A Generalized Kalman Filter and Smoother with Application to Mixed-Frequency Data*

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1 Introduction

This technical report contains a unified treatment of the Kalman filter and smoother (KFS) techniques that account for missing data, mixed-frequency/temporal aggregation, time-varying system parameters, and shifts in the mean and/or variance of either the vector of observables or the latent state variables. It serves as a fully contained treatment of existing methods, and borrows greatly in both notational choices and derivations from the more thorough treatments in Harvey (1989) and Durbin and Koopman (2012).

Additionally, this note can serve as documentation for the Matlab code *GeneralizedKFilterSmoother.m*, a function that we have written to implement our derived techniques and equations. This function is currently used as part of the estimation process for the National Financial Conditions Index (NFCI) produced by the Federal Reserve Bank of Chicago and described in Brave and Butters (2012) and utilizes a similar framework as the Chicago Fed Dynamic Stochastic General Equilibrium (DSGE) model of Brave et al. (2012).

In the following sections, we present a generalized state space framework and discuss its relevant parameters. After describing the model, we develop a thorough characterization of the **Harvey accumulator**, which nests the treatment of many common data irregularities in our state space framework. Then, we address the initial conditions needed for the KFS methods. Next, we provide details for the fully flexible KFS recursion equations. Finally, we offer some concluding remarks as to how our Matlab function is used to estimate the NFCI.

2 The Model

The underlying model for the Kalman filter and smoother is the state space framework discussed here. The system comprises two equations: (i) a *state* equation, eq. (1), and (ii) a *measurement* equation, eq. (2). The former governs the evolution of the vector of latent state variables, while the latter characterizes how the unobserved states map into the vector of observed variables potentially with measurement error. Formally, the state space system is given by

$$
\begin{aligned}
\alpha_t &= T_{\tau_t^T}\alpha_{t-1} + c_{\tau_t^c} + R_{\tau_t^R}\eta_t, \;\; \eta_t \sim \mathcal{N}(0, Q_{\tau_t^Q}), && (1)\\
y_t &= Z_{\tau_t^Z}\alpha_t + d_{\tau_t^d} + \epsilon_t, \;\; \epsilon_t \sim \mathcal{N}(0, H_{\tau_t^H}) && (2)\\
t = 1,\ldots,n & \qquad \text{and } \alpha_0 \sim \mathcal{N}(a_0, P_0)
\end{aligned}
$$

*This note combines material from Durbin and Koopman (2012) and Harvey (1989) in addition to the work of the authors. The views expressed herein are those of the authors and do not necessarily represent the views of the Federal Reserve Bank of Chicago or the Federal Reserve System.

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Table 1: Dimensions of the state space model

Vector		Matrix	
y_t	$p \times 1$	Z_t	$p \times m$
α_t	$m \times 1$	T_t	$p \times m$
d_t	$p \times 1$	H_t	$p \times p$
ϵ_t	$p \times 1$	R_t	$m \times g$
c_t	$m \times 1$	Q_t	$g \times g$
η_t	$g \times 1$		
a_0	$m \times 1$	P_0	$m \times m$

where the system matrices (vectors) in red denote exogenous parameters taken as given and subsequently passed as inputs to *GeneralizedKFilterSmoother.m*.[1] Alternatively, the vector of observables, y_t, is given in blue to signify that it will also necessarily serve as an input to *GeneralizedKFilterSmoother.m*.[2] The dimension of the matrices are given in table (1).

Though the choice of this particular state space representation seems innocuous, it is not without some subtleties worth noting. First, the evolution of the state variable, α_t, is not a universal characterization. Durbin and Koopman (2012) have a somewhat different representation, letting $\alpha_t = T_{t-1}\alpha_{t-1} + R_{t-1}\eta_{t-1}$.[3] Their departure from the characterization given here, and in Harvey (1989), can be isolated into two parts: (i) the calendar indexing of the dynamics (T_{t-1} instead of T_t) and (ii) the particular η-shock contributing to the state at any calendar date t (η_{t-1} instead of η_t). Additionally, the particular interpretation of a_0 and P_0 proves to vary across characterizations. In our case, we will adopt the convention that $a_0 \equiv \mathbb{E}[\alpha_0|\Omega_0]$ and $P_0 \equiv \mathbb{V}[\alpha_0|\Omega_0]$, where Ω_0 denotes the entire information set at time 0. In this set up, a_0 serves as the unconditional *inference* of the initial state.[4]

