Modeling Projectile Motion with and without Air Resistance: Numerical Implementation and Analysis using Python and the Euler Method[*]

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In this study, we address the problem of projectile motion through the numerical resolution of differential equations using the Euler method implemented in Python. The investigation explores the impact of air resistance on the trajectory and maximum horizontal range of a projectile. Two scenarios were analyzed: one without air resistance and another with a specific drag coefficient of $B/m = 0.00004\,\mathrm{s}^{-1}$. Our results show that, in the absence of air resistance, the theoretical angle for achieving the maximum horizontal range is 45 degrees, consistent with fundamental physics principles. However, when considering air resistance, the optimal angle for maximum range reduces to approximately 38.9 degrees. This deviation highlights the significant influence of air resistance on projectile motion, emphasizing the need to account for drag effects in practical applications.

I. INTRODUCTION

The study of projectile motion is a fundamental topic in classical physics and various engineering applications. Solving the differential equations that describe a projectile's motion allows us to predict its trajectory and range based on initial conditions and environmental effects. In this context, the Euler method serves as a useful numerical tool for approximating solutions in problems where analytical solutions may be complex or unattainable.

The main objective of this study is to analyze the numerical solution of projectile motion using the Euler method. To achieve this, the numerical solution of the problem is obtained through a simple implementation of the Euler method, aiming to evaluate its accuracy and efficiency in simulating projectile motion.

A key aspect of the present work is the comparison of the projectile's trajectory in two scenarios: one without air resistance and another with a specific drag force. In the first case, it is verified that the optimal angle for achieving maximum horizontal range is 45 degrees, consistent with basic physics theory. This validation is essential to ensure the reliability of the numerical solution obtained.

In the second scenario, air resistance is introduced, represented by a specific drag coefficient. This analysis allows for determining the angle at which maximum horizontal range is achieved when considering drag effects. Comparing both cases provides a comprehensive view of how real-world conditions affect a projectile's behavior and highlights the importance of accounting for air resistance in practical applications.

Through this analysis, the study aims not only to validate the numerical method used but also to provide a basis for understanding the effects of air resistance on projectile motion. This contributes to improved application and adjustment of simulation models in real-world contexts.

II. BACKGROUND

The study of projectile motion has been a topic of interest since the early days of classical physics. The theory of projectile motion, based on Newton's laws, establishes that in an environment without air resistance, the optimal angle for achieving maximum horizontal range is 45 degrees. This is supported by basic physics texts such as Sears & Zemansky [2].
However, in practice, it is essential to consider air resistance when analyzing projectile motion and incorporate it into our models to achieve a more realistic approximation of the situation. In this context, it is reasonable to anticipate that, since air opposes the projectile's motion, the maximum range will be reduced, and the optimal angle for achieving that range may also vary. Therefore, it becomes interesting to investigate the optimal angle in the presence of resistance to maximize the horizontal range.
In this context, the Euler method emerges as a valuable tool for finding an accurate approximation of the optimal angle in the presence of resistance. This numerical method allows solving differential equations iteratively, providing an approximation of projectile trajectories under the influence of air resistance.

A. Euler's Method

Euler's method helps us find approximate solutions to first-order ordinary differential equations of the form (1). The derivation of Euler's method is based on class notes by Juan Diego Chang [1].

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0 \tag{1}$$

This method works by approximating the derivative discretely as follows:

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$$

Using the definition of the derivative (without employing the limit), equation (1) leads to the following recurrence rela-

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tion:

$$\frac{y_{n+1}-y_n}{\Delta x}=f(x_n,y_n)$$

This can be rewritten in a more useful form for our implementation as:

$$y_{n+1}=y_n+f(x_n,y_n)\Delta x \tag{2}$$

This is known as the Euler approximation or method. It simply consists of converting a continuous function into a discrete one.

B. Equation of Motion for a Projectile

For practical purposes, we study the motion of a projectile in two spatial dimensions, x and y. The motion of the projectile is described according to Newton's second law:

$$\sum \vec{F}=m\vec{a}$$

In this case, we consider two forces: gravity and air resistance. The latter is assumed to be proportional to the square of the velocity, as supported by various experimental observations.

$$m\frac{d\vec{v}}{dt}=m\vec{g}-B_2 v\vec{v} \tag{3}$$

In equation (3), B_2 is the proportionality constant for the air resistance term.

