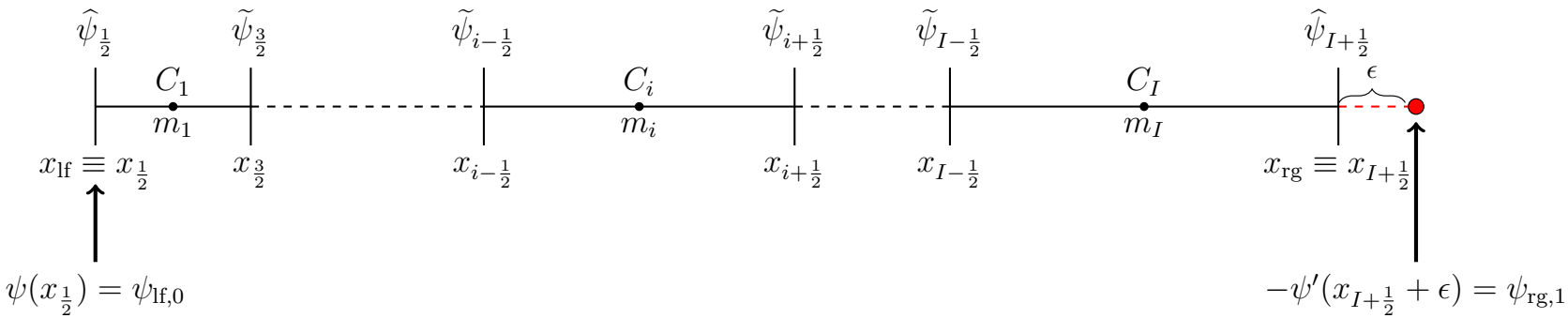
Pure Diffusion | Formulation

$$\begin{cases} -\psi'' = s & \text{in} \quad \Omega =]x_{\text{lf}}, x_{\text{rg}} + \epsilon[\\ \psi = \psi_{\text{lf},0} & \text{on} \quad x = x_{\text{lf}} \\ -\psi' = \psi_{\text{rg},1} & \text{on} \quad x = x_{\text{rg}} + \epsilon \end{cases}$$

Mesh



- C_i — cell i
- I — number of cells
- $x_{i-\frac{1}{2}}$, $x_{i+\frac{1}{2}}$ — boundary points of cell i
- h_i — length of cell i
- m_i — centroid of cell i

Polynomial Reconstructions | Inner Vertices

$$\psi_{i+\frac{1}{2},\mathrm{d}}(x) = \sum_{\alpha=0}^{\mathrm{d}} \mathcal{R}_{i+\frac{1}{2},\alpha}(x - x_{i+\frac{1}{2}})^{\alpha}$$

$$\min_{\mathcal{R}_{i+\frac{1}{2},0},\dots,\mathcal{R}_{i+\frac{1}{2},\mathrm{d}}} \quad \sum_{j \in \widehat{S}_{i+\frac{1}{2}}} \omega_j \left[\frac{1}{h_j} \int_{c_j} \psi_{i+\frac{1}{2},\mathrm{d}}(x)\mathrm{d}x - \psi_j \right]^2$$

This will be needed to approximate $\mathbf{F}_{i+\frac{1}{2}} \approx \mathcal{F}_{i+\frac{1}{2}} = \widetilde{\psi}'_{i+\frac{1}{2}}(x_{i+\frac{1}{2}})$

Polynomial Reconstructions | Left Boundary

$$\psi_{\frac{1}{2},\mathrm{d}}(x) = \sum_{\alpha=0}^{\mathrm{d}} \mathcal{R}_{\frac{1}{2},\alpha}(x - x_{\mathrm{lf}})^{\alpha}$$

$$\min_{\mathcal{R}_{\frac{1}{2},0},\ldots,\mathcal{R}_{\frac{1}{2},\mathrm{d}}} \quad \sum_{j \in \widehat{S}_{\frac{1}{2}}} \omega_j \left[\frac{1}{h_j} \int_{c_j} \psi_{\frac{1}{2},\mathrm{d}}(x)\mathrm{d}x - \psi_j \right]^2$$

$$\text{s.t.} \quad \psi_{\frac{1}{2},\mathrm{d}}(x_{\mathrm{lf}}) = \psi_{\mathrm{lf},0}$$

