Robust Testing and Variable Selection for High-Dimensional Time Series

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Outline

1. Focus on high-dimensional linear time series
2. Basic theory: Extreme value theory in the domain of Gumble distribution
3. Testing for serial correlations
4. Finite-sample adjustments and simulations
5. A simple example
6. Variable selection: statistics and procedure
7. Simulation and comparison

Objectives

Part of an ongoing project on analysis of big dependent data

1. Develop statistical methods for analysis of high-dimensional time series
2. Focus on simple methods if possible
3. Make analysis of big dependent data easy, including statistical inference.

Extreme Value Theory: (Maximum and minimum)
Let $x_1, \ldots, x_k$ be a random sample (iid) from a standard Gaussian distribution. Define the order statistics

$$x_{k,k} \le x_{k-1,k} \le \cdots \le x_{2,k} \le x_{1,k}.$$

Also, define the norming constants

$$c_k = \frac{1}{\sqrt{2\ln(k)}}, \quad d_k = \sqrt{2\ln(k)} - \frac{\ln(4\pi) + \ln(\ln(k))}{2\sqrt{2\ln(k)}}.$$

Then, as $k \to \infty$

$$T_{1,k} = \frac{x_{1,k} - d_k}{c_k} \to_d X, \quad T_{k,k} = \frac{-x_{k,k} - d_k}{c_k} \to_d X,$$

and $T_{1,k}$ and $T_{k,k}$ are asymptotically independent, where the CDF of X is $\Lambda(x) = \exp(-e^{-x})$.
See, for instance, the classical book by Embrechts, et al. (2001).

Basic tool and its property

Spearman's rank correlation:
Consider two random variables X and Y with *continuous* marginal distributions.
Let $\{(x_i, y_i)\}$ be a sample of size n from (X, Y).
Let r_i^x be the rank of x_i and r_i^y be the rank of y_i in the sample.
The Spearman's rank correlation is defined as

$$\hat{\rho} = \frac{\sum_{i=1}^{n}(r_i^x - \bar{r})(r_i^y - \bar{r})}{n(n^2-1)/12},$$

where $\bar{r} = (n+1)/2$.
Limiting property: $\sqrt{n}\hat{\rho} \rightarrow_d N(0,1)$ as $n \rightarrow \infty$, if X and Y are independent.

High-dimensional time series

Let $\boldsymbol{X}_t = (x_{1t}, \dots, x_{pt})'$ be a p-dimensional stationary time series.
Let $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_n\}$ be a realization of n observations of $\boldsymbol{X}_t$. Let $\boldsymbol{r}_t = (r_{1t}, \dots, r_{pt})'$ be the corresponding tth observation of ranks. That is, r_{it} denotes the rank series of x_{it}.
The lag-ℓ Spearman's rank cross-correlation matrix is defined by

$$\widehat{\boldsymbol{\Gamma}}_\ell = [\widehat{\Gamma}_{\ell,ij}] = \frac{12}{n(n^2-1)} \sum_{t=\ell+1}^{n} (\boldsymbol{r}_t - \bar{\boldsymbol{r}})(\boldsymbol{r}_{t-\ell} - \bar{\boldsymbol{r}})',$$

where $\bar{\boldsymbol{r}}$ denotes the constant vector of $(n+1)/2$.

The robustness of the rank correlation has been widely studied in the literature.

A fundamenal question

Are there serial correlations in a given high-dimensional time series?
$H_0 : \mathbf{\Gamma}_1 = \cdots = \mathbf{\Gamma}_m = \mathbf{0}$ vs $H_1 : \mathbf{\Gamma}_i \neq \mathbf{0}$ for some i, where, for simplicity, $\mathbf{\Gamma}_\ell = E[\widehat{\mathbf{\Gamma}}_\ell]$.

The question also applies to the residuals of a fitted model, e.g. Is the model adequate for the data?

The well-known Ljung-Box $Q(m)$ statistic is not useful as the dimension p increases.
Some work available, e.g. Chang, Yao and Zhou (2016) and Li et al. (2016).