To determine the position function, equation (3) is transformed into a second-order differential equation. Since Euler's method requires first-order differential equations, equation (3) is broken down into a system of four first-order differential equations as follows:

$$\begin{cases} \frac{dx}{dt} &= v_x \\ \frac{dv_x}{dt} &= -kvv_x \\ \frac{dy}{dt} &= v_y \\ \frac{dv_y}{dt} &= -g-kvv_y \end{cases} \tag{4}$$

Here, $k=\frac{B_2}{m}$ and $v=\sqrt{v_x^2+v_y^2}$.

C. Euler Method for Projectile Motion

At this point, we apply the Euler method to the system of equations (4), resulting in the following recurrence relations. These will be used for implementation in Python to solve the equations iteratively.

$$\begin{cases} x_{n+1} &= x_n+v_n^x\Delta t \\ v_{n+1}^x &= v_n^x-kv_nv_n^x\Delta t \\ y_{n+1} &= y_n+v_n^y\Delta t \\ v_{n+1}^y &= v_n^y-g\Delta t-kv_nv_n^y\Delta t \end{cases} \tag{5}$$

with $v_n=\sqrt{(v_n^x)^2+(v_n^y)^2}$.

Parameters to Use

A Python program is developed to solve the projectile's differential equations using the Euler method as described in (5). These recurrence equations are solved iteratively starting from initial conditions.

The initial values for the projectile's motion are as follows:

$$\begin{aligned} \frac{B_2}{m} &= 0.00004\,\mathrm{s}^{-1}, \\ g &= 9.8\,\mathrm{m/s}^2, \\ v_0 &= 700\,\mathrm{m/s}, \\ \Delta t &= 0.01, \\ \theta &= \frac{\pi}{6}. \end{aligned} \tag{6}$$

Additionally, it is assumed that the projectile starts from an initial position of $(x,y)=(0,0)$.

III. RESULTS

These results provide, first, a comparison between a projectile experiencing air resistance and one that does not. The behavior of both projectiles is examined, and the impact of air resistance on the affected projectile is evaluated. Subsequently, a graphical analysis is carried out to determine the optimal angle that maximizes the horizontal range for both the projectile with air resistance and the one without.

A. Effect of Air Resistance on Projectiles

In Figure (1), the trajectory of the projectile without air resistance retains a "parabolic"shape due to the absence of resistance, implying that its velocity remains unaltered. On the horizontal axis, the projectile follows uniform linear motion, while on the vertical axis, it maintains accelerated motion due to gravity. As a result, the theoretical range of this projectile tends to infinity.

However, for a projectile experiencing air resistance, as shown in Figure (2), it can be observed that at a certain horizontal distance, the projectile's motion tends to adopt a vertical trajectory, resembling free-fall motion. This is expected, as air resistance acts as a force opposing the motion, gradually reducing the projectile's velocity in both axes. In this context, since horizontal motion is uniform linear motion (i.e., without acceleration), air resistance will decrease the horizontal velocity of the projectile until it is reduced to zero. On the other hand, vertical motion, being accelerated, will reach a terminal velocity, in accordance with the principles studied in classical mechanics. Consequently, the object will continue its descent due to gravity, but at a certain horizontal distance, its motion will become predominantly vertical, as air resistance will have nullified the horizontal velocity component.

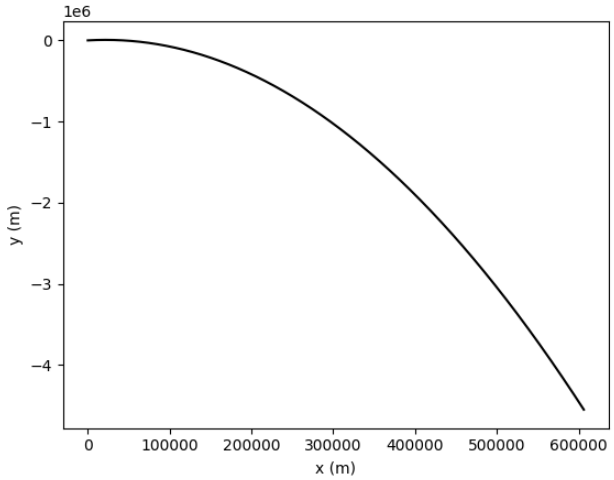

Figura 1: Trajectory of a projectile without air resistance for a time $t = 1000$ s.

