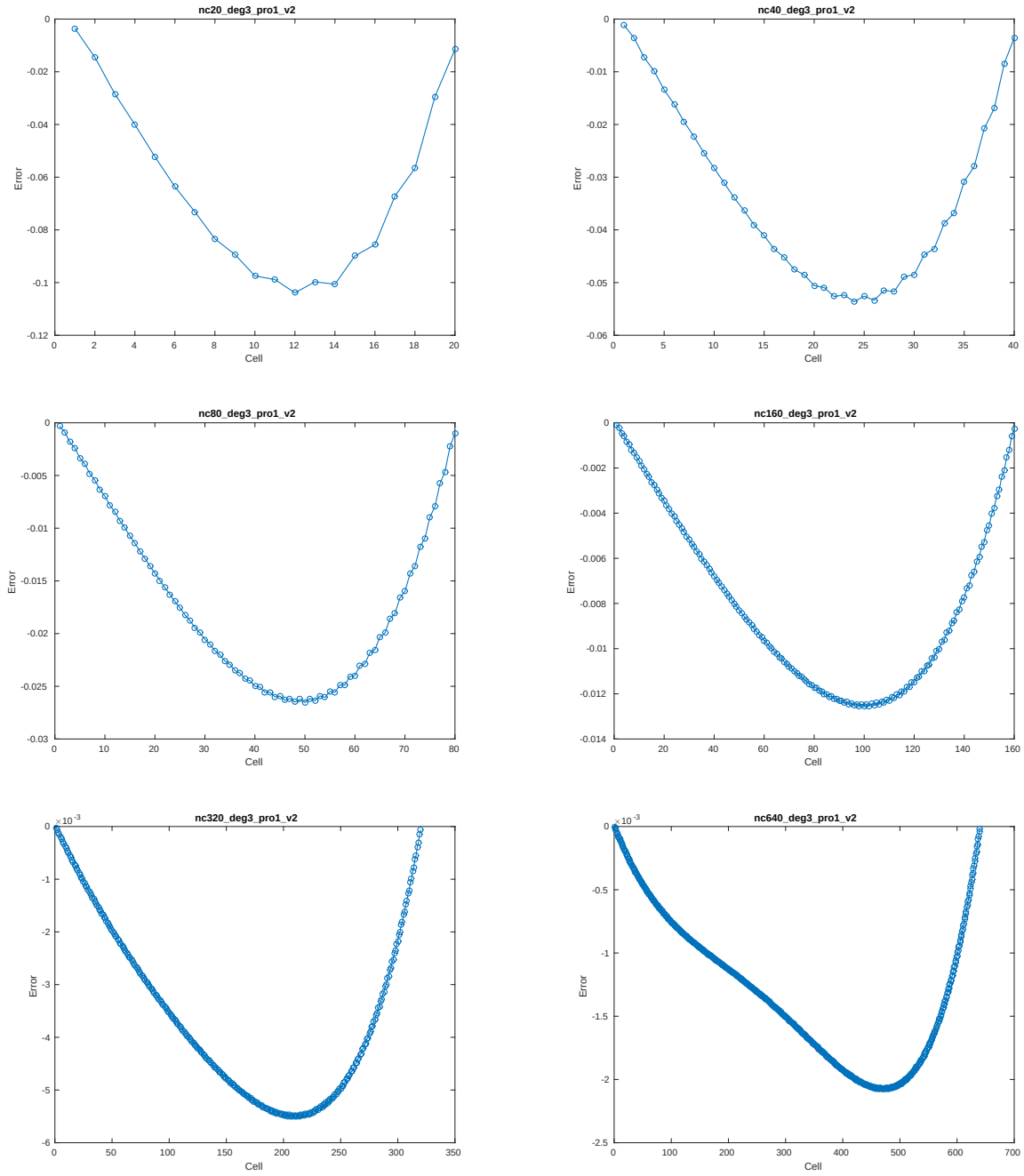

Figure 15: $\omega = 1|1,1$, d=3 $\left(\frac{x-\overline{x}}{h^2}\right)$