Basic properties of rank correlations: A theorem

Conditions: Continuous random varaibles (no moment requirements)

1. $\boldsymbol{X}_t$ is a white noise (no serial correlations)
2. Components are independent: X_{it} and X_{jt} are independent for $i \neq j$.

The rank cross-correlation matrix $\widehat{\mathbf{\Gamma}}_\ell$ satisfies

(a) $E(\widehat{\Gamma}_{\ell,ii}) = -\frac{n-\ell}{n(n-1)}, \quad 1 \leq \ell \leq n-1,$

(b) $E(\widehat{\Gamma}_{\ell,ij}) = 0,\ i \neq j,$

(c) $Var(\widehat{\Gamma}_{\ell,ii}) = \frac{5(n-\ell)-4}{5n^2} + O(n^{-3}),$

(d) $Cov(\widehat{\Gamma}_{\ell,ii}, \widehat{\Gamma}_{h,ii}) = -\frac{2}{n^2} + O(n^{-3}), \quad 1 \leq \ell < h \leq n-1,$

(e) $Var(\widehat{\Gamma}_{\ell,ij}) = \frac{1}{n} + O(n^{-2}),\ i \neq j,$

(f) $Cov(\widehat{\Gamma}_{\ell,ij}, \widehat{\Gamma}_{\ell,uv}) = 0 + O(n^{-2}), \quad (i,j) \neq (u,v),$

where $1 \leq i,j \leq p$.

(a), (c), and (d) are shown by Dufour and Roy (1986). Others by independence condition.

Summary of the theorem

Let vec($\boldsymbol{A}$) be the column-stacking vector of matrix $\boldsymbol{A}$. Then,

$$\sqrt{n} \times \text{vec}(\widehat{\Gamma}_\ell) \to_d N(0, I), \quad n \to \infty,$$

where $N(0, I)$ denotes the p^2-dimensional standard Gaussian distribution, provided that the conditions hold.

Testing: single cross-correlation matrix

$H_0 : \mathbf{\Gamma}_\ell = \mathbf{0}$ versus $H_1 : \mathbf{\Gamma}_\ell \neq \mathbf{0}$.
Define

$$T_{\ell,max} = \sqrt{n} \times \max\{\widehat{\mathbf{\Gamma}}_\ell\} \quad T_{\ell,min} = -\sqrt{n} \times \min\{\widehat{\mathbf{\Gamma}}_\ell\}.$$

Let c_p and d_p be the norming constants with $k = p^2$.
Under H_0,

$$T^*_{\ell,max} = \frac{T_{\ell,max} - d_p}{c_p} \to_d X, \quad T^*_{\ell,min} = \frac{T_{\ell,min} - d_p}{c_p} \to_d X,$$

where X denotes the Gumble distribution, if $n \to \infty$ and $p \to \infty$.

For type-I error α, H_0 is rejected if either $T^*_{\ell,max} \geq v_\alpha$ or $T^*_{\ell,min} \geq v_\alpha$, where $v_\alpha = -\ln(-\ln(1 - \frac{\alpha}{2}))$.

Single test statistic

Define

$$T_\ell = \sqrt{n} \times \max\{T_{\ell,max}, T_{\ell,min}\} = \sqrt{n} \times \max|\widehat{\mathbf{\Gamma}}_\ell|.$$

Then, under H_0, $(T_\ell - d_p)/c_p \to_d \max\{X_1, X_2\}$ as $n \to \infty$ and $p \to \infty$, where X_1 and X_2 are two independent Gumble random variates.

The limiting distribution of T_ℓ is available.