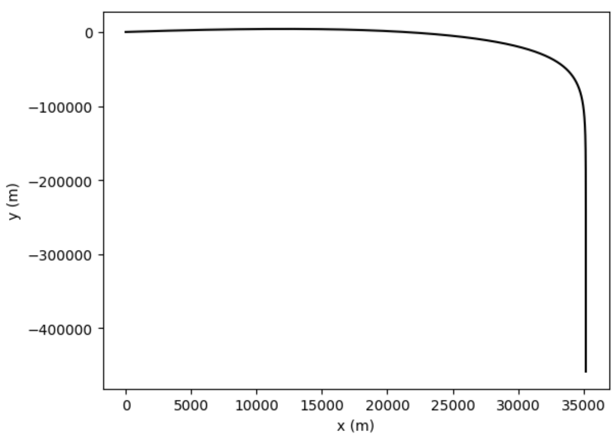

Figura 2: Trajectory of a projectile with air resistance for a time $t = 1000$ s.

In the graphs shown in Figures (1) and (2), for the Euler method, a total of $N = 10^5$ iterations was performed.

Maximum Range

Another crucial aspect to compare between a projectile experiencing air resistance and one that does not is the significant difference in their maximum horizontal and vertical ranges. As illustrated in the graph in Figure (3), the projectile without air resistance reaches greater distances compared to the projectile facing resistance. It is important to note that both projectiles were launched under the same initial conditions, namely, with the same angle and initial velocity. The graph depicts the trajectory of the projectiles from the moment of launch, starting from a reference height of zero, until their descent and return to that reference height.

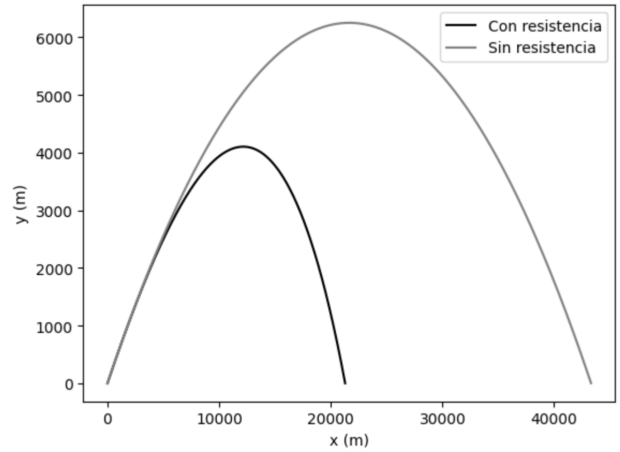

Figura 3: Comparison of the trajectories of two projectiles: one with air resistance and one without, starting from the same initial conditions.

B. Maximum Range Angle for a Projectile Without Air Resistance

After simulating various projectile trajectories, we now focus our attention on those that do not experience air resistance. We determined the maximum ranges for projectiles launched at different angles, specifically varying the angle from 0 to 90 degrees in increments of 3 degrees. The initial velocity is kept constant for all projectiles, as described in (6).

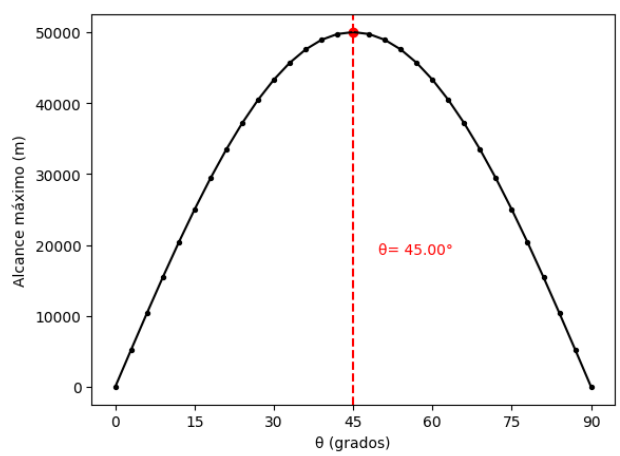

Figura 4: Optimal angle for maximum horizontal range of a projectile without air resistance.

