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In this document we distinguish three schemes:

- PRO1 — this is considered the “old” method because we aren’t using the least squares with the constrains. To calculate the $\hat{\mathcal{R}}$ matrix (matrix of the coefficients) we use $\hat{\mathcal{R}} = \text{pinv}(A) \times B$
- PRO2 — in this scheme, to calculate the $\hat{\mathcal{R}}$ matrix we resort to the least squares with constrains and the degree of the reconstruction $\hat{\psi}_1$ and $\hat{\psi}_I$ is equal to the degree setted;
- PRO3 — in this scheme, to calculate the $\hat{\mathcal{R}}$ matrix we resort to the least squares with constrains and the degree of the reconstruction $\hat{\psi}_1$ and $\hat{\psi}_I$ is equal to the degree setted plus:
 - in 01_01 problems, degree+1
 - in 02_02 problems, degree+2
 - in 01_23 problems, degree+1 at the left and degree+3 at the right.

1 01_01

Example 1.1. In this tests we consider:

- $\psi(x) = x^4$
- $\psi_l = 0$
- $\psi_r = 1$
- $\psi_{ll} = 0$
- $\psi_{rr} = 4$
- $g(x) = -24$

Table 1: Numerical results of PRO1 scheme to the example ??.

I	$\omega = 1 1, 1$		$\omega = 1 3, 1$		$\omega = 1 3, 3$		$\omega = 1 3, 10$		
	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	
$\mathbb{P}_3(4)$	20	3.33E-03	—	2.51E-03	—	2.51E-03	—	2.51E-03	—
	40	4.31E-04	2.95	3.21E-04	2.97	3.21E-04	2.97	3.21E-04	2.97
	80	5.46E-05	2.98	4.04E-05	2.99	4.04E-05	2.99	4.04E-05	2.99
	160	6.86E-06	2.99	5.07E-06	2.99	5.07E-06	2.99	5.07E-06	2.99
	240	2.04E-06	3.00	1.50E-06	3.00	1.50E-06	3.00	1.50E-06	3.00
	360	6.04E-07	3.00	4.46E-07	3.00	4.46E-07	3.00	4.46E-07	3.00
	540	1.79E-07	3.00	1.32E-07	3.00	1.32E-07	3.00	1.32E-07	3.00
$\mathbb{P}_5(6)$	20	2.59E-14	—	1.30E-14	—	3.94E-14	—	1.04E-14	—
	40	7.90E-14	↑	6.93E-14	↑	7.13E-14	↑	7.84E-14	↑
	80	4.14E-13	↑	6.67E-13	↑	2.12E-13	↑	4.74E-13	↑
	160	2.53E-11	↑	4.92E-12	↑	1.36E-12	↑	3.40E-12	↑
	240	1.99E-11	0.59	3.56E-11	↑	9.60E-12	↑	8.67E-12	↑
	360	2.30E-10	↑	1.17E-10	↑	7.53E-11	↑	7.46E-11	↑
	540	1.05E-09	↑	2.50E-10	↑	1.14E-10	↑	1.75E-10	↑

Table 2: Numerical results of pro2 scheme to the example ??.

I	$\omega = 1 1, 1$		$\omega = 1 3, 1$		$\omega = 1 3, 3$		$\omega = 1 3, 10$		
	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	
$\mathbb{P}_3(4)$	20	2.60E-04	—	2.06E-04	—	2.06E-04	—	2.06E-04	—
	40	3.35E-05	2.95	2.65E-05	2.96	2.65E-05	2.96	2.65E-05	2.96
	80	4.14E-06	3.02	3.27E-06	3.02	3.27E-06	3.02	3.27E-06	3.02
	160	4.90E-07	3.08	3.82E-07	3.10	3.82E-07	3.10	3.82E-07	3.10
	240	1.36E-07	3.16	1.05E-07	3.19	1.05E-07	3.19	1.05E-07	3.19
	360	3.62E-08	3.27	2.78E-08	3.27	2.75E-08	3.29	2.79E-08	3.26
	540	8.97E-09	3.44	5.18E-09	4.14	4.97E-09	4.22	4.71E-09	4.39
$\mathbb{P}_5(6)$	20	1.78E-07	—	1.48E-07	—	1.48E-07	—	1.48E-07	—
	40	5.36E-09	5.05	4.45E-09	5.06	4.45E-09	5.06	4.46E-09	5.06
	80	1.51E-10	5.15	1.37E-10	5.03	1.26E-10	5.15	1.37E-10	5.03
	160	7.06E-11	1.10	1.70E-10	↑	5.08E-11	1.30	1.70E-10	↑
	240	6.83E-11	0.08	1.12E-10	1.02	1.12E-10	↑	7.20E-11	2.11
	360	3.65E-10	↑	7.68E-10	↑	1.34E-09	↑	4.83E-10	↑
	540	2.68E-09	↑	4.76E-09	↑	4.78E-09	↑	6.20E-10	↑