Joint test for multiple lags

$H_0 : \mathbf{\Gamma}_1 = \cdots = \mathbf{\Gamma}_m = \mathbf{0}$ versus $H_1 : \mathbf{\Gamma}_i \neq \mathbf{0}$ for some $1 \leq i \leq m$. Define

$$\begin{aligned} T_{max}(m) &= \max\{T_{\ell,max} | \ell = 1, \ldots, m\} \\ T_{min}(m) &= \max\{T_{\ell,min} | \ell = 1, \ldots, m\} \end{aligned}$$

and the norming constants $c_{p,m}$ and $d_{p,m}$ as before with $k = mp^2$. Then,

$$\frac{T_{max} - d_{m,p}}{c_{m,p}} \rightarrow_d X, \quad \frac{T_{min} - d_{m,p}}{c_{m,p}} \rightarrow_d X.$$

Empirical sizes: 10,000 realizations

Test Statistic	$p = 300, n = 3000$			$p = 300, n = 5000$			$p = 500, n = 5000$		
	10%	5%	1%	10%	5%	1%	10%	5%	1%
(a) Standard Gaussian distribution									
T_1	8.48	4.08	0.64	9.32	4.62	0.94	9.06	4.76	0.92
T_2	9.10	4.56	0.68	9.20	3.83	0.54	8.48	4.02	0.54
T_3	8.76	4.30	0.64	8.82	4.06	0.62	9.40	4.42	0.78
T_4	9.18	4.52	0.82	9.28	4.22	0.56	9.48	4.36	0.74
T_5	8.86	4.10	0.84	9.66	4.38	0.86	9.38	4.60	0.84
T_{10}	9.00	3.96	0.64	9.26	4.58	0.62	9.62	5.12	1.06
$T(5)$	9.40	4.38	0.86	9.16	4.50	0.68	9.30	4.86	0.96
$T(10)$	9.36	4.96	0.66	9.62	4.12	0.72	9.96	4.76	0.82

Empirical sizes continued

Test Statistic	$p = 300, n = 3000$ 10%	5%	1%	$p = 300, n = 5000$ 10%	5%	1%	$p = 500, n = 5000$ 10%	5%	1%
(b) Cauchy distribution, i.e. t_1									
T_1	8.86	4.72	0.82	8.68	4.16	0.68	9.80	5.16	0.82
T_2	8.90	4.48	0.48	8.66	4.06	0.66	8.78	4.10	0.78
T_3	9.12	4.18	0.86	9.32	4.14	0.72	9.12	4.44	0.62
T_4	9.06	4.54	0.56	9.50	4.80	0.72	9.02	4.08	0.92
T_5	9.04	4.04	0.80	8.76	4.46	0.68	9.06	4.74	0.74
T_{10}	9.42	4.32	0.72	9.06	4.50	0.38	9.40	4.44	0.78
$T(5)$	9.24	4.36	0.66	9.44	4.42	0.62	9.10	4.42	0.84
$T(10)$	9.08	4.20	0.84	8.84	4.10	0.72	9.48	4.52	0.76

Finite-sample adjustments

Adjust for actual data used:

$$\widehat{\boldsymbol{\Gamma}}_\ell^* = \frac{12}{(n-\ell)[(n-\ell)^2-1]} \sum_{t=\ell+1}^{n} (\boldsymbol{r}_t^* - \tilde{r})(\boldsymbol{r}_{t-\ell}^* - \tilde{r})',$$

where $\tilde{\boldsymbol{r}}$ is a p-dimensional constant vector of $(n-\ell+1)/2$, $\boldsymbol{r}_t^*$ is the rank matrix of $\boldsymbol{X}[(\ell+1):n,]$ and $\boldsymbol{r}_{t-\ell}^*$ is the rank matrix of $\boldsymbol{X}[1:(n-\ell),]$.

