In this tests we introduced the finite differences with the unknowns (ϕ_i) in the centroids of the cells.

First, we set the four boundary conditions, two of them in each boundary of the domain of the mesh: ϕ_{l}, ϕ_{ll}, ϕ_{r}, ϕ_{rr}.

Follow that, we have a fourth order differential equation (biharmonic operator):

$$\begin{aligned}
-\phi^{(4)} &= s && \text{in } \Omega =]x_{\mathrm{lf}}, x_{\mathrm{rg}}[\\
\phi &= \phi_{\mathrm{lf},0} && \text{on } x = x_{\mathrm{lf}} \\
\phi^{(1)} &= \phi_{\mathrm{lf},1} && \text{on } x = x_{\mathrm{lf}} \\
\phi &= \phi_{\mathrm{rg},0} && \text{on } x = x_{\mathrm{rg}} \\
\phi^{(1)} &= \phi_{\mathrm{rg},1} && \text{on } x = x_{\mathrm{rg}}
\end{aligned}$$

where the ϕ function is the exact function and the s is the source term. Following, we approximate the exact function for a polynomial of degree 4 that needs 5 points,

$$\phi(x) \approx p_4(x) = ax^4 + bx^3 + cx^2 + dx + e$$

$$\begin{aligned}
& -\phi^{(4)} = s \\
\Leftrightarrow & -(ax^4 + bx^3 + cx^2 + dx + e)^{(4)} = s(m_i) \\
\Leftrightarrow & -24a = s(m_i)
\end{aligned}$$

To calculate the a value we need to solve a linear system that:

- for the fist cell:

  Note: consider that $x_{\mathrm{lf}} = 0$

$$\begin{aligned}
& EQ1 : p_4(0) = \phi_{\mathrm{l}} \Leftrightarrow e = \phi_{\mathrm{l}} \\
& EQ2 : p_4^{(1)}(0) = \phi_{\mathrm{ll}} \Leftrightarrow d = \phi_{\mathrm{ll}} \\
& EQ3 : p_4\left(\frac{h}{2}\right) = \phi_1 \Leftrightarrow a\left(\frac{h}{2}\right)^4 + b\left(\frac{h}{2}\right)^3 + c\left(\frac{h}{2}\right)^2 + d\left(\frac{h}{2}\right) + e = \phi_1 \\
& EQ4 : p_4\left(\frac{3h}{2}\right) = \phi_2 \Leftrightarrow a\left(\frac{3h}{2}\right)^4 + b\left(\frac{3h}{2}\right)^3 + c\left(\frac{3h}{2}\right)^2 + d\left(\frac{3h}{2}\right) + e = \phi_2 \\
& EQ5 : p_4\left(\frac{5h}{2}\right) = \phi_3 \Leftrightarrow a\left(\frac{5h}{2}\right)^4 + b\left(\frac{5h}{2}\right)^3 + c\left(\frac{5h}{2}\right)^2 + d\left(\frac{5h}{2}\right) + e = \phi_3
\end{aligned}$$

- for the second cell:

$$
\begin{aligned}
&EQ1: p_4(0) = \phi_1 \Leftrightarrow e = \phi_1\\
&EQ2: p_4\left(\frac{h}{2}\right) = \phi_1 \Leftrightarrow a\left(\frac{h}{2}\right)^4 + b\left(\frac{h}{2}\right)^3 + c\left(\frac{h}{2}\right)^2 + d\left(\frac{h}{2}\right) + e = \phi_1\\
&EQ3: p_4\left(\frac{3h}{2}\right) = \phi_2 \Leftrightarrow a\left(\frac{3h}{2}\right)^4 + b\left(\frac{3h}{2}\right)^3 + c\left(\frac{3h}{2}\right)^2 + d\left(\frac{3h}{2}\right) + e = \phi_2\\
&EQ4: p_4\left(\frac{5h}{2}\right) = \phi_3 \Leftrightarrow a\left(\frac{5h}{2}\right)^4 + b\left(\frac{5h}{2}\right)^3 + c\left(\frac{5h}{2}\right)^2 + d\left(\frac{5h}{2}\right) + e = \phi_3\\
&EQ5: p_4\left(\frac{7h}{2}\right) = \phi_4 \Leftrightarrow a\left(\frac{7h}{2}\right)^4 + b\left(\frac{7h}{2}\right)^3 + c\left(\frac{7h}{2}\right)^2 + d\left(\frac{7h}{2}\right) + e = \phi_4
\end{aligned}
$$

- for the $i = 3, \ldots, I-2$ cells:

$$
\begin{aligned}
&EQ1: p_4(m_i - 2h) = \phi_{i-2} \Leftrightarrow a(m_i - 2h)^4 + b(m_i - 2h)^3 + c(m_i - 2h)^2 + d(m_i - 2h) + e = \phi_{i-2}\\
&EQ2: p_4(m_i - h) = \phi_{i-1} \Leftrightarrow a(m_i - h)^4 + b(m_i - h)^3 + c(m_i - h)^2 + d(m_i - h) + e = \phi_{i-1}\\
&EQ3: p_4(m_i) = \phi_i \Leftrightarrow a(m_i)^4 + b(m_i)^3 + c(m_i)^2 + d(m_i) + e = \phi_i\\
&EQ4: p_4(m_i + h) = \phi_{i+1} \Leftrightarrow a(m_i + h)^4 + b(m_i + h)^3 + c(m_i + h)^2 + d(m_i + h) + e = \phi_{i+1}\\
&EQ5: p_4(m_i + 2h) = \phi_{i+2} \Leftrightarrow a(m_i + 2h)^4 + b(m_i + 2h)^3 + c(m_i + 2h)^2 + d(m_i + 2h) + e = \phi_{i+2}
\end{aligned}
$$