The resulting graph, shown in (4), reveals that the optimal angle for achieving the maximum horizontal distance is 45 degrees, as anticipated. This finding not only confirms the theoretical value of the optimal angle for maximum range but also validates the efficacy and accuracy of the Euler numerical method in analyzing projectile motion.

C. Maximum Range Angle for a Projectile With Air Resistance

Analogous to the previous analysis, we have conducted tests for projectiles that experience air resistance, with resistance values and initial velocity set as described in (6). In this case, launch angles ranging from 0 to 90 degrees, with increments of 0.1 degrees, have been considered, allowing the analysis of 900 different angles. This approach aims to determine the optimal angle for achieving the maximum horizontal distance with greater precision.

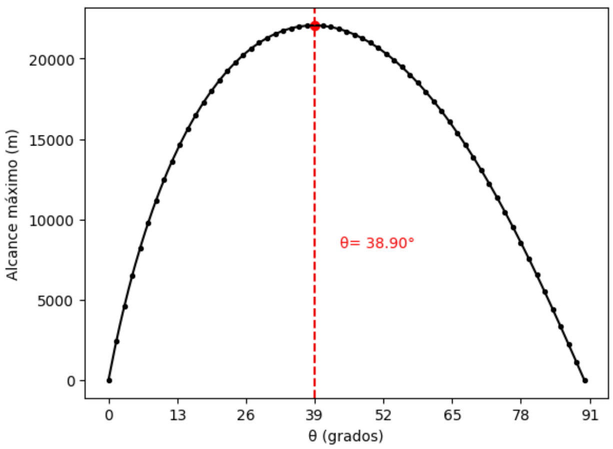

Figura 5: Optimal angle for maximum horizontal range of a projectile with air resistance.

The results, presented in the graph in figure (5), indicate that the optimal angle for maximum horizontal range is 38.9 degrees, which represents a notable difference compared to the projectile without air resistance. Additionally, the maximum ranges have been reduced by approximately half in the presence of air resistance.

IV. CONCLUSIONS

From the results obtained, it is observed that air resistance has a significant impact on the behavior of projectiles. For projectiles without air resistance, the trajectory follows a parabolic shape, with horizontal motion being uniform and undisturbed. This behavior results in a theoretical infinite horizontal range, as air resistance does not act to reduce the velocity.

When air resistance is introduced, the situation changes significantly. Projectiles with resistance present trajectories that deviate from the ideal parabolic shape as they progress. At a certain horizontal distance, the projectile begins to fall vertically, as air resistance progressively decreases the horizontal velocity until it is completely nullified. Unlike the uniform horizontal motion without resistance, the vertical motion reaches a terminal velocity, thus limiting the horizontal range of the projectile.

Regarding launch angles, it was found that for projectiles without air resistance, the optimal angle for maximizing horizontal range is 45 degrees, a result that matches theoretical predictions. On the other hand, when considering an air resistance of $B_2/m = 0.00004$ s^{-1} and an initial velocity of $v_0 = 700$ m/s, the optimal angle is reduced to 38.9 degrees. This difference underscores the significant effect of air resistance on the distance traveled by the projectile. Additionally, the maximum ranges are reduced by approximately half compared to projectiles without resistance.

Finally, the validated results confirm that the Euler numerical method is effective and accurate for analyzing the motion of projectiles with air resistance. The comparison between theoretical and numerical results reinforces the validity of this methodological approach in simulating and studying projectile behavior.

V. APPENDIX

The Python program used for this analysis can be found **here**.

[1] Juan Diego Chang. Física computacional. Notas de clase en Física computacional, Escuela de Ciencias Físicas y Matemáticas, USAC, 2024. Accedido en: Septiembre 2024.

[2] Young Freedman and Sears Zemansky. Física universitaria. *Editorial. Prentice Hall. México. Decimosegunda edición*, 2009.