Bias adjustment of auto-correlations

$$\widehat{\Gamma}_{\ell,ii}^a = \widehat{\Gamma}_{\ell,ii}^* + \frac{n-\ell}{n(n-1)}, \quad i = 1, \ldots, p$$

Finite-sample adjustments continued

Variance adjustment

$$\sqrt{5n^2/[5(n-\ell)-4]} \times \mathrm{vec}(\widehat{\mathbf{\Gamma}}_\ell^a) \to_d N(0, I), \quad n \to \infty.$$

The test statistic for a single lag matrix

$$T_\ell^a = \sqrt{5n^2/[5(n-\ell)-4]} \times \max\{|\widehat{\mathbf{\Gamma}}_{\ell,ij}^a|\}.$$

Block size adjustment: for $n < 3000$

Adjustmented norming constants:

$$c_p^a = [2\ln(p^2\xi)]^{-1/2}, \quad d_p^a = \sqrt{2\ln(p^2\xi)} - \frac{\ln(4\pi) + \ln\ln(p^2\xi)}{2[2\ln(p^2\xi)]^{1/2}},$$

where

$$\xi = \begin{cases} 0.78^{\eta} & \text{if } n < 3000, \\ 1 & \text{otherwise,} \end{cases}$$

where $\eta = \min\{5, (n+p)/n\}$.

Empircal sizes: individual lag, 30,000 realizations

	$n = 100$			$n = 300$			$n = 500$		
p	10%	5%	1%	10%	5%	1%	10%	5%	1%
(a) Lag-1									
10	9.62	3.94	0.35	9.61	4.07	0.36	9.86	4.21	0.42
30	9.37	3.95	0.42	10.3	4.57	0.61	10.6	4.81	0.67
50	9.17	3.90	0.49	10.5	4.71	0.68	10.6	4.66	0.59
100	9.24	4.01	0.46	10.3	4.55	0.63	10.8	5.02	0.79
300	11.5	4.77	0.55	11.7	5.46	0.83	11.6	5.42	0.90
500	12.8	5.39	0.55	13.0	6.01	0.87	12.2	5.68	0.85
(b) Lag-2									
10	9.54	3.86	0.30	9.76	4.12	0.41	9.73	4.10	0.41
30	9.31	3.92	0.39	10.2	4.57	0.60	10.9	4.86	0.63
50	9.21	3.90	0.49	10.5	4.78	0.66	10.7	4.91	0.67
100	9.26	3.85	0.46	10.5	4.85	0.71	10.7	4.94	0.76
300	11.9	4.85	0.52	11.9	5.69	0.80	11.6	5.67	0.92
500	13.2	5.74	0.60	13.4	6.31	0.99	12.3	5.73	0.89

p	$n = 100$ 10%	5%	1%	$n = 300$ 10%	5%	1%	$n = 500$ 10%	5%	1%
(c) Lag-5									
10	9.41	3.88	0.39	10.0	4.07	0.42	10.1	4.16	0.43
30	9.61	4.07	0.40	10.4	4.54	0.56	10.4	4.75	0.64
50	9.02	3.88	0.39	10.4	4.74	0.65	10.8	4.98	0.67
100	9.49	4.11	0.39	10.6	4.90	0.65	10.8	4.97	0.75
300	11.6	5.03	0.51	11.6	5.49	0.79	11.5	5.41	0.83
500	13.1	5.48	0.64	13.0	5.95	0.98	12.7	6.03	0.93
(d) Lag-10									
10	9.38	3.94	0.34	9.56	4.00	0.45	9.59	4.12	0.56
30	9.41	4.07	0.44	10.7	4.77	0.59	10.4	4.76	0.64
50	8.86	3.82	0.34	10.3	4.68	0.64	10.7	4.85	0.73
100	9.24	3.83	0.44	10.5	4.76	0.73	10.9	5.03	0.76
300	11.7	5.14	0.60	11.5	5.20	0.74	12.0	5.61	0.94
500	13.0	5.56	0.60	13.5	6.18	0.93	12.4	6.02	0.92

Number of lags adjustment: for $p < 300$

Effective lags used

$$m^* = \begin{cases} m \times 0.9^{(n+p)/n} & \text{if } p < 300, \\ m & \text{otherwise,} \end{cases}$$

and adjust the scale and location parameters as

$$\begin{aligned} c^a_{p,m} &= [2\ln(p^2\xi m^*)]^{-1/2} \\ d^a_{p,m} &= \sqrt{2\ln(p^2\xi m^*)} - \frac{\ln(4\pi) + \ln\ln(p^2\xi m^*)}{2[2\ln(p^2\xi m^*)]^{1/2}}. \end{aligned}$$