- for the penultimate cell:

$$
\begin{aligned}
&EQ1: p_4(m_{I-3}) = \phi_{I-3} \Leftrightarrow a(m_{I-3})^4 + b(m_{I-3})^3 + c(m_{I-3})^2 + d(m_{I-3}) + e = \phi_{I-3}\\
&EQ2: p_4(m_{I-2}) = \phi_{I-2} \Leftrightarrow a(m_{I-2})^4 + b(m_{I-2})^3 + c(m_{I-2})^2 + d(m_{I-2}) + e = \phi_{I-2}\\
&EQ3: p_4(m_{I-1}) = \phi_{I-1} \Leftrightarrow a(m_{I-1})^4 + b(m_{I-1})^3 + c(m_{I-1})^2 + d(m_{I-1}) + e = \phi_{I-1}\\
&EQ4: p_4(m_I) = \phi_I \Leftrightarrow a(m_I)^4 + b(m_I)^3 + c(m_I)^2 + d(m_I) + e = \phi_I\\
&EQ5: p_4\left(m_I + \frac{h}{2}\right) = \phi_{\mathrm{r}} \Leftrightarrow a\left(m_I + \frac{h}{2}\right)^4 + b\left(m_I + \frac{h}{2}\right)^3 + c\left(m_I + \frac{h}{2}\right)^2 + d\left(m_I + \frac{h}{2}\right) + e = \phi_{\mathrm{r}}
\end{aligned}
$$

- for the last cell:

$$
\begin{aligned}
&EQ1: p_4(m_{I-2}) = \phi_{I-2} \Leftrightarrow a(m_{I-2})^4 + b(m_{I-2})^3 + c(m_{I-2})^2 + d(m_{I-2}) + e \\
&EQ2: p_4(m_{I-1}) = \phi_{I-1} \Leftrightarrow a(m_{I-1})^4 + b(m_{I-1})^3 + c(m_{I-1})^2 + d(m_{I-1}) + e \\
&EQ3: p_4(m_I) = \phi_I \Leftrightarrow a(m_I)^4 + b(m_I)^3 + c(m_I)^2 + d(m_I) + e \\
&EQ4: p_4\left(m_I + \frac{h}{2}\right) = \phi_{\mathrm{r}} \Leftrightarrow a\left(m_I + \frac{h}{2}\right)^4 + b\left(m_I + \frac{h}{2}\right)^3 + c\left(m_I + \frac{h}{2}\right)^2 + d\left(m_I + \frac{h}{2}\right) + e = \phi_{\mathrm{r}} \\
&EQ5: p_4^{(1)}\left(m_I + \frac{h}{2}\right) = \phi_{\mathrm{rr}} \Leftrightarrow 4a\left(m_I + \frac{h}{2}\right)^3 + 3b\left(m_I + \frac{h}{2}\right)^2 + 2c\left(m_I + \frac{h}{2}\right) + d = \phi_{\mathrm{rr}}
\end{aligned}
$$

And, for the first cell we have:

$$-\frac{48}{h^4}\phi_1 + \frac{32}{3h^4}\phi_2 - \frac{48}{25h^4}\phi_3 = S_1 - \frac{64}{5h^3}\phi_{\mathrm{ll}} - \frac{2944}{75h^4}\phi_{\mathrm{l}}$$

for the second cell we have:

$$\frac{8}{h^4}\phi_1 - \frac{8}{h^4}\phi_2 + \frac{24}{5h^4}\phi_3 - \frac{8}{7h^4}\phi_4 = S_2 + \frac{128}{35h^4}\phi_{\mathrm{l}}$$

for the $i = 3, \ldots, I-2$ cells we have:

$$-\frac{1}{h^4}\phi_{i-2} + \frac{4}{h^4}\phi_{i-1} - \frac{6}{h^4}\phi_i + \frac{4}{h^4}\phi_{i+1} - \frac{1}{h^4}\phi_{i+2} = S_i$$

for the penultimate cell we have:

$$-\frac{8}{7h^4}\phi_{I-3} + \frac{24}{5h^4}\phi_{I-2} - \frac{8}{h^4}\phi_{I-1} + \frac{8}{h^4}\phi_I = S_{I-1} + \frac{128}{35h^4}\phi_{\mathrm{r}}$$

for the last cell we have:

$$-\frac{48}{25h^4}\phi_{I-2} + \frac{32}{3h^4}\phi_{I-1} - \frac{48}{h^4}\phi_I = S_I + \frac{64}{5h^3}\phi_{\mathrm{rr}} - \frac{2944}{75h^4}\phi_{\mathrm{r}}$$

In this test we will consider:

- $\phi(x) = \exp(x)$
- $\phi_{\mathrm{l}} = 1$;
- $\phi_{\mathrm{ll}} = 1$;
- $\phi_{\mathrm{r}} = \exp(1)$;
- $\phi_{\mathrm{rr}} = \exp(1)$;
- $g(x) = -\exp(x)$.

Table 1: Table of errors and convergence order of this test.

I	$\mathrm{E}_{0,I}(E_1)$	$\mathrm{E}_{0,I}(O_1)$	$\mathrm{E}_{0,I}(E_\infty)$	$\mathrm{E}_{0,I}(O_\infty)$	$\mathrm{E}_{0,I}(E_c)$	$\mathrm{E}_{0,I}(O_c)$
10	7.19E−04	—	1.08E−03	—	9.95E−02	—
20	1.80E−04	2.00	2.76E−04	1.96	5.20E−02	0.94
40	4.50E−05	2.00	6.99E−05	1.98	2.66E−02	0.97
80	1.12E−05	2.00	1.76E−05	1.99	1.34E−02	0.98
160	2.81E−06	2.00	4.41E−06	2.00	6.76E−03	0.99