This will be needed to approximate $\mathbf{F}_{\frac{1}{2}} \approx \mathcal{F}_{\frac{1}{2}} = \psi'_{\frac{1}{2}}(x_{\mathrm{lf}})$

Polynomial Reconstructions | Right Boundary

$$\psi_{I+\frac{1}{2},\mathrm{d}}(x) = \sum_{\alpha=0}^{\mathrm{d}} \mathcal{R}_{I+\frac{1}{2},\alpha}(x - x_{\mathrm{rg}})^{\alpha}$$

$$\min_{\mathcal{R}_{I+\frac{1}{2},0},\ldots,\mathcal{R}_{I+\frac{1}{2},\mathrm{d}}} \quad \sum_{j \in \widehat{S}_{I+\frac{1}{2}}} \omega_j \left[\frac{1}{h_j} \int_{c_j} \psi_{I+\frac{1}{2},\mathrm{d}}(x)\mathrm{d}x - \psi_j \right]^2$$

$$\text{s.t.} \quad -\psi'_{I+\frac{1}{2},\mathrm{d}}(x_{\mathrm{rg}} + \epsilon) = \psi_{\mathrm{rg},1}$$

This will be needed to approximate $\mathbf{F}_{I+\frac{1}{2}} \approx \mathcal{F}_{I+\frac{1}{2}} = \widehat{\psi}'_{I+\frac{1}{2}}(x_{\mathrm{rg}})$

Tests

In this test we will consider:

- $\overline{\Omega} = [0, 1 + \epsilon]$
- $\psi(x) = \exp(x)$
- $\psi(0) = 1$
- $\varphi_{n2} = -\exp(1 + \epsilon)$

Tests | $\epsilon = 0$ | degree d

uniform mesh

	I	$E_{0,\infty}$	$O_{0,\infty}$
$\mathbb{P}_1$	10	1.09E−02	—
	20	2.68E−03	2.02
	30	1.19E−03	2.01
	40	6.67E−04	2.01
$\mathbb{P}_2$	10	4.93E−03	—
	20	1.38E−03	1.83
	30	8.01E−04	1.35
	40	3.88E−04	2.52
$\mathbb{P}_3$	10	2.99E−05	—
	20	1.93E−06	3.95
	30	3.86E−07	3.97
	40	1.23E−07	3.98
$\mathbb{P}_4$	10	1.15E−05	—
	20	1.20E−06	3.26
	30	2.00E−07	4.42
	40	6.69E−08	3.80
$\mathbb{P}_5$	10	9.53E−08	—
	20	2.00E−09	5.58
	30	1.91E−10	5.79
	40	3.53E−11	5.86

non-uniform mesh

	I	$E_{0,\infty}$	$O_{0,\infty}$
$\mathbb{P}_1$	10	2.18E−02	—
	20	5.50E−03	1.99
	30	2.56E−03	1.88
	40	1.60E−03	1.64
$\mathbb{P}_2$	10	4.28E−03	—
	20	2.05E−03	1.06
	30	1.70E−03	0.45
	40	4.96E−04	4.29
$\mathbb{P}_3$	10	4.11E−05	—
	20	2.29E−06	4.16
	30	7.21E−07	2.85
	40	1.76E−07	4.91
$\mathbb{P}_4$	10	1.07E−05	—
	20	8.77E−07	3.61
	30	2.26E−07	3.34
	40	6.37E−08	4.40
$\mathbb{P}_5$	10	7.38E−07	—
	20	1.01E−08	6.20
	30	1.05E−09	5.58
	40	2.04E−10	5.69