Empirical sizes: joint test, 30,000 realizations

	$n = 100$			$n = 300$			$n = 500$		
p	10%	5%	1%	10%	5%	1%	10%	5%	1%
(a) $m = 5$, the first 5 lags									
10	10.5	4.39	0.46	11.0	4.69	0.63	11.3	4.99	0.67
30	9.58	3.99	0.46	11.5	5.20	0.78	11.8	5.41	0.82
50	9.36	3.93	0.45	11.5	5.38	0.72	11.8	5.35	0.83
100	9.47	3.95	0.49	11.4	5.18	0.82	11.7	5.68	0.83
300	8.93	3.66	0.38	10.6	4.98	0.76	11.2	5.40	0.81
500	9.67	3.99	0.38	12.0	5.66	0.86	11.9	5.59	0.93
(b) $m = 10$, the first 10 lags									
10	10.2	4.37	0.53	11.1	4.98	0.63	11.3	5.13	0.68
30	9.07	3.70	0.48	11.3	5.12	0.72	11.7	5.38	0.77
50	8.54	3.50	0.38	11.6	5.24	0.79	11.8	5.66	0.83
100	8.47	3.54	0.40	11.1	5.04	0.80	12.0	5.65	0.83
300	7.84	3.24	0.31	10.2	4.77	0.71	11.1	5.19	0.82
500	8.35	3.38	0.38	11.5	5.33	0.77	11.7	5.52	0.86

General covariance matrix

1. $p < n$: Apply PCA to $\boldsymbol{x}_t$. Apply tests to the principal component series
2. $p \geq n$: Select $p^* = 0.75n$ series for testing

Selection: $H_0 : \boldsymbol{\Gamma}_1 = \cdots = \boldsymbol{\Gamma}_m = \mathbf{0}$.

- For each $\widehat{\boldsymbol{\Gamma}}_\ell$, define a p-dimensional weight vector $\boldsymbol{w}_\ell = (w_{\ell,1}, \ldots, w_{\ell,p})'$ such that

$$w_{\ell,i} = \max_{1 \leq j \leq p} |\widehat{\Gamma}_{\ell,i.}|$$

- The weights for each component x_{it} is

$$w_i = \sum_{\ell=1}^{m} w_{\ell,i}, \quad i = 1, \ldots, p.$$

- Select the p^* sub-series with the highest weights.

Correlated series

$$\boldsymbol{x}_t = \boldsymbol{A}^{1/2}\epsilon_t$$

where $\boldsymbol{A}$ is a symmetric matrix with

$$A_{ij} = 0.9^{|i-j|}$$

and elements of ϵ_t are iid t_3 random variates.

The results are based on 10,000 realizations.

General covariance matrix: size simulation

p	$n = 100$			$n = 300$			$n = 500$		
	10%	5%	1%	10%	5%	1%	10%	5%	1%
(a) Lag-1									
10	9.25	3.74	0.30	9.89	4.25	0.46	9.80	4.03	0.42
30	9.36	3.90	0.39	10.5	4.79	0.69	10.7	4.76	0.72
50	9.05	3.86	0.41	10.6	4.81	0.64	10.7	4.79	0.63
100	9.24	4.01	0.36	10.6	4.79	0.69	11.3	5.28	0.75
300	9.33	4.07	0.37	11.3	5.40	0.77	11.7	5.42	0.93
500	8.79	3.67	0.39	10.6	4.74	0.69	11.4	5.07	0.84
(b) Lag-2									
10	9.34	3.70	0.33	9.63	3.96	0.37	9.82	4.08	0.43
30	9.62	3.81	0.37	10.3	4.46	0.55	10.8	5.08	0.72
50	9.21	4.11	0.47	10.4	4.89	0.71	10.8	4.72	0.60
100	9.08	3.83	0.51	10.7	4.95	0.71	11.0	5.07	0.70
300	9.12	3.64	0.42	11.4	5.13	0.68	11.4	5.54	0.85
500	8.94	3.67	0.27	11.2	4.99	0.75	11.8	5.45	0.85

