

In this tests we consider:

- $\psi(x) = \sin(6\pi x) \exp(x)$
- $\psi_l = 0$
- $\psi_{ll} = 6\pi$
- $\psi_r = 0$
- $\psi_{rr} = 6e\pi$
- $g(x) = \exp(x) (24\pi(36\pi^2 - 1) \cos(6\pi x) - (1296\pi^4 - 216\pi^2 + 1) \sin(6\pi x))$
- the different PRO schemes are:
  - PRO1 — weak ( $R = A^\dagger B$ ) and degree  $d$
  - PRO2 — strong (constrained least squares) and degree  $d$
  - PRO3 — weak ( $R = A^\dagger B$ ) and degree  $d + 1$
  - PRO4 — strong (constrained least squares) and degree  $d + 1$
  - PRO5 — weak ( $R = A^\dagger B$ ) and degree  $d + 2$
  - PRO6 — strong (constrained least squares) and degree  $d + 2$

Table 1: Numerical results of PRO1 scheme.

	$I$	$\omega = 1 1, 1$		$\omega = 1 3, 1$		$\omega = 1 3, 3$		$\omega = 1 3, 10$	
		$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$
$\mathbb{P}_3(4)$	20	1.91E+01	—	1.64E+01	—	1.64E+01	—	1.64E+01	—
	40	1.98E+00	3.27	1.57E+00	3.38	1.57E+00	3.38	1.57E+00	3.38
	80	1.30E−01	3.92	9.14E−02	4.11	9.14E−02	4.11	9.14E−02	4.11
	160	2.95E−02	2.14	3.33E−02	1.46	3.33E−02	1.46	3.33E−02	1.46
	320	1.04E−02	1.51	1.08E−02	1.63	1.08E−02	1.63	1.08E−02	1.63
	640	2.89E−03	1.85	2.93E−03	1.88	2.93E−03	1.88	2.93E−03	1.88
	1280	7.54E−04	1.94	7.59E−04	1.95	7.59E−04	1.95	7.59E−04	1.95
	2560	1.92E−04	1.97	1.92E−04	1.98	1.93E−04	1.98	1.93E−04	1.98
$\mathbb{P}_5(6)$	20	1.82E+00	—	1.24E+00	—	1.24E+00	—	1.24E+00	—
	40	2.02E−01	3.18	1.99E−01	2.64	1.99E−01	2.64	1.99E−01	2.64
	80	8.66E−03	4.54	7.57E−03	4.71	7.57E−03	4.71	7.57E−03	4.71
	160	3.28E−04	4.72	3.05E−04	4.63	3.05E−04	4.63	3.05E−04	4.63
	320	1.49E−05	4.46	1.52E−05	4.32	1.52E−05	4.32	1.52E−05	4.32
	640	9.65E−07	3.95	8.75E−07	4.12	8.73E−07	4.13	8.78E−07	4.12
	1280	3.40E−08	4.83	1.06E−07	3.04	6.29E−08	3.79	6.02E−08	3.87
	2560	2.49E−07	↑	1.20E−06	↑	6.22E−07	↑	1.02E−06	↑

Table 2: Numerical results of PRO2 scheme.

	$I$	$\omega = 1 1, 1$		$\omega = 1 3, 1$		$\omega = 1 3, 3$		$\omega = 1 3, 10$	
		$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$
$\mathbb{P}_3(4)$	20	1.91E+01	—	1.64E+01	—	1.64E+01	—	1.64E+01	—
	40	1.98E+00	3.27	1.57E+00	3.38	1.57E+00	3.38	1.57E+00	3.38
	80	1.30E−01	3.92	9.14E−02	4.11	9.14E−02	4.11	9.14E−02	4.11
	160	2.95E−02	2.14	3.33E−02	1.46	3.33E−02	1.46	3.33E−02	1.46
	320	1.04E−02	1.51	1.08E−02	1.63	1.08E−02	1.63	1.08E−02	1.63
	640	2.89E−03	1.85	2.93E−03	1.88	2.93E−03	1.88	2.93E−03	1.88
	1280	7.54E−04	1.94	7.59E−04	1.95	7.59E−04	1.95	7.59E−04	1.95
	2560	1.92E−04	1.97	1.92E−04	1.98	1.92E−04	1.98	1.92E−04	1.98
$\mathbb{P}_4(6)$	20	1.01E+01	—	8.51E+00	—	8.51E+00	—	8.51E+00	—
	40	1.27E+00	2.99	1.11E+00	2.94	1.11E+00	2.94	1.11E+00	2.94
	80	2.96E−01	2.10	2.59E−01	2.10	2.59E−01	2.10	2.59E−01	2.10
	160	6.44E−02	2.20	5.54E−02	2.23	5.54E−02	2.23	5.54E−02	2.23
	320	1.53E−02	2.08	1.30E−02	2.09	1.30E−02	2.09	1.30E−02	2.09
	640	3.77E−03	2.02	3.21E−03	2.02	3.21E−03	2.02	3.21E−03	2.02
	1280	9.39E−04	2.01	7.98E−04	2.01	7.98E−04	2.01	7.98E−04	2.01
	2560	2.35E−04	2.00	2.00E−04	2.00	2.00E−04	2.00	2.00E−04	2.00
$\mathbb{P}_5(6)$	20	1.82E+00	—	1.24E+00	—	1.24E+00	—	1.24E+00	—
	40	2.02E−01	3.18	1.99E−01	2.64	1.99E−01	2.64	1.99E−01	2.64
	80	8.66E−03	4.54	7.57E−03	4.71	7.57E−03	4.71	7.57E−03	4.71
	160	3.28E−04	4.72	3.05E−04	4.63	3.05E−04	4.63	3.05E−04	4.63
	320	1.49E−05	4.46	1.52E−05	4.32	1.52E−05	4.32	1.52E−05	4.32
	640	9.63E−07	3.95	8.68E−07	4.13	8.68E−07	4.13	8.68E−07	4.13
	1280	5.70E−07	0.76	4.96E−07	0.81	4.80E−07	0.86	5.28E−07	0.72
	2560	4.49E−07	0.34	8.62E−07	↑	8.62E−07	↑	2.05E−06	↑