Table 3: Numerical results of pro3 scheme to the example ??.

I	$\omega = 1 1, 1$		$\omega = 1 3, 1$		$\omega = 1 3, 3$		$\omega = 1 3, 10$		
	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	
$\mathbb{P}_3(4)$	20	6.89E-05	—	4.01E-05	—	4.01E-05	—	4.01E-05	—
	40	8.09E-06	3.09	4.87E-06	3.04	4.87E-06	3.04	4.87E-06	3.04
	80	8.53E-07	3.25	4.91E-07	3.31	4.91E-07	3.31	4.91E-07	3.31
	160	7.67E-08	3.48	3.66E-08	3.74	3.66E-08	3.74	3.66E-08	3.74
	240	1.71E-08	3.70	7.42E-09	3.94	7.42E-09	3.94	7.44E-09	3.93
	360	3.64E-09	3.81	5.51E-09	0.74	4.88E-09	1.03	4.88E-09	1.04
	540	4.81E-09	↑	7.41E-09	↑	7.70E-09	↑	7.41E-09	↑
$\mathbb{P}_5(6)$	20	1.17E-07	—	1.08E-07	—	1.08E-07	—	1.08E-07	—
	40	2.98E-09	5.30	3.13E-09	5.11	3.13E-09	5.11	3.13E-09	5.11
	80	8.58E-11	5.12	8.33E-11	5.23	8.59E-11	5.19	8.26E-11	5.24
	160	2.69E-11	1.67	4.08E-11	1.03	2.00E-11	2.10	4.08E-11	1.02
	240	6.25E-11	↑	4.40E-10	↑	4.40E-10	↑	3.81E-10	↑
	360	1.58E-09	↑	1.03E-09	↑	1.85E-10	2.13	3.66E-10	0.09
	540	7.53E-09	↑	1.00E-08	↑	8.10E-09	↑	5.13E-09	↑

Example 1.2. In this tests we consider:

- $\psi(x) = \exp(x)$
- $\psi_l = 1$
- $\psi_r = e$
- $\psi_{ll} = 1$
- $\psi_{rr} = e$
- $g(x) = -\exp(x)$

Table 4: Numerical results of PRO1 scheme to the example ??.

	I	$\omega = 1 1, 1$		$\omega = 1 3, 1$		$\omega = 1 3, 3$		$\omega = 1 3, 10$	
		$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$
$\mathbb{P}_3(4)$	20	2.60E-04	—	2.07E-04	—	2.07E-04	—	2.06E-04	—
	40	3.35E-05	2.95	2.65E-05	2.96	2.65E-05	2.96	2.65E-05	2.96
	80	4.14E-06	3.02	3.27E-06	3.02	3.27E-06	3.02	3.27E-06	3.02
	160	4.90E-07	3.08	3.82E-07	3.10	3.82E-07	3.10	3.82E-07	3.10
	240	1.36E-07	3.16	1.05E-07	3.19	1.05E-07	3.19	1.05E-07	3.19
	360	3.64E-08	3.26	2.79E-08	3.27	2.74E-08	3.31	2.76E-08	3.30
	540	9.11E-09	3.41	6.16E-09	3.72	6.98E-09	3.38	7.26E-09	3.29
$\mathbb{P}_5(6)$	20	1.78E-07	—	1.48E-07	—	1.48E-07	—	1.48E-07	—
	40	5.36E-09	5.05	4.45E-09	5.06	4.45E-09	5.06	4.45E-09	5.06
	80	1.56E-10	5.11	1.40E-10	4.99	1.38E-10	5.01	1.35E-10	5.04
	160	1.45E-11	3.42	9.88E-11	0.51	9.13E-11	0.60	4.14E-11	1.71
	240	9.29E-11	↑	3.94E-10	↑	3.02E-10	↑	4.94E-10	↑
	360	1.25E-09	↑	8.54E-10	↑	8.91E-10	↑	1.15E-09	↑
	540	1.44E-09	↑	5.59E-09	↑	2.64E-09	↑	2.11E-09	↑