General covariance matrix: size simulation continued

p	$n = 100$ 10%	5%	1%	$n = 300$ 10%	5%	1%	$n = 500$ 10%	5%	1%
(c) Lag-5									
10	9.52	3.88	0.26	9.77	4.04	0.45	9.70	4.23	0.43
30	9.51	3.97	0.38	10.4	4.49	0.55	10.5	4.73	0.60
50	9.04	3.87	0.43	10.8	5.00	0.68	10.9	5.05	0.74
100	8.86	3.72	0.36	10.4	4.67	0.70	10.9	5.15	0.74
300	9.38	3.84	0.43	11.3	5.51	0.75	11.0	5.20	0.86
500	9.60	4.15	0.53	11.3	5.26	0.76	12.1	5.64	0.82
(d) Lag-10									
10	9.31	3.70	0.30	9.33	4.02	0.41	9.63	4.09	0.40
30	9.13	3.87	0.40	10.6	4.62	0.66	10.7	4.80	0.69
50	9.28	3.95	0.38	10.5	4.79	0.66	10.5	4.83	0.67
100	9.60	4.30	0.52	10.4	4.82	0.59	11.0	4.94	0.72
300	9.17	3.68	0.36	11.6	5.32	0.86	11.5	5.36	0.84
500	9.29	3.68	0.36	11.4	5.28	0.76	11.7	4.96	0.83

p	$n = 100$			$n = 300$			$n = 500$		
	10%	5%	1%	10%	5%	1%	10%	5%	1%
(a) $m = 5$, the first 5 lags									
10	10.0	4.09	0.39	11.5	4.85	0.58	11.4	4.87	0.61
30	9.48	3.80	0.41	11.2	5.04	0.66	12.1	5.57	0.85
50	9.31	3.93	0.42	11.6	5.44	0.76	11.9	5.48	0.84
100	9.05	3.87	0.37	11.5	5.44	0.81	12.3	5.65	0.89
300	9.38	3.80	0.36	12.1	5.65	0.82	11.4	5.47	0.83
500	9.13	3.74	0.45	12.2	5.75	0.76	11.4	5.20	0.80
(b) $m = 10$, the first 10 lags									
10	9.49	3.81	0.40	11.4	5.08	0.68	11.3	5.31	0.64
30	8.39	3.59	0.39	11.2	5.17	0.67	11.9	5.63	0.83
50	8.43	3.48	0.39	11.5	5.37	0.79	11.7	5.37	0.78
100	8.48	3.63	0.36	11.2	5.19	0.76	11.8	5.63	0.95
300	7.96	3.42	0.27	12.2	5.65	0.76	11.4	5.31	0.86
500	8.01	3.28	0.39	12.3	5.42	0.65	11.1	4.94	0.83

Power study

Two data generating processes

1. VAR(1): $\boldsymbol{x}_t = \boldsymbol{\Phi}\boldsymbol{x}_{t-1} + \boldsymbol{e}_t$
2. VMA(1): $\boldsymbol{x}_t = \boldsymbol{e}_t - \boldsymbol{\Theta}\boldsymbol{e}_{t-1}$

where $\boldsymbol{e}_t$ are iid $N(0, \boldsymbol{I})$, $\boldsymbol{x}_0 = \boldsymbol{0}$.
For VAR(1) models, let ζ is the sparsity parameter. Non-zero elements of $\boldsymbol{\Phi}$ is $N = \lfloor p^2\zeta \rfloor$. For each realization, $\boldsymbol{\Phi}$ is obtained using

- Random sample N from $1 : p^2$ without replacement
- Draw N uniform random variates from $[-0.95, 0.95]$
- Assign (ii) to the N positions in vec($\boldsymbol{\Phi}$).