Tests | $\epsilon = \frac{h}{2}$ | degree d

uniform mesh

	I	$E_{0,\infty}$	$O_{0,\infty}$
$\mathbb{P}_1$	10	1.22E−01	—
	20	6.44E−02	0.92
	30	4.37E−02	0.95
	40	3.31E−02	0.97
$\mathbb{P}_2$	10	5.10E−03	—
	20	8.54E−04	2.58
	30	4.07E−04	1.83
	40	1.33E−04	3.89
$\mathbb{P}_3$	10	1.33E−04	—
	20	1.02E−05	3.70
	30	2.16E−06	3.83
	40	7.08E−07	3.88
$\mathbb{P}_4$	10	1.62E−05	—
	20	9.43E−07	4.10
	30	9.41E−08	5.69
	40	2.36E−08	4.81
$\mathbb{P}_5$	10	3.29E−07	—
	20	6.16E−09	5.74
	30	5.75E−10	5.85
	40	1.06E−10	5.89

non-uniform mesh

	I	$E_{0,\infty}$	$O_{0,\infty}$
$\mathbb{P}_1$	10	1.22E−01	—
	20	6.44E−02	0.92
	30	4.37E−02	0.95
	40	3.30E−02	0.98
$\mathbb{P}_2$	10	4.58E−03	—
	20	2.19E−03	1.07
	30	1.83E−03	0.44
	40	1.46E−04	8.79
$\mathbb{P}_3$	10	6.19E−05	—
	20	5.52E−06	3.49
	30	7.02E−07	5.09
	40	2.41E−07	3.72
$\mathbb{P}_4$	10	6.79E−06	—
	20	3.29E−07	4.37
	30	5.89E−08	4.25
	40	1.03E−08	6.07
$\mathbb{P}_5$	10	5.66E−07	—
	20	8.38E−09	6.08
	30	9.61E−10	5.34
	40	1.87E−10	5.68

Tests | $\epsilon = \frac{h}{2}$ | degree d+1

uniform mesh

	I	$E_{0,\infty}$	$O_{0,\infty}$
$\mathbb{P}_1/\mathbb{P}_2$	10	9.06E−03	—
	20	2.28E−03	1.99
	30	1.01E−03	2.00
	40	5.71E−04	2.00
$\mathbb{P}_2/\mathbb{P}_3$	10	2.36E−03	—
	20	3.87E−04	2.61
	30	2.44E−04	1.14
	40	6.54E−05	4.57
$\mathbb{P}_3/\mathbb{P}_4$	10	1.91E−05	—
	20	1.14E−06	4.07
	30	2.21E−07	4.04
	40	6.94E−08	4.03
$\mathbb{P}_4/\mathbb{P}_5$	10	5.81E−06	—
	20	5.99E−07	3.28
	30	3.38E−08	7.09
	40	7.70E−09	5.14
$\mathbb{P}_5/\mathbb{P}_6$	10	1.50E−07	—
	20	2.73E−09	5.78
	30	2.54E−10	5.86
	40	4.64E−11	5.90

non-uniform mesh

	I	$E_{0,\infty}$	$O_{0,\infty}$
$\mathbb{P}_1/\mathbb{P}_2$	10	2.02E−02	—
	20	5.28E−03	1.94
	30	2.45E−03	1.89
	40	1.46E−03	1.80
$\mathbb{P}_2/\mathbb{P}_3$	10	3.29E−03	—
	20	2.15E−03	0.61
	30	1.82E−03	0.40
	40	1.30E−04	9.19
$\mathbb{P}_3/\mathbb{P}_4$	10	4.21E−05	—
	20	2.34E−06	4.17
	30	7.18E−07	2.92
	40	1.77E−07	4.87
$\mathbb{P}_4/\mathbb{P}_5$	10	5.24E−06	—
	20	2.05E−07	4.68
	30	6.60E−08	2.79
	40	1.22E−08	5.86
$\mathbb{P}_5/\mathbb{P}_6$	10	5.45E−07	—
	20	8.16E−09	6.06
	30	9.52E−10	5.30
	40	1.86E−10	5.68