Table 3: Numerical results of PRO3 scheme.

	$I$	$\omega = 1 1, 1$		$\omega = 1 3, 1$		$\omega = 1 3, 3$		$\omega = 1 3, 10$	
		$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$
$\mathbb{P}_3(4)$	20	4.23E+00	—	1.93E+00	—	1.93E+00	—	1.93E+00	—
	40	3.90E−01	3.44	6.95E−01	1.48	6.95E−01	1.48	6.95E−01	1.48
	80	1.60E−01	1.29	1.86E−01	1.91	1.86E−01	1.91	1.86E−01	1.91
	160	4.59E−02	1.80	4.80E−02	1.95	4.80E−02	1.95	4.80E−02	1.95
	320	1.20E−02	1.93	1.22E−02	1.97	1.22E−02	1.97	1.22E−02	1.97
	640	3.07E−03	1.97	3.09E−03	1.98	3.09E−03	1.98	3.09E−03	1.98
	1280	7.75E−04	1.99	7.77E−04	1.99	7.77E−04	1.99	7.77E−04	1.99
	2560	1.94E−04	1.99	1.95E−04	2.00	1.95E−04	2.00	1.95E−04	2.00
$\mathbb{P}_4(6)$	20	6.37E+00	—	1.33E+00	—	1.33E+00	—	1.33E+00	—
	40	1.24E+00	2.36	1.05E+00	0.34	1.05E+00	0.34	1.05E+00	0.34
	80	2.87E−01	2.11	2.30E−01	2.19	2.30E−01	2.19	2.30E−01	2.19
	160	6.26E−02	2.20	5.29E−02	2.12	5.29E−02	2.12	5.29E−02	2.12
	320	1.51E−02	2.05	1.29E−02	2.04	1.29E−02	2.04	1.29E−02	2.04
	640	3.76E−03	2.00	3.20E−03	2.01	3.20E−03	2.01	3.20E−03	2.01
	1280	9.38E−04	2.00	7.97E−04	2.00	7.97E−04	2.00	7.98E−04	2.00
	2560	2.35E−04	2.00	2.00E−04	1.99	2.00E−04	2.00	2.00E−04	1.99
$\mathbb{P}_5(6)$	20	1.39E+00	—	1.27E+00	—	1.27E+00	—	1.27E+00	—
	40	1.48E−01	3.24	1.47E−01	3.11	1.47E−01	3.11	1.47E−01	3.11
	80	6.20E−03	4.57	6.03E−03	4.60	6.03E−03	4.60	6.03E−03	4.60
	160	2.79E−04	4.48	2.69E−04	4.49	2.69E−04	4.49	2.69E−04	4.49
	320	1.58E−05	4.14	1.44E−05	4.23	1.44E−05	4.23	1.44E−05	4.23
	640	9.64E−07	4.04	8.68E−07	4.05	8.61E−07	4.06	8.68E−07	4.05
	1280	1.21E−07	2.99	8.40E−08	3.37	7.47E−08	3.53	1.98E−08	5.45
	2560	2.24E−06	↑	8.94E−07	↑	6.56E−07	↑	5.27E−07	↑

Table 4: Numerical results of PRO4 scheme.