Furthermore, in the case in which y_t contains series realized at different frequencies, the $d_{\tau_t^d}$ and ϵ_t contribute *only during periods of observation*. In the traditional Kalman filter convention, $d_{\tau_t^d}$ represents the deterministic portion of the measurement error and ϵ_t represents the stochastic portion.[5] Note that this choice of interpretation is without loss of generality. For series in which the econometrician wishes to have the deterministic trend component characterized in the highest frequency of observation, the series must enter into both equations after the appropriate changes in the Z and T system matrices are made so that the deterministic trend enters into $c_{\tau_t^c}$.

2.1 Harvey Accumulator

In cases where k elements of the observation vector y_t are realized at different frequencies, this must be accommodated in the framework described previously with the inclusion of Harvey (1989) accumulators for each of the lower frequencies than the highest, or base, frequency of observation. The reason for this is straightforward: The latent state variables inherit the base frequency of observation. Matching

[1] It should be noted that for the sake of consistency the system parameters a_0, P_0 are given in red above. However, users of the *GeneralizedKFilterSmoother.m* code will not in fact have to pass these parameters given the code's appropriate use of the diffuse prior initialization of the KFS equations. See the next section for more details. Furthermore, each system matrix, e.g. Z and its particular timing within the time series, e.g. τ_t^Z, will be passed to the function as Matlab *structures*. See *GeneralizedKFilterSmoother.m* for more details.

[2] It is assumed that the history $\{y_t\}_{t=1}^n$ that is passed to the function will include *NaN* in any period where a particular series is missing/not observed, allowing for the full cross section of time series to be balanced. Note that this convention does allow for the number of elements of the cross section to vary over time, as long as upon dropping out of the sample a time series' position in the matrix has *NaN*s as place holders.

[3] Durbin and Koopman (2012) do not allow for time-varying means, c_t or d_t, to enter into their model. In favor of extrapolating how they might have extended their model to include such parameters, they are suppressed in this discussion.

[4] There exist two potential alternative interpretations of the initial conditions: (i) the initial condition of the state is at a period where y_t is observed, i.e. a_1 and (ii) the initial condition is a *prediction* of the initial state, i.e. $a_1 = \mathbb{E}[\alpha_1|\Omega_0]$. Obviously, these two departures from the convention we use are not mutually exclusive, and yield slightly different recursive equations for the Kalman filter/smoother involving the initial time periods.

[5] In many financial and economic applications, the $d_{\tau_t^d}$ is often thought of as the deterministic trend in the observables. This interpretation is suitable within this framework as long as the trend is characterized at the frequency of the particular series and not the potentially higher base frequency.

temporal aggregation properties between states and observables is, thus, essential to characterizing the parameters.

There are three types of temporal aggregation that we consider in *GeneralizedKFilterSmoother.m*: *sums*, *averages* and the *triangle average*. Given that the triangle average and simple average are similar in their construction, we describe the first two of these accumulators within the state space framework leaving the derivation of the triangle average to the next section. Without loss of generality, denote the particular series that are defined to be *sum* variables as $h = 1, \dots, k^{\star}$, leaving $h = k^{\star} + 1, \dots, k$ to be the *averaged* variables.[6] There is a particular number of accumulators that need to be constructed that depends on the number of state variables that each of these series loads onto and the particular combination of types of aggregation for each of these series. In cases where multiple series with different sizes/types of temporal aggregation load onto the same state variable, multiple accumulators will need to be constructed for the same original state variable. Conversely, even with several series that are observed at different frequencies/aggregations than the base frequency, it is conceivable that fewer accumulators may need to be constructed than observed series with frequency/aggegation adjustments.