Table 5: Numerical results of pro2 scheme to the example ??.

	I	$\omega = 1 1, 1$		$\omega = 1 3, 1$		$\omega = 1 3, 3$		$\omega = 1 3, 10$	
		$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$
$\mathbb{P}_3(4)$	20	2.60E-04	—	2.06E-04	—	2.06E-04	—	2.06E-04	—
	40	3.35E-05	2.95	2.65E-05	2.96	2.65E-05	2.96	2.65E-05	2.96
	80	4.14E-06	3.02	3.27E-06	3.02	3.27E-06	3.02	3.27E-06	3.02
	160	4.90E-07	3.08	3.82E-07	3.10	3.82E-07	3.10	3.82E-07	3.10
	240	1.36E-07	3.16	1.05E-07	3.19	1.05E-07	3.19	1.05E-07	3.19
	360	3.62E-08	3.27	2.78E-08	3.27	2.75E-08	3.29	2.79E-08	3.26
	540	8.97E-09	3.44	5.18E-09	4.14	4.97E-09	4.22	4.71E-09	4.39
$\mathbb{P}_5(6)$	20	1.78E-07	—	1.48E-07	—	1.48E-07	—	1.48E-07	—
	40	5.36E-09	5.05	4.45E-09	5.06	4.45E-09	5.06	4.46E-09	5.06
	80	1.51E-10	5.15	1.37E-10	5.03	1.26E-10	5.15	1.37E-10	5.03
	160	7.06E-11	1.10	1.70E-10	↑	5.08E-11	1.30	1.70E-10	↑
	240	6.83E-11	0.08	1.12E-10	1.02	1.12E-10	↑	7.20E-11	2.11
	360	3.65E-10	↑	7.68E-10	↑	1.34E-09	↑	4.83E-10	↑
	540	2.68E-09	↑	4.76E-09	↑	4.78E-09	↑	6.20E-10	↑

Table 6: Numerical results of pro3 scheme to the example ??.

	I	$\omega = 1 1, 1$		$\omega = 1 3, 1$		$\omega = 1 3, 3$		$\omega = 1 3, 10$	
		$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$
$\mathbb{P}_3(4)$	20	6.89E-05	—	4.01E-05	—	4.01E-05	—	4.01E-05	—
	40	8.09E-06	3.09	4.87E-06	3.04	4.87E-06	3.04	4.87E-06	3.04
	80	8.53E-07	3.25	4.91E-07	3.31	4.91E-07	3.31	4.91E-07	3.31
	160	7.67E-08	3.48	3.66E-08	3.74	3.66E-08	3.74	3.66E-08	3.74
	240	1.71E-08	3.70	7.42E-09	3.94	7.42E-09	3.94	7.44E-09	3.93
	360	3.64E-09	3.81	5.51E-09	0.74	4.88E-09	1.03	4.88E-09	1.04
	540	4.81E-09	↑	7.41E-09	↑	7.70E-09	↑	7.41E-09	↑
$\mathbb{P}_5(6)$	20	1.17E-07	—	1.08E-07	—	1.08E-07	—	1.08E-07	—
	40	2.98E-09	5.30	3.13E-09	5.11	3.13E-09	5.11	3.13E-09	5.11
	80	8.58E-11	5.12	8.33E-11	5.23	8.59E-11	5.19	8.26E-11	5.24
	160	2.69E-11	1.67	4.08E-11	1.03	2.00E-11	2.10	4.08E-11	1.02
	240	6.25E-11	↑	4.40E-10	↑	4.40E-10	↑	3.81E-10	↑
	360	1.58E-09	↑	1.03E-09	↑	1.85E-10	2.13	3.66E-10	0.09
	540	7.53E-09	↑	1.00E-08	↑	8.10E-09	↑	5.13E-09	↑