	$I$	$\omega = 1 1, 1$		$\omega = 1 3, 1$		$\omega = 1 3, 3$		$\omega = 1 3, 10$	
		$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$
$\mathbb{P}_3(4)$	20	4.23E+00	—	1.93E+00	—	1.93E+00	—	1.93E+00	—
	40	3.90E−01	3.44	6.95E−01	1.48	6.95E−01	1.48	6.95E−01	1.48
	80	1.60E−01	1.29	1.86E−01	1.91	1.86E−01	1.91	1.86E−01	1.91
	160	4.59E−02	1.80	4.80E−02	1.95	4.80E−02	1.95	4.80E−02	1.95
	320	1.20E−02	1.93	1.22E−02	1.97	1.22E−02	1.97	1.22E−02	1.97
	640	3.07E−03	1.97	3.09E−03	1.98	3.09E−03	1.98	3.09E−03	1.98
	1280	7.75E−04	1.99	7.77E−04	1.99	7.77E−04	1.99	7.77E−04	1.99
	2560	1.95E−04	1.99	1.95E−04	2.00	1.95E−04	2.00	1.95E−04	2.00
$\mathbb{P}_4(6)$	20	6.37E+00	—	1.33E+00	—	1.33E+00	—	1.33E+00	—
	40	1.24E+00	2.36	1.05E+00	0.34	1.05E+00	0.34	1.05E+00	0.34
	80	2.87E−01	2.11	2.30E−01	2.19	2.30E−01	2.19	2.30E−01	2.19
	160	6.26E−02	2.20	5.29E−02	2.12	5.29E−02	2.12	5.29E−02	2.12
	320	1.51E−02	2.05	1.29E−02	2.04	1.29E−02	2.04	1.29E−02	2.04
	640	3.76E−03	2.00	3.20E−03	2.01	3.20E−03	2.01	3.20E−03	2.01
	1280	9.39E−04	2.00	7.98E−04	2.00	7.98E−04	2.00	7.98E−04	2.00
	2560	2.35E−04	2.00	2.00E−04	2.00	2.00E−04	2.00	2.00E−04	2.00
$\mathbb{P}_5(6)$	20	1.39E+00	—	1.27E+00	—	1.27E+00	—	1.27E+00	—
	40	1.48E−01	3.24	1.47E−01	3.11	1.47E−01	3.11	1.47E−01	3.11
	80	6.20E−03	4.57	6.03E−03	4.60	6.03E−03	4.60	6.03E−03	4.60
	160	2.79E−04	4.48	2.69E−04	4.49	2.69E−04	4.49	2.69E−04	4.49
	320	1.49E−05	4.22	1.57E−04	0.77	1.57E−04	0.77	1.57E−04	0.77
	640	8.61E−06	0.79	9.26E−06	4.09	9.26E−06	4.09	9.26E−06	4.09
	1280	5.52E−07	3.96	4.81E−07	4.27	5.11E−07	4.18	4.76E−07	4.28
	2560	1.11E−06	↑	5.97E−07	↑	5.97E−07	↑	5.53E−07	↑

Table 5: Numerical results of PRO5 scheme.

	$I$	$\omega = 1 1, 1$		$\omega = 1 3, 1$		$\omega = 1 3, 3$		$\omega = 1 3, 10$	
		$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$
$\mathbb{P}_3(4)$	20	1.28E+01	—	3.77E+00	—	3.77E+00	—	3.77E+00	—
	40	1.60E+00	2.99	3.39E−01	3.47	3.39E−01	3.47	3.39E−01	3.47
	80	1.05E−01	3.93	1.35E−01	1.33	1.35E−01	1.33	1.35E−01	1.33
	160	3.11E−02	1.76	4.35E−02	1.63	4.35E−02	1.63	4.35E−02	1.63
	320	1.06E−02	1.56	1.18E−02	1.88	1.18E−02	1.88	1.18E−02	1.88
	640	2.91E−03	1.86	3.05E−03	1.95	3.05E−03	1.95	3.05E−03	1.95
	1280	7.57E−04	1.95	7.72E−04	1.98	7.77E−04	1.97	7.77E−04	1.97
	2560	1.94E−04	1.96	1.95E−04	1.99	1.95E−04	2.00	1.95E−04	2.00
$\mathbb{P}_4(6)$	20	6.51E+00	—	1.86E+00	—	1.86E+00	—	1.86E+00	—
	40	1.15E+00	2.50	9.97E−01	0.90	9.97E−01	0.90	9.97E−01	0.90
	80	2.64E−01	2.12	2.30E−01	2.12	2.30E−01	2.12	2.30E−01	2.12
	160	6.24E−02	2.08	5.31E−02	2.12	5.31E−02	2.12	5.31E−02	2.12
	320	1.52E−02	2.04	1.29E−02	2.04	1.29E−02	2.04	1.29E−02	2.04
	640	3.76E−03	2.01	3.20E−03	2.01	3.20E−03	2.01	3.20E−03	2.01
	1280	9.38E−04	2.00	7.97E−04	2.00	7.97E−04	2.00	7.97E−04	2.00
	2560	2.35E−04	2.00	2.00E−04	1.99	2.00E−04	1.99	2.00E−04	1.99
$\mathbb{P}_5(6)$	20	1.83E+00	—	1.55E+00	—	1.55E+00	—	1.55E+00	—
	40	1.08E−01	4.08	1.79E−01	3.12	1.79E−01	3.12	1.79E−01	3.12
	80	6.78E−03	4.00	6.79E−03	4.72	6.79E−03	4.72	6.79E−03	4.72
	160	1.46E−04	5.54	2.87E−04	4.56	2.87E−04	4.56	2.87E−04	4.56
	320	1.59E−05	3.20	1.44E−05	4.32	1.44E−05	4.32	1.44E−05	4.32
	640	9.59E−07	4.05	8.68E−07	4.05	8.65E−07	4.05	8.69E−07	4.05
	1280	1.32E−07	2.86	1.61E−07	2.43	8.08E−08	3.42	2.58E−08	5.07
	2560	2.45E−06	↑	6.20E−07	↑	1.20E−06	↑	1.35E−06	↑