With the *GeneralizedKFilterSmoother.m* function, we identify the minimum number of accumulators, τ, that need to be constructed and build them into the measurement and state equations by augmenting the Z, T, R, and c system matrices, where again without loss of generality we assume that $s = 1, \dots, \tau^{\star}$ are accumulators for state variables mapping to observed series interpreted as *sums*, while $s = \tau^{\star} + 1, \dots, \tau$ are accumulators for state variables mapping to the observed series interpreted as *averages*. Beyond the system matrices typically provided to a Kalman filter, the only additional inputs required are ξ and ψ. ξ is a $k \times 1$ vector that includes the linear indices of the series within y_t that require accumulation. ψ is a $k \times (n+1)$ matrix that provides the full time series calendar indicator for each series that needs temporal aggregation.[7]

For series defined to be *sum* variables, ψ^i will be a vector of mostly ones with zeros at time periods that represent the first period within the lower frequency. For series defined to be *average* variables, ψ^i will be a vector consisting of repeating cycles indicating the first, second, third, etc. period of the base frequency within the lower frequency. Equations (3)-(6) provide an example of ψ for the case where the calendar base frequency is weekly while series within y_t are observed at the monthly and quarterly frequencies. This set of temporal aggregations is exactly the one necessary to estimate the NFCI of Brave and Butters (2012) and is discussed further in section (5).

$$\psi_t(\text{monthly sum}) = \begin{cases} 0 & \text{1st week of the month} \\ 1 & \text{otherwise} \end{cases} \tag{3}$$

$$\psi_t(\text{monthly average}) = \left\{ j \quad j\text{th week of the month} \right. \tag{4}$$

$$\psi_t(\text{quarterly sum}) = \begin{cases} 0 & \text{1st week of the quarter} \\ 1 & \text{otherwise} \end{cases} \tag{5}$$

$$\psi_t(\text{quarterly average}) = \left\{ j \quad j\text{th week of the quarter} \right. \tag{6}$$

Once we have determined with *GeneralizedKFilterSmoother.m* the accumulators, ζ_t^i, that need to be created, we construct a selection matrix A in order to better facilitate the augmentation of the T, c and R system matrices.[8] The function then addresses the determination of the necessary number of particular system matrices given any number of exogenous structural "breaks" within the sample period. This number is automatically determined by utilizing the matrix ψ and vector $\tau_t^{(\cdot)}$ for each system matrix.

[6] Series that are interpreted as *instantaneous* variables do not need temporal aggregation even in the event that they are observed at a different frequency than the base frequency.

[7] The econometrician will need to provide the *GeneralizedKFilterSmoother.m* code with the ψ_t for the $n+1$ period in order for the filter to forecast the one-step-ahead state vector for the last period.

[8] This selection matrix differs slightly from the selection matrix used by Harvey (1989). Our selection matrix pre-multiplies the transition matrix T, as opposed to the measurement system matrix Z as in Harvey (1989). This allows for the most concise treatment of the number of accumulator variables constructed in the event that the measurement system matrix Z has structural parameters. This is often the case in dynamic factor models such as the NFCI of Brave and Butters (2012).

$$
\begin{aligned}
\begin{bmatrix} \alpha_t \\ \zeta_t^1 \\ \vdots \\ \zeta_t^{\tau^\star} \\ \zeta_t^{\tau^\star+1} \\ \vdots \\ \zeta_t^{\tau} \end{bmatrix}
&=
\begin{bmatrix}
T_{\tau_t^T} & & & & & & \\
A_1 T_{\tau_t^T} & \psi_t^1 & & & & & \\
\vdots & & \ddots & & & & \\
A_{\tau^\star} T_{\tau_t^T} & & & \psi_t^{\tau^\star} & & & \\
\frac{1}{\psi_t^{\tau^\star+1}} A_{\tau^\star+1} T_{\tau_t^T} & & & & \frac{\psi_t^{\tau^\star+1}-1}{\psi_t^{\tau^\star+1}} & & \\
\vdots & & & & & \ddots & \\
\frac{1}{\psi_t^{\tau}} A_{\tau} T_{\tau_t^T} & & & & & & \frac{\psi_t^{\tau}-1}{\psi_t^{\tau}}
\end{bmatrix}
\begin{bmatrix} \alpha_{t-1} \\ \zeta_{t-1}^1 \\ \vdots \\ \zeta_{t-1}^{\tau^\star} \\ \zeta_{t-1}^{\tau^\star+1} \\ \vdots \\ \zeta_{t-1}^{\tau} \end{bmatrix} \\
&\quad + \begin{bmatrix} c_{\tau_t^c} \\ A_1 c_{\tau_t^c} \\ \vdots \\ A_{\tau^\star} c_{\tau_t^c} \\ \frac{1}{\psi_t^{\tau^\star+1}} A_{\tau^\star+1} c_{\tau_t^c} \\ \vdots \\ \frac{1}{\psi_t^{\tau}} A_{\tau} c_{\tau_t^c} \end{bmatrix}
+ \begin{bmatrix} R_{\tau_t^R} \\ A_1 R_{\tau_t^R} \\ \vdots \\ A_{\tau^\star} R_{\tau_t^R} \\ \frac{1}{\psi_t^{\tau^\star+1}} A_{\tau^\star+1} R_{\tau_t^R} \\ \vdots \\ \frac{1}{\tau_t^{\tau}} A_{\tau^\star+1} R_{\tau_t^R} \end{bmatrix} \eta_t
\end{aligned}
\tag{7}
$$