Example 1.3. In this tests we consider:

- $\psi(x) = -\exp(x) + x^3(3 - e) + x^2(2e - 5) + x + 1$
- $\psi_l = 0$
- $\psi_r = 0$
- $\psi_{ll} = 0$
- $\psi_{rr} = 0$
- $g(x) = \exp(x)$

Table 7: Numerical results of PRO1 scheme to the example ??.

	I	$\omega = 1 1, 1$		$\omega = 1 3, 1$		$\omega = 1 3, 3$		$\omega = 1 3, 10$	
		$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$
$\mathbb{P}_3(4)$	20	2.60E-04	—	2.07E-04	—	2.07E-04	—	2.06E-04	—
	40	3.35E-05	2.95	2.65E-05	2.96	2.65E-05	2.96	2.65E-05	2.96
	80	4.14E-06	3.02	3.27E-06	3.02	3.27E-06	3.02	3.27E-06	3.02
	160	4.90E-07	3.08	3.82E-07	3.10	3.82E-07	3.10	3.82E-07	3.10
	240	1.36E-07	3.16	1.05E-07	3.19	1.05E-07	3.19	1.05E-07	3.19
	360	3.64E-08	3.25	2.75E-08	3.30	2.75E-08	3.30	2.75E-08	3.30
	540	9.21E-09	3.39	6.78E-09	3.45	6.79E-09	3.45	6.78E-09	3.45
$\mathbb{P}_5(6)$	20	1.78E-07	—	1.48E-07	—	1.48E-07	—	1.48E-07	—
	40	5.36E-09	5.05	4.46E-09	5.06	4.46E-09	5.06	4.46E-09	5.06
	80	1.55E-10	5.11	1.41E-10	4.98	1.41E-10	4.98	1.41E-10	4.98
	160	5.68E-12	4.77	4.98E-12	4.82	4.75E-12	4.89	4.60E-12	4.94
	240	1.02E-12	4.23	1.59E-12	2.82	1.73E-12	2.49	1.84E-12	2.26
	360	1.46E-12	↑	5.19E-12	↑	1.73E-12	0.00	3.84E-12	↑
	540	1.96E-11	↑	8.55E-12	↑	8.17E-12	↑	1.03E-11	↑

Table 8: Numerical results of pro2 scheme to the example ??.

	I	$\omega = 1 1, 1$		$\omega = 1 3, 1$		$\omega = 1 3, 3$		$\omega = 1 3, 10$	
		$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$
$\mathbb{P}_3(4)$	20	2.60E-04	—	2.06E-04	—	2.06E-04	—	2.06E-04	—
	40	3.35E-05	2.95	2.65E-05	2.96	2.65E-05	2.96	2.65E-05	2.96
	80	4.14E-06	3.02	3.27E-06	3.02	3.27E-06	3.02	3.27E-06	3.02
	160	4.90E-07	3.08	3.82E-07	3.10	3.82E-07	3.10	3.82E-07	3.10
	240	1.36E-07	3.16	1.05E-07	3.19	1.05E-07	3.19	1.05E-07	3.19
	360	3.64E-08	3.25	2.75E-08	3.30	2.75E-08	3.30	2.75E-08	3.30
	540	9.21E-09	3.39	6.78E-09	3.45	6.78E-09	3.45	6.78E-09	3.45
$\mathbb{P}_5(6)$	20	1.78E-07	—	1.48E-07	—	1.48E-07	—	1.48E-07	—
	40	5.36E-09	5.05	4.46E-09	5.06	4.46E-09	5.06	4.46E-09	5.06
	80	1.55E-10	5.11	1.41E-10	4.98	1.41E-10	4.98	1.41E-10	4.98
	160	5.39E-12	4.85	4.65E-12	4.92	4.59E-12	4.94	4.65E-12	4.92
	240	1.82E-12	2.68	2.82E-13	6.91	2.82E-13	6.88	4.93E-13	5.54
	360	2.51E-12	↑	2.13E-12	↑	1.74E-12	↑	5.69E-13	↑
	540	7.75E-12	↑	6.97E-12	↑	6.87E-12	↑	4.86E-12	↑

Table 9: Numerical results of pro3 scheme to the example ??.