Table 6: Numerical results of PRO6 scheme.

	$I$	$\omega = 1 1, 1$		$\omega = 1 3, 1$		$\omega = 1 3, 3$		$\omega = 1 3, 10$	
		$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$	$E_{\infty,0}$	$O_{\infty,0}$
$\mathbb{P}_3(4)$	20	1.28E+01	—	3.77E+00	—	3.77E+00	—	3.77E+00	—
	40	1.60E+00	2.99	3.39E−01	3.47	3.39E−01	3.47	3.39E−01	3.47
	80	1.05E−01	3.93	1.35E−01	1.33	1.35E−01	1.33	1.35E−01	1.33
	160	3.11E−02	1.76	4.35E−02	1.63	4.35E−02	1.63	4.35E−02	1.63
	320	1.06E−02	1.56	1.18E−02	1.88	1.18E−02	1.88	1.18E−02	1.88
	640	3.07E−03	1.78	3.09E−03	1.93	3.09E−03	1.93	3.09E−03	1.93
	1280	7.75E−04	1.99	7.77E−04	1.99	7.77E−04	1.99	7.77E−04	1.99
	2560	1.95E−04	1.99	1.95E−04	2.00	1.95E−04	2.00	1.95E−04	2.00
$\mathbb{P}_4(6)$	20	6.51E+00	—	1.86E+00	—	1.86E+00	—	1.86E+00	—
	40	1.15E+00	2.50	9.97E−01	0.90	9.97E−01	0.90	9.97E−01	0.90
	80	2.64E−01	2.12	2.30E−01	2.12	2.30E−01	2.12	2.30E−01	2.12
	160	6.24E−02	2.08	5.31E−02	2.12	5.31E−02	2.12	5.31E−02	2.12
	320	1.51E−02	2.05	1.31E−02	2.02	1.31E−02	2.02	1.31E−02	2.02
	640	3.77E−03	2.00	3.21E−03	2.03	3.21E−03	2.03	3.21E−03	2.03
	1280	9.39E−04	2.01	7.98E−04	2.01	7.98E−04	2.01	7.98E−04	2.01
	2560	2.35E−04	2.00	2.00E−04	2.00	2.00E−04	2.00	1.99E−04	2.00
$\mathbb{P}_5(6)$	20	1.83E+00	—	1.55E+00	—	1.55E+00	—	1.55E+00	—
	40	1.08E−01	4.08	1.79E−01	3.12	1.79E−01	3.12	1.79E−01	3.12
	80	6.78E−03	4.00	6.79E−03	4.72	6.79E−03	4.72	6.79E−03	4.72
	160	3.29E−04	4.37	3.05E−04	4.47	3.05E−04	4.47	3.05E−04	4.47
	320	1.49E−05	4.46	1.57E−04	0.96	1.57E−04	0.96	1.57E−04	0.96
	640	8.61E−06	0.79	9.26E−06	4.09	9.26E−06	4.09	9.26E−06	4.09
	1280	5.71E−07	3.91	4.22E−07	4.46	5.38E−07	4.11	5.15E−07	4.17
	2560	4.29E−07	0.41	5.08E−07	↑	5.08E−07	0.08	8.81E−07	↑