To build the measurement equation, one just needs to replace each corresponding row in $Z_{\tau_t^Z}$, ξ_h for $h = 1, \ldots, k$ with a row of zeros and put the corresponding nonzero elements of the original $Z_{\tau_t^Z}$ matrix into the columns $m + i(1), m + i(2)$, etc., associated with the particular accumulators $\zeta_t^{i(1)}$, $\zeta_t^{i(2)}$, etc., that correspond to that series' temporal aggregation requirements. This, too, is accomplished within the *GeneralizedKFilterSmoother.m* function.

2.2 Derivation for Triangle Averaging

This section derives the recursive formulation of the triangle average accumulator. The triangle average will be described as an S average of H over H growth. Unlike the other two accumulators, this accumulator has two distinct timing characteristics. The first component, S, denotes the period over which the observation is being averaged. For example, in the case of a monthly (base frequency) model, S could be 3 to denote that the average is done over a quarter. The second component, H, denotes the horizon over which the growth rate is constructed. For example, in the case of a monthly (base frequency) model, H could be 12 to denote that the growth rate is a *year over year* growth rate. Taken together, the two components lead to the construction of the growth rate observed. In the example given above $(S, H) = (3, 12)$ over a monthly base frequency, the variable would constitute a *quarterly average of year over year growth.* Similarly, if $(S, H) = (12, 3)$ the variable would constitute a *yearly average of quarter over quarter growth.*

Our derivation will characterize the recursive formulation of the triangle average accumulator necessary to aggregate a series of base frequency growth rates for any particular type of S average H over H growth. This aggregation will exactly implement the triangle weighting scheme, and do so in the most efficient manner possible, i.e. only storing H lags of the base frequency growth rates.

For the purpose of the derivation, take the level of some variable that one hopes to aggregate as Y_t, where t is given in the base frequency. Furthermore, define any H over H growth for the variable as, $y_t^H = Y_t - Y_{t-H}$. Similarly, the growth rate in the base frequency (or $H = 1$) will be defined as $y_t = Y_t - Y_{t-1}$. Typically, the data we observe is the H over H growth rate averaged over some period longer the base frequency of the model we hope to describe. For example, it might be the case that we observe quarterly averages, though we hope to create a monthly frequency model. For our purposes, a S-ly average of H over H growth will be the data we observe: $y_t^{S,H} = \frac{1}{S} \sum_{i=0}^{S-1} y_{t-i}^H$. For notational ease define $D = \min\{S, H\}$. The triangle weighting yields a useful equality of this S-ly average of H over H growth, summing over the last $H+S-1$ terms in a triangle fashion where the largest weighting D is given to the middle $H+S-2D+1$ terms cascading down from there:

$$
\begin{aligned}
y_t^{S,H} &= \frac{1}{S}\Big(y_t + 2y_{t-1} + \cdots + \\
&\quad T\sum_{i=0}^{H+S-2D} y_{t-D+1-i} + \cdots + 2y_{t-H-S+3} + y_{t-H-S+2}\Big)
\end{aligned} \tag{8}
$$

It is important to note at this point that equation (8) denotes the *rolling* average of the S-ly H over H growth rates. As we did to characterize the other average accumulator, we will want to describe how the average *within* the S period of lower frequency evolves. Consequently, the triangle average accumulator indicator will be defined as:

$$
\psi_t(S \text{ avg. of } H \text{ growth}) = \begin{cases} 1 & \text{first period within } S \text{ period} \\ 2 & \text{second period within } S \text{ period} \\ \vdots & \vdots \\ S & \text{last period within } S \text{ period} \end{cases} \tag{9}
$$

where the triangle average accumulator will be given by:

$$
\zeta_t = \frac{1}{\psi_t}(y_t + y_{t-1} + \cdots + y_{t-H+1}) + \frac{(\psi_t - 1)}{\psi_t}\zeta_{t-1}. \tag{10}
$$