	I	$\omega = 1 1, 1$		$\omega = 1 3, 1$		$\omega = 1 3, 3$		$\omega = 1 3, 10$	
		$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$
$\mathbb{P}_3(4)$	20	6.89E-05	—	4.01E-05	—	4.01E-05	—	4.01E-05	—
	40	8.09E-06	3.09	4.87E-06	3.04	4.87E-06	3.04	4.87E-06	3.04
	80	8.53E-07	3.25	4.91E-07	3.31	4.91E-07	3.31	4.91E-07	3.31
	160	7.67E-08	3.48	3.66E-08	3.74	3.66E-08	3.74	3.66E-08	3.74
	240	1.71E-08	3.70	7.45E-09	3.93	7.45E-09	3.93	7.45E-09	3.93
	360	3.64E-09	3.82	5.54E-09	0.73	5.54E-09	0.73	5.54E-09	0.73
	540	3.00E-09	0.48	3.72E-09	0.98	3.72E-09	0.98	3.72E-09	0.98
$\mathbb{P}_5(6)$	20	1.17E-07	—	1.08E-07	—	1.08E-07	—	1.08E-07	—
	40	2.98E-09	5.30	3.13E-09	5.11	3.13E-09	5.11	3.13E-09	5.11
	80	8.88E-11	5.07	9.82E-11	4.99	9.82E-11	5.00	9.82E-11	4.99
	160	3.13E-12	4.82	3.15E-12	4.96	3.29E-12	4.90	3.15E-12	4.96
	240	1.65E-12	1.58	5.61E-13	4.25	5.61E-13	4.37	2.74E-13	6.02
	360	1.81E-12	↑	4.66E-11	↑	4.71E-11	↑	4.50E-11	↑
	540	1.25E-11	↑	5.63E-12	5.21	2.29E-11	1.77	2.76E-11	1.21

2 02_02

Example 2.1. In this tests we consider:

- $\psi(x) = x^4$
- $\psi_l = 0$
- $\psi_r = 1$
- $M_l = 0$
- $M_r = -12$
- $g(x) = -24$

Example 2.2. In this tests we consider:

- $\psi(x) = \exp(x)$
- $\psi_l = 1$
- $\psi_r = e$
- $M_l = -1$
- $M_r = -e$
- $g(x) = -\exp(x)$

Example 2.3. In this tests we consider:

- $\psi(x) = -\exp(x) + \left(\frac{e-1}{6}\right)x^3 + \frac{x^2}{2} + \left(\frac{5e-8}{6}\right)x + 1$
- $\psi_l = 0$
- $\psi_r = 0$
- $M_l = 0$
- $M_r = 0$
- $g(x) = \exp(x)$

3 01_23

Example 3.1. In this tests we consider:

- $\psi(x) = x^4$
- $\psi_1 = 0$
- $\psi_{11} = 0$
- $M_r = -12$
- $G = -24$
- $g(x) = 24$

Example 3.2. In this tests we consider:

- $\psi(x) = \exp(x)$
- $\psi_I = 1$
- $\psi_{II} = 1$
- $M_r = -e$
- $G = -e$
- $g(x) = -\exp(x)$

Example 3.3. In this tests we consider:

- $\psi(x) = -\exp(x) + x^3 \left(\frac{e-1}{6}\right) + \frac{x^2}{2} + x + 1$
- $\psi_1 = 0$
- $\psi_{11} = 0$
- $M_r = 0$
- $G = 1$
- $g(x) = \exp(x)$