Tests | $\epsilon = h$ | degree d

uniform mesh

	I	$E_{0,\infty}$	$O_{0,\infty}$
$\mathbb{P}_1$	10	2.61E−01	—
	20	1.33E−01	0.97
	30	8.94E−02	0.98
	40	6.73E−02	0.99
$\mathbb{P}_2$	10	9.72E−03	—
	20	1.38E−03	2.81
	30	5.72E−04	2.17
	40	2.01E−04	3.64
$\mathbb{P}_3$	10	1.78E−04	—
	20	1.30E−05	3.78
	30	2.70E−06	3.87
	40	8.78E−07	3.91
$\mathbb{P}_4$	10	1.90E−05	—
	20	1.03E−06	4.20
	30	1.05E−07	5.62
	40	2.63E−08	4.82
$\mathbb{P}_5$	10	4.00E−07	—
	20	7.42E−09	5.75
	30	6.91E−10	5.86
	40	1.27E−10	5.90

non-uniform mesh

	I	$E_{0,\infty}$	$O_{0,\infty}$
$\mathbb{P}_1$	10	2.66E−01	—
	20	1.34E−01	0.98
	30	9.01E−02	0.99
	40	6.75E−02	1.00
$\mathbb{P}_2$	10	5.97E−03	—
	20	2.26E−03	1.40
	30	1.84E−03	0.51
	40	1.90E−04	7.89
$\mathbb{P}_3$	10	8.01E−05	—
	20	6.90E−06	3.54
	30	7.87E−07	5.35
	40	2.99E−07	3.37
$\mathbb{P}_4$	10	8.09E−06	—
	20	3.70E−07	4.45
	30	5.78E−08	4.58
	40	1.09E−08	5.80
$\mathbb{P}_5$	10	5.69E−07	—
	20	8.42E−09	6.08
	30	9.63E−10	5.35
	40	1.88E−10	5.69

Tests | $\epsilon = h$ | degree d+1

	uniform mesh			non-uniform mesh			uniform mesh		
	I	$E_{0,\infty}$	$O_{0,\infty}$	I	$E_{0,\infty}$	$O_{0,\infty}$	I	$E_{c,\infty}$	$O_{c,\infty}$
$\mathbb{P}_1/\mathbb{P}_2$	10	8.99E−03	—	10	2.02E−02	—	10	7.64E−02	—
	20	2.27E−03	1.98	20	5.27E−03	1.93	20	3.74E−02	1.03
	30	1.01E−03	1.99	30	2.45E−03	1.89	30	2.48E−02	1.02
	40	5.71E−04	2.00	40	1.46E−03	1.80	40	1.85E−02	1.01
$\mathbb{P}_2/\mathbb{P}_3$	10	2.42E−03	—	10	3.33E−03	—	10	1.10E−02	—
	20	4.02E−04	2.59	20	2.15E−03	0.63	20	3.05E−03	1.84
	30	2.55E−04	1.13	30	1.83E−03	0.41	30	1.41E−03	1.91
	40	6.98E−05	4.50	40	1.31E−04	9.15	40	8.05E−04	1.94
$\mathbb{P}_3/\mathbb{P}_4$	10	1.90E−05	—	10	4.22E−05	—	10	5.29E−04	—
	20	1.13E−06	4.07	20	2.35E−06	4.17	20	6.13E−05	3.11
	30	2.21E−07	4.04	30	7.18E−07	2.92	30	1.77E−05	3.06
	40	6.93E−08	4.02	40	1.77E−07	4.87	40	7.37E−06	3.04
$\mathbb{P}_4/\mathbb{P}_5$	10	5.93E−06	—	10	5.23E−06	—	10	9.27E−05	—
	20	5.85E−07	3.34	20	2.07E−07	4.66	20	5.12E−06	4.18
	30	3.34E−08	7.06	30	6.60E−08	2.81	30	9.71E−07	4.10
	40	7.62E−09	5.13	40	1.24E−08	5.82	40	3.01E−07	4.07
$\mathbb{P}_5/\mathbb{P}_6$	10	1.62E−07	—	10	5.45E−07	—	10	2.96E−06	—
	20	2.96E−09	5.78	20	8.16E−09	6.06	20	8.15E−08	5.18
	30	2.75E−10	5.86	30	9.52E−10	5.30	30	1.03E−08	5.10
	40	5.03E−11	5.90	40	1.86E−10	5.68	40	2.39E−09	5.07