p	$n = 100$			$n = 300$			$n = 500$		
	ζ	10%	5%	ζ	10%	5%	ζ	10%	5%
(a) Individual test: T_1^a									
10	0.05	90.0	83.3	0.01	63.1	57.0	0.01	72.3	67.3
30	0.05	99.8	99.3	0.01	98.3	96.5	0.005	92.9	90.3
50	0.05	100.	100.	0.01	99.9	99.5	0.005	99.5	99.0
100	0.01	84.4	76.4	0.005	98.3	95.9	0.001	68.5	57.7
300	0.01	100.	100.	0.005	100.	100.	0.001	80.8	72.8
500	0.01	100.	100.	0.005	100.	100.	0.001	96.0	92.5
(b) Individual test: T_2^a									
10	0.05	32.1	24.0	0.01	14.2	8.51	0.01	14.5	9.00
30	0.05	78.3	70.5	0.01	30.9	23.6	0.005	19.2	13.4
50	0.05	99.0	98.0	0.01	46.4	38.1	0.005	30.7	23.6
100	0.01	33.2	26.0	0.005	34.3	27.3	0.001	14.6	8.38
300	0.01	99.9	99.7	0.005	94.0	90.3	0.001	22.1	15.4
500	0.01	100.	100.	0.005	100.	100.	0.001	28.9	21.9

Power study: Joint statistics

p	$n = 100$ ζ	10%	5%	$n = 300$ ζ	10%	5%	$n = 500$ ζ	10%	5%
(c) Joint test: $T^a(5)$									
10	0.05	80.9	72.8	0.01	58.1	52.1	0.01	68.1	63.1
30	0.05	99.0	97.6	0.01	95.0	92.1	0.005	89.0	86.0
50	0.05	100.	100.	0.01	99.1	98.1	0.005	98.3	96.9
100	0.01	73.4	65.2	0.005	93.8	89.4	0.001	54.7	44.9
300	0.01	100.	100.	0.005	100.	100.	0.001	67.7	60.0
500	0.01	100.	100.	0.005	100.	100.	0.001	89.3	84.2
(d) Joint test: $T^a(10)$									
10	0.05	75.9	67.6	0.01	56.0	50.0	0.01	66.3	61.7
30	0.05	97.9	96.1	0.01	92.9	89.3	0.005	87.2	84.0
50	0.05	100.	99.9	0.01	98.4	96.9	0.005	97.3	95.9
100	0.01	67.3	59.1	0.005	90.3	85.8	0.001	49.0	39.8
300	0.01	100.	100.	0.005	100.	100.	0.001	62.8	56.1
500	0.01	100.	100.	0.005	100.	100.	0.001	85.4	80.1

VMA(1) models: power study

$\mathbf{\Theta} = [\Theta_{ij}]$ matrix is defined as

$$\Theta_{ij} = \begin{cases} 0 & \text{if } |i-j| > 1 \\ g_{ij} & \text{Otherwise} \end{cases}$$

with

$$g_{ij} = \begin{cases} 0 & \text{with prob } 1-\omega \\ U[-0.95, 0.95] & \text{with prob } \omega \end{cases}$$