2.3 Initial Conditions

GeneralizedKfilterSmoother.m automatically implements the proper initial conditions of the state, a_0 as described by Harvey (1989). In general, the proper initial conditions allow the variance of the state $P_0 \to \infty$ in the limit for the elements of the state that have non-stationary components.[9] In the case of stationary series, the initial condition amounts to the solution of a linear system of equations governed by the system matrices $T_{\tau_1^T}$, $R_{\tau_1^R}$, and $Q_{\tau_1^Q}$. Harvey (1989) and Hamilton (1994) suggest that one checks that all the absolute values of the eigenvalues of the system matrix T fall within the unit circle to ensure stationarity. This is the procedure followed within *GeneralizedKFilterSmoother.m* to construct initial conditions for the state.

For the stationary case, the initial conditions are given by:

$$
\begin{aligned}
a_0 &= \left(I_m - T_{\tau_1^T}\right)^{-1} c_{\tau_1^c}, \\
P_0 &= \left(I_{m^2} - T_{\tau_1^T} \otimes T_{\tau_1^T}\right)^{-1} \operatorname{vec}(R_{\tau_1^R} Q_{\tau_1^Q} R'_{\tau_1^R})
\end{aligned}
$$

For the non-stationary case, the initial conditions are set according to Harvey (1989):

$$
\begin{aligned}
a_0 &= 0 \\
P_0 &= \kappa I_m
\end{aligned}
$$

for $\kappa \to \infty$.[10]

[9] We adopt the convention of Harvey (1989) and drive up the variance of the initial state by a scalar κ, using a particularly large κ. For a derivation of the exact diffuse prior initial conditions of the KFS equations, see Durbin and Koopman (2012).

[10] There is an option in *GeneralizedKfilterSmoother.m* for the user to input a_0 and P_0. For a discussion on how one might want to set initial conditions for cases where the state has a mix of stationary and non-stationary elements see Durbin and Koopman (2012).

3 KFS Equations

Given the inputs to the state space model discussed in section (2), we can proceed to extract the latent state variables. At any period t, missing values can be dealt with by simply constructing new system matrices $Z^{\star}_{\tau^{Z}_{t}}$, $d^{\star}_{\tau^{d}_{t}}$, and $H^{\star}_{\tau^{H}_{t}}$ by multiplying the original system matrix by W_t. To build the W_t matrix, simply drop the rows of a $p \times p$ identity matrix corresponding to the particular series missing at time t. The new system matrices for the measurement equation become:

$$
\begin{aligned}
y^{\star}_{t} &= W_t y_t && (11)\\
Z^{\star}_{t} &= W_t Z_{\tau^{Z}_{t}}, && (12)\\
d^{\star}_{t} &= W_t d_{\tau^{d}_{t}}, && (13)\\
H^{\star}_{t} &= W_t H_{\tau^{H}_{t}} W'_t && (14)
\end{aligned}
$$

3.1 Kalman Filter

With this small modification, the standard Kalman filter equations can be utilized, fully accommodating both temporal aggregation and missing observations

$$
\begin{aligned}
a_1 &= T_{\tau^{T}_{1}} a_0 + c_{\tau^{c}_{1}}, && (15)\\
P_1 &= T_{\tau^{T}_{1}} P_0 T'_{\tau^{T}_{1}} + R_{\tau^{R}_{1}} Q_{\tau^{Q}_{1}} R'_{\tau^{R}_{1}}, && (16)\\
v_t &= y^{\star}_{t} - Z^{\star}_{t} a_t - d^{\star}_{t}, && (17)\\
F_t &= Z^{\star}_{t} P_t Z^{\star\prime}_{t} + H^{\star}_{t}, && (18)\\
K_t &= T_{\tau^{T}_{t+1}} P_t Z^{\star\prime}_{t} F^{-1}_{t}, && (19)\\
L_t &= T_{\tau^{T}_{t+1}} - K_t Z^{\star}_{t}, && (20)\\
a_{t+1} &= T_{\tau^{T}_{t+1}} a_t + c_{\tau^{c}_{t+1}} + K_t v_t, && (21)\\
P_{t+1} &= T_{\tau^{T}_{t+1}} P_t L'_t + R_{\tau^{R}_{t+1}} Q_{\tau^{Q}_{t+1}} R'_{\tau^{R}_{t+1}}, \text{ for } t = 1, \ldots, n, && (22)
\end{aligned}
$$

where a_0 and P_0 are given and the dimensions of the system matrices are found in table (2).