Tests | $\epsilon = h^2$ | degree d

uniform mesh

	I	$E_{0,\infty}$	$O_{0,\infty}$
$\mathbb{P}_1$	10	1.51E−02	—
	20	3.95E−03	1.93
	30	1.78E−03	1.96
	40	1.01E−03	1.97
$\mathbb{P}_2$	10	2.56E−03	—
	20	9.50E−04	1.43
	30	6.20E−04	1.05
	40	2.93E−04	2.60
$\mathbb{P}_3$	10	6.39E−05	—
	20	4.51E−06	3.82
	30	9.26E−07	3.91
	40	2.99E−07	3.93
$\mathbb{P}_4$	10	1.34E−05	—
	20	9.97E−07	3.75
	30	1.29E−07	5.05
	40	3.95E−08	4.11
$\mathbb{P}_5$	10	2.40E−07	—
	20	4.23E−09	5.83
	30	3.86E−10	5.90
	40	7.02E−11	5.93

non-uniform mesh

	I	$E_{0,\infty}$	$O_{0,\infty}$
$\mathbb{P}_1$	10	1.94E−02	—
	20	5.19E−03	1.90
	30	2.43E−03	1.87
	40	1.45E−03	1.78
$\mathbb{P}_2$	10	3.10E−03	—
	20	1.95E−03	0.67
	30	1.74E−03	0.27
	40	3.56E−04	5.53
$\mathbb{P}_3$	10	4.11E−05	—
	20	3.47E−06	3.56
	30	7.10E−07	3.91
	40	1.74E−07	4.90
$\mathbb{P}_4$	10	7.07E−06	—
	20	4.68E−07	3.92
	30	9.43E−08	3.95
	40	2.54E−08	4.55
$\mathbb{P}_5$	10	5.80E−07	—
	20	8.49E−09	6.09
	30	9.69E−10	5.35
	40	1.89E−10	5.69

Tests | $\epsilon = h^2$ | degree d+1

uniform mesh

	I	$E_{0,\infty}$	$O_{0,\infty}$
$\mathbb{P}_1/\mathbb{P}_2$	10	9.29E−03	—
	20	2.31E−03	2.01
	30	1.02E−03	2.01
	40	5.75E−04	2.00
$\mathbb{P}_2/\mathbb{P}_3$	10	2.89E−03	—
	20	7.20E−04	2.00
	30	4.91E−04	0.95
	40	1.97E−04	3.17
$\mathbb{P}_3/\mathbb{P}_4$	10	1.94E−05	—
	20	1.15E−06	4.08
	30	2.22E−07	4.04
	40	6.98E−08	4.03
$\mathbb{P}_4/\mathbb{P}_5$	10	6.57E−06	—
	20	7.40E−07	3.15
	30	6.79E−08	5.89
	40	2.03E−08	4.20
$\mathbb{P}_5/\mathbb{P}_6$	10	1.27E−07	—
	20	2.18E−09	5.86
	30	1.98E−10	5.92
	40	3.57E−11	5.94

non-uniform mesh

	I	$E_{0,\infty}$	$O_{0,\infty}$
$\mathbb{P}_1/\mathbb{P}_2$	10	2.06E−02	—
	20	5.34E−03	1.95
	30	2.46E−03	1.91
	40	1.47E−03	1.79
$\mathbb{P}_2/\mathbb{P}_3$	10	3.09E−03	—
	20	2.05E−03	0.60
	30	1.79E−03	0.33
	40	1.81E−04	7.97
$\mathbb{P}_3/\mathbb{P}_4$	10	4.20E−05	—
	20	2.34E−06	4.17
	30	7.18E−07	2.91
	40	1.77E−07	4.88
$\mathbb{P}_4/\mathbb{P}_5$	10	5.34E−06	—
	20	2.27E−07	4.55
	30	5.74E−08	3.39
	40	1.42E−08	4.85
$\mathbb{P}_5/\mathbb{P}_6$	10	5.53E−07	—
	20	8.23E−09	6.07
	30	9.56E−10	5.31
	40	1.87E−10	5.68