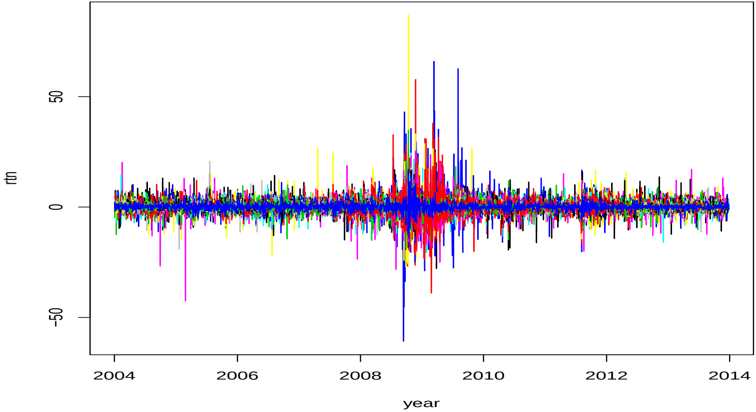
p	$n = 100, \omega = 0.3$			$n = 300, \omega = 0.2$			$n = 500, \omega = 0.1$		
	10%	5%	1%	10%	5%	1%	10%	5%	1%
(a) Indivudal test: T_1^a									
10	95.9	89.8	63.4	99.8	99.6	98.6	97.8	96.7	94.1
30	66.5	48.7	20.3	1.00	99.8	98.2	99.8	99.6	98.8
50	43.4	28.3	9.48	99.7	98.5	90.2	99.8	99.5	98.0
100	27.4	16.4	4.65	91.9	83.3	57.9	98.5	96.4	86.7
300	20.7	12.2	3.48	51.9	38.3	19.1	62.4	50.6	31.0
500	20.8	11.9	3.54	43.7	31.1	13.9	49.6	37.5	21.0
(b) Individual test: T_2^a									
10	9.62	3.84	0.33	9.62	4.00	0.48	10.4	4.67	0.57
30	9.64	3.85	0.40	10.3	4.42	0.47	10.8	5.03	0.70
50	9.34	4.16	0.40	10.1	4.49	0.69	10.8	4.88	0.69
100	9.26	3.69	0.43	10.7	4.79	0.55	11.0	5.16	0.71
300	9.51	3.89	0.45	10.8	4.99	0.70	11.4	5.45	0.81
500	9.23	3.69	0.49	11.5	5.35	0.82	12.3	5.83	0.91

Joint test: power

p	$n = 100, \omega = 0.3$			$n = 300, \omega = 0.2$			$n = 500, \omega = 0.1$		
	10%	5%	1%	10%	5%	1%	10%	5%	1%
	(c) Joint test: $T^a(5)$								
10	86.0	75.7	47.4	99.4	99.1	97.9	96.3	95.3	92.7
30	42.2	29.0	11.2	99.7	99.0	94.6	99.6	99.2	97.9
50	26.6	15.7	4.73	97.1	93.7	78.7	99.3	98.7	95.5
100	17.2	9.21	2.22	76.7	65.0	42.3	94.3	90.1	77.0
300	14.9	7.60	1.86	36.7	25.8	12.3	46.0	36.0	22.7
500	15.0	7.87	2.24	31.7	20.8	8.57	35.2	26.3	14.4
	(d) Joint test: $T^a(10)$								
10	79.0	67.4	40.5	99.3	98.8	97.2	95.7	94.6	92.0
30	33.9	22.4	8.31	99.1	98.1	91.7	99.3	98.8	97.3
50	20.5	11.6	3.48	94.6	89.4	72.2	98.9	97.8	93.8
100	13.9	7.22	1.53	68.2	56.6	36.3	91.2	85.6	72.5
300	12.2	6.20	1.42	31.2	21.9	9.99	40.2	31.6	19.9
500	12.5	6.31	1.79	26.4	17.0	6.98	31.0	22.5	12.2

An example

- Daily returns, in percentages, of 92 component series of the S&P 100 index from January 2, 2004 to December 31, 2013.
- Sample size $n = 2517$


50
0
-50
rtn
2004
2006
2008
2010
2012
2014
year

	Individual Test				Joint test	
T_1^a	T_2^a	T_3^a	T_4^a	T_5^a	$T^a(5)$	$T^a(10)$
(a) Original return series						
5.86	3.86	4.11	3.92	3.68	5.86	5.86
(b) Residuals of the VAR(1) model						
3.74	3.80	3.82	3.81	3.80	3.82	4.34

critical values: (10%,5%,1%)

1. Individual lag: 4.34, 4.51, 4.89
2. Joint 5 lags: 4.65, 4.81, 5.17
3. Joint 10 lags: 4.79, 4.94, 5.29.