3.2 Kalman Smoother

Once a single pass through the Kalman filter has been made, the *smoothed* estimates of the unobserved state can be found according to one of two methods described as follows.[11]

3.2.1 State Smoother

The standard Kalman smoother equations yield a smoothed estimate of the unobserved state given the entire history of the data $\mathbb{E}\left[\alpha_t | y_1, \ldots, y_n\right]$. For the exact derivation of the state smoother equations see Durbin and Koopman (2012). The state smoother equations are as follows

$$
\begin{aligned}
r_t &= Z^{\star\prime}_{t} F^{-1}_{t} v_t + L'_t r_{t+1}, && (23)\\
\hat{\alpha}_t &= a_t + P_t r_t, && (24)
\end{aligned}
$$

where $r_{n+1} = 0$ and $\hat{a}_0 = a_0 + P_0 T'_{\tau^{T}_{1}} r_1$.

3.2.2 Disturbance Smoother

In many cases the size of the state is much larger than the number of shocks in the state equation, i.e., $m >> g$. In this case, a much more efficient smoother routine involves smoothing the disturbances, after which one can recover the state from the standard recursion equation given by equation (1). For

[11] Calculating the smoothed estimates of the state with *GeneralizedKfilterSmoother.m* is optional so that computational time is saved in instances where the code is being used for simply evaluating the likelihood of a set of parameter values.

Table 2: Dimensions of the Kalman filter and smoother

Vector		Matrix	
v_t	$p \times 1$	F_t	$p \times p$
a_t	$m \times 1$	K_t	$m \times p$
r_t	$m \times 1$	L_t	$m \times m$
$\hat{\eta}_t$	$g \times 1$	P_t	$m \times m$
$\hat{\alpha}_t$	$m \times 1$		

the exact derivation of the disturbance smoother see Durbin and Koopman (2012). The disturbance smoother equations are as follows

$$r_t = Z_t^{\star\prime} F_t^{-1} v_t + L_t' r_{t+1}, \tag{25}$$
$$\hat{\eta}_t = Q_{\tau_t^Q} R'_{\tau_t^R} r_t, \text{ for } t = n, \ldots, 1, \tag{26}$$

where $r_{n+1} = 0$ and $\hat{a}_0 = a_0 + P_0 T'_{\tau_1^T} r_1$.[12]

4 Univariate KFS Equations with Diffuse initialization

For both computational reasons, the univariate characterization of the Kalman filter and smoother are often used in the literature. Below is a unified characterization of the univariate filter and smoother recursive equations.

4.1 Missing observations and non-diagonal H_t matrix

While the univariate filter and smoother equations are relatively straightforward, they will not hold for non-diagonal covariance matrices of the error term in the measurement equation. In these cases, the standard Cholesky factorization, C_t, can be used to diagonalize the $W_t H_{\tau_t^H} W_t'$. In our particular setting, including this factorization in the presence of missing observations leads to the following characterization:

$$\begin{aligned} C_t H_t^\star C_t' &= W_t H_{\tau_t^H} W_t' \\ y_t^\star &= C_t^{-1} W_t y_t \\ Z_t^\star &= C_t^{-1} W_t Z_{\tau_t^Z} \\ d_t^\star &= C_t^{-1} W_t d_{\tau_t^d} \end{aligned}$$

4.2 Univariate filter

The univariate recursive filter equations then follow straightforwardly as

[12] Implementing the expectation-maximization algorithm as in Brave and Butters (2012) requires the recursive equations to be augmented with two additional moment matrices: $N_{t-1} = Z_t' F_t^{-1} Z_t + L_t' N_t L_t$ and $J_t = P_{t-1} L_t' (I_m - N_{t-1} P_t)$. In the case where one is using the univariate filter (see section (4)), an additional matrix F_t will have to be calculated that is otherwise not created in the standard univariate recursive equations.

$$
\begin{aligned}
v_{t,i} &= y_t^{\star} - Z_{t,i}^{\star} a_{t,i} - d_t^{\star} \\
F_{t,i} &= Z_{t,i}^{\star} P_{t,i} Z_{t,i}^{\star\prime} + H_{t,i}^{\star} \\
M_{t,i} &= P_{t,i} Z_{t,i}^{\star\prime} \\
a_{t,i+1} &= a_{t,i} + M_{t,i} F_{t,i}^{-1} v_{t,i} \\
P_{t,i+1} &= P_{t,i} - M_{t,i} F_{t,i}^{-1} M_{t,i}'
\end{aligned}
$$

for $i = 1, \ldots, p_t$.

$$
\begin{aligned}
a_{t+1,1} &= T_{\tau_{t+1}^T} a_{t,p_t+1} + c_{\tau_{t+1}^c} \\
P_{t+1,1} &= T_{\tau_{t+1}^T} P_{t,p_t+1} T_{\tau_{t+1}^T}' + R_{\tau_{t+1}^R} Q_{\tau_{t+1}^Q} R_{\tau_{t+1}^R}'
\end{aligned}
$$

for $t = 1, \ldots, n$.

4.3 Univariate smoother

The standard univariate smoother is given by the following recursive equations

$$
\begin{aligned}
L_{t,i} &= I - M_{t,i} Z_{t,i}^{\star} F_{i,t}^{-1} \\
r_{t,i-1} &= Z_{t,i}^{\star\prime} F_{t,i}^{-1} v_{t,i} + L_{t,i}' r_{t,i} \\
N_{t,i-1} &= Z_{t,i}' F_{t,i}^{-1} Z_{t,i} + L_{t,i}' N_{t,i} L_{t,i}
\end{aligned}
$$

for $i = p_t, \ldots, 1$

$$
\begin{aligned}
r_{t-1,p_t} &= T_{\tau_t^T}' r_{t,0} \\
N_{t,p_t} &= T_{\tau_t^T}' N_{t+1,0} T_{\tau_t^T}
\end{aligned}
$$

where $r_{n+1} = 0$ and $\hat{a}_0 = a_0 + P_0 T_{\tau_1^T}' r_1$.

5 Conclusion

The appendix of Brave and Butters (2012) describes how our generalized state space framework is restricted to arrive at the model underlying the NFCI. We repeat the state space representation of the dynamic factor model for the NFCI here, where y_t is a vector of stationary financial variables that have been demeaned and standardized, ε_t and η_t are independently distributed idiosyncratic error vectors with $\varepsilon_t \sim N(0, H)$ and $\eta_t \sim N(0, Q)$; α_t is a vector made up of a latent coincident factor f_t, the weekly NFCI, and its $L-1$ lags; and $t = 1, \ldots, n$, where n is the longest time series length of the collection of p variables in the NFCI.

$$
\begin{aligned}
y_t &= Z\alpha_t + \varepsilon_t, && (27) \\
\alpha_{t+1} &= T\alpha_t + R\eta_t, && (28) \\
a_0 &= \mathbb{E}[\alpha_0] \text{ and } P_0 = \text{Var}[\alpha_0] \text{ given}
\end{aligned}
$$

We briefly describe how *GeneralizedKFilterSmoother.m* may be used to estimate this model. Aside from the data and system parameters, our code requires that we specify the temporal aggregation properties of every data series in-line with the calendar dating established in *example.m*. The weekly model for the NFCI includes three Harvey accumulators capturing monthly and quarterly *sums* and monthly *averages*. Missing from here are quarterly *averages*. *GeneralizedKFilterSmoother.m*, however, can accommodate any combination of frequencies and aggregation types.

Brave and Butters (2012) further detail how the KFS methods described in this technical report can be used as part of the estimation based on an expectation-maximization (EM) algorithm for the system matrices of the NFCI. *GeneralizedKFilterSmoother.m* greatly simplifies the E-step of this algorithm and can be nested within a larger section of code that estimates the system parameters (M-step) and tracks the improvement in log-likelihood with each successive iteration of the